

ABSTRACT

PONGPRASERT, SUCHADA. $D_6^{(1)}$ -Geometric Crystal Corresponding to the Dynkin Spin Node $i = 6$ and Its Ultra-discretization. (Under the direction of Dr. Kailash C. Misra).

Kac-Moody Lie algebras were discovered independently by Victor G. Kac and Robert V. Moody around 1968. These algebras are infinite dimensional analogs of finite dimensional semisimple Lie algebras. There are three types of Kac-Moody Lie algebras: finite type, affine type, and indefinite type. These algebras, especially affine Lie algebras, have various applications in physics and mathematics. In 1985, Michio Jimbo and Vladimir G. Drinfeld introduced the notion of the quantum group which are deformations of the universal enveloping algebras of symmetrizable Kac-Moody Lie algebras. In 1988, George Lusztig showed that the representation theory of a Kac-Moody Lie algebra is parallel to that of its quantum group in the generic case. In 1990, Masaki Kashiwara developed the crystal basis as a nice combinatorial tool to study the irreducible highest weight modules over a quantum group. Then, in 1999, the notion of geometric crystal is introduced by Arkady Berenstein and David Kazhdan as a geometric analog to Kashiwara's crystal (or algebraic crystal). A remarkable relation between positive geometric crystals and algebraic crystals is the ultra-discretization functor \mathcal{UD} between them. Applying this functor, positive rational functions are transferred to piecewise linear functions.

In 2008, Masaki Kashiwara, Toshiki Nakashima and Masato Okado gave a conjecture that for each affine Lie algebra \mathfrak{g} and each Dynkin index $i \in I \setminus \{0\}$, there exists a positive geometric crystal $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ whose ultra-discretization $\mathcal{UD}(\mathcal{V})$ is isomorphic to the limit B^∞ of a coherent family of perfect crystals $\{B^l\}_{l \geq 1}$ for the Langlands dual \mathfrak{g}^L . So far this conjecture has been proved for the Dynkin index $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ by Masaki Kashiwara, Toshiki Nakashima and Masato Okado in 2008, $\mathfrak{g} = G_2^{(1)}$ by Toshiki Nakashima in 2010, $\mathfrak{g} = D_4^{(3)}$ by Mana Igarashi, Kailash C. Misra and Toshiki Nakashima in 2012. For $\mathfrak{g} = A_n^{(1)}, i > 1$, Kailash C. Misra and Toshiki Nakashima showed the conjecture to be true in 2018. For $\mathfrak{g} = D_5^{(1)}$, this conjecture has been shown to hold for $i = 5$ by Mana Igarashi, Kailash C. Misra and myself in 2019.

In this thesis we prove the conjecture for $\mathfrak{g} = D_6^{(1)}$ and Dynkin index $i = 6$, the spin node. We construct a positive geometric crystal $\mathcal{V}(D_6^{(1)})$ in the level zero fundamental spin module $W(\omega_6)$, and for $l \geq 1$, we coordinatize the perfect crystal $B^{6,l}$ for $D_6^{(1)}$ given by Seok-Jin Kang, Masaki Kashiwara, Kailash C. Misra, Tetsuji Miwa, Toshiki Nakashima and Atsushi Nakayashiki in 1992 and define an explicit 0-action. Then we show that the family of perfect crystals $\{B^{6,l}\}_{l \geq 1}$ is a coherent family and determine its limit $B^{6,\infty}$. Finally, we ultra-discretize the positive geometric crystal $\mathcal{V}(D_6^{(1)})$ and show that it is isomorphic to $B^{6,\infty}$ as crystals.

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$D_6^{(1)}$ -Geometric Crystal Corresponding to the Dynkin
Spin Node $i = 6$ and Its Ultra-discretization

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2019

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DEDICATION

To Jo and Sky.

BIOGRAPHY

Suchada Pongprasert was born in Ratchaburi, Thailand, in 1987. She received a scholarship from the Development and Promotion in Science and Technology Talents Project (DPST) in 2002 for high school, undergraduate and graduate studies in Thailand. In 2009, she earned her Bachelor of Science in Mathematics from Silpakorn University. In 2010, she attended the Master program in Mathematics at Chulalongkorn University for one semester before she received a scholarship from the Ministry of Science and Technology, Royal Thai Government to study abroad. She attended North Carolina State University where she obtained her Master of Science in Mathematics in 2014 and Master of Operations Research in 2016 before completing her Doctor of Philosophy in Mathematics in 2019. She will then return to Thailand to begin her career as a lecturer and researcher at Srinakharinwirot University.

ACKNOWLEDGEMENTS

Firstly, I would like to thank my advisor, Dr. Kailash C. Misra. Without his help and never-ending patience, I would have accomplished nothing. I am deeply grateful for the time he took to teach, guide, and mentor me, both personally and academically through the rough time of my graduate career. He was truly the best advisor and mentor I could have asked for and he is going to be my role model throughout the rest of my career.

I would also like to thank my committee members for their support and flexibility: Dr. Naihuan Jing, for his dedicated time and helpful advisement, Dr. Mohan S. Putcha, for great algebra classes and having an answer to all of my questions, and Dr. Ernest L. Stitzinger, for always having a smile, and all of those thoughtful and encouraging talks.

In addition, I would like to thank my family for their unconditional love and support, especially to my husband Kittipong Pongprasert, my son Napan Sky Pongprasert, my mother Suwanna Mookayapanit, my father Channarong Pathomkamnoed, my stepfather Chainarong Chokmor, my brother Supachok Chokmor as well as my parents in-law Wichian and Noi Pongprasert. I know I would not be where I am today without each and every one of you.

Last but not least, I am very thankful for all of my friends, both in the math department and community, both in the US and in Thailand who have supported me and who have helped create wonderful memories for the past seven years. I give my thanks for all of you (you know who you are).

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CHAPTER

1

INTRODUCTION

In the late eighteenth century, Sophus Lie was trying to develop a Galois theory for differential equations. This led to his discovery of continuous transformation groups and infinitesimal groups which today are what we call Lie groups and Lie algebras, respectively. By 1893, Lie (together with Friedrich Engel) had completed the final, third volume of the massive treatise *Theorie der Transformationsgruppen*. Also by 1890, William Killing had succeeded in classifying the finite dimensional complex simple Lie algebras. Killing's work was rigorously treated and extended in Élie Cartan's 1894 thesis. The finite dimensional simple Lie algebras over \mathbb{C} fall into four families A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$) respectively corresponding to the groups $SL(n+1, \mathbb{C})$, $SO(2n+1, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(2n, \mathbb{C})$, and five exceptional ones denoted by G_2, F_4, E_6, E_7, E_8 with dimensions 14, 52, 78, 133, 248 respectively. Later, Claude Chevalley extended this classification to algebras over fields of characteristic 0 (cf. [2]). Through the works of Chevalley and Jean-Pierre Serre, we learn that we can realize these algebras in terms of generators and relations (cf. [6], [10], [16]).

An $n \times n$ integral matrix $A = (a_{ij})$ is a generalized Cartan matrix (GCM) if $a_{ii} = 2$, $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. A GCM A is indecomposable if it is not equivalent to a matrix in block form and A is symmetrizable if there exists a nonsingular diagonal matrix D such that DA is symmetric. Indecomposable symmetrizable GCMs are classified into three types: finite, affine, and indefinite. If we delete the first row and the first column of an indecomposable affine GCM A , the remaining matrix is a Cartan matrix for a finite

dimensional simple Lie algebra. In 1967-1968, Victor Kac ([14], [15]) and Robert Moody ([31], [32]) independently defined Lie algebras associated with any GCM A via generators and relations that we now call Kac-Moody algebras. Finite type GCMs are positive definite and produce finite dimensional simple Kac-Moody algebras while the other types are positive semidefinite and negative definite GCMs and yield infinite dimensional Kac-Moody algebras. Kac-Moody algebras have grown into an important field due to their numerous applications in physics, including conformal field theory and statistical physics, as well as in various fields of mathematics, including combinatorics, algebraic geometry and number theory.

In this thesis, we focus on the affine Kac-Moody algebra $D_6^{(1)} = \widehat{\mathfrak{so}}(12, \mathbb{C}) = \mathfrak{so}(12, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C}d$ where \mathbf{c} is the canonical central element and d is a degree derivation [16]. The index set is $I = \{0, 1, 2, 3, 4, 5, 6\}$ and its GCM is $A = (a_{ij})_{i,j \in I}$ where $a_{ii} = 2, a_{j,j+1} = -1 = a_{j+1,j}$ for $j = 1, 2, 3, 4, a_{02} = a_{20} = a_{46} = a_{64} = -1$ and $a_{ij} = 0$ otherwise. In Chapter 2, we review some of the fundamental definitions and theorems of Lie algebras, Kac Moody algebras as well as their representation theory.

In 1985, Michio Jimbo [13] and Vladimir Drinfel'd [3] independently introduced the notion of the quantum group $U_q(\mathfrak{g})$ as a q -deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a symmetrizable Kac-Moody algebra \mathfrak{g} . Then in 1988, George Lusztig [27] showed that for the generic q , the $U_q(\mathfrak{g})$ representation theory parallels that of $U(\mathfrak{g})$ representation theory and hence provides a new tool for studying representations of Kac-Moody algebras. Around 1990, Masaki Kashiwara [22] (also see ([23], [28])) introduced the notion of crystal bases which can be viewed as a basis at $q = 0$ and provide combinatorial tools to study the structure of integrable $U_q(\mathfrak{g})$ -module. In Chapter 3, we review some basic definitions and properties about quantum groups and crystal bases.

For a dominant weight λ of level $l = \lambda(\mathbf{c})$, Kashiwara defined the crystal base $(L^q(\lambda), B^q(\lambda))$ [22] for the integrable highest weight $U_q(\mathfrak{g})$ -module $V^q(\lambda)$. As shown in [20], the crystal $B^q(\lambda)$ can be realized as a set of paths in the semi-infinite tensor product $\cdots \otimes B^l \otimes B^l \otimes B^l$ where B^l is a perfect crystal of level l . This is called the path realization of the crystal $B^q(\lambda)$. A perfect crystal is indeed a crystal for certain finite dimensional module called Kirillov-Reshetikhin module (KR-module for short) of the quantum affine algebra $U_q(\mathfrak{g})$ ([7], [8], [26]). The KR-modules are parametrized by two integers (i, l) where $i \in I \setminus \{0\}$ and l any positive integer. Let $\{\varpi\}_{i \in I \setminus \{0\}}$ be the set of level 0 fundamental weights [24]. Goro Hatayama et al ([7], [8]) conjectured that any KR-module $W(l\varpi_i)$ admit a crystal base $B^{i,l}$ in the sense of Kashiwara and $B^{i,l}$ is perfect if l is a multiple of $\max(1, \frac{2}{(\alpha_i, \alpha_i)})$ where $\alpha_i \in \Pi$, the set of simple roots of \mathfrak{g} . This conjecture has been proved for quantum affine algebras $U_q(\mathfrak{g})$ of classical types ([4], [5], [36]). When $\{B^{i,l}\}_{l \geq 1}$ is a coherent family of perfect crystals [19], we denote its limit by $B^{i,\infty}$. In Chapter 4, we recall the concepts of quantum affine algebra, perfect crystals and the path realization.

In 1999, the notion of geometric crystal was introduced by Arkady Berenstein and David Kazhdan [1] as a geometric analog to Kashiwara's crystal (or algebraic crystal) [22]. In fact, geometric crystal is defined in [1] for reductive groups and is extended to general Kac-Moody groups in [33]. Let I be the index set of simple roots of \mathfrak{g} . A geometric crystal consists of a variety X , rational \mathbb{C}^\times -actions $e_i : \mathbb{C}^\times \times X \rightarrow X$ and rational functions $\gamma_i, \varepsilon_i : X \rightarrow \mathbb{C}$ ($i \in I$) which satisfy certain conditions (see Definition 5.2.2). A geometric crystal is said to be a positive geometric crystal if it admits a positive structure (see Definition 5.3.2).

The positive geometric crystals are related to Kashiwara crystals via the ultra-discretization functor \mathcal{UD} ([1], [33]) which transforms positive rational functions to piecewise-linear functions by the simple correspondence:

$$x \times y \mapsto x + y, \quad \frac{x}{y} \mapsto x - y, \quad x + y \mapsto \max\{x, y\}.$$

It was conjectured in [25] that for each affine Lie algebra \mathfrak{g} and each Dynkin index $i \in I \setminus \{0\}$, there exists a positive geometric crystal $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ whose ultra-discretization $\mathcal{UD}(\mathcal{V})$ is isomorphic to the limit B^∞ of a coherent family of perfect crystals for the Langlands dual \mathfrak{g}^L . If \mathfrak{g} is simply laced then the Langland dual is \mathfrak{g} itself. So far this conjecture has been proved for the Dynkin index $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ [25], $\mathfrak{g} = D_4^{(3)}$ [11] and $\mathfrak{g} = G_2^{(1)}$ [35]. For $i > 1$, this conjecture has been shown to hold for $\mathfrak{g} = A_n^{(1)}$ ([29], [30]). For $\mathfrak{g} = D_5^{(1)}$, we showed the conjecture to be true for $i = 5$ [12].

In this thesis we prove the conjecture in [25] for $\mathfrak{g} = D_6^{(1)}$ and Dynkin index $i = 6$, the spin node. Let \mathfrak{g}_i denote the subalgebra of \mathfrak{g} with index set $I_i = I \setminus \{i\}$. In Chapter 6, we construct a positive geometric crystals $\mathcal{V}_1 = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k = 1, 2, 3, 4, 5, 6\}$ and $\mathcal{V}_2 = \{V_2(y), \bar{e}_k^c, \bar{\gamma}_k, \bar{\varepsilon}_k \mid k = 0, 2, 3, 4, 5, 6\}$ for $D_6 = \mathfrak{g}_0, \mathfrak{g}_1$ respectively in the fundamental representation $W(\omega_6)$. Then we define a birational isomorphism $\bar{\sigma}$ between \mathcal{V}_1 and \mathcal{V}_2 , and using this isomorphism we define the 0-action on the geometric crystal \mathcal{V}_1 and show that it is a positive geometric crystal $\mathcal{V}(D_6^{(1)})$ for the quantum affine algebra $U_q(D_6^{(1)})$ (Theorem 6.2.6). In Chapter 7, for $l \geq 1$ we coordinatize the perfect crystal $B^{6,l}$ for $D_6^{(1)}$ given in [21] and define an explicit 0-action. Then we show that the family of perfect crystals $\{B^{6,l}\}_{l \geq 1}$ is a coherent family and determine its limit $B^{6,\infty}$ (Theorem 7.0.1). Finally in Chapter 8, we ultra-discretize the positive geometric crystal $\mathcal{V}(D_6^{(1)})$ and show that it is isomorphic to $B^{6,\infty}$ as crystals (Theorem 8.0.1). This proves the conjecture of Masaki Kashiwara, Toshiki Nakashima and Masato Okado [25] in this case.

CHAPTER

2

KAC-MOODY ALGEBRAS AND INTEGRABLE REPRESENTATIONS

We begin this chapter with the definition of a Lie algebra, and recall background information about Lie algebra representation theory. We then review the basic theory of a particular class of Lie algebras called Kac-Moody algebras and their representation theory.

2.1 Lie algebras

Definition 2.1.1. A *Lie Algebra* \mathbf{L} is a vector space over the field \mathbb{C} together with an operation (called the *bracket*), $[,] : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{L}$ such that for all $x, y, z \in \mathbf{L}$ and $a, b \in \mathbb{C}$,

1. $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$,
2. $[x, x] = 0$,
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Note that the first two axioms in Definition 2.1.1 imply that the bracket operation is *anticommutative*:

$$[x, y] = -[y, x] \text{ for all } x, y \in \mathbf{L}.$$

A subspace \mathbf{L}' of a Lie algebra \mathbf{L} is a (*Lie*) *subalgebra* of \mathbf{L} if $[x, y] \in \mathbf{L}'$ for all $x, y \in \mathbf{L}'$. A subalgebra I of \mathbf{L} is an *ideal* of \mathbf{L} if $[x, y] \in I$ for all $x \in \mathbf{L}$, $y \in I$. If I is an ideal of \mathbf{L} , the quotient space \mathbf{L}/I becomes a Lie algebra with the bracket defined by

$$[x + I, y + I] = [x, y] + I \text{ for all } x, y \in \mathbf{L}.$$

A Lie algebra \mathbf{L} is *simple* if it is nonabelian, i.e. $[\mathbf{L}, \mathbf{L}] \neq \{0\}$, and its only ideals are $\{0\}$ and itself. A Lie algebra \mathbf{L} is *solvable* if $\mathbf{L}^{(m)} = \{0\}$ for some $m \in \mathbb{Z}_{\geq 0}$ where $\mathbf{L}^{(0)} = \mathbf{L}$ and $\mathbf{L}^{(m)} = [\mathbf{L}^{(m-1)}, \mathbf{L}^{(m-1)}]$ for $m \in \mathbb{Z}_{>0}$. The $\mathbf{L}^{(m)}$ are ideals of \mathbf{L} and the series

$$\mathbf{L} \supseteq \mathbf{L}^{(1)} \supseteq \mathbf{L}^{(2)} \supseteq \mathbf{L}^{(3)} \supseteq \dots$$

is called the *derived series* of \mathbf{L} . If \mathbf{L} contains no nonzero proper solvable ideals, then we say \mathbf{L} is *semisimple*. Equivalently, \mathbf{L} is semisimple if we can write it as a direct sum of simple ideals. Note that a simple Lie algebra is also a semisimple Lie algebra but the converse is not necessarily true.

Let \mathbf{L} and \mathbf{L}' be Lie algebras. A *homomorphism* of \mathbf{L} into \mathbf{L}' is a linear map $\phi : \mathbf{L} \rightarrow \mathbf{L}'$ satisfying

$$\phi([x, y]) = [\phi(x), \phi(y)] \text{ for all } x, y \in \mathbf{L}.$$

The *kernel* of ϕ , $\ker \phi = \{x \in \mathbf{L} \mid \phi(x) = 0\}$ is an ideal of \mathbf{L} . Note that every ideal of \mathbf{L} is the kernel of the canonical homomorphism $\pi : \mathbf{L} \rightarrow \mathbf{L}/I$.

Definition 2.1.2. An *associative algebra* A over \mathbb{C} is a ring A which can also be viewed as a vector space over \mathbb{C} , such that the underlying addition and the zero element 0 are the same in the ring and vector space, and $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$ for all $x, y \in A, a \in \mathbb{C}$.

Example 2.1.3. 1. Let A be an associative algebra over \mathbb{C} and define the bracket on A by

$$[a, b] = ab - ba \text{ for all } a, b \in A.$$

Then the bracket operation satisfies the anticommutativity and the Jacobi identity and hence the pair $(A, [,])$ becomes a Lie algebra.

2. Let V be a vector space over \mathbb{C} . Then the set of all linear transformations on V denoted by $\text{End } V$ with the bracket defined by

$$[x, y] = xy - yx \text{ for all } x, y \in \text{End } V$$

is a Lie algebra called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}(V)$. If $V = \mathbb{C}^n$, the general linear Lie algebra is denoted by $\mathfrak{gl}(n, \mathbb{C})$.

3. Let \mathbf{L} be a subspace of $\mathfrak{gl}(n, \mathbb{C})$ consisting of $n \times n$, trace-zero matrices. Then \mathbf{L} is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ called *special linear Lie algebra*. This Lie algebra is simple and denoted by $\mathfrak{sl}(n, \mathbb{C})$ (or A_{n-1}). Note that the set $\{E_{ii} - E_{i+1,i+1}, E_{jk} \mid 1 \leq i \leq n-1, 1 \leq j \neq k \leq n\}$, where E_{ij} denote the $n \times n$ matrix with a one in the (i, j) -entry and zeros elsewhere is a basis for $\mathfrak{sl}(n, \mathbb{C})$ and

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

4. Let $n = 2l$ and $J = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. Then

$$\mathbf{L} = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid JA + A^T J = 0\}$$

with $[A, B] = AB - BA$ for all $A, B \in \mathbf{L}$ is a Lie algebra which is denoted by $\mathfrak{so}(2l, \mathbb{C})$ (or D_l) and called (*even*) *special orthogonal Lie algebra*

2.2 Representations of Lie algebras

Definition 2.2.1. Let \mathbf{L} be a Lie algebra and let V be a vector space over \mathbb{C} .

1. A *representation* of \mathbf{L} on V is a Lie algebra homomorphism $\varphi : \mathbf{L} \rightarrow \mathfrak{gl}(V)$.
2. A vector space V is called an **L-module** if there is an operation $\mathbf{L} \times V \rightarrow V$, denoted by $(x, v) \mapsto x \cdot v$, such that for all $x, y \in \mathbf{L}$, $u, v \in V$ and $a, b \in \mathbb{C}$
 - (a) $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$,
 - (b) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,
 - (c) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.
3. A subspace W of an **L-module** V is called a **submodule** of V if

$$x \cdot W \subset W \text{ for all } x \in \mathbf{L}.$$

4. An **L-module** V is *irreducible* if the only submodules of V are $\{0\}$ and V itself.
5. Let V and W be two **L-modules**. A linear transformation $\phi : V \rightarrow W$ is a **L-module homomorphism** if $\phi(x \cdot v) = x \cdot \phi(v)$ for all $x \in \mathbf{L}$, $v \in V$.

Suppose V is an \mathbf{L} -module. Then the map $\varphi : \mathbf{L} \rightarrow \mathfrak{gl}(V)$ defined by $\varphi(x)(v) = x \cdot v$ for all $x \in \mathbf{L}$ and $v \in V$ is a representation. Conversely, if $\varphi : \mathbf{L} \rightarrow \mathfrak{gl}(V)$ is a representation, then by defining $x \cdot v = \varphi(x)(v)$ for all $x \in \mathbf{L}$ and $v \in V$, we see that V is an \mathbf{L} -module. Due to this equivalence, representations and \mathbf{L} -modules can be used interchangeably.

Example 2.2.2. 1. Let $\mathbf{L} = \mathfrak{gl}(n, \mathbb{C})$ be the general linear Lie algebra and $V = \mathbb{C}^n$. Define a map $\mathbf{L} \times V \rightarrow V$, $(x, v) \mapsto xv$, by matrix multiplication. Then V is an \mathbf{L} -module called the *vector representation* of $\mathfrak{gl}(n, \mathbb{C})$.

2. Let \mathbf{L} be a Lie algebra. Define a map $\text{ad} : \mathbf{L} \rightarrow \mathfrak{gl}(L)$ by

$$\text{ad } x(y) = [x, y] \text{ for all } x, y \in \mathbf{L}.$$

Then ad is a Lie algebra homomorphism called the *adjoint representation* of \mathbf{L} .

3. Let V and W be \mathbf{L} -modules, then $V \oplus W$ is an \mathbf{L} -module with action given by

$$x \cdot (v + w) = x \cdot v + x \cdot w \text{ for all } x \in \mathbf{L}, v \in V, w \in W.$$

4. Let V and W be \mathbf{L} -modules, then $V \otimes W$ is an \mathbf{L} -module with action given by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \text{ for all } x \in \mathbf{L}, v \in V, w \in W.$$

5. Let V be an \mathbf{L} -module over field \mathbb{C} , and consider the dual space $V^* = \{f : V \rightarrow \mathbb{C} \mid f \text{ is linear}\}$. Then V^* is an \mathbf{L} -module under the action $x \cdot f$ satisfying

$$(x \cdot f)(v) = -f(x \cdot v) \text{ for all } v \in V, x \in \mathbf{L}, f \in V^*.$$

Note that not every Lie algebra is an associative algebra. However, we can construct an associative algebra from a Lie algebra, called a universal enveloping algebra.

Definition 2.2.3. Let \mathbf{L} be a Lie algebra. A *universal enveloping algebra* of \mathbf{L} is a pair $(U(\mathbf{L}), \iota)$ such that $U(\mathbf{L})$ is an associative algebra over \mathbb{C} with unity, $\iota : \mathbf{L} \rightarrow U(\mathbf{L})$ is a linear map satisfying

$$\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x) \text{ for all } x, y \in \mathbf{L}$$

and satisfying the following universal property. For any associative algebra \mathcal{A} and any linear map $j : \mathbf{L} \rightarrow \mathcal{A}$ satisfying

$$j([x, y]) = j(x)j(y) - j(y)j(x) \text{ for all } x, y \in \mathbf{L},$$

there exists a unique homomorphism of associative algebra $\phi : U(\mathbf{L}) \rightarrow \mathcal{A}$ such that $\phi \circ \iota = j$. The universal property of $U(\mathbf{L})$ can be shown in the commutative diagram as follows.

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\iota} & U(\mathbf{L}) \\ & \searrow j & \swarrow \exists! \phi \\ & \mathcal{A} & \end{array}$$

Figure 2.1: The universal property of the universal enveloping algebra

The universal enveloping algebra can be constructed as $U(\mathbf{L}) = \mathcal{T}(\mathbf{L})/\mathcal{I}$ where $\mathcal{T}(\mathbf{L}) = \bigoplus_{k=0}^{\infty} \mathbf{L}^{\otimes k}$ is the tensor algebra of \mathbf{L} and \mathcal{I} is the two-sided ideal of $\mathcal{T}(\mathbf{L})$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathbf{L}$. The linear map $\iota : \mathbf{L} \rightarrow U(\mathbf{L})$ is constructed by composing the natural maps $\mathbf{L} \hookrightarrow \mathcal{T}(\mathbf{L})$ and $\pi : \mathcal{T}(\mathbf{L}) \rightarrow U(\mathbf{L})$. Thus, we can view $U(\mathbf{L})$ as the maximal associative algebra over \mathbb{C} with unity generated by \mathbf{L} satisfying the relation

$$xy - yx = [x, y] \text{ for all } x, y \in \mathbf{L}.$$

The following theorem known as the *Poincare-Birkhoff-Witt (PBW) Theorem* proves that ι is injective which allows us to view \mathbf{L} as a subspace of $U(\mathbf{L})$ and also provides a basis for $U(\mathbf{L})$.

Theorem 2.2.4. [9]

1. The map $\iota : \mathbf{L} \rightarrow U(\mathbf{L})$ is injective.
2. Let $\{x_i \mid i \in I\}$ be an ordered basis for \mathbf{L} where I is an index set. Then the set $\{x_{i_1} x_{i_2} \cdots x_{i_k} \mid i_1 \leq i_2 \leq \cdots \leq i_k, k \geq 0\}$ forms a basis for $U(\mathbf{L})$.

Suppose V is an \mathbf{L} -module. We can define the action of $U(\mathbf{L})$ on V inductively by setting

$$(x_1 x_2 \cdots x_r) \cdot v = x_1 \cdot ((x_2 \cdots x_r) \cdot v) = x_1 \cdot (x_2 \cdots (x_r \cdot v))$$

for all $x_1, \dots, x_r \in \mathbf{L}$, $v \in V$ which implies that a representation of \mathbf{L} naturally extends to that of $U(\mathbf{L})$. Conversely, by PBW Theorem, a representation of $U(\mathbf{L})$ is also a representation of \mathbf{L} . Hence, the representation theory of a Lie algebra \mathbf{L} is parallel to the representation theory of its universal enveloping algebra $U(\mathbf{L})$.

2.3 Kac-Moody algebras

In this section, we discuss Kac-Moody algebras which are generalization of finite dimensional semisimple Lie algebras and may or may not be infinite dimensional. These algebras can be constructed from a special matrix called a generalized Cartan matrix.

Definition 2.3.1. Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j \in I}$ with integer entries is a **generalized Cartan matrix (GCM)** if it satisfies

1. $a_{ii} = 2$ for all $i \in I$,
2. $a_{ij} \leq 0$ if $i \neq j$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Definition 2.3.2. 1. A GCM A is **symmetrizable** if there is a diagonal matrix $D = \text{diag}(d_i)_{i \in I}$ such that $d_i \in \mathbb{Z}_{>0}$ and DA is symmetric.

2. A GCM A is **indecomposable** if for every pair of nonempty subsets $I_1, I_2 \subset I$ such that $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ with $a_{ij} \neq 0$.

Theorem 2.3.3. [9] Let $A = (a_{ij})_{i,j \in I}$ be an indecomposable $n \times n$ GCM. Then one and only one of the following three possibilities hold for both A and A^T .

(Finite) $\det A \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.

(Affine) $\text{corank } A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.

(Indefinite) There exists $u > 0$ such that $Au < 0$; $Av \geq 0$ and $v \geq 0$ implies $v = 0$.

Here u, v are column vectors in \mathbb{R}^n and we say $u > 0$ (respectively, $u \geq 0$) if $u_i > 0$ (respectively, $u_i \geq 0$) for all $i = 1, \dots, n$. Also, A is said to be **finite** (respectively, **affine** or **indefinite**) type if A satisfies the corresponding condition.

Definition 2.3.4. Let A be an indecomposable GCM. The **Dynkin diagram** of A is the diagram consists of vertices indexed by I and edges defined using the following rule, for $i \neq j$

- If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertices i and j are connected with $|a_{ij}|$ edges and has an arrow pointing toward i if $|a_{ij}| > 1$.
- If $a_{ij}a_{ji} > 4$, then the vertices i and j are connected with a bold edge labeled with the ordered pair $(|a_{ij}|, |a_{ji}|)$.

Definition 2.3.5. The **Cartan datum** associated with the symmetrizable GCM $A = (a_{ij})_{i,j \in I}$ is a quintuple $(A, \Pi, \check{\Pi}, P, \check{P})$ where

$\check{P} = \text{span}_{\mathbb{Z}}\{\{\check{\alpha}_1, \dots, \check{\alpha}_n\} \cup \{d_s | s = 1, \dots, |I| - \text{rank } A\}\}$ is a free abelian group

of rank $2|I| - \text{rank } A$ called the **coweight lattice**,

Define $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ to be the complex extension of \check{P} called the **Cartan subalgebra**,

$P = \{\lambda \in \mathfrak{t}^* | \lambda(\check{P}) \subset \mathbb{Z}\}$ is called the *weight lattice*,

$\check{\Pi} = \{\check{\alpha}_1, \dots, \check{\alpha}_n\} \subset \mathfrak{t}$ is called the set of *simple coroots*, and

$\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{t}^*$ is called the set of *simple roots* which satisfy $\alpha_j(\check{\alpha}_i) = a_{ij}$ and $\alpha_j(d_s) = \delta_{sj}$.

Definition 2.3.6. The *fundamental weights* $\Lambda_i \in \mathfrak{t}^*$ are linear functionals on \mathfrak{t} given by

$$\Lambda_i(\check{\alpha}_j) = \delta_{ij} \text{ and } \Lambda_i(d_s) = 0$$

for $i, j \in I$ and $s = 1, 2, \dots, |I| - \text{rank } A$.

With the Cartan datum, we can construct a Kac-Moody algebra as follows.

Definition 2.3.7. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ associated with the Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is a Lie algebra with generators e_i, f_i ($i \in I$) and $h \in \check{P}$ satisfying the following relations.

1. $[h, h'] = 0$ for $h, h' \in \check{P}$,
2. $[e_i, f_i] = \delta_{ij} \check{\alpha}_i$,
3. $[h, e_i] = \alpha_i(h) e_i$ for $h \in \check{P}$,
4. $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \check{P}$,
5. $(\text{ad } e_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$,
6. $(\text{ad } f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$.

The generators e_i and f_i are called *Chevalley generators*. Also, the relation (1)-(4) are known as the *Weyl relations* and relation (5)-(6) are known as the *Serre relations*.

If A is a GCM of finite (respectively, affine, indefinite) type, then we call the quintuple $(A, \Pi, \check{\Pi}, P, \check{P})$ an *finite* (respectively, *affine*, *indefinite*) *Cartan datum* and to each Cartan datum we can associate the *finite* (respectively, *affine*, *indefinite*) *Kac-Moody algebra* \mathfrak{g} .

Definition 2.3.8. Let \mathfrak{g} be an affine Kac-Moody algebra with index set $I = \{0, 1, \dots, n\}$. Then there exists a vector $u = (a_0, a_1, \dots, a_n)^T$ such that $Au = 0$, $a_i \in \mathbb{Z}_{>0}$ and $\text{gcd}(a_0, a_1, \dots, a_n) = 1$. The element $\delta = \sum_{i=0}^n a_i \check{\alpha}_i$ is called the *null root*. Dually, there exists a vector $v = (\check{a}_0, \check{a}_1, \dots, \check{a}_n)^T$ such that $A^T v = 0$, $\check{a}_i \in \mathbb{Z}_{>0}$ and $\text{gcd}(\check{a}_0, \check{a}_1, \dots, \check{a}_n) = 1$. The element $\mathbf{c} = \sum_{i=0}^n \check{a}_i \check{\alpha}_i$ is called the *canonical central element*.

Example 2.3.9. The special linear algebra $A_{n-1} = \mathfrak{sl}(n, \mathbb{C})$ is a Kac-Moody algebra and has the following GCM and corresponding Dynkin diagram.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & -1 & 2 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{ccccccc} & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\ 1 & 2 & & & n-2 & n-1 & \end{array}$$

Figure 2.2: Generalized Cartan Matrix and Dynkin diagram for A_{n-1}

The Chevalley generators are $\{e_i = E_{i,i+1}, f_i = E_{i+1,i} \mid 1 \leq i \leq n-1\}$ and the Cartan subalgebra \mathfrak{t} is $\text{span}\{h_i = E_{ii} - E_{i+1,i+1} \mid 1 \leq i \leq n-1\}$.

Example 2.3.10. The special affine linear algebra $A_{n-1}^{(1)} = \widehat{\mathfrak{sl}}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C}d$ where \mathbf{c} is the canonical central element and d is a degree derivation is a Kac-Moody algebra and has the following GCM and corresponding Dynkin diagram.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & -1 & 2 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{ccccc} & & \bullet & & \\ & & 0 & & \\ & \swarrow & & \searrow & \\ 1 & \bullet & \bullet & \cdots & \bullet & n-2 & n-1 \end{array}$$

Figure 2.3: Generalized Cartan Matrix and Dynkin diagram for $A_{n-1}^{(1)}$

The free abelian $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*, $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is called the *positive root lattice* and $Q_- = -Q_+$ is called the *negative root lattice*

Definition 2.3.11. For each $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{t}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is called a *root* of \mathfrak{g} and \mathfrak{g}_α is called the *root space* attached to α . The dimension of \mathfrak{g}_α is called the *root multiplicity* of α . We denote the set of all roots by Δ and denote the set of positive (respectively, negative) roots by $\Delta_+ = \Delta \cap Q_+$ (respectively, $\Delta_- = \Delta \cap Q_-$). The subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is called the *derived subalgebra*.

Denote by \mathfrak{g}_+ (respectively, \mathfrak{g}_-) the subalgebra of \mathfrak{g} generated by the element e_i (respectively, f_i) with $i \in I$. The basic properties of Kac-Moody algebras are given in the following proposition.

Proposition 2.3.12. [16]

1. We have the *triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{t} \oplus \mathfrak{g}_+ \quad (\text{direct sum of vector spaces}).$$

2. \mathfrak{g}_+ (respectively, \mathfrak{g}_-) is the Lie algebra generated by the element e_i , ($i \in I$) (respectively, f_i , ($i \in I$)) with the defining relation (5) (respectively, (6)) in Definition 2.3.7.
3. There exists an involution $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$, called the **Chevalley involution**, such that $e_i \mapsto f_i$, $f_i \mapsto e_i$ and $h \mapsto -h$.

4. We have the *root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \quad \text{with } \dim \mathfrak{g}_\alpha < \infty \text{ for all } \alpha \in Q.$$

5. If the generalized Cartan matrix A is indecomposable, then every ideal of the Kac-Moody algebra \mathfrak{g} either contains its derived subalgebra \mathfrak{g}' or is contained in its center $Z(\mathfrak{g}) = \{h \in \mathfrak{t} \mid \alpha_i(h) = 0 \text{ for all } i \in I\}$.

The triangle decomposition given in Proposition 2.3.12 implies that if α is a positive root, then we have $\mathfrak{g}_\alpha \in \mathfrak{g}_+$ and if α is a negative root, then $\mathfrak{g}_\alpha \in \mathfrak{g}_-$. Also, by the Chevalley involution, we have the multiplicity of α equals to the multiplicity of $-\alpha$.

Definition 2.3.13. Let $\text{Aut } \mathfrak{t}$ be the set of all automorphisms on \mathfrak{t} . For each $i \in I$, define the *simple reflections* $s_i \in \text{Aut } \mathfrak{t}$ by

$$s_i(h) = h - \alpha_i(h)\check{\alpha}_i.$$

The group W generated by all simple reflections is called the **Weyl group**. It induces the action of W on \mathfrak{t}^* by $s_i(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i$.

Definition 2.3.14. Let $w \in W$ and choose l to be the smallest integer such that $w = s_{i_1}s_{i_2} \cdots s_{i_l}$. Then l is called the *length* of w and the expression $w = s_{i_1}s_{i_2} \cdots s_{i_l}$ is called the *reduced expression* of w .

Definition 2.3.15. A root $\alpha \in \Delta$ is a *real root* if it is Weyl group conjugate to a simple root, i.e. if there exists $w \in W$ such that $w(\alpha_i) = \alpha$ for some $i \in I$. Otherwise, α is called an *imaginary root*.

Denote the set of real roots as Δ^{re} and the set of imaginary roots as Δ^{im} . Then we have $\Delta = \Delta^{\text{re}} \cup \Delta^{\text{im}}$.

Corresponding to a Kac-Moody algebra \mathfrak{g} , we can construct its universal enveloping algebra $U(\mathfrak{g})$ as follows.

Definition 2.3.16. The *universal enveloping algebra* $U(\mathfrak{g})$ of \mathfrak{g} is the associative algebra over \mathbb{C} with unity generated by e_i, f_i ($i \in I$) and t satisfying the following relations.

1. $hh' = h'h$ for $h, h' \in t$,
2. $e_i f_j - f_j e_i = \delta_{ij} h_i$ for $i, j \in I$,
3. $h e_i - e_i h = \alpha_i(h) e_i$ for $h \in t, i \in I$,
4. $h f_i - f_i h = -\alpha_i(h) f_i$ for $h \in t, i \in I$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Let U^+ (respectively, U^0 and U^-) be the subalgebra of $U(\mathfrak{g})$ generated by the elements e_i (respectively, t and f_i) for $i \in I$. We define the root space of $U(g)$ as follows.

$$U_\beta = \{u \in U(\mathfrak{g}) \mid hu - uh = \beta(h)u \text{ for all } h \in t\} \text{ for } \beta \in Q,$$

$$U_\beta^\pm = \{u \in U^\pm \mid hu - uh = \beta(h)u \text{ for all } h \in t\} \text{ for } \beta \in Q_\pm.$$

Using the PBW Theorem, we can extend the triangular decomposition and the root space decomposition to the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} as follows.

Proposition 2.3.17. Let \mathfrak{g} be a Kac-Moody algebra and let $U(\mathfrak{g})$ be the associated universal enveloping algebra. Then we have

1. $U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$,
2. $U(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_\beta$,
3. $U^\pm = \bigoplus_{\beta \in Q_\pm} U_\beta^\pm$.

2.4 Representation theory of Kac-Moody algebras

For this section, let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra and let V be a \mathfrak{g} -module. Recall that, by the PBW Theorem, the representation theory of \mathfrak{g} is parallel to the representation theory of its universal enveloping algebra $U(\mathfrak{g})$. Thus, the definitions and properties in this section can be extended for $U(\mathfrak{g})$.

Definition 2.4.1. For any $\mu \in \mathfrak{t}^*$, the μ -weight space is

$$V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{t}\}.$$

If $V_\mu \neq \{0\}$, μ is called a *weight* of V and the vector $v \in V_\mu$ is called a *weight vector* of weight μ . The dimension of V_μ is called *weight multiplicity* of μ and the set of weights of the \mathfrak{g} -module V is denoted by $\text{wt}(V)$.

Definition 2.4.2. A \mathfrak{g} -module V is a *weight module* if it admits a *weight space decomposition*

$$V = \bigoplus_{\mu \in \mathfrak{t}^*} V_\mu.$$

Recall the positive root lattice $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Define a partial ordering on \mathfrak{t}^* by

$$\lambda \geq \mu \text{ if and only if } \lambda - \mu \in Q_+$$

for $\lambda, \mu \in \mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^*$, set $D(\lambda) = \{\mu \in \mathfrak{t}^* \mid \lambda \geq \mu\}$.

Definition 2.4.3. [Category \mathcal{O}]

Objects: weight modules V over \mathfrak{g} with $\dim V_\lambda < \infty$ for all $\lambda \in \mathfrak{t}^*$ and for which there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathfrak{t}^*$ such that

$$\text{wt}(V) \subset D(\lambda_1) \cup \dots \cup D(\lambda_s).$$

Morphisms: \mathfrak{g} -module homomorphisms

Closure: the category is closed under finite direct sums or finite tensor products

The important examples of \mathfrak{g} -modules in the category \mathcal{O} are highest weight modules given in the following definition.

Definition 2.4.4. A weight module V is a *highest weight module* of *highest weight* $\lambda \in \mathfrak{t}^*$ if there exists a nonzero vector $v_\lambda \in V$, called a *highest weight vector*, satisfying

$$\begin{aligned} e_i v_\lambda &= 0 \text{ for all } i \in I, \\ hv_\lambda &= \lambda(h)v_\lambda \text{ for all } h \in \mathfrak{t}, \\ V &= U(\mathfrak{g})v_\lambda. \end{aligned}$$

We now consider a specific type of highest weight module called a Verma module given in the following definition.

Definition 2.4.5. Fix $\lambda \in \mathfrak{t}^*$ and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by all e_i and $h - \lambda(h)1$ ($i \in I, h \in \mathfrak{t}$). Set $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ and give $M(\lambda)$ a $U(\mathfrak{g})$ -module structure by left multiplication. Then $M(\lambda)$ is called **Verma module**.

Proposition 2.4.6. [9]

1. $M(\lambda)$ is a highest weight \mathfrak{g} -module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$.
2. Every highest weight \mathfrak{g} -module with highest weight λ is a homomorphic image of $M(\lambda)$.
3. As U^- -module, $M(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$.
4. $M(\lambda)$ has a unique maximal submodule.

Definition 2.4.7. Let $M(\lambda)$ be a Verma module and $N(\lambda)$ be its unique maximal submodule. Then $M(\lambda)/N(\lambda)$ is the **irreducible highest weight module** denoted by $V(\lambda)$.

Proposition 2.4.8. [16] Every irreducible \mathfrak{g} -module in the category \mathcal{O} is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{t}^*$.

Definition 2.4.9. A weight module V over a Kac-Moody Lie algebra \mathfrak{g} is called **integrable** if all e_i and f_i ($i \in I$) are locally nilpotent on V , i.e. for each $i \in I$, there exist positive integer N_1, N_2 such that $e_i^{N_1} \cdot v = 0$ and $f_i^{N_2} \cdot v = 0$ for all $v \in V$.

We call the elements in the weight lattice $P = \{\lambda \in \mathfrak{t}^* | \lambda(\check{P}) \subset \mathbb{Z}\}$ **integral weights** and the set of **dominant integral weights** is denoted by

$$P^+ = \{\lambda \in P | \lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

Definition 2.4.10. [Category \mathcal{O}_{int}] The objects in this category are integrable \mathfrak{g} -modules in category \mathcal{O} such that $\text{wt}(V) \subset P$.

Note that, by the definition, any \mathfrak{g} -module V in the category \mathcal{O} has a weight space decomposition

$$V = \bigoplus_{\lambda \in P} V_\lambda, \text{ where } V_\lambda = \{v \in V | hv = \lambda(h)v \text{ for all } h \in \check{P}\}.$$

Also, we have the following property.

Proposition 2.4.11. [16] Let $V(\lambda)$ be the irreducible highest weight \mathfrak{g} -module with highest weight $\lambda \in \mathfrak{t}^*$. Then $V(\lambda)$ is in the category \mathcal{O}_{int} if and only if $\lambda \in P^+$.

We end this section with a complete reducibility theorem of \mathfrak{g} -modules in the category \mathcal{O}_{int} .

Theorem 2.4.12. [16] *Let \mathfrak{g} be a symmetrizable Kac-Moody algebra associated with a Cartan datum $\{A, \Pi, \check{\Pi}, P, \check{P}\}$. Then every \mathfrak{g} -module in the category \mathcal{O}_{int} is isomorphic to a direct sum of irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

CHAPTER

3

QUANTUM GROUPS AND CRYSTAL BASES

In this chapter, we introduce quantum deformations of the universal enveloping algebra of a Kac-Moody algebra \mathfrak{g} . These deformations are known as quantum groups and denoted by $U_q(\mathfrak{g})$. We also review the crystal basis theory which provides a very powerful combinatorial tool for studying the structure of integrable representations of quantum groups in the category $\mathcal{O}_{\text{int}}^q$.

3.1 Quantum groups

For $m, n \in \mathbb{Z}$ and q any indeterminate, define the following:

- $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ is a *q -integer*,
- $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ ($n > 0$) and $[0]_q! = 1$ are *q -factorials*,
- $\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$ ($m \geq n \geq 0$) is the *q -binomial coefficient*.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GCM with a symmetrizing matrix $D = \text{diag } (d_i)_{i \in I}$ such that $d_i \in \mathbb{Z}_{>0}$ and let $(A, \Pi, \check{\Pi}, P, \check{P})$ be a Cartan datum associated with A .

Definition 3.1.1. The *quantum group* or the *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is the associative algebra over $\mathbb{C}(q)$ with 1 generated by the elements e_i, f_i ($i \in I$) and q^h ($h \in \check{P}$) satisfying the following relations.

1. $q^0 = 1, q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in \check{P},$
2. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i \check{\alpha}_i} - q^{-d_i \check{\alpha}_i}}{q^{d_i} - q^{-d_i}} \text{ for } i, j \in I,$
3. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \text{ for } h \in \check{P}, i \in I,$
4. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \text{ for } h \in \check{P}, i \in I,$
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \text{ for } i \neq j,$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \text{ for } i \neq j.$

Note that $U_q(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ as $q \rightarrow 1$.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then the quantum group $U_q(\mathfrak{sl}(2, \mathbb{C}))$ is the associative algebra generated by $\{e, f, q^h\}$ such that the following relations hold.

1. $q^h e q^{-h} = q^2 e,$
2. $q^h f q^{-h} = q^{-2} f,$
3. $ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}.$

Since the defining relations for a quantum group $U_q(\mathfrak{g})$ are analogous to those for the universal enveloping algebra $U(\mathfrak{g})$, the quantum group $U_q(\mathfrak{g})$ has the *triangular decomposition*

$$U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+$$

where U_q^+ (respectively, U_q^-) is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (respectively, f_i) for $i \in I$ and U_q^0 is the subalgebra of $U_q(\mathfrak{g})$ generated by q^h ($h \in \check{P}$), and the *root space decomposition*

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q} (U_q)_\beta$$

where $(U_q)_\beta = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\beta(h)u} \text{ for all } h \in \check{P}\}$.

3.2 Representation theory of quantum groups

In this section, we study the representation theory of quantum group which is quite parallel to that of Kac-Moody algebras discussed in the previous chapter.

Definition 3.2.1. Let V^q be a $U_q(\mathfrak{g})$ -module. For any $\mu \in P$, the μ -weight space is

$$V_\mu^q = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in \check{P}\}.$$

If $V_\mu^q \neq \{0\}$, μ is called a *weight* of V^q and the vector $v \in V_\mu^q$ is called a *weight vector* of weight μ . The dimension of V_μ^q is called *weight multiplicity* of μ and the set of weights of the $U_q(\mathfrak{g})$ -module V^q is denoted by $\text{wt}(V^q)$.

Definition 3.2.2. A $U_q(\mathfrak{g})$ -module V^q is a *weight module* if it admits a *weight space decomposition*

$$V^q = \bigoplus_{\mu \in P} V_\mu^q.$$

For $\lambda \in P$, set $D(\lambda) = \{\mu \in P \mid \lambda \geq \mu\}$.

Definition 3.2.3. [Category \mathcal{O}^q] The objects in this category are weight modules V^q over $U_q(\mathfrak{g})$ with $\dim V_\lambda^q < \infty$ for all $\lambda \in P$ and for which there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_s \in P$ such that

$$\text{wt}(V^q) \subset D(\lambda_1) \cup \dots \cup D(\lambda_s).$$

As is the case with Kac-Moody algebras, the important examples of $U_q(\mathfrak{g})$ -modules in the category \mathcal{O}^q are highest weight modules given in the following definition.

Definition 3.2.4. A weight module V^q is a *highest weight module* of *highest weight* $\lambda \in P$ if there exists a nonzero vector $v_\lambda \in V^q$, called a *highest weight vector*, satisfying

$$\begin{aligned} e_i v_\lambda &= 0 \text{ for all } i \in I, \\ q^h v_\lambda &= q^{\lambda(h)} v_\lambda \text{ for all } h \in \check{P}, \\ V^q &= U_q(\mathfrak{g}) v_\lambda. \end{aligned}$$

Consider a specific type of highest weight module called a Verma module given in the following definition.

Definition 3.2.5. Fix $\lambda \in P$ and let $J^q(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by all e_i and $q^h - q^{\lambda(h)} 1$ ($i \in I, h \in \check{P}$). Set $M^q(\lambda) = U_q(\mathfrak{g}) / J^q(\lambda)$ and give $M^q(\lambda)$ a $U_q(\mathfrak{g})$ -module structure by left multiplication. Then $M^q(\lambda)$ is called *Verma module*.

Proposition 3.2.6. [9]

1. $M^q(\lambda)$ is a highest weight $U_q(\mathfrak{g})$ -module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$.
2. Every highest weight $U_q(\mathfrak{g})$ -module with highest weight λ is a homomorphic image of $M^q(\lambda)$.
3. As U_q^- -module, $M^q(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J^q(\lambda)$.
4. $M^q(\lambda)$ has a unique maximal submodule.

Definition 3.2.7. Let $M^q(\lambda)$ be a Verma module and $N^q(\lambda)$ be its unique maximal submodule. Then $M^q(\lambda)/N^q(\lambda)$ is the *irreducible highest weight module* denoted by $V^q(\lambda)$.

Definition 3.2.8. A weight module V^q over the quantum group $U_q(\mathfrak{g})$ is called *integrable* if all e_i and f_i ($i \in I$) are locally nilpotent on V^q .

Definition 3.2.9. [Category $\mathcal{O}_{\text{int}}^q$] The objects in this category are integrable $U_q(\mathfrak{g})$ -modules in category \mathcal{O}^q .

Proposition 3.2.10. [16] Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight $\lambda \in P$. Then $V^q(\lambda)$ is in the $\mathcal{O}_{\text{int}}^q$ if and only if $\lambda \in P^+$.

Also, similar to category \mathcal{O}_{int} , any $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{\text{int}}^q$ is completely reducible and has a weight space decomposition.

3.3 Crystal bases

The crystal basis theory for quantum groups was introduced by Kashiwara around 1990 [22]. Crystal bases can be viewed as bases at $q = 0$ and they have nice combinatorial features reflecting the internal structure of integrable $U_q(\mathfrak{g})$ -module V^q in the category $\mathcal{O}_{\text{int}}^q$.

Lemma 3.3.1. [9] Let $V^q = \bigoplus_{\lambda \in P} V_\lambda^q$ be a $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{\text{int}}^q$. For each $i \in I$, every weight vector $v \in V_\lambda^q$ ($\lambda \in \text{wt}(V^q)$) may be written in the form

$$v = v_0 + f_i v_1 + \cdots + f_i^{(N)} v_N,$$

where $N \in \mathbb{Z}_{\geq 0}$ and $v_k \in V_{\lambda+k\alpha_i}^q \cap \ker e_i$ and $f_i^{(k)} = \frac{f_i^k}{[k]_q!}$. Furthermore, v_k in the above expression is uniquely determined by v and $v_k \neq 0$ only if $\lambda(\check{\alpha}_i) + k \geq 0$.

Definition 3.3.2. The *Kashiwara operators* \tilde{e}_i and \tilde{f}_i ($i \in I$) are endomorphisms on V^q such that, for $v \in V^q$,

$$\tilde{e}_i v = \sum_{k=1}^N f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k=0}^N f_i^{(k+1)} v_k.$$

Note that $\tilde{e}_i V_\lambda^q = e_i V_\lambda^q \subset V_{\lambda+\alpha_i}^q$ and $\tilde{f}_i V_\lambda^q = f_i V_\lambda^q \subset V_{\lambda-\alpha_i}^q$ for all $i \in I$ and $\lambda \in P$. Let

$$\mathbf{A}_0 = \left\{ \frac{g(q)}{h(q)} \middle| g(q), h(q) \in \mathbb{C}[q], h(0) \neq 0 \right\}.$$

Then \mathbf{A}_0 is a principal ideal domain with $\mathbb{C}(q)$ as its field of quotients.

Definition 3.3.3. Let V^q be a $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{\text{int}}^q$. A *crystal lattice* L is a free \mathbf{A}_0 -submodule of V^q satisfying the following relations.

1. L generates V^q as a vector space over $\mathbb{C}(q)$,
2. $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap V_\lambda^q$ for all $\lambda \in P$,
3. $\tilde{e}_i L \subset L, \tilde{f}_i L \subset L$ for all $i \in I$.

Remark 3.3.4. Let J_0 be the unique maximal ideal of \mathbf{A}_0 generated by q . Then there exists an isomorphism of fields from \mathbf{A}_0/J_0 to \mathbb{C} given by $\frac{g(q)}{h(q)} + J_0 \mapsto \frac{g(0)}{h(0)}$ and hence $\mathbb{C} \otimes_{\mathbf{A}_0} L \cong L/J_0 L = L/qL$.

Definition 3.3.5. [9] Let V^q be a $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{\text{int}}^q$. A *crystal base* of V^q is a pair (L, B) such that

1. L is a crystal lattice for V^q ,
2. B is a \mathbb{C} -basis of $L/qL \cong \mathbb{C} \otimes_{\mathbf{A}_0} L$,
3. $B = \sqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
4. $\tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
5. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for any $b, b' \in B$ and $i \in I$.

Theorem 3.3.6. [9] Let $\lambda \in P^+$ be a dominant integral weight and $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight λ and highest weight vector v_λ . Let

$$L(\lambda) = \sum_{r \geq 0, i_k \in I} \mathbf{A}_0 \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda,$$

and set

$$B(\lambda) = \{ \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda) \mid r \geq 0, i_k \in I \} \setminus \{0\}.$$

Then the pair $(L(\lambda), B(\lambda))$ is a crystal base for $V^q(\lambda)$.

Definition 3.3.7. Given a crystal base (L, B) for $V^q \in \mathcal{O}_{\text{int}}^q$, we can define a *crystal graph* for V^q in the following way. The vertex set consists of all elements of B and the edge set consists of i -colored arrows. Two vertices $b, b' \in B$ are joined by an i -colored arrow, $b \xrightarrow{i} b'$, if and only if $\tilde{f}_i b = b'$ for $i \in I$.

For $i \in I$, we define the maps $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}$ by

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\}.$$

Then ε_i denotes the number of i -colored arrows coming into the vertex b and φ_i denotes the number of i -colored arrows going out of the vertex b . Hence we have $\varphi_i(b) - \varepsilon_i(b) = \lambda(\check{\alpha}_i)$.

Next we will develop the tensor product rule which is one of the nicest features of crystal bases.

Definition 3.3.8. [9] Let V_j^q be a $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{\text{int}}^q$ and let (L_j, B_j) be a crystal base of V_j^q ($j = 1, 2$). Set $L = L_1 \otimes_{\mathbf{A}_0} L_2$ and $B = B_1 \times B_2$. Then (L, B) is a crystal base of $V_1^q \otimes_{\mathbf{C}(q)} V_2^q$ where the action of Kashiwara operators \tilde{e}_i and \tilde{f}_i on B ($i \in I$) are given as follows.

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

Hence we have

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \check{\alpha}_i, \text{wt}(b_1) \rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle \check{\alpha}_i, \text{wt}(b_2) \rangle\}. \end{aligned}$$

Here we understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$ and we write $b_1 \otimes b_2$ instead of (b_1, b_2) . Also we denote the crystal graph of $V_1^q \otimes V_2^q$ as $B_1 \otimes B_2$.

Definition 3.3.9. [9] Let I be an finite index set and let $A = (a_{ij})_{i,j \in I}$ be a GCM with the Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$. A *crystal* associated with the Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is a set B together with the maps $\text{wt} : B \rightarrow P$, $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{0\}$ and $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$ ($i \in I$) satisfying the following properties.

1. $\varphi_i(b) = \varepsilon_i(b) + \langle \check{\alpha}_i, \text{wt}(b) \rangle$ for all $i \in I$,
2. $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$,

3. $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
4. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in B$,
5. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in B$,
6. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for any $b, b' \in B$ and $i \in I$,
7. if $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 3.3.10. Let B_1, B_2 be crystals associated with the Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$. A *crystal morphism* $\Omega : B_1 \rightarrow B_2$ is a map $\Omega : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ such that

1. $\Omega(0) = 0$,
2. if $b \in B_1$ and $\Omega(b) \in B_2$, then $\text{wt}(\Omega(b)) = \text{wt}(b)$, $\varepsilon_i(\Omega(b)) = \varepsilon_i(b)$ and $\varphi_i(\Omega(b)) = \varphi_i(b)$ for all $i \in I$,
3. if $b, b' \in B_1$, $\Omega(b), \Omega(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i(\Omega(b)) = \Omega(b')$ and $\Omega(b) = \tilde{e}_i(\Omega(b'))$ for all $i \in I$.

A crystal morphism is called *strict* if it commutes with all \tilde{e}_i and \tilde{f}_i ($i \in I$). Also, a crystal morphism $\Omega : B_1 \rightarrow B_2$ is called an *isomorphism* if it is both one-to-one and onto from $B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$.

CHAPTER

4

QUANTUM AFFINE LIE ALGEBRAS AND PERFECT CRYSTALS

For this chapter, we introduce the notion of a perfect crystal for a quantum affine algebra. We begin by recalling basic definition related to quantum affine algebras.

4.1 Quantum affine algebras

Let \mathfrak{g} be an affine Kac-Moody algebra with affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ and index set $I = \{0, 1, \dots, n\}$ where $A = (a_{ij})_{i,j \in I}$ is the affine GCM, $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots, $\check{\Pi} = \{\check{\alpha}_i \mid i \in I\}$ is the set of simple coroots, P and \check{P} are the affine weight lattice, and coweight lattice respectively. Let $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$, \mathbf{c} , δ , and $\{\Lambda_i \mid i \in I\}$ denote the Cartan subalgebra, the canonical central element, the null root and the set of fundamental weights respectively. Note that $\alpha_j(\check{\alpha}_i) = a_{ij}$ and $\Lambda_j(\check{\alpha}_i) = \delta_{ij}$ and $\mathfrak{t} = \text{span}_{\mathbb{C}}\{\check{\alpha}_i, d \mid i \in I\}$ where d is a degree derivation. Then

$$P = \bigoplus_{j \in I} \mathbb{Z}\Lambda_j \oplus \mathbb{Z}\delta \subset \mathfrak{t}^* \quad \text{and} \quad \check{P} = \bigoplus_{i \in I} \mathbb{Z}\check{\alpha}_i \oplus \mathbb{Z}d \subset \mathfrak{t}.$$

Note that the set of affine dominant integral weights is

$$P^+ = \{\lambda \in P \mid \lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

The quantum group associated with the affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is called a *quantum affine algebra*, denoted by $U_q(\mathfrak{g})$.

Definition 4.1.1. Let $U'_q(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, f_i, q^{\pm d_i \check{\alpha}_i} \mid i \in I\}$. This subalgebra is also called a *quantum affine algebra*. Let

$$P_{cl} = \bigoplus_{j \in I} \mathbb{Z}\Lambda_j \quad \text{and} \quad \check{P}_{cl} = \bigoplus_{i \in I} \mathbb{Z}\check{\alpha}_i.$$

The set P_{cl} is called the *classical weight lattice* and the elements of P_{cl} are called the *classical weights*. The quintuple $(A, \Pi, \check{\Pi}, P_{cl}, \check{P}_{cl})$ is called a *classical Cartan datum* and the quantum affine algebra $U'_q(\mathfrak{g})$ can be regarded as the quantum group associated with the classical Cartan datum. Define

$$P_{cl}^+ = \{\lambda \in P_{cl} \mid \lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

The elements of P_{cl}^+ are called the *classical dominant integral weights* and we say that $\lambda \in P_{cl}^+$ has *level* $l = \lambda(\mathbf{c})$. We denote $(P_{cl}^+)_l$ to be the set of classical dominant integral weights of level l .

Note that $U'_q(\mathfrak{g})$ can have finite dimensional irreducible modules while all the nontrivial irreducible $U_q(\mathfrak{g})$ -modules are infinite dimensional.

4.2 Perfect crystals

Definition 4.2.1. A *classical crystal* is a crystal associated with the classical Cartan datum $(A, \Pi, \check{\Pi}, P_{cl}, \check{P}_{cl})$. This crystal is also known as a $U'_q(\mathfrak{g})$ -crystal.

Let B be a classical crystal. For $b \in B$, define

$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i.$$

Definition 4.2.2. [9] For a positive integer l , a finite classical crystal B^l is a *perfect crystal of level l* if it satisfies the following properties.

1. there exists a finite dimensional $U'_q(\mathfrak{g})$ -module with a crystal basis whose crystal graph is isomorphic to B^l .

- 2. $B^l \otimes B^l$ is connected,
- 3. there exists a classical weight $\lambda_0 \in P_{cl}$ such that

$$\text{wt}(B^l) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i, \quad \#(B_{\lambda_0}^l) = 1,$$

- 4. for any $b \in B^l$, we have $\langle \mathbf{c}, \varepsilon(b) \rangle \geq l$
- 5. for each $\lambda \in (P_{cl}^+)_l$, there exist unique vectors $b^\lambda \in B^l$ and $b_\lambda \in B^l$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

For a perfect crystal B^l , we define

$$(B^l)_{\min} = \{b \in B^l \mid \langle \mathbf{c}, \varepsilon(b) \rangle = l\}.$$

Then the maps $\varepsilon, \varphi : (B^l)_{\min} \rightarrow (P_{cl}^+)_l$ are bijections.

We now present a crystal isomorphism theorem that is essential for understanding path realization which we discuss in the next section. For the rest of the chapter, assume that \mathfrak{g} is an affine Kac-Moody algebra, $l \in \mathbb{Z}_{>0}$, B^l is a perfect crystal of level l , $\lambda \in (P_{cl}^+)_l$ is a classical dominant integral weights of level l , b_λ is a unique vector in B^l such that $\varphi(b_\lambda) = \lambda$, $V(\lambda)$ is the highest weight $U'_q(\mathfrak{g})$ -module with highest weight λ and $B(\lambda)$ is the crystal graph of $V(\lambda)$.

Theorem 4.2.3. *The map*

$$\Psi : B(\lambda) \rightarrow B(\varepsilon(b_\lambda)) \otimes B^l \quad \text{given by} \quad u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where u_λ is the highest weight vector of $B(\lambda)$ and $u_{\varepsilon(b_\lambda)}$ is the highest weight vector of $B(\varepsilon(b_\lambda))$, is a strict crystal isomorphism.

4.3 Path realization of crystal graphs

In this section, we will use Theorem 4.2.3 to develop path realization. We first set

$$\lambda_0 = \lambda, \quad \lambda_{k+1} = \varepsilon(\lambda_k); \quad b_0 = b_\lambda, \quad b_{k+1} = b_{\lambda_{k+1}}.$$

By Theorem 4.2.3, we have a crystal isomorphism

$$\Psi : B(\lambda_j) \xrightarrow{\sim} B(\lambda_{j+1}) \otimes B^l \quad \text{given by} \quad u_{\lambda_j} \mapsto u_{\lambda_{j+1}} \otimes b_{\lambda_j}$$

such that $\varphi(b_j) = \lambda_j$ and $\varepsilon(b_j) = \lambda_{j+1}$. Composing the Ψ 's yields a sequence of crystal isomorphisms

$$B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes B^l \xrightarrow{\sim} B(\lambda_2) \otimes B^l \otimes B^l \xrightarrow{\sim} \dots$$

given by

$$u_\lambda \mapsto u_{\lambda_1} \otimes b_0 \mapsto u_{\lambda_2} \otimes b_1 \otimes b_0 \mapsto \dots.$$

Then we have two infinite sequences

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^\infty = (\dots, \lambda_{k+1}, \lambda_k, \dots, \lambda_1, \lambda_0) \in ((P_{cl}^+)_l)^\infty$$

and

$$\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \dots \otimes b_{k+1} \otimes b_k \otimes \dots \otimes b_1 \otimes b_0 \in (B^l)^{\otimes \infty}.$$

Since there are only finitely many elements in $(P_{cl}^+)_l$, there exist $N > 0$ and $k \geq 0$ such that $\lambda_{k+N} = \lambda_k$. Since φ and ε are bijective, we have

$$b_{k+N} = \varphi^{-1}(\lambda_{k+N}) = \varphi^{-1}(\lambda_k) = b_k$$

and the following equations

$$\begin{aligned} \varepsilon(b_{k-1}) &= \lambda_k = \lambda_{k+N} = \varepsilon(b_{k+N-1}), \\ b_{k-1} &= \varepsilon^{-1}(\varepsilon(b_{k-1})) = \varepsilon^{-1}(\varepsilon(b_{k+N-1})) = b_{k+N-1}, \\ \lambda_{k-1} &= \varphi(b_{k-1}) = \varphi(b_{k+N-1}) = \lambda_{k+N-1}, \\ &\vdots \\ \varepsilon(b_0) &= \lambda_1 = \lambda_{1+N} = \varepsilon(b_N), \\ b_0 &= \varepsilon^{-1}(\varepsilon(b_0)) = \varepsilon^{-1}(\varepsilon(b_N)) = b_N, \\ \lambda_0 &= \varphi(b_0) = \varphi(b_N) = \lambda_N, \end{aligned}$$

and

$$\begin{aligned} \lambda_{k+1} &= \varepsilon(b_k) = \varepsilon(b_{k+N}) = \lambda_{k+N+1}, \\ \varphi(b_{k+1}) &= \lambda_{k+1} = \lambda_{k+N+1} = \varphi(b_{k+N+1}) \\ b_{k+1} &= \varphi^{-1}(\varphi(b_{k+1})) = \varphi^{-1}(\varphi(b_{k+N+1})) = b_{k+N+1} \\ &\vdots \end{aligned}$$

Hence \mathbf{w}_λ and \mathbf{p}_λ are periodic sequences, each with the period $N > 0$.

Definition 4.3.1. [9]

1. The sequence $\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$ is called the *ground-state path* of weight λ .
2. A λ -*path* in B^l is a sequence $\mathbf{p} = (\mathbf{p}_k)_{k=0}^\infty = \cdots \otimes \mathbf{p}_{k+1} \otimes \mathbf{p}_k \otimes \cdots \otimes \mathbf{p}_1 \otimes \mathbf{p}_0$ with $\mathbf{p}_k \in B^l$ such that $\mathbf{p}_k = b_k$ for all $k \gg 0$

Let $\mathcal{P}(\lambda)$ be the set of all λ -paths in B^l . We can define a crystal structure on $\mathcal{P}(\lambda)$ as shown in the following proposition.

Proposition 4.3.2. [9] Let $\mathbf{p} = (\mathbf{p}_k)_{k=0}^\infty$ be a λ -path in B^l and let $N > 0$ be the smallest positive integer such that $\mathbf{p}_k = b_k$ for all $k \geq N$. For each $i \in I$, we define

$$\begin{aligned} wt_{cl}\mathbf{p} &= \lambda_N + \sum_{k=0}^{N-1} wt_{cl}\mathbf{p}_k, \\ \tilde{e}_i\mathbf{p} &= \cdots \otimes \mathbf{p}_{N+1} \otimes \tilde{e}_i(\mathbf{p}_N \otimes \cdots \otimes \mathbf{p}_0), \\ \tilde{f}_i\mathbf{p} &= \cdots \otimes \mathbf{p}_{N+1} \otimes \tilde{f}_i(\mathbf{p}_N \otimes \cdots \otimes \mathbf{p}_0), \\ \varepsilon_i(\mathbf{p}) &= \max\{\varepsilon_i(\mathbf{p}') - \varphi_i(b_N), 0\}, \\ \varphi_i(\mathbf{p}) &= \varphi_i(\mathbf{p}') + \max\{\varphi_i(b_N) - \varepsilon_i(\mathbf{p}'), 0\}, \end{aligned}$$

where $\mathbf{p}' = \mathbf{p}_{N-1} \otimes \cdots \otimes \mathbf{p}_1 \otimes \mathbf{p}_0$ and wt_{cl} denotes the classical weights. Then the maps $wt_{cl} : \mathcal{P}(\lambda) \rightarrow P_{cl}$, $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda) \sqcup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{P}(\lambda) \rightarrow \mathbb{Z}$ define a $U'_q(\mathfrak{g})$ -crystal structure on $\mathcal{P}(\lambda)$.

We can now present the main result of this section: the path realization of $U'_q(\mathfrak{g})$ -crystal $B(\lambda)$.

Theorem 4.3.3. The map

$$\Psi : B(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda) \quad \text{given by} \quad u_\lambda \mapsto \mathbf{p}_\lambda,$$

is a crystal isomorphism.

CHAPTER

5

GEOMETRIC CRYSTALS AND ULTRA-DISCRETIZATION

In this chapter, we review Kac-Moody groups and necessary definitions and facts about geometric crystals and ultra-discretization.

5.1 Kac-Moody groups

Let \mathfrak{g} be an affine Kac-Moody algebra with affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ and index set $I = \{0, 1, \dots, n\}$ where $A = (a_{ij})_{i,j \in I}$ is the affine GCM, $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots, $\check{\Pi} = \{\check{\alpha}_i \mid i \in I\}$ is the set of simple coroots, P and \check{P} are the affine weight lattice, and coweight lattice respectively. Let $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$, \mathbf{c} , δ , and $\{\Lambda_i \mid i \in I\}$ denote the Cartan subalgebra, the canonical central element, the null root and the set of fundamental weights respectively. There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_{\alpha}$ where $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ is the root lattice. The Weyl group W of \mathfrak{g} is generated by the simple reflections $\{s_i \mid i \in I\}$. The sets Δ , $\Delta_+ = \Delta \cap Q_+ = \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $\Delta^{\text{re}} = \{w(\alpha_i) \mid w \in W, i \in I\}$ are the set of roots, positive roots, and real roots respectively. We denote $\mathfrak{t}_{cl}^* = \mathfrak{t}^*/\mathbb{C}\delta$ and $(\mathfrak{t}_{cl}^*)_0 = \{\lambda \in \mathfrak{t}_{cl}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$. We also denote \mathfrak{g}_i to be the subalgebra of \mathfrak{g} with index set $I_i = I \setminus \{i\}$ which is a finite dimensional semisimple Lie algebra.

Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived Lie algebra of \mathfrak{g} and G^* be the free group generated by the

free product of the additive group \mathfrak{g}_α ($\alpha \in \Delta^{\text{re}}$), with the canonical inclusion $i_\alpha : \mathfrak{g}_\alpha \hookrightarrow G^*$. For any integrable \mathfrak{g}' -module (V, π) , a homomorphism

$$\pi_V^* : G^* \rightarrow \text{Aut}_{\mathbb{C}}(V) \quad \text{is defined by} \quad \pi_V^*(i_\alpha(e)) = \exp \pi(e).$$

Set $N^* := \cap_{V:\text{integrable}} \ker(\pi_V^*)$ and $G := G^*/N^*$ which is called a **Kac-Moody group** associated with the Kac-Moody Lie algebra \mathfrak{g}' ([18], [17]). Let $\rho : G^* \rightarrow G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta^{\text{re}}$), define $\exp e := \rho(i_\alpha(e))$ and $S_\alpha := \exp \mathfrak{g}_\alpha$ which is an one-parameter subgroup of G . The group G is generated by S_α ($\alpha \in \Delta^{\text{re}}$). Let S^\pm be the subgroups generated by $S_{\pm\alpha}$ ($\alpha \in \Delta_+^{\text{re}} := \Delta^{\text{re}} \cap Q_+$), i.e. $S^\pm := \langle S_{\pm\alpha} \mid \alpha \in \Delta_+^{\text{re}} \rangle$.

For any $i \in I$, there exist a unique homomorphism $\phi_i : SL(2, \mathbb{C}) \rightarrow G$ such that

$$\phi_i \left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right) = c^{\check{\alpha}_i}, \quad \phi_i \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \quad \phi_i \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i).$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\check{\alpha}_i(c) := c^{\check{\alpha}_i}$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL(2, \mathbb{C}))$, $H_i := \phi_i(\{\text{diag}(c, c^{-1}) \mid c \in \mathbb{C}^\times\})$. Let H be the subgroup of G generated by H_i 's with the Lie algebra \mathfrak{t} . Then H is called a **maximal torus** in G , and $B^\pm = S^\pm H$ are called the **Borel subgroups** of G . The element $\bar{s}_i := x_i(-1)y_i(1)x_i(-1) \in \text{norm}_G(H)$ is a representative of $s_i \in W = \text{norm}_G(H)/H$. Define $R(w)$ for each $w \in W$ by

$$R(w) := \{(i_1, i_2, \dots, i_l) \in I^l \mid w = s_{i_1}s_{i_2}\cdots s_{i_l}\},$$

where l is the length of w . We can associate to each $w \in W$ its standard representative $\bar{w} \in \text{norm}_G(H)$ by

$$\bar{w} = \bar{s}_{i_1}\bar{s}_{i_2}\cdots\bar{s}_{i_l},$$

for any $(i_1, i_2, \dots, i_l) \in R(w)$.

5.2 Geometric crystals

Definition 5.2.1. [33] A set X is an *ind-variety* over \mathbb{C} if there exists a filtration $X_0 \subset X_1 \subset X_2 \subset \dots$ such that

1. $\cup_{n \geq 0} X_n = X$,
2. Each X_n is a finite dimensional variety over \mathbb{C} such that the inclusion $X_n \hookrightarrow X_{n+1}$ is a closed embedding.

The geometric crystal for the simply laced affine Lie algebra \mathfrak{g} is defined as follows.

Definition 5.2.2. ([1],[33]) The geometric crystal for the simply laced affine Lie algebra \mathfrak{g} is a quadruple $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$, where X is an ind-variety, $e_i : \mathbb{C}^\times \times X \rightarrow X$ ($(c, x) \mapsto e_i^c(x)$) are rational \mathbb{C}^\times -actions and $\gamma_i, \varepsilon_i : X \rightarrow \mathbb{C}$ ($i \in I$) are rational functions satisfying the following:

1. $\{1\} \times X \subset \text{dom}(e_i)$ for any $i \in I$ where $\text{dom}(e_i)$ is the maximal open subset of $\mathbb{C}^\times \times X$ on which e_i is defined.
2. $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
3. $\{e_i\}_{i \in I}$ satisfy the following relations :
$$e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} \quad \text{if } a_{ij} = a_{ji} = 0,$$

$$e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} \quad \text{if } a_{ij} = a_{ji} = -1,$$
4. $\varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x) \quad \text{if } a_{i,j} = a_{j,i} = 0$.

For fixed $i \in I$, let G^i be the reductive algebraic group with Lie algebra \mathfrak{g}_i and B^i, W^i be its Borel subgroup, Weyl group respectively. We consider the flag variety $X^i := G^i/B^i$. For $w \in W^i$, the Schubert cell X_w^i associated with w has a natural \mathfrak{g}_i -geometric crystal structure [1, 33]. Let $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ be a reduced expression. For $\mathbf{i} := (i_1, i_2, \dots, i_l)$, set

$$B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, c_2, \dots, c_l) := Y_{i_1}(c_1) Y_{i_2}(c_2) \cdots Y_{i_l}(c_l) \mid c_1, c_2, \dots, c_l \in \mathbb{C}^\times\} \subset B^-,$$

where $Y_j(c) := y_j(\frac{1}{c}) \check{\alpha}_j(c) = y_j(\frac{1}{c}) c^{\check{\alpha}_j}$. Then we have the following result.

Theorem 5.2.3. ([1], [33]) The set $B_{\mathbf{i}}^-$ with the explicit actions of e_k^c, ε_k , and γ_k , for $k \in I_i, c \in \mathbb{C}^\times$ given by:

$$e_k^c(Y_{\mathbf{i}}(c_1, \dots, c_l)) = Y_{\mathbf{i}}(c_1, \dots, c_l),$$

where

$$\begin{aligned} \mathcal{C}_j &:= c_j \cdot \frac{\sum_{1 \leq m \leq j, i_m=k} \frac{c}{c_1^{a_{i_1,k}} \cdots c_{m-1}^{a_{i_{m-1},k}} c_m}}{\sum_{1 \leq m < j, i_m=k} \frac{c}{c_1^{a_{i_1,k}} \cdots c_{m-1}^{a_{i_{m-1},k}} c_m}} + \frac{\sum_{j < m \leq k, i_m=k} \frac{1}{c_1^{a_{i_1,k}} \cdots c_{m-1}^{a_{i_{m-1},k}} c_m}}{\sum_{j \leq m \leq k, i_m=k} \frac{1}{c_1^{a_{i_1,k}} \cdots c_{m-1}^{a_{i_{m-1},k}} c_m}}, \\ \varepsilon_k(Y_{\mathbf{i}}(c_1, \dots, c_l)) &= \sum_{1 \leq m \leq l, i_m=k} \frac{1}{c_1^{a_{i_1,k}} \cdots c_{m-1}^{a_{i_{m-1},k}} c_m}, \\ \gamma_k(Y_{\mathbf{i}}(c_1, \dots, c_l)) &= c_1^{a_{i_1,k}} \cdots c_l^{a_{i_l,k}}, \end{aligned}$$

is a geometric crystal isomorphic to X_w^i .

The geometric crystal $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is said to be *positive* if it has a positive structure ([1], [25], [33]). Roughly speaking this means that each of the rational maps e_i^c , ε_i and γ_i are given by ratio of polynomial functions with positive coefficients. For example, B_i^- is a positive geometric crystal.

5.3 Positive structure and Ultra-discretization

Set $R := \mathbb{C}(c)$ and define

$$\begin{aligned} v : R \setminus \{0\} &\rightarrow \mathbb{Z} \\ f(c) &\mapsto \deg(f(c)). \end{aligned}$$

where \deg is the degree of poles at $c = \infty$. For any algebraic torus $T = (\mathbb{C}^\times)^l$ over \mathbb{C} , let $X^*(T) := \text{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l$ (respectively $X_*(T) := \text{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l$) be the *lattice of characters* (respectively *co-characters*) of T .

We define a category \mathcal{T}_+ whose objects are algebraic tori over \mathbb{C} and whose arrows are positive rational mappings. Let $f : T \rightarrow T'$ be a positive rational mapping of algebraic tori T and T' . We define a map $\widehat{f} : X_*(T) \rightarrow X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Lemma 5.3.1. [1] If T_1, T_2, T_3 are algebraic tori and $f \in \text{Mor}^+(T_1, T_2)$ and $g \in \text{Mor}^+(T_2, T_3)$ are positive rational mappings, then we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

By Lemma 5.3.1, we obtain a functor

$$\begin{aligned} \mathcal{UD} : \quad \mathcal{T}_+ &\rightarrow \text{Set} \\ T &\mapsto X_*(T) \\ (f : T \rightarrow T') &\mapsto (\widehat{f} : X_*(T) \rightarrow X_*(T')). \end{aligned}$$

Definition 5.3.2. [1] Let $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ be a geometric crystal, T' an algebraic torus and $\theta : T' \rightarrow X$ a birational mapping. The mapping θ is called a *positive structure* on χ if it satisfies

1. for any $i \in I$, the rational functions $\gamma_i \circ \theta : T' \rightarrow \mathbb{C}$ and $\varepsilon \circ \theta : T' \rightarrow \mathbb{C}$ are positive.
2. for any $i \in I$, the rational mapping $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta : T' \rightarrow X$ be a positive structure on a geometric crystal χ . Applying the functor \mathcal{UD} to positive rational mappings $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta : T' \rightarrow \mathbb{C}$ (the notations are as above), we obtain

$$\begin{aligned}\tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T') \rightarrow X_*(T') \\ \text{wt}_i &:= \mathcal{UD}(\gamma_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z} \\ \varepsilon_i &:= \mathcal{UD}(\varepsilon_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z}.\end{aligned}$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a free pre-crystal (see [1], Sections 7) and we denote it by $\mathcal{UD}_{\theta, T'}(\chi)$. We thus have the following theorem.

Theorem 5.3.3. ([1], [33]) *For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and positive structure $\theta : T' \rightarrow X$, the associated pre-crystal $\mathcal{UD}_{\theta, T'}(\chi) = (X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a crystal.*

Now, let \mathcal{GC}^+ be a category whose object is a triplet (χ, T', θ) where $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a geometric crystal and $\theta : T' \rightarrow X$ is a positive structure on χ , and morphism $f : (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)$ is given by a rational map $\varphi : X_1 \rightarrow X_2$ ($\chi_i = (X_i, \dots)$) such that

$$\begin{aligned}\varphi \circ e_i^{X_1} &= e_i^{X_2} \circ \varphi, \quad \gamma_i^{X_2} \circ \varphi = \gamma_i^{X_1}, \quad \varepsilon_i^{X_2} \circ \varphi = \varepsilon_i^{X_1}, \\ \text{and } f &:= \theta_2^{-1} \circ \varphi \circ \theta_1 : T'_1 \rightarrow T'_2,\end{aligned}$$

is a positive rational mapping. Let \mathcal{CR} be the category of crystals. Then by the theorem above, we have

Corollary 5.3.4. *\mathcal{UD} defines a functor*

$$\begin{array}{ccc}\mathcal{UD} : & \mathcal{GC}^+ & \rightarrow \mathcal{CR} \\ & (\chi, T', \theta) & \mapsto X_*(T') \\ & (f : (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)) & \mapsto (\hat{f} : X_*(T'_1) \rightarrow X_*(T'_2)).\end{array}$$

We call the functor \mathcal{UD} *ultra-discretization* as in ([33], [34]), while it is called *tropicalization* in [1].

CHAPTER

6

AFFINE GEOMETRIC CRYSTAL
 $\mathcal{V}(D_6^{(1)})$

From now on, we assume \mathfrak{g} to be the affine Lie algebra $D_6^{(1)}$ with index set $I = \{0, 1, 2, 3, 4, 5, 6\}$ and the following GCM and corresponding Dynkin diagram.

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

```

graph LR
    0((0)) ---|2| 1((1))
    1 ---|2| 2((2))
    2 ---|1| 3((3))
    3 ---|1| 4((4))
    4 ---|1| 5((5))
    4 ---|2| 6((6))
    5 ---|2| 4
  
```

Figure 6.1: Generalized Cartan Matrix and Dynkin diagram for $\mathfrak{g} = D_6^{(1)}$

Let $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\check{\alpha}_0, \check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\}$ denote the set of simple roots, simple coroots and fundamental weights, respectively. Then $c = \check{\alpha}_0 + \check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + 2\check{\alpha}_4 + \check{\alpha}_5 + \check{\alpha}_6$ and $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$ are the central element and null root respectively. The sets $P_{cl} = \bigoplus_{j=0}^6 \mathbb{Z}\Lambda_j$ and $P = P_{cl} \oplus \mathbb{Z}\delta$ are called classical weight lattice and weight lattice respectively.

We consider the Dynkin diagram automorphism σ defined by

$$\sigma : 0 \mapsto 6, 1 \mapsto 5, 2 \mapsto 4, 3 \mapsto 3, 4 \mapsto 2, 5 \mapsto 1, 6 \mapsto 0.$$

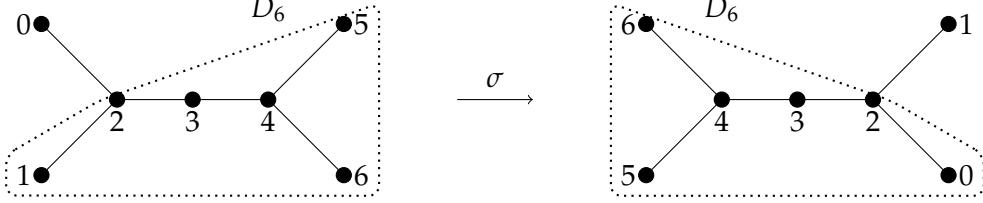


Figure 6.2: \mathfrak{g} and $\sigma(\mathfrak{g})$ Dynkin diagrams

Let $I_j = I \setminus \{j\}$ and \mathfrak{g}_j (respectively $\sigma(\mathfrak{g})_j$) be the subalgebra of \mathfrak{g} (resp. $\sigma(\mathfrak{g})$) with index set I_j . Then observe that \mathfrak{g}_0 as well as \mathfrak{g}_1 and $\sigma(\mathfrak{g})_1$ are isomorphic to D_6 .

6.1 Fundamental representation $W(\varpi_6)$ for $D_6^{(1)}$

Let $W(\varpi_6)$ be the level 0 fundamental $U'_q(\mathfrak{g})$ -module associated with the level 0 weight $\varpi_6 = \Lambda_6 - \Lambda_0$ [24]. By ([24], Theorem 5.17), $W(\varpi_6)$ is a finite dimensional irreducible integrable $U'_q(\mathfrak{g})$ -module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite dimensional $D_6^{(1)}$ -module $W(\varpi_6)$, which we call the fundamental $D_6^{(1)}$ -module.

The fundamental $D_6^{(1)}$ -module $W(\varpi_6)$ is a 32-dimensional module with the basis

$$\{(i_1, i_2, i_3, i_4, i_5, i_6) | i_k \in \{+, -\}, i_1 i_2 i_3 i_4 i_5 i_6 = +\}.$$

The actions of e_k , f_k and $\check{\alpha}_k(c) = c^{\check{\alpha}_k}$ ($c \in \mathbb{C}^\times$) on these basis vectors are given as follows [21].

For $0 \leq k \leq 6$, we have

$$e_k(i_1, i_2, i_3, i_4, i_5, i_6) = \begin{cases} (-, -, i_3, i_4, i_5, i_6) & \text{if } k = 0, (i_1, i_2) = (+, +) \\ (i_1, \dots, +, -, \dots, i_6) & \text{if } k \neq 0, k \neq 6, \\ & k \quad k+1 \quad (i_k, i_{k+1}) = (-, +) \\ (i_1, i_2, i_3, i_4, +, +) & \text{if } k = 6, (i_5, i_6) = (-, -) \\ 0 & \text{otherwise.} \end{cases}$$

$$f_k(i_1, i_2, i_3, i_4, i_5, i_6) = \begin{cases} (+, +, i_3, i_4, i_5, i_6) & \text{if } k = 0, (i_1, i_2) = (-, -) \\ (i_1, \dots, -, +, \dots, i_6) & \text{if } k \neq 0, k \neq 6, \\ & k \quad k+1 \quad (i_k, i_{k+1}) = (+, -) \\ (i_1, i_2, i_3, i_4, -, -) & \text{if } k = 6, (i_5, i_6) = (+, +) \\ 0 & \text{otherwise.} \end{cases}$$

$$\check{\alpha}_k(c)(i_1, i_2, i_3, i_4, i_5, i_6) = \begin{cases} c(i_1, i_2, i_3, i_4, i_5, i_6) & \text{if } k = 0, (i_1, i_2) = (-, -) \\ & \text{or } k \neq 0, k \neq 6, \\ & (i_k, i_{k+1}) = (+, -) \\ & \text{or } k = 6, (i_5, i_6) = (+, +) \\ c^{-1}(i_1, i_2, i_3, i_4, i_5, i_6) & \text{if } k = 0, (i_1, i_2) = (+, +) \\ & \text{or } k \neq 0, k \neq 6, \\ & (i_k, i_{k+1}) = (-, +) \\ & \text{or } k = 6, (i_5, i_6) = (-, -) \\ (i_1, i_2, i_3, i_4, i_5, i_6) & \text{otherwise.} \end{cases}$$

Note that in $W(\omega_6)$, we have $(+, +, +, +, +, +)$ (respectively $(-, +, +, +, +, -)$) is a \mathfrak{g}_0 (respectively \mathfrak{g}_1) highest weight vector with weight $\omega_6 = \Lambda_6 - \Lambda_0$ (respectively $\check{\omega}_6 := \Lambda_5 - \Lambda_1$). The crystal graph for $W(\omega_6)$ is explicitly shown below.

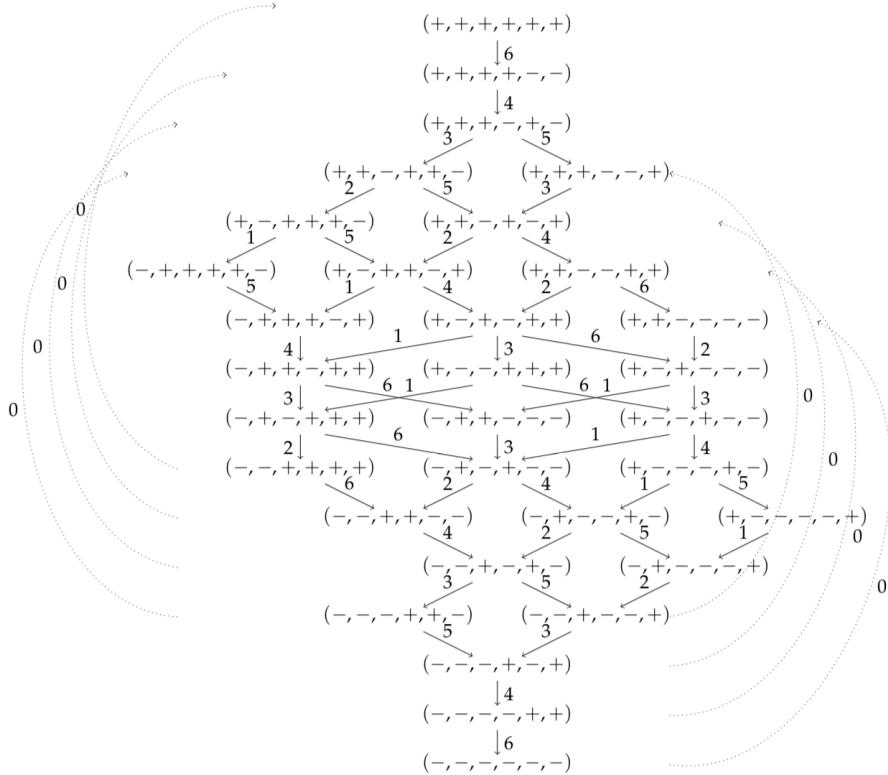


Figure 6.3: $W(\omega_6)$ $U'_q(\mathfrak{g})$ -module crystal graph

We define $\sigma(\Lambda_k) = \Lambda_{\sigma(k)}$ for all $k \in I$. Then we define the action σ on $W(\omega_6)$ by $\sigma(v) = v'$ if $\sigma(\text{wt}(v)) = \text{wt}(v')$. Thus we have

$$\begin{aligned}
\sigma(+, +, +, +, +, +) &= (-, -, -, -, -, -), & \sigma(+, +, +, +, -, -) &= (+, +, -, -, -, -), \\
\sigma(+, +, +, -, +, -) &= (+, -, +, -, -, -), & \sigma(+, +, -, +, +, -) &= (+, -, -, +, -, -), \\
\sigma(+, +, +, -, -, +) &= (-, +, +, -, -, -), & \sigma(+, -, +, +, +, -) &= (+, -, -, -, +, -), \\
\sigma(+, +, -, +, -, +) &= (-, +, -, +, -, -), & \sigma(-, +, +, +, +, -) &= (+, -, -, -, -, +), \\
\sigma(+, -, +, +, -, +) &= (-, +, -, -, +, -), & \sigma(+, +, -, -, +, +) &= (-, -, +, +, -, -), \\
\sigma(-, +, +, +, -, +) &= (-, +, -, -, -, +), & \sigma(+, -, +, -, +, +) &= (-, -, +, -, +, -), \\
\sigma(+, +, -, -, -, -) &= (+, +, +, +, -, -), & \sigma(-, +, +, -, +, +) &= (-, -, +, -, -, +), \\
\sigma(+, -, -, +, +, +) &= (-, -, -, +, +, -), & \sigma(+, -, +, -, -, -) &= (+, +, +, -, +, -), \\
\sigma(-, +, -, +, +, +) &= (-, -, -, +, -, +), & \sigma(-, +, +, -, -, -) &= (+, +, +, -, -, +), \\
\sigma(+, -, -, +, -, -) &= (+, +, -, +, +, -), & \sigma(-, -, +, +, +, +) &= (-, -, -, -, +, +), \\
\sigma(-, +, -, +, -, -) &= (+, +, -, +, -, +), & \sigma(+, -, -, -, +, -) &= (+, -, +, +, +, -), \\
\sigma(-, -, +, +, -, -) &= (+, +, -, -, +, +), & \sigma(-, +, -, +, -, -) &= (+, -, +, +, -, +), \\
\sigma(+, -, -, -, -, +) &= (-, +, +, +, +, -), & \sigma(-, -, +, -, +, -) &= (+, -, +, -, +, +), \\
\sigma(-, +, -, -, -, +) &= (-, +, +, +, -, +), & \sigma(-, -, -, +, +, -) &= (+, -, -, +, +, +), \\
\sigma(-, -, +, -, -, +) &= (-, +, +, -, +, +), & \sigma(-, -, -, +, -, +) &= (-, +, -, +, +, +), \\
\sigma(-, -, -, -, +, +) &= (-, -, +, +, +, +), & \sigma(-, -, -, -, +, -) &= (+, +, +, +, +, +).
\end{aligned}$$

6.2 Affine Geometric Crystal $\mathcal{V}(D_6^{(1)})$ in $W(\omega_6)$

Now we will construct the affine geometric crystal $\mathcal{V}(D_6^{(1)})$ in $W(\omega_6)$ explicitly. Denote $\mathfrak{t}_{cl}^* = \mathfrak{t}^*/\mathbb{C}\delta$ and $(\mathfrak{t}_{cl}^*)_0 = \{\lambda \in \mathfrak{t}_{cl}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$. For $\xi \in (\mathfrak{t}_{cl}^*)_0$, let $t(\xi)$ be the translation as in ([24], Sect 4). Define simple reflections $s_k(\lambda) := \lambda - \lambda(\check{\alpha}_k)\alpha_k$, $k \in I$ and let $W = \langle s_k \mid k \in I \rangle$ be the Weyl group for $D_6^{(1)}$. We denote W^j to be the Weyl group of \mathfrak{g}_j with index set I_j . Let $w_1 = s_6s_4s_3s_2s_5s_4s_3s_6s_4s_5s_1s_2s_3s_4s_6 \in W^0$ and $w_2 = s_5s_4s_3s_2s_6s_4s_3s_5s_4s_6s_0s_2s_3s_4s_5 \in W^1$. Then we have the following propositions.

Proposition 6.2.1. $t(\omega_6) = \sigma w_1$.

Proof. Let $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \lambda_2\Lambda_2 + \lambda_3\Lambda_3 + \lambda_4\Lambda_4 + \lambda_5\Lambda_5 + \lambda_6\Lambda_6$

$$= (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6).$$

Then we have

$$s_6(\lambda) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 + \lambda_6, \lambda_5, -\lambda_6)$$

$$s_4s_6(\lambda) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3 + \lambda_4 + \lambda_6, -\lambda_4 - \lambda_6, \lambda_4 + \lambda_5 + \lambda_6, \lambda_4)$$

$$s_3 s_4 s_6(\lambda) = (\lambda_0, \lambda_1, \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_3 - \lambda_4 - \lambda_6, \lambda_3, \lambda_4 + \lambda_5 + \lambda_6, \lambda_4)$$

$$s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_2, \lambda_3, \lambda_4 + \lambda_5 + \lambda_6, \lambda_4)$$

$$s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1, \\ \lambda_2, \lambda_3, \lambda_4 + \lambda_5 + \lambda_6, \lambda_4)$$

$$s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1, \\ \lambda_2, \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, -\lambda_4 - \lambda_5 - \lambda_6, \lambda_4)$$

$$s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1, \\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, -\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_3, \lambda_3 \\ + 2\lambda_4 + \lambda_5 + \lambda_6)$$

$$s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1, \\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, \lambda_3, -\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6)$$

$$s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, -\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, \lambda_3, -\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6)$$

$$s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, -\lambda_2 - \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, \lambda_2)$$

$$s_5 s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, \lambda_3, -\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_2)$$

$$s_2 s_5 s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6, \lambda_5, -\lambda_1 - \lambda_2 \\ - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, \\ \lambda_3, -\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_2)$$

$$\begin{aligned} s_3s_2s_5s_4s_3s_6s_4s_5s_1s_2s_3s_4s_6(\lambda) = & (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6, \lambda_5, \lambda_4, -\lambda_1 \\ & - \lambda_2 - \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 \\ & + \lambda_6, -\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_2) \end{aligned}$$

$$\begin{aligned} s_4 s_3 s_2 s_5 s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) = & (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6, \lambda_5, \lambda_4, \lambda_3, -\lambda_1 \\ & - \lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_1, \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 \\ & + \lambda_5 + \lambda_6) \end{aligned}$$

$$\begin{aligned}
s_6 s_4 s_3 s_2 s_5 s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) &= (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6, \lambda_5, \lambda_4, \lambda_3, \lambda_2, \\
&\quad \lambda_1, -\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6) \\
\sigma s_6 s_4 s_3 s_2 s_5 s_4 s_3 s_6 s_4 s_5 s_1 s_2 s_3 s_4 s_6(\lambda) &= (-\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \\
&\quad \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_6) \\
&= \lambda + (-\lambda_0 - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, 0, 0, 0, 0, 0, \\
&\quad \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6) \\
&= \lambda + (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6)(\Lambda_6 - \Lambda_0) \\
&= \lambda + (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6)\varpi_6.
\end{aligned}$$

□

Proposition 6.2.2. $t(\check{\omega}_6) = \sigma w_2$.

Proof. Let $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \lambda_2\Lambda_2 + \lambda_3\Lambda_3 + \lambda_4\Lambda_4 + \lambda_5\Lambda_5 + \lambda_6\Lambda_6$
 $= (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$.

Then we have

$$\begin{aligned}
s_5(\lambda) &= (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 + \lambda_5, -\lambda_5, \lambda_6) \\
s_4 s_5(\lambda) &= (\lambda_0, \lambda_1, \lambda_2, \lambda_3 + \lambda_4 + \lambda_5, -\lambda_4 - \lambda_5, \lambda_4, \lambda_4 + \lambda_5 + \lambda_6) \\
s_3 s_4 s_5(\lambda) &= (\lambda_0, \lambda_1, \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, -\lambda_3 - \lambda_4 - \lambda_5, \lambda_3, \lambda_4, \lambda_4 \\
&\quad + \lambda_5 + \lambda_6) \\
s_2 s_3 s_4 s_5(\lambda) &= (\lambda_0 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, -\lambda_2 \\
&\quad - \lambda_3 - \lambda_4 - \lambda_5, \lambda_2, \lambda_3, \lambda_4, \lambda_4 + \lambda_5 + \lambda_6) \\
s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0, \\
&\quad \lambda_2, \lambda_3, \lambda_4, \lambda_4 + \lambda_5 + \lambda_6) \\
s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0, \\
&\quad \lambda_2, \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, -\lambda_4 - \lambda_5 - \lambda_6) \\
s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0, \\
&\quad \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, -\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_3 + 2\lambda_4 \\
&\quad + \lambda_5 + \lambda_6, \lambda_3) \\
s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0, \\
&\quad \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, -\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_3) \\
s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0 \\
&\quad + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, -\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_2 \\
&\quad + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, -\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_3)
\end{aligned}$$

$$\begin{aligned}
s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0 \\
&\quad + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, -\lambda_2 - \lambda_3 - 2\lambda_4 - \lambda_5 \\
&\quad - \lambda_6, \lambda_2, \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6) \\
s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (-\lambda_0 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \lambda_0 \\
&\quad + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_4, \lambda_3, \lambda_2, -\lambda_2 - 2\lambda_3 - 2\lambda_4 \\
&\quad - \lambda_5 - \lambda_6) \\
s_2 s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (\lambda_6, \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6, -\lambda_0 - \lambda_2 \\
&\quad - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \lambda_0 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, \\
&\quad \lambda_3, \lambda_2, -\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6) \\
s_3 s_2 s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (\lambda_6, \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6, \lambda_4, -\lambda_0 \\
&\quad - \lambda_2 - \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_0 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 \\
&\quad + \lambda_6, \lambda_2, -\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6) \\
s_4 s_3 s_2 s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (\lambda_6, \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6, \lambda_4, \lambda_3, -\lambda_0 \\
&\quad - \lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_0 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 \\
&\quad + \lambda_6, \lambda_0) \\
s_5 s_4 s_3 s_2 s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (\lambda_6, \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6, \lambda_4, \lambda_3, \lambda_2, \\
&\quad - \lambda_0 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_0) \\
\sigma s_5 s_4 s_3 s_2 s_6 s_4 s_3 s_5 s_4 s_6 s_0 s_2 s_3 s_4 s_5(\lambda) &= (\lambda_0, -\lambda_0 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, \lambda_2, \lambda_3, \lambda_4, \lambda_0 + \lambda_1 \\
&\quad + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6, \lambda_6) \\
&= \lambda + (0, -\lambda_0 - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6, 0, 0, 0, \\
&\quad \lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6, 0) \\
&= \lambda + (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6)(\Lambda_5 - \Lambda_1) \\
&= \lambda + (\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6)\check{\omega}_6.
\end{aligned}$$

□

Associated with these Weyl group elements $w_1, w_2 \in W$, we define algebraic varieties $\mathcal{V}_1, \mathcal{V}_2 \subset W(\varpi_6)$ as follows.

$$\begin{aligned}
\mathcal{V}_1 &= \{ V_1(x) := Y_6(x_6^{(3)})Y_4(x_4^{(4)})Y_3(x_3^{(3)})Y_2(x_2^{(2)})Y_5(x_5^{(2)})Y_4(x_4^{(3)})Y_3(x_3^{(2)})Y_6(x_6^{(2)})Y_4(x_4^{(2)}) \\
&\quad Y_5(x_5^{(1)})Y_1(x_1^{(1)})Y_2(x_2^{(1)})Y_3(x_3^{(1)})Y_4(x_4^{(1)})Y_6(x_6^{(1)})(+, +, +, +, +, +) | x_m^{(l)} \in \mathbb{C}^\times \}, \\
\mathcal{V}_2 &= \{ V_2(y) := Y_5(y_5^{(3)})Y_4(y_4^{(4)})Y_3(y_3^{(3)})Y_2(y_2^{(2)})Y_6(y_6^{(2)})Y_4(y_4^{(3)})Y_3(y_3^{(2)})Y_5(y_5^{(2)})Y_4(y_4^{(2)}) \\
&\quad Y_6(y_6^{(1)})Y_0(y_0^{(1)})Y_2(y_2^{(1)})Y_3(y_3^{(1)})Y_4(y_4^{(1)})Y_5(y_5^{(1)})(-, +, +, +, +, -) | y_m^{(l)} \in \mathbb{C}^\times \}
\end{aligned}$$

where $x = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)})$ and $y = (y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, y_2^{(2)}, y_5^{(2)}, y_4^{(2)}, y_3^{(1)}, y_5^{(1)}, y_4^{(1)}, y_0^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)}, y_5^{(1)})$.

From the explicit actions of f_k 's on $W(\omega_6)$, we observe that $f_k^2 = 0$ for all $k \in I$. Therefore, we have

$$Y_k(c) = (1 + \frac{f_k}{c})\check{\alpha}_k(c) \text{ for all } k \in I.$$

Thus we have the explicit forms of $V_1(x)$ and $V_2(y)$ as follows.

$$\begin{aligned} V_1(x) = & x_6^{(3)}x_6^{(2)}x_6^{(1)}(+, +, +, +, +, +) + (x_6^{(2)}x_6^{(1)} + \frac{x_4^{(4)}x_4^{(3)}x_6^{(1)}}{x_6^{(3)}} + \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(3)}x_6^{(2)}}) \\ & \times (+, +, +, +, -, -) + (x_4^{(3)}x_6^{(1)} + \frac{x_3^{(3)}x_5^{(2)}x_6^{(1)}}{x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} + \frac{x_3^{(3)}x_5^{(2)}x_3^{(2)}x_4^{(1)}}{x_4^{(4)}x_4^{(3)}} \\ & + \frac{x_3^{(3)}x_5^{(2)}x_4^{(2)}x_4^{(1)}}{x_4^{(4)}x_6^{(2)}} + \frac{x_3^{(3)}x_5^{(2)}x_3^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(4)}x_4^{(3)}x_4^{(2)}})(+, +, +, -, +, -) + (x_5^{(2)}x_6^{(1)} + \frac{x_2^{(2)}x_5^{(2)}x_4^{(1)}}{x_3^{(3)}} \\ & + \frac{x_5^{(2)}x_3^{(2)}x_4^{(1)}}{x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)}x_5^{(2)}x_5^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_2^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(1)}}{x_3^{(3)}x_4^{(2)}}) \\ & \times (+, +, -, +, +, -) + (x_3^{(3)}x_6^{(1)} + \frac{x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_3^{(3)}x_3^{(2)}x_4^{(1)}}{x_4^{(3)}} + \frac{x_3^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} \\ & + \frac{x_3^{(3)}x_3^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}})(+, +, +, -, -, +) + (x_5^{(2)}x_4^{(1)} + \frac{x_5^{(2)}x_5^{(1)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_5^{(2)}x_5^{(1)}x_2^{(1)}}{x_3^{(2)}} \\ & + \frac{x_5^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(2)}})(+, -, +, +, +, -) + (x_4^{(4)}x_6^{(1)} + \frac{x_4^{(4)}x_2^{(2)}x_4^{(1)}}{x_3^{(3)}} + \frac{x_4^{(4)}x_3^{(2)}x_3^{(1)}}{x_5^{(2)}} \\ & + \frac{x_4^{(4)}x_3^{(2)}x_4^{(1)}}{x_4^{(3)}} + \frac{x_4^{(4)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} + \frac{x_4^{(4)}x_2^{(2)}x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_5^{(2)}} + \frac{x_4^{(4)}x_2^{(2)}x_5^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_4^{(4)}x_2^{(2)}x_5^{(1)}x_3^{(1)}}{x_3^{(3)}x_4^{(2)}}) \\ & + \frac{x_4^{(4)}x_3^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(4)}x_2^{(2)}x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_3^{(3)}x_5^{(2)}x_3^{(2)}})(+, +, -, +, -, +) + x_5^{(2)}x_5^{(1)} \\ & \times (-, +, +, +, +, -) + (x_4^{(4)}x_4^{(1)} + \frac{x_4^{(4)}x_5^{(1)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_4^{(4)}x_4^{(3)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_4^{(4)}x_5^{(1)}x_2^{(1)}}{x_3^{(2)}} \\ & + \frac{x_4^{(4)}x_5^{(1)}x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_1^{(1)}}{x_2^{(2)}x_5^{(2)}} + \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(2)}})(+, -, +, +, -, +) + (x_6^{(3)}x_6^{(1)} \\ & + \frac{x_6^{(3)}x_2^{(2)}x_3^{(1)}}{x_4^{(4)}} + \frac{x_6^{(3)}x_2^{(2)}x_4^{(1)}}{x_3^{(3)}} + \frac{x_6^{(3)}x_3^{(2)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_6^{(3)}x_3^{(2)}x_4^{(1)}}{x_4^{(3)}} + \frac{x_6^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}}) \end{aligned}$$

$$\begin{aligned}
& + \frac{x_6^{(3)}x_2^{(2)}x_6^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_6^{(3)}x_2^{(2)}x_4^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_6^{(3)}x_2^{(2)}x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_5^{(2)}} + \frac{x_6^{(3)}x_2^{(2)}x_5^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} \\
& + \frac{x_6^{(3)}x_2^{(2)}x_5^{(1)}x_3^{(1)}}{x_3^{(3)}x_4^{(2)}} + \frac{x_6^{(3)}x_3^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_6^{(3)}x_2^{(2)}x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_3^{(3)}x_5^{(2)}x_3^{(2)}})(+, +, -, -, +, +) \\
& + (x_4^{(4)}x_5^{(1)} + \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}}{x_5^{(2)}})(-, +, +, +, -, +) + (x_6^{(3)}x_4^{(1)} + \frac{x_6^{(3)}x_3^{(3)}x_3^{(1)}}{x_4^{(4)}} + \frac{x_6^{(3)}x_5^{(1)}x_1^{(1)}}{x_2^{(2)}} \\
& + \frac{x_6^{(3)}x_4^{(3)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_6^{(3)}x_5^{(1)}x_2^{(1)}}{x_3^{(2)}} + \frac{x_6^{(3)}x_5^{(1)}x_3^{(1)}}{x_4^{(2)}} + \frac{x_6^{(3)}x_3^{(3)}x_4^{(2)}x_1^{(1)}}{x_4^{(4)}x_2^{(2)}} + \frac{x_6^{(3)}x_3^{(3)}x_6^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} \\
& + \frac{x_6^{(3)}x_3^{(3)}x_4^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_6^{(3)}x_4^{(3)}x_4^{(2)}x_1^{(1)}}{x_2^{(2)}x_5^{(2)}} + \frac{x_6^{(3)}x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(2)}} + \frac{x_6^{(3)}x_3^{(3)}x_3^{(2)}x_6^{(2)}x_1^{(1)}}{x_4^{(4)}x_2^{(2)}x_4^{(3)}}) \\
& \times (+, -, +, -, +, +) + (x_6^{(1)} + \frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_2^{(2)}x_3^{(1)}}{x_4^{(4)}} + \frac{x_2^{(2)}x_4^{(1)}}{x_3^{(3)}} + \frac{x_2^{(2)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_2^{(2)}x_4^{(1)}}{x_4^{(3)}} \\
& + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)}x_6^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_2^{(2)}x_4^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_2^{(2)}x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_5^{(2)}} + \frac{x_2^{(2)}x_5^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_2^{(2)}x_5^{(1)}x_3^{(1)}}{x_3^{(3)}x_4^{(2)}} \\
& + \frac{x_3^{(2)}x_5^{(1)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_2^{(2)}x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_3^{(3)}x_5^{(2)}x_3^{(2)}})(+, +, -, -, -, -) + (x_6^{(3)}x_5^{(1)} + \frac{x_6^{(3)}x_3^{(3)}x_4^{(2)}}{x_4^{(4)}} \\
& + \frac{x_6^{(3)}x_4^{(3)}x_4^{(2)}}{x_5^{(2)}} + \frac{x_6^{(3)}x_3^{(3)}x_3^{(2)}x_6^{(2)}}{x_4^{(4)}x_4^{(3)}}) \times (-, +, +, -, +, +) + (x_6^{(3)}x_3^{(1)} + \frac{x_6^{(3)}x_6^{(2)}x_1^{(1)}}{x_3^{(3)}} \\
& + \frac{x_6^{(3)}x_4^{(2)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_6^{(3)}x_6^{(2)}x_2^{(1)}}{x_4^{(3)}} + \frac{x_6^{(3)}x_4^{(2)}x_2^{(1)}}{x_3^{(2)}} + \frac{x_6^{(3)}x_3^{(2)}x_6^{(2)}x_1^{(1)}}{x_2^{(2)}x_4^{(3)}})(+, -, -, +, +, +) \\
& + (x_4^{(1)} + \frac{x_3^{(3)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_3^{(3)}x_3^{(1)}}{x_4^{(4)}} + \frac{x_5^{(1)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_5^{(1)}x_2^{(1)}}{x_3^{(2)}} + \frac{x_5^{(1)}x_3^{(1)}}{x_4^{(2)}} + \frac{x_3^{(3)}x_3^{(2)}x_1^{(1)}}{x_6^{(3)}x_2^{(2)}} \\
& + \frac{x_3^{(3)}x_4^{(2)}x_1^{(1)}}{x_4^{(4)}x_2^{(2)}} + \frac{x_3^{(3)}x_6^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_3^{(3)}x_4^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_1^{(1)}}{x_2^{(2)}x_5^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(2)}} \\
& + \frac{x_3^{(3)}x_3^{(2)}x_6^{(2)}x_1^{(1)}}{x_4^{(4)}x_2^{(2)}x_4^{(3)}})(+, -, +, -, -, -) + (x_6^{(3)}x_4^{(2)} + \frac{x_6^{(3)}x_2^{(2)}x_6^{(2)}}{x_3^{(3)}} + \frac{x_6^{(3)}x_3^{(2)}x_6^{(2)}}{x_4^{(3)}}) \\
& \times (-, +, -, +, +, +) + (x_5^{(1)} + \frac{x_3^{(3)}x_3^{(2)}}{x_6^{(3)}} + \frac{x_3^{(3)}x_4^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_5^{(2)}} + \frac{x_3^{(3)}x_3^{(2)}x_6^{(2)}}{x_4^{(4)}x_4^{(3)}}) \\
& \times (-, +, +, -, -, -) + (x_3^{(1)} + \frac{x_4^{(4)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_1^{(1)}}{x_3^{(3)}} + \frac{x_4^{(2)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_6^{(2)}x_2^{(1)}}{x_4^{(3)}} + \frac{x_4^{(2)}x_2^{(1)}}{x_3^{(2)}} \\
& + \frac{x_4^{(4)}x_4^{(3)}x_1^{(1)}}{x_6^{(3)}x_3^{(3)}} + \frac{x_4^{(4)}x_3^{(2)}x_1^{(1)}}{x_6^{(3)}x_2^{(2)}} + \frac{x_3^{(2)}x_6^{(2)}x_1^{(1)}}{x_2^{(2)}x_4^{(3)}})(+, -, -, +, -, -) + x_6^{(3)}x_6^{(2)}
\end{aligned}$$

$$\begin{aligned}
& \times (-, -, +, +, +, +) + (x_4^{(2)} + \frac{x_4^{(4)}x_3^{(2)}}{x_6^{(3)}} + \frac{x_2^{(2)}x_6^{(2)}}{x_3^{(3)}} + \frac{x_3^{(2)}x_6^{(2)}}{x_4^{(3)}} + \frac{x_4^{(4)}x_2^{(2)}x_4^{(3)}}{x_6^{(3)}x_3^{(3)}}) \\
& \times (-, +, -, +, -, -) + (x_2^{(1)} + \frac{x_5^{(2)}x_1^{(1)}}{x_4^{(4)}} + \frac{x_4^{(3)}x_1^{(1)}}{x_3^{(3)}} + \frac{x_3^{(2)}x_1^{(1)}}{x_2^{(2)}})(+, -, -, -, +, -) \\
& + (x_6^{(2)} + \frac{x_4^{(4)}x_4^{(3)}}{x_6^{(3)}})(-, -, +, +, -, -) + (x_3^{(2)} + \frac{x_2^{(2)}x_5^{(2)}}{x_4^{(4)}} + \frac{x_2^{(2)}x_4^{(3)}}{x_3^{(3)}})(-, +, -, -, +, -) \\
& + x_1^{(1)}(+, -, -, -, -, +) + (x_4^{(3)} + \frac{x_3^{(3)}x_5^{(2)}}{x_4^{(4)}})(-, -, +, -, +, -) + x_2^{(2)} \\
& \times (-, +, -, -, -, +) + x_5^{(2)}(-, -, -, +, +, -) + x_3^{(3)}(-, -, +, -, -, +) + x_4^{(4)} \\
& \times (-, -, -, +, -, +) + x_6^{(3)}(-, -, -, -, +, +) + (-, -, -, -, -, -), \\
V_2(y) &= y_5^{(3)}y_5^{(2)}y_5^{(1)}(-, +, +, +, +, -) + (y_5^{(2)}y_5^{(1)} + \frac{y_4^{(4)}y_4^{(3)}y_5^{(1)}}{y_5^{(3)}} + \frac{y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}{y_5^{(3)}y_5^{(2)}}) \\
& \times (-, +, +, +, -, +) + (y_4^{(3)}y_5^{(1)} + \frac{y_3^{(3)}y_6^{(2)}y_5^{(1)}}{y_4^{(4)}} + \frac{y_4^{(3)}y_4^{(2)}y_4^{(1)}}{y_5^{(2)}} + \frac{y_3^{(3)}y_6^{(2)}y_3^{(2)}y_4^{(1)}}{y_4^{(4)}y_4^{(3)}} \\
& + \frac{y_3^{(3)}y_6^{(2)}y_4^{(2)}y_4^{(1)}}{y_4^{(4)}y_5^{(2)}} + \frac{y_3^{(3)}y_6^{(2)}y_3^{(2)}y_6^{(1)}y_3^{(1)}}{y_4^{(4)}y_4^{(3)}y_4^{(2)}})(-, +, +, -, +, +) + (y_6^{(2)}y_5^{(1)} + \frac{y_2^{(2)}y_6^{(2)}y_4^{(1)}}{y_3^{(3)}}) \\
& + \frac{y_6^{(2)}y_3^{(2)}y_4^{(1)}}{y_4^{(3)}} + \frac{y_6^{(2)}y_4^{(2)}y_4^{(1)}}{y_5^{(2)}} + \frac{y_2^{(2)}y_6^{(2)}y_6^{(1)}y_2^{(1)}}{y_3^{(3)}y_3^{(2)}} + \frac{y_2^{(2)}y_6^{(2)}y_6^{(1)}y_3^{(1)}}{y_3^{(3)}y_4^{(2)}} + \frac{y_6^{(2)}y_3^{(2)}y_6^{(1)}y_3^{(1)}}{y_4^{(3)}y_4^{(2)}} \\
& \times (-, +, -, +, +, +) + (y_3^{(3)}y_5^{(1)} + \frac{y_3^{(3)}y_3^{(2)}y_3^{(1)}}{y_6^{(2)}} + \frac{y_3^{(3)}y_3^{(2)}y_4^{(1)}}{y_4^{(3)}} + \frac{y_3^{(3)}y_4^{(2)}y_4^{(1)}}{y_5^{(2)}} \\
& + \frac{y_3^{(3)}y_3^{(2)}y_6^{(1)}y_3^{(1)}}{y_4^{(3)}y_4^{(2)}})(-, +, +, -, -, -) + (y_6^{(2)}y_4^{(1)} + \frac{y_6^{(2)}y_6^{(1)}y_0^{(1)}}{y_2^{(2)}} + \frac{y_6^{(2)}y_6^{(1)}y_2^{(1)}}{y_3^{(2)}}) \\
& + \frac{y_6^{(2)}y_6^{(1)}y_3^{(1)}}{y_4^{(2)}})(-, -, +, +, +, +) + (y_4^{(4)}y_5^{(1)} + \frac{y_4^{(4)}y_2^{(2)}y_4^{(1)}}{y_3^{(3)}} + \frac{y_4^{(4)}y_3^{(2)}y_3^{(1)}}{y_6^{(2)}} + \frac{y_4^{(4)}y_3^{(2)}y_4^{(1)}}{y_4^{(3)}}) \\
& + \frac{y_4^{(4)}y_4^{(2)}y_4^{(1)}}{y_5^{(2)}} + \frac{y_4^{(4)}y_2^{(2)}y_4^{(3)}y_3^{(1)}}{y_3^{(3)}y_6^{(2)}} + \frac{y_4^{(4)}y_2^{(2)}y_6^{(1)}y_2^{(1)}}{y_3^{(3)}y_3^{(2)}} + \frac{y_4^{(4)}y_2^{(2)}y_6^{(1)}y_3^{(1)}}{y_3^{(3)}y_4^{(2)}} + \frac{y_4^{(4)}y_3^{(2)}y_6^{(1)}y_3^{(1)}}{y_4^{(3)}y_4^{(2)}} \\
& + \frac{y_4^{(4)}y_2^{(2)}y_4^{(3)}y_4^{(2)}y_2^{(1)}}{y_3^{(3)}y_6^{(2)}y_3^{(2)}})(-, +, -, +, -, -) + y_6^{(2)}y_6^{(1)}(+, +, +, +, +) + (y_4^{(4)}y_4^{(1)} \\
& + \frac{y_4^{(4)}y_6^{(1)}y_0^{(1)}}{y_2^{(2)}} + \frac{y_4^{(4)}y_4^{(3)}y_3^{(1)}}{y_6^{(2)}} + \frac{y_4^{(4)}y_6^{(1)}y_2^{(1)}}{y_3^{(2)}} + \frac{y_4^{(4)}y_6^{(1)}y_3^{(1)}}{y_4^{(2)}} + \frac{y_4^{(4)}y_4^{(3)}y_4^{(2)}y_0^{(1)}}{y_2^{(2)}y_6^{(2)}})
\end{aligned}$$

$$\begin{aligned}
& + \frac{y_4^{(4)} y_4^{(3)} y_4^{(2)} x_2^{(1)}}{y_6^{(2)} y_3^{(2)}}) (-, -, +, +, -, -) + (y_5^{(3)} y_5^{(1)} + \frac{y_5^{(3)} y_2^{(2)} y_3^{(1)}}{y_4^{(4)}} + \frac{y_5^{(3)} y_2^{(2)} y_4^{(1)}}{y_3^{(3)}} \\
& + \frac{y_5^{(3)} y_3^{(2)} y_3^{(1)}}{y_6^{(2)}} + \frac{y_5^{(3)} y_3^{(2)} y_4^{(1)}}{y_4^{(3)}} + \frac{y_5^{(3)} y_4^{(2)} y_4^{(1)}}{y_5^{(2)}} + \frac{y_5^{(3)} y_2^{(2)} y_5^{(1)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_5^{(3)} y_2^{(2)} y_4^{(2)} y_2^{(1)}}{y_4^{(4)} y_3^{(2)}} \\
& + \frac{y_5^{(3)} y_2^{(2)} y_4^{(3)} y_3^{(1)}}{y_3^{(3)} y_6^{(2)}} + \frac{y_5^{(3)} y_2^{(2)} y_6^{(1)} y_2^{(1)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_5^{(3)} y_2^{(2)} y_6^{(1)} y_3^{(1)}}{y_3^{(3)} y_4^{(2)}} + \frac{y_5^{(3)} y_3^{(2)} y_6^{(1)} y_3^{(1)}}{y_4^{(3)} y_4^{(2)}} \\
& + \frac{y_5^{(3)} y_2^{(2)} y_4^{(3)} y_2^{(1)}}{y_3^{(3)} y_6^{(2)} y_3^{(2)}}) (-, +, -, -, +, -) + (y_4^{(4)} y_6^{(1)} + \frac{y_4^{(4)} y_4^{(3)} y_4^{(2)}}{y_6^{(2)}}) (+, +, +, +, -, -) \\
& + (y_5^{(3)} y_4^{(1)} + \frac{y_5^{(3)} y_3^{(3)} y_3^{(1)}}{y_4^{(4)}} + \frac{y_5^{(3)} y_6^{(1)} y_0^{(1)}}{y_2^{(2)}} + \frac{y_5^{(3)} y_4^{(3)} y_3^{(1)}}{y_6^{(2)}} + \frac{y_5^{(3)} y_6^{(1)} y_2^{(1)}}{y_3^{(2)}} + \frac{y_5^{(3)} y_6^{(1)} y_3^{(1)}}{y_4^{(2)}} \\
& + \frac{y_5^{(3)} y_3^{(3)} y_4^{(2)} y_0^{(1)}}{y_4^{(4)} y_2^{(2)}} + \frac{y_5^{(3)} y_3^{(3)} y_5^{(2)} y_2^{(1)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_5^{(3)} y_3^{(3)} y_4^{(2)} y_2^{(1)}}{y_4^{(4)} y_3^{(2)}} + \frac{y_5^{(3)} y_4^{(3)} y_4^{(2)} y_0^{(1)}}{y_2^{(2)} y_6^{(2)}} \\
& + \frac{y_5^{(3)} y_4^{(3)} y_4^{(2)} y_2^{(1)}}{y_6^{(2)} y_3^{(2)}} + \frac{y_5^{(3)} y_3^{(3)} y_3^{(2)} y_5^{(1)}}{y_4^{(4)} y_2^{(2)} y_4^{(3)}}) (-, -, +, -, +, -) + (y_5^{(1)} + \frac{y_2^{(2)} y_2^{(1)}}{y_5^{(3)}} + \frac{y_2^{(2)} y_3^{(1)}}{y_4^{(4)}} \\
& + \frac{y_2^{(2)} y_4^{(1)}}{y_3^{(3)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_6^{(2)}} + \frac{y_3^{(2)} y_4^{(1)}}{y_4^{(3)}} + \frac{y_4^{(2)} y_4^{(1)}}{y_5^{(2)}} + \frac{y_2^{(2)} y_5^{(2)} y_2^{(1)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_2^{(2)} y_4^{(2)} y_2^{(1)}}{y_4^{(4)} y_3^{(2)}} + \frac{y_2^{(2)} y_4^{(3)} y_3^{(1)}}{y_3^{(3)} y_6^{(2)}} \\
& + \frac{y_2^{(2)} y_6^{(1)} y_2^{(1)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_2^{(2)} y_6^{(1)} y_3^{(1)}}{y_3^{(3)} y_4^{(2)}} + \frac{y_3^{(2)} y_6^{(1)} y_3^{(1)}}{y_4^{(3)} y_4^{(2)}} + \frac{y_2^{(2)} y_4^{(3)} y_4^{(2)} y_2^{(1)}}{y_3^{(3)} y_6^{(2)} y_3^{(2)})} (-, +, -, -, -, +) \\
& + (y_5^{(3)} y_6^{(1)} + \frac{y_5^{(3)} y_3^{(3)} y_4^{(2)}}{y_4^{(4)}} + \frac{y_5^{(3)} y_4^{(3)} y_4^{(2)}}{y_6^{(2)}} + \frac{y_5^{(3)} y_3^{(3)} y_3^{(2)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)}}) (+, +, +, -, +, -) \\
& + (y_5^{(3)} y_3^{(1)} + \frac{y_5^{(3)} y_5^{(2)} y_0^{(1)}}{y_3^{(3)}} + \frac{y_5^{(3)} y_4^{(2)} y_0^{(1)}}{y_2^{(2)}} + \frac{y_5^{(3)} y_5^{(2)} y_2^{(1)}}{y_4^{(3)}} + \frac{y_5^{(3)} y_4^{(2)} y_2^{(1)}}{y_3^{(2)}} + \frac{y_5^{(3)} y_3^{(2)} y_5^{(2)} y_0^{(1)}}{y_2^{(2)} y_4^{(3)}}) \\
& \times (-, -, -, +, +, -) + (y_4^{(1)} + \frac{y_3^{(3)} y_2^{(1)}}{y_5^{(3)}} + \frac{y_3^{(3)} y_3^{(1)}}{y_4^{(4)}} + \frac{y_6^{(1)} y_0^{(1)}}{y_2^{(2)}} + \frac{y_4^{(1)} y_3^{(1)}}{y_6^{(2)}} + \frac{y_6^{(1)} y_2^{(1)}}{y_3^{(2)}} \\
& + \frac{y_6^{(1)} y_3^{(1)}}{y_4^{(2)}} + \frac{y_3^{(3)} y_3^{(2)} y_0^{(1)}}{y_5^{(3)} y_2^{(2)}} + \frac{y_3^{(3)} y_4^{(2)} y_0^{(1)}}{y_4^{(4)} y_2^{(2)}} + \frac{y_3^{(3)} y_5^{(2)} y_2^{(1)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_3^{(3)} y_4^{(2)} y_2^{(1)}}{y_4^{(4)} y_3^{(2)}} + \frac{y_4^{(3)} y_4^{(2)} y_0^{(1)}}{y_2^{(2)} y_6^{(2)}} \\
& + \frac{y_4^{(3)} y_4^{(2)} y_2^{(1)}}{y_6^{(2)} y_3^{(2)}} + \frac{y_3^{(3)} y_3^{(2)} y_5^{(2)} y_0^{(1)}}{y_4^{(4)} y_2^{(2)} y_4^{(3)}}) (-, -, +, -, -, +) + (y_5^{(3)} y_4^{(2)} + \frac{y_5^{(3)} y_2^{(2)} y_5^{(2)}}{y_3^{(3)}} \\
& + \frac{y_5^{(3)} y_3^{(2)} y_5^{(2)}}{y_4^{(3)}}) (+, +, -, +, +, -) + (y_6^{(1)} + \frac{y_3^{(3)} y_3^{(2)}}{y_5^{(3)}} + \frac{y_3^{(3)} y_4^{(2)}}{y_4^{(4)}} + \frac{y_4^{(3)} y_4^{(2)}}{y_6^{(2)}} \\
& + \frac{y_3^{(3)} y_3^{(2)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)}}) (+, +, +, -, +) + (y_3^{(1)} + \frac{y_4^{(4)} y_2^{(1)}}{y_5^{(3)}} + \frac{y_5^{(2)} y_0^{(1)}}{y_3^{(3)}} + \frac{y_4^{(2)} y_0^{(1)}}{y_2^{(2)}} + \frac{y_5^{(2)} y_2^{(1)}}{y_4^{(3)}})
\end{aligned}$$

$$\begin{aligned}
& + \frac{y_4^{(2)} y_2^{(1)}}{y_3^{(2)}} + \frac{y_4^{(4)} y_4^{(3)} y_0^{(1)}}{y_5^{(3)} y_3^{(3)}} + \frac{y_4^{(4)} y_3^{(2)} y_0^{(1)}}{y_5^{(3)} y_2^{(2)}} + \frac{y_3^{(2)} y_5^{(2)} y_0^{(1)}}{y_2^{(2)} y_4^{(3)}}) (-, -, -, +, -, +) + y_5^{(3)} y_5^{(2)} \\
& \times (+, -, +, +, +, -) + (y_4^{(2)} + \frac{y_4^{(4)} y_3^{(2)}}{y_5^{(3)}} + \frac{y_2^{(2)} y_5^{(2)}}{y_3^{(3)}} + \frac{y_3^{(2)} y_5^{(2)}}{y_4^{(3)}} + \frac{y_4^{(4)} y_2^{(2)} y_4^{(3)}}{y_5^{(3)} y_3^{(3)}}) \\
& \times (+, +, -, +, -, +) + (y_2^{(1)} + \frac{y_6^{(2)} y_0^{(1)}}{y_4^{(4)}} + \frac{y_4^{(3)} y_0^{(1)}}{y_3^{(3)}} + \frac{y_3^{(2)} y_0^{(1)}}{y_2^{(2)}}) (-, -, -, -, +, +) \\
& + (y_5^{(2)} + \frac{y_4^{(4)} y_4^{(3)}}{y_5^{(3)}}) (+, -, +, +, -, +) + (y_3^{(2)} + \frac{y_2^{(2)} y_6^{(2)}}{y_4^{(4)}} + \frac{y_2^{(2)} y_4^{(3)}}{y_3^{(3)}}) (+, +, -, -, +, +) \\
& + y_0^{(1)} (-, -, -, -, -, -) + (y_4^{(3)} + \frac{y_3^{(3)} y_6^{(2)}}{y_4^{(4)}}) (+, -, +, -, +, +) + y_2^{(2)} (+, +, -, -, -, -) \\
& + y_6^{(2)} (+, -, -, +, +, +) + y_3^{(3)} (+, -, +, -, -, -) + y_4^{(4)} (+, -, -, +, -, -) + y_5^{(3)} \\
& \times (+, -, -, -, +, -) + (+, -, -, -, -, +).
\end{aligned}$$

Now for a given x we solve the equation

$$V_2(y) = a(x)\sigma(V_1(x)). \quad (6.2.1)$$

where $a(x)$ is a rational function in x and the action of σ on $V_1(x)$ is induced by its action on $W(\omega_6)$. Though this equation is over-determined, it can be solved uniquely by comparing the coefficients of the basis vectors of $W(\omega_6)$. We give the explicit solutions of $a(x)$, and the variables $y_m^{(l)}$ below.

Lemma 6.2.3. *The rational function $a(x)$ and the complete solution of (6.2.1) is:*

$$\begin{aligned}
a(x) &= \frac{1}{x_5^{(2)} x_5^{(1)}}, \\
y_0^{(1)} &= \frac{x_6^{(3)} x_6^{(2)} x_6^{(1)}}{x_5^{(2)} x_5^{(1)}}, \quad y_6^{(1)} = \frac{1}{x_5^{(2)}}, \quad y_6^{(2)} = \frac{1}{x_5^{(1)}}, \\
y_2^{(1)} &= \left(\frac{x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_6^{(2)}} + \frac{x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_4^{(2)} x_4^{(1)}} + \frac{x_6^{(2)} x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \right)^{-1}, \\
y_2^{(2)} &= \frac{x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{(x_5^{(2)})^2 (x_5^{(1)})^2} \left(\frac{x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_6^{(2)}} + \frac{x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_4^{(2)} x_4^{(1)}} + \frac{x_6^{(2)} x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \right), \\
y_3^{(1)} &= \left(\frac{x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_4^{(2)}} + \frac{x_6^{(2)} x_5^{(2)} x_5^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)}} + \frac{x_5^{(2)} x_4^{(1)}}{x_6^{(3)} x_3^{(1)}} + \frac{x_6^{(2)} x_5^{(2)} x_4^{(1)}}{x_4^{(4)} x_4^{(3)} x_3^{(1)}} + \frac{x_5^{(2)} x_4^{(2)} x_4^{(1)}}{x_4^{(4)} x_3^{(2)} x_3^{(1)}} + \frac{x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_3^{(3)} x_3^{(2)} x_3^{(1)}} \right)^{-1},
\end{aligned}$$

$$\begin{aligned}
y_3^{(2)} &= \frac{x_3^{(3)}x_3^{(2)}x_3^{(1)}}{(x_5^{(2)})^2(x_5^{(1)})^2} \left(\frac{x_5^{(2)}x_5^{(1)}}{x_6^{(3)}x_4^{(2)}} + \frac{x_6^{(2)}x_5^{(2)}x_5^{(1)}}{x_4^{(4)}x_4^{(3)}x_4^{(2)}} + \frac{x_5^{(2)}x_4^{(1)}}{x_6^{(3)}x_3^{(1)}} + \frac{x_6^{(2)}x_5^{(2)}x_4^{(1)}}{x_4^{(4)}x_4^{(3)}x_3^{(1)}} + \frac{x_5^{(2)}x_4^{(2)}x_4^{(1)}}{x_4^{(4)}x_3^{(2)}x_3^{(1)}} \right. \\
&\quad \left. + \frac{x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_3^{(3)}x_3^{(2)}x_3^{(1)}} \right) \left(\frac{x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_4^{(4)}x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(3)}x_3^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}x_3^{(3)}}{x_6^{(2)}x_5^{(1)}x_4^{(4)}} + \frac{x_6^{(1)}x_3^{(3)}}{x_5^{(1)}x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(2)}x_5^{(1)}} \right. \\
&\quad \left. + \frac{x_6^{(1)}x_4^{(3)}}{x_5^{(2)}x_5^{(1)}} \right)^{-1}, \\
y_3^{(3)} &= \frac{x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_4^{(4)}x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(3)}x_3^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}x_3^{(3)}}{x_6^{(2)}x_5^{(1)}x_4^{(4)}} + \frac{x_6^{(1)}x_3^{(3)}}{x_5^{(1)}x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(2)}x_5^{(1)}}, \\
y_4^{(1)} &= \left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_5^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}} + \frac{x_2^{(2)}x_3^{(1)}}{x_2^{(2)}x_2^{(1)}} \right)^{-1}, \\
y_4^{(2)} &= \frac{x_2^{(2)}x_2^{(1)}}{(x_5^{(2)})^2(x_5^{(1)})^2} \left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_5^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_2^{(2)}x_2^{(1)}} \right) \\
&\quad \times \left(\frac{x_2^{(2)}x_2^{(1)}}{x_5^{(1)}x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(2)}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{(x_5^{(1)})^2x_4^{(4)}} \right. \\
&\quad \left. + \frac{x_4^{(3)}x_4^{(1)}x_2^{(2)}}{x_5^{(2)}(x_5^{(1)})^2x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(2)}(x_5^{(1)})^2} \right)^{-1}, \\
y_4^{(3)} &= \left(\frac{x_2^{(2)}x_2^{(1)}}{x_5^{(1)}x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(2)}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{(x_5^{(1)})^2x_4^{(4)}} \right. \\
&\quad \left. + \frac{x_4^{(3)}x_4^{(1)}x_2^{(2)}}{x_5^{(2)}(x_5^{(1)})^2x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(2)}(x_5^{(1)})^2} \right) \left(\frac{x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} \right. \\
&\quad \left. + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right)^{-1}, \\
y_4^{(4)} &= \frac{x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}}, \\
y_5^{(1)} &= \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_1^{(1)}} \right)^{-1}, \\
y_5^{(2)} &= \frac{x_1^{(1)}}{x_5^{(2)}x_5^{(1)}} \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_1^{(1)}} \right) \left(\frac{x_1^{(1)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_3^{(2)}} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}} \right)^{-1}, \\
y_5^{(3)} &= \frac{x_1^{(1)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_3^{(2)}} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}}.
\end{aligned}$$

Using Lemma 6.2.3 we define the map

$$\bar{\sigma} : \mathcal{V}_1 \rightarrow \mathcal{V}_2,$$

$$V_1(x) \mapsto V_2(y).$$

Now we have the following result.

Proposition 6.2.4. *The map $\bar{\sigma} : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is a bi-positive birational isomorphism with the inverse positive rational map*

$$\bar{\sigma}^{-1} : \mathcal{V}_2 \rightarrow \mathcal{V}_1,$$

$$V_2(y) \mapsto V_1(x)$$

given by

$$\begin{aligned} x_1^{(1)} &= \frac{y_5^{(3)} y_5^{(2)} y_5^{(1)}}{y_6^{(2)} y_6^{(1)}}, & x_5^{(1)} &= \frac{1}{y_6^{(2)}}, & x_5^{(2)} &= \frac{1}{y_6^{(1)}}, \\ x_2^{(1)} &= \left(\frac{y_6^{(2)} y_6^{(1)}}{y_5^{(3)} y_5^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(1)}}{y_5^{(3)} y_4^{(2)} y_4^{(1)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)} y_5^{(1)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)} y_4^{(1)}} \right)^{-1}, \\ x_2^{(2)} &= \frac{y_4^{(4)} y_4^{(3)} y_4^{(2)} y_4^{(1)}}{(y_6^{(2)})^2 (y_6^{(1)})^2} \left(\frac{y_6^{(2)} y_6^{(1)}}{y_5^{(3)} y_5^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(1)}}{y_5^{(3)} y_4^{(2)} y_4^{(1)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)} y_5^{(1)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)} y_4^{(1)}} \right), \\ x_3^{(1)} &= \left(\frac{y_6^{(2)} y_6^{(1)}}{y_5^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_4^{(1)}}{y_5^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_5^{(2)} y_4^{(1)}}{y_4^{(4)} y_4^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_4^{(2)} y_4^{(1)}}{y_4^{(4)} y_3^{(2)} y_3^{(1)}} + \frac{y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_3^{(3)} y_3^{(2)} y_3^{(1)}} \right)^{-1}, \\ x_3^{(2)} &= \frac{y_3^{(3)} y_3^{(2)} y_3^{(1)}}{(y_6^{(2)})^2 (y_6^{(1)})^2} \left(\frac{y_6^{(2)} y_6^{(1)}}{y_5^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_4^{(1)}}{y_5^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_5^{(2)} y_4^{(1)}}{y_4^{(4)} y_4^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_4^{(2)} y_4^{(1)}}{y_4^{(4)} y_3^{(2)} y_3^{(1)}} \right. \\ &\quad \left. + \frac{y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_3^{(3)} y_3^{(2)} y_3^{(1)}} \right) \left(\frac{y_3^{(3)} y_3^{(2)} y_3^{(1)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(3)} y_3^{(2)}}{y_6^{(1)} y_4^{(4)} y_4^{(3)}} + \frac{y_4^{(2)} y_4^{(1)} y_3^{(3)}}{y_6^{(1)} y_5^{(2)} y_4^{(4)}} + \frac{y_5^{(1)} y_3^{(3)}}{y_6^{(1)} y_4^{(4)}} + \frac{y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_6^{(2)} y_6^{(1)} y_5^{(2)}} \right. \\ &\quad \left. + \frac{y_5^{(1)} y_4^{(3)}}{y_6^{(2)} y_6^{(1)}} \right)^{-1}, \\ x_3^{(3)} &= \frac{y_3^{(3)} y_3^{(2)} y_3^{(1)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(3)} y_3^{(2)}}{y_6^{(1)} y_4^{(4)} y_4^{(3)}} + \frac{y_4^{(2)} y_4^{(1)} y_3^{(3)}}{y_6^{(1)} y_5^{(2)} y_4^{(4)}} + \frac{y_5^{(1)} y_3^{(3)}}{y_6^{(1)} y_4^{(4)}} + \frac{y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_6^{(2)} y_6^{(1)} y_5^{(2)}} + \frac{y_5^{(1)} y_4^{(3)}}{y_6^{(2)} y_6^{(1)}}, \\ x_4^{(1)} &= \left(\frac{y_6^{(2)}}{y_5^{(3)}} + \frac{y_6^{(2)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_6^{(2)} y_4^{(2)}}{y_4^{(4)} y_3^{(2)}} + \frac{y_4^{(3)} y_4^{(2)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_6^{(2)} y_3^{(1)}}{y_4^{(4)} y_2^{(1)}} + \frac{y_4^{(3)} y_3^{(1)}}{y_3^{(3)} y_2^{(1)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_2^{(2)} y_2^{(1)}} \right)^{-1}, \\ x_4^{(2)} &= \frac{y_2^{(2)} y_2^{(1)}}{(y_6^{(2)})^2 (y_6^{(1)})^2} \left(\frac{y_6^{(2)}}{y_5^{(3)}} + \frac{y_6^{(2)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_6^{(2)} y_4^{(2)}}{y_4^{(4)} y_3^{(2)}} + \frac{y_4^{(3)} y_4^{(2)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_6^{(2)} y_3^{(1)}}{y_4^{(4)} y_2^{(1)}} + \frac{y_4^{(3)} y_3^{(1)}}{y_3^{(3)} y_2^{(1)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_2^{(2)} y_2^{(1)}} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{y_2^{(2)} y_2^{(1)}}{y_6^{(1)} y_4^{(4)} y_3^{(2)}} + \frac{y_4^{(3)} y_2^{(2)} y_2^{(1)}}{y_6^{(2)} y_6^{(1)} y_3^{(3)} y_3^{(2)}} + \frac{y_3^{(1)} y_2^{(2)}}{y_6^{(1)} y_4^{(4)} y_4^{(2)}} + \frac{y_4^{(3)} y_3^{(1)} y_2^{(2)}}{y_6^{(2)} y_6^{(1)} y_4^{(2)} y_3^{(3)}} + \frac{y_4^{(1)} y_2^{(2)}}{(y_6^{(1)})^2 y_4^{(4)}} \right. \\
& \left. + \frac{y_4^{(3)} y_4^{(1)} y_2^{(2)}}{y_6^{(2)} (y_6^{(1)})^2 y_3^{(3)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_6^{(2)} y_6^{(1)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(2)}}{y_6^{(2)} (y_6^{(1)})^2} \right)^{-1}, \\
x_4^{(3)} &= \left(\frac{y_2^{(2)} y_2^{(1)}}{y_6^{(1)} y_4^{(4)} y_3^{(2)}} + \frac{y_4^{(3)} y_2^{(2)} y_2^{(1)}}{y_6^{(2)} y_6^{(1)} y_3^{(3)} y_3^{(2)}} + \frac{y_3^{(1)} y_2^{(2)}}{y_6^{(1)} y_4^{(4)} y_4^{(2)}} + \frac{y_4^{(3)} y_3^{(1)} y_2^{(2)}}{y_6^{(2)} y_6^{(1)} y_4^{(2)} y_3^{(3)}} + \frac{y_4^{(1)} y_2^{(2)}}{(y_6^{(1)})^2 y_4^{(4)}} \right. \\
& \left. + \frac{y_4^{(3)} y_4^{(1)} y_2^{(2)}}{y_6^{(2)} (y_6^{(1)})^2 y_3^{(3)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_6^{(2)} y_6^{(1)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(2)}}{y_6^{(2)} (y_6^{(1)})^2} \right) \left(\frac{y_2^{(2)} y_2^{(1)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_3^{(1)} y_2^{(2)}}{y_4^{(2)} y_3^{(3)}} + \frac{y_4^{(1)} y_2^{(2)}}{y_6^{(1)} y_3^{(3)}} \right. \\
& \left. + \frac{y_3^{(2)} y_3^{(1)}}{y_4^{(3)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(2)}}{y_6^{(1)} y_4^{(3)}} + \frac{y_4^{(2)} y_4^{(1)}}{y_6^{(1)} y_5^{(2)}} + \frac{y_5^{(1)}}{y_6^{(1)}} \right)^{-1}, \\
x_4^{(4)} &= \frac{y_2^{(2)} y_2^{(1)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_3^{(1)} y_2^{(2)}}{y_4^{(2)} y_3^{(3)}} + \frac{y_4^{(1)} y_2^{(2)}}{y_6^{(1)} y_3^{(3)}} + \frac{y_3^{(2)} y_3^{(1)}}{y_4^{(3)} y_4^{(2)}} + \frac{y_4^{(1)} y_3^{(2)}}{y_6^{(1)} y_4^{(3)}} + \frac{y_4^{(2)} y_4^{(1)}}{y_6^{(1)} y_5^{(2)}} + \frac{y_5^{(1)}}{y_6^{(1)}}, \\
x_6^{(1)} &= \left(\frac{y_6^{(2)}}{y_4^{(4)}} + \frac{y_4^{(3)}}{y_3^{(3)}} + \frac{y_3^{(2)}}{y_2^{(2)}} + \frac{y_2^{(1)}}{y_0^{(1)}} \right)^{-1}, \\
x_6^{(2)} &= \frac{y_0^{(1)}}{y_6^{(2)} y_6^{(1)}} \left(\frac{y_6^{(2)}}{y_4^{(4)}} + \frac{y_4^{(3)}}{y_3^{(3)}} + \frac{y_3^{(2)}}{y_2^{(2)}} + \frac{y_2^{(1)}}{y_0^{(1)}} \right) \left(\frac{y_0^{(1)}}{y_2^{(2)}} + \frac{y_2^{(1)}}{y_3^{(2)}} + \frac{y_3^{(1)}}{y_4^{(2)}} + \frac{y_4^{(1)}}{y_6^{(1)}} \right)^{-1}, \\
x_6^{(3)} &= \frac{y_0^{(1)}}{y_2^{(2)}} + \frac{y_2^{(1)}}{y_3^{(2)}} + \frac{y_3^{(1)}}{y_4^{(2)}} + \frac{y_4^{(1)}}{y_6^{(1)}}.
\end{aligned}$$

Proof. The fact that $\bar{\sigma}$ is a bi-positive birational map follows from the explicit formulas. The rest follows by direct calculations. \square

It is known that \mathcal{V}_1 (respectively \mathcal{V}_2) has the structure of a \mathfrak{g}_0 (respectively \mathfrak{g}_1) positive geometric crystal ([1], [25], [33]). Taking the sequence $\mathbf{i} = (6, 4, 3, 2, 5, 4, 3, 6, 4, 5, 1, 2, 3, 4, 6)$, the explicit actions of e_k^c , γ_k , ε_k on $V_1(x)$ for $k = 1, 2, 3, 4, 5, 6$ are given by Theorem 5.2.3 as follows.

$$e_k^c(V_1(x)) = \begin{cases} V_1(x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, cx_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \\ x_4^{(1)}, x_6^{(1)}), \quad k = 1, \\ V_1(x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, c_2 x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, \frac{c}{c_2} x_2^{(1)}, \\ x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), \quad k = 2, \\ V_1(x_6^{(3)}, x_4^{(4)}, c_{3_1} x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, c_{3_2} x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, \\ \frac{c}{c_{3_1} c_{3_2}} x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), \quad k = 3, \end{cases}$$

$$e_k^c(V_1(x)) = \begin{cases} V_1(x_6^{(3)}, c_{4_1}x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, c_{4_2}x_4^{(3)}, x_3^{(2)}, c_{4_3}x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, \\ x_2^{(1)}, x_3^{(1)}, \frac{c}{c_{4_1}c_{4_2}c_{4_3}}x_4^{(1)}, x_6^{(1)}), k = 4, \\ V_1(x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, c_{5_1}x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, \frac{c}{c_5}x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, \\ x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), k = 5, \\ V_1(c_{6_1}x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, c_{6_2}x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, \\ x_3^{(1)}, x_4^{(1)}, \frac{c}{c_{6_1}c_{6_2}}x_6^{(1)}), k = 6, \end{cases}$$

where

$$\begin{aligned} c_2 &= \frac{cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}}{x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}}, \\ c_{3_1} &= \frac{cx_3^{(3)}(x_3^{(2)})^2x_3^{(1)} + x_4^{(3)}x_3^{(2)}x_3^{(1)}x_2^{(2)} + x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_3^{(3)}(x_3^{(2)})^2x_3^{(1)} + x_4^{(3)}x_3^{(2)}x_3^{(1)}x_2^{(2)} + x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}, \\ c_{3_2} &= \frac{cx_3^{(3)}(x_3^{(2)})^2x_3^{(1)} + cx_4^{(3)}x_3^{(2)}x_3^{(1)}x_2^{(2)} + x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{cx_3^{(3)}(x_3^{(2)})^2x_3^{(1)} + x_4^{(3)}x_3^{(2)}x_3^{(1)}x_2^{(2)} + x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}, \\ c_{4_1} &= \frac{cc_4^1 + cc_4^2 + cc_4^3 + cc_4^4}{c_4^1 + c_4^2 + c_4^3 + c_4^4}, \\ c_{4_2} &= \frac{cc_4^1 + cc_4^2 + cc_4^3 + cc_4^4}{cc_4^1 + cc_4^2 + cc_4^3 + cc_4^4}, \\ c_{4_3} &= \frac{cc_4^1 + cc_4^2 + cc_4^3 + cc_4^4}{cc_4^1 + cc_4^2 + cc_4^3 + cc_4^4}, \\ c_4^1 &= x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)}, \\ c_4^2 &= x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)}, \\ c_4^3 &= x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)}, \\ c_4^4 &= x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}, \\ c_5 &= \frac{cx_5^{(2)}x_5^{(1)} + x_4^{(3)}x_4^{(2)}}{x_5^{(2)}x_5^{(1)} + x_4^{(3)}x_4^{(2)}}, \\ c_{6_1} &= \frac{cx_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}, \\ c_{6_2} &= \frac{cx_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + cx_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{cx_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}. \end{aligned}$$

$$\gamma_k(V_1(x)) = \begin{cases} \frac{(x_1^{(1)})^2}{x_2^{(2)}x_2^{(1)}}, & k = 1, \\ \frac{(x_2^{(2)})^2(x_2^{(1)})^2}{x_3^{(3)}x_3^{(2)}x_3^{(1)}x_1^{(1)}}, & k = 2, \\ \frac{(x_3^{(3)})^2(x_3^{(2)})^2(x_3^{(1)})^2}{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}x_2^{(2)}x_2^{(1)}}, & k = 3, \\ \frac{(x_4^{(4)})^2(x_4^{(3)})^2(x_4^{(2)})^2(x_4^{(1)})^2}{x_5^{(5)}x_5^{(4)}x_5^{(3)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}}, & k = 4, \\ \frac{(x_5^{(5)})^2(x_5^{(4)})^2}{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}, & k = 5, \\ \frac{(x_6^{(6)})^2(x_6^{(5)})^2(x_6^{(4)})^2}{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}, & k = 6. \end{cases}$$

$$\varepsilon_k(V_1(x)) = \begin{cases} \frac{x_2^{(2)}}{x_1^{(1)}}, & k = 1, \\ \frac{x_3^{(3)}}{x_2^{(2)}}\left(1 + \frac{x_3^{(2)}x_1^{(1)}}{x_2^{(2)}x_2^{(1)}}\right), & k = 2, \\ \frac{x_4^{(4)}}{x_3^{(3)}}\left(1 + \frac{x_4^{(3)}x_2^{(2)}}{x_3^{(2)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_3^{(3)}(x_2^{(2)})^2x_3^{(1)}}\right), & k = 3, \\ \frac{x_5^{(5)}}{x_4^{(4)}}\left(1 + \frac{x_5^{(3)}x_3^{(3)}}{x_4^{(4)}x_3^{(3)}} + \frac{x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_4^{(4)}(x_4^{(3)})^2x_4^{(2)}} + \frac{x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)}}\right), & k = 4, \\ \frac{x_4^{(4)}}{x_5^{(2)}}\left(1 + \frac{x_4^{(3)}x_1^{(2)}}{x_5^{(2)}x_5^{(1)}}\right), & k = 5, \\ \frac{1}{x_6^{(3)}}\left(1 + \frac{x_4^{(4)}x_4^{(3)}}{x_6^{(3)}x_6^{(2)}} + \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{x_6^{(3)}(x_6^{(2)})^2x_6^{(1)}}\right), & k = 6. \end{cases}$$

By taking the sequence $\mathbf{i} = (5, 4, 3, 2, 6, 4, 3, 5, 4, 6, 0, 2, 3, 4, 5)$, we also have the following explicit actions of \bar{e}_k^c , $\bar{\gamma}_k$, $\bar{\varepsilon}_k$ on $V_2(y)$ for $k = 0, 2, 3, 4, 5, 6$ by Theorem 5.2.3.

$$\bar{e}_k^c(V_2(y)) = \begin{cases} V_2(y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, y_3^{(2)}, y_5^{(2)}, y_4^{(2)}, y_6^{(1)}, cy_0^{(1)}, y_2^{(1)}, y_3^{(1)}, \\ y_4^{(1)}, y_5^{(1)}), k = 0, \\ V_2(y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, \bar{c}_2y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, y_3^{(2)}, y_5^{(2)}, y_4^{(2)}, y_6^{(1)}, y_0^{(1)}, \frac{c}{\bar{c}_2}y_2^{(1)}, \\ y_3^{(1)}, y_4^{(1)}, y_5^{(1)}), k = 2, \\ V_2(y_5^{(3)}, y_4^{(4)}, \bar{c}_{3_1}y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, \bar{c}_{3_2}y_3^{(2)}, y_5^{(2)}, y_4^{(2)}, y_6^{(1)}, y_0^{(1)}, y_2^{(1)}, \\ \frac{c}{\bar{c}_{3_1}\bar{c}_{3_2}}y_3^{(1)}, y_4^{(1)}, y_5^{(1)}), k = 3, \\ V_2(y_5^{(3)}, \bar{c}_{4_1}y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, \bar{c}_{4_2}y_4^{(3)}, y_3^{(2)}, y_5^{(2)}, \bar{c}_{4_3}y_4^{(2)}, y_6^{(1)}, y_0^{(1)}, \\ y_2^{(1)}, y_3^{(1)}, \frac{c}{\bar{c}_{4_1}\bar{c}_{4_2}\bar{c}_{4_3}}y_4^{(1)}, y_5^{(1)}), k = 4, \\ V_2(\bar{c}_{5_1}y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, y_3^{(2)}, \bar{c}_{5_2}y_5^{(2)}, y_4^{(2)}, y_6^{(1)}, y_0^{(1)}, y_2^{(1)}, \\ y_3^{(1)}, y_4^{(1)}, \frac{c}{\bar{c}_{5_1}\bar{c}_{5_2}}y_5^{(1)}), k = 5, \\ V_2(y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, \bar{c}_6y_6^{(2)}, y_4^{(3)}, y_3^{(2)}, y_5^{(2)}, y_4^{(2)}, \frac{c}{\bar{c}_6}y_6^{(1)}, y_0^{(1)}, y_2^{(1)}, \\ y_3^{(1)}, y_4^{(1)}, y_5^{(1)}), k = 6, \end{cases}$$

where

$$\begin{aligned}
\bar{c}_2 &= \frac{cy_2^{(2)}y_2^{(1)} + y_3^{(2)}y_0^{(1)}}{y_2^{(2)}y_2^{(1)} + y_3^{(2)}y_0^{(1)}}, \\
\bar{c}_{3_1} &= \frac{cy_3^{(3)}(y_3^{(2)})^2y_3^{(1)} + y_4^{(3)}y_3^{(2)}y_3^{(1)}y_2^{(2)} + y_4^{(3)}y_4^{(2)}y_2^{(2)}y_2^{(1)}}{y_3^{(3)}(y_3^{(2)})^2y_3^{(1)} + y_4^{(3)}y_3^{(2)}y_3^{(1)}y_2^{(2)} + y_4^{(3)}y_4^{(2)}y_2^{(2)}y_2^{(1)}}, \\
\bar{c}_{3_2} &= \frac{cy_3^{(3)}(y_3^{(2)})^2y_3^{(1)} + cy_4^{(3)}y_3^{(2)}y_3^{(1)}y_2^{(2)} + y_4^{(3)}y_4^{(2)}y_2^{(2)}y_2^{(1)}}{cy_3^{(3)}(y_3^{(2)})^2y_3^{(1)} + y_4^{(3)}y_3^{(2)}y_3^{(1)}y_2^{(2)} + y_4^{(3)}y_4^{(2)}y_2^{(2)}y_2^{(1)}}, \\
\bar{c}_{4_1} &= \frac{c\bar{c}_4^1 + \bar{c}_4^2 + \bar{c}_4^3 + \bar{c}_4^4}{\bar{c}_4^1 + \bar{c}_4^2 + \bar{c}_4^3 + \bar{c}_4^4}, \\
\bar{c}_{4_2} &= \frac{c\bar{c}_4^1 + c\bar{c}_4^2 + \bar{c}_4^3 + \bar{c}_4^4}{c\bar{c}_4^1 + \bar{c}_4^2 + \bar{c}_4^3 + \bar{c}_4^4}, \\
\bar{c}_{4_3} &= \frac{c\bar{c}_4^1 + c\bar{c}_4^2 + c\bar{c}_4^3 + \bar{c}_4^4}{c\bar{c}_4^1 + c\bar{c}_4^2 + \bar{c}_4^3 + \bar{c}_4^4}, \\
\bar{c}_4^1 &= y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)}, \\
\bar{c}_4^2 &= y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)}, \\
\bar{c}_4^3 &= y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)}, \\
\bar{c}_4^4 &= y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)}, \\
\bar{c}_{5_1} &= \frac{cy_5^{(3)}(y_5^{(2)})^2y_5^{(1)} + y_5^{(2)}y_5^{(1)}y_4^{(4)}y_4^{(3)} + y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}{y_5^{(3)}(y_5^{(2)})^2y_5^{(1)} + y_5^{(2)}y_5^{(1)}y_4^{(4)}y_4^{(3)} + y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}, \\
\bar{c}_{5_2} &= \frac{cy_5^{(3)}(y_5^{(2)})^2y_5^{(1)} + cy_5^{(2)}y_5^{(1)}y_4^{(4)}y_4^{(3)} + y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}{cy_5^{(3)}(y_5^{(2)})^2y_5^{(1)} + y_5^{(2)}y_5^{(1)}y_4^{(4)}y_4^{(3)} + y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}, \\
\bar{c}_6 &= \frac{cy_6^{(2)}y_6^{(1)} + y_4^{(3)}y_4^{(2)}}{y_6^{(2)}y_6^{(1)} + y_4^{(3)}y_4^{(2)}}.
\end{aligned}$$

$$\bar{\gamma}_k(V_2(y)) = \begin{cases} \frac{(y_0^{(1)})^2}{y_2^{(2)}y_2^{(1)}}, & k = 0, \\ \frac{(y_2^{(2)})^2(y_2^{(1)})^2}{y_3^{(3)}y_3^{(2)}y_3^{(1)}y_0^{(1)}}, & k = 2, \\ \frac{(y_3^{(3)})^2(y_3^{(2)})^2(y_3^{(1)})^2}{y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}y_2^{(2)}y_2^{(1)}}, & k = 3, \\ \frac{(y_4^{(4)})^2(y_4^{(3)})^2(y_4^{(2)})^2(y_4^{(1)})^2}{y_6^{(2)}y_6^{(1)}y_5^{(3)}y_5^{(2)}y_5^{(1)}y_3^{(3)}y_3^{(2)}y_3^{(1)}}, & k = 4, \\ \frac{(y_5^{(3)})^2(y_5^{(2)})^2(y_5^{(1)})^2}{y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}, & k = 5, \\ \frac{(y_6^{(2)})^2(y_6^{(1)})^2}{y_4^{(4)}y_4^{(3)}y_4^{(2)}y_4^{(1)}}, & k = 6. \end{cases}$$

$$\bar{e}_k(V_2(y)) = \begin{cases} \frac{y_2^{(2)}}{y_0^{(1)}}, & k = 0, \\ \frac{y_3^{(3)}}{y_2^{(2)}} \left(1 + \frac{y_3^{(2)} y_0^{(1)}}{y_2^{(2)} y_2^{(1)}}\right), & k = 2, \\ \frac{y_4^{(4)}}{y_3^{(3)}} \left(1 + \frac{y_4^{(3)} y_2^{(2)}}{y_3^{(3)} y_3^{(2)}} + \frac{y_4^{(3)} y_4^{(2)} y_2^{(2)} y_2^{(1)}}{y_3^{(3)} (y_3^{(2)})^2 y_3^{(1)}}\right), & k = 3, \\ \frac{y_5^{(5)}}{y_4^{(4)}} \left(1 + \frac{y_6^{(3)} y_3^{(3)}}{y_4^{(4)} y_4^{(3)}} + \frac{y_6^{(2)} y_5^{(2)} y_3^{(3)} y_3^{(2)}}{y_4^{(4)} (y_4^{(3)})^2 y_4^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)} y_3^{(3)} y_3^{(2)} y_3^{(1)}}{y_4^{(4)} (y_4^{(3)})^2 (y_4^{(2)})^2 y_4^{(1)}}\right), & k = 4, \\ \frac{1}{y_5^{(3)}} \left(1 + \frac{y_4^{(4)} y_4^{(3)}}{y_5^{(3)} y_5^{(2)}} + \frac{y_4^{(4)} y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_5^{(3)} (y_5^{(2)})^2 y_5^{(1)}}\right), & k = 5, \\ \frac{y_4^{(4)}}{y_6^{(2)}} \left(1 + \frac{y_4^{(3)} y_4^{(2)}}{y_6^{(2)} y_6^{(1)}}\right), & k = 6. \end{cases}$$

The following proposition shows the relation between $\bar{\sigma}, e_2^c$ and e_4^c which we use in the proof of the next theorem

Proposition 6.2.5. *The relations $\bar{\sigma}e_2^c = e_4^c\bar{\sigma}$ holds.*

Proof. Set $e_2^c(V_1(x)) = V_1(z)$, $\bar{\sigma}(V_1(z)) = V_2(y')$, $\bar{\sigma}(V_1(x)) = V_2(y)$ and $e_4^c(V_2(y)) = V_2(w)$. We need to show that $y_m^{(l)'} = w_m^{(l)}$ for $(l, m) \in \{(1, 0), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (4, 4)\}$. The proof follows from straightforward calculations.

$$\begin{aligned} y_0^{(1)'} &= \frac{z_6^{(3)} z_6^{(2)} z_6^{(1)}}{z_5^{(2)} z_5^{(1)}} = \frac{x_6^{(3)} x_6^{(2)} x_6^{(1)}}{x_5^{(2)} x_5^{(1)}} = y_0^{(1)} = w_0^{(1)} \\ y_2^{(1)'} &= \left(\frac{z_5^{(2)} z_5^{(1)}}{z_6^{(3)} z_6^{(2)}} + \frac{z_6^{(1)} z_5^{(2)} z_5^{(1)}}{z_6^{(3)} z_4^{(2)} z_4^{(1)}} + \frac{z_6^{(2)} z_6^{(1)} z_5^{(2)} z_5^{(1)}}{z_4^{(4)} z_4^{(3)} z_4^{(2)} z_4^{(1)}} \right)^{-1} \\ &= \left(\frac{x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_6^{(2)}} + \frac{x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_4^{(2)} x_4^{(1)}} + \frac{x_6^{(2)} x_6^{(1)} x_5^{(2)} x_5^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \right)^{-1} = y_2^{(1)} = w_2^{(1)} \\ y_3^{(1)'} &= \left(\frac{z_6^{(2)} z_6^{(1)}}{z_5^{(3)} z_4^{(2)}} + \frac{z_6^{(2)} z_6^{(1)} z_5^{(2)}}{z_4^{(4)} z_4^{(3)} z_4^{(2)}} + \frac{z_6^{(2)} z_4^{(1)}}{z_5^{(3)} z_3^{(1)}} + \frac{z_6^{(2)} z_5^{(2)} z_4^{(1)}}{z_4^{(4)} z_4^{(3)} z_3^{(1)}} + \frac{z_6^{(2)} z_4^{(2)} z_4^{(1)}}{z_4^{(4)} z_3^{(2)} z_3^{(1)}} + \frac{z_4^{(3)} z_4^{(2)} z_4^{(1)}}{z_3^{(3)} z_3^{(2)} z_3^{(1)}} \right)^{-1} \\ &= \left(\frac{y_6^{(2)} y_6^{(1)}}{y_5^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_6^{(1)} y_5^{(2)}}{y_4^{(4)} y_4^{(3)} y_4^{(2)}} + \frac{y_6^{(2)} y_4^{(1)}}{y_5^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_5^{(2)} y_4^{(1)}}{y_4^{(4)} y_4^{(3)} y_3^{(1)}} + \frac{y_6^{(2)} y_4^{(2)} y_4^{(1)}}{y_4^{(4)} y_3^{(2)} y_3^{(1)}} + \frac{y_4^{(3)} y_4^{(2)} y_4^{(1)}}{y_3^{(3)} y_3^{(2)} y_3^{(1)}} \right)^{-1} \\ &= y_3^{(1)} = w_3^{(1)} \\ y_4^{(1)'} &= \left(\frac{z_5^{(2)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_5^{(2)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_5^{(2)} z_4^{(2)}}{z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_4^{(2)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_5^{(2)} z_3^{(1)}}{z_4^{(4)} z_2^{(1)}} + \frac{z_4^{(3)} z_3^{(1)}}{z_3^{(3)} z_2^{(1)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_2^{(2)} z_2^{(1)}} \right)^{-1} \\ &= \left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_5^{(2)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_5^{(2)} x_4^{(2)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(2)} x_3^{(1)} (cx_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_4^{(4)} x_2^{(1)} (cx_2^{(2)} x_2^{(1)} + cx_3^{(2)} x_1^{(1)})} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{x_4^{(3)}x_3^{(1)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_3^{(3)}x_2^{(1)}(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)})} + \frac{x_3^{(2)}x_3^{(1)}}{cx_2^{(2)}x_2^{(1)}} \Big)^{-1} \\
& = (cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)}) \left(\left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_5^{(2)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{cx_2^{(2)}x_2^{(1)}} \right) (cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right)^{-1} \\
& = \left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_5^{(2)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_2^{(2)}x_2^{(1)}} \right)^{-1} \left(\left(\frac{x_5^{(2)}}{x_6^{(3)}} \right. \right. \\
& \quad \left. \left. + \frac{x_6^{(2)}x_5^{(2)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_2^{(2)}x_2^{(1)}} \right) (cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)}) \right) \\
& \quad \left(\left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_5^{(2)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_5^{(2)}x_4^{(2)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{cx_2^{(2)}x_2^{(1)}} \right) (cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)}) + \left(\frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} \right. \right. \\
& \quad \left. \left. + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}} \right) (cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right)^{-1} \\
& = y_4^{(1)} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + cy_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \quad \left. + cy_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
& \quad \left. + cy_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \text{ by Maple} \\
& = w_4^{(1)}
\end{aligned}$$

$$\begin{aligned}
y_5^{(1)'} &= \left(\frac{z_5^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)}}{z_3^{(3)}} + \frac{z_3^{(2)}}{z_2^{(2)}} + \frac{z_2^{(1)}}{z_1^{(1)}} \right)^{-1} \\
&= \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_2^{(1)}(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)})}{x_1^{(1)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} \right)^{-1} \\
&= \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_1^{(1)}} \right)^{-1} = y_5^{(1)} = w_5^{(1)} \\
y_6^{(1)'} &= \frac{1}{z_5^{(2)}} = \frac{1}{x_5^{(2)}} = y_6^{(1)} = w_6^{(1)} \\
y_2^{(2)'} &= \frac{z_4^{(4)}z_4^{(3)}z_4^{(2)}z_4^{(1)}}{(z_5^{(2)})^2(z_5^{(1)})^2} \left(\frac{z_5^{(2)}z_5^{(1)}}{z_6^{(3)}z_6^{(2)}} + \frac{z_6^{(1)}z_5^{(2)}z_5^{(1)}}{z_6^{(3)}z_4^{(2)}z_4^{(1)}} + \frac{z_6^{(2)}z_6^{(1)}z_5^{(2)}z_5^{(1)}}{z_4^{(4)}z_4^{(3)}z_4^{(2)}z_4^{(1)}} \right) \\
&= \frac{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{(x_5^{(2)})^2(x_5^{(1)})^2} \left(\frac{x_5^{(2)}x_5^{(1)}}{x_6^{(3)}x_6^{(2)}} + \frac{x_6^{(1)}x_5^{(2)}x_5^{(1)}}{x_6^{(3)}x_4^{(2)}x_4^{(1)}} + \frac{x_6^{(2)}x_6^{(1)}x_5^{(2)}x_5^{(1)}}{x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} \right) = y_2^{(2)} = w_2^{(2)}
\end{aligned}$$

$$\begin{aligned}
y_3^{(2)'} &= \frac{z_3^{(3)} z_3^{(2)} z_3^{(1)}}{(z_5^{(2)})^2 (z_5^{(1)})^2} \left(\frac{z_5^{(2)} z_5^{(1)}}{z_6^{(3)} z_4^{(2)}} + \frac{z_6^{(2)} z_5^{(2)} z_5^{(1)}}{z_4^{(4)} z_4^{(3)} z_4^{(2)}} + \frac{z_5^{(2)} z_4^{(1)}}{z_6^{(3)} z_3^{(1)}} + \frac{z_6^{(2)} z_5^{(2)} z_4^{(1)}}{z_4^{(4)} z_4^{(3)} z_3^{(1)}} + \frac{z_5^{(2)} z_4^{(2)} z_4^{(1)}}{z_4^{(4)} z_3^{(2)} z_3^{(1)}} \right. \\
&\quad \left. + \frac{z_4^{(3)} z_4^{(2)} z_4^{(1)}}{z_3^{(3)} z_3^{(2)} z_3^{(1)}} \right) \left(\frac{z_3^{(3)} z_3^{(2)} z_3^{(1)}}{z_4^{(4)} z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(3)} z_3^{(2)}}{z_5^{(1)} z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_4^{(1)} z_3^{(3)}}{z_6^{(2)} z_5^{(1)} z_4^{(4)}} + \frac{z_6^{(1)} z_3^{(3)}}{z_5^{(1)} z_4^{(4)}} + \frac{z_4^{(3)} z_4^{(2)} z_4^{(1)}}{z_6^{(2)} z_5^{(2)} z_5^{(1)}} \right. \\
&\quad \left. + \frac{z_6^{(1)} z_4^{(3)}}{z_5^{(2)} z_5^{(1)}} \right)^{-1} \\
&= \frac{x_3^{(3)} x_3^{(2)} x_3^{(1)}}{(x_5^{(2)})^2 (x_5^{(1)})^2} \left(\frac{x_5^{(2)} x_5^{(1)}}{x_6^{(3)} x_4^{(2)}} + \frac{x_6^{(2)} x_5^{(2)} x_5^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)}} + \frac{x_5^{(2)} x_4^{(1)}}{x_6^{(3)} x_3^{(1)}} + \frac{x_6^{(2)} x_5^{(2)} x_4^{(1)}}{x_4^{(4)} x_4^{(3)} x_3^{(1)}} + \frac{x_5^{(2)} x_4^{(2)} x_4^{(1)}}{x_4^{(4)} x_3^{(2)} x_3^{(1)}} \right. \\
&\quad \left. + \frac{x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_3^{(3)} x_3^{(2)} x_3^{(1)}} \right) \left(\frac{x_3^{(3)} x_3^{(2)} x_3^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(3)} x_3^{(2)}}{x_5^{(1)} x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)} x_3^{(3)}}{x_6^{(2)} x_5^{(1)} x_4^{(4)}} + \frac{x_6^{(1)} x_3^{(3)}}{x_5^{(1)} x_4^{(4)}} + \frac{x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_6^{(2)} x_5^{(2)} x_5^{(1)}} \right. \\
&\quad \left. + \frac{x_6^{(1)} x_4^{(3)}}{x_5^{(2)} x_5^{(1)}} \right)^{-1} = y_3^{(2)} = w_3^{(2)} \\
y_4^{(2)'} &= \frac{z_2^{(2)} z_2^{(1)}}{(z_5^{(2)})^2 (z_5^{(1)})^2} \left(\frac{z_5^{(2)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_5^{(2)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_5^{(2)} z_4^{(2)}}{z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_4^{(2)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_5^{(2)} z_3^{(1)}}{z_4^{(4)} z_2^{(1)}} + \frac{z_4^{(3)} z_3^{(1)}}{z_3^{(3)} z_2^{(1)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_2^{(2)} z_2^{(1)}} \right) \\
&\quad \times \left(\frac{z_2^{(2)} z_2^{(1)}}{z_5^{(1)} z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_5^{(1)} z_3^{(3)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_5^{(1)} z_4^{(4)} z_4^{(2)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_5^{(1)} z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{(z_5^{(1)})^2 z_4^{(4)}} \right. \\
&\quad \left. + \frac{z_4^{(3)} z_4^{(1)} z_2^{(2)}}{z_5^{(2)} (z_5^{(1)})^2 z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)} z_5^{(1)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(2)}}{z_5^{(2)} (z_5^{(1)})^2} \right)^{-1} \\
&= \frac{c x_2^{(2)} x_2^{(1)}}{(x_5^{(2)})^2 (x_5^{(1)})^2} \left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_5^{(2)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_5^{(2)} x_4^{(2)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(2)} x_3^{(1)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_4^{(4)} x_2^{(1)} (c x_2^{(2)} x_2^{(1)} + c x_3^{(2)} x_1^{(1)})} \right. \\
&\quad \left. + \frac{x_4^{(3)} x_3^{(1)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_3^{(3)} x_2^{(1)} (c x_2^{(2)} x_2^{(1)} + c x_3^{(2)} x_1^{(1)})} + \frac{x_3^{(2)} x_3^{(1)}}{c x_2^{(2)} x_2^{(1)}} \right) \left(\frac{c x_2^{(2)} x_2^{(1)}}{x_5^{(1)} x_4^{(4)} x_3^{(2)}} + \frac{c x_4^{(3)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)}} \right. \\
&\quad \left. + \frac{x_3^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(1)} x_4^{(4)} x_4^{(2)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(2)} x_5^{(1)} x_4^{(2)} x_3^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} \right. \\
&\quad \left. + \frac{x_4^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{(x_5^{(1)})^2 x_4^{(4)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_4^{(3)} x_4^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(2)} (x_5^{(1)})^2 x_3^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_5^{(1)} x_4^{(2)}} \right. \\
&\quad \left. + \frac{x_4^{(1)} x_3^{(2)}}{x_5^{(2)} (x_5^{(1)})^2} \right)^{-1} \\
&= \frac{c x_2^{(2)} x_2^{(1)}}{(x_5^{(2)})^2 (x_5^{(1)})^2} \cdot \frac{1}{(c x_2^{(2)} x_2^{(1)} + c x_3^{(2)} x_1^{(1)})} \left(\left(\frac{x_5^{(2)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_5^{(2)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_5^{(2)} x_4^{(2)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)}}{x_3^{(3)} x_3^{(2)}} \right. \right. \\
&\quad \left. \left. + \frac{x_3^{(2)} x_3^{(1)}}{c x_2^{(2)} x_2^{(1)}} \right) (c x_2^{(2)} x_2^{(1)} + c x_3^{(2)} x_1^{(1)}) + \left(\frac{x_5^{(2)} x_3^{(1)}}{x_4^{(4)} x_2^{(1)}} + \frac{x_4^{(3)} x_3^{(1)}}{x_3^{(3)} x_2^{(1)}} \right) (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{x_4^{(3)}x_4^{(2)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{cx_2^{(2)}x_2^{(1)}})(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)}) + \left(\frac{x_5^{(2)}x_3^{(1)}}{x_4^{(4)}x_2^{(1)}} + \frac{x_4^{(3)}x_3^{(1)}}{x_3^{(3)}x_2^{(1)}}\right)(cx_2^{(2)}x_2^{(1)} \\
& + x_3^{(2)}x_1^{(1)})\left(\left(\frac{x_2^{(2)}x_2^{(1)}}{x_5^{(1)}x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(2)}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}x_3^{(3)}}\right.\right. \\
& \left.\left. + \frac{x_4^{(1)}x_2^{(2)}}{(x_5^{(1)})^2x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(1)}x_2^{(2)}}{x_5^{(2)}(x_5^{(1)})^2x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(2)}(x_5^{(1)})^2}\right)(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})\right) \\
& \times \left(\left(\frac{cx_2^{(2)}x_2^{(1)}}{x_5^{(1)}x_4^{(4)}x_3^{(2)}} + \frac{cx_4^{(3)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(2)}(x_5^{(1)})^2}\right)(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})\right. \\
& \left. + \left(\frac{x_3^{(1)}x_2^{(2)}}{x_5^{(1)}x_4^{(4)}x_4^{(2)}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_5^{(1)}x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{(x_5^{(1)})^2x_4^{(4)}} + \frac{x_4^{(3)}x_4^{(1)}x_2^{(2)}}{x_5^{(2)}(x_5^{(1)})^2x_3^{(3)}}\right)(cx_2^{(2)}x_2^{(1)}\right. \\
& \left. + x_3^{(2)}x_1^{(1)})\right)^{-1} \\
& = y_4^{(2)} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + cy_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
& \left. + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \left(y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} \right. \\
& \left. + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \\
& \times \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \text{ by Maple} \\
& = y_4^{(2)} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + cy_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
& \left. + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} = w_4^{(2)} \\
y_5^{(2)'} & = \frac{z_1^{(1)}}{z_5^{(2)}z_5^{(1)}} \left(\frac{z_5^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)}}{z_3^{(3)}} + \frac{z_3^{(2)}}{z_2^{(2)}} + \frac{z_2^{(1)}}{z_1^{(1)}} \right) \left(\frac{z_1^{(1)}}{z_2^{(2)}} + \frac{z_2^{(1)}}{z_3^{(2)}} + \frac{z_3^{(1)}}{z_4^{(2)}} + \frac{z_4^{(1)}}{z_5^{(1)}} \right)^{-1} \\
& = \frac{x_1^{(1)}}{x_5^{(2)}x_5^{(1)}} \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_2^{(1)}(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)})}{x_1^{(1)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} \right) \\
& \times \left(\frac{x_1^{(1)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_2^{(1)}(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)})}{x_3^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}} \right)^{-1} \\
& = \frac{x_1^{(1)}}{x_5^{(2)}x_5^{(1)}} \left(\frac{x_5^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}}{x_3^{(3)}} + \frac{x_3^{(2)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_1^{(1)}} \right) \left(\frac{x_1^{(1)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_3^{(2)}} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}} \right)^{-1} = y_5^{(2)} = w_5^{(2)}
\end{aligned}$$

$$\begin{aligned}
y_6^{(2)'} &= \frac{z_6^{(3)} z_6^{(2)} z_6^{(1)}}{z_5^{(2)} z_5^{(1)}} = \frac{x_6^{(3)} x_6^{(2)} x_6^{(1)}}{x_5^{(2)} x_5^{(1)}} = y_0^{(1)} = w_6^{(2)} \\
y_3^{(3)'} &= \frac{z_3^{(3)} z_3^{(2)} z_3^{(1)}}{z_4^{(4)} z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(3)} z_3^{(2)}}{z_5^{(1)} z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_4^{(1)} z_3^{(3)}}{z_6^{(2)} z_5^{(1)} z_4^{(4)}} + \frac{z_6^{(1)} z_3^{(3)}}{z_5^{(1)} z_4^{(4)}} + \frac{z_4^{(3)} z_4^{(2)} z_4^{(1)}}{z_6^{(2)} z_5^{(2)} z_5^{(1)}} + \frac{z_6^{(1)} z_4^{(3)}}{z_5^{(2)} z_5^{(1)}} \\
&= \frac{x_3^{(3)} x_3^{(2)} x_3^{(1)}}{x_4^{(4)} x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(3)} x_3^{(2)}}{x_5^{(1)} x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)} x_3^{(3)}}{x_6^{(2)} x_5^{(1)} x_4^{(4)}} + \frac{x_6^{(1)} x_3^{(3)}}{x_5^{(1)} x_4^{(4)}} + \frac{x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_6^{(2)} x_5^{(2)} x_5^{(1)}} + \frac{x_6^{(1)} x_4^{(3)}}{x_5^{(2)} x_5^{(1)}} \\
&= y_3^{(3)} = w_3^{(3)} \\
y_4^{(3)'} &= \left(\frac{z_2^{(2)} z_2^{(1)}}{z_5^{(1)} z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_5^{(1)} z_3^{(3)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_5^{(1)} z_4^{(4)} z_4^{(2)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_5^{(1)} z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{(z_5^{(1)})^2 z_4^{(4)}} \right. \\
&\quad \left. + \frac{z_4^{(3)} z_4^{(1)} z_2^{(2)}}{z_5^{(2)} (z_5^{(1)})^2 z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)} z_5^{(1)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(2)}}{z_5^{(2)} (z_5^{(1)})^2} \right) \left(\frac{z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_5^{(1)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} \right. \\
&\quad \left. + \frac{z_4^{(1)} z_3^{(2)}}{z_5^{(1)} z_4^{(3)}} + \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)} z_5^{(1)}} + \frac{z_6^{(1)}}{z_5^{(1)}} \right)^{-1} \\
&= \left(\frac{c x_2^{(2)} x_2^{(1)}}{x_5^{(1)} x_4^{(4)} x_3^{(2)}} + \frac{c x_4^{(3)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(1)} x_4^{(4)} x_4^{(2)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} \right. \\
&\quad \left. + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(2)} x_5^{(1)} x_4^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_4^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{(x_5^{(1)})^2 x_4^{(4)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} \right. \\
&\quad \left. + \frac{x_4^{(3)} x_4^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(2)} (x_5^{(1)})^2 x_3^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_5^{(1)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_5^{(2)} (x_5^{(1)})^2} \right) \left(\frac{c x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} \right. \\
&\quad \left. + \frac{x_3^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_4^{(2)} x_3^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_4^{(1)} x_2^{(2)} (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})}{x_5^{(1)} x_3^{(3)} (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} + \frac{x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_5^{(1)} x_4^{(3)}} \right. \\
&\quad \left. + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)} x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right)^{-1} \\
&= \frac{1}{(x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)})} \left(\left(\frac{c x_2^{(2)} x_2^{(1)}}{x_5^{(1)} x_4^{(4)} x_3^{(2)}} + \frac{c x_4^{(3)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_5^{(1)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_5^{(2)} (x_5^{(1)})^2} \right) \right. \\
&\quad \times (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) + \left(\frac{x_3^{(1)} x_2^{(2)}}{x_5^{(1)} x_4^{(4)} x_4^{(2)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_5^{(1)} x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{(x_5^{(1)})^2 x_4^{(4)}} + \frac{x_4^{(3)} x_4^{(1)} x_2^{(2)}}{x_5^{(2)} (x_5^{(1)})^2 x_3^{(3)}} \right) \\
&\quad \times (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) \left((x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) \left(\frac{x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_5^{(1)} x_3^{(3)}} \right) (c x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) \right. \\
&\quad \left. + \left(\frac{c x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_5^{(1)} x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)} x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) (x_2^{(2)} x_2^{(1)} + x_3^{(2)} x_1^{(1)}) \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
& \times (x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \Big) \left(\left(\frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} \right) (cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) + \left(\frac{cx_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} \right. \right. \\
& \left. \left. + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) (x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right)^{-1} \\
= & y_4^{(3)} \left(y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
& \left. + y_6^{(2)}y_5^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} \right. \\
& \left. + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \\
& \times \left(y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \text{by Maple} \\
= & y_4^{(3)} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + cy_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
& \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
& \left. + y_6^{(2)}y_5^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} = w_4^{(3)} \\
y_5^{(3)'} = & \frac{z_1^{(1)}}{z_2^{(2)}} + \frac{z_2^{(1)}}{z_3^{(2)}} + \frac{z_3^{(1)}}{z_4^{(2)}} + \frac{z_4^{(1)}}{z_5^{(1)}} \\
= & \frac{x_1^{(1)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_2^{(1)}(cx_2^{(2)}x_2^{(1)} + cx_3^{(2)}x_1^{(1)})}{x_3^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}} \\
= & \frac{x_1^{(1)}}{x_2^{(2)}} + \frac{x_2^{(1)}}{x_3^{(2)}} + \frac{x_3^{(1)}}{x_4^{(2)}} + \frac{x_4^{(1)}}{x_5^{(1)}} = y_5^{(3)} = w_5^{(3)} \\
y_4^{(4)'} = & \frac{z_2^{(2)}z_2^{(1)}}{z_3^{(3)}z_3^{(2)}} + \frac{z_3^{(1)}z_2^{(2)}}{z_4^{(2)}z_3^{(3)}} + \frac{z_4^{(1)}z_2^{(2)}}{z_5^{(1)}z_3^{(2)}} + \frac{z_3^{(2)}z_3^{(1)}}{z_4^{(1)}z_3^{(2)}} + \frac{z_4^{(1)}z_3^{(2)}}{z_5^{(1)}z_4^{(3)}} + \frac{z_4^{(2)}z_4^{(1)}}{z_6^{(2)}z_5^{(1)}} + \frac{z_6^{(1)}}{z_5^{(1)}} \\
= & \frac{cx_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_4^{(2)}x_3^{(3)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_4^{(1)}x_2^{(2)}(cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})}{x_5^{(1)}x_3^{(3)}(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} \\
& + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \\
= & \frac{1}{(x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)})} \left(\left(\frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} \right) (cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) + \left(\frac{cx_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} \right. \right. \\
& \left. \left. + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) (x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) \left(\left(\frac{x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} \right. \right. \\
&\quad \left. \left. + \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) (x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right)^{-1} \\
&\quad \times \left(\left(\frac{x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_5^{(1)}x_3^{(3)}} \right) (cx_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) + \left(\frac{cx_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_5^{(1)}x_4^{(3)}} \right. \right. \\
&\quad \left. \left. + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}x_5^{(1)}} + \frac{x_6^{(1)}}{x_5^{(1)}} \right) (x_2^{(2)}x_2^{(1)} + x_3^{(2)}x_1^{(1)}) \right) \\
&= y_4^{(4)} \left(cy_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} \right. \\
&\quad \left. + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right) \left(y_4^{(4)}(y_4^{(3)})^2(y_4^{(2)})^2y_4^{(1)} + y_6^{(2)}y_4^{(3)}(y_4^{(2)})^2y_4^{(1)}y_3^{(3)} \right. \\
&\quad \left. + y_6^{(2)}y_5^{(2)}y_4^{(2)}y_4^{(1)}y_3^{(3)}y_3^{(2)} + y_6^{(2)}y_6^{(1)}y_5^{(2)}y_3^{(3)}y_3^{(2)}y_3^{(1)} \right)^{-1} \text{ by Maple} \\
&= w_4^{(4)}.
\end{aligned}$$

□

In order to give \mathcal{V}_1 a $\mathfrak{g} = D_6^{(1)}$ -geometric crystal structure, we need to define the actions of e_0^c , γ_0 , and ε_0 on $V_1(x)$. We use the \mathfrak{g}_1 -geometric crystal structure on \mathcal{V}_2 to define the action of e_0^c , γ_0 , and ε_0 on $V_1(x)$ as follows.

$$e_0^c(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{e_{\sigma(0)}^c} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \bar{e}_6^c(V_2(y)), \quad (6.2.2)$$

$$\gamma_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\gamma_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\gamma_6}(V_2(y)), \quad (6.2.3)$$

$$\varepsilon_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\varepsilon_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\varepsilon_6}(V_2(y)). \quad (6.2.4)$$

Theorem 6.2.6. *The algebraic variety $\mathcal{V} = \mathcal{V}_1 = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k \in I\}$ is a positive geometric crystal for the affine Lie algebra $\mathfrak{g} = D_6^{(1)}$ with the e_0^c , γ_0 , and ε_0 actions on $V_1(x)$ given by:*

$$\begin{aligned}
\gamma_0(V_1(x)) &= \frac{1}{x_2^{(2)}x_2^{(1)}}, \\
\varepsilon_0(V_1(x)) &= x_6^{(1)} + \frac{x_5^{(1)}x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_5^{(1)}x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} + \frac{x_4^{(1)}x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)}x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_4^{(3)}} \\
&\quad + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(3)}x_3^{(2)}} \\
&\quad + \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_3^{(3)}} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}},
\end{aligned}$$

$e_0^c(V_1(x)) = V_1(x') = V_1(x_6^{(3)'}, x_4^{(4)'}, \dots, x_6^{(1)'})$ where

$$\begin{aligned}
x_1^{(1)'} &= \frac{x_1^{(1)}}{c}, & x_2^{(1)'} &= \frac{x_2^{(1)}}{c}, & x_2^{(2)'} &= \frac{x_2^{(2)}}{c}, \\
x_3^{(1)'} &= x_3^{(1)} \frac{K}{K_{3_1}}, & x_3^{(2)'} &= x_3^{(2)} \frac{K_{3_1}}{cK_{3_2}}, & x_3^{(3)'} &= x_3^{(3)} \frac{K_{3_2}}{cK}, \\
x_4^{(1)'} &= x_4^{(1)} \frac{K}{K_{4_1}}, & x_4^{(2)'} &= x_4^{(2)} \frac{KK_{4_1}}{K_{4_2}}, & x_4^{(3)'} &= x_4^{(3)} \frac{K_{4_2}}{cKK_{4_3}}, \\
x_4^{(4)'} &= x_4^{(4)} \frac{K_{4_3}}{cK}, & x_5^{(1)'} &= x_5^{(1)} \frac{K}{K_{5_1}}, & x_5^{(2)'} &= x_5^{(2)} \frac{K_{5_1}}{cK}, \\
x_6^{(1)'} &= x_6^{(1)} \frac{K}{K_{6_1}}, & x_6^{(2)'} &= x_6^{(2)} \frac{K_{6_1}}{K_{6_2}}, & x_6^{(3)'} &= x_6^{(3)} \frac{K_{6_2}}{cK},
\end{aligned}$$

$$\begin{aligned}
K &= x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
&\quad + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
&\quad + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{3_1} &= cx_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
&\quad + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + c \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
&\quad + c \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{3_2} &= cx_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
&\quad + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
&\quad + c \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{4_1} &= cx_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
&\quad + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}}
\end{aligned}$$

$$\begin{aligned}
& + c \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + c \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + c \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}) K, \\
K_{4_3} = & c x_6^{(1)} + c \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
& + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + c \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& + c \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{5_1} = & c x_6^{(1)} + c \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
& + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{6_1} = & c x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
& + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{6_2} = & c x_6^{(1)} + c \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + c \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
& + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + c \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + c \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + c \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + c \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& + c \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}.
\end{aligned}$$

Proof. Since $\mathcal{V} = \mathcal{V}_1$ is a positive geometric crystal for \mathfrak{g}_0 , to show that it is a positive geometric crystal for $\mathfrak{g} = D_6^{(1)}$ by Definition 5.2.2, it suffices to show that the following relations involving the 0-action hold:

1. $\gamma_0(e_k^c(V_1(x))) = c^{a_{k0}} \gamma_0(V_1(x))$ for all $k \in I$
2. $\gamma_k(e_0^c(V_1(x))) = c^{a_{0k}} \gamma_k(V_1(x))$ for all $k \in I$

$$3. \ \varepsilon_0(e_0^c(V_1(x))) = c^{-1}\varepsilon_0(V_1(x))$$

$$4. \ e_0^{c_1}e_k^{c_2} = e_k^{c_2}e_0^{c_1} \text{ for all } k \in \{1, 3, 4, 5, 6\}$$

$$5. \ e_0^{c_1}e_2^{c_1c_2}e_0^{c_2} = e_2^{c_2}e_0^{c_1c_2}e_2^{c_1}$$

Note that $x_1^{(1)'} = c^{-1}x_1^{(1)}$, $x_2^{(2)'}x_2^{(1)'} = c^{-2}x_2^{(2)}x_2^{(1)}$, $x_3^{(3)'}x_3^{(2)'}x_3^{(1)'} = c^{-2}x_3^{(3)}x_3^{(2)}x_3^{(1)}$, $x_4^{(4)'}x_4^{(3)'}x_4^{(2)'}x_4^{(1)'} = c^{-2}x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}$, $x_5^{(2)'}x_5^{(1)'} = c^{-1}x_5^{(2)}x_5^{(1)}$, $x_6^{(3)'}x_6^{(2)'}x_6^{(1)'} = c^{-1}x_6^{(3)}x_6^{(2)}x_6^{(1)}$. Relations (1) and (2) follows easily from the defined actions.

$$\gamma_0(e_0^c(V_1(x))) = \frac{1}{x_2^{(2)'}x_2^{(1)'}} = \frac{1}{c^{-2}x_2^{(2)}x_2^{(1)}} = c^2\gamma_0(V_1(x)) = c^{a_{0,0}}\gamma_0(V_1(x)),$$

$$\gamma_0(e_2^c(V_1(x))) = \frac{1}{c_2x_2^{(2)}cc_2^{-1}x_2^{(1)'}} = c^{-1}\gamma_0(V_1(x)) = c^{a_{2,0}}\gamma_0(V_1(x)),$$

$$\gamma_0(e_k^c(V_1(x))) = \frac{1}{x_2^{(2)'}x_2^{(1)'}} = c^0\gamma_0(V_1(x)) = c^{a_{k,0}}\gamma_0(V_1(x)) \text{ for all } k = \{1, 3, 4, 5, 6\},$$

$$\gamma_1(e_0^c(V_1(x))) = \frac{(x_1^{(1)'})^2}{x_2^{(2)'}x_2^{(1)'}} = \frac{(c^{-1})^2(x_1^{(1)})^2}{c^{-2}x_2^{(2)}x_2^{(1)}} = c^0\gamma_1(V_1(x)) = c^{a_{0,1}}\gamma_1(V_1(x)),$$

$$\gamma_2(e_0^c(V_1(x))) = \frac{(x_2^{(2)'})^2(x_3^{(1)'})^2}{x_3^{(3)'}x_3^{(2)'}x_3^{(1)'}x_1^{(1)'}} = \frac{(c^{-2})^2(x_2^{(2)})^2(x_3^{(1)})^2}{c^{-2}x_3^{(3)}x_3^{(2)}x_3^{(1)}c^{-1}x_1^{(1)}} = c^{-1}\gamma_2(V_1(x)) = c^{a_{0,2}}\gamma_2(V_1(x)),$$

$$\begin{aligned} \gamma_3(e_0^c(V_1(x))) &= \frac{(x_3^{(3)'})^2(x_3^{(2)'})^2(x_3^{(1)'})^2}{x_4^{(4)'}x_4^{(3)'}x_4^{(2)'}x_4^{(1)'}x_2^{(2)'}x_2^{(1)'}} = \frac{(c^{-2})^2(x_3^{(3)})^2(x_3^{(2)})^2(x_3^{(1)})^2}{c^{-2}x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}c^{-2}x_2^{(2)}x_2^{(1)}} = c^0\gamma_3(V_1(x)) \\ &= c^{a_{0,3}}\gamma_3(V_1(x)), \end{aligned}$$

$$\begin{aligned} \gamma_4(e_0^c(V_1(x))) &= \frac{(x_4^{(4)'})^2(x_4^{(3)'})^2(x_4^{(2)'})^2(x_4^{(1)'})^2}{x_6^{(3)'}x_6^{(2)'}x_6^{(1)'}x_5^{(2)'}x_5^{(1)'}x_3^{(3)'}x_3^{(2)'}x_3^{(1)'}} = \frac{(c^{-2})^2(x_4^{(4)})^2(x_4^{(3)})^2(x_4^{(2)})^2(x_4^{(1)})^2}{c^{-1}x_6^{(3)}x_6^{(2)}x_6^{(1)}c^{-1}x_5^{(2)}x_5^{(1)}c^{-2}x_3^{(3)}x_3^{(2)}x_3^{(1)}} \\ &= c^0\gamma_4(V_1(x)) = c^{a_{0,4}}\gamma_4(V_1(x)), \end{aligned}$$

$$\gamma_5(e_0^c(V_1(x))) = \frac{(x_5^{(2)'})^2(x_5^{(1)'})^2}{x_4^{(4)'}x_4^{(3)'}x_4^{(2)'}x_4^{(1)'}} = \frac{(c^{-1})^2(x_5^{(2)})^2(x_5^{(1)})^2}{c^{-2}x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} = c^0\gamma_5(V_1(x)) = c^{a_{0,5}}\gamma_5(V_1(x)),$$

$$\begin{aligned} \gamma_6(e_0^c(V_1(x))) &= \frac{(x_6^{(3)'})^2(x_6^{(2)'})^2(x_6^{(1)'})^2}{x_4^{(4)'}x_4^{(3)'}x_4^{(2)'}x_4^{(1)'}} = \frac{(c^{-1})^2(x_6^{(3)})^2(x_6^{(2)})^2(x_6^{(1)})^2}{c^{-2}x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} = c^0\gamma_6(V_1(x)) \\ &= c^{a_{0,6}}\gamma_6(V_1(x)). \end{aligned}$$

Now we consider relation (3). We have

$$\begin{aligned}
\varepsilon_0(e_0^c(V_1(x))) &= x_6^{(1)'} + \frac{x_5^{(1)'}x_2^{(2)'}x_2^{(1)'}}{x_3^{(3)'}x_3^{(2)'}} + \frac{x_5^{(1)'}x_3^{(1)'}x_2^{(2)'}}{x_4^{(2)'}x_3^{(3)'}} + \frac{x_4^{(1)'}x_2^{(2)'}}{x_3^{(3)'}} + \frac{x_5^{(1)'}x_3^{(2)'}x_3^{(1)'}}{x_4^{(3)'}x_4^{(2)'}} + \frac{x_4^{(1)'}x_3^{(2)'}}{x_4^{(3)'}} \\
&\quad + \frac{x_4^{(2)'}x_4^{(1)'}}{x_6^{(2)'}} + \frac{x_2^{(2)'}x_2^{(1)'}}{x_6^{(3)'}} + \frac{x_6^{(2)'}x_2^{(2)'}x_2^{(1)'}}{x_4^{(4)'}x_4^{(3)'}} + \frac{x_4^{(2)'}x_2^{(2)'}x_2^{(1)'}}{x_4^{(4)'}x_3^{(2)'}} + \frac{x_4^{(3)'}x_4^{(2)'}x_2^{(2)'}x_2^{(1)'}}{x_5^{(2)'}x_3^{(3)'}x_3^{(2)'}} + \frac{x_3^{(1)'}x_2^{(2)'}}{x_4^{(4)'}} \\
&\quad + \frac{x_4^{(3)'}x_3^{(1)'}x_2^{(2)'}}{x_5^{(2)'}x_3^{(3)'}} + \frac{x_3^{(2)'}x_3^{(1)'}}{x_5^{(2)'}} \\
&= x_6^{(1)} \cdot \frac{K}{K_{6_1}} + \frac{x_5^{(1)}x_2^{(2)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} \cdot \frac{K^2}{K_{5_1}K_{3_1}} + \frac{x_5^{(1)}x_3^{(1)}x_2^{(2)}}{x_4^{(2)}x_3^{(3)}} \cdot \frac{K^2K_{4_2}}{K_{5_1}K_{4_1}K_{3_2}K_{3_1}} + \frac{x_4^{(1)}x_2^{(2)}}{x_3^{(3)'}} \cdot \frac{K^2}{K_{4_1}K_{3_2}} \\
&\quad + \frac{x_5^{(1)}x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)'}} \cdot \frac{K^2K_{4_3}}{K_{5_1}K_{4_1}K_{3_2}} + \frac{x_4^{(1)}x_3^{(2)'}}{x_4^{(3)'}} \cdot \frac{K^2K_{4_3}K_{3_1}}{K_{4_2}K_{4_1}K_{3_2}} + \frac{x_4^{(2)}x_4^{(1)'}}{x_6^{(2)'}} \cdot \frac{K^2K_{6_2}}{K_{6_1}K_{4_2}} + \frac{x_2^{(2)}x_2^{(1)'}}{x_6^{(3)'}} \cdot \frac{K}{cK_{6_2}} \\
&\quad + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)'}}{x_4^{(4)}x_4^{(3)'}} \cdot \frac{K^2K_{6_1}}{K_{6_2}K_{4_2}} + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)'}}{x_4^{(4)'}x_3^{(2)'}} \cdot \frac{K^2K_{4_1}K_{3_2}}{K_{4_3}K_{4_2}K_{3_1}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)'}}{x_5^{(2)}x_3^{(3)}x_3^{(2)'}} \cdot \frac{K^2K_{4_1}}{K_{5_1}K_{4_3}K_{3_1}} \\
&\quad + \frac{x_3^{(1)}x_2^{(2)'}}{x_4^{(4)'}} \cdot \frac{K^2}{K_{4_3}K_{3_1}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)'}}{x_5^{(2)}x_3^{(3)'}} \cdot \frac{K^2K_{4_2}}{K_{5_1}K_{4_3}K_{3_2}K_{3_1}} + \frac{x_3^{(2)}x_3^{(1)'}}{x_5^{(2)'}} \cdot \frac{K^2}{K_{5_1}K_{3_2}} \\
&= c^{-1}K \left(x_6^{(1)} \cdot \frac{c}{K_{6_1}} + \frac{x_5^{(1)}x_2^{(2)}x_2^{(1)'}}{x_3^{(3)}x_3^{(2)'}} \cdot \frac{cK}{K_{5_1}K_{3_1}} + \frac{x_5^{(1)}x_3^{(1)}x_2^{(2)'}}{x_4^{(2)}x_3^{(3)'}} \cdot \frac{cKK_{4_2}}{K_{5_1}K_{4_1}K_{3_2}K_{3_1}} + \frac{x_4^{(1)}x_2^{(2)'}}{x_3^{(3)'}} \cdot \frac{cK}{K_{4_1}K_{3_2}} \right. \\
&\quad + \frac{x_5^{(1)}x_3^{(2)}x_3^{(1)'}}{x_4^{(3)}x_4^{(2)'}} \cdot \frac{cKK_{4_3}}{K_{5_1}K_{4_1}K_{3_2}} + \frac{x_4^{(1)}x_3^{(2)'}}{x_4^{(3)'}} \cdot \frac{cKK_{4_3}K_{3_1}}{K_{4_2}K_{4_1}K_{3_2}} + \frac{x_4^{(2)}x_4^{(1)'}}{x_6^{(2)'}} \cdot \frac{cKK_{6_2}}{K_{6_1}K_{4_2}} + \frac{x_2^{(2)}x_2^{(1)'}}{x_6^{(3)'}} \cdot \frac{1}{K_{6_2}} \\
&\quad + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)'}}{x_4^{(4)}x_4^{(3)'}} \cdot \frac{cKK_{6_1}}{K_{6_2}K_{4_2}} + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)'}}{x_4^{(4)'}x_3^{(2)'}} \cdot \frac{cKK_{4_1}K_{3_2}}{K_{4_3}K_{4_2}K_{3_1}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)'}}{x_5^{(2)}x_3^{(3)}x_3^{(2)'}} \cdot \frac{cKK_{4_1}}{K_{5_1}K_{4_3}K_{3_1}} \\
&\quad \left. + \frac{x_3^{(1)}x_2^{(2)'}}{x_4^{(4)'}} \cdot \frac{cK}{K_{4_3}K_{3_1}} + \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)'}}{x_5^{(2)}x_3^{(3)'}} \cdot \frac{cKK_{4_2}}{K_{5_1}K_{4_3}K_{3_2}K_{3_1}} + \frac{x_3^{(2)}x_3^{(1)'}}{x_5^{(2)'}} \cdot \frac{cK}{K_{5_1}K_{3_2}} \right) \\
&= c^{-1}K \left(\left(x_6^{(1)}(cK_{6_2}K_{5_2}K_{5_1}K_{4_3}K_{4_2}K_{4_1}K_{3_2}K_{3_1}) + \frac{x_5^{(1)}x_2^{(2)}x_2^{(1)'}}{x_3^{(3)}x_3^{(2)'}}(cKK_{6_2}K_{6_1}K_{5_2}K_{4_3}K_{4_2}K_{4_1}K_{3_2}) \right. \right. \\
&\quad + \frac{x_5^{(1)}x_3^{(1)}x_2^{(2)'}}{x_4^{(2)}x_3^{(3)'}}(cKK_{6_2}K_{6_1}K_{5_2}K_{4_3}(K_{4_2})^2) + \frac{x_4^{(1)}x_2^{(2)'}}{x_3^{(3)'}}(cKK_{6_2}K_{6_1}K_{5_2}K_{5_1}K_{4_3}K_{4_2}K_{3_1}) \\
&\quad + \frac{x_5^{(1)}x_3^{(2)}x_3^{(1)'}}{x_4^{(3)}x_4^{(2)'}}(cKK_{6_2}K_{6_1}K_{5_2}(K_{4_3})^2K_{4_2}K_{3_1}) + \frac{x_4^{(1)}x_3^{(2)'}}{x_4^{(3)'}}(cKK_{6_2}K_{6_1}K_{5_2}K_{5_1}(K_{4_3})^2(K_{3_1})^2) \\
&\quad \left. \left. + \frac{x_4^{(2)}x_4^{(1)'}}{x_6^{(2)'}}(cK(K_{6_2})^2K_{5_2}K_{5_1}K_{4_3}K_{4_1}K_{3_2}K_{3_1}) + \frac{x_2^{(2)}x_2^{(1)'}}{x_6^{(3)'}}(K_{6_1}K_{5_2}K_{5_1}K_{4_3}K_{4_2}K_{4_1}K_{3_2}K_{3_1}) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} (c K(K_{61})^2 K_{52} K_{51} K_{43} K_{41} K_{32} K_{31}) + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} (c K K_{62} K_{61} K_{52} K_{51} (K_{41})^2 (K_{32})^2) \\
& + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} (c K K_{62} K_{61} K_{52} K_{42} (K_{41})^2 K_{32}) + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} (c K K_{62} K_{61} K_{52} K_{51} K_{42} K_{41} K_{32}) \\
& + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} (c K K_{62} K_{61} K_{52} (K_{42})^2 K_{41}) + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} (c K K_{62} K_{61} K_{52} K_{43} K_{42} K_{41} K_{31}) \\
& \times \left(K_{62} K_{61} K_{52} K_{51} K_{43} K_{42} K_{41} K_{32} K_{31} \right)^{-1} \\
& = c^{-1} K \left((K_{62} K_{61} K_{52} K_{51} K_{43} K_{42} K_{41} K_{32} K_{31}) (K_{62} K_{61} K_{52} K_{51} K_{43} K_{42} K_{41} K_{32} K_{31})^{-1} \right) \text{ by Maple} \\
& = c^{-1} (x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} \\
& + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}) = c^{-1} \varepsilon_0(V_1(x)).
\end{aligned}$$

Next we show relation (4). Let $k \in \{1, 3, 4, 5, 6\}$, we need to show that

$$e_0^{c_1} e_k^{c_2}(V_1(x)) = e_k^{c_2} e_0^{c_1}(V_1(x)).$$

Set $e_k^{c_2}(V_1(x)) = V_1(z)$, $e_0^{c_1}(V_1(z)) = V_1(z')$, $e_0^{c_1}(V_1(x)) = V_1(x')$ and $e_k^{c_2}(V_1(x')) = V_1(u)$. We have to show that

$$z_m^{(l)'} = u_m^{(l)}$$

for $(l, m) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (4, 4)\}$. Set

$$\begin{aligned}
K_z &= z_6^{(1)} + \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} \\
&+ \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} + \frac{z_2^{(2)} z_2^{(1)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} \\
&+ \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}, \\
K_{z3_1} &= c_1 z_6^{(1)} + \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + c_1 \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + c_1 \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + c_1 \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}}
\end{aligned}$$

$$\begin{aligned}
& + c_1 \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} + c_1 \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} + \frac{z_2^{(2)} z_2^{(1)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} \\
& + c_1 \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + c_1 \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + c_1 \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}, \\
K_{z5_1} = & c_1 z_6^{(1)} + c_1 \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + c_1 \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + c_1 \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + c_1 \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} \\
& + c_1 \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} + c_1 \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} + \frac{z_2^{(2)} z_2^{(1)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} \\
& + \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}, \\
K_{z6_1} = & c_1 z_6^{(1)} + \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} \\
& + \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} + \frac{z_2^{(2)} z_2^{(1)}}{z_6^{(3)}} + \frac{z_6^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_4^{(3)}} + \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} + \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} \\
& + \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}, \\
K_{z6_2} = & c_1 z_6^{(1)} + c_1 \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + c_1 \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + c_1 \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + c_1 \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} \\
& + c_1 \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} + c_1 \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} + \frac{z_2^{(2)} z_2^{(1)}}{z_6^{(3)}} + c_1 \frac{z_6^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_4^{(3)}} + c_1 \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} \\
& + c_1 \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} + c_1 \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + c_1 \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + c_1 \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}, \\
K_x = & x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} \\
& + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} \\
& + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{x3_1} = & c_1 x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\
& + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}}
\end{aligned}$$

$$+ \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(3)}x_3^{(2)}} + c_1 \frac{x_3^{(1)}x_2^{(2)}}{x_4^{(4)}} + c_1 \frac{x_4^{(3)}x_3^{(1)}x_2^{(2)}}{x_5^{(2)}x_3^{(3)}} + c_1 \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}},$$

$$K_{x3_2} = c_1 x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\ + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} \\ + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + c_1 \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}},$$

$$\begin{aligned}
K_{x4_1} = & c_1 x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\
& + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} \\
& + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}},
\end{aligned}$$

$$\begin{aligned}
K_{x4_2} = & \left(c_1 x_6^{(1)} + c_1 \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \right. \\
& + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} \\
& + c_1 \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c_1 \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + c_1 \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \left(c_1 x_6^{(1)} \right. \\
& + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} \\
& + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} \\
& + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \left. \right) + c_1^2 \frac{x_4^{(2)} x_4^{(1)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_4^{(1)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} \\
& - 2 \frac{x_4^{(2)} x_4^{(1)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}},
\end{aligned}$$

$$K_{x4_3} = c_1 x_6^{(1)} + c_1 \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\ + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}}$$

$$\begin{aligned}
& + c_1 \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c_1 \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + c_1 \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{x5_1} = & c_1 x_6^{(1)} + c_1 \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\
& + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} \\
& + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{x6_1} = & c_1 x_6^{(1)} + \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} \\
& + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} \\
& + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, \\
K_{x6_2} = & c_1 x_6^{(1)} + c_1 \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \\
& + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + c_1 \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} \\
& + c_1 \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + c_1 \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + c_1 \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + c_1 \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}.
\end{aligned}$$

For $k = 1$, observe that there is no term involving $z_1^{(1)}$ in $K_z, K_{z3_1}, K_{z3_2}, K_{z4_1}, K_{z4_2}, K_{z4_3}, K_{z5_1}, K_{z6_1}$ and K_{z6_2} . Similarly there is no term involving $x_1^{(1)}$ in $K_x, K_{x3_1}, K_{x3_2}, K_{x4_1}, K_{x4_2}, K_{x4_3}, K_{x5_1}, K_{x6_1}$ and K_{x6_2} . Hence $K_z = K_x, K_{z3_1} = K_{x3_1}, K_{z3_2} = K_{x3_2}, K_{z4_1} = K_{x4_1}, K_{z4_2} = K_{x4_2}, K_{z4_3} = K_{x4_3}, K_{z5_1} = K_{x5_1}, K_{z6_1} = K_{x6_1}$ and $K_{z6_2} = K_{x6_2}$. Now we have the following.

- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{c_2 x_1^{(1)}}{c_1} = c_2 x_1^{(1)'} = u_1^{(1)}$.
- $z_2^{(1)'} = \frac{z_2^{(1)}}{c_1} = \frac{x_2^{(1)}}{c_1} = x_2^{(1)'} = u_2^{(1)}$.
- $z_2^{(2)'} = \frac{z_2^{(2)}}{c_1} = \frac{x_2^{(2)}}{c_1} = x_2^{(2)'} = u_2^{(2)}$.
- $z_3^{(1)'} = z_3^{(1)} \frac{K_z}{K_{z3_1}} = x_3^{(1)} \frac{K_x}{K_{x3_1}} = x_3^{(1)'} = u_3^{(1)}$.
- $z_3^{(2)'} = z_3^{(2)} \frac{K_{z3_1}}{c_1 K_{z3_2}} = x_3^{(2)} \frac{K_{x3_1}}{c_1 K_{x3_2}} = x_3^{(2)'} = u_3^{(2)}$.

- $z_3^{(3)'} = z_3^{(3)} \frac{K_{z3}}{c_1 K_z} = x_3^{(3)} \frac{K_{x3}}{c_1 K_x} = x_3^{(3)'} = u_3^{(3)}$.
- $z_4^{(1)'} = z_4^{(1)} \frac{K_z}{K_{z4_1}} = x_4^{(1)} \frac{K_x}{K_{x4_1}} = x_4^{(1)'} = u_4^{(1)}$.
- $z_4^{(2)'} = z_4^{(2)} \frac{K_z K_{z4_1}}{K_{z4_2}} = x_4^{(2)} \frac{K_x K_{x4_1}}{K_{x4_2}} = x_4^{(2)'} = u_4^{(2)}$.
- $z_4^{(3)'} = z_4^{(3)} \frac{K_{z4_2}}{c_1 K_z K_{z4_3}} = \frac{K_{x4_2}}{c_1 K_x K_{x4_3}} = x_4^{(3)'} = u_4^{(3)}$.
- $z_4^{(4)'} = z_4^{(4)} \frac{K_{z4_3}}{c_1 K_z} = x_4^{(4)} \frac{K_{x4_3}}{c_1 K_x} = x_4^{(4)'} = u_4^{(4)}$.
- $z_5^{(1)'} = z_5^{(1)} \frac{K_z}{K_{z5_1}} = x_5^{(1)} \frac{K_x}{K_{x5_1}} = x_5^{(1)'} = u_5^{(1)}$.
- $z_5^{(2)'} = z_5^{(2)} \frac{K_{z5_1}}{c_1 K_z} = x_5^{(2)} \frac{K_{x5_1}}{c_1 K_x} = x_5^{(2)'} = u_5^{(2)}$.
- $z_6^{(1)'} = z_6^{(1)} \frac{K_z}{K_{z6_1}} = x_6^{(1)} \frac{K_x}{K_{x6_1}} = x_6^{(1)'} = u_6^{(1)}$.
- $z_6^{(2)'} = z_6^{(2)} \frac{K_{z6_1}}{K_{z6_2}} = x_6^{(2)} \frac{K_{x6_1}}{K_{x6_2}} = x_6^{(2)'} = u_6^{(2)}$.
- $z_6^{(3)'} = z_6^{(3)} \frac{K_{z6_2}}{c_1 K_z} = x_6^{(3)} \frac{K_{x6_2}}{c_1 K_x} = x_6^{(3)'} = u_6^{(3)}$.

For $k = 3$, we first observe that

$$\begin{aligned} \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)}) &= \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}), \\ \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} (x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) &= \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} (x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} &= \left(\frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)}) + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right) \\ &\quad + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \\ &\quad \times \left(c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right)^{-1} \\ &= \left(\left(\frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} \right) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right) \\ &\quad \times \left(c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right)^{-1} = \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}}, \end{aligned}$$

and

$$\begin{aligned}
& \frac{z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_4^{(4)} z_3^{(2)}} + \frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} = \left(\frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right. \\
& \quad \left. + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right) \\
& \quad \times \left(c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right)^{-1} \\
& = \left(\left(\frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} \right) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right) \\
& \quad \times \left(c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right)^{-1} = \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}}.
\end{aligned}$$

We also have

$$\begin{aligned}
& \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} = \left(\frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \right. \\
& \quad + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} \\
& \quad \times (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \\
& \quad + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \\
& \quad \times (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \left((c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} \right. \\
& \quad + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \\
& \quad \left. + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} \right)^{-1} \\
& = \left(\left(\frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} \right) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \right. \\
& \quad + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \left. \right) \\
& \quad \times ((c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} \right. \\
& \quad + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \left. \right)^{-1} \text{ by Maple} \\
& = \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}} = \left(\frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \right. \\
& \quad + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \\
& \quad \times (x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \\
& \quad + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \\
& \quad \times (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \Big) \left((c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} \right. \\
& \quad + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} \\
& \quad \left. + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right)^{-1} \\
& = \left(\left(\frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \right) \times (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)}) \right. \\
& \quad + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \Big) \\
& \quad \times \left((c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) (c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} \right. \\
& \quad \left. + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}) \right)^{-1} \text{ by Maple,} \\
& = \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}.
\end{aligned}$$

These imply $K_z = K_x$, $K_{z4_1} = K_{x4_1}$, $K_{z4_2} = K_{x4_2}$, $K_{z4_3} = K_{x4_3}$, $K_{z5_1} = K_{x5_1}$, $K_{z6_1} = K_{x6_1}$ and $K_{z6_2} = K_{x6_2}$. Now we have the following.

- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{x_1^{(1)}}{c_1} = x_1^{(1)'} = u_1^{(1)}$.
- $z_2^{(1)'} = \frac{z_2^{(1)}}{c_1} = \frac{x_2^{(1)}}{c_1} = x_2^{(1)'} = u_2^{(1)}$.
- $z_2^{(2)'} = \frac{z_2^{(2)}}{c_1} = \frac{x_2^{(2)}}{c_1} = x_2^{(2)'} = u_2^{(2)}$.
- $z_4^{(1)'} = z_4^{(1)} \frac{K_z}{K_{z4_1}} = x_4^{(1)} \frac{K_x}{K_{x4_1}} = x_4^{(1)'} = u_4^{(1)}$.
- $z_4^{(2)'} = z_4^{(2)} \frac{K_z K_{z4_1}}{K_{z4_2}} = x_4^{(2)} \frac{K_x K_{x4_1}}{K_{x4_2}} = x_4^{(2)'} = u_4^{(2)}$.
- $z_4^{(3)'} = z_4^{(3)} \frac{K_{z4_2}}{c_1 K_z K_{z4_3}} = \frac{K_{x4_2}}{c_1 K_x K_{x4_3}} = x_4^{(3)'} = u_4^{(3)}$.
- $z_4^{(4)'} = z_4^{(4)} \frac{K_{z4_3}}{c_1 K_z} = x_4^{(4)} \frac{K_{x4_3}}{c_1 K_x} = x_4^{(4)'} = u_4^{(4)}$.

- $z_5^{(1)'} = z_5^{(1)} \frac{K_z}{K_{z5_1}} = x_5^{(1)} \frac{K_x}{K_{x5_1}} = x_5^{(1)'} = u_5^{(1)}$.
- $z_5^{(2)'} = z_5^{(2)} \frac{K_{z5_1}}{c_1 K_z} = x_5^{(2)} \frac{K_{x5_1}}{c_1 K_x} = x_5^{(2)'} = u_5^{(2)}$.
- $z_6^{(1)'} = z_6^{(1)} \frac{K_z}{K_{z6_1}} = x_6^{(1)} \frac{K_x}{K_{x6_1}} = x_6^{(1)'} = u_6^{(1)}$.
- $z_6^{(2)'} = z_6^{(2)} \frac{K_{z6_1}}{K_{z6_2}} = x_6^{(2)} \frac{K_{x6_1}}{K_{x6_2}} = x_6^{(2)'} = u_6^{(2)}$.
- $z_6^{(3)'} = z_6^{(3)} \frac{K_{z6_2}}{c_1 K_z} = x_6^{(3)} \frac{K_{x6_2}}{c_1 K_x} = x_6^{(3)'} = u_6^{(3)}$.

Also, note that

$$\begin{aligned}
& \left(\frac{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}} \right) K_{z3_1} \\
&= K_{x3_1} (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} + c_1^{-3} c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1}) (c_1^{-2} x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1})^{-1} \text{ by Maple}, \\
& \left(\frac{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}} \right) K_{z3_2} \\
&= K_{x3_2} (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1}) (c_1^{-2} x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1})^{-1} \text{ by Maple}.
\end{aligned}$$

Thus we have

- $z_3^{(1)'} = z_3^{(1)} \frac{K_z}{K_{z3_1}} = x_3^{(1)} \cdot \frac{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}} \cdot \frac{K_x}{K_{z3_1}}$
 $= x_3^{(1)} \frac{K_x}{K_{x3_1}} (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} + c_1^{-3} c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1}$
 $\quad + c_1^{-3} c_2 x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1}) (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1}$
 $\quad + c_1^{-3} c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1})^{-1}$
 $= x_3^{(1)} \cdot \frac{c_2 x_3^{(3)'} (x_3^{(2)'})^2 x_3^{(1)'} + c_2 x_4^{(3)'} x_3^{(2)'} x_3^{(1)'} x_2^{(2)'} + c_2 x_4^{(3)'} x_4^{(2)'} x_2^{(2)'} x_2^{(1)'}}{c_2 x_3^{(3)'} (x_3^{(2)'})^2 x_3^{(1)'} + c_2 x_4^{(3)'} x_3^{(2)'} x_3^{(1)'} x_2^{(2)'} + x_4^{(3)'} x_4^{(2)'} x_2^{(2)'} x_2^{(1)'}} = u_3^{(1)},$
- $z_3^{(2)'} = z_3^{(2)} \frac{K_{z3_1}}{c_1 K_{z3_2}} = x_3^{(2)} \cdot \frac{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}} \cdot \frac{K_{z3_1}}{c_1 K_{z3_2}}$
 $= x_3^{(2)} \frac{K_{x3_1}}{c_1 K_{x3_2}} (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} + c_1^{-3} c_2 x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1}$
 $\quad + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1}) (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1}$
 $\quad + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1})^{-1}$
 $= x_3^{(2)} \cdot \frac{c_2 x_3^{(3)'} (x_3^{(2)'})^2 x_3^{(1)'} + c_2 x_4^{(3)'} x_3^{(2)'} x_3^{(1)'} x_2^{(2)'} + x_4^{(3)'} x_4^{(2)'} x_2^{(2)'} x_2^{(1)'}}{c_2 x_3^{(3)'} (x_3^{(2)'})^2 x_3^{(1)'} + x_4^{(3)'} x_3^{(2)'} x_3^{(1)'} x_2^{(2)'} + x_4^{(3)'} x_4^{(2)'} x_2^{(2)'} x_2^{(1)'}} = u_3^{(2)},$

$$\begin{aligned}
\bullet \quad z_3^{(3)'} &= z_3^{(3)} \frac{K_{z3_2}}{c_1 K_z} = x_3^{(3)} \cdot \frac{c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} + x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} + x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}} \cdot \frac{K_{z3_2}}{c_1 K_x} \\
&= x_3^{(3)} \frac{K_{x3_2}}{c_1 K_x} (c_1^{-2} c_2 x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1} (c_1^{-2} x_3^{(3)} (x_3^{(2)})^2 x_3^{(1)} K_{x3_1} (K_{x3_2})^{-1} \\
&\quad + c_1^{-3} x_4^{(3)} x_3^{(2)} x_3^{(1)} x_2^{(2)} K_{x4_2} (K_{x4_3} K_{x3_2})^{-1} + c_1^{-3} x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)} K_{x4_1} (K_{x4_3})^{-1})^{-1} \\
&= x_3^{(3)} \cdot \frac{c_2 x_3^{(3)}' (x_3^{(2)})' x_3^{(1)'} + x_4^{(3)}' x_3^{(2)}' x_3^{(1)'} x_2^{(2)'} + x_4^{(3)}' x_4^{(2)}' x_2^{(2)'} x_2^{(1)'}}{x_3^{(3)'} (x_3^{(2)})' x_3^{(1)'} + x_4^{(3)'} x_3^{(2)'} x_3^{(1)'} x_2^{(2)'} + x_4^{(3)'} x_4^{(2)'} x_2^{(2)'} x_2^{(1)'}} = u_3^{(3)}.
\end{aligned}$$

For $k = 4$, we first observe that

$$\begin{aligned}
\frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} (x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)}) &= \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}), \\
\frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} (x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)}) &= \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} (x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)}).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}} &= \left(\frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
&\quad + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \\
&\quad + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + c_2 x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \Big) \\
&\quad \times \left(c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \right. \\
&\quad \left. + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} \right)^{-1} \\
&= \left(\left(\frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} \right) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
&\quad \left. + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \right) \left(c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \right. \\
&\quad \left. + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} \right)^{-1} \\
&= \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{z_4^{(1)} z_2^{(2)}}{z_3^{(3)}},
\end{aligned}$$

and

$$\frac{z_3^{(1)} z_2^{(2)}}{z_4^{(4)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} = \left(\frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} (x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)}) \right.$$

$$\begin{aligned}
& + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \\
& + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \\
& \times \left(c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \right. \\
& \left. + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} \right)^{-1} \\
& = \left(\left(\frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} \right) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
& \left. + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \right) \left(c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \right. \\
& \left. + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} \right)^{-1} \\
& = \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}}.
\end{aligned}$$

We also have

$$\begin{aligned}
& \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(1)} z_3^{(2)}}{z_4^{(3)}} + \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} = \left(\frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
& \left. + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \right. \\
& \left. + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) + \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} \right. \\
& \left. \times (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \right. \\
& \left. + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
& \left. + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + c_2 x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) + \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \right. \\
& \left. + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \right. \\
& \left. \times (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \right. \\
& \left. + c_2 x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \right) \left((c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \right. \\
& \left. + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} \right. \\
& \left. + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \left(\left(\frac{x_5^{(1)}x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} \right) (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \right. \\
&\quad + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \\
&\quad \times (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + c_2x_6^{(2)}x_5^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} \\
&\quad \left. + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \right) \left(\left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \right. \right. \\
&\quad \left. + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \right) \left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \right. \\
&\quad \left. \left. + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + c_2x_6^{(2)}x_5^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \right) \right)^{-1}
\end{aligned}$$

by Maple

$$= \frac{x_5^{(1)}x_3^{(2)}x_3^{(1)}}{x_4^{(3)}x_4^{(2)}} + \frac{x_4^{(1)}x_3^{(2)}}{x_4^{(3)}} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}},$$

and

$$\begin{aligned}
&\frac{z_6^{(2)}z_2^{(2)}z_2^{(1)}}{z_4^{(4)}z_4^{(3)}} + \frac{z_4^{(2)}z_2^{(2)}z_2^{(1)}}{z_4^{(4)}z_3^{(2)}} + \frac{z_4^{(3)}z_4^{(2)}z_2^{(2)}z_2^{(1)}}{z_5^{(2)}z_3^{(3)}z_3^{(2)}} = \left(\frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} (x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \right. \\
&\quad + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \\
&\quad \times (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} \\
&\quad + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} (x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \\
&\quad + x_6^{(2)}x_5^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \\
&\quad + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + c_2x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \\
&\quad + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(3)}x_3^{(2)}} (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \\
&\quad + x_6^{(2)}x_5^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \\
&\quad + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + c_2x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \\
&\quad \times \left(\left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} \right. \right. \\
&\quad \left. + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \right) \left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \right. \\
&\quad \left. \left. + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \right) \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \left(\left(\frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(3)}x_3^{(2)}} \right) (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \right. \\
&\quad + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \\
&\quad \times (c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} \\
&\quad \left. + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)}) \right) \left(\left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} + x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} \right. \right. \\
&\quad + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)}x_3^{(2)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \left. \right) \left(c_2x_4^{(4)}(x_4^{(3)})^2(x_4^{(2)})^2x_4^{(1)} \right. \\
&\quad \left. + c_2x_5^{(2)}x_4^{(3)}(x_4^{(2)})^2x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_4^{(2)}x_4^{(1)}x_3^{(3)} + x_6^{(2)}x_5^{(2)}x_5^{(1)}x_3^{(3)}x_3^{(2)}x_3^{(1)} \right) \right)^{-1}
\end{aligned}$$

by Maple

$$= \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} + \frac{x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_3^{(2)}} + \frac{x_4^{(3)}x_4^{(2)}x_2^{(2)}x_2^{(1)}}{x_5^{(2)}x_3^{(3)}x_3^{(2)}}.$$

These imply $K_z = K_x$, $K_{z3_1} = K_{x3_1}$, $K_{z3_2} = K_{x3_2}$, $K_{z5_1} = K_{x5_1}$, $K_{z6_1} = K_{x6_1}$ and $K_{z6_2} = K_{x6_2}$. Now we have the following.

- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{x_1^{(1)}}{c_1} = x_1^{(1)'} = u_1^{(1)}$.
- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{c_2x_1^{(1)}}{c_1} = c_2x_1^{(1)'} = u_1^{(1)}$.
- $z_2^{(1)'} = \frac{z_2^{(1)}}{c_1} = \frac{x_2^{(1)}}{c_1} = x_2^{(1)'} = u_2^{(1)}$.
- $z_2^{(2)'} = \frac{z_2^{(2)}}{c_1} = \frac{x_2^{(2)}}{c_1} = x_2^{(2)'} = u_2^{(2)}$.
- $z_3^{(1)'} = z_3^{(1)} \frac{K_z}{K_{z3_1}} = x_3^{(1)} \frac{K_x}{K_{x3_1}} = x_3^{(1)'} = u_3^{(1)}$.
- $z_3^{(2)'} = z_3^{(2)} \frac{K_{z3_1}}{c_1 K_{z3_2}} = x_3^{(2)} \frac{K_{x3_1}}{c_1 K_{x3_2}} = x_3^{(2)'} = u_3^{(2)}$.
- $z_3^{(3)'} = z_3^{(3)} \frac{K_{z3_2}}{c_1 K_z} = x_3^{(3)} \frac{K_{x3_2}}{c_1 K_x} = x_3^{(3)'} = u_3^{(3)}$.
- $z_5^{(1)'} = z_5^{(1)} \frac{K_z}{K_{z5_1}} = x_5^{(1)} \frac{K_x}{K_{x5_1}} = x_5^{(1)'} = u_5^{(1)}$.
- $z_5^{(2)'} = z_5^{(2)} \frac{K_{z5_1}}{c_1 K_z} = x_5^{(2)} \frac{K_{x5_1}}{c_1 K_x} = x_5^{(2)'} = u_5^{(2)}$.
- $z_6^{(1)'} = z_6^{(1)} \frac{K_z}{K_{z6_1}} = x_6^{(1)} \frac{K_x}{K_{x6_1}} = x_6^{(1)'} = u_6^{(1)}$.
- $z_6^{(2)'} = z_6^{(2)} \frac{K_{z6_1}}{K_{z6_2}} = x_6^{(2)} \frac{K_{x6_1}}{K_{x6_2}} = x_6^{(2)'} = u_6^{(2)}$.
- $z_6^{(3)'} = z_6^{(3)} \frac{K_{z6_2}}{c_1 K_z} = x_6^{(3)} \frac{K_{x6_2}}{c_1 K_x} = x_6^{(3)'} = u_6^{(3)}$.

Also, note that, by Maple, we have

Thus we have

$$\begin{aligned} \bullet \quad & z_4^{(1)'} = z_4^{(1)} \frac{K_z}{K_{z4_1}} \\ &= x_4^{(1)} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \\ &+ c_2 x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} \\ &+ c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)})^{-1} \cdot \frac{K_x}{K_{z4_1}} \\ &= x_4^{(1)} \frac{K_x}{K_{x4_1}} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x4_1} (K_{x4_3})^{-1} \\ &+ c_1^{-3} c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x5_1} K_{x4_1} K_{x3_2} (K_{x4_3} K_{x4_2})^{-1} \end{aligned}$$

$$\begin{aligned}
& + c_1^{-3} c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_3^{(1)} x_3^{(3)} x_3^{(2)} K_{x61} K_{x51} K_{x31} (K_{x62} K_{x42})^{-1} \\
& + c_1^{-3} c_2 x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x61} (K_{x62})^{-1} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x41} (K_{x43})^{-1} \\
& + c_1^{-3} c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x51} K_{x41} K_{x32} (K_{x43} K_{x42})^{-1} \\
& + c_1^{-3} c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x61} K_{x51} K_{x31} (K_{x62} K_{x42})^{-1} \\
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x61} (K_{x62})^{-1} -1 \\
& = x_4^{(1)'} (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + c_2 x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'} + c_2 x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'} \\
& + c_2 x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'}) (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + c_2 x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'} \\
& + c_2 x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'} + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'})^{-1} \\
& = u_4^{(1)},
\end{aligned}$$

$$\begin{aligned}
z_4^{(2)'} &= z_4^{(2)} \frac{K_x K_{z41}}{K_{z42}} \\
&= x_4^{(2)} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \\
&\quad + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)})^{-1} \cdot \frac{K_x K_{z41}}{K_{z42}} \\
&\quad + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)})^{-1} \cdot \frac{K_x K_{z41}}{K_{z42}} \\
&= x_4^{(2)} \frac{K_x K_{x41}}{K_{x42}} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x41} (K_{x43})^{-1} \\
&\quad + c_1^{-3} c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x51} K_{x41} K_{x32} (K_{x43} K_{x42})^{-1} \\
&\quad + c_1^{-3} c_2 x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x61} K_{x51} K_{x31} (K_{x62} K_{x42})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x61} (K_{x62})^{-1} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x41} (K_{x43})^{-1} \\
&\quad + c_1^{-3} c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x51} K_{x41} K_{x32} (K_{x43} K_{x42})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x61} K_{x51} K_{x31} (K_{x62} K_{x42})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x61} (K_{x62})^{-1})^{-1} \\
&= x_4^{(2)'} (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + c_2 x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'} + c_2 x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'} \\
&\quad + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'}) (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + c_2 x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'})^{-1} \\
&\quad + x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'})^{-1} + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'})^{-1} \\
&= u_4^{(2)},
\end{aligned}$$

$$\begin{aligned}
z_4^{(3)'} &= z_4^{(3)} \frac{K_{z4_2}}{c_1 K_x K_{z4_3}} \\
&= x_4^{(3)} (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} \\
&\quad + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} + x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)})^{-1} \cdot \frac{K_{z4_2}}{c_1 K_x K_{z4_3}} \\
&= x_4^{(3)} \frac{K_{x4_2}}{c_1 K_x K_{x4_3}} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x4_1} (K_{x4_3})^{-1} \\
&\quad + c_1^{-3} c_2 x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x5_1} K_{x4_1} K_{x3_2} (K_{x4_3} K_{x4_2})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x6_1} K_{x5_1} K_{x3_1} (K_{x6_2} K_{x4_2})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x6_1} (K_{x6_2})^{-1}) (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)})^2 (x_4^{(2)})^2 x_4^{(1)} K_{x4_1} (K_{x4_3})^{-1} \\
&\quad + c_1^{-3} x_5^{(2)} x_4^{(3)} (x_4^{(2)})^2 x_4^{(1)} x_3^{(3)} K_{x5_1} K_{x4_1} K_{x3_2} (K_{x4_3} K_{x4_2})^{-1} \\
&\quad + c_1^{-3} x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} K_{x6_1} K_{x5_1} K_{x3_1} (K_{x6_2} K_{x4_2})^{-1})
\end{aligned}$$

$$\begin{aligned}
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x6_1} (K_{x6_2})^{-1} \Big)^{-1} \\
& = x_4^{(3)'} (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + c_2 x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'} + x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'} \\
& + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'}) (c_2 x_4^{(4)'} (x_4^{(3)'})^2 (x_4^{(2)'})^2 x_4^{(1)'} + x_5^{(2)'} x_4^{(3)'} (x_4^{(2)'})^2 x_4^{(1)'} x_3^{(3)'} \\
& + x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)'} x_3^{(3)'} x_3^{(2)'} + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'})^{-1} \\
& = u_4^{(3)},
\end{aligned}$$

$$\begin{aligned}
\bullet \quad & z_4^{(4)'} = z_4^{(4) \frac{K_{z4_3}}{c_1 K_z}} \\
& = x_4^{(4)} (c_2 x_4^{(4)} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) + x_5^{(2)} x_4^{(3)} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)}) \\
& + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)}) (x_4^{(4)} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) + x_5^{(2)} x_4^{(3)} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) \\
& + x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)}) + x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)})^{-1} \cdot \frac{K_{z4_3}}{c_1 K_x} \\
& = x_4^{(4) \frac{K_{x4_3}}{c_1 K_x}} (c_1^{-3} c_2 x_4^{(4)} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) K_{x4_1} (K_{x4_3})^{-1} \\
& + c_1^{-3} x_5^{(2)} x_4^{(3)} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) K_{x5_1} K_{x4_1} K_{x3_2} (K_{x4_3} K_{x4_2})^{-1} \\
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x6_1} K_{x5_1} K_{x3_1} (K_{x6_2} K_{x4_2})^{-1} \\
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x6_1} (K_{x6_2})^{-1}) (c_1^{-3} x_4^{(4)} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) K_{x4_1} (K_{x4_3})^{-1} \\
& + c_1^{-3} x_5^{(2)} x_4^{(3)} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) K_{x5_1} K_{x4_1} K_{x3_2} (K_{x4_3} K_{x4_2})^{-1} \\
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_4^{(2)} x_4^{(1)} x_3^{(3)} x_3^{(2)} K_{x6_1} K_{x5_1} K_{x3_1} (K_{x6_2} K_{x4_2})^{-1} \\
& + c_1^{-3} x_6^{(2)} x_5^{(2)} x_5^{(1)} x_3^{(3)} x_3^{(2)} x_3^{(1)} K_{x6_1} (K_{x6_2})^{-1})^{-1} \\
& = x_4^{(4)'} (c_2 x_4^{(4)'} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) + x_5^{(2)'} x_4^{(3)'} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) + x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)} x_3^{(3)} x_3^{(2)'} \\
& + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'}) (x_4^{(4)'} (x_4^{(3)')^2} (x_4^{(2)')^2} x_4^{(1)}) + x_5^{(2)'} x_4^{(3)'} (x_4^{(2)')^2} x_4^{(1)} x_3^{(3)}) \\
& + x_6^{(2)'} x_5^{(2)'} x_4^{(2)'} x_4^{(1)} x_3^{(3)'} x_3^{(2)'} + x_6^{(2)'} x_5^{(2)'} x_5^{(1)'} x_3^{(3)'} x_3^{(2)'} x_3^{(1)'})^{-1} \\
& = u_4^{(4)}.
\end{aligned}$$

For $k = 5$, we first observe that

$$\begin{aligned}
\frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} & = \left(\frac{c_2 x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} \right) \left(\frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
& = \left(\frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} \right) \left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
& = \frac{x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}}, \\
\frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} & = \left(\frac{c_2 x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \right) \left(\frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
& = \left(\frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \right) \left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
& = \frac{x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}},
\end{aligned}$$

$$\begin{aligned}
\frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}} &= \left(\frac{c_2 x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \right) \left(\frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
&= \left(\frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \right) \left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \\
&= \frac{x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}.
\end{aligned}$$

These imply $K_z = K_x$, $K_{z3_1} = K_{x3_1}$, $K_{z3_2} = K_{x3_2}$, $K_{z4_1} = K_{x4_1}$, $K_{z4_2} = K_{x4_2}$, $K_{z4_3} = K_{x4_3}$, $K_{z6_1} = K_{x6_1}$ and $K_{z6_2} = K_{x6_2}$. Now we have the following.

- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{x_1^{(1)}}{c_1} = x_1^{(1)'} = u_1^{(1)}$.
- $z_2^{(1)'} = \frac{z_2^{(1)}}{c_1} = \frac{x_2^{(1)}}{c_1} = x_2^{(1)'} = u_2^{(1)}$.
- $z_2^{(2)'} = \frac{z_2^{(2)}}{c_1} = \frac{x_2^{(2)}}{c_1} = x_2^{(2)'} = u_2^{(2)}$.
- $z_3^{(1)'} = z_3^{(1)} \frac{K_z}{K_{z3_1}} = x_3^{(1)} \frac{K_x}{K_{x3_1}} = x_3^{(1)'} = u_3^{(1)}$.
- $z_3^{(2)'} = z_3^{(2)} \frac{K_{z3_1}}{c_1 K_{z3_2}} = x_3^{(2)} \frac{K_{x3_1}}{c_1 K_{x3_2}} = x_3^{(2)'} = u_3^{(2)}$.
- $z_3^{(3)'} = z_3^{(3)} \frac{K_{z3_2}}{c_1 K_z} = x_3^{(3)} \frac{K_{x3_2}}{c_1 K_x} = x_3^{(3)'} = u_3^{(3)}$.
- $z_4^{(1)'} = z_4^{(1)} \frac{K_z}{K_{z4_1}} = x_4^{(1)} \frac{K_x}{K_{x4_1}} = x_4^{(1)'} = u_4^{(1)}$.
- $z_4^{(2)'} = z_4^{(2)} \frac{K_z K_{z4_1}}{K_{z4_2}} = x_4^{(2)} \frac{K_x K_{x4_1}}{K_{x4_2}} = x_4^{(2)'} = u_4^{(2)}$.
- $z_4^{(3)'} = z_4^{(3)} \frac{K_{z4_2}}{c_1 K_z K_{z4_3}} = \frac{K_{x4_2}}{c_1 K_x K_{x4_3}} = x_4^{(3)'} = u_4^{(3)}$.
- $z_4^{(4)'} = z_4^{(4)} \frac{K_{z4_3}}{c_1 K_z} = x_4^{(4)} \frac{K_{x4_3}}{c_1 K_x} = x_4^{(4)'} = u_4^{(4)}$.
- $z_6^{(1)'} = z_6^{(1)} \frac{K_z}{K_{z6_1}} = x_6^{(1)} \frac{K_x}{K_{x6_1}} = x_6^{(1)'} = u_6^{(1)}$.
- $z_6^{(2)'} = z_6^{(2)} \frac{K_{z6_1}}{K_{z6_2}} = x_6^{(2)} \frac{K_{x6_1}}{K_{x6_2}} = x_6^{(2)'} = u_6^{(2)}$.
- $z_6^{(3)'} = z_6^{(3)} \frac{K_{z6_2}}{c_1 K_z} = x_6^{(3)} \frac{K_{x6_2}}{c_1 K_x} = x_6^{(3)'} = u_6^{(3)}$.

We also have

$$c_1 \frac{z_5^{(1)} z_2^{(2)} z_2^{(1)}}{z_3^{(3)} z_3^{(2)}} + c_1 \frac{z_5^{(1)} z_3^{(1)} z_2^{(2)}}{z_4^{(2)} z_3^{(3)}} + c_1 \frac{z_5^{(1)} z_3^{(2)} z_3^{(1)}}{z_4^{(3)} z_4^{(2)}} + \frac{z_4^{(3)} z_4^{(2)} z_2^{(2)} z_2^{(1)}}{z_5^{(2)} z_3^{(3)} z_3^{(2)}} + \frac{z_4^{(3)} z_3^{(1)} z_2^{(2)}}{z_5^{(2)} z_3^{(3)}} + \frac{z_3^{(2)} z_3^{(1)}}{z_5^{(2)}}$$

$$\begin{aligned}
&= \left(\frac{c_1 c_2 x_5^{(1)} x_2^{(2)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}} + \frac{c_1 c_2 x_5^{(1)} x_3^{(1)} x_2^{(2)}}{x_4^{(2)} x_3^{(3)}} + \frac{c_1 c_2 x_5^{(1)} x_3^{(2)} x_3^{(1)}}{x_4^{(3)} x_4^{(2)}} + \frac{x_4^{(3)} x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_4^{(3)} x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_3^{(3)}} \right. \\
&\quad \left. + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}} \right) \frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \\
&= \left((c_1 c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}) \left(\frac{x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_4^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_4^{(3)} x_4^{(2)}} \right) \right) \left(\frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right),
\end{aligned}$$

then,

$$\begin{aligned}
&\left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) K_{z5_1} = \left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \left(c_1 x_6^{(1)} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} \right. \\
&\quad \left. + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} + \left((c_1 c_2 x_5^{(2)} x_5^{(1)} \right. \right. \\
&\quad \left. \left. + x_4^{(3)} x_4^{(2)}) \left(\frac{x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_4^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_4^{(3)} x_4^{(2)}} \right) \left(\frac{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \right) \right) \\
&= \left(\frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \right) \left(c_1 x_6^{(1)} + c_1 \frac{x_4^{(1)} x_2^{(2)}}{x_3^{(3)}} + c_1 \frac{x_4^{(1)} x_3^{(2)}}{x_4^{(3)}} + c_1 \frac{x_4^{(2)} x_4^{(1)}}{x_6^{(2)}} + \frac{x_2^{(2)} x_2^{(1)}}{x_6^{(3)}} \right. \\
&\quad \left. + \frac{x_6^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_4^{(3)}} + \frac{x_4^{(2)} x_2^{(2)} x_2^{(1)}}{x_4^{(4)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_4^{(4)}} \right) + \left((c_1 c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}) \right. \\
&\quad \times \left. \left(\frac{x_2^{(2)} x_2^{(1)}}{x_5^{(2)} x_3^{(3)} x_3^{(2)}} + \frac{x_3^{(1)} x_2^{(2)}}{x_5^{(2)} x_4^{(2)} x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)} x_4^{(3)} x_4^{(2)}} \right) \right) \\
&= K_{x5_1} \cdot \frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} K_{x4_1} (K_{x4_3})^{-1} \text{ by Maple.}
\end{aligned}$$

Thus we have

- $z_5^{(1)'} = z_5^{(1)} \frac{K_z}{K_{z5_1}} = x_5^{(1)} \cdot \frac{c_2 x_5^{(2)} x_5^{(1)} + c_2 x_4^{(3)} x_4^{(2)}}{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \cdot \frac{K_x}{K_{z5_1}} = x_5^{(1)} \frac{K_x}{K_{x5_1}} \cdot \frac{c_1^{-1} c_2 x_5^{(2)} x_5^{(1)} + c_1^{-1} c_2 x_4^{(3)} x_4^{(2)} K_{x4_1} (K_{x4_3})^{-1}}{c_1^{-1} c_2 x_5^{(2)} x_5^{(1)} + c_1^{-1} x_4^{(3)} x_4^{(2)} K_{x4_1} (K_{x4_3})^{-1}}$
 $= x_5^{(1)} \cdot \frac{c_2 x_5^{(2)'} x_5^{(1)'} + c_2 x_4^{(3)'} x_4^{(2)'}}{c_2 x_5^{(2)'} x_5^{(1)'} + x_4^{(3)'} x_4^{(2)'}} = u_5^{(1)},$
- $z_5^{(2)'} = z_5^{(2)} \frac{K_{z5_1}}{c_1 K_z} = x_5^{(2)} \cdot \frac{c_2 x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}}{x_5^{(2)} x_5^{(1)} + x_4^{(3)} x_4^{(2)}} \cdot \frac{K_{z5_1}}{c_1 K_x} = x_5^{(2)} \frac{K_{z5_1}}{c_1 K_x} \cdot \frac{c_1^{-1} c_2 x_5^{(2)} x_5^{(1)} + c_1^{-1} x_4^{(3)} x_4^{(2)} K_{x4_1} (K_{x4_3})^{-1}}{c_1^{-1} x_5^{(2)} x_5^{(1)} + c_1^{-1} x_4^{(3)} x_4^{(2)} K_{x4_1} (K_{x4_3})^{-1}}$
 $= x_5^{(2)} \cdot \frac{c_2 x_5^{(2)'} x_5^{(1)'} + x_4^{(3)'} x_4^{(2)'}}{x_5^{(2)'} x_5^{(1)'} + x_4^{(3)'} x_4^{(2)'}} = u_5^{(2)}.$

For $k = 6$, we first observe that

$$z_6^{(1)} + \frac{z_4^{(2)} z_4^{(1)}}{z_6^{(2)}} = x_6^{(1)} \cdot \frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + c_2 x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}$$

$$\begin{aligned}
& + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} \cdot \frac{c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} \\
& = (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})^{-1} \\
& \quad \times (x_6^{(1)}(c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + c_2x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})) \\
& \quad + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}} \cdot (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})) \\
& = (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})^{-1} \\
& \quad \times \left((x_6^{(1)} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}})(c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}) \right) \\
& = x_6^{(1)} + \frac{x_4^{(2)}x_4^{(1)}}{x_6^{(2)}}, \\
\frac{z_2^{(2)}z_2^{(1)}}{z_6^{(3)}} + \frac{z_6^{(2)}z_2^{(2)}z_2^{(1)}}{z_4^{(4)}z_4^{(3)}} & = \frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} \cdot \frac{x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} \\
& \quad + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} \cdot \frac{c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}}{c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}} \\
& = (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})^{-1} \\
& \quad \times \left(\frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} \cdot (x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + c_2x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}) \right. \\
& \quad \left. + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} \cdot (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + c_2x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}) \right) \\
& = (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)})^{-1} \\
& \quad \times \left(\left(\frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}} \right) (c_2x_6^{(3)}(x_6^{(2)})^2x_6^{(1)} + x_6^{(2)}x_6^{(1)}x_4^{(4)}x_4^{(3)} + c_2x_4^{(4)}x_4^{(3)}x_4^{(2)}x_4^{(1)}) \right) \\
& = \frac{x_2^{(2)}x_2^{(1)}}{x_6^{(3)}} + \frac{x_6^{(2)}x_2^{(2)}x_2^{(1)}}{x_4^{(4)}x_4^{(3)}}.
\end{aligned}$$

These imply $K_z = K_x, K_{z3_1} = K_{x3_1}, K_{z3_2} = K_{x3_2}, K_{z4_1} = K_{x4_1}, K_{z4_2} = K_{x4_2}, K_{z4_3} = K_{x4_3}$ and $K_{z5_1} = K_{x5_1}$. Now we have the following.

- $z_1^{(1)'} = \frac{z_1^{(1)}}{c_1} = \frac{x_1^{(1)}}{c_1} = x_1^{(1)'} = u_1^{(1)}$.
- $z_2^{(1)'} = \frac{z_2^{(1)}}{c_1} = \frac{x_2^{(1)}}{c_1} = x_2^{(1)'} = u_2^{(1)}$.
- $z_2^{(2)'} = \frac{z_2^{(2)}}{c_1} = \frac{x_2^{(2)}}{c_1} = x_2^{(2)'} = u_2^{(2)}$.

- $z_3^{(1)'} = z_3^{(1)} \frac{K_z}{K_{z3_1}} = x_3^{(1)} \frac{K_x}{K_{x3_1}} = x_3^{(1)'} = u_3^{(1)}$.
- $z_3^{(2)'} = z_3^{(2)} \frac{K_{z3_1}}{c_1 K_{z3_2}} = x_3^{(2)} \frac{K_{x3_1}}{c_1 K_{x3_2}} = x_3^{(2)'} = u_3^{(2)}$.
- $z_3^{(3)'} = z_3^{(3)} \frac{K_{z3_2}}{c_1 K_z} = x_3^{(3)} \frac{K_{x3_2}}{c_1 K_x} = x_3^{(3)'} = u_3^{(3)}$.
- $z_4^{(1)'} = z_4^{(1)} \frac{K_z}{K_{z4_1}} = x_4^{(1)} \frac{K_x}{K_{x4_1}} = x_4^{(1)'} = u_4^{(1)}$.
- $z_4^{(2)'} = z_4^{(2)} \frac{K_z K_{z4_1}}{K_{z4_2}} = x_4^{(2)} \frac{K_x K_{x4_1}}{K_{x4_2}} = x_4^{(2)'} = u_4^{(2)}$.
- $z_4^{(3)'} = z_4^{(3)} \frac{K_{z4_2}}{c_1 K_z K_{z4_3}} = \frac{K_{x4_2}}{c_1 K_x K_{x4_3}} = x_4^{(3)'} = u_4^{(3)}$.
- $z_4^{(4)'} = z_4^{(4)} \frac{K_{z4_3}}{c_1 K_z} = x_4^{(4)} \frac{K_{x4_3}}{c_1 K_x} = x_4^{(4)'} = u_4^{(4)}$.
- $z_5^{(1)'} = z_5^{(1)} \frac{K_z}{K_{z5_1}} = x_5^{(1)} \frac{K_x}{K_{x5_1}} = x_5^{(1)'} = u_5^{(1)}$.
- $z_5^{(2)'} = z_5^{(2)} \frac{K_{z5_1}}{c_1 K_z} = x_5^{(2)} \frac{K_{x5_1}}{c_1 K_x} = x_5^{(2)'} = u_5^{(2)}$.

Also, note that

$$\begin{aligned}
& \left(\frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \right) K_{z6_1} \\
&= K_{x6_1} (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} + c_1^{-2} c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} \\
&\quad + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}) (c_1^{-1} x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} \\
&\quad + c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)})^{-1} \text{ by Maple,} \\
& \left(\frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \right) K_{z6_2} \\
&= K_{x6_2} (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} + c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} \\
&\quad + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}) (c_1^{-1} x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} \\
&\quad + c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)})^{-1} \text{ by Maple.}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\bullet \quad & z_6^{(1)'} = z_6^{(1)} \frac{K_z}{K_{z6_1}} = x_6^{(1)} \cdot \frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + c_2 x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \cdot \frac{K_x}{K_{z6_1}} \\
&= x_6^{(1)} \frac{K_x}{K_{x6_1}} (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} + c_1^{-2} c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} \\
&\quad + c_1^{-2} c_2 x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}) (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x6_1} (K_{x6_2})^{-1} \\
&\quad + c_1^{-2} c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x4_2} (K K_{x6_2})^{-1} + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)})^{-1} \\
&= x_6^{(1)} \cdot \frac{c_2 x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + c_2 x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + c_2 x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}}{c_2 x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + c_2 x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}} = u_6^{(1)},
\end{aligned}$$

- $z_6^{(2)'} = z_6^{(2)} \frac{K_{z61}}{K_{z62}} = x_6^{(2)} \cdot \frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \cdot \frac{K_{z61}}{K_{z62}}$
 $= x_6^{(2)} \frac{K_{x61}}{K_{x62}} (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x61} (K_{x62})^{-1} + c_1^{-2} c_2 x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x42} (K K_{x62})^{-1}$
 $+ c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}) (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x61} (K_{x62})^{-1}$
 $+ c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x42} (K K_{x62})^{-1} + c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)})^{-1}$
 $= x_6^{(2)'} \cdot \frac{c_2 x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + c_2 x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}}{c_2 x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}} = u_6^{(2)},$
- $z_6^{(3)'} = z_6^{(3)} \frac{K_{z62}}{c_1 K_z} = x_6^{(3)} \cdot \frac{c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}}{x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} + x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} + x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}} \cdot \frac{K_{z62}}{c_1 K_z}$
 $= x_6^{(3)} \frac{K_{x62}}{c_1 K_x} (c_1^{-1} c_2 x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x61} (K_{x62})^{-1} + c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x42} (K K_{x62})^{-1}$
 $+ c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)}) (c_1^{-1} x_6^{(3)} (x_6^{(2)})^2 x_6^{(1)} K_{x61} (K_{x62})^{-1} + c_1^{-2} x_6^{(2)} x_6^{(1)} x_4^{(4)} x_4^{(3)} K_{x42} (K K_{x62})^{-1}$
 $+ c_1^{-2} x_4^{(4)} x_4^{(3)} x_4^{(2)} x_4^{(1)})^{-1}$
 $= x_6^{(3)'} \cdot \frac{c_2 x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}}{x_6^{(3)'} (x_6^{(2)'})^2 x_6^{(1)'} + x_6^{(2)'} x_6^{(1)'} x_4^{(4)'} x_4^{(3)'} + x_4^{(4)'} x_4^{(3)'} x_4^{(2)'} x_4^{(1)'}} = u_6^{(3)}.$

Finally to show relation (5) we observe that $\bar{e}_6^{c_1} \bar{e}_4^{c_1 c_2} \bar{e}_6^{c_2} = \bar{e}_4^{c_2} \bar{e}_6^{c_1 c_2} \bar{e}_4^{c_1}$ since \mathcal{V}_2 is a \mathfrak{g}_1 -geometric crystal. Hence by Proposition 6.2.5, we have

$$\begin{aligned} e_0^{c_1} e_2^{c_1 c_2} e_0^{c_2} &= \bar{\sigma}^{-1} \bar{e}_6^{c_1} \bar{\sigma} e_2^{c_1 c_2} \bar{\sigma}^{-1} \bar{e}_6^{c_2} \bar{\sigma} \\ &= \bar{\sigma}^{-1} \bar{e}_6^{c_1} \bar{e}_4^{c_1 c_2} \bar{e}_6^{c_2} \bar{\sigma} = \bar{\sigma}^{-1} \bar{e}_4^{c_2} \bar{e}_6^{c_1 c_2} \bar{e}_4^{c_1} \bar{\sigma} \\ &= e_2^{c_2} \bar{\sigma}^{-1} \bar{e}_6^{c_1 c_2} \bar{\sigma} e_2^{c_1} = e_2^{c_2} e_0^{c_1 c_2} e_2^{c_1}, \end{aligned}$$

which completes the proof. \square

CHAPTER

7

PERFECT CRYSTALS OF TYPE $D_6^{(1)}$

For a positive integer l , we consider the sets $B^{6,l}$ and $B^{6,\infty}$ as follows.

$$B^{6,l} = \left\{ b = (b_{ij}) \begin{array}{c} i \leq j \leq i+5, \\ 1 \leq i \leq 6 \end{array} \middle| \begin{array}{l} b_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{j=i}^{i+5} b_{ij} = l, 1 \leq i \leq 6, \\ \sum_{j=i}^{6-t} b_{ij} = \sum_{j=i+t}^{5+t} b_{i+t,j}, 1 \leq i, t \leq 5, \\ \sum_{j=i}^t b_{ij} \geq \sum_{j=i+1}^{t+1} b_{i+1,j}, 1 \leq i \leq t \leq 5 \end{array} \right\},$$

$$B^{6,\infty} = \left\{ b = (b_{ij}) \begin{array}{c} i \leq j \leq i+5, \\ 1 \leq i \leq 6 \end{array} \middle| \begin{array}{l} b_{ij} \in \mathbb{Z}, \sum_{j=i}^{i+5} b_{ij} = 0, 1 \leq i \leq 6, \\ \sum_{j=i}^{6-t} b_{ij} = \sum_{j=i+t}^{5+t} b_{i+t,j}, 1 \leq i, t \leq 5 \end{array} \right\}.$$

For $\mathcal{B} = B^{6,l}$ or $B^{6,\infty}$ we define the maps $\tilde{e}_k, \tilde{f}_k : \mathcal{B} \rightarrow \mathcal{B} \cup \{0\}$, $\varepsilon_k, \varphi_k : \mathcal{B} \rightarrow \mathbb{Z}$, $0 \leq k \leq 6$ and $\text{wt} : \mathcal{B} \rightarrow P_{cl}$, as follows. First we define conditions (E_j) , $1 \leq j \leq 14$:

$$(E_1) \quad -b_{13} - b_{14} - b_{15} + b_{22} > b_{55},$$

$$\begin{aligned}
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} + b_{44}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{24} + b_{44}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{35} + b_{44}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{14} + b_{33}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{13} - b_{25} + b_{33}, \\
& -b_{13} - b_{14} - b_{15} + b_{22} > -b_{24} - b_{25} + b_{33}, \\
(E_2) \quad & -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > b_{55}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} + b_{44}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{24} + b_{44}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{35} + b_{44}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{14} + b_{33}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{13} - b_{25} + b_{33}, \\
& -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35} > -b_{24} - b_{25} + b_{33}, \\
(E_3) \quad & -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > b_{55}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} + b_{44}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{24} + b_{44},
\end{aligned}$$

$$\begin{aligned}
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{35} + b_{44}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} - b_{14} + b_{33}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{13} - b_{25} + b_{33}, \\
& -b_{13} - b_{14} - b_{23} + b_{33} + b_{34} > -b_{24} - b_{25} + b_{33}, \\
(E_4) \quad & -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > b_{55}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{13} + b_{44}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{24} + b_{44}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{35} + b_{44}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{13} - b_{14} + b_{33}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{13} - b_{25} + b_{33}, \\
& -b_{13} - b_{23} - b_{25} + b_{33} + b_{34} > -b_{24} - b_{25} + b_{33}, \\
(E_5) \quad & -b_{13} - b_{14} + b_{33} > b_{55}, \\
& -b_{13} - b_{14} + b_{33} > -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} + b_{33} > -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} + b_{33} > -b_{13} + b_{44}, \\
& -b_{13} - b_{14} + b_{33} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{14} + b_{33} > -b_{24} + b_{44}, \\
& -b_{13} - b_{14} + b_{33} > -b_{35} + b_{44}, \\
& -b_{13} - b_{14} + b_{33} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} - b_{14} + b_{33} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{13} - b_{14} + b_{33} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} - b_{14} + b_{33} > -b_{13} - b_{23} - b_{25} + b_{33} + b_{34},
\end{aligned}$$

- $-b_{13} - b_{14} + b_{33} > -b_{13} - b_{25} + b_{33},$
- $-b_{13} - b_{14} + b_{33} > -b_{24} - b_{25} + b_{33},$
- (E₆) $-b_{13} - b_{23} + b_{44} + b_{45} > b_{55},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{13} - b_{34} + b_{44} + b_{45},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{13} + b_{44},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{24} - b_{34} + b_{44} + b_{45},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{24} + b_{44},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{35} + b_{44},$
- $-b_{13} - b_{23} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{15} + b_{22},$
- $-b_{13} - b_{23} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35},$
- $-b_{13} - b_{23} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34},$
- $-b_{13} - b_{23} + b_{44} + b_{45} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{13} - b_{14} + b_{33},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{13} - b_{25} + b_{33},$
- $-b_{13} - b_{23} + b_{44} + b_{45} > -b_{24} - b_{25} + b_{33},$
- (E₇) $-b_{13} - b_{25} + b_{33} > b_{55},$
- $-b_{13} - b_{25} + b_{33} > -b_{13} - b_{23} + b_{44} + b_{45},$
- $-b_{13} - b_{25} + b_{33} > -b_{13} - b_{34} + b_{44} + b_{45},$
- $-b_{13} - b_{25} + b_{33} > -b_{13} + b_{44},$
- $-b_{13} - b_{25} + b_{33} > -b_{24} - b_{34} + b_{44} + b_{45},$
- $-b_{13} - b_{25} + b_{33} > -b_{24} + b_{44},$
- $-b_{13} - b_{25} + b_{33} > -b_{35} + b_{44},$
- $-b_{13} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{15} + b_{22},$
- $-b_{13} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35},$
- $-b_{13} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34},$
- $-b_{13} - b_{25} + b_{33} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34},$
- $-b_{13} - b_{25} + b_{33} \geq -b_{13} - b_{14} + b_{33},$
- $-b_{13} - b_{25} + b_{33} > -b_{24} - b_{25} + b_{33},$
- (E₈) $-b_{24} - b_{25} + b_{33} > b_{55},$
- $-b_{24} - b_{25} + b_{33} > -b_{13} - b_{23} + b_{44} + b_{45},$
- $-b_{24} - b_{25} + b_{33} > -b_{13} - b_{34} + b_{44} + b_{45},$

$$\begin{aligned}
& -b_{24} - b_{25} + b_{33} > -b_{13} + b_{44}, \\
& -b_{24} - b_{25} + b_{33} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{24} - b_{25} + b_{33} > -b_{24} + b_{44}, \\
& -b_{24} - b_{25} + b_{33} > -b_{35} + b_{44}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{14} + b_{33}, \\
& -b_{24} - b_{25} + b_{33} \geq -b_{13} - b_{25} + b_{33},
\end{aligned}$$

$$(E_9) \quad -b_{13} - b_{34} + b_{44} + b_{45} > b_{55},$$

$$\begin{aligned}
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} > -b_{13} + b_{44}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} > -b_{24} + b_{44}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} > -b_{35} + b_{44}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} + b_{33}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{25} + b_{33}, \\
& -b_{13} - b_{34} + b_{44} + b_{45} > -b_{24} - b_{25} + b_{33},
\end{aligned}$$

$$(E_{10}) \quad -b_{13} + b_{44} > b_{55},$$

$$\begin{aligned}
& -b_{13} + b_{44} \geq -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} + b_{44} > -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{13} + b_{44} > -b_{24} + b_{44}, \\
& -b_{13} + b_{44} > -b_{35} + b_{44}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35},
\end{aligned}$$

$$\begin{aligned}
& -b_{13} + b_{44} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{14} + b_{33}, \\
& -b_{13} + b_{44} \geq -b_{13} - b_{25} + b_{33}, \\
& -b_{13} + b_{44} > -b_{24} - b_{25} + b_{33}, \\
(E_{11}) \quad & -b_{24} - b_{34} + b_{44} + b_{45} > b_{55}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} > -b_{13} + b_{44}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} > -b_{24} + b_{44}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} > -b_{35} + b_{44}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{14} + b_{33}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{13} - b_{25} + b_{33}, \\
& -b_{24} - b_{34} + b_{44} + b_{45} \geq -b_{24} - b_{25} + b_{33}, \\
(E_{12}) \quad & -b_{24} + b_{44} > b_{55}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{23} + b_{44} + b_{45}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{34} + b_{44} + b_{45}, \\
& -b_{24} + b_{44} \geq -b_{13} + b_{44}, \\
& -b_{24} + b_{44} \geq -b_{24} - b_{34} + b_{44} + b_{45}, \\
& -b_{24} + b_{44} > -b_{35} + b_{44}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{14} - b_{15} + b_{22}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{14} + b_{33}, \\
& -b_{24} + b_{44} \geq -b_{13} - b_{25} + b_{33}, \\
& -b_{24} + b_{44} \geq -b_{24} - b_{25} + b_{33},
\end{aligned}$$

- (E₁₃) $-b_{35} + b_{44} > b_{55},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{23} + b_{44} + b_{45},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{34} + b_{44} + b_{45},$
 $-b_{35} + b_{44} \geq -b_{13} + b_{44},$
 $-b_{35} + b_{44} \geq -b_{24} - b_{34} + b_{44} + b_{45},$
 $-b_{35} + b_{44} \geq -b_{24} + b_{44},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{14} - b_{15} + b_{22},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{14} + b_{33},$
 $-b_{35} + b_{44} \geq -b_{13} - b_{25} + b_{33},$
 $-b_{35} + b_{44} \geq -b_{24} - b_{25} + b_{33},$
- (E₁₄) $b_{55} \geq -b_{13} - b_{23} + b_{44} + b_{45},$
 $b_{55} \geq -b_{13} - b_{34} + b_{44} + b_{45},$
 $b_{55} \geq -b_{13} + b_{44},$
 $b_{55} \geq -b_{24} - b_{34} + b_{44} + b_{45},$
 $b_{55} \geq -b_{24} + b_{44},$
 $b_{55} \geq -b_{35} + b_{44},$
 $b_{55} \geq -b_{13} - b_{14} - b_{15} + b_{22},$
 $b_{55} \geq -b_{13} - b_{14} - b_{23} - b_{24} + b_{33} + b_{34} + b_{35},$
 $b_{55} \geq -b_{13} - b_{14} - b_{23} + b_{33} + b_{34},$
 $b_{55} \geq -b_{13} - b_{23} - b_{25} + b_{33} + b_{34},$
 $b_{55} \geq -b_{13} - b_{14} + b_{33},$
 $b_{55} \geq -b_{13} - b_{25} + b_{33},$
 $b_{55} \geq -b_{24} - b_{25} + b_{33}.$

Then we define conditions (F_j) ($1 \leq j \leq 14$) by replacing $>$ (resp. \geq) with \geq (resp. $>$) in (E_j) . Let $b = (b_{ij}) \in \mathcal{B}$. Then for $\tilde{e}_k(b) = (b'_{ij})$ where

$$\left. \begin{array}{l}
b'_{11} = b_{11} - 1, b'_{16} = b_{16} + 1, b'_{22} = b_{22} - 1, b'_{27} = b_{27} + 1, b'_{36} = b_{36} - 1, \\
b'_{38} = b_{38} + 1, b'_{47} = b_{47} - 1, b'_{49} = b_{49} + 1, b'_{59} = b_{59} - 1, b'_{5,10} = b_{5,10} + 1, \\
b'_{69} = b_{69} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_1) \\
b'_{11} = b_{11} - 1, b'_{15} = b_{15} + 1, b'_{22} = b_{22} - 1, b'_{26} = b_{26} + 1, b'_{35} = b_{35} - 1, \\
b'_{38} = b_{38} + 1, b'_{46} = b_{46} - 1, b'_{49} = b_{49} + 1, b'_{58} = b_{58} - 1, b'_{5,10} = b_{5,10} + 1, \\
b'_{69} = b_{69} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_2) \\
b'_{11} = b_{11} - 1, b'_{15} = b_{15} + 1, b'_{22} = b_{22} - 1, b'_{24} = b_{24} + 1, b'_{25} = b_{25} - 1, \\
b'_{26} = b_{26} + 1, b'_{34} = b_{34} - 1, b'_{38} = b_{38} + 1, b'_{46} = b_{46} - 1, b'_{47} = b_{47} + 1, \\
b'_{48} = b_{48} - 1, b'_{49} = b_{49} + 1, b'_{57} = b_{57} - 1, b'_{5,10} = b_{5,10} + 1, b'_{69} = b_{69} - 1, \\
b'_{6,11} = b_{6,11} + 1 \text{ if } (E_3) \\
b'_{11} = b_{11} - 1, b'_{14} = b_{14} + 1, b'_{22} = b_{22} - 1, b'_{26} = b_{26} + 1, b'_{34} = b_{34} - 1, \\
b'_{37} = b_{37} + 1, b'_{46} = b_{46} - 1, b'_{49} = b_{49} + 1, b'_{57} = b_{57} - 1, b'_{5,10} = b_{5,10} + 1, \\
b'_{69} = b_{69} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_4) \\
b'_{11} = b_{11} - 1, b'_{15} = b_{15} + 1, b'_{22} = b_{22} - 1, b'_{23} = b_{23} + 1, b'_{25} = b_{25} - 1, \\
b'_{26} = b_{26} + 1, b'_{33} = b_{33} - 1, b'_{38} = b_{38} + 1, b'_{46} = b_{46} - 1, b'_{47} = b_{47} + 1, \\
b'_{48} = b_{48} - 1, b'_{49} = b_{49} + 1, b'_{57} = b_{57} - 1, b'_{58} = b_{58} + 1, b'_{59} = b_{59} - 1, \\
b'_{5,10} = b_{5,10} + 1, b'_{68} = b_{68} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_5) \\
b'_{11} = b_{11} - 1, b'_{14} = b_{14} + 1, b'_{22} = b_{22} - 1, b'_{25} = b_{25} + 1, b'_{34} = b_{34} - 1, \\
b'_{36} = b_{36} + 1, b'_{45} = b_{45} - 1, b'_{49} = b_{49} + 1, b'_{56} = b_{56} - 1, b'_{5,10} = b_{5,10} + 1, \\
b'_{69} = b_{69} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_6) \\
b'_{11} = b_{11} - 1, b'_{14} = b_{14} + 1, b'_{22} = b_{22} - 1, b'_{23} = b_{23} + 1, b'_{24} = b_{24} - 1, \\
b'_{26} = b_{26} + 1, b'_{33} = b_{33} - 1, b'_{37} = b_{37} + 1, b'_{46} = b_{46} - 1, b'_{48} = b_{48} + 1, \\
b'_{57} = b_{57} - 1, b'_{58} = b_{58} + 1, b'_{59} = b_{59} - 1, b'_{5,10} = b_{5,10} + 1, b'_{68} = b_{68} - 1, \\
b'_{6,11} = b_{6,11} + 1 \text{ if } (E_7) \\
b'_{11} = b_{11} - 1, b'_{13} = b_{13} + 1, b'_{22} = b_{22} - 1, b'_{26} = b_{26} + 1, b'_{33} = b_{33} - 1, \\
b'_{37} = b_{37} + 1, b'_{46} = b_{46} - 1, b'_{48} = b_{48} + 1, b'_{57} = b_{57} - 1, b'_{5,10} = b_{5,10} + 1, \\
b'_{68} = b_{68} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_8) \\
b'_{11} = b_{11} - 1, b'_{14} = b_{14} + 1, b'_{22} = b_{22} - 1, b'_{23} = b_{23} + 1, b'_{24} = b_{24} - 1, \\
b'_{25} = b_{25} + 1, b'_{33} = b_{33} - 1, b'_{36} = b_{36} + 1, b'_{45} = b_{45} - 1, b'_{49} = b_{49} + 1, \\
b'_{56} = b_{56} - 1, b'_{58} = b_{58} + 1, b'_{59} = b_{59} - 1, b'_{5,10} = b_{5,10} + 1, b'_{68} = b_{68} - 1, \\
b'_{6,11} = b_{6,11} + 1 \text{ if } (E_9)
\end{array} \right\} k=0$$

$$\begin{aligned}
& \left. \begin{array}{l} b'_{11} = b_{11} - 1, b'_{14} = b_{14} + 1, b'_{22} = b_{22} - 1, b'_{23} = b_{23} + 1, b'_{24} = b_{24} - 1, \\ b'_{25} = b_{25} + 1, b'_{33} = b_{33} - 1, b'_{34} = b_{34} + 1, b'_{35} = b_{35} - 1, b'_{36} = b_{36} + 1, \\ b'_{44} = b_{44} - 1, b'_{49} = b_{49} + 1, b'_{56} = b_{56} - 1, b'_{57} = b_{57} + 1, b'_{59} = b_{59} - 1, \\ b'_{5,10} = b_{5,10} + 1, b'_{67} = b_{67} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_{10}) \end{array} \right\} \\
& k = 0 : \left. \begin{array}{l} b'_{11} = b_{11} - 1, b'_{13} = b_{13} + 1, b'_{22} = b_{22} - 1, b'_{25} = b_{25} + 1, b'_{33} = b_{33} - 1, \\ b'_{34} = b_{34} + 1, b'_{35} = b_{35} - 1, b'_{36} = b_{36} + 1, b'_{44} = b_{44} - 1, b'_{48} = b_{48} + 1, \\ b'_{56} = b_{56} - 1, b'_{57} = b_{57} + 1, b'_{58} = b_{58} - 1, b'_{5,10} = b_{5,10} + 1, b'_{67} = b_{67} - 1, \\ b'_{6,11} = b_{6,11} + 1 \text{ if } (E_{12}) \end{array} \right\} \\
& \left. \begin{array}{l} b'_{11} = b_{11} - 1, b'_{13} = b_{13} + 1, b'_{22} = b_{22} - 1, b'_{24} = b_{24} + 1, b'_{33} = b_{33} - 1, \\ b'_{36} = b_{36} + 1, b'_{44} = b_{44} - 1, b'_{47} = b_{47} + 1, b'_{56} = b_{56} - 1, b'_{5,10} = b_{5,10} + 1, \\ b'_{67} = b_{67} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_{13}) \end{array} \right\} \\
& \left. \begin{array}{l} b'_{11} = b_{11} - 1, b'_{13} = b_{13} + 1, b'_{22} = b_{22} - 1, b'_{24} = b_{24} + 1, b'_{33} = b_{33} - 1, \\ b'_{35} = b_{35} + 1, b'_{44} = b_{44} - 1, b'_{46} = b_{46} + 1, b'_{56} = b_{56} - 1, b'_{5,10} = b_{5,10} + 1, \\ b'_{67} = b_{67} - 1, b'_{6,11} = b_{6,11} + 1 \text{ if } (E_{14}) \end{array} \right\} \\
& k = 1 : b'_{11} = b_{11} + 1, b'_{12} = b_{12} - 1, b'_{6,10} = b_{6,10} + 1, b'_{6,11} = b_{6,11} - 1 \\
& k = 2 : \left. \begin{array}{l} b'_{12} = b_{12} + 1, b'_{13} = b_{13} - 1, b'_{59} = b_{59} + 1, b'_{5,10} = b_{5,10} - 1 \text{ if } b_{12} \geq b_{23} \\ b'_{22} = b_{22} + 1, b'_{23} = b_{23} - 1, b'_{69} = b_{69} + 1, b'_{6,10} = b_{6,10} - 1 \text{ if } b_{12} < b_{23} \end{array} \right\} \\
& \left. \begin{array}{l} b'_{13} = b_{13} + 1, b'_{14} = b_{14} - 1, b'_{48} = b_{48} + 1, b'_{49} = b_{49} - 1 \\ \text{if } b_{13} \geq b_{24}, b_{13} + b_{23} \geq b_{24} + b_{34} \end{array} \right\} \\
& k = 3 : \left. \begin{array}{l} b'_{23} = b_{23} + 1, b'_{24} = b_{24} - 1, b'_{58} = b_{58} + 1, b'_{59} = b_{59} - 1 \\ \text{if } b_{13} < b_{24}, b_{23} \geq b_{34} \end{array} \right\} \\
& \left. \begin{array}{l} b'_{33} = b_{33} + 1, b'_{34} = b_{34} - 1, b'_{68} = b_{68} + 1, b'_{69} = b_{69} - 1 \\ \text{if } b_{13} + b_{23} < b_{24} + b_{34}, b_{23} < b_{34} \end{array} \right\} \\
& k = 4 : \left. \begin{array}{l} b'_{14} = b_{14} + 1, b'_{15} = b_{15} - 1, b'_{37} = b_{37} + 1, b'_{38} = b_{38} - 1 \\ \text{if } b_{14} \geq b_{25}, b_{14} + b_{24} \geq b_{25} + b_{35}, b_{14} + b_{24} + b_{34} \geq b_{25} + b_{35} + b_{45} \\ b'_{24} = b_{24} + 1, b'_{25} = b_{25} - 1, b'_{47} = b_{47} + 1, b'_{48} = b_{48} - 1 \\ \text{if } b_{14} < b_{25}, b_{24} \geq b_{35}, b_{24} + b_{34} \geq b_{35} + b_{45} \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
k = 4 : & \left\{ \begin{array}{l} b'_{34} = b_{34} + 1, b'_{35} = b_{35} - 1, b'_{57} = b_{57} + 1, b'_{58} = b_{58} - 1 \\ \quad \text{if } b_{14} + b_{24} < b_{25} + b_{35}, b_{24} < b_{35}, b_{34} \geq b_{45} \\ b'_{44} = b_{44} + 1, b'_{45} = b_{45} - 1, b'_{67} = b_{67} + 1, b'_{68} = b_{68} - 1 \\ \quad \text{if } b_{14} + b_{24} + b_{34} < b_{25} + b_{35} + b_{45}, b_{24} + b_{34} < b_{35} + b_{45}, b_{34} < b_{45} \end{array} \right. \\
k = 5 : & \left\{ \begin{array}{l} b'_{25} = b_{25} + 1, b'_{26} = b_{26} - 1, b'_{36} = b_{36} + 1, b'_{37} = b_{37} - 1 \\ \quad \text{if } b_{25} + b_{44} + b_{45} \geq b_{33} + b_{34} \\ b'_{45} = b_{45} + 1, b'_{46} = b_{46} - 1, b'_{56} = b_{56} + 1, b'_{57} = b_{57} - 1 \\ \quad \text{if } b_{25} + b_{44} + b_{45} < b_{33} + b_{34} \end{array} \right. \\
k = 6 : & \left\{ \begin{array}{l} b'_{15} = b_{15} + 1, b'_{16} = b_{16} - 1, b'_{26} = b_{26} + 1, b'_{27} = b_{27} - 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + b_{35} \geq b_{22} + b_{23} + b_{24}, \\ \quad \quad b_{15} + b_{33} + b_{34} + 2b_{35} + b_{55} \geq b_{22} + b_{23} + b_{24} + b_{44} \\ b'_{35} = b_{35} + 1, b'_{36} = b_{36} - 1, b'_{46} = b_{46} + 1, b'_{47} = b_{47} - 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + b_{35} < b_{22} + b_{23} + b_{24}, b_{35} + b_{55} \geq b_{44} \\ b'_{55} = b_{55} + 1, b'_{56} = b_{56} - 1, b'_{66} = b_{66} + 1, b'_{67} = b_{67} - 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + 2b_{35} + b_{55} < b_{22} + b_{23} + b_{24} + b_{44}, b_{35} + b_{55} < b_{44} \end{array} \right.
\end{aligned}$$

and $b'_{ij} = b_{ij}$ otherwise.

Also $\tilde{f}_k(b) = (b'_{ij})$ where

$$\begin{aligned}
& \left\{ \begin{array}{l} b'_{11} = b_{11} + 1, b'_{16} = b_{16} - 1, b'_{22} = b_{22} + 1, b'_{27} = b_{27} - 1, b'_{36} = b_{36} + 1, \\ b'_{38} = b_{38} - 1, b'_{47} = b_{47} + 1, b'_{49} = b_{49} - 1, b'_{59} = b_{59} + 1, b'_{5,10} = b_{5,10} - 1, \\ b'_{69} = b_{69} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_1) \end{array} \right. \\
& \left\{ \begin{array}{l} b'_{11} = b_{11} + 1, b'_{15} = b_{15} - 1, b'_{22} = b_{22} + 1, b'_{26} = b_{26} - 1, b'_{35} = b_{35} + 1, \\ b'_{38} = b_{38} - 1, b'_{46} = b_{46} + 1, b'_{49} = b_{49} - 1, b'_{58} = b_{58} + 1, b'_{5,10} = b_{5,10} - 1, \\ b'_{69} = b_{69} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_2) \end{array} \right. \\
k = 0 : & \left\{ \begin{array}{l} b'_{11} = b_{11} + 1, b'_{15} = b_{15} - 1, b'_{22} = b_{22} + 1, b'_{24} = b_{24} - 1, b'_{25} = b_{25} + 1, \\ b'_{26} = b_{26} - 1, b'_{34} = b_{34} + 1, b'_{38} = b_{38} - 1, b'_{46} = b_{46} + 1, b'_{47} = b_{47} - 1, \\ b'_{48} = b_{48} + 1, b'_{49} = b_{49} - 1, b'_{57} = b_{57} + 1, b'_{5,10} = b_{5,10} - 1, b'_{69} = b_{69} + 1, \\ b'_{6,11} = b_{6,11} - 1 \text{ if } (F_3) \end{array} \right. \\
& \left\{ \begin{array}{l} b'_{11} = b_{11} + 1, b'_{14} = b_{14} - 1, b'_{22} = b_{22} + 1, b'_{26} = b_{26} - 1, b'_{34} = b_{34} + 1, \\ b'_{37} = b_{37} - 1, b'_{46} = b_{46} + 1, b'_{49} = b_{49} - 1, b'_{57} = b_{57} + 1, b'_{5,10} = b_{5,10} - 1, \\ b'_{69} = b_{69} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_4) \end{array} \right.
\end{aligned}$$

$$\left\{ \begin{array}{l}
\begin{aligned}
& b'_{11} = b_{11} + 1, b'_{15} = b_{15} - 1, b'_{22} = b_{22} + 1, b'_{23} = b_{23} - 1, b'_{25} = b_{25} + 1, \\
& b'_{26} = b_{26} - 1, b'_{33} = b_{33} + 1, b'_{38} = b_{38} - 1, b'_{46} = b_{46} + 1, b'_{47} = b_{47} - 1, \\
& b'_{48} = b_{48} + 1, b'_{49} = b_{49} - 1, b'_{57} = b_{57} + 1, b'_{58} = b_{58} - 1, b'_{59} = b_{59} + 1, \\
& b'_{5,10} = b_{5,10} - 1, b'_{68} = b_{68} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_5) \\
& b'_{11} = b_{11} + 1, b'_{14} = b_{14} - 1, b'_{22} = b_{22} + 1, b'_{25} = b_{25} - 1, b'_{34} = b_{34} + 1, \\
& b'_{36} = b_{36} - 1, b'_{45} = b_{45} + 1, b'_{49} = b_{49} - 1, b'_{56} = b_{56} + 1, b'_{5,10} = b_{5,10} - 1, \\
& b'_{69} = b_{69} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_6) \\
& b'_{11} = b_{11} + 1, b'_{14} = b_{14} - 1, b'_{22} = b_{22} + 1, b'_{23} = b_{23} - 1, b'_{24} = b_{24} + 1, \\
& b'_{26} = b_{26} - 1, b'_{33} = b_{33} + 1, b'_{37} = b_{37} - 1, b'_{46} = b_{46} + 1, b'_{48} = b_{48} - 1, \\
& b'_{57} = b_{57} + 1, b'_{58} = b_{58} - 1, b'_{59} = b_{59} + 1, b'_{5,10} = b_{5,10} - 1, b'_{68} = b_{68} + 1, \\
& b'_{6,11} = b_{6,11} - 1 \text{ if } (F_7) \\
& b'_{11} = b_{11} + 1, b'_{13} = b_{13} - 1, b'_{22} = b_{22} + 1, b'_{26} = b_{26} - 1, b'_{33} = b_{33} + 1, \\
& b'_{37} = b_{37} - 1, b'_{46} = b_{46} + 1, b'_{48} = b_{48} - 1, b'_{57} = b_{57} + 1, b'_{5,10} = b_{5,10} - 1, \\
& b'_{68} = b_{68} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_8) \\
& b'_{11} = b_{11} + 1, b'_{14} = b_{14} - 1, b'_{22} = b_{22} + 1, b'_{23} = b_{23} - 1, b'_{24} = b_{24} + 1, \\
& b'_{25} = b_{25} - 1, b'_{33} = b_{33} + 1, b'_{36} = b_{36} - 1, b'_{45} = b_{45} + 1, b'_{49} = b_{49} - 1, \\
& b'_{56} = b_{56} + 1, b'_{58} = b_{58} - 1, b'_{59} = b_{59} + 1, b'_{5,10} = b_{5,10} - 1, b'_{68} = b_{68} + 1, \\
k=0 : & b'_{6,11} = b_{6,11} - 1 \text{ if } (F_9) \\
& b'_{11} = b_{11} + 1, b'_{14} = b_{14} - 1, b'_{22} = b_{22} + 1, b'_{23} = b_{23} - 1, b'_{24} = b_{24} + 1, \\
& b'_{25} = b_{25} - 1, b'_{33} = b_{33} + 1, b'_{34} = b_{34} - 1, b'_{35} = b_{35} + 1, b'_{36} = b_{36} - 1, \\
& b'_{44} = b_{44} + 1, b'_{49} = b_{49} - 1, b'_{56} = b_{56} + 1, b'_{57} = b_{57} - 1, b'_{59} = b_{59} + 1, \\
& b'_{5,10} = b_{5,10} - 1, b'_{67} = b_{67} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_{10}) \\
& b'_{11} = b_{11} + 1, b'_{13} = b_{13} - 1, b'_{22} = b_{22} + 1, b'_{25} = b_{25} - 1, b'_{33} = b_{33} + 1, \\
& b'_{36} = b_{36} - 1, b'_{45} = b_{45} + 1, b'_{48} = b_{48} - 1, b'_{56} = b_{56} + 1, b'_{5,10} = b_{5,10} - 1, \\
& b'_{68} = b_{68} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_{11}) \\
& b'_{11} = b_{11} + 1, b'_{13} = b_{13} - 1, b'_{22} = b_{22} + 1, b'_{25} = b_{25} - 1, b'_{33} = b_{33} + 1, \\
& b'_{34} = b_{34} - 1, b'_{35} = b_{35} + 1, b'_{36} = b_{36} - 1, b'_{44} = b_{44} + 1, b'_{48} = b_{48} - 1, \\
& b'_{56} = b_{56} + 1, b'_{57} = b_{57} - 1, b'_{58} = b_{58} + 1, b'_{5,10} = b_{5,10} - 1, b'_{67} = b_{67} + 1, \\
& b'_{6,11} = b_{6,11} - 1 \text{ if } (F_{12}) \\
& b'_{11} = b_{11} + 1, b'_{13} = b_{13} - 1, b'_{22} = b_{22} + 1, b'_{24} = b_{24} - 1, b'_{33} = b_{33} + 1, \\
& b'_{36} = b_{36} - 1, b'_{44} = b_{44} + 1, b'_{47} = b_{47} - 1, b'_{56} = b_{56} + 1, b'_{5,10} = b_{5,10} - 1, \\
& b'_{67} = b_{67} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_{13}) \\
& b'_{11} = b_{11} + 1, b'_{13} = b_{13} - 1, b'_{22} = b_{22} + 1, b'_{24} = b_{24} - 1, b'_{33} = b_{33} + 1, \\
& b'_{35} = b_{35} - 1, b'_{44} = b_{44} + 1, b'_{46} = b_{46} - 1, b'_{56} = b_{56} + 1, b'_{5,10} = b_{5,10} - 1, \\
& b'_{67} = b_{67} + 1, b'_{6,11} = b_{6,11} - 1 \text{ if } (F_{14})
\end{aligned}
\right.$$

$$\begin{aligned}
k = 1 : & b'_{11} = b_{11} - 1, b'_{12} = b_{12} + 1, b'_{6,10} = b_{6,10} - 1, b'_{6,11} = b_{6,11} + 1 \\
k = 2 : & \begin{cases} b'_{12} = b_{12} - 1, b'_{13} = b_{13} + 1, b'_{59} = b_{59} - 1, b'_{5,10} = b_{5,10} + 1 \text{ if } b_{12} > b_{23} \\ b'_{22} = b_{22} - 1, b'_{23} = b_{23} + 1, b'_{69} = b_{69} - 1, b'_{6,10} = b_{6,10} + 1 \text{ if } b_{12} \leq b_{23} \end{cases} \\
k = 3 : & \begin{cases} b'_{13} = b_{13} - 1, b'_{14} = b_{14} + 1, b'_{48} = b_{48} - 1, b'_{49} = b_{49} + 1 \\ \quad \text{if } b_{13} > b_{24}, b_{13} + b_{23} > b_{24} + b_{34} \\ b'_{23} = b_{23} - 1, b'_{24} = b_{24} + 1, b'_{58} = b_{58} - 1, b'_{59} = b_{59} + 1 \\ \quad \text{if } b_{13} \leq b_{24}, b_{23} > b_{34} \\ b'_{33} = b_{33} - 1, b'_{34} = b_{34} + 1, b'_{68} = b_{68} - 1, b'_{69} = b_{69} + 1 \\ \quad \text{if } b_{13} + b_{23} \leq b_{24} + b_{34}, b_{23} \leq b_{34} \end{cases} \\
k = 4 : & \begin{cases} b'_{14} = b_{14} - 1, b'_{15} = b_{15} + 1, b'_{37} = b_{37} - 1, b'_{38} = b_{38} + 1 \\ \quad \text{if } b_{14} > b_{25}, b_{14} + b_{24} > b_{25} + b_{35}, b_{14} + b_{24} + b_{34} > b_{25} + b_{35} + b_{45} \\ b'_{24} = b_{24} - 1, b'_{25} = b_{25} + 1, b'_{47} = b_{47} - 1, b'_{48} = b_{48} + 1 \\ \quad \text{if } b_{14} \leq b_{25}, b_{24} > b_{35}, b_{24} + b_{34} > b_{35} + b_{45} \\ b'_{34} = b_{34} - 1, b'_{35} = b_{35} + 1, b'_{57} = b_{57} - 1, b'_{58} = b_{58} + 1 \\ \quad \text{if } b_{14} + b_{24} \leq b_{25} + b_{35}, b_{24} \leq b_{35}, b_{34} > b_{45} \\ b'_{44} = b_{44} - 1, b'_{45} = b_{45} + 1, b'_{67} = b_{67} - 1, b'_{68} = b_{68} + 1 \\ \quad \text{if } b_{14} + b_{24} + b_{34} \leq b_{25} + b_{35} + b_{45}, b_{24} + b_{34} \leq b_{35} + b_{45}, b_{34} \leq b_{45} \end{cases} \\
k = 5 : & \begin{cases} b'_{25} = b_{25} - 1, b'_{26} = b_{26} + 1, b'_{36} = b_{36} - 1, b'_{37} = b_{37} + 1 \\ \quad \text{if } b_{25} + b_{44} + b_{45} > b_{33} + b_{34} \\ b'_{45} = b_{45} - 1, b'_{46} = b_{46} + 1, b'_{56} = b_{56} - 1, b'_{57} = b_{57} + 1 \\ \quad \text{if } b_{25} + b_{44} + b_{45} \leq b_{33} + b_{34} \end{cases} \\
k = 6 : & \begin{cases} b'_{15} = b_{15} - 1, b'_{16} = b_{16} + 1, b'_{26} = b_{26} - 1, b'_{27} = b_{27} + 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + b_{35} > b_{22} + b_{23} + b_{24}, \\ \quad \quad b_{15} + b_{33} + b_{34} + 2b_{35} + b_{55} > b_{22} + b_{23} + b_{24} + b_{44} \\ b'_{35} = b_{35} - 1, b'_{36} = b_{36} + 1, b'_{46} = b_{46} - 1, b'_{47} = b_{47} + 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + b_{35} \leq b_{22} + b_{23} + b_{24}, b_{35} + b_{55} > b_{44} \\ b'_{55} = b_{55} - 1, b'_{56} = b_{56} + 1, b'_{66} = b_{66} - 1, b'_{67} = b_{67} + 1 \\ \quad \text{if } b_{15} + b_{33} + b_{34} + 2b_{35} + b_{55} \leq b_{22} + b_{23} + b_{24} + b_{44}, b_{35} + b_{55} \leq b_{44} \end{cases}
\end{aligned}$$

and $b'_{ij} = b_{ij}$ otherwise.

For $b \in B^{6,l}$ if $\tilde{e}_k(b)$ or $\tilde{f}_k(b)$ does not belong to $B^{6,l}$, then we assume it to be 0. The maps $\varepsilon_k(b)$, $\varphi_k(b)$ and $\text{wt}_k(b)$ for $k = 0, 1, 2, 3, 4, 5, 6$ are given as follows. We observe that $\text{wt}_k(b) = \varphi_k(b) - \varepsilon_k(b)$, $\varphi(b) = \sum_{k=0}^6 \varphi_k(b) \Lambda_k$, $\varepsilon(b) = \sum_{k=0}^6 \varepsilon_k(b) \Lambda_k$ and $\text{wt}(b) = \varphi(b) - \varepsilon(b)$.

$$\varepsilon_0(b) = \begin{cases} l + \max\{-b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, -b_{13} - b_{23} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{13} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{24} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{24} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{35} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{14} - b_{15} - b_{23} - b_{24} - b_{25} \\ -b_{26} - b_{27}, -b_{13} - b_{14} - b_{23} - b_{24} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{14} - b_{23} \\ -b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{23} - b_{25} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} \\ -b_{14} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} \\ -b_{38}, -b_{24} - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}\} \text{ if } b \in B^{5,l}, \\ \max\{-b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, -b_{13} - b_{23} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{13} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{24} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{24} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \\ -b_{35} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{14} - b_{15} - b_{23} - b_{24} - b_{25} \\ -b_{26} - b_{27}, -b_{13} - b_{14} - b_{23} - b_{24} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{14} - b_{23} \\ -b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{23} - b_{25} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} \\ -b_{14} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} \\ -b_{38}, -b_{24} - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}\} \text{ if } b \in B^{5,\infty}, \end{cases}$$

$$\varepsilon_1(b) = b_{12},$$

$$\varepsilon_2(b) = \max\{b_{13}, -b_{12} + b_{13} + b_{23}\},$$

$$\varepsilon_3(b) = \max\{b_{14}, -b_{13} + b_{14} + b_{24}, -b_{13} + b_{14} - b_{23} + b_{24} + b_{34}\},$$

$$\varepsilon_4(b) = \max\{b_{15}, -b_{14} + b_{15} + b_{25}, -b_{14} + b_{15} - b_{24} + b_{25} + b_{35},$$

$$-b_{14} + b_{15} - b_{24} + b_{25} - b_{34} + b_{35} + b_{45}\},$$

$$\varepsilon_5(b) = \max\{b_{11} + b_{12} + b_{13} + b_{14} - b_{22} - b_{23} - b_{24} - b_{25},$$

$$b_{11} + b_{12} + b_{13} + b_{14} - b_{22} - b_{23} - b_{24} - 2b_{25} + b_{33} + b_{34} - b_{44} - b_{45}\},$$

$$\varepsilon_6(b) = \begin{cases} l + \max\{-b_{11} - b_{12} - b_{13} - b_{14} - b_{15}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} \\ + b_{22} + b_{23} + b_{24} - b_{33} - b_{34} - b_{35}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} \\ + b_{22} + b_{23} + b_{24} - b_{33} - b_{34} - 2b_{35} + b_{44} - b_{55}\} \text{ if } b \in B^{6,l}, \end{cases}$$

$$\begin{aligned} \varepsilon_6(b) &= \left\{ \begin{array}{l} \max\{-b_{11} - b_{12} - b_{13} - b_{14} - b_{15}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} \\ + b_{22} + b_{23} + b_{24} - b_{33} - b_{34} - b_{35}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} \\ + b_{22} + b_{23} + b_{24} - b_{33} - b_{34} - 2b_{35} + b_{44} - b_{55}\} \text{ if } b \in B^{6,\infty}, \end{array} \right. \\ &\quad \left. l + \max\{-b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{24} + b_{25} + b_{26} + b_{27} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} + b_{23} + b_{25} + b_{26} + b_{27} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} + b_{23} + b_{25} + b_{26} + b_{27} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{35} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} - b_{14} - b_{15}, -b_{11} - b_{12} - b_{13} - b_{14} + b_{25} + b_{26} + b_{27} - b_{36} - b_{37} \right. \\ &\quad \left. -b_{38}, -b_{11} - b_{12} - b_{13} - b_{14} + b_{24} + b_{25} + b_{26} + b_{27} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{24} + b_{26} + b_{27} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{11} - b_{12} - b_{13} \right. \\ &\quad \left. -b_{14} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{26} + b_{27} b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} + b_{23} + b_{26} + b_{27} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}\} \text{ if } b \in B^{6,l}, \right. \\ \varphi_0(b) &= \left\{ \begin{array}{l} \max\{-b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, \right. \\ \left. -b_{11} - b_{12} - b_{13} + b_{24} + b_{25} + b_{26} + b_{27} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} + b_{23} + b_{25} + b_{26} + b_{27} - b_{34} - b_{46} - b_{47} b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} + b_{23} + b_{25} + b_{26} + b_{27} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{35} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, \right. \\ \left. -b_{11} - b_{12} - b_{13} - b_{14} - b_{15}, -b_{11} - b_{12} - b_{13} - b_{14} + b_{25} + b_{26} + b_{27} - b_{36} - b_{37} \right. \\ &\quad \left. -b_{38}, -b_{11} - b_{12} - b_{13} - b_{14} + b_{24} + b_{25} + b_{26} + b_{27} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{24} + b_{26} + b_{27} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{11} - b_{12} - b_{13} \right. \\ &\quad \left. -b_{14} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} - b_{13} + b_{23} + b_{24} + b_{26} + b_{27} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, \right. \\ &\quad \left. -b_{11} - b_{12} + b_{23} + b_{26} + b_{27} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}\} \text{ if } b \in B^{6,\infty}, \right. \\ \varphi_1(b) &= b_{11} - b_{22}, \end{array} \right. \end{aligned}$$

$$\begin{aligned}\varphi_2(b) &= \max\{b_{22} - b_{33}, b_{12} + b_{22} - b_{23} - b_{33}\}, \\ \varphi_3(b) &= \max\{b_{33} - b_{44}, b_{23} + b_{33} - b_{34} - b_{44}, b_{13} + b_{23} - b_{24} + b_{33} - b_{34} - b_{44}\}, \\ \varphi_4(b) &= \max\{b_{44} - b_{55}, b_{34} + b_{44} - b_{45} - b_{55}, b_{24} + b_{34} - b_{35} + b_{44} - b_{45} - b_{55}, \\ &\quad b_{14} + b_{24} - b_{25} + b_{34} - b_{35} + b_{44} - b_{45} - b_{55}\}, \\ \varphi_5(b) &= \max\{b_{45}, b_{25} - b_{33} - b_{34} + b_{44} + 2b_{45}\},\end{aligned}$$

$$\varphi_6(b) = \begin{cases} l + \max\{-b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, b_{35} - b_{44} + b_{55} - b_{56} - b_{57} \\ \quad - b_{58} - b_{59} - b_{5,10}, b_{15} - b_{22} - b_{23} - b_{24} + b_{33} + b_{34} + 2b_{35} - b_{44} \\ \quad + b_{55} - b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}\} \text{ if } b \in B^{6,l}, \\ \max\{-b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, b_{35} - b_{44} + b_{55} - b_{56} - b_{57} \\ \quad - b_{58} - b_{59} - b_{5,10}, b_{15} - b_{22} - b_{23} - b_{24} + b_{33} + b_{34} + 2b_{35} - b_{44} \\ \quad + b_{55} - b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}\} \text{ if } b \in B^{6,\infty}, \end{cases}$$

$$\text{wt}_0(b) = -b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{26} + b_{27},$$

$$\text{wt}_1(b) = b_{11} - b_{12} - b_{22},$$

$$\text{wt}_2(b) = b_{12} - b_{13} + b_{22} - b_{23} - b_{33},$$

$$\text{wt}_3(b) = b_{13} - b_{14} + b_{23} - b_{24} + b_{33} - b_{34} - b_{44},$$

$$\text{wt}_4(b) = b_{14} - b_{15} + b_{24} - b_{25} + b_{34} - b_{35} + b_{44} - b_{45} - b_{55},$$

$$\text{wt}_5(b) = -b_{11} - b_{12} - b_{13} - b_{14} + b_{22} + b_{23} + b_{24} + 2b_{25} - b_{33} - b_{34} + b_{44} + 2b_{45},$$

$$\begin{aligned}\text{wt}_6(b) &= b_{11} + b_{12} + b_{13} + b_{14} + 2b_{15} - b_{22} - b_{23} - b_{24} + b_{33} + b_{34} + 2b_{35} - b_{44} + b_{55} \\ &\quad - b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}.\end{aligned}$$

Choose elements $b_0^0, b_1^0, b_2^0, b_3^0, b_4^0, b_5^0, b_6^0$ where

$$\begin{aligned}(b_0^0)_{ij} &= 1 && \text{if } (i, j) = (1, 6), (2, 7), (3, 8), (4, 9), (5, 10), (6, 11), \\ (b_1^0)_{ij} &= 1 && \text{if } (i, j) = (1, 1), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10), \\ (b_2^0)_{ij} &= 1 && \text{if } (i, j) = (1, 1), (1, 5), (2, 2), (2, 6), (3, 6), (3, 8), (4, 7), (4, 9), (5, 8), (5, 10), \\ &&& (6, 9), (6, 11), \\ (b_3^0)_{ij} &= 1 && \text{if } (i, j) = (1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (3, 6), (4, 6), (4, 9), (5, 7), (5, 10), \\ &&& (6, 8), (6, 11), \\ (b_4^0)_{ij} &= 1 && \text{if } (i, j) = (1, 1), (1, 3), (2, 2), (2, 4), (3, 3), (3, 5), (4, 4), (4, 6), (5, 6), (5, 10), \\ &&& (6, 7), (6, 11), \\ (b_5^0)_{ij} &= 1 && \text{if } (i, j) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 11), \\ (b_6^0)_{ij} &= 1 && \text{if } (i, j) = (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6),\end{aligned}$$

and $(b_{\mathbf{k}}^0)_{ij} = 0$ otherwise, for $0 \leq k \leq 6$.

As shown in [21], the crystal $B^{6,l}$ is a perfect crystal with the set of minimal elements:

$$\begin{aligned} (B^{6,l})_{\min} &= \{b \in B^{6,l} \mid \langle \mathbf{c}, \varepsilon(b) \rangle = l\} \\ &= \left\{ \sum_{k=0}^6 a_k b_{\mathbf{k}}^0 \mid a_k \in \mathbb{Z}_{\geq 0}, a_0 + a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 + a_6 = l \right\}. \end{aligned}$$

For $\lambda \in P_{cl}$, consider the crystal $T_\lambda = \{t_\lambda\}$ with

$$\begin{aligned} \tilde{e}_k(t_\lambda) &= \tilde{f}_k(t_\lambda) = 0, & \varepsilon_k(t_\lambda) &= \varphi_k(t_\lambda) = -\infty, \\ \text{wt}(t_\lambda) &= \lambda, \end{aligned}$$

for $k = 0, 1, 2, 3, 4, 5, 6$. Then for $\lambda, \mu \in P_{cl}$, $T_\lambda \otimes B^{6,l} \otimes T_\mu$ is a crystal with the structure given by

$$\begin{aligned} \tilde{e}_k(t_\lambda \otimes b \otimes t_\mu) &= t_\lambda \otimes \tilde{e}_k b \otimes t_\mu, & \tilde{f}_k(t_\lambda \otimes b \otimes t_\mu) &= t_\lambda \otimes \tilde{f}_k b \otimes t_\mu, \\ \varepsilon_k(t_\lambda \otimes b \otimes t_\mu) &= \varepsilon_k(b) - \langle \check{\alpha}_k, \lambda \rangle, & \varphi_k(t_\lambda \otimes b \otimes t_\mu) &= \varphi_k(b) + \langle \check{\alpha}_k, \mu \rangle, \\ \text{wt}(t_\lambda \otimes b \otimes t_\mu) &= \lambda + \mu + \text{wt}(b) \end{aligned}$$

where $t_\lambda \otimes b \otimes t_\mu \in T_\lambda \otimes B^{6,l} \otimes T_\mu$.

The notion of a coherent family of perfect crystals and its limit is defined in [19]. In the following theorem we prove that the family of $D_6^{(1)}$ crystals $\{B^{6,l}\}_{l \geq 1}$ form a coherent family with limit $B^{6,\infty}$ containing the special vector $b^\infty = \mathbf{0}$ (i.e. $(b^\infty)_{ij} = 0$ for $i \leq j \leq i+5$, $1 \leq i \leq 6$).

Theorem 7.0.1. *The family of the perfect crystal $\{B^{6,l}\}_{l \geq 1}$ forms a coherent family and the crystal $B^{6,\infty}$ is its limit with the vector b^∞ .*

Proof. Set $J = \{(l, b) \mid l \in \mathbb{Z}_{>0}, b \in (B^{6,l})_{\min}\}$. By ([19], Definition 4.1), we need to show that

1. $\text{wt}(b^\infty) = 0, \varepsilon(b^\infty) = \varphi(b^\infty) = 0$,
2. for any $(l, b) \in J$, there exists an embedding of crystals

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes B^{6,l} \otimes T_{-\varphi(b)} \longrightarrow B^{6,\infty}$$

where $f_{(l,b)}(t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}) = b^\infty$,

3. $B^{6,\infty} = \cup_{(l,b) \in J} \text{Im } f_{(l,b)}$.

Since $\varepsilon_k(b^\infty) = 0, \varphi_k(b^\infty) = 0, 0 \leq k \leq 6$, we have $\varepsilon(b^\infty) = 0, \varphi(b^\infty) = 0$ and hence $\text{wt}(b^\infty) = 0$ which proves (1).

Let $l \in \mathbb{Z}_{>0}$ and $b^0 = (b_{ij}^0)$ be an element of $(B^{6,l})_{\min}$. Then there exist $a_k \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq 6$ such that $a_0 + a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 + a_6 = l$ and

$$\begin{aligned} b_{11}^0 &= a_1 + a_2 + a_3 + a_4 + a_6, & b_{12}^0 &= a_5, & b_{13}^0 &= a_4, & b_{14}^0 &= a_3, & b_{15}^0 &= a_2, & b_{16}^0 &= a_0, \\ b_{22}^0 &= a_2 + a_3 + a_4 + a_6, & b_{23}^0 &= a_5, & b_{24}^0 &= a_4, & b_{25}^0 &= a_3, & b_{26}^0 &= a_1 + a_2, & b_{27}^0 &= a_0, \\ b_{33}^0 &= a_3 + a_4 + a_6, & b_{34}^0 &= a_5, & b_{35}^0 &= a_4, & b_{36}^0 &= a_2 + a_3, & b_{37}^0 &= a_1, & b_{38}^0 &= a_0 + a_2, \\ b_{44}^0 &= a_4 + a_6, & b_{45}^0 &= a_5, & b_{46}^0 &= a_3 + a_4, & b_{47}^0 &= a_2, & b_{48}^0 &= a_1, & b_{49}^0 &= a_0 + a_2 + a_3, \\ b_{55}^0 &= a_6, & b_{56}^0 &= a_4 + a_5, & b_{57}^0 &= a_3, & b_{58}^0 &= a_2, & b_{59}^0 &= a_1, & b_{5,10}^0 &= a_0 + a_2 + a_3 + a_4, \\ b_{66}^0 &= a_6, & b_{67}^0 &= a_4, & b_{68}^0 &= a_3, & b_{69}^0 &= a_2, & b_{6,10}^0 &= a_1, & b_{6,11}^0 &= a_0 + a_2 + a_3 + a_4 + a_5, \text{ and} \\ \varepsilon(b^0) &= a_6\Lambda_0 + a_5\Lambda_1 + a_4\Lambda_2 + a_3\Lambda_3 + a_2\Lambda_4 + a_1\Lambda_5 + a_0\Lambda_6, \\ \varphi(b^0) &= a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3 + a_4\Lambda_4 + a_5\Lambda_5 + a_6\Lambda_6. \end{aligned}$$

For any $b = (b_{ij}) \in B^{6,l}$, we define a map

$$f_{(l,b^0)} : T_{\varepsilon(b^0)} \otimes B^{6,l} \otimes T_{-\varphi(b^0)} \longrightarrow B^{6,\infty}$$

by $f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}) = b' = (b'_{ij})$ where $b'_{ij} = b_{ij} - b_{ij}^0$ for all $i \leq j \leq i+5, 1 \leq i \leq 6$. Then it is easy to see that

$$\begin{aligned} \varepsilon_k(b') &= \varepsilon_k(b) - a_{6-k} = \varepsilon_k(b) - \langle \check{\alpha}_k, \varepsilon(b^0) \rangle \text{ for } 0 \leq k \leq 6, \\ \varphi_k(b') &= \varphi_k(b) - a_k = \varphi_k(b) + \langle \check{\alpha}_k, -\varphi(b^0) \rangle \text{ for } 0 \leq k \leq 6. \end{aligned}$$

Hence we have

$$\begin{aligned} \varepsilon_k(b') &= \varepsilon_k(b) - \langle \check{\alpha}_k, \varepsilon(b^0) \rangle = \varepsilon_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}), \\ \varphi_k(b') &= \varphi_k(b) + \langle \check{\alpha}_k, -\varphi(b^0) \rangle = \varphi_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}), \\ \text{wt}(b') &= \sum_{k=0}^6 (\varphi_k(b') - \varepsilon_k(b')) \Lambda_k = \text{wt}(b) + \sum_{k=0}^6 \langle \check{\alpha}_k, -\varphi(b^0) \rangle \Lambda_k + \sum_{k=0}^6 \langle \check{\alpha}_k, \varepsilon(b^0) \rangle \Lambda_k \\ &= \text{wt}(b) - \varphi(b^0) + \varepsilon(b^0) = \text{wt}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}). \end{aligned}$$

For $0 \leq k \leq 6, b \in B^{6,l}$, it can be checked easily that the conditions for the action of $\tilde{\varepsilon}_k$ on $b' = b - b^0$ hold if and only if the conditions for the action of $\tilde{\varepsilon}_k$ on b hold. Hence from the

defined action of \tilde{e}_k , we see that $\tilde{e}_k(b') = \tilde{e}_k(b) - b^0, 0 \leq k \leq 6$. This implies that

$$\begin{aligned} f_{(l,b^0)}(\tilde{e}_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})) &= f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes \tilde{e}_k(b) \otimes t_{-\varphi(b^0)}) \\ &= \tilde{e}_k(b) - b^0 = \tilde{e}_k(b') = \tilde{e}_k(f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})). \end{aligned}$$

Similarly, we have $f_{(l,b^0)}(\tilde{f}_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})) = \tilde{f}_k(f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}))$. Clearly the map $f_{(l,b^0)}$ is injective with $f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b^0 \otimes t_{-\varphi(b^0)}) = b^\infty$. This proves (2).

We observe that $\sum_{j=i}^{i+5} b'_{ij} = \sum_{j=i}^{i+5} b_{ij} - \sum_{j=i}^{i+5} b_{ij}^0 = l - l = 0$ for all $1 \leq i \leq 6$. Also,

$$\begin{aligned} b'_{11} &= b_{11} - b_{11}^0 \\ &= b_{66} + b_{67} + b_{68} + b_{69} + b_{6,10} - a_1 - a_2 - a_3 - a_4 - a_6 \\ &= b'_{66} + b'_{67} + b'_{68} + b'_{69} + b'_{6,10}, \\ b'_{11} + b'_{12} &= b_{11} - b_{11}^0 + b_{12} - b_{12}^0 \\ &= b_{55} + b_{56} + b_{57} + b_{58} + b_{59} - a_1 - a_2 - a_3 - a_4 - a_5 - a_6 \\ &= b'_{55} + b'_{56} + b'_{57} + b'_{58} + b'_{59}, \\ b'_{11} + b'_{12} + b'_{13} &= b_{11} - b_{11}^0 + b_{12} - b_{12}^0 + b_{13} - b_{13}^0 \\ &= b_{44} + b_{45} + b_{46} + b_{47} + b_{48} - a_1 - a_2 - a_3 - 2a_4 - a_5 - a_6 \\ &= b'_{44} + b'_{45} + b'_{46} + b'_{47} + b'_{48}, \\ b'_{11} + b'_{12} + b'_{13} + b'_{14} &= b_{11} - b_{11}^0 + b_{12} - b_{12}^0 + b_{13} - b_{13}^0 + b_{14} - b_{14}^0 \\ &= b_{33} + b_{34} + b_{35} + b_{36} + b_{37} - a_1 - a_2 - 2a_3 - 2a_4 - a_5 - a_6 \\ &= b'_{33} + b'_{34} + b'_{35} + b'_{36} + b'_{37}, \\ b'_{11} + b'_{12} + b'_{13} + b'_{14} + b'_{15} &= b_{11} - b_{11}^0 + b_{12} - b_{12}^0 + b_{13} - b_{13}^0 + b_{14} - b_{14}^0 + b_{15} - b_{15}^0 \\ &= b_{22} + b_{23} + b_{24} + b_{25} + b_{26} - a_1 - 2a_2 - 2a_3 - 2a_4 - a_5 - a_6 \\ &= b'_{22} + b'_{23} + b'_{24} + b'_{25} + b'_{26}, \\ b'_{22} &= b_{22} - b_{22}^0 \\ &= b_{66} + b_{67} + b_{68} + b_{69} - a_2 - a_3 - a_4 - a_6 \\ &= b'_{66} + b'_{67} + b'_{68} + b'_{69}, \\ b'_{22} + b'_{23} &= b_{22} - b_{22}^0 + b_{23} - b_{23}^0 \\ &= b_{55} + b_{56} + b_{57} + b_{58} - a_2 - a_3 - a_4 - a_5 - a_6 \\ &= b'_{55} + b'_{56} + b'_{57} + b'_{58}, \\ b'_{22} + b'_{23} + b'_{24} &= b_{22} - b_{22}^0 + b_{23} - b_{23}^0 + b_{24} - b_{24}^0 \\ &= b_{44} + b_{45} + b_{46} + b_{47} - a_2 - a_3 - 2a_4 - a_5 - a_6 \end{aligned}$$

$$\begin{aligned}
&= b'_{44} + b'_{45} + b'_{46} + b'_{47}, \\
b'_{22} + b'_{23} + b'_{24} + b'_{25} &= b_{22} - b^0_{22} + b_{23} - b^0_{23} + b_{24} - b^0_{24} + b_{25} - b^0_{25} \\
&= b_{33} + b_{34} + b_{35} + b_{36} - a_2 - 2a_3 - 2a_4 - a_5 - a_6 \\
&= b'_{33} + b'_{34} + b'_{35} + b'_{36}, \\
b'_{33} &= b_{33} - b^0_{33} \\
&= b_{66} + b_{67} + b_{68} - a_3 - a_4 - a_6 = b'_{66} + b'_{67} + b'_{68}, \\
b'_{33} + b'_{34} &= b_{33} - b^0_{33} + b_{34} - b^0_{34} \\
&= b_{55} + b_{56} + b_{57} - a_3 - a_4 - a_5 - a_6 = b'_{55} + b'_{56} + b'_{57}, \\
b'_{33} + b'_{34} + b'_{35} &= b_{33} - b^0_{33} + b_{34} - b^0_{34} + b_{35} - b^0_{35} \\
&= b_{44} + b_{45} + b_{46} - a_3 - 2a_4 - a_5 - a_6 = b'_{44} + b'_{45} + b'_{46}, \\
b'_{44} &= b_{44} - b^0_{44} = b_{66} + b_{67} - a_4 - a_6 = b'_{66} + b'_{67}, \\
b'_{44} + b'_{45} &= b_{44} - b^0_{44} + b_{45} - b^0_{45} = b_{55} + b_{56} - a_4 - a_5 - a_6 = b'_{55} + b'_{56}, \\
b'_{55} &= b_{55} - b^0_{55} = b_{66} - a_6 = b'_{66}.
\end{aligned}$$

Hence we have $B^{6,\infty} \supseteq \cup_{(l,b) \in J} \text{Im } f_{(l,b)}$. To prove (3) we also need to show that $B^{6,\infty} \subseteq \cup_{(l,b) \in J} \text{Im } f_{(l,b)}$. Let $b' = (b'_{ij}) \in B^{6,\infty}$. By (2), we can assume that $b' \neq b^\infty$. Set

$$\begin{aligned}
a_1 &= \max\{-b'_{11} + b'_{22}, -b'_{11} - b'_{12} + b'_{22} + b'_{23}, -b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24}, \\
&\quad -b'_{11} - b'_{12} - b'_{13} - b'_{14} + b'_{22} + b'_{23} + b'_{24} + b'_{25}, 0\}, \\
a_2 &= \max\{-b'_{22} + b'_{33}, -b'_{22} - b'_{23} + b'_{33} + b'_{34}, -b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34} + b'_{35}, \\
&\quad -b'_{15}, -b'_{26} - a_1, 0\}, \\
a_3 &= \max\{-b'_{33} + b'_{44}, -b'_{33} - b'_{34} + b'_{44} + b'_{45}, -b'_{14}, -b'_{25}, -b'_{36} - a_2, 0\}, \\
a_4 &= \max\{-b'_{44} + b'_{55}, -b'_{13}, -b'_{24}, -b'_{35}, -b'_{46} - a_3, 0\}, \\
a_5 &= \max\{-b'_{12}, -b'_{23}, -b'_{34}, -b'_{45}, -b'_{56} - a_4, 0\}, \\
a_6 &= \max\{-b'_{11} - a_1 - a_2 - a_3 - a_4, -b'_{22} - a_2 - a_3 - a_4, -b'_{33} - a_3 - a_4, -b'_{44} - a_4, -b'_{55}, 0\}, \\
a_0 &= \max\{b'_{11} - a_2 - a_3 - a_4 - a_5, b'_{11} + b'_{12} - a_2 - a_3 - a_4, b'_{11} + b'_{12} + b'_{13} - a_2 - a_3, \\
&\quad b'_{11} + b'_{12} + b'_{13} + b'_{14} - a_2, b'_{11} + b'_{12} + b'_{13} + b'_{14} + b'_{15}, 0\}.
\end{aligned}$$

Let $l = a_0 + a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 + a_6$. Let $b^0 = (b^0_{ij})$ where

$$\begin{aligned}
b^0_{11} &= a_1 + a_2 + a_3 + a_4 + a_6, b^0_{12} = a_5, b^0_{13} = a_4, b^0_{14} = a_3, b^0_{15} = a_2, b^0_{16} = a_0, \\
b^0_{22} &= a_2 + a_3 + a_4 + a_6, b^0_{23} = a_5, b^0_{24} = a_4, a^0_{25} = a_3, b^0_{26} = a_1 + a_2, b^0_{27} = a_0, \\
b^0_{33} &= a_3 + a_4 + a_6, b^0_{34} = a_5, b^0_{35} = a_4, b^0_{36} = a_2 + a_3, b^0_{37} = a_1, b^0_{38} = a_0 + a_2,
\end{aligned}$$

$$\begin{aligned}
b_{44}^0 &= a_4 + a_6, \quad b_{45}^0 = a_5, \quad b_{46}^0 = a_3 + a_4, \quad b_{47}^0 = a_2, \quad b_{48}^0 = a_1, \quad b_{49}^0 = a_0 + a_2 + a_3, \\
b_{55}^0 &= a_6, \quad b_{56}^0 = a_4 + a_5, \quad b_{57}^0 = a_3, \quad b_{58}^0 = a_2, \quad b_{5,9}^0 = a_1, \quad b_{5,10}^0 = a_0 + a_2 + a_3 + a_4, \\
b_{66}^0 &= a_6, \quad b_{67}^0 = a_4, \quad b_{68}^0 = a_3, \quad b_{69}^0 = a_2, \quad b_{6,10}^0 = a_1, \quad b_{6,11}^0 = a_0 + a_2 + a_3 + a_4 + a_5.
\end{aligned}$$

Then $\varepsilon(b^0) = a_6\Lambda_0 + a_5\Lambda_1 + a_4\Lambda_2 + a_3\Lambda_3 + a_2\Lambda_4 + a_1\Lambda_5 + a_0\Lambda_6$ and $\varphi(b^0) = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3 + a_4\Lambda_4 + a_5\Lambda_5 + a_6\Lambda_6$. It is easy to see that $b^0 \in (B^{6,l})_{\min}$.

Set $b = (b_{ij})$ where $b_{ij} = b'_{ij} + b_{ij}^0$. Then $\sum_{j=i}^{i+5} b_{ij} = \sum_{j=i}^{i+5} b'_{ij} + \sum_{j=i}^{i+5} b_{ij}^0 = 0 + l = l$, $1 \leq i \leq 6$ and we observe that

$$\begin{aligned}
b_{11} &= b'_{11} + b_{11}^0 = b'_{11} + a_1 + a_2 + a_3 + a_4 + a_6 \geq 0, \text{ since } a_6 \geq -b'_{11} - a_1 - a_2 - a_3 - a_4, \\
b_{12} &= b'_{12} + b_{12}^0 = b'_{12} + a_5 \geq 0, \text{ since } a_5 \geq -b'_{12}, \\
b_{13} &= b'_{13} + b_{13}^0 = b'_{13} + a_4 \geq 0, \text{ since } a_4 \geq -b'_{13}, \\
b_{14} &= b'_{14} + b_{14}^0 = b'_{14} + a_3 \geq 0, \text{ since } a_3 \geq -b'_{14}, \\
b_{15} &= b'_{15} + b_{15}^0 = b'_{14} + a_2 \geq 0, \text{ since } a_2 \geq -b'_{15}, \\
b_{16} &= b'_{16} + b_{16}^0 = b'_{15} + a_0 = -b'_{11} - b'_{12} - b'_{13} - b'_{14} - b'_{15} + a_0 \geq 0, \\
&\quad \text{since } a_0 \geq b'_{11} + b'_{12} + b'_{13} + b'_{14} + b'_{15}, \\
b_{22} &= b'_{22} + b_{22}^0 = b'_{22} + a_2 + a_3 + a_4 + a_6 \geq 0, \text{ since } a_6 \geq -b'_{22} - a_2 - a_3 - a_4, \\
b_{23} &= b'_{23} + b_{23}^0 = b'_{23} + a_5 \geq 0, \text{ since } a_5 \geq -b'_{23}, \\
b_{24} &= b'_{24} + b_{24}^0 = b'_{24} + a_4 \geq 0, \text{ since } a_4 \geq -b'_{24}, \\
b_{25} &= b'_{25} + b_{25}^0 = b'_{25} + a_3 \geq 0, \text{ since } a_3 \geq -b'_{25}, \\
b_{26} &= b'_{26} + b_{26}^0 = b'_{26} + a_1 + a_2 \geq 0, \text{ since } a_2 \geq -b'_{26} - a_1, \\
b_{27} &= b'_{27} + b_{27}^0 = b'_{27} + a_0 = -b'_{22} - b'_{23} - b'_{24} - b'_{25} - b'_{26} + a_0 \geq 0, \\
&\quad \text{since } a_0 \geq b'_{11} + b'_{12} + b'_{13} + b'_{14} + b'_{15} = b'_{22} + b'_{23} + b'_{24} + b'_{25} + b'_{26}, \\
b_{33} &= b'_{33} + b_{33}^0 = b'_{33} + a_3 + a_4 + a_6 \geq 0, \text{ since } a_6 \geq -b'_{33} - a_3 - a_4, \\
b_{34} &= b'_{34} + b_{34}^0 = b'_{34} + a_5 \geq 0, \text{ since } a_5 \geq -b'_{34}, \\
b_{35} &= b'_{35} + b_{35}^0 = b'_{35} + a_4 \geq 0, \text{ since } a_4 \geq -b'_{35}, \\
b_{36} &= b'_{36} + b_{36}^0 = b'_{36} + a_2 + a_3 \geq 0, \text{ since } a_3 \geq -b'_{36} - a_2, \\
b_{37} &= b'_{37} + b_{37}^0 = b'_{37} + a_1 \geq 0, \text{ since } a_1 \geq -b'_{11} - b'_{12} - b'_{13} - b'_{14} + b'_{22} + b'_{23} + b'_{24} + b'_{25} \\
&\quad = -b'_{33} - b'_{34} - b'_{35} - b'_{36} - b'_{37} + b'_{33} + b'_{34} + b'_{35} + b'_{36} \\
&\quad = -b'_{37}, \\
b_{38} &= b'_{38} + b_{38}^0 = b'_{38} + a_0 + a_2 = -b'_{33} - b'_{34} - b'_{35} - b'_{36} - b'_{37} + a_0 + a_2 \geq 0, \\
&\quad \text{since } a_0 \geq b'_{11} + b'_{12} + b'_{13} + b'_{14} - a_2
\end{aligned}$$

$$= b'_{33} + b'_{34} + b'_{35} + b'_{36} + b'_{37} - a_2,$$

$$b_{44} = b'_{44} + b^0_{44} = b'_{44} + a_4 + a_6 \geq 0, \text{ since } a_6 \geq -b'_{44} - a_4,$$

$$b_{45} = b'_{45} + b^0_{45} = b'_{45} + a_5 \geq 0, \text{ since } a_5 \geq -b'_{45},$$

$$b_{46} = b'_{46} + b^0_{46} = b'_{46} + a_3 + a_4 \geq 0, \text{ since } a_4 \geq -b'_{46} - a_3,$$

$$\begin{aligned} b_{47} = b'_{47} + b^0_{47} &= b'_{47} + a_2 \geq 0, \text{ since } a_2 \geq -b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34} + b'_{35} \\ &= -b'_{44} - b'_{45} - b'_{46} - b'_{47} + b'_{44} + b'_{45} + b'_{46} = -b'_{47}, \end{aligned}$$

$$\begin{aligned} b_{48} = b'_{48} + b^0_{48} &= b'_{48} + a_1 \geq 0, \text{ since } a_1 \geq -b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24} \\ &= -b'_{44} - b'_{45} - b'_{46} - b'_{47} - b'_{48} + b'_{44} + b'_{45} + b'_{46} + b'_{47} \\ &= -b'_{48}, \end{aligned}$$

$$\begin{aligned} b_{49} = b'_{49} + b^0_{49} &= b'_{49} + a_0 + a_2 + a_3 = -b'_{44} - b'_{45} - b'_{46} - b'_{47} - b'_{48} + a_0 + a_2 + a_3 \geq 0, \\ &\text{since } a_0 \geq b'_{11} + b'_{12} + b'_{13} - a_2 - a_3 \\ &= b'_{44} + b'_{45} + b'_{46} + b'_{47} + b'_{48} - a_2 - a_3, \end{aligned}$$

$$b_{55} = b'_{55} + b^0_{55} = b'_{55} + a_6 \geq 0, \text{ since } a_6 \geq -b'_{55},$$

$$b_{56} = b'_{56} + b^0_{56} = b'_{56} + a_4 + a_5 \geq 0, \text{ since } a_5 \geq -b'_{56} - a_4,$$

$$\begin{aligned} b_{57} = b'_{57} + b^0_{57} &= b'_{57} + a_3 \geq 0, \text{ since } a_3 \geq -b'_{33} - b'_{34} + b'_{44} + b'_{45} \\ &= -b'_{55} - b'_{56} - b'_{57} + b'_{55} + b'_{56} = -b'_{57}, \end{aligned}$$

$$\begin{aligned} b_{58} = b'_{58} + b^0_{58} &= b'_{58} + a_2 \geq 0, \text{ since } a_2 \geq -b'_{22} - b'_{23} + b'_{33} + b'_{34} \\ &= -b'_{55} - b'_{56} - b'_{57} - b'_{58} + b'_{55} + b'_{56} + b'_{57} = -b'_{58}, \end{aligned}$$

$$\begin{aligned} b_{59} = b'_{59} + b^0_{59} &= b'_{59} + a_1 \geq 0, \text{ since } a_1 \geq -b'_{11} - b'_{12} + b'_{22} + b'_{23} \\ &= -b'_{55} - b'_{56} - b'_{57} - b'_{58} - b'_{59} + b'_{55} + b'_{56} + b'_{57} + b'_{58} \\ &= -b'_{59}, \end{aligned}$$

$$\begin{aligned} b_{5,10} = b'_{5,10} + b^0_{5,10} &= b'_{5,10} + a_0 + a_2 + a_3 + a_4 \\ &= -b'_{55} - b'_{56} - b'_{57} - b'_{58} - b'_{59} + a_0 + a_2 + a_3 + a_4 \geq 0, \\ &\text{since } a_0 \geq b'_{11} + b'_{12} - a_2 - a_3 - a_4 \\ &= b'_{55} + b'_{56} + b'_{57} + b'_{58} + b'_{59} - a_2 - a_3 - a_4, \end{aligned}$$

$$b_{66} = b'_{66} + b^0_{66} = b'_{66} + a_6 \geq 0, \text{ since } a_6 \geq -b'_{55} = -b'_{66},$$

$$b_{67} = b'_{67} + b^0_{67} = b'_{67} + a_4 \geq 0, \text{ since } a_4 \geq -b'_{44} + b'_{55} = -b'_{66} - b'_{67} + b'_{66} = -b'_{67},$$

$$b_{68} = b'_{68} + b^0_{68} = b'_{68} + a_3 \geq 0, \text{ since } a_3 \geq -b'_{33} + b'_{44} = -b'_{66} - b'_{67} - b'_{68} + b'_{66} + b'_{67} = -b'_{68},$$

$$\begin{aligned} b_{69} = b'_{69} + b^0_{69} &= b'_{69} + a_2 \geq 0, \text{ since } a_2 \geq -b'_{22} + b'_{33} \\ &= -b'_{66} - b'_{67} - b'_{68} - b'_{69} + b'_{66} + b'_{67} + b'_{68} = -b'_{69}, \end{aligned}$$

$$b_{6,10} = b'_{6,10} + b^0_{6,10} = b'_{6,10} + a_1 \geq 0, \text{ since } a_1 \geq -b'_{11} + b'_{22}$$

$$= -b'_{66} - b'_{67} - b'_{68} - b'_{69} - b'_{6,10} + b'_{66} + b'_{67} + b'_{68} \\ + b'_{69} = -b'_{6,10},$$

$$b_{6,11} = b'_{6,11} + b^0_{6,11} = b'_{6,11} + a_0 + a_2 + a_3 + a_4 + a_5 \\ = -b'_{66} - b'_{67} - b'_{68} - b'_{69} - b'_{6,10} + a_0 + a_2 + a_3 + a_4 + a_5 \geq 0, \\ \text{since } a_0 \geq b'_{11} - a_2 - a_3 - a_4 - a_5.$$

We also have

$$\begin{aligned} b_{11} &= b'_{11} + b^0_{11} \\ &= b'_{66} + b'_{67} + b'_{68} + b'_{69} + b'_{6,10} + a_1 + a_2 + a_3 + a_4 + a_6 \\ &= b_{66} + b_{67} + b_{68} + b_{69} + b_{6,10}, \\ b_{11} + b_{12} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} \\ &= b'_{55} + b'_{56} + b'_{57} + b'_{58} + b'_{59} + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= b_{55} + b_{56} + b_{57} + b_{58} + b_{59}, \\ b_{11} + b_{12} + b_{13} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} + b'_{13} + b^0_{13} \\ &= b'_{44} + b'_{45} + b'_{46} + b'_{47} + b'_{48} + a_1 + a_2 + a_3 + 2a_4 + a_5 + a_6 \\ &= b_{44} + b_{45} + b_{46} + b_{47} + b_{48}, \\ b_{11} + b_{12} + b_{13} + b_{14} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} + b'_{13} + b^0_{13} + b'_{14} + b^0_{14} \\ &= b'_{33} + b'_{34} + b'_{35} + b'_{36} + b'_{37} + a_1 + a_2 + 2a_3 + 2a_4 + a_5 + a_6 \\ &= b_{33} + b_{34} + b_{35} + b_{36} + b_{37}, \\ b_{11} + b_{12} + b_{13} + b_{14} + b_{15} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} + b'_{13} + b^0_{13} + b'_{14} + b^0_{14} + b'_{15} + b^0_{15} \\ &= b'_{22} + b'_{23} + b'_{24} + b'_{25} + b'_{26} + a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 + a_6 \\ &= b_{22} + b_{23} + b_{24} + b_{25} + b_{26}, \\ b_{22} &= b'_{22} + b^0_{22} \\ &= b'_{66} + b'_{67} + b'_{68} + b'_{69} + a_2 + a_3 + a_4 + a_6 \\ &= b_{66} + b_{67} + b_{68} + b_{69}, \\ b_{22} + b_{23} &= b'_{22} + b^0_{22} + b'_{23} + b^0_{23} \\ &= b'_{55} + b'_{56} + b'_{57} + b'_{58} + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= b_{55} + b_{56} + b_{57} + b_{58}, \\ b_{22} + b_{23} + b_{24} &= b'_{22} + b^0_{22} + b'_{23} + b^0_{23} + b'_{24} + b^0_{24} \\ &= b'_{44} + b'_{45} + b'_{46} + b'_{47} + a_2 + a_3 + 2a_4 + a_5 + a_6 \\ &= b_{44} + b_{45} + b_{46} + b_{47}, \end{aligned}$$

$$\begin{aligned}
b_{22} + b_{23} + b_{24} + b_{25} &= b'_{22} + b^0_{22} + b'_{23} + b^0_{23} + b'_{24} + b^0_{24} + b'_{25} + b^0_{25} \\
&= b'_{33} + b'_{34} + b'_{35} + b'_{36} + a_2 + 2a_3 + 2a_4 + a_5 + a_6 \\
&= b_{33} + b_{34} + b_{35} + b_{36}, \\
b_{33} &= b'_{33} + b^0_{33} \\
&= b'_{66} + b'_{67} + b'_{68} + a_3 + a_4 + a_6 = b_{66} + b_{67} + b_{68}, \\
b_{33} + b_{34} &= b'_{33} + b^0_{33} + b'_{34} + b^0_{34} \\
&= b'_{55} + b'_{56} + b'_{57} + a_3 + a_4 + a_5 + a_6 = b_{55} + b_{56} + b_{57}, \\
b_{33} + b_{34} + b_{35} &= b'_{33} + b^0_{33} + b'_{34} + b^0_{34} + b'_{35} + b^0_{35} \\
&= b'_{44} + b'_{45} + b'_{46} + a_3 + 2a_4 + a_5 + a_6 = b_{44} + b_{45} + b_{46}, \\
b_{44} &= b'_{44} + b^0_{44} = b'_{66} + b'_{67} + a_4 + a_6 = b_{66} + b_{67}, \\
b_{44} + b_{45} &= b'_{44} + b^0_{44} + b'_{45} + b^0_{45} = b'_{55} + b'_{56} + a_4 + a_5 + a_6 = b_{55} + b_{56}, \\
b_{55} &= b'_{55} + b^0_{55} = b'_{66} + a_6 = b_{66}.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
b_{11} &= b'_{11} + b^0_{11} \geq b'_{22} + b^0_{22} = b_{22}, \text{ since } b^0_{11} - b^0_{22} = a_1 \geq -b'_{11} + b'_{22}, \\
b_{22} &= b'_{22} + b^0_{22} \geq b'_{33} + b^0_{33} = b_{33}, \text{ since } b^0_{22} - b^0_{33} = a_2 \geq -b'_{22} + b'_{33}, \\
b_{33} &= b'_{33} + b^0_{33} \geq b'_{44} + b^0_{44} = b_{44}, \text{ since } b^0_{33} - b^0_{44} = a_3 \geq -b'_{33} + b'_{44}, \\
b_{44} &= b'_{44} + b^0_{44} \geq b'_{55} + b^0_{55} = b_{55}, \text{ since } b^0_{44} - b^0_{55} = a_4 \geq -b'_{44} + b'_{55}, \\
b_{11} + b_{12} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} \geq b'_{22} + b^0_{22} + b'_{23} + b^0_{23} = b_{22} + b_{23}, \\
&\quad \text{since } b^0_{11} + b^0_{12} - b^0_{22} - b^0_{23} = a_1 \geq -b'_{11} - b'_{12} + b'_{22} + b'_{23}, \\
b_{22} + b_{23} &= b'_{22} + b^0_{22} + b'_{23} + b^0_{23} \geq b'_{33} + b^0_{33} + b'_{34} + b^0_{34} = b_{33} + b_{34}, \\
&\quad \text{since } b^0_{22} + b^0_{23} - b^0_{33} - b^0_{34} = a_2 \geq -b'_{22} - b'_{23} + b'_{33} + b'_{34}, \\
b_{33} + b_{34} &= b'_{33} + b^0_{33} + b'_{34} + b^0_{34} \geq b'_{44} + b^0_{44} + b'_{45} + b^0_{45} = b_{44} + b_{45}, \\
&\quad \text{since } b^0_{33} + b^0_{34} - b^0_{44} - b^0_{45} = a_3 \geq -b'_{33} - b'_{34} + b'_{44} + b'_{45}, \\
b_{11} + b_{12} + b_{13} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} + b'_{13} + b^0_{13} \\
&\geq b'_{22} + b^0_{22} + b'_{23} + b^0_{23} + b'_{24} + b^0_{24} = b_{22} + b_{23} + b_{24}, \\
&\quad \text{since } b^0_{11} + b^0_{12} + b^0_{13} - b^0_{22} - b^0_{23} - b^0_{24} \\
&\quad \quad = a_1 \geq -b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24}, \\
b_{22} + b_{23} + b_{24} &= b'_{22} + b^0_{22} + b'_{23} + b^0_{23} + b'_{24} + b^0_{24} \\
&\geq b'_{33} + b^0_{33} + b'_{34} + b^0_{34} + b'_{35} + b^0_{35} = b_{33} + b_{34} + b_{35}, \\
&\quad \text{since } b^0_{22} + b^0_{23} + b^0_{24} - b^0_{33} - b^0_{34} - b^0_{35}
\end{aligned}$$

$$\begin{aligned}
&= a_2 \geq -b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34} + b'_{35}, \\
b_{11} + b_{12} + b_{13} + b_{14} &= b'_{11} + b^0_{11} + b'_{12} + b^0_{12} + b'_{13} + b^0_{13} + b'_{14} + b^0_{14} \\
&\geq b'_{22} + b^0_{22} + b'_{23} + b^0_{23} + b'_{24} + b^0_{24} + b'_{25} + b^0_{25} = b_{22} + b_{23} + b_{24} + b_{25}, \\
\text{since } b^0_{11} + b^0_{12} + b^0_{13} + b^0_{14} - b^0_{22} - b^0_{23} - b^0_{24} - b^0_{25} \\
&= a_1 \geq -b'_{11} - b'_{12} - b'_{13} - b'_{14} + b'_{22} + b'_{23} + b'_{24} + b'_{25}.
\end{aligned}$$

Thus $b \in B^{6,l}$. Since $f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}) = b'$, we have $b' \in \cup_{(l,b) \in J} \text{Im } f_{(l,b)}$ which proves (3). \square

CHAPTER

8

ULTRA-DISCRETIZATION OF $\mathcal{V}(D_6^{(1)})$

It is known that the ultra-discretization of a positive geometric crystal is a Kashiwara's crystal ([1], [33]). In this chapter we apply the ultra-discretization functor \mathcal{UD} to the positive geometric crystal $\mathcal{V} = \mathcal{V}(D_6^{(1)})$ constructed in Chapter 6. Then we show that as crystal it is isomorphic to the crystal $B^{6,\infty}$ given in Chapter 7 which proves the conjecture in [25] for this case.

As a set $\mathcal{X} = \mathcal{UD}(\mathcal{V}) = \mathbb{Z}^{15}$. We denote the variables $x_m^{(l)}$ in \mathcal{V} by the same notation $x_m^{(l)}$ in $\mathcal{UD}(\mathcal{V}) = \mathcal{X}$.

Let $x = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \in \mathcal{X}$. By applying the ultra-discretization functor \mathcal{UD} to the positive geometric crystal \mathcal{V} in Chapter 6, we have,

$$\mathcal{UD}(\gamma_k)(x) = \begin{cases} -x_2^{(2)} - x_2^{(1)}, & k = 0, \\ 2x_1^{(1)} - x_2^{(2)} - x_2^{(1)}, & k = 1, \\ -x_1^{(1)} + 2x_2^{(2)} + 2x_2^{(1)} - x_3^{(3)} - x_3^{(2)} - x_3^{(1)}, & k = 2, \\ -x_2^{(2)} - x_2^{(1)} + 2x_3^{(3)} + 2x_3^{(2)} + 2x_3^{(1)} - x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)}, & k = 3, \end{cases}$$

$$\begin{aligned} \mathcal{UD}(\gamma_k)(x) &= \begin{cases} -x_3^{(3)} - x_3^{(2)} - x_3^{(1)} + 2x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + 2x_4^{(1)} - x_5^{(2)} - x_5^{(1)} \\ \quad -x_6^{(3)} - x_6^{(2)} - x_6^{(1)}, & k = 4, \\ -x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)} + 2x_5^{(2)} + 2x_5^{(1)}, & k = 5, \\ -x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)} + 2x_6^{(3)} + 2x_6^{(2)} + 2x_6^{(1)}, & k = 6. \end{cases} \\ \mathcal{UD}(\varepsilon_k)(x) &= \begin{cases} \max\{x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} \\ \quad -x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} \\ \quad +x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\ \quad x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} \\ \quad +x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} \\ \quad -x_4^{(4)} x_2^{(2)} - x_3^{(3)} + x_3^{(1)}, +x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, & k = 0, \\ -x_1^{(1)} + x_2^{(2)}, & k = 1, \\ \max\{-x_2^{(2)} + x_3^{(3)}, x_1^{(1)} - 2x_2^{(2)} - x_2^{(1)} + x_3^{(3)} + x_3^{(2)}\}, & k = 2, \\ \max\{-x_3^{(3)} + x_4^{(4)}, x_2^{(2)} - 2x_3^{(3)} - x_3^{(2)} + x_4^{(4)} + x_4^{(3)}, \\ \quad x_2^{(2)} + x_2^{(1)} - 2x_3^{(3)} - 2x_3^{(2)} - x_3^{(1)} + x_4^{(4)} + x_4^{(3)} + x_4^{(2)}\}, & k = 3, \\ \max\{-x_4^{(4)} + x_6^{(3)}, x_3^{(3)} - 2x_4^{(4)} - x_4^{(3)} + x_5^{(2)} + x_6^{(3)}, x_3^{(3)} + x_3^{(2)} \\ \quad - 2x_4^{(4)} - 2x_4^{(3)} - x_4^{(2)} + x_5^{(3)} + x_6^{(2)} + x_6^{(3)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} \\ \quad - 2x_4^{(4)} - 2x_4^{(3)} - 2x_4^{(2)} - x_4^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(3)} + x_6^{(2)}\}, & k = 4, \\ \max\{x_4^{(4)} - x_5^{(2)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} - 2x_5^{(2)} - x_5^{(1)}\}, & k = 5, \\ \max\{-x_6^{(3)}, x_4^{(4)} + x_4^{(3)} - 2x_6^{(3)} - x_6^{(2)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} \\ \quad - 2x_6^{(3)} - 2x_6^{(2)} - x_6^{(1)}\}, & k = 6. \end{cases} \end{aligned}$$

We define

$$\begin{aligned} \check{c}_2 &= \max\{c + x_2^{(2)} + x_2^{(1)}, x_3^{(2)} + x_1^{(1)}\} - \max\{x_2^{(2)} + x_2^{(1)}, x_3^{(2)} + x_1^{(1)}\}, \\ \check{c}_{3_1} &= \max\{c + x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)}, x_2^{(2)} + x_3^{(2)} + x_3^{(1)} + x_4^{(3)}, x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}\} \\ &\quad - \max\{x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)}, x_2^{(2)} + x_3^{(2)} + x_3^{(1)} + x_4^{(3)}, x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}\}, \\ \check{c}_{3_2} &= \max\{c + x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)}, c + x_2^{(2)} + x_3^{(2)} + x_3^{(1)} + x_4^{(3)}, x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}\} \\ &\quad - \max\{c + x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)}, x_2^{(2)} + x_3^{(2)} + x_3^{(1)} + x_4^{(3)}, x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}\}, \\ \check{c}_{4_1} &= \max\{c + x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, x_3^{(3)} + x_3^{(2)} \\ &\quad + x_4^{(2)} + x_4^{(1)} + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}\} - \max\{x_4^{(4)} \\ &\quad + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, x_3^{(3)} + x_3^{(2)} + x_4^{(2)} + x_4^{(1)}\} \end{aligned}$$

$$\begin{aligned}
& + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)} \}, \\
c_{\check{4}_2} &= \max \{ c + x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, c + x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, x_3^{(3)} \\
& + x_3^{(2)} + x_4^{(2)} + x_4^{(1)} + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)} \} \\
& - \max \{ c + x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, \\
& x_3^{(3)} + x_3^{(2)} + x_4^{(2)} + x_4^{(1)} + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)} \}, \\
c_{\check{4}_3} &= \max \{ c + x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, c + x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, \\
& c + x_3^{(3)} + x_3^{(2)} + x_4^{(2)} + x_4^{(1)} + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)} \} \\
& - \max \{ c + x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)}, c + x_3^{(3)} + x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} + x_5^{(2)}, \\
& x_3^{(3)} + x_3^{(2)} + x_4^{(2)} + x_4^{(1)} + x_5^{(2)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)} \}, \\
\check{c}_5 &= \max \{ c + x_5^{(2)} + x_5^{(1)}, x_4^{(3)} + x_4^{(2)} \} - \max \{ x_5^{(2)} + x_5^{(1)}, x_4^{(3)} + x_4^{(2)} \}, \\
c_{\check{6}_1} &= \max \{ c + x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} \} \\
& - \max \{ x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} \}, \\
c_{\check{6}_2} &= \max \{ c + x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)}, c + x_4^{(4)} + x_4^{(3)} + x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} \} \\
& - \max \{ c + x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_6^{(2)} + x_6^{(1)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} \}, \\
\check{K} &= \max \{ x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} \\
& - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} \\
& + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \},
\end{aligned}$$

Then we have

$$\mathcal{UD}(e_k^c)(x) = \begin{cases} (x_6^{(3)'}, x_4^{(4)'}, x_3^{(3)'}, x_2^{(2)} - c, x_5^{(2)'}, x_4^{(3)'}, x_3^{(2)'}, x_6^{(2)'}, x_4^{(2)'}, x_5^{(1)'}, \\ x_1^{(1)} - c, x_2^{(1)} - c, x_3^{(1)'}, x_4^{(1)'}, x_6^{(1)'}), & k = 0, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} + c, \\ x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), & k = 1, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + \check{c}_2, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, \\ x_1^{(1)}, x_2^{(1)} + c - \check{c}_2, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), & k = 2, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + c_{\check{3}_1}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + c_{\check{3}_2}, x_6^{(2)}, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)} + c - c_{\check{3}_1} - c_{\check{3}_2}, x_4^{(1)}, x_6^{(1)}), & k = 3, \end{cases}$$

$$\mathcal{UD}(e_k^c)(x) = \begin{cases} (x_6^{(3)}, x_4^{(4)} + c_{\check{4}_1}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)} + c_{\check{4}_2}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)} \\ + c_{\check{4}_3}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)} + c - c_{\check{4}_1} - c_{\check{4}_2} - c_{\check{4}_3}, x_6^{(1)}), & k = 4, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)} + c_{\check{5}}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)} \\ + c - c_{\check{5}}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), & k = 5, \\ (x_6^{(3)} + c_{\check{6}_1}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + c_{\check{6}_2}, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} + c - c_{\check{6}_1} - c_{\check{6}_2}), & k = 6, \end{cases}$$

where

$$\begin{aligned} x_3^{(1)'} &= x_3^{(1)} + \check{K} - \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\ &\quad - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\ &\quad - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\ &\quad - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\ &\quad x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, c \\ &\quad + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, \\ x_3^{(2)'} &= -c + x_3^{(2)} + \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\ &\quad - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\ &\quad - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\ &\quad - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\ &\quad x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\ &\quad c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\} - \max\{c \\ &\quad + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} \\ &\quad + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} \\ &\quad - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, c + x_2^{(2)} \\ &\quad + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} \\ &\quad + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} \\ &\quad + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, \\ x_3^{(3)'} &= -c + x_3^{(3)} + \max\{c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} \\ &\quad - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\ &\quad - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, c + x_2^{(2)} \\ &\quad + x_2^{(1)} - x_6^{(3)}, c + x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} \end{aligned}$$

$$\begin{aligned}
& -x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} \\
& - x_5^{(2)} \} - \check{K}, \\
x_4^{(1)'} &= x_4^{(1)} + \check{K} - \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} \\
& + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(2)} + x_4^{(2)}, x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, \\
x_4^{(2)'} &= x_4^{(2)} + \check{K} + \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} \\
& + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(2)} + x_4^{(2)}, x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \} - \max\{c + x_6^{(1)} + \max\{c \\
& + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} \\
& - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} \\
& + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c \\
& + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, c + x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} + \check{K}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \\
& + \check{K}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)} + \check{K}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} \\
& + \check{K}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)} + \max\{x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
& + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} \\
& - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\
& - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} \\
& + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, c + x_4^{(2)} + x_4^{(1)}
\end{aligned}$$

$$\begin{aligned}
& -x_6^{(2)} + \max\{c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\
& - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\
& - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} \\
& + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} \\
& - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}, x_2^{(2)} \\
& + x_2^{(1)} - x_6^{(3)} + \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} \\
& + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \\
& x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} \\
& - x_4^{(3)} + \max\{c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} \\
& + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} \\
& - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} \\
& - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} + \check{K}, \\
& c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} + \check{K}, c + x_2^{(2)} + x_3^{(1)} \\
& - x_4^{(4)} + \check{K}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} + \check{K}, c + x_3^{(2)} + x_3^{(1)} \\
& - x_5^{(2)} + \check{K}\},
\end{aligned}$$

$$\begin{aligned}
x_4^{(3)'} = & -c + x_4^{(3)} + \max\{c + x_6^{(1)} + \max\{c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
& + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c \\
& + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} \\
& + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c \\
& + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} \\
& + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} \\
& - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \\
& + \check{K}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} + \check{K}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}
\end{aligned}$$

$$\begin{aligned}
& + \check{K}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} + \check{K}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)} \\
& + \max\{x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} \\
& + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} \\
& + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} \\
& - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
& + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)} + \max\{c + x_6^{(1)}, c + x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c \\
& + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} \\
& + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} \\
& - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} \\
& - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)} + \max\{c \\
& + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\
& c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} \\
& - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} \\
& + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} \\
& + x_3^{(1)} - x_5^{(2)}\}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)} + \max\{c + x_6^{(1)}, x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
& + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} \\
& - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} \\
& - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} \\
& + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\}, c \\
& + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} + \check{K}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
& + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} + \check{K}, c + x_2^{(2)} + x_3^{(1)} - x_5^{(2)} + \check{K} - \check{K} - \max\{c \\
& + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} \\
& + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c
\end{aligned}$$

$$\begin{aligned}
& + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} \\
& + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} \\
& + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, \\
x_4^{(4)'} & = -c + x_4^{(4)} + \max \{ c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} \\
& - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} \\
& + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} \\
& + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} \\
& - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \} - \check{K}, \\
x_5^{(1)'} & = x_5^{(1)} + \check{K} - \max \{ c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\
& - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\
& - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, \\
x_5^{(2)'} & = -c + x_5^{(2)} + \max \{ c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\
& - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\
& - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \} - \check{K}, \\
x_6^{(1)'} & = x_6^{(1)} + \check{K} - \max \{ c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
& + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} \\
& + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} \\
& - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} \\
& - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \}, \\
x_6^{(2)'} & = x_6^{(2)} + \max \{ c + x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} \}
\end{aligned}$$

$$\begin{aligned}
& -x_4^{(2)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, x_3^{(3)} + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} \\
& -x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, x_2^{(2)} + x_2^{(1)} + x_6^{(2)} \\
& -x_4^{(4)} - x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
& + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_5^{(2)} \} - \max\{c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\
& c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} \\
& + x_3^{(2)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, c + x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} \\
& - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& c + x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} \\
& + x_3^{(1)} - x_5^{(2)} \}, \\
x_6^{(3)'} & = -c + x_6^{(3)} + \max\{c + x_6^{(1)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, c + x_2^{(2)} \\
& - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, c + x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, c + x_3^{(3)} + x_3^{(2)} - x_4^{(3)} \\
& - x_4^{(2)} + x_5^{(1)}, c + x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, c + x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} \\
& - x_6^{(3)}, c + x_2^{(2)} + x_2^{(1)} + x_6^{(2)} - x_4^{(4)} - x_4^{(3)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(2)} \\
& - x_4^{(4)} + x_4^{(2)}, c + x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, c + x_2^{(2)} \\
& + x_3^{(1)} - x_4^{(4)}, c + x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, c + x_3^{(2)} + x_3^{(1)} \\
& - x_5^{(2)} \} - \check{K}.
\end{aligned}$$

As shown in ([1], [33]), \mathcal{X} with maps $\tilde{e}_k, \tilde{f}_k : \mathcal{X} \longrightarrow \mathcal{X} \cup \{0\}$, $\varepsilon_k, \varphi_k : \mathcal{X} \longrightarrow \mathbb{Z}$, $0 \leq k \leq 6$ and $\text{wt} : \mathcal{X} \longrightarrow P_{cl}$ is a Kashiwara's crystal where for $x \in \mathcal{X}$,

$$\begin{aligned}
\tilde{e}_k(x) &= \mathcal{UD}(e_k^c)(x)|_{c=1}, \quad \tilde{f}_k(x) = \mathcal{UD}(e_k^c)(x)|_{c=-1}, \\
\text{wt}(x) &= \sum_{k=0}^6 \text{wt}_k(x) \Lambda_k \text{ where } \text{wt}_k(x) = \mathcal{UD}(\gamma_k)(x), \\
\varepsilon_k(x) &= \mathcal{UD}(\varepsilon_k)(x), \quad \varphi_k(x) = \text{wt}_k(x) + \varepsilon_k(x).
\end{aligned}$$

In particular, the explicit actions of \tilde{f}_k and \tilde{e}_k for $1 \leq k \leq 6$ on \mathcal{X} is given as follows.

$$\tilde{f}_1(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}),$$

$$\begin{aligned}
\tilde{f}_2(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_2^{(2)} + x_2^{(1)} > x_1^{(1)} + x_3^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_2^{(2)} + x_2^{(1)} \leq x_1^{(1)} + x_3^{(2)}, \end{cases} \\
\tilde{f}_3(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(3)} + x_3^{(2)} > x_2^{(2)} + x_4^{(3)}, x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} > x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(3)} + x_3^{(2)} \leq x_2^{(2)} + x_4^{(3)}, x_3^{(2)} + x_3^{(1)} > x_2^{(1)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(2)} + x_3^{(1)} \leq x_2^{(1)} + x_4^{(2)}, x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} \leq x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}, \end{cases} \\
\tilde{f}_4(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + x_4^{(3)} > x_3^{(3)} + x_5^{(2)}, x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} > x_3^{(3)} + x_3^{(2)} + x_5^{(2)} + x_6^{(2)}, \\ x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} > x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + x_4^{(3)} \leq x_3^{(3)} + x_5^{(2)}, x_4^{(3)} + x_4^{(2)} > x_3^{(2)} + x_6^{(2)}, \\ x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} > x_3^{(2)} + x_3^{(1)} + x_5^{(1)} + x_6^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} \leq x_3^{(3)} + x_3^{(2)} + x_5^{(2)} + x_6^{(2)}, \\ x_4^{(3)} + x_4^{(2)} \leq x_3^{(2)} + x_6^{(2)}, x_4^{(2)} + x_4^{(1)} > x_3^{(1)} + x_5^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)} - 1, x_6^{(1)}) \\ \text{if } x_4^{(2)} + x_4^{(1)} \leq x_3^{(1)} + x_5^{(1)}, x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \leq x_3^{(2)} + x_3^{(1)} + x_5^{(1)} + x_6^{(2)}, \\ x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \leq x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}, \end{cases} \\
\tilde{f}_5(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_5^{(2)} + x_5^{(1)} > x_4^{(3)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)} - 1, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_5^{(2)} + x_5^{(1)} \leq x_4^{(3)} + x_4^{(2)}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_6(x) &= \begin{cases} (x_6^{(3)} - 1, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_6^{(3)} + x_6^{(2)} > x_4^{(4)} + x_4^{(3)}, x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} > x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_6^{(3)} + x_6^{(2)} \leq x_4^{(4)} + x_4^{(3)}, x_6^{(2)} + x_6^{(1)} > x_4^{(2)} + x_4^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} - 1) \\ \text{if } x_6^{(2)} + x_6^{(1)} \leq x_4^{(2)} + x_4^{(1)}, x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} \leq x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}. \end{cases} \\
\tilde{e}_1(x) &= (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}), \\
\tilde{e}_2(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_2^{(2)} + x_2^{(1)} \geq x_1^{(1)} + x_3^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_2^{(2)} + x_2^{(1)} < x_1^{(1)} + x_3^{(2)}, \end{cases} \\
\tilde{e}_3(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(3)} + x_3^{(2)} \geq x_2^{(2)} + x_4^{(3)}, x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} \geq x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(3)} + x_3^{(2)} < x_2^{(2)} + x_4^{(3)}, x_3^{(2)} + x_3^{(1)} \geq x_2^{(1)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_3^{(2)} + x_3^{(1)} < x_2^{(1)} + x_4^{(2)}, x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} < x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}, \end{cases} \\
\tilde{e}_4(x) &= \begin{cases} (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + x_4^{(3)} \geq x_3^{(3)} + x_5^{(2)}, x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} \geq x_3^{(3)} + x_3^{(2)} + x_5^{(2)} + x_6^{(2)}, \\ x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \geq x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + x_4^{(3)} < x_3^{(3)} + x_5^{(2)}, x_4^{(3)} + x_4^{(2)} \geq x_3^{(2)} + x_6^{(2)}, \\ x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \geq x_3^{(2)} + x_3^{(1)} + x_5^{(1)} + x_6^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)} + 1, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} < x_3^{(3)} + x_3^{(2)} + x_5^{(2)} + x_6^{(2)}, \\ x_4^{(3)} + x_4^{(2)} < x_3^{(2)} + x_6^{(2)}, x_4^{(2)} + x_4^{(1)} \geq x_3^{(1)} + x_5^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} + 1, x_6^{(1)}) \\ \text{if } x_4^{(2)} + x_4^{(1)} < x_3^{(1)} + x_5^{(1)}, x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} < x_3^{(2)} + x_3^{(1)} + x_5^{(1)} + x_6^{(2)}, \\ x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} < x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}, \end{cases} \end{aligned}$$

$$\tilde{e}_5(x) = \begin{cases} (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_5^{(2)} + x_5^{(1)} \geq x_4^{(3)} + x_4^{(2)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)} + 1, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_5^{(2)} + x_5^{(1)} < x_4^{(3)} + x_4^{(2)}, \end{cases}$$

$$\tilde{e}_6(x) = \begin{cases} (x_6^{(3)} + 1, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_6^{(3)} + x_6^{(2)} \geq x_4^{(4)} + x_4^{(3)}, x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} \geq x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \\ \text{if } x_6^{(3)} + x_6^{(2)} < x_4^{(4)} + x_4^{(3)}, x_6^{(2)} + x_6^{(1)} \geq x_4^{(2)} + x_4^{(1)}, \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} + 1) \\ \text{if } x_6^{(2)} + x_6^{(1)} < x_4^{(2)} + x_4^{(1)}, x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} < x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}. \end{cases}$$

To determine the explicit action of $\tilde{f}_0(x)$ we define conditions $(\check{F}1) - (\check{F}14)$ as follows.

$$\begin{aligned} (\check{F}1) \quad & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_6^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\ & x_2^{(2)} + x_2^{(1)} - x_6^{(3)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)}, \\ (\check{F}2) \quad & x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_6^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\ & x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \end{aligned}$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} \geq x_6^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} \geq x_2^{(2)}$$

$$x_2^{(2)} + x_3^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} + x_3^{(1)} - x_2^{(2)} - x_4^{(4)} + x_4^{(2)} > x_2^{(2)} - x_2^{(3)} + x_4^{(1)},$$

$$x_{\hat{s}}^{(2)} + x_{\hat{s}}^{(1)} - x_{\hat{s}}^{(2)} - x_{\hat{s}}^{(4)} + x_{\hat{s}}^{(2)} \geq x_{\hat{s}}^{(2)} + x_{\hat{s}}^{(1)} - x_{\hat{s}}^{(3)}$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_3^{(4)} + x_4^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}.$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_4^{(2)}$$

$$x_2 + x_2 - x_3 - x_4 + x_4 \leq x_4 + x_4 - x_6, \\ x^{(2)} + x^{(1)} - x^{(2)} - x^{(4)} + x^{(2)} \leq x^{(2)} + x^{(1)} - x^{(3)}$$

$$x_2 + x_2 - x_3 - x_4 + x_4 > x_2 + x_2 - x_6, \\ x^{(2)}_1 + x^{(1)}_1 - x^{(2)}_2 - x^{(4)}_3 + x^{(2)}_4 > x^{(2)}_1 + x^{(1)}_1 - x^{(4)}_3$$

$$x_2 + x_2 - x_3 - x_4 + x_4 > x_2 + x_2 - x_4$$

$$\gamma^{(2)} + \gamma^{(1)} - \gamma^{(2)} - \gamma^{(4)} + \gamma^{(2)} > \gamma^{(2)} + \gamma^{(1)} - \gamma^{(3)}$$

$$x_2' + x_2'' - x_3' - x_4' + x_4'' \geq x_2' + x_2'' - x_3' - x_3'' + x_4''$$

$$(2) + (1) - (2) - (4) + (2) > (2) + (1) - (4)$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(1)} \geq x_2^{(2)} + x_3^{(1)} - x_4^{(2)},$$

$$x_2^{(\zeta)} + x_2^{(1)} - x_3^{(\zeta)} - x_4^{(1)} + x_4^{(\zeta)} \geq x_2^{(\zeta)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(\zeta)},$$

$$\stackrel{(2)}{=} \stackrel{(1)}{=} \stackrel{(2)}{=} \stackrel{(4)}{=} \stackrel{(2)}{=} \stackrel{(2)}{=} \stackrel{(3)}{=} \stackrel{(2)}{=}$$

$$x_2^{(z)} + x_2^{(x)} - x_3^{(z)} - x_4^{(x)} + x_4^{(z)} \geq x_3^{(z)} + x_3^{(x)} - x_5^{(z)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_6^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$\begin{aligned}
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},
\end{aligned}$$

$$\begin{aligned}
(F5) \quad & x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_6^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_3^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_2^{(2)} + x_3^{(1)} - x_4^{(4)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},
\end{aligned}$$

$$\begin{aligned}
(F6) \quad & x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_6^{(1)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},
\end{aligned}$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)},$$

$$x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},$$

$$(F7) \quad x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_6^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_4^{(2)} + x_4^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},$$

$$(F8) \quad x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_6^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_3^{(2)} + x_3^{(3)} - x_5^{(2)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)},$$

(F9) $x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_6^{(1)}$,

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},$$

(F10) $x_2^{(2)} - x_3^{(3)} + x_4^{(1)} \geq x_6^{(1)}$,

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} \geq x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)},$$

$$x_2^{(2)} - x_3^{(3)} + x_4^{(1)} \geq x_3^{(2)} + x_3^{(3)} - x_5^{(2)},$$

(F11) $x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} \geq x_6^{(1)}$,

$$x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)},$$

$$x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)},$$

$$\begin{aligned}
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} \geq x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} \geq x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_3^{(3)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)} > x_3^{(2)} + x_3^{(3)} - x_5^{(2)},
\end{aligned}$$

$$\begin{aligned}
(F12) \quad & x_3^{(2)} - x_4^{(3)} + x_4^{(1)} \geq x_6^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} \geq x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_3^{(2)} - x_4^{(3)} + x_4^{(1)} > x_3^{(2)} + x_3^{(3)} - x_5^{(2)},
\end{aligned}$$

$$\begin{aligned}
(F13) \quad & x_4^{(2)} + x_4^{(1)} - x_6^{(2)} \geq x_6^{(1)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_3^{(2)} - x_4^{(3)} + x_4^{(1)},
\end{aligned}$$

$$\begin{aligned}
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_4^{(2)} + x_4^{(1)} - x_6^{(2)} > x_3^{(2)} + x_3^{(3)} - x_5^{(2)}, \\
(F14) \quad & x_6^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_5^{(1)}, \\
& x_6^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} - x_4^{(2)} + x_5^{(1)}, \\
& x_6^{(1)} > x_2^{(2)} - x_3^{(3)} + x_4^{(1)}, \\
& x_6^{(1)} > x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, \\
& x_6^{(1)} > x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, \\
& x_6^{(1)} > x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, \\
& x_6^{(1)} > x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
& x_6^{(1)} > x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, \\
& x_6^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_4^{(4)} + x_4^{(2)}, \\
& x_6^{(1)} > x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, \\
& x_6^{(1)} > x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, \\
& x_6^{(1)} > x_2^{(2)} - x_3^{(3)} + x_3^{(1)} + x_4^{(3)} - x_5^{(2)}, \\
& x_6^{(1)} > x_3^{(2)} + x_3^{(3)} - x_5^{(2)},
\end{aligned}$$

Then we define conditions $(\check{E}j)$ ($1 \leq j \leq 14$) by replacing $>$ (resp. \geq) with \geq (resp. $>$) in $(\check{F}j)$.

Let $x \in \mathcal{X}$. Then $\tilde{f}_0(x) = \mathcal{UD}(e_0^c)(x)|_{c=-1}$ given by

$$\tilde{f}_0(x) = \begin{cases} (x_6^{(3)} + 1, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}1), \\ (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}2), \\ (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}3), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}4), \\ (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}5), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}6), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}7), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}8), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}9), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)}, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) \text{ if } (\check{F}10), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}11), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)}, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{F}12), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) \text{ if } (\check{F}13), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)} + 1, \\ x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)} + 1) \text{ if } (\check{F}14). \end{cases}$$

And, $\tilde{e}_0(x) = \mathcal{UD}(e_0^c)(x)|_{c=1}$ given by

$$\tilde{e}_0(x) = \begin{cases} (x_6^{(3)} - 1, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}1), \\ (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)}, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}2), \\ (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}3), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}4), \\ (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}5), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}6), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}7), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}8), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (\check{E}9), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)}, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (E\check{10}), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) \text{ if } (E\check{11}), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)}, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)}) \text{ if } (E\check{12}), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)}) \text{ if } (E\check{13}), \\ (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)} - 1, \\ x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)} - 1) \text{ if } (E\check{14}). \end{cases}$$

The following theorem gives an affirmative answer to the conjecture in [25] for $D_6^{(1)}$ and Dynkin index $i = 6$.

Theorem 8.0.1. *The map*

$$\begin{aligned} \Omega : \quad B^{6,\infty} &\rightarrow \mathcal{X}, \\ b = (b_{ij})_{i \leq j \leq i+5, 1 \leq i \leq 6} &\mapsto x = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, \\ &\quad x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) \end{aligned}$$

defined by

$$x_m^{(l)} = \begin{cases} \sum_{j=m-l+1}^m b_{m-l+1,j}, & \text{for } m = 1, 2, 3, 4, \\ \sum_{j=m-2l+1}^m b_{m-2l+1,j}, & \text{for } m = 5, \\ \sum_{j=m-2l+1}^{m-1} b_{m-2l+1,j}, & \text{for } m = 6. \end{cases}$$

is an isomorphism of crystals.

Proof. First we observe that the map $\Omega^{-1} : \mathcal{X} \rightarrow B^{6,\infty}$ is given by $\Omega^{-1}(x) = b$ where

$$\begin{array}{llll} b_{11} = x_1^{(1)}, & b_{12} = x_2^{(2)} - x_1^{(1)}, & b_{13} = x_3^{(3)} - x_2^{(2)}, & b_{14} = x_4^{(4)} - x_3^{(3)}, \\ b_{15} = x_6^{(3)} - x_4^{(4)}, & b_{16} = -x_6^{(3)}, & b_{22} = x_2^{(1)}, & b_{23} = x_3^{(2)} - x_2^{(1)}, \\ b_{24} = x_4^{(3)} - x_3^{(2)}, & b_{25} = x_5^{(2)} - x_4^{(3)}, & b_{26} = x_6^{(3)} - x_5^{(2)}, & b_{27} = -x_6^{(3)}, \\ b_{33} = x_3^{(1)}, & b_{34} = x_4^{(2)} - x_3^{(1)}, & b_{35} = x_6^{(2)} - x_4^{(2)}, & b_{36} = x_5^{(2)} - x_6^{(2)}, \\ b_{37} = x_4^{(4)} - x_5^{(2)}, & b_{38} = -x_4^{(4)}, & b_{44} = x_4^{(1)}, & b_{45} = x_5^{(1)} - x_4^{(1)}, \\ b_{46} = x_6^{(2)} - x_5^{(1)}, & b_{47} = x_4^{(3)} - x_6^{(2)}, & b_{48} = x_3^{(3)} - x_4^{(3)}, & b_{49} = -x_3^{(3)}, \\ b_{55} = x_6^{(1)}, & b_{56} = x_5^{(1)} - x_6^{(1)}, & b_{57} = x_4^{(2)} - x_5^{(1)}, & b_{58} = x_3^{(2)} - x_4^{(2)}, \\ b_{59} = x_2^{(2)} - x_3^{(2)}, & b_{5,10} = -x_2^{(2)}, & b_{66} = x_6^{(1)}, & b_{67} = x_4^{(1)} - x_6^{(1)}, \\ b_{68} = x_3^{(1)} - x_4^{(1)}, & b_{69} = x_2^{(1)} - x_3^{(1)}, & b_{6,10} = x_1^{(1)} - x_2^{(1)}, & b_{6,11} = -x_1^{(1)}. \end{array}$$

Hence the map Ω is bijective. To prove that Ω is an isomorphism of crystals we need to show that for $b \in B^{6,\infty}$ and $0 \leq k \leq 6$ we have:

$$\begin{aligned} \Omega(\tilde{f}_k(b)) &= \tilde{f}_k(\Omega(b)), \\ \Omega(\tilde{e}_k(b)) &= \tilde{e}_k(\Omega(b)), \\ \text{wt}_k(\Omega(b)) &= \text{wt}_k(b), \\ \varepsilon_k(\Omega(b)) &= \varepsilon_k(b). \end{aligned}$$

Hence $\varphi_k(\Omega(b)) = \text{wt}_k(\Omega(b)) + \varepsilon_k(\Omega(b)) = \text{wt}_k(b) + \varepsilon_k(b) = \varphi_k(b)$. We first observe that the conditions for the action of \tilde{f}_k on $\Omega(b)$ in \mathcal{X} hold if and only if the corresponding conditions for the action of \tilde{f}_k on b in $B^{6,\infty}$ hold for all $0 \leq k \leq 6$. Let $b \in B^{6,\infty}$. Suppose $\Omega(b) = x$. Then we have the followings.

- If x satisfies condition $(\check{F}1)$, then b satisfies condition (F_1) and $\tilde{f}_0(x) = (x_6^{(3)} + 1, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}2)$, then b satisfies condition (F_2) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}3)$, then b satisfies condition (F_3) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(1)}, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}4)$, then b satisfies condition (F_4) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}5)$, then b satisfies condition (F_5) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}6)$, then b satisfies condition (F_6) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}7)$, then b satisfies condition (F_7) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}8)$, then b satisfies condition (F_8) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)} + 1, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition $(\check{F}9)$, then b satisfies condition (F_9) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.

- If x satisfies condition (F10), then b satisfies condition (F_{10}) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)}, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition (F11), then b satisfies condition (F_{11}) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition (F12), then b satisfies condition (F_{12}) and $\tilde{f}_0(x) = x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)} + 1, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)}, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition (F13), then b satisfies condition (F_{13}) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)} + 1, x_4^{(2)} + 1, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- If x satisfies condition (F14), then b satisfies condition (F_{14}) and $\tilde{f}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} + 1, x_6^{(2)}, x_4^{(2)} + 1, x_5^{(1)} + 1, x_1^{(1)} + 1, x_2^{(1)} + 1, x_3^{(1)} + 1, x_4^{(1)} + 1, x_6^{(1)}) = \Omega(\tilde{f}_0(b))$.
- $\tilde{f}_1(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_1(b))$.
- If $x_2^{(2)} + x_2^{(1)} > x_1^{(1)} + x_3^{(2)}$, then $b_{11} + b_{12} + b_{22} > b_{11} + b_{22} + b_{23}$, hence $\tilde{f}_2(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_2(b))$.
- If $x_2^{(2)} + x_2^{(1)} \leq x_1^{(1)} + x_3^{(2)}$, then $b_{11} + b_{12} + b_{22} \leq b_{11} + b_{22} + b_{23}$, hence $\tilde{f}_2(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_2(b))$.
- If $x_3^{(3)} + x_3^{(2)} > x_2^{(2)} + x_4^{(3)}$ and $x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} > x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{22} + b_{23} > b_{11} + b_{12} + b_{22} + b_{23} + b_{24}$ and $b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{33} > b_{11} + b_{12} + 2b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{f}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_3(b))$.
- If $x_3^{(3)} + x_3^{(2)} \leq x_2^{(2)} + x_4^{(3)}$ and $x_3^{(2)} + x_3^{(1)} > x_2^{(1)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{22} + b_{23} \leq b_{11} + b_{12} + b_{22} + b_{23} + b_{24}$ and $b_{22} + b_{23} + b_{33} > b_{22} + b_{33} + b_{34}$, hence $\tilde{f}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_3(b))$.
- If $x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} \leq x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}$ and $x_3^{(2)} + x_3^{(1)} \leq x_2^{(1)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{33} \leq b_{11} + b_{12} + 2b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$ and $b_{22} + b_{23} + b_{33} \leq b_{22} + b_{33} + b_{34}$, hence $\tilde{f}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_3(b))$.

- If $x_4^{(4)} + x_4^{(3)} > x_3^{(3)} + x_5^{(2)}$, $x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} > x_3^{(3)} + x_5^{(2)} + x_6^{(2)}$, and $x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} > x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} > b_{11} + b_{12} + b_{13} + b_{22} + b_{23} + b_{24} + b_{25}$, $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + b_{33} + b_{34} > b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35}$, $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + 2b_{33} + 2b_{34} + b_{44} > b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + 2b_{33} + b_{34} + b_{35} + b_{44} + b_{45} + b_{46}$, hence $f_4(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(2)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_4(b))$.
- If $x_4^{(4)} + x_4^{(3)} \leq x_3^{(3)} + x_5^{(2)}$, $x_4^{(3)} + x_4^{(2)} > x_3^{(2)} + x_6^{(2)}$, and $x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} > x_3^{(2)} + x_3^{(1)} + x_5^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} \leq b_{11} + b_{12} + b_{13} + b_{22} + b_{23} + b_{24} + b_{25}$, $b_{22} + b_{23} + b_{24} + b_{33} + b_{34} > b_{22} + b_{23} + b_{33} + b_{34} + b_{35}$, and $b_{22} + b_{23} + b_{24} + 2b_{33} + 2b_{34} + b_{44} > b_{22} + b_{23} + 2b_{33} + b_{34} + b_{35} + b_{44} + b_{45}$, hence $f_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(2)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_4(b))$.
- If $x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} \leq x_3^{(3)} + x_3^{(2)} + x_5^{(2)}$, $x_4^{(3)} + x_4^{(2)} \leq x_3^{(2)} + x_6^{(2)}$, and $x_4^{(2)} + x_4^{(1)} > x_3^{(1)} + x_5^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + b_{33} + b_{34} \leq b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35}$, $b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{35} > b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{35} + b_{44} + b_{45}$, hence $f_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(2)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_4(b))$.
- If $x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \leq x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}$, $x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \leq x_3^{(1)} + x_5^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + 2b_{33} + 2b_{34} + b_{44} \leq b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + 2b_{33} + b_{34} + b_{35} + b_{44} + b_{45} + b_{46}$, $b_{22} + b_{23} + b_{24} + 2b_{33} + 2b_{34} + b_{44} \leq b_{22} + b_{23} + 2b_{33} + b_{34} + b_{35} + b_{44} + b_{45}$, and $b_{33} + b_{34} + b_{44} \leq b_{33} + b_{44} + b_{45}$, hence $f_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(2)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_4(b))$.
- If $x_5^{(2)} + x_5^{(1)} > x_4^{(3)} + x_4^{(2)}$, then $b_{22} + b_{23} + b_{24} + b_{25} + b_{44} + b_{45} > b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{f}_5(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_5(b))$.
- If $x_5^{(2)} + x_5^{(1)} \leq x_4^{(3)} + x_4^{(2)}$, then $b_{22} + b_{23} + b_{24} + b_{25} + b_{44} + b_{45} \leq b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{f}_5(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)} - 1, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_5(b))$.
- If $x_6^{(3)} + x_6^{(2)} > x_4^{(4)} + x_4^{(3)}$ and $x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} > x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + b_{33} + b_{34} + b_{35} > b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24}$ and $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + 2b_{33} + 2b_{34} + 2b_{35} + b_{55} > b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{44}$, hence $\tilde{f}_6(x) = (x_6^{(3)} - 1, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_6(b))$.

- If $x_6^{(3)} + x_6^{(2)} \leq x_4^{(4)} + x_4^{(3)}$ and $x_6^{(2)} + x_6^{(1)} > x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + b_{33} + b_{34} + b_{35} \leq b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24}$ and $b_{33} + b_{34} + b_{35} + b_{55} > b_{33} + b_{34} + b_{44}$, hence $\tilde{f}_6(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{f}_6(b))$.
- If $x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} \leq x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}$ and $x_6^{(2)} + x_6^{(1)} \leq x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + 2b_{33} + 2b_{34} + 2b_{35} + b_{55} \leq b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{44}$ and $b_{33} + b_{34} + b_{35} + b_{55} \leq b_{33} + b_{34} + b_{44}$, hence $\tilde{f}_6(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(1)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)} - 1) = \Omega(\tilde{f}_6(b))$.

Hence $\Omega(\tilde{f}_k(b)) = \tilde{f}_k(\Omega(b))$ for all $0 \leq k \leq 6$. Similarly, we have

- If x satisfies condition $(\check{E}1)$, then b satisfies condition (E_1) and $\tilde{e}_0(x) = (x_6^{(3)} - 1, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}2)$, then b satisfies condition (E_2) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}3)$, then b satisfies condition (E_3) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}4)$, then b satisfies condition (E_4) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}5)$, then b satisfies condition (E_5) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}6)$, then b satisfies condition (E_6) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}7)$, then b satisfies condition (E_7) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition $(\check{E}8)$, then b satisfies condition (E_8) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)} - 1, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)} - 1, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)}, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.

- If x satisfies condition (E9), then b satisfies condition (E9) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition (E10), then b satisfies condition (E10) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} - 1, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)}, x_6^{(2)} - 1, x_4^{(2)}, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition (E11), then b satisfies condition (E11) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition (E12), then b satisfies condition (E12) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)} - 1, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)}, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition (E13), then b satisfies condition (E13) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)} - 1, x_4^{(2)} - 1, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)}) = \Omega(\tilde{e}_0(b))$.
- If x satisfies condition (E14), then b satisfies condition (E14) and $\tilde{e}_0(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} - 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)} - 1, x_6^{(2)}, x_4^{(2)} - 1, x_5^{(1)} - 1, x_1^{(1)} - 1, x_2^{(1)} - 1, x_3^{(1)} - 1, x_4^{(1)} - 1, x_6^{(1)} - 1) = \Omega(\tilde{e}_0(b))$.
- $\tilde{e}_1(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)} + 1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_1(b))$.
- If $x_2^{(2)} + x_2^{(1)} \geq x_1^{(1)} + x_3^{(2)}$, then $b_{11} + b_{12} + b_{22} \geq b_{11} + b_{22} + b_{23}$, hence $\tilde{e}_2(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)} + 1, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_2(b))$.
- If $x_2^{(2)} + x_2^{(1)} < x_1^{(1)} + x_3^{(2)}$, then $b_{11} + b_{12} + b_{22} < b_{11} + b_{22} + b_{23}$, hence $\tilde{e}_2(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)} + 1, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_2(b))$.
- If $x_3^{(3)} + x_3^{(2)} \geq x_2^{(2)} + x_4^{(3)}$ and $x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} \geq x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{22} + b_{23} \geq b_{11} + b_{12} + b_{22} + b_{23} + b_{24}$ and $b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{33} \geq b_{11} + b_{12} + 2b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{e}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)} + 1, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_3(b))$.
- If $x_3^{(3)} + x_3^{(2)} < x_2^{(2)} + x_4^{(3)}$ and $x_3^{(2)} + x_3^{(1)} \geq x_2^{(1)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{22} + b_{23} < b_{11} + b_{12} + b_{22} + b_{23} + b_{24}$ and $b_{22} + b_{23} + b_{33} \geq b_{22} + b_{33} + b_{34}$, hence $\tilde{e}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_3(b))$.

- If $x_3^{(3)} + 2x_3^{(2)} + x_3^{(1)} < x_2^{(2)} + x_2^{(1)} + x_4^{(3)} + x_4^{(2)}$ and $x_3^{(2)} + x_3^{(1)} < x_2^{(1)} + x_4^{(2)}$, then $b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{33} < b_{11} + b_{12} + 2b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$ and $b_{22} + b_{23} + b_{33} < b_{22} + b_{33} + b_{34}$, hence $\tilde{e}_3(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_3(b))$.
- If $x_4^{(4)} + x_4^{(3)} \geq x_3^{(3)} + x_5^{(2)}$, $x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} \geq x_3^{(3)} + x_5^{(2)} + x_6^{(2)}$, and $x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \geq x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} \geq b_{11} + b_{12} + b_{13} + b_{22} + b_{23} + b_{24} + b_{25}$, $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + b_{33} \geq b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35}$, $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + 2b_{33} + b_{34} + b_{35} + b_{36} \geq b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35} + b_{36} + b_{44} + b_{45} + b_{46}$, hence $e_4(x) = (x_6^{(3)}, x_4^{(4)} + 1, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_4(b))$.
- If $x_4^{(4)} + x_4^{(3)} < x_3^{(3)} + x_5^{(2)}$, $x_4^{(3)} + x_4^{(2)} \geq x_3^{(2)} + x_6^{(2)}$, and $x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} \geq x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_6^{(2)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} < b_{11} + b_{12} + b_{13} + b_{22} + b_{23} + b_{24} + b_{25}$, $b_{22} + b_{23} + b_{24} + b_{33} + b_{34} \geq b_{22} + b_{23} + b_{33} + b_{34} + b_{35}$, and $b_{22} + b_{23} + b_{24} + 2b_{33} + 2b_{34} + b_{44} + b_{45} \geq b_{22} + b_{23} + 2b_{33} + b_{34} + b_{35} + b_{44} + b_{45}$, hence $e_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_4(b))$.
- If $x_4^{(4)} + 2x_4^{(3)} + x_4^{(2)} < x_3^{(3)} + x_3^{(2)} + x_5^{(2)} + x_6^{(2)}$, $x_4^{(3)} + x_4^{(2)} < x_3^{(2)} + x_6^{(2)}$, and $x_4^{(2)} + x_4^{(1)} \geq x_3^{(1)} + x_5^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + b_{33} + b_{34} < b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35}$, $b_{22} + b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35} + b_{36} \geq b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{35} + b_{44} + b_{45}$, hence $e_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_4(b))$.
- If $x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} < x_3^{(3)} + x_3^{(2)} + x_3^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(2)}$, $x_4^{(3)} + 2x_4^{(2)} + x_4^{(1)} < x_3^{(1)} + x_5^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + 2b_{22} + 2b_{23} + 2b_{24} + b_{25} + b_{33} + b_{34} < b_{11} + b_{12} + b_{13} + 2b_{22} + 2b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35} + b_{36}$, $b_{22} + b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35} + b_{36} + b_{44} + b_{45} + b_{46}$, $b_{22} + b_{23} + b_{24} + b_{25} + b_{33} + b_{34} + b_{35} + b_{36} + b_{44} + b_{45} + b_{46} < b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{35} + b_{36} + b_{44} + b_{45} + b_{46}$, hence $e_4(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_4(b))$.
- If $x_5^{(2)} + x_5^{(1)} \geq x_4^{(3)} + x_4^{(2)}$, then $b_{22} + b_{23} + b_{24} + b_{25} + b_{44} + b_{45} \geq b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{e}_5(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)} + 1, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_5(b))$.
- If $x_5^{(2)} + x_5^{(1)} < x_4^{(3)} + x_4^{(2)}$, then $b_{22} + b_{23} + b_{24} + b_{25} + b_{44} + b_{45} < b_{22} + b_{23} + b_{24} + b_{33} + b_{34}$, hence $\tilde{e}_5(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_5(b))$.

- If $x_6^{(3)} + x_6^{(2)} \geq x_4^{(4)} + x_4^{(3)}$ and $x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} \geq x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + b_{33} + b_{34} + b_{35} \geq b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24}$ and $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + 2b_{33} + 2b_{34} + 2b_{35} + b_{55} \geq b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{44}$, hence $\tilde{e}_6(x) = (x_6^{(3)} + 1, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_6(b))$.
- If $x_6^{(3)} + x_6^{(2)} < x_4^{(4)} + x_4^{(3)}$ and $x_6^{(2)} + x_6^{(1)} \geq x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + b_{33} + b_{34} + b_{35} < b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24}$ and $b_{33} + b_{34} + b_{35} + b_{55} \geq b_{33} + b_{34} + b_{44}$, hence $\tilde{e}_6(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_6(b))$.
- If $x_6^{(3)} + 2x_6^{(2)} + x_6^{(1)} < x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)}$ and $x_6^{(2)} + x_6^{(1)} < x_4^{(2)} + x_4^{(1)}$, then $b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + 2b_{33} + 2b_{34} + 2b_{35} + b_{55} < b_{11} + b_{12} + b_{13} + b_{14} + b_{22} + b_{23} + b_{24} + b_{33} + b_{34} + b_{44}$ and $b_{33} + b_{34} + b_{35} + b_{55} < b_{33} + b_{34} + b_{44}$, hence $\tilde{e}_6(x) = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)} + 1, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)}) = \Omega(\tilde{e}_6(b))$.

Therefore $\Omega(\tilde{e}_k(b)) = \tilde{e}_k(\Omega(b))$ for all $0 \leq k \leq 6$. We also have

- $\text{wt}_0(x) = -x_2^{(2)} - x_2^{(1)} = -b_{11} - b_{12} - b_{22} = -b_{11} - b_{12} + b_{23} + b_{24} + b_{25} + b_{27} = \text{wt}_0(b)$.
- $\text{wt}_1(x) = 2x_1^{(1)} - x_2^{(2)} - x_2^{(1)} = 2b_{11} - b_{11} - b_{12} - b_{22} = b_{11} - b_{12} - b_{22} = \text{wt}_1(b)$.
- $\begin{aligned} \text{wt}_2(x) &= -x_1^{(1)} + 2x_2^{(2)} + 2x_2^{(1)} - x_3^{(3)} - x_3^{(2)} - x_3^{(1)} \\ &= -b_{11} + 2b_{11} + 2b_{12} + 2b_{22} - b_{11} - b_{12} - b_{13} - b_{22} - b_{23} - b_{33} \\ &= b_{12} - b_{13} + b_{22} - b_{23} - b_{33} = \text{wt}_2(b). \end{aligned}$
- $\begin{aligned} \text{wt}_3(x) &= -x_2^{(2)} - x_2^{(1)} + 2x_3^{(3)} + 2x_3^{(2)} + 2x_3^{(1)} - x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)} \\ &= -b_{11} - b_{12} - b_{22} + 2b_{11} + 2b_{12} + 2b_{13} + 2b_{22} + 2b_{23} + 2b_{33} - b_{11} - b_{12} - b_{13} - b_{14} - b_{22} \\ &\quad - b_{23} - b_{24} - b_{33} - b_{34} - b_{44} \\ &= b_{13} - b_{14} + b_{23} - b_{24} + b_{33} - b_{34} - b_{44} = \text{wt}_3(b). \end{aligned}$
- $\begin{aligned} \text{wt}_4(x) &= -x_3^{(3)} - x_3^{(2)} - x_3^{(1)} + 2x_4^{(4)} + 2x_4^{(3)} + 2x_4^{(2)} + 2x_4^{(1)} - x_5^{(2)} - x_5^{(1)} - x_6^{(3)} - x_6^{(2)} - x_6^{(1)} \\ &= -b_{11} - b_{12} - b_{13} - b_{22} - b_{23} - b_{33} + 2b_{11} + 2b_{12} + 2b_{13} + 2b_{14} + 2b_{22} + 2b_{23} + 2b_{24} \\ &\quad + 2b_{33} + 2b_{34} + 2b_{44} - b_{22} - b_{23} - b_{24} - b_{25} - b_{44} - b_{45} - b_{11} - b_{12} - b_{13} - b_{14} - b_{15} - b_{33} \\ &\quad - b_{34} - b_{35} - b_{55} \\ &= b_{14} - b_{15} + b_{24} - b_{25} + b_{34} - b_{35} + b_{44} - b_{45} - b_{55} = \text{wt}_4(b). \end{aligned}$
- $\begin{aligned} \text{wt}_5(x) &= -x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)} + 2x_5^{(2)} + 2x_5^{(1)} \\ &= -b_{11} - b_{12} - b_{13} - b_{14} - b_{22} - b_{23} - b_{24} - b_{33} - b_{34} - b_{44} + 2b_{22} + 2b_{23} + 2b_{24} + 2b_{25} \\ &\quad + 2b_{44} + 2b_{45} \\ &= -b_{11} - b_{12} - b_{13} - b_{14} + b_{22} + b_{23} + b_{24} + 2b_{25} - b_{33} - b_{34} + b_{44} + 2b_{45} = \text{wt}_5(b). \end{aligned}$

- $$\begin{aligned}
\text{wt}_6(x) &= -x_4^{(4)} - x_4^{(3)} - x_4^{(2)} - x_4^{(1)} + 2x_6^{(3)} + 2x_6^{(2)} + 2x_6^{(1)} \\
&= -b_{11} - b_{12} - b_{13} - b_{14} - b_{22} - b_{23} - b_{24} - b_{33} - b_{34} - b_{44} + 2b_{11} + 2b_{12} + 2b_{13} + 2b_{14} \\
&\quad + 2b_{15} + 2b_{33} + 2b_{34} + 2b_{35} + 2b_{55} \\
&= b_{11} + b_{12} + b_{13} + b_{14} + 2b_{15} - b_{22} - b_{23} - b_{24} + b_{33} + b_{34} + 2b_{35} - b_{44} + 2b_{55} \\
&= b_{11} + b_{12} + b_{13} + b_{14} + 2b_{15} - b_{22} - b_{23} - b_{24} + b_{33} + b_{34} + 2b_{35} - b_{44} + b_{55} - b_{56} - b_{57} \\
&\quad - b_{58} - b_{59} - b_{5,10} \\
&= \text{wt}_6(b).
\end{aligned}$$

Hence, $\text{wt}_k(\Omega(b)) = \text{wt}_k(b)$ for all $0 \leq k \leq 6$. Also,

- $$\begin{aligned}
\varepsilon_0(x) &= \max\{x_6^{(1)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} - x_3^{(1)} + x_5^{(1)}, x_3^{(1)} + x_2^{(2)} - x_4^{(2)} - x_3^{(3)} + x_5^{(1)}, x_2^{(2)} - x_3^{(3)} \\
&\quad + x_4^{(1)}, x_3^{(2)} + x_3^{(1)} - x_4^{(3)} - x_4^{(2)} + x_5^{(1)}, x_3^{(2)} - x_4^{(3)} + x_4^{(1)}, x_4^{(2)} + x_4^{(1)} - x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_6^{(3)}, \\
&\quad x_2^{(2)} + x_2^{(1)} - x_4^{(4)} - x_4^{(3)} + x_6^{(2)}, x_2^{(2)} + x_2^{(1)} - x_3^{(2)} + x_4^{(2)} - x_4^{(4)}, x_2^{(2)} + x_2^{(1)} - x_3^{(3)} - x_3^{(2)} \\
&\quad + x_4^{(3)} + x_4^{(2)} - x_5^{(2)}, x_2^{(2)} + x_3^{(1)} - x_4^{(4)}, x_2^{(2)} + x_3^{(1)} - x_3^{(3)} + x_4^{(2)} - x_5^{(2)}, x_3^{(2)} + x_3^{(1)} - x_5^{(2)}\} \\
&= \max\{-b_{56} - b_{57} - b_{58} - b_{59} - b_{5,10}, -b_{13} - b_{23} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{34} - b_{46} \\
&\quad - b_{47} - b_{48} - b_{49}, -b_{13} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{24} - b_{34} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{24} \\
&\quad - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{35} - b_{45} - b_{46} - b_{47} - b_{48} - b_{49}, -b_{13} - b_{14} - b_{15} - b_{23} - b_{24} \\
&\quad - b_{25} - b_{26} - b_{27}, -b_{13} - b_{14} - b_{23} - b_{24} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{14} - b_{23} - b_{35} - b_{36} - b_{37} \\
&\quad - b_{38}, -b_{13} - b_{23} - b_{25} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} - b_{14} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{13} \\
&\quad - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}, -b_{24} - b_{25} - b_{34} - b_{35} - b_{36} - b_{37} - b_{38}\} \\
&= \varepsilon_0(b).
\end{aligned}$$
- $$\varepsilon_1(x) = x_2^{(2)} - x_1^{(1)} = b_{12} = \varepsilon_1(b).$$
- $$\begin{aligned}
\varepsilon_2(x) &= \max\{-x_2^{(2)} + x_3^{(3)}, x_1^{(1)} - 2x_2^{(2)} - x_2^{(1)} + x_3^{(3)} + x_3^{(2)}\} \\
&= \max\{b_{13}, -b_{12} + b_{13} + b_{23}\} = \varepsilon_2(b).
\end{aligned}$$
- $$\begin{aligned}
\varepsilon_3(x) &= \max\{-x_3^{(3)} + x_4^{(4)}, x_2^{(2)} - 2x_3^{(3)} - x_3^{(2)} + x_4^{(4)} + x_4^{(3)}, x_2^{(2)} + x_2^{(1)} - 2x_3^{(3)} - 2x_3^{(2)} - x_3^{(1)} \\
&\quad + x_4^{(4)} + x_4^{(3)} + x_4^{(2)}\} \\
&= \max\{b_{14}, -b_{13} + b_{14} + b_{24}, -b_{13} + b_{14} - b_{23} + b_{24} + b_{34}\} = \varepsilon_3(b).
\end{aligned}$$
- $$\begin{aligned}
\varepsilon_4(x) &= \max\{-x_4^{(4)} + x_6^{(3)}, x_3^{(3)} - 2x_4^{(4)} - x_4^{(3)} + x_5^{(2)} + x_6^{(3)}, x_3^{(3)} + x_3^{(2)} - 2x_4^{(4)} - 2x_4^{(3)} - x_4^{(2)} \\
&\quad + x_5^{(2)} + x_6^{(3)} + x_6^{(2)}, x_3^{(3)} + x_3^{(2)} + x_3^{(1)} - 2x_4^{(4)} - 2x_4^{(3)} - 2x_4^{(2)} - x_4^{(1)} + x_5^{(2)} + x_5^{(1)} + x_6^{(3)} \\
&\quad + x_6^{(2)}\} \\
&= \max\{b_{15}, -b_{14} + b_{15} + b_{25}, -b_{14} + b_{15} - b_{24} + b_{25} + b_{35}, -b_{14} + b_{15} - b_{24} + b_{25} - b_{34} \\
&\quad + b_{35} + b_{45}\} \\
&= \varepsilon_4(b).
\end{aligned}$$
- $$\begin{aligned}
\varepsilon_5(x) &= \max\{x_4^{(4)} - x_5^{(2)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} - 2x_5^{(2)} - x_5^{(1)}\} \\
&= \max\{b_{11} + b_{12} + b_{13} + b_{14} - b_{22} - b_{23} - b_{24} - b_{25}, b_{11} + b_{12} + b_{13} + b_{14} - b_{22} - b_{23} - b_{24}\}
\end{aligned}$$

- $$\begin{aligned}
& - 2b_{25} + b_{33} + b_{34} - b_{44} - b_{45} \} \\
& = \varepsilon_5(b). \\
\bullet \quad \varepsilon_6(x) & = \max\{-x_6^{(3)}, x_4^{(4)} + x_4^{(3)} - 2x_6^{(3)} - x_6^{(2)}, x_4^{(4)} + x_4^{(3)} + x_4^{(2)} + x_4^{(1)} - 2x_6^{(3)} - 2x_6^{(2)} - x_6^{(1)}\} \\
& = \max\{-b_{11} - b_{12} - b_{13} - b_{14} - b_{15}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} + b_{22} + b_{23} + b_{24} - b_{33} \\
& \quad - b_{34} - b_{35}, -b_{11} - b_{12} - b_{13} - b_{14} - 2b_{15} + b_{22} + b_{23} + b_{24} - b_{33} - b_{34} - 2b_{35} + b_{44} - b_{55}\} \\
& = \varepsilon_6(b).
\end{aligned}$$

Thus $\varepsilon_k(\Omega(b)) = \varepsilon_k(b)$ for $0 \leq k \leq 6$ which completes the proof. \square

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