

ABSTRACT

YADAVALLI, ANILA. Darboux Transformations and Fay Identities of the Extended Bigraded Toda Hierarchy. (Under the direction of Bojko Bakalov).

An integrable hierarchy is an infinite set of nonlinear evolution equations that can be solved simultaneously. They arise by starting with a base differential equation that models a physical process and then adding evolution equations that are symmetries of it and each other. Some classical examples are the Kadomstev-Petviashvili (KP), Korteweg-de Vries (KdV), and Toda hierarchies. The extended Toda hierarchy (ETH), was introduced as a hierarchy that governs the Gromov-Witten invariants of $\mathbb{C}P^1$. Its generalization, the extended bigraded Toda hierarchy (EBTH), was introduced as a hierarchy that would encode the relations between the Gromov-Witten invariants of some $\mathbb{C}P^1$ orbifolds.

A special feature of integrable hierarchies is that there exist soliton solutions, or solutions whose motion can be described by (solitary) waves that maintain their shape as they travel through a narrow channel. An n -soliton solution is expressed as a polynomial in exponentials with n distinct exponents. Darboux transformations were introduced in 1882 as an algebraic method of generating new solutions to nonlinear evolution equations from existing ones, and can be used to find soliton solutions to integrable hierarchies.

We begin by reviewing some standard but nontrivial facts about integrable hierarchies in general. In particular we will review the KP hierarchy and its Darboux transformations and Fay identities. Next, we introduce the EBTH in Lax form. We write a bilinear equation for the EBTH in terms of the tau function using notation that is more convenient than that of the existing bilinear equation and requires a shorter proof. We use the bilinear equation to write two Fay identities for the EBTH. We then review Darboux transformations on the EBTH and show that the action of the Darboux transformations on a tau function of the EBTH is given by certain vertex operators. Finally, we use the aforementioned vertex operators and bilinear equation to provide generalized Fay identities for the EBTH.

© Copyright 2019 by Anila Yadavalli

All Rights Reserved

Darboux Transformations and Fay Identities of the Extended Bigraded Toda Hierarchy

by
Anila Yadavalli

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2019

APPROVED BY:

Irina Kogan

Kailash Misra

Ernest Stitzinger

Bojko Bakalov
Chair of Advisory Committee

DEDICATION

To Samir.

BIOGRAPHY

The author was born in Milwaukee, WI where her grandfather was a mathematics professor at Marquette University. She was raised in San Jose, CA and graduated from University of California, San Diego in 2012 with a Bachelor's of Science in mathematics. Upon graduation, she served as an Indicorps Fellow in Ahmedabad, Gujarat, India for one year. In January 2014, she moved to Raleigh, NC to pursue doctoral studies in mathematics at North Carolina State University. She enjoys yoga, indoor rock climbing, hiking, and reading. At the time of this writing, she looks forward to a teaching postdoctoral appointment in the School of Mathematics at University of Minnesota.

ACKNOWLEDGEMENTS

I have many people to thank, without whom I would not have been able to reach this point.

First and foremost, to my advisor, Bojko Bakalov, thank you for your constant support, always encouraging me to work hard towards my goals, and pushing me to become a better mathematician. Your mentorship has helped me grow in many ways, and I will always value the things I have learned from you.

To my committee members, Irina Kogan, Kailash Misra, and Ernest Stitzinger thank you for introducing me to several new and interesting areas of mathematics as well as providing me with guidance on various aspects of graduate school.

To the other faculty members who have served as mentors to me, Alina Duca, Molly Fenn, and Cynthia Vinzant, thank you for being inspiring examples of women in mathematics and lifting me up so I could reach so many of my goals.

To my parents and grandparents, thank you for always encouraging me to choose my own path, instilling in me the importance of education, and taking interest in my work, regardless of how foreign it may have been.

To Vamsi, Paula and Anika Yadavalli, thank you for being my home away from home. Your tough love, thoughtful advice, and home-cooked meals have truly gotten me through many of the ups and downs of graduate school.

To Michael Ruddy, thank you for teaching me how to acknowledge my strengths and convincing me to become a confident person—a very difficult task, but you never gave up! You never failed to support me and make me laugh on top of juggling your own commitments as a doctoral student, and for that I will always be grateful.

TABLE OF CONTENTS

Chapter 1 Introduction and Preliminaries	1
1.1 Introduction	1
1.2 Vertex Operators	3
1.3 Darboux Transformations	5
1.3.1 Darboux transformations for a differential operator	5
1.3.2 The Original Darboux Transformation	7
1.3.3 Lax Pairs and the KdV Equation	9
Chapter 2 The KP Hierarchy	11
2.1 Definition of the KP hierarchy	11
2.1.1 Pseudo-differential operators	11
2.1.2 The KdV hierarchy	13
2.1.3 Lax form of the KP hierarchy	13
2.1.4 Wave functions and bilinear equation	14
2.1.5 Tau function	16
2.2 Fay identity for the KP Hierarchy	17
2.3 Darboux Transformations of the KP Hierarchy	18
2.3.1 Generalized Fay identities	20
Chapter 3 The Extended Bigraded Toda Hierarchy (EBTH)	22
3.1 Definition of the EBTH	22
3.1.1 Spaces of difference and differential-difference operators	22
3.1.2 Definition of the EBTH	23
3.2 Bilinear equation for the EBTH	28
3.2.1 Dual wave functions	28
3.2.2 Bilinear equation for the wave functions	30
3.2.3 Bilinear equation for the tau-function	36
3.3 Two difference Fay identities for the EBTH	39
Chapter 4 Darboux Transformations of the EBTH	43
4.1 Darboux Transformations of the EBTH	43
4.1.1 Factoring difference operators	43
4.1.2 Darboux transformations of L and ψ	45
4.1.3 Action of Darboux transformations on τ	52
4.2 Generalized Fay identities	57
Chapter 5 Conclusion	60
5.1 Conclusion	60
5.2 Future Directions	60
5.2.1 Tau function for the EBTH	60
5.2.2 Equivalence of Fay identities to ETH and EBTH	61
5.2.3 W -Algebras	61
5.2.4 Bispectral problem	61

5.2.5	Cluster Algebras	61
5.2.6	Random Matrices	62
	References	63

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Integrable hierarchies arise by starting with a differential equation that models a physical process and constructing a system of infinitely many symmetries of the equation. They admit *soliton solutions*, or solutions whose motion can be described by a (solitary) wave that maintains its shape as it travels through a narrow canal. Some classical examples of integrable hierarchies are the KP, KdV and Toda hierarchies. Such systems can be studied using methods from representation theory [34], combinatorics [6, 48], and algebraic geometry [48]. Studying integrable hierarchies has led to algebraic methods of finding soliton solutions to nonlinear differential equations and has also led to numerous applications in other areas of mathematics such as random matrices and representation theory.

The main focus of this thesis is the use of Darboux transformations to generate soliton solutions of the *extended Toda hierarchy* (ETH) and the *extended bigraded Toda hierarchy* (EBTH). The Toda hierarchies arise from the Lax form of the Toda lattice equation [59] which describes the motion of charged particles on an infinite lattice. In 1984, Ueno and Takasaki introduced the 2D Toda hierarchy and its reduction, the 1D Toda hierarchy [60]. In 1986, ten Kroode and Bergvelt used the representation theory of $sl_2(\mathbb{C})$ to show that the KdV and 1D Toda hierarchies were closely related on an algebraic level [57].

The *extended Toda hierarchy* (ETH) was originally introduced by Getzler [30] and Zhang [63] in its bihamiltonian form, and later in its Lax form by Carlet, Dubrovin and Zhang [16].

It is obtained by adding an extra set of commuting flows to the 1D Toda hierarchy, which are given in terms of a “logarithm” of the Lax operator. It was shown in [30, 24, 45, 50] that the Gromov–Witten total descendant potential of \mathbb{CP}^1 is a tau-function of the ETH.

The *extended bigraded Toda hierarchy* (EBTH) was introduced by Carlet [15] as a generalization of the extended Toda hierarchy related to the Frobenius manifolds from [23]. The EBTH is defined for every pair (k, m) of positive integers, and it coincides with the ETH for $k = m = 1$. The total descendant potential of \mathbb{CP}^1 with two orbifold points of orders k and m is a tau-function of the EBTH (see [47, 18]). The EBTH contains the bigraded Toda hierarchy, which is a reduction of the 2D Toda hierarchy (see [54, 60]).

Various applications of the Toda hierarchies, including matrix factorization problems for $\mathbb{Z} \times \mathbb{Z}$ matrices and a connection between solutions to the Toda hierarchies and partition functions for melting crystal models are discussed in [55].

In this thesis, we investigate how Darboux transformations of the EBTH affect the tau-function. We start by using the approach of Takasaki [54] to derive a *bilinear equation* for the EBTH, which is equivalent to the one from [18] after a change of variables. From this we obtain a difference Fay identity, similar to what was done in [53, 58] for the the 2D Toda hierarchy. Some Fay identities for the EBTH were given in [41], but writing the Fay identity in our notation allows us to study the action of Darboux transformations on the tau-function. In [14], Carlet defined *Darboux transformations* on the wave functions for the ETH, and in [42], Li and Song generalized them to the EBTH.

Our main result is that the action of Darboux transformations on the tau-function is given by applying the vertex operators

$$\Gamma_+(z) = e^{-\partial_s} \exp\left(\sum_{j=1}^{\infty} t_j z^j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{\partial t_j}{j} z^{-j}\right)$$

and

$$\Gamma_-(z) = z^s e^{\partial_s} \exp\left(-\sum_{j=1}^{\infty} \bar{t}_j z^{-j}\right) \exp\left(\sum_{j=1}^{\infty} \frac{\partial \bar{t}_j}{j} z^j\right).$$

Thus, new tau-functions for the EBTH can be obtained from existing ones by applying a product of Γ_+ and Γ_- evaluated at different values $z_i \in \mathbb{C}^*$. As an application, we derive *generalized Fay identities* for the EBTH.

The outline of this thesis is as follows.

In the remainder of Chapter 1, we provide some background on vertex operators and Darboux transformations which will be important for stating the main results of this thesis.

In Chapter 2, we define pseudo-differential operators and use the KP and KdV hierarchies as examples to introduce the concept of an integrable hierarchy. We define the wave function, wave

operators, and tau function for the KP hierarchy. Then, following [61], we review Fay identities and Darboux transformations for the KP hierarchy.

In Chapter 3, we define the main subject of this thesis, the extended bigraded Toda hierarchy (EBTH). We give an explicit bilinear equation for the EBTH that is equivalent to the one from [18], in the notation introduced by Takasaki. We provide a shorter proof than what was done in [18]. From the bilinear equation written in this form, we get two difference Fay identities satisfied by the tau function of the EBTH (cf. [53]). This is similar to what was done in [41], but we are following Takasaki's notation, and our proof is shorter.

In Chapter 4 we review the Darboux transformations on L and ψ from [42]. We apply results from [13] to factoring difference operators, and use this to provide an alternate proof of [42], Theorem 3.4. Then we show the main result of this thesis: the action of the Darboux transformations on the tau-function is given by the vertex operators $\Gamma_+(z)$ and $\Gamma_-(z)$. This result is new even in the case $k = m = 1$ corresponding to the extended Toda hierarchy. We use it to conclude that new tau-functions can be found by acting on an existing tau-function τ with a product of $\Gamma_+(z_i)$ and $\Gamma_-(z_i)$ for certain $z_i \in \mathbb{C}^*$. As an application, we derive generalized Fay identities for the EBTH.

Finally, Chapter 5 contains concluding remarks and future directions.

1.2 Vertex Operators

The main results presented in this thesis rely on the use of *vertex operators* to produce new solutions to integrable hierarchies starting from known ones. Vertex operators were discovered by physicists in string theory, but they appear in the representation theory of infinite dimensional Lie algebras and as an application, the theory of integrable hierarchies (see [34, Chapter 14]). In general, a vertex operator is an operator of the form

$$\Gamma(\mathbf{a}, \mathbf{b}) = \exp\left(\sum_{j=1}^{\infty} a_j x_j\right) \exp\left(-\sum_{j=1}^{\infty} b_j \partial_{x_j}\right)$$

where $\mathbf{a} = (a_1, a_2, a_3, \dots)$, $\mathbf{b} = (b_1, b_2, b_3, \dots)$ and $\partial_{x_j} = \frac{\partial}{\partial x_j}$. Using the property that

$$e^{c\partial_x} f(x) = f(x + c),$$

we have that the action of the vertex operator $\Gamma(\mathbf{a}, \mathbf{b})$ above on a function $f(\mathbf{x}) = f(x_1, x_2, x_3, \dots)$ is

$$\Gamma(\mathbf{a}, \mathbf{b})f(\mathbf{x}) = \exp\left(\sum_{j=1}^{\infty} a_j x_j\right) f(x_1 - b_1, x_2 - b_2, x_3 - b_3, \dots).$$

The product of two vertex operators is itself a vertex operator. To multiply two vertex operators, we will use the well known Campbell-Baker-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]}$$

for operators A and B such that $[A, B]$ commutes with A and B . Here, $[A, B]$ refers to the commutator bracket

$$[A, B] = AB - BA.$$

We will also use the property of formal power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n}{n} = \log(1 + a)$$

to write

$$\exp\left(-\sum_{j=1}^{\infty} \frac{a^j}{j}\right) = \exp(\log(1 - a)) = 1 - a. \quad (1.2.1)$$

Example 1.2.1. *Let*

$$\begin{aligned} \Gamma_+(z) &= \exp\left(\sum_{j=1}^{\infty} z^j x_j\right) \exp\left(-2 \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \partial_{x_j}\right) \\ \Gamma_-(z) &= \exp\left(-\sum_{j=1}^{\infty} z^j x_j\right) \exp\left(2 \sum_{j=1}^{\infty} \frac{\partial_{x_j}}{j} z^{-j}\right). \end{aligned}$$

The product $\Gamma_-(z_1)\Gamma_+(z_2)$ is defined for $z_1 \neq z_2$ and is given by

$$\begin{aligned} &\Gamma_-(z_1)\Gamma_+(z_2) \\ &= \exp\left(-\sum_{j=1}^{\infty} z_1^j x_j\right) \exp\left(2 \sum_{j=1}^{\infty} \frac{\partial_{x_j}}{j} z_1^{-j}\right) \\ &\quad \times \exp\left(\sum_{j=1}^{\infty} z_2^j x_j\right) \exp\left(-2 \sum_{j=1}^{\infty} \frac{\partial_{x_j}}{j} z_2^{-j}\right) \\ &= \left(\frac{1}{1 - z_2 z_1^{-1}}\right)^2 \exp\left(-\sum_{j=1}^{\infty} z_1^j x_j\right) \exp\left(\sum_{j=1}^{\infty} z_2^j x_j\right) \\ &\quad \times \exp\left(2 \sum_{j=1}^{\infty} \frac{\partial_{x_j}}{j} z_1^{-j}\right) \exp\left(-2 \sum_{j=1}^{\infty} \frac{\partial_{x_j}}{j} z_2^{-j}\right) \end{aligned}$$

We can see that because of the appearance of the term $\left(\frac{1}{1 - z_2 z_1^{-1}}\right)^2$, this product is not defined when $z_1 = z_2$.

When writing vertex operators, or differential operators in general, we use *normal ordering*, in which all differentiation operators appear on the right and multiplication operators appear on the left. The notation $: p :$ is used to denote the normal product. For example,

$$: x_n \partial_{x_n} : = : \partial_{x_n} x_n : = x_n \partial_{x_n}$$

Normal ordering is important in ensuring that a product of operators is well defined. For instance, consider the Euler operator $\sum_{n=1}^{\infty} n x_n \partial_{x_n}$ acting on weighted homogenous polynomials $f(\mathbf{x})$, where $\deg(x_n) = n$. Then

$$\left(\sum_{n=1}^{\infty} n x_n \partial_{x_n}\right) f(\mathbf{x}) = \deg(f) f(\mathbf{x}).$$

However, if we apply instead the operator $\sum_{n=1}^{\infty} n \partial_{x_n} x_n$ to $f(\mathbf{x})$, using the product rule from calculus, we get

$$\left(\sum_{n=1}^{\infty} n \partial_{x_n} x_n\right) f(\mathbf{x}) = \left(\sum_{n=1}^{\infty} n (1 + x_n \partial_{x_n})\right) f(\mathbf{x}).$$

In the case, of $f(\mathbf{x}) = 1$, this gives ∞ , so we see that this operator is not well defined.

1.3 Darboux Transformations

Another important tool used in this thesis are *Darboux transformations*. Darboux transformations were originally introduced by Darboux in 1882 and later generalized by Crum. Matveev and Salle realized that Darboux's theorem could be applied to nonlinear differential equations as a completely algebraic tool to generate new solutions starting from known ones. In this section, we provide an overview of Darboux transformations and then review the original Darboux transformations and its generalization following [44].

1.3.1 Darboux transformations for a differential operator

Classically, a *Darboux transformation* for a differential operator L is defined by factoring $L = QR$ and defining a new differential operator $L^{[1]}$ by switching the two factors: $L^{[1]} = RQ$. Notice that if Ψ is an eigenfunction for L , then

$$L^{[1]}(R\Psi) = RQ(R\Psi) = R(QR\Psi) = R\lambda\Psi = \lambda R\Psi$$

so that the new function $\Psi^{[1]} = R\Psi$ is an eigenfunction for $L^{[1]}$.

The following lemma gives a condition for when such a factorization is possible (see e.g. [38]).

Lemma 1.3.1. *A differential operator L can be factorized as $L = QR$ if and only if $\ker(R) \subset \ker(L)$.*

We provide a sketch of the proof here since it will be useful in later sections. For more details, see [38, Section 5.4].

Proof. If $L = QR$ and $R\psi = 0$, then clearly $L\psi = QR\psi = Q(0) = 0$. On the other hand, we will suppose $R\psi = 0$ implies $L\psi = 0$. Suppose the order of L is n and the order of R is m (where $m < n$). Then we can find some Q and P such that

$$L = QR + P$$

where Q is of order $n - m$ and P is of order at most $m - 1$. Suppose P is not identically zero. By assumption, every $\psi \in \ker(R)$ must also satisfy $P\psi = 0$, so that P has at least m linearly independent solutions. This is a contradiction since P can have at most $m - 1$ linearly independent solutions. \square

Such an R can be found using the Wronskian determinant

$$\text{Wr}(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Suppose Ψ is a function satisfying $L\Psi = \lambda\Psi$. Then letting $\Psi_1 = \Psi(x, \lambda) \Big|_{\lambda=\lambda_1}$ for some fixed λ_1 , we define R by

$$Rf = \begin{vmatrix} \Psi_1 & f \\ \Psi_1' & f' \end{vmatrix} = \text{Wr}(\Psi_1, f).$$

Since $\Psi_1 \in \ker R$ and $\Psi_1 \in \ker(L - \lambda_1 I)$, we can write $L - \lambda_1 I = QR$, where $Q = (L - \lambda_1 I)R^{-1}$. Switching these two factors, we get

$$RQ = R(L - \lambda_1 I)R^{-1} = RLR^{-1} - \lambda_1 I = L^{[1]} - \lambda_1 I,$$

and

$$L^{[1]}\Psi^{[1]} = RLR^{-1}R\Psi = R\lambda\Psi = \lambda\Psi^{[1]}.$$

This process can be repeated N times to obtain a new $L^{[N]}$ and $\Psi^{[N]}$ satisfying $L^{[N]}\Psi^{[N]} = \lambda\Psi^{[N]}$. A formula for $L^{[N]}$ and $\Psi^{[N]}$ can be found in terms of the Wronskian of the original solution Ψ

evaluated at various values λ_i (see Theorem 1.3.4 below).

1.3.2 The Original Darboux Transformation

We continue using the notation from the previous subsection, and will let $u_x = \frac{\partial u}{\partial x}$. Suppose $u(x)$ and $\Psi(x, \lambda)$ satisfy the Sturm-Liouville equation

$$-\Psi_{xx} + u(x)\Psi = \lambda\Psi, \quad (1.3.1)$$

for some constant λ . The equation (1.3.1) is equivalent to the eigenvalue problem $L\Psi = \lambda\Psi$ if L is the *Schrödinger operator*

$$L = -\partial_x^2 + u(x). \quad (1.3.2)$$

Remark 1.3.2. *The coefficients of ∂_x and $u(x)$ in L can be made arbitrary by rescaling. Later, we will take both coefficients to be 1.*

The Darboux transformation on L and Ψ is defined explicitly by:

$$\begin{aligned} \Psi &\mapsto \Psi^{[1]} = \left(\frac{\partial}{\partial x} - \frac{\Psi_{1x}}{\Psi_1} \right) \Psi = \frac{\text{Wr}(\Psi_1, \Psi)}{\Psi_1}, \\ u &\mapsto u^{[1]} = u - 2 \frac{d^2}{dx^2} \ln \Psi_1, \\ L &\mapsto L^{[1]} = -\partial_x^2 + u^{[1]}. \end{aligned}$$

We can now state Darboux's theorem.

Theorem 1.3.3. *The function $\Psi^{[1]}$ satisfies the differential equation*

$$-\Psi_{xx}^{[1]} + u^{[1]}\Psi^{[1]} = \lambda\Psi^{[1]}. \quad (1.3.3)$$

The proof is by direct computation and can be found in [44]. A generalization of Darboux's theorem is obtained by applying the Darboux transformation to (1.3.1) N times for any $N \in \mathbb{Z}^{>0}$. For example, when $N = 2$, we obtain

$$\Psi^{[2]} = \left(\frac{d}{dx} - \frac{\Psi_{2x}^{[1]}}{\Psi_2^{[1]}} \right) \left(\frac{d}{dx} - \frac{\Psi_{1x}}{\Psi_1} \right) \Psi$$

where $\Psi_2^{[1]} = \Psi^{[1]}(x, \lambda) \Big|_{\lambda=\lambda_2}$ for some fixed λ_2 , and

$$u^{[2]} = u^{[1]} - 2 \frac{d^2}{dx^2} \ln (\text{Wr}(\Psi_1, \Psi_2)).$$

Then $\Psi^{[2]}$ and $u^{[2]}$ satisfy the differential equation

$$-\Psi_{xx}^{[2]} + u^{[2]}\Psi^{[2]} = \lambda\Psi^{[2]}.$$

An important fact about such Darboux transformations is that a formula for $\Psi^{[N]}$ can be obtained completely in terms of the initial solution Ψ evaluated at various values $\lambda_1, \lambda_2, \dots, \lambda_N$ (see e.g. [44]).

Theorem 1.3.4. *The function*

$$\Psi^{[N]} = \frac{\text{Wr}(\Psi_1, \Psi_2, \dots, \Psi_N, \Psi)}{\text{Wr}(\Psi_1, \Psi_2, \dots, \Psi_N)}$$

satisfies the differential equation

$$-\Psi_{xx}^{[N]} + u^{[N]}\Psi^{[N]} = \lambda\Psi^{[N]}, \quad (1.3.4)$$

with

$$u^{[N]} = u - 2\frac{d^2}{dx^2} \ln(\text{Wr}(\Psi_1, \Psi_2, \dots, \Psi_N)).$$

Proof. Let $R^{[k]}$ be the differential operator $\left(\frac{d}{dx} - \frac{\Psi_k'^{[k-1]}}{\Psi_k^{[k-1]}}\right)$, where prime denotes the derivative with respect to x . Then

$$\begin{aligned} \Psi^{[N]} &= R^{[N]} \dots R^{[2]} R^{[1]} \Psi \\ &= \Psi^{(N)} + s_{N-1} \Psi^{(N-1)} + \dots + s_0 \Psi. \end{aligned} \quad (1.3.5)$$

Observe that $R^{[k]} \dots R^{[2]} R^{[1]} \Psi_k = 0$; for example, when $N = 2$, we have

$$\begin{aligned} R^{[2]} R^{[1]} \Psi_2 &= R^{[2]}(R^{[1]} \Psi_2) \\ &= R^{[2]} \left(\Psi^{[1]} \Big|_{\lambda=\lambda_2} \right) \\ &= R^{[2]} \Psi_2^{[1]} \\ &= 0. \end{aligned}$$

Thus, we have that $R^{[N]} \dots R^{[2]} R^{[1]} \Psi_k = 0$ for every $k = 1, 2, \dots, N$. This fact, along with (1.3.5) can be used to obtain a system of linear algebraic equations of the form

$$\Psi_k^{(N)} + s_{N-1} \Psi_k^{(N-1)} + \dots + s_1 \Psi_k' + s_0 \Psi_k = 0$$

for $k = 1 \dots N$.

Cramer's rule can be used to explicitly find each coefficient s_i . The rest of the theorem is proved by substituting the expression (1.3.5) into (1.3.4) with the explicit s_i and solving for $u^{[N]}$. \square

1.3.3 Lax Pairs and the KdV Equation

To illustrate an application of how Darboux transformations can be applied to finding solutions to nonlinear differential equations, we consider the Korteweg-de Vries (KdV) equation which describes the motion of water waves in a narrow canal. For $u = u(x, t)$ and any constants c_1 and c_2 , the KdV equation is defined by:

$$\frac{\partial u}{\partial t} = c_1 u \frac{\partial u}{\partial x} + c_2 \frac{\partial^3 u}{\partial x^3}.$$

Again, we note that the constants can be rescaled arbitrarily, but for convenience, we will take $c_1 = \frac{3}{2}$ and $c_2 = \frac{1}{4}$. Direct computation shows that the KdV equation is equivalent to the *Lax equation*

$$\frac{\partial L}{\partial t} = [B, L] \tag{1.3.6}$$

where B and L are the operators

$$\begin{aligned} B &= \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x \\ L &= \partial_x^2 + u \end{aligned} \tag{1.3.7}$$

(see [48, Chapter 1]). We call B and L a *Lax pair* for the KdV equation. Again we can show through direct calculation that a solution ϕ to the system of equations

$$\begin{aligned} L\phi &= \lambda\phi \\ \phi_t &= B\phi \end{aligned} \tag{1.3.8}$$

satisfies $(\phi_{xx})_t = (\phi_t)_{xx}$ if and only if

$$\frac{\partial u}{\partial t} = \frac{3}{2}uu_x + \frac{1}{4}u_{3x}$$

(cf. [33, 48]). In other words, the KdV equation is the *compatibility condition* for the system (1.3.8). Observing that the first equation of (1.3.8) is equivalent to the Sturm-Liouville equation (1.3.1) for which we defined Darboux transformations, we can conclude that if one knows a solution u to the KdV equation and therefore a solution ϕ to (1.3.8), then the Darboux

transformation can be applied successively to produce new solutions to the KdV equation [33]:

$$(u, \phi) \mapsto (u^{[1]}, \phi^{[1]}) \mapsto (u^{[2]}, \phi^{[2]}) \mapsto \dots$$

As a further application, we consider the following question posed in 1986 by Duistermaat and Grünbaum [25]: for which linear ordinary differential operators $L(x, \partial_x)$ does there exist a non-zero family of eigenfunctions $\Psi(x, z)$ such that

$$\begin{aligned} L(x, \partial_x)\Psi(x, z) &= f(z)\Psi(x, z), \\ M(z, \partial_z)\Psi(x, z) &= g(x)\Psi(x, z) \end{aligned} \tag{1.3.9}$$

for some other differential operator $M(z, \partial_z)$? This question was initially studied due to its applications to “limited angle tomography” [31], but eventually was shown to be connected to integrable hierarchies and soliton mathematics [8, 9, 10]. When L is of order 2, the question can be answered using the Darboux transformations discussed for the KdV equation. In this case, L can be written in the Schrödinger form (1.3.2) and the potentials $u(x)$ are found according to the following theorem.

Theorem 1.3.5 ([25]). *The potentials $u(x)$ for which the equations 1.3.9 hold (for non-zero Ψ and M of positive order) are given by:*

1. $u(x) = ax + b, \quad a, b \in \mathbb{C},$
2. $u(x) = \frac{c}{(x-a)^2} + b, \quad a, b, c, \in \mathbb{C},$
3. $u(x)$ obtained from finitely many Darboux transformations starting from $u(x) = 0,$
4. $u(x)$ obtained from finitely many Darboux transformations starting from $u(x) = -\frac{1}{4x^2}.$

The first two cases correspond to the Airy and Bessel functions respectively. These results were generalized in [10].

CHAPTER 2

THE KP HIERARCHY

In this chapter, we will introduce integrable hierarchies by presenting known results for the KP hierarchy following [61, 48]. We start by defining the hierarchy in *Lax form*. We then introduce the concepts of wave functions, wave operators, and tau functions, and finally show how Darboux transformations can be used to generate soliton solutions to the KP Hierarchy.

2.1 Definition of the KP hierarchy

2.1.1 Pseudo-differential operators

If L is a differential operator, we can define $L^{1/n}$ for any $n \in \mathbb{Z}$ in terms of *pseudo-differential operators*. Such operators contain negative powers of $\partial = \partial_x$, and their action is defined on functions of the form $f(x) = e^{kx}$ using

$$\partial^n(e^{kx}) = k^n e^{kx}$$

for $n \in \mathbb{Z}$.

Definition 2.1.1. A *pseudo-differential operator* (of order $n \in \mathbb{Z}$) is an operator of the form

$$A = \sum_{i=-\infty}^n a_i \partial^i.$$

for $a_i = a_i(x)$ and $a_n \neq 0$.

We will let \mathcal{D}^- denote the space of pseudo-differential operators of the form

$$A = \sum_{i=-\infty}^{-1} a_i \partial^i.$$

For any pseudo-differential operator $A = \sum_{i=-\infty}^n a_i \partial^i$ we define the decomposition of A into its “plus” and “minus” parts by

$$A_+ = \sum_{i=0}^n a_i \partial^i, \quad A_- = \sum_{i=-\infty}^{-1} a_i \partial^i.$$

Multiplication of pseudo-differential operators is defined according to the *Leibniz rule*:

$$\partial^n \circ f = \sum_{i=0}^{\infty} \binom{n}{i} (\partial^i f) \partial^{n-i}.$$

If A is a pseudo-differential operator of the form $A = \sum_{i=-\infty}^n a_i \partial^i$, then there exists a unique A^{-1} of the form $\sum_{i=-\infty}^{-n} \tilde{a}_i \partial^i$ such that $A \circ A^{-1} = A^{-1} \circ A = I$. We also define the *symbol of A* by

$$\hat{A} = \sum_{i=-\infty}^n a_i z^i,$$

or equivalently,

$$\hat{A} e^{xz} = A e^{xz}.$$

Finally, the *formal adjoint of A* is given by

$$A^* = \sum_{i=-\infty}^n (-\partial)^i \circ a_i$$

and satisfies the properties

$$(AB)^* = B^* A^*, \quad (A^*)^{-1} = (A^{-1})^*, \quad (A^*)^* = A$$

(here, B is some other pseudo-differential operator).

2.1.2 The KdV hierarchy

We can use pseudo-differential operators to compute the square root of the Schrödinger operator $L = \partial^2 + u$ that appears in the Sturm-Liouville equation (1.3.1). If we let $P = \partial + \sum_{i=-\infty}^{-1} p_i \partial^i$ for some unknown $p_i = p_i(x)$, we can set $P^2 = \partial^2 + u$ and compare coefficients to determine the p_i (see [48]). Notice we can use this to determine $(\partial^2 + u)^{n/2} = L^{n/2}$ for any positive, odd integer n . One can show that the operator B from (1.3.7) can be expressed as

$$B = \partial_x^3 + \frac{3}{2}u + \frac{3}{4}u_x \partial_x = (L^{3/2})_+.$$

Thus, we can rewrite the Lax representation for the KdV equation as

$$\frac{\partial L}{\partial t} = [(L^{3/2})_+, L]. \quad (2.1.1)$$

In general, we have that

$$[L, (L^{n/2})_+] = [L, L^{n/2} - (L^{n/2})_-] = -[L, (L^{n/2})_-]. \quad (2.1.2)$$

To define the KdV hierarchy, we will introduce an infinite sequence of variables (t_1, t_3, t_5, \dots) .

Definition 2.1.2. *The system of Lax equations*

$$\frac{\partial L}{\partial t_n} = [(L^{n/2})_+, L], \quad n = 1, 3, 5, \dots \quad (2.1.3)$$

*is known as the **KdV hierarchy**.*

All of the equations in the KdV hierarchy are pairwise compatible, or can be solved simultaneously. When $n = 1$, we have

$$\frac{\partial L}{\partial t_1} = [(L^{1/2})_+, L] = [\partial, L] = \frac{\partial L}{\partial x}$$

so we can identify x with t_1 . Similarly when $n = 3$, using (2.1.1), we can identify t with t_3 .

2.1.3 Lax form of the KP hierarchy

We can now proceed with the definition of the KP Hierarchy. Instead of the Schrödinger operator, we start with a pseudo-differential operator $L = \partial + a_{-1}\partial^{-1} + a_{-2}\partial^{-2} + \dots$, where each $a_i = a_i(x, t_1, t_2, \dots)$.

Definition 2.1.3. *The system of Lax equations*

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, \dots \quad (2.1.4)$$

is known as the **KP hierarchy**.

Observe that

$$[(L^n)_+, L] = [L^n - (L^n)_-, L] = -[(L^n)_-, L] \in \partial + \mathcal{D}^-$$

since $[L^n, L] = 0$. If we impose the restriction that L^2 is a strictly differential operator, then the system of equations (2.1.3) reduces to the KdV hierarchy. In this case $L^2 = \partial^2 + u$ and

$$\frac{\partial L}{\partial t_{2n}} = [(L^{2n})_+, L] = [L^{2n}, L] = 0.$$

If we consider the equation of the KP hierarchy corresponding to $n = 1$ and use

$$\begin{aligned} [\partial, a_j \partial^j] f &= (\partial \circ a_j \partial^j - a_j \partial^j \circ \partial) f \\ &= \frac{\partial a_j}{\partial x} (\partial^j f) + a_j \partial^{j+1} f - a_j \partial^{j+1} f \\ &= \frac{\partial a_j}{\partial x} (\partial^j f), \end{aligned}$$

we have

$$\frac{\partial L}{\partial t_1} = [L_+, L] = [\partial, L] = \sum_{j=-\infty}^{-1} \frac{\partial a_j}{\partial x} \partial^j.$$

On the other hand, direct computation shows that

$$\frac{\partial L}{\partial t_1} = \sum_{j=-\infty}^{-1} \frac{\partial a_j}{\partial t_1} \partial^j,$$

so we can identify t_1 with x and write

$$a_j(x, t_1, t_2, \dots) = a_j(t_1 + x, t_2, \dots) = a_j(t_1, t_2, \dots).$$

From now on, we will use boldface notation to refer to the infinite sequence of variables indexed by the integers; for instance, $\mathbf{t} = (t_1, t_2, t_3, \dots)$.

2.1.4 Wave functions and bilinear equation

If L is a Lax operator for the KP hierarchy, there exists a wave operator $W \in I + \mathcal{D}^-$ and a wave function $\psi = W e^{\sum_{j=1}^{\infty} t_j z^j}$ on which the equations (2.1.4) induce actions. The following

theorem summarizes these actions and gives an explicit definition of W and ψ in terms of L . First, we introduce the notation

$$\xi(\mathbf{t}, z) = \sum_{i=1}^{\infty} t_i z^i.$$

Theorem 2.1.4 ([61]). *The following are equivalent:*

- (a) $\frac{\partial L}{\partial t_n} = [(L^n)_+, L]$ for $n = 1, 2, \dots$
- (b) *There exists a wave operator*

$$W = 1 + \sum_{j=1}^{\infty} w_j \partial^{-j} \in I + \mathcal{D}^-, \quad w_j = w_j(\mathbf{t})$$

such that $L = W \partial W^{-1}$ and

$$\frac{\partial W}{\partial t_n} = -(L^n)_- W, \quad n = 1, 2, 3, \dots \quad (2.1.5)$$

- (c) *There exists a wave function*

$$\psi = \psi(\mathbf{t}, z) = W e^{\xi(\mathbf{t}, z)}$$

satisfying

$$\begin{aligned} L\psi &= z\psi, \\ \frac{\partial \psi}{\partial t_n} &= (L^n)_+ \psi, \quad n = 1, 2, 3, \dots \end{aligned} \quad (2.1.6)$$

The wave function can be used to write a single *bilinear equation* that encodes the entire system (2.1.4) into a single equation. First, we define the *adjoint wave function* ψ^* by

$$\psi^*(\mathbf{t}, z) = (W^*)^{-1} e^{-\sum_{j=1}^{\infty} t_j z^j}.$$

Following Theorem 1.1 from [61], we can determine if ψ is a wave function for the KP hierarchy from the following bilinear equation. Here we use the notation

$$\oint \frac{dz}{2\pi i} \left(\sum a_i z^i \right) = \text{Res}_z \sum a_i z^i = a_{-1}$$

to denote the residue of a formal power series (see e.g. [19]).

Theorem 2.1.5. *A function ψ is a wave function for the KP hierarchy if and only if*

$$\oint \frac{dz}{2\pi i} \psi(\mathbf{t}', z) \psi^*(\mathbf{t}, z) = 0 \quad (2.1.7)$$

for all t and t' .

The proof of this theorem uses the Taylor expansion of $\psi(\mathbf{t}', z)$ about $t'_i = t_i$ along with the following lemma (cf. [22, 61]).

Lemma 2.1.6. *If A and B are two pseudo-differential operators, then*

$$\text{Res}_z ((Ae^{xz}) \cdot (Be^{-xz})) = \text{Res}_\partial AB^*.$$

2.1.5 Tau function

The final object that will be of use for us when studying integrable hierarchies is the tau function. As defined, the KP hierarchy is an equation in infinitely many variables t_1, t_2, \dots for infinitely many functions w_1, w_2, \dots .

Sato showed that the system (2.1.4) can be expressed in terms of a single unknown function $\tau = \tau(\mathbf{t})$, called the *tau function* [51].

Theorem 2.1.7. *There exists a function $\tau(\mathbf{t})$ such that the wave functions ψ and ψ^* can be represented as*

$$\psi(\mathbf{t}, z) = W e^{\sum_{i=1}^{\infty} t_j z^j} = \frac{\tau(\mathbf{t} - [z^{-1}])}{\tau(\mathbf{t})} e^{\xi(\mathbf{t}, z)} \quad (2.1.8)$$

and

$$\psi^*(\mathbf{t}, z) = \frac{\tau(\mathbf{t} + [z^{-1}])}{\tau(\mathbf{t})} e^{-\xi(\mathbf{t}, z)} \quad (2.1.9)$$

where

$$[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots \right).$$

Based on this definition, it is helpful to define the vertex operator $\Gamma(z)$ by

$$\Gamma(z) = e^{\xi(\mathbf{t}, z)} \exp \left(- \sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_{t_i} \right) \quad (2.1.10)$$

and write

$$\psi = \frac{\Gamma(z)\tau}{\tau}. \quad (2.1.11)$$

This allows us to explicitly solve for the first few coefficients of \hat{W} in terms of τ by expanding (2.1.11) and comparing coefficients. For example,

$$w_1 = \frac{-\partial_{t_1} \tau}{\tau},$$

$$w_2 = \frac{1}{2} \left(\frac{\partial_{t_1}^2 \tau - \partial_{t_2} \tau}{\tau} \right).$$

Using the equations (2.1.8) and (2.1.9), we can rewrite the bilinear equation (2.1.7) in terms of only the tau function:

$$\oint \frac{dz}{2\pi i} \tau(\mathbf{t} - [z^{-1}])\tau(\mathbf{t}' + [z^{-1}])e^{\xi(\mathbf{t}-\mathbf{t}',z)} = 0, \quad (2.1.12)$$

giving us a single equation in a single function that is equivalent to the KP hierarchy. As with the wave functions ψ and ψ^* , we can state the following theorem.

Theorem 2.1.8 ([61]). *A function τ is a tau function for the KP hierarchy if and only if it satisfies the bilinear equation (2.1.12) for all \mathbf{t}, \mathbf{t}' .*

2.2 Fay identity for the KP Hierarchy

Using the bilinear equation (2.1.12), we can arrive at certain identities satisfied by any tau function for the KP hierarchy, known as the *Fay identities* [6].

Shifting the variables \mathbf{t} and \mathbf{t}' in the bilinear equation and computing the residue, we obtain the following Fay identity for the KP hierarchy, originally proved by Shiota [52].

Theorem 2.2.1. *Any tau function for the KP hierarchy satisfies the identity*

$$\begin{aligned} & (s_0 - s_1)(s_2 - s_3)\tau(\mathbf{t} + [s_0] + [s_1])\tau(\mathbf{t} + [s_2] + [s_3]) \\ & + (s_0 - s_2)(s_3 - s_1)\tau(\mathbf{t} + [s_0] + [s_2])\tau(\mathbf{t} + [s_3] + [s_1]) \\ & + (s_0 - s_3)\tau(\mathbf{t} + [s_0] + [s_3])\tau(\mathbf{t} + [s_1] + [s_2]) = 0. \end{aligned} \quad (2.2.1)$$

Proof. We start by letting $\mathbf{t} = \mathbf{t} - [s_1] - [s_2] - [s_3]$ and $\mathbf{t}' = \mathbf{t} + [s_0]$ in (2.1.12). Then by (1.2.1), the bilinear equation becomes

$$\begin{aligned} & \oint \frac{dz}{2\pi i} \frac{(1 - zs_0)}{(1 - zs_1)(1 - zs_2)(1 - zs_3)} \\ & \times \tau(\mathbf{t} + [s_1] + [s_2] + [s_3] - [z^{-1}])\tau(\mathbf{t} + [s_0] + [z^{-1}]) = 0. \end{aligned} \quad (2.2.2)$$

To compute this residue, we use partial fractions to write

$$\begin{aligned} & \frac{(1 - zs_0)g(z)}{(1 - zs_1)(1 - zs_2)(1 - zs_3)} = \\ & \frac{(s_0 - s_1)s_1}{(s_1 - s_2)(s_3 - s_1)} \cdot \frac{g(z)}{1 - zs_1} + \frac{(s_0 - s_2)s_2}{(s_1 - s_2)(s_2 - s_3)} \cdot \frac{g(z)}{1 - zs_2} \\ & + \frac{(s_0 - s_3)s_3}{(s_3 - s_1)(s_2 - s_3)} \cdot \frac{g(z)}{1 - zs_3}, \end{aligned}$$

and then use the property that if $g(z) = \sum_{i=0}^{\infty} g_i z^{-i}$, then

$$\begin{aligned} \operatorname{Res}_z \frac{g(z)}{1 - z\lambda} &= \operatorname{Res}_z \sum_{i,j=0}^{\infty} g_i \lambda^j z^{-i+j} \\ &= \sum_{i=0}^{\infty} g_i \lambda^{i-1} \\ &= \lambda^{-1} g(\lambda). \end{aligned} \tag{2.2.3}$$

Doing so, we get

$$(s_1 - s_2)^{-1} (s_2 - s_3)^{-1} (s_3 - s_1)^{-1} \times \text{left hand side of (2.2.1)} = 0,$$

so we have proved the theorem. □

Differentiating both sides of (2.2.1) and making a certain reduction, we can also prove the following *differential Fay identity* satisfied by any tau function for the KP hierarchy [61].

Theorem 2.2.2. *Any tau function $\tau(\mathbf{t})$ for the KP hierarchy satisfies*

$$\operatorname{Wr}(\tau(\mathbf{t} - [s_1]), \tau(\mathbf{t} - [s_2])) = (s_1^{-1} - s_2^{-1}) (\tau(\mathbf{t} - [s_1])\tau(\mathbf{t} - [s_2]) - \tau(\mathbf{t})\tau(\mathbf{t} - [s_1] - [s_2])). \tag{2.2.4}$$

Conversely, the differential Fay identity was shown to be equivalent to the bilinear equation (2.1.12) in [56].

2.3 Darboux Transformations of the KP Hierarchy

In this section, we use the ideas from Section 1.3.1 to define Darboux transformations for the KP hierarchy. Further details and proofs of the theorems stated in this section can be found in [6] (see also [61, 8]).

Recall that any wave function ψ for the KP hierarchy satisfies

$$L\psi = z\psi.$$

For a function ϕ , let A_ϕ be the operator defined by

$$A_\phi = \partial - \frac{\phi_x}{\phi}.$$

A Darboux transformation on a wave function ψ for the KP hierarchy and its corresponding Lax operator L is defined by

$$L^{[1]} = A_\phi L A_\phi^{-1}, \quad \psi^{[1]} = A_\phi \psi = \frac{\text{Wr}(\phi, \psi)}{\phi}.$$

Then just as with any classical Darboux transformation, we have

$$L^{[1]} \psi^{[1]} = A_\phi L A_\phi^{-1} A_\phi \psi = A_\phi L \psi = A_\phi(z\psi) = z A_\phi \psi = z \psi^{[1]},$$

so that $\psi^{[1]}$ is an eigenfunction for $L^{[1]}$. If we pick $\phi = \psi_1 = \psi|_{z=z_1}$ for some known wave function ψ of the KP hierarchy, we have

$$\psi^{[1]} = A_{\psi_1} \psi = \frac{\text{Wr}(\psi, \psi_1)}{\psi} \quad (2.3.1)$$

also satisfies the equations (2.1.6), and is itself a wave function for the KP hierarchy. Furthermore, one can show using the Fay identity (2.2.1) that the action of the above Darboux transformation on the tau function is given by the vertex operator

$$X(\mathbf{t}, \lambda) = \exp\left(\sum_{i=1}^{\infty} t_i \lambda^i\right) \exp\left(-\sum_{i=1}^{\infty} \frac{\lambda^{-i}}{i} \partial_{t_i}\right).$$

In other words, we have the following theorem.

Theorem 2.3.1 ([6]). *If ψ is a known solution to the KP hierarchy with corresponding tau function τ , and $\psi^{[1]}$ is found as in (2.3.1), then*

$$\psi^{[1]} = \frac{\tau^{[1]}(\mathbf{t} - [z^{-1}])}{\tau^{[1]}(\mathbf{t})}$$

where $\tau^{[1]}(\mathbf{t}) = X(\mathbf{t}, z_1)\tau(\mathbf{t})$.

That $\tau^{[1]}$ is a tau function for the KP hierarchy can be verified by plugging

$$X(\mathbf{t}, z_1)\tau(\mathbf{t}) = \exp\left(\sum_{i=1}^{\infty} t_i z_1^i\right) \tau(\mathbf{t} - [z_1^{-1}])$$

into (2.1.12).

Since Darboux transformations can be repeated, we can explicitly find $\tau^{[N]}$, the tau function corresponding to the wave function $\psi^{[N]}$ obtained from N iterations of the Darboux

transformation:

$$\begin{aligned}\tau^{[N]}(\mathbf{t}) &= X(\mathbf{t}, z_N)X(\mathbf{t}, z_{N-1}) \cdots X(\mathbf{t}, z_1)\tau(\mathbf{t}) \\ &= \prod_{1 \leq i < j \leq N} \left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{i=1}^{\infty} t_i (z_1^{-i} + z_2^{-i} + \cdots + z_N^{-i})\right) \tau(\mathbf{t} - [z_1] - [z_2] - \cdots - [z_N]).\end{aligned}$$

2.3.1 Generalized Fay identities

As a consequence of Theorem 2.3.1, we obtain the *generalized Fay identities* satisfied by any tau function for the KP hierarchy. To state the theorem, we will use the notation consistent with [6]:

$$\Delta(s_1, \dots, s_n) = \prod_{1 \leq j < i \leq n} (s_i^{-1} - s_j^{-1}).$$

Theorem 2.3.2 ([6]). *Any tau function for the KP hierarchy satisfies the generalized Fay identities*

$$\begin{aligned}\tau(\mathbf{t} - [s_1] - [s_2] - \cdots - [s_n])\Delta(s_1, \dots, s_n)(\tau(\mathbf{t} - [r_1] - [r_2] - \cdots - [r_n])\Delta(r_1, \dots, r_n))^{n-1} \\ = \det[\tau(\mathbf{t} - \sum_{l=1}^n [r_l] + [r_i] - [s_j])\Delta(r_1, \dots, r_{i-1}, s_j, r_{i+1}, \dots, r_n)]_{1 \leq i, j \leq n}.\end{aligned}$$

We also state the differential version of the generalized Fay identities.

Theorem 2.3.3 ([6]). *Any tau function for the KP hierarchy satisfies the generalized differential Fay identities*

$$\begin{aligned}\text{Wr}(\psi(\mathbf{t}, z_1^{-1}), \dots, \psi(\mathbf{t}, z_N^{-1})) = \\ \left(\prod_{1 \leq j < i \leq n} (z_i^{-1} - z_j^{-1})\right) \exp\left(\sum_{i=1}^{\infty} t_i (z_1^{-i} + \cdots + z_N^{-i})\right) \frac{\tau(\mathbf{t} - [z_1] - [z_2] - \cdots - [z_N])}{\tau(\mathbf{t})}.\end{aligned}\quad (2.3.2)$$

It is also useful to consider the vertex operator

$$\tilde{X}(\mathbf{t}, \mu) = \exp\left(-\sum_{i=1}^{\infty} t_i \mu^i\right) \exp\left(\sum_{i=1}^{\infty} \frac{\mu^{-i}}{i} \partial_{t_i}\right),$$

often referred to as an “inverse” of the X . However it is not exactly an inverse since we see that the expression $\tilde{X}(\mathbf{t}, \mu)X(\mathbf{t}, \lambda)$ has a singularity at $\lambda = \mu$ (cf. Example 1.2.1). The function $\tilde{X}\tau$ is also a tau function for the KP hierarchy. Furthermore, for some tau function $\tau(\mathbf{t})$ of the KP

hierarchy we have

$$\begin{aligned}\tilde{X}(\mathbf{t}, \lambda)X(\mathbf{t}, \mu)\tau &= \frac{\lambda}{\lambda - \mu} \exp\left(\sum_{i=1}^{\infty} t_i(\mu^i - \lambda^i)\right) \tau(\mathbf{t} + [\lambda^{-1}] - [\mu^{-1}]) \\ &= \frac{\lambda}{\lambda - \mu} \exp\left(\sum_{i=1}^{\infty} t_i(\mu^i - \lambda^i)\right) \exp\left(\sum_{i=1}^{\infty} (\lambda^{-i} - \mu^{-i}) \frac{\partial t_i}{i}\right) \tau(\mathbf{t}).\end{aligned}$$

If we expand the last line about positive powers of $(\mu - \lambda)$ and all powers of λ , we get

$$\frac{\lambda}{\lambda - \mu} \sum_{k=0}^{\infty} \frac{(\mu - \lambda)^k}{k!} \left(\sum_{l=-\infty}^{\infty} \lambda^{-l-k} W_l^{(k)}(\tau) \right) \quad (2.3.3)$$

where the operators $W_l^{(k)}$ are defined recursively with the first few given by

$$\begin{aligned}W_n^{(1)} &= J_n^{(1)} = \partial_{t_n} + (-n)t_{-n}, \quad t_{-n} = 0 \quad \text{for } n > 0, \\ W_n^{(2)} &= J_n^{(2)} - (n+1)J_n^{(1)}, \\ W_n^{(3)} &= J_n^{(3)} - \frac{3}{2}(n+2)J_n^{(2)} + (n+1)(n+2)J_n^{(1)}, \\ W_n^{(4)} &= J_n^{(4)} - 2(n+3)J_n^{(3)} + (2n^2 + 9n + 11)J_n^{(2)} - (n+1)(n+2)(n+3)J_n^{(1)},\end{aligned}$$

where

$$J_n^{(k)} = \sum_{i_1+i_2+\dots+i_k=n} : J_{i_1}^{(1)} J_{i_2}^{(1)} \dots J_{i_k}^{(1)} : .$$

To illustrate an application of Theorems 2.3.1 and 2.3.2, we introduce the Lie algebra

$$w_{\infty} = \text{span}\{z^{\alpha} \partial_z^{\beta} \mid \alpha, \beta \in \mathbb{Z}, \beta \geq 0\}$$

of differential operators on the circle S^1 , equipped with the commutator bracket. The central extension of w_{∞} is known as the Lie algebra $W_{1+\infty}$, and the operators $W_l^{(k)}$ above are generators for it. The Fay identities can be used to show that the action of w_{∞} on the Lax operator corresponds to the action of $W_{1+\infty}$ on the tau function for the KP hierarchy [6, 3, 21].

The Fay identities can also be used to connect the theory of the KP hierarchy to Jacobian varieties of smooth curves (see [52]).

CHAPTER 3

THE EXTENDED BIGRADED TODA HIERARCHY (EBTH)

3.1 Definition of the EBTH

The extended bigraded Toda hierarchy (EBTH) was introduced by Carlet in 2006 [15], and is the main subject of this thesis. In this chapter, we introduce the EBTH following [12] and using the notation from [54]. While this notation differs from the one originally presented in [15, 18], that version can be obtained from a change of variables which we discuss below.

We begin this chapter by discussing the spaces of difference and differential-difference operators. Then we present a definition of the EBTH, its Lax operator, wave operators, wave functions, and tau-function. Next, we derive a bilinear equation for the EBTH using Takasaki's approach from [54]. Our equation is equivalent to the bilinear equation from [18], but we provide a shorter proof. As a consequence, we obtain two difference Fay identities satisfied by tau-functions of the EBTH.

3.1.1 Spaces of difference and differential-difference operators

Consider functions of a variable s , and the shift operator $\Lambda = e^{\partial_s}$ defined by $(\Lambda f)(s) = f(s+1)$. The space \mathcal{A} of (formal) *difference operators* consists of all expressions of the form

$$A = \sum_{i \in \mathbb{Z}} a_i(s) \Lambda^i.$$

We have $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ where \mathcal{A}_+ (respectively, \mathcal{A}_-) consists of $A \in \mathcal{A}$ such that $a_i = 0$ for all $i < 0$ (respectively, $i \geq 0$). For $A \in \mathcal{A}$, we define its projections

$$A_+ = \sum_{i \geq 0} a_i(s) \Lambda^i \in \mathcal{A}_+, \quad A_- = \sum_{i < 0} a_i(s) \Lambda^i \in \mathcal{A}_-.$$

We let \mathcal{A}_{++} be the space of difference operators $A \in \mathcal{A}$ such that $a_i = 0$ for $i \ll 0$ (i.e., the powers of Λ are bounded from below), and \mathcal{A}_{--} be the space of $A \in \mathcal{A}$ such that $a_i = 0$ for $i \gg 0$ (i.e., the powers of Λ are bounded from above). Both \mathcal{A}_{++} and \mathcal{A}_{--} are associative algebras, where the product is defined by linearity and

$$(a(s)\Lambda^i)(b(s)\Lambda^j) = a(s)b(s+i)\Lambda^{i+j}.$$

Let $\mathcal{A}_{\text{fin}} = \mathcal{A}_{++} \cap \mathcal{A}_{--}$. The product of a difference operator $A \in \mathcal{A}$ by an element of \mathcal{A}_{fin} is defined, but in general, the product of an element of \mathcal{A}_{++} and an element of \mathcal{A}_{--} is not well defined.

We will also consider the space $\mathcal{A}[\partial_s]$ of (formal) *differential-difference operators*, where $\Lambda \partial_s = \partial_s \Lambda$. Note that such operators depend polynomially on ∂_s . Again, there is a splitting $\mathcal{A}[\partial_s] = \mathcal{A}_+[\partial_s] \oplus \mathcal{A}_-[\partial_s]$, and we have the associative algebras $\mathcal{A}_{++}[\partial_s]$ and $\mathcal{A}_{--}[\partial_s]$, where the product is defined by linearity and

$$(a(s)\Lambda^i \partial_s^n)(b(s)\Lambda^j \partial_s^m) = \sum_{k=0}^n \binom{n}{k} a(s) \frac{\partial^k b}{\partial s^k}(s+i) \Lambda^{i+j} \partial_s^{m+n-k}.$$

Differential-difference operators can be applied to z^s so that

$$(a(s)\Lambda^i \partial_s^n) z^s = a(s) z^i (\log z)^n z^s.$$

3.1.2 Definition of the EBTH

The EBTH is defined similarly to the KP hierarchy, but in terms of difference operators instead of pseudo-differential operators.

For fixed, positive integers k and m , consider a *Lax operator* of the form

$$L = \Lambda^k + u_{k-1}(s)\Lambda^{k-1} + \cdots + u_{-m}(s)\Lambda^{-m} \in \mathcal{A}_{\text{fin}}, \quad u_{-m}(s) \neq 0.$$

There exist *wave operators* (also called dressing operators):

$$\begin{aligned} W &= 1 + \sum_{i=1}^{\infty} w_i(s) \Lambda^{-i} \in 1 + \mathcal{A}_- \subset \mathcal{A}_{--}, \\ \bar{W} &= \sum_{i=0}^{\infty} \bar{w}_i(s) \Lambda^i \in \mathcal{A}_+, \quad \bar{w}_0(s) \neq 0, \end{aligned} \tag{3.1.1}$$

such that

$$L = W \Lambda^k W^{-1} = \bar{W} \Lambda^{-m} \bar{W}^{-1}. \tag{3.1.2}$$

This allows us to define fractional powers of L for any integer n by

$$L^{\frac{n}{k}} = W \Lambda^n W^{-1} \in \mathcal{A}_{--}, \quad L^{\frac{n}{m}} = \bar{W} \Lambda^{-n} \bar{W}^{-1} \in \mathcal{A}_{++}, \tag{3.1.3}$$

which commute with L and satisfy

$$\left(L^{\frac{n}{k}}\right)^k = \left(L^{\frac{n}{m}}\right)^m = L^n, \quad n \in \mathbb{Z}_{\geq 0}.$$

However, observe that $L^{\frac{n}{k}} \neq L^{\frac{n}{m}}$, unless $\frac{n}{k} = \frac{n}{m} \in \mathbb{Z}_{\geq 0}$. We define $\log L \in \mathcal{A}$ by

$$\log L = \frac{1}{2} W \partial_s W^{-1} - \frac{1}{2} \bar{W} \partial_s \bar{W}^{-1} = -\frac{1}{2} \frac{\partial W}{\partial s} W^{-1} + \frac{1}{2} \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}.$$

Then $\log L$ commutes with all L^n for $n \in \mathbb{Z}_{\geq 0}$, but the composition of $\log L$ with a fractional power of L is not well defined in general.

Definition 3.1.1 ([15]). *The extended bigraded Toda hierarchy (abbreviated EBTH) in Lax form is given by:*

$$\begin{aligned} \partial_{t_n} L &= [(L^{\frac{n}{k}})_+, L], & n \geq 1, \\ \partial_{\bar{t}_n} L &= [(L^{\frac{n}{m}})_+, L], & n \geq 1, \\ \partial_{x_n} L &= [(2L^n \log L)_+, L], & n \geq 0. \end{aligned} \tag{3.1.4}$$

The first two equations in (3.1.4) describe the *bigraded Toda hierarchy*, which is a reduction of the 2D Toda hierarchy (see [54, 60]). For $k = m = 1$, the EBTH is equivalent to the *extended Toda hierarchy* (ETH) [16, 54].

The flows of the EBTH induce flows on the dressing operators:

$$\begin{aligned} \partial_{t_n} W &= -(L^{\frac{n}{k}})_- W, & \partial_{t_n} \bar{W} &= (L^{\frac{n}{k}})_+ \bar{W}, \\ \partial_{\bar{t}_n} W &= -(L^{\frac{n}{m}})_- W, & \partial_{\bar{t}_n} \bar{W} &= (L^{\frac{n}{m}})_+ \bar{W}, \\ \partial_{x_n} W &= -(2L^n \log L)_- W, & \partial_{x_n} \bar{W} &= (2L^n \log L)_+ \bar{W}. \end{aligned} \tag{3.1.5}$$

Remark 3.1.2. Since $\partial_{x_0} - \partial_s$ and $\partial_{t_{nk}} - \partial_{\bar{t}_{nm}}$ act trivially on L , W and \bar{W} , it follows that L , W and \bar{W} depend on $x_0 + s$ and $t_{nk} + \bar{t}_{nm}$ for $n \geq 1$. Without loss of generality, we can assume $x_0 = s$ and $t_{nk} = \bar{t}_{nm}$.

Remark 3.1.3. To compare our version of the EBTH to the one from [18], we need to change there $\epsilon \mapsto -\epsilon$, which leads to $\Lambda \mapsto \Lambda^{-1}$ and $\zeta \mapsto \zeta^{-1}$ (z here), and then apply the following change of variables:

$$\begin{aligned} x &= \epsilon s, \\ q_n^{k-\alpha} &= \epsilon k \binom{n + \frac{\alpha}{k}}{n+1} t_{nk+\alpha}, \quad \alpha = 1, 2, \dots, k-1, \\ q_n^{k+\beta} &= \epsilon m \binom{n + \frac{\beta}{m}}{n+1} \bar{t}_{nm+\beta}, \quad \beta = 1, 2, \dots, m-1, \\ q_n^{k+m} &= \epsilon m(n+1)! \left(t_{(n+1)k} + \bar{t}_{(n+1)m} + c_{n+1} \left(\frac{1}{k} + \frac{1}{m} \right) x_{k+1} \right), \\ q_n^k &= \epsilon n! x_n, \quad n \geq 0. \end{aligned}$$

Here c_n are the harmonic numbers

$$c_0 = 0, \quad c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n},$$

and $(p)_n$ denotes the Pochhammer symbol,

$$\begin{aligned} (p)_0 &= 1, \\ (p)_n &= \prod_{i=1}^n (p - i + 1), \quad n \geq 1, \\ (p)_{-n} &= \prod_{i=-n+1}^0 (p - i + 1)^{-1} = \frac{1}{(p+n)_n}. \end{aligned}$$

Due to Remark 3.1.2, from now on we will always assume $x_0 = s$ and $t_{nk} = \bar{t}_{nm}$ for $n \geq 1$. Recall that in our notation,

$$\mathbf{t} = (t_1, t_2, \dots), \quad \bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots), \quad \mathbf{x} = (x_1, x_2, \dots),$$

and

$$\xi(\mathbf{t}, z) = \sum_{i=1}^{\infty} t_i z^i.$$

We will also introduce the new notation

$$\xi_k(\mathbf{t}, z) = \sum_{n=1}^{\infty} t_{nk} z^{nk} = \sum_{n=1}^{\infty} \bar{t}_{nm} z^{nk}.$$

Then

$$\xi_m(\bar{\mathbf{t}}, z^{-1}) = \sum_{n=1}^{\infty} \bar{t}_{nm} z^{-nm} = \sum_{n=1}^{\infty} t_{nk} z^{-nm}.$$

We let

$$\begin{aligned} \chi &= z^{s+\xi(\mathbf{x}, z^k)} e^{\xi(\mathbf{t}, z) - \frac{1}{2}\xi_k(\mathbf{t}, z)}, \\ \bar{\chi} &= z^{s+\xi(\mathbf{x}, z^{-m})} e^{-\xi(\bar{\mathbf{t}}, z^{-1}) + \frac{1}{2}\xi_m(\bar{\mathbf{t}}, z^{-1})}. \end{aligned} \quad (3.1.6)$$

Observe that, by definition,

$$\begin{aligned} \partial_{t_i} \chi &= z^i \chi, & \partial_{\bar{t}_j} \bar{\chi} &= -z^{-j} \bar{\chi} & \text{if } k \nmid i, \ m \nmid j, \\ \partial_{t_{nk}} \chi &= \partial_{\bar{t}_{nm}} \chi = \frac{1}{2} z^{nk} \chi, & \partial_{t_{nk}} \bar{\chi} &= \partial_{\bar{t}_{nm}} \bar{\chi} = -\frac{1}{2} z^{-nm} \bar{\chi}. \end{aligned} \quad (3.1.7)$$

The *wave functions* ψ and $\bar{\psi}$ of the EBTH are defined by:

$$\begin{aligned} \psi &= \psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = W\chi = w\chi, \\ \bar{\psi} &= \bar{\psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \bar{W}\bar{\chi} = \bar{w}\bar{\chi}, \end{aligned} \quad (3.1.8)$$

where

$$w = 1 + \sum_{i=1}^{\infty} w_i(s) z^{-i}, \quad \bar{w} = \sum_{i=0}^{\infty} \bar{w}_i(s) z^i \quad (3.1.9)$$

are the (left) symbols of W and \bar{W} respectively. Here we view w and \bar{w} as formal power series of z^{-1} and z ; however, when we state our results, we will assume that $w(z)$ is convergent for z in some domain $\mathcal{U} \subset \mathbb{C}$.

The wave functions satisfy

$$L\psi = z^k \psi, \quad L\bar{\psi} = z^{-m} \bar{\psi}. \quad (3.1.10)$$

We have:

$$\begin{aligned}
\partial_{t_n} \psi &= (L^{\frac{n}{k}})_+ \psi, & n \in \mathbb{Z}_{\geq 1} \setminus k\mathbb{Z}, \\
\partial_{\bar{t}_n} \psi &= -(L^{\frac{n}{m}})_- \psi, & n \in \mathbb{Z}_{\geq 1} \setminus m\mathbb{Z}, \\
\partial_{t_{nk}} \psi &= \partial_{\bar{t}_{nm}} \psi = A_n \psi, & n \in \mathbb{Z}_{\geq 1}, \\
\partial_{x_n} \psi &= (L^n \partial_s + P_n) \psi, & n \in \mathbb{Z}_{\geq 0},
\end{aligned} \tag{3.1.11}$$

and exactly the same equations hold for $\bar{\psi}$, where

$$A_n = \frac{1}{2}(L^n)_+ - \frac{1}{2}(L^n)_- = (L^n)_+ - \frac{1}{2}L^n = \frac{1}{2}L^n - (L^n)_- \tag{3.1.12}$$

and

$$\begin{aligned}
P_n &= -\left(L^n \frac{\partial W}{\partial s} W^{-1}\right)_+ - \left(L^n \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}\right)_- \\
&= L^n W \partial_s W^{-1} - (2L^n \log L)_- - L^n \partial_s \\
&= L^n \bar{W} \partial_s \bar{W}^{-1} + (2L^n \log L)_+ - L^n \partial_s.
\end{aligned} \tag{3.1.13}$$

Observe that, due to (3.1.1) and (3.1.3), we have

$$(L^{\frac{n}{k}})_+, (L^{\frac{n}{m}})_-, A_n, P_n \in \mathcal{A}_{\text{fin}}, \quad P_0 = 0.$$

Finally, by [18, 41], there exists a *tau-function* τ such that

$$\psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \chi, \tag{3.1.14}$$

$$\bar{\psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \bar{\chi}. \tag{3.1.15}$$

Remark 3.1.4. *Since $t_{nk} = \bar{t}_{nm}$, we need to specify how to do the shifts $\mathbf{t} - [z^{-1}]$ in (3.1.14) and $\bar{\mathbf{t}} + [z]$ in (3.1.15). Here and further, our convention is that in (3.1.14), $\mathbf{t} - [z^{-1}]$ includes all variables t_1, t_2, \dots , while $\bar{\mathbf{t}}$ only includes \bar{t}_i such that $m \nmid i$. Similarly, in (3.1.15), all $\bar{t}_1, \bar{t}_2, \dots$ are shifted, while \mathbf{t} only includes t_i such that $k \nmid i$.*

3.2 Bilinear equation for the EBTH

3.2.1 Dual wave functions

Just as in the case of pseudo-differential operators, we can define the *formal adjoint* of a difference operator $A = \sum_{i \in \mathbb{Z}} a_i(s) \Lambda^i \in \mathcal{A}$ by

$$A^* = \sum_{i \in \mathbb{Z}} \Lambda^{-i} \circ a_i(s) = \sum_{i \in \mathbb{Z}} a_i(s-i) \Lambda^{-i}.$$

It satisfies the properties:

$$(AB)^* = B^*A^*, \quad (A^*)^* = A, \quad (A^{-1})^* = (A^*)^{-1}. \quad (3.2.1)$$

For given wave operators W and \bar{W} , we define the *dual wave functions* ψ^* and $\bar{\psi}^*$ by:

$$\begin{aligned} \psi^* &= (W^*)^{-1} \chi^{-1} = (W^*)^{-1} z^{-s-\xi(\mathbf{x}, z^k)} e^{-\xi(\bar{\mathbf{t}}, z) + \frac{1}{2} \xi_k(\mathbf{t}, z)}, \\ \bar{\psi}^* &= (\bar{W}^*)^{-1} \bar{\chi}^{-1} = (\bar{W}^*)^{-1} z^{-s-\xi(\mathbf{x}, z^{-m})} e^{\xi(\bar{\mathbf{t}}, z^{-1}) - \frac{1}{2} \xi_m(\bar{\mathbf{t}}, z^{-1})}. \end{aligned} \quad (3.2.2)$$

If W and \bar{W} satisfy (3.1.2), (3.1.5), then it is easy to derive equations satisfied by ψ^* and $\bar{\psi}^*$. For example, observing that

$$\begin{aligned} L^* &= (W \Lambda^k W^{-1})^* \\ &= (W^{-1})^* (\Lambda^k)^* W^* \\ &= (W^*)^{-1} \Lambda^{-k} W^*, \end{aligned}$$

we have

$$\begin{aligned} L^* \psi^* &= (W^*)^{-1} \Lambda^{-k} (W^*) (W^*)^{-1} \chi^{-1} \\ &= (W^*)^{-1} \Lambda^{-k} \chi^{-1} \\ &= z^k \psi^*. \end{aligned}$$

Doing the same for $\bar{\psi}^*$, we have (cf. (3.1.10)):

$$L^* \psi^* = z^k \psi^*, \quad L^* \bar{\psi}^* = z^{-m} \bar{\psi}^*. \quad (3.2.3)$$

We will not list all the other equations, which are similar to (3.1.11), but we will need the following lemma.

Lemma 3.2.1. *For every solution of the EBTH, the dual wave functions satisfy*

$$\begin{aligned}(\partial_{x_n} - z^{nk}\partial_s)\psi^* &= -P_n^*\psi^*, \\(\partial_{x_n} - z^{-nm}\partial_s)\bar{\psi}^* &= -P_n^*\bar{\psi}^*,\end{aligned}$$

for all $n \in \mathbb{Z}_{\geq 1}$, where P_n is given by (3.1.13).

Proof. First, since $(\partial_{x_n} - z^{nk}\partial_s)\chi = 0$, we have

$$(\partial_{x_n} - z^{nk}\partial_s)\psi^* = \left((\partial_{x_n} - z^{nk}\partial_s)(W^*)^{-1} \right) \chi^{-1}. \quad (3.2.4)$$

Using (3.1.5) and the product rule on $WW^{-1} = 1$, we find

$$\begin{aligned}\partial_{x_n} W^{-1} &= -W^{-1}(\partial_{x_n} W)W^{-1} = W^{-1}(2L^n \log L)_-, \\ \partial_s W^{-1} &= \partial_{x_0} W^{-1} = W^{-1}(2 \log L)_-.\end{aligned}$$

Note that taking formal adjoint commutes with taking derivative with respect to x_n , because the latter is done coefficient by coefficient. Hence,

$$(\partial_{x_n} - z^{nk}\partial_s)(W^*)^{-1} = (2L^n \log L)_-^*(W^*)^{-1} - z^{nk}(2 \log L)_-^*(W^*)^{-1}.$$

Using (3.2.4) and (3.2.3), we can write

$$\begin{aligned}(\partial_{x_n} - z^{nk}\partial_s)\psi^* &= (2L^n \log L)_-^* - z^{nk}(2 \log L)_-^* \psi^* \\ &= (2L^n \log L)_-^* \psi^* - (2 \log L)_-^* z^{nk} \psi^* \\ &= (2L^n \log L)_-^* \psi^* - (2 \log L)_-^* (L^n)^* \psi^* \\ &= (2L^n \log L)_-^* \psi^* - (2L^n (\log L)_-)^* \psi^* \\ &= ((2L^n \log L)_- - 2L^n (\log L)_-)^* \psi^*.\end{aligned} \quad (3.2.5)$$

Finally, using (3.1.13) and observing that $P_0 = 0$ we get

$$-(2 \log L)_- = -W\partial_s W^{-1} + \partial_s,$$

so that (3.2.5) becomes

$$((2L^n \log L)_- - L^n W\partial_s W^{-1} + L^n \partial_s)^* \psi^* = -P_n^* \psi^*.$$

The equation for $\bar{\psi}^*$ is proved in the exact same way. □

3.2.2 Bilinear equation for the wave functions

The next result provides bilinear equations satisfied by the wave functions and dual wave functions of the EBTH.

Theorem 3.2.2. *The wave functions $\psi = W\chi$ and $\bar{\psi} = \bar{W}\bar{\chi}$ solve the EBTH if and only if they satisfy the bilinear equation*

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} \psi(s' - \xi(\mathbf{a}, z^k), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \psi^*(s - \xi(\mathbf{b}, z^k), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm} \bar{\psi}(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \\ & \quad \times \bar{\psi}^*(s - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \end{aligned} \quad (3.2.6)$$

for all $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_1, b_2, \dots)$, $n \in \mathbb{Z}_{\geq 0}$ and $s - s' \in \mathbb{Z}$.

Remark 3.2.3. *By Taylor expansions of ψ and $\bar{\psi}$ about $\mathbf{t}' = \mathbf{t}$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}}$, the bilinear equation (3.2.6) is equivalent to:*

$$\begin{aligned} & \oint \frac{dz}{2\pi i} (\partial_{\mathbf{t}}^\alpha \partial_{\bar{\mathbf{t}}}^\beta \psi(s' - \xi(\mathbf{a}, z^k), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{a}, z)) \psi^*(s - \xi(\mathbf{b}, z^k), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \\ &= \oint \frac{dz}{2\pi i} (\partial_{\mathbf{t}}^\alpha \partial_{\bar{\mathbf{t}}}^\beta \bar{\psi}(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{a}, z)) \\ & \quad \times \bar{\psi}^*(s - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \end{aligned}$$

for all multi-indices α, β , where $\partial_{\mathbf{t}}^\alpha = \partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2} \dots$ and $\partial_{\bar{\mathbf{t}}}^\beta = \partial_{\bar{t}_1}^{\beta_1} \partial_{\bar{t}_2}^{\beta_2} \dots$.

Remark 3.2.4. *By taking a linear combination of equations (3.2.6) for different $n \in \mathbb{Z}_{\geq 0}$, we can replace z^{nk} by $f(z^k)$ on the left side of (3.2.6) and z^{-nm} by $f(z^{-m})$ on the right side, for any formal power series $f(z) \in \mathbb{C}[[z]]$.*

The following lemma from [49] will be useful in the proof of the above theorem. In this lemma and below, we will use the notation $(A)_j = a_j(s)$ for the coefficient of Λ^j in a difference operator $A = \sum_{j \in \mathbb{Z}} a_j(s) \Lambda^j$.

Lemma 3.2.5. *Let A and B be difference operators such that the product BA^* is well defined. Then*

$$(BA^*)_j = \oint \frac{dz}{2\pi i} (\Lambda^j A z^s)(B z^{-s}), \quad j \in \mathbb{Z}.$$

In particular, suppose that \bar{A}, \bar{B} are two other difference operators such that $\bar{B}\bar{A}^$ is well defined. Then*

$$\oint \frac{dz}{2\pi i} (\Lambda^j A z^s)(B z^{-s}) = \oint \frac{dz}{2\pi i} (\Lambda^j \bar{A} z^s)(\bar{B} z^{-s})$$

for all $j \in \mathbb{Z}$, if and only if $BA^* = \bar{B}\bar{A}^*$.

Proof of Theorem 3.2.2. First, following the approach of [54], we will prove that the equations of the EBTH imply the bilinear equation (3.2.6). By (3.1.8), (3.2.2) and Lemma 3.2.5, we have

$$\begin{aligned} & \oint \frac{dz}{2\pi i} (\Lambda^j \psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z)) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} (\Lambda^j \bar{\psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z)) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \end{aligned} \quad (3.2.7)$$

for all $j \in \mathbb{Z}$. Therefore,

$$\begin{aligned} & \oint \frac{dz}{2\pi i} \psi(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} \bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \end{aligned} \quad (3.2.8)$$

for all s, s' with $s - s' \in \mathbb{Z}$.

Now applying L^n as a difference operator with respect to s' to both sides of (3.2.8) and using (3.1.10), we obtain

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} \psi(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm} \bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \end{aligned} \quad (3.2.9)$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $s - s' \in \mathbb{Z}$. Recall that the action of the derivatives with respect to \mathbf{t} and $\bar{\mathbf{t}}$ on the wave functions is given by difference operators (see (3.1.11)). We can apply the generating function $\exp(\sum_{i=1}^{\infty} c_i \partial_{t_i})$ to ψ and $\bar{\psi}$ in the above equation, thus shifting \mathbf{t} by a constant \mathbf{c} . Let us denote $\mathbf{t} + \mathbf{c}$ by \mathbf{t}' . Doing the same for $\bar{\mathbf{t}}$, we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} \psi(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm} \bar{\psi}(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned} \quad (3.2.10)$$

Notice that, by (3.1.10) and (3.1.11),

$$\begin{aligned} & (\partial_{x_\ell} - z^{\ell k} \partial_s) \psi = Q_\ell \psi, \\ & (\partial_{x_\ell} - z^{-\ell m} \partial_s) \bar{\psi} = Q_\ell \bar{\psi}, \quad Q_\ell = P_\ell - \frac{\partial(L^\ell)}{\partial s}, \end{aligned}$$

where P_ℓ is given by (3.1.13). We can apply the difference operator Q_ℓ to the variable s' on

both sides of (3.2.10) to obtain

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} ((\partial_{x_\ell} - z^{\ell k} \partial_{s'}) \psi(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z)) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm} ((\partial_{x_\ell} - z^{-\ell m} \partial_{s'}) \bar{\psi}(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z)) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \end{aligned}$$

for all $n \geq 0$, $\ell \geq 1$. Using the generating function

$$\begin{aligned} & \exp\left(\sum_{\ell=1}^{\infty} a_\ell (\partial_{x_\ell} - z^{\ell k} \partial_{s'})\right) \psi(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \\ &= \psi(s' - \xi(\mathbf{a}, z^k), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z), \end{aligned}$$

we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} \psi(s' - \xi(\mathbf{a}, z^k), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm} \bar{\psi}(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

Similarly, by acting with $-P_\ell^*$ on s in both sides of this equation and using Lemma 3.2.1, we obtain the bilinear equation (3.2.6).

Conversely, we have to prove that if ψ and $\bar{\psi}$ satisfy the bilinear equation (3.2.6), then they obey the equations of the EBTH. More precisely, suppose that the functions

$$\psi = W\chi, \quad \psi^* = T\chi^{-1}, \quad \bar{\psi} = \bar{W}\bar{\chi}, \quad \bar{\psi}^* = \bar{T}\bar{\chi}^{-1}$$

satisfy (3.2.6), where W, \bar{W}, T, \bar{T} are difference operators such that

$$W, T^* \in 1 + \mathcal{A}_-, \quad \bar{W}, \bar{T}^* \in \mathcal{A}_+$$

(cf. (3.1.1), (3.1.6), (3.1.8), (3.2.2)). Then we will prove that $\psi, \bar{\psi}$ are the wave functions and $\psi^*, \bar{\psi}^*$ are the dual wave functions of a solution of the EBTH.

First, setting $\mathbf{a} = \mathbf{b} = \mathbf{0}$, $\mathbf{t} = \mathbf{t}'$, $\bar{\mathbf{t}} = \bar{\mathbf{t}}'$ in (3.2.6), we obtain (3.2.9) as a special case. Then putting $n = 0$ gives (3.2.8), and equivalently, (3.2.7). By Lemma 3.2.5, equation (3.2.7) implies that $TW^* = \bar{T}\bar{W}^*$. Since $(TW^*)^* = WT^* \in 1 + \mathcal{A}_-$ and $(\bar{T}\bar{W}^*)^* = \bar{W}\bar{T}^* \in \mathcal{A}_+$, we conclude that

$$T = (W^*)^{-1}, \quad \bar{T} = (\bar{W}^*)^{-1},$$

and (3.2.2) holds.

Second, we define $L = W\Lambda^k W^{-1}$ and want to prove (3.1.2). Notice that $L\psi = W\Lambda^k W^{-1}W\chi =$

$z^k\psi$. Applying L with respect to s' to both sides of (3.2.8) and using (3.2.9) for $n = 1$, we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} (L\bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z))\bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-m}\bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z)\bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

For $s' = s + j$ with $j \in \mathbb{Z}$, we have:

$$\begin{aligned} L|_{s=s'}\bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) &= \Lambda^j L\bar{W}\bar{\chi}, \\ z^{-m}\bar{\psi}(s', \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) &= \Lambda^j \bar{W}\Lambda^{-m}\bar{\chi}. \end{aligned}$$

From Lemma 3.2.5, it follows that

$$(\bar{W}^*)^{-1}(L\bar{W})^* = (\bar{W}^*)^{-1}(\bar{W}\Lambda^{-m})^*,$$

from which we can conclude

$$(L\bar{W})^* = (\bar{W}\Lambda^{-m})^*.$$

By (3.2.1), this is equivalent to

$$\bar{W}^*L^* = (\Lambda^{-m})^*\bar{W}^*,$$

which simplifies to

$$L = \bar{W}\Lambda^{-m}\bar{W}^{-1},$$

thus proving (3.1.2) and (3.1.10).

Next, we will show that we can identify t_{nk} with \bar{t}_{nm} in L , W and \bar{W} for $n \in \mathbb{Z}_{\geq 1}$ (cf. Remark 3.1.2). Observe that, by (3.1.7) and (3.1.8),

$$\begin{aligned} \frac{\partial\psi}{\partial t_{nk}} &= \frac{\partial W}{\partial t_{nk}}\chi + \frac{1}{2}z^{nk}W\chi, & \frac{\partial\bar{\psi}}{\partial t_{nk}} &= \frac{\partial\bar{W}}{\partial t_{nk}}\bar{\chi} - \frac{1}{2}z^{-nm}\bar{W}\bar{\chi}, \\ \frac{\partial\psi}{\partial \bar{t}_{nm}} &= \frac{\partial W}{\partial \bar{t}_{nm}}\chi + \frac{1}{2}z^{nk}W\chi, & \frac{\partial\bar{\psi}}{\partial \bar{t}_{nm}} &= \frac{\partial\bar{W}}{\partial \bar{t}_{nm}}\bar{\chi} - \frac{1}{2}z^{-nm}\bar{W}\bar{\chi}; \end{aligned}$$

hence,

$$\begin{aligned} \left(\frac{\partial}{\partial t_{nk}} - \frac{\partial}{\partial \bar{t}_{nm}}\right)\psi &= \left(\frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial \bar{t}_{nm}}\right)\chi, \\ \left(\frac{\partial}{\partial t_{nk}} - \frac{\partial}{\partial \bar{t}_{nm}}\right)\bar{\psi} &= \left(\frac{\partial\bar{W}}{\partial t_{nk}} - \frac{\partial\bar{W}}{\partial \bar{t}_{nm}}\right)\bar{\chi}. \end{aligned}$$

By Remark 3.2.3, we can apply $\partial_{t_{nk}} - \partial_{\bar{t}_{nm}}$ to ψ and $\bar{\psi}$ in the bilinear equation (3.2.8) to obtain

$$\begin{aligned} & \oint \frac{dz}{2\pi i} \left(\left(\frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial \bar{t}_{nm}} \right) \chi(s') \right) (W^*)^{-1} \chi^{-1}(s) \\ &= \oint \frac{dz}{2\pi i} \left(\left(\frac{\partial \bar{W}}{\partial t_{nk}} - \frac{\partial \bar{W}}{\partial \bar{t}_{nm}} \right) \bar{\chi}(s') \right) (\bar{W}^*)^{-1} \bar{\chi}^{-1}(s) \end{aligned}$$

for $s - s' \in \mathbb{Z}$. Using Lemma 3.2.5 as before, we get

$$(W^*)^{-1} \left(\frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial \bar{t}_{nm}} \right)^* = (\bar{W}^*)^{-1} \left(\frac{\partial \bar{W}}{\partial t_{nk}} - \frac{\partial \bar{W}}{\partial \bar{t}_{nm}} \right)^*,$$

or equivalently,

$$\left(\frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial \bar{t}_{nm}} \right) W^{-1} = \left(\frac{\partial \bar{W}}{\partial t_{nk}} - \frac{\partial \bar{W}}{\partial \bar{t}_{nm}} \right) \bar{W}^{-1}.$$

By (3.1.1), the left-hand side of this equation lies in \mathcal{A}_- , while the right-hand side in \mathcal{A}_+ . Therefore, both sides vanish.

To finish the proof of the theorem, it is left to show that if ψ and $\bar{\psi}$ satisfy the bilinear equation (3.2.6), then they satisfy (3.1.11). First, consider the derivatives with respect to t_{nk} and \bar{t}_{nm} for $n \in \mathbb{Z}_{\geq 1}$. As above, we have

$$\frac{\partial \psi}{\partial t_{nk}} = \frac{\partial W}{\partial t_{nk}} \chi + \frac{1}{2} z^{nk} W \chi = \frac{\partial W}{\partial t_{nk}} \chi + \frac{1}{2} L^n W \chi,$$

which implies

$$\left(\frac{\partial}{\partial t_{nk}} - A_n \right) \psi = \left(\frac{\partial W}{\partial t_{nk}} + (L^n)_- W \right) \chi,$$

where A_n is given by (3.1.12). Similarly,

$$\left(\frac{\partial}{\partial t_{nk}} - A_n \right) \bar{\psi} = \left(\frac{\partial \bar{W}}{\partial t_{nk}} - (L^n)_+ \bar{W} \right) \bar{\chi}.$$

We can apply the operator $\partial_{t_{nk}} - A_n$ to ψ and $\bar{\psi}$ in the bilinear equation (3.2.8). By Lemma 3.2.5 again, we obtain

$$\frac{\partial W}{\partial t_{nk}} = \frac{\partial W}{\partial \bar{t}_{nm}} = -(L^n)_- W, \quad \frac{\partial \bar{W}}{\partial t_{nk}} = \frac{\partial \bar{W}}{\partial \bar{t}_{nm}} = (L^n)_+ W,$$

as claimed.

Next, let n be such that k does not divide n . Using (3.1.10), we get

$$\begin{aligned} \left(\frac{\partial}{\partial t_n} - (L^{\frac{n}{k}})_+ \right) \psi &= \left(\frac{\partial W}{\partial t_n} \chi + z^n W \chi - (L^{\frac{n}{k}})_+ W \chi \right) \\ &= \left(\frac{\partial W}{\partial t_n} + (L^{\frac{n}{k}})_- W \right) \chi, \end{aligned}$$

and similarly,

$$\left(\frac{\partial}{\partial t_n} - (L^{\frac{n}{k}})_+ \right) \bar{\psi} = \left(\frac{\partial \bar{W}}{\partial t_n} - (L^{\frac{n}{k}})_+ \bar{W} \right) \bar{\chi}.$$

Applying the operator $\partial_{t_n} - (L^{\frac{n}{k}})_+$ to ψ and $\bar{\psi}$ in (3.2.8) and using Lemma 3.2.5 gives

$$\frac{\partial W}{\partial t_n} + (L^{\frac{n}{k}})_- W = \frac{\partial \bar{W}}{\partial t_n} - (L^{\frac{n}{k}})_+ \bar{W} = 0.$$

Finally, consider the derivatives with respect to the logarithmic variables x_n . By (3.1.13) and $\psi = W\chi$, we see that

$$\begin{aligned} \left(\frac{\partial}{\partial x_n} - (L^n \partial_s + P_n) \right) \psi &= \frac{\partial W}{\partial x_n} \chi + W \frac{\partial \chi}{\partial x_n} - (L^n W \partial_s W^{-1} - (2L^n \log L)_-) \psi \\ &= \frac{\partial W}{\partial x_n} \chi + z^{nk} \log(z) W \chi - z^{nk} \log(z) W \chi + (2L^n \log L)_- W \chi \\ &= \left(\frac{\partial W}{\partial x_n} + (2L^n \log L)_- W \right) \chi. \end{aligned}$$

Similarly,

$$\left(\frac{\partial}{\partial x_n} - (L^n \partial_s + P_n) \right) \bar{\psi} = \left(\frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L)_+ \bar{W} \right) \bar{\chi}.$$

Applying the operator $\partial_{x_n} - (L^n \partial_s + P_n)$ to ψ and $\bar{\psi}$ in (3.2.8) gives

$$\begin{aligned} \oint \frac{dz}{2\pi i} \left(\left(\frac{\partial W}{\partial x_n} + (2L^n \log L)_- W \right) \chi(s') \right) (W^*)^{-1} \chi^{-1}(s) \\ = \oint \frac{dz}{2\pi i} \left(\left(\frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L)_+ \bar{W} \right) \bar{\chi}(s') \right) (\bar{W}^*)^{-1} \bar{\chi}^{-1}(s). \end{aligned}$$

By Lemma 3.2.5, this implies

$$\left(\frac{\partial W}{\partial x_n} + (2L^n \log L)_- W \right) W^{-1} = \left(\frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L)_+ \bar{W} \right) \bar{W}^{-1}.$$

Since the left side is in \mathcal{A}_- and the right side is in \mathcal{A}_+ , both sides must vanish. This completes

the proof of Theorem 3.2.2. □

3.2.3 Bilinear equation for the tau-function

In this subsection, we will derive a bilinear equation satisfied by the tau-function τ of the EBTH. Recall that the wave functions ψ and $\bar{\psi}$ can be expressed in terms of τ by (3.1.14), (3.1.15). Next, we do it for the dual wave functions defined by (3.2.2).

Proposition 3.2.6. *The dual wave functions ψ^* and $\bar{\psi}^*$ of the EBTH can be expressed in terms of the tau-function τ as follows:*

$$\psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \chi^{-1}, \quad (3.2.11)$$

$$\bar{\psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [z], \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \bar{\chi}^{-1}, \quad (3.2.12)$$

where we use the convention of Remark 3.1.4.

Proof. Let us write

$$\psi = w\chi, \quad \bar{\psi} = \bar{w}\bar{\chi}, \quad \psi^* = w^*\chi^{-1}, \quad \bar{\psi}^* = \bar{w}^*\bar{\chi}^{-1},$$

for some functions $w, \bar{w}, w^*, \bar{w}^*$ (cf. (3.1.8), (3.2.2)). Then we can rewrite the bilinear equation (3.2.6) as

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk+s'-s} e^{\xi(\mathbf{t}'-\mathbf{t}, z) - \frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t}, z)} \\ & \quad \times w(s' - \xi(\mathbf{a}, z^k), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \\ & \quad \times w^*(s - \xi(\mathbf{b}, z^k), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \\ & = \oint \frac{dz}{2\pi i} z^{-nm+s'-s} e^{-\xi(\bar{\mathbf{t}}'-\bar{\mathbf{t}}, z^{-1}) + \frac{1}{2}\xi_m(\bar{\mathbf{t}}'-\bar{\mathbf{t}}, z^{-1})} \\ & \quad \times \bar{w}(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}, z) \\ & \quad \times \bar{w}^*(s - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}, z) \end{aligned} \quad (3.2.13)$$

for all $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_1, b_2, \dots)$, $n \in \mathbb{Z}_{\geq 0}$ and $s - s' \in \mathbb{Z}$. Setting $s' = s$, $\mathbf{a} = \mathbf{b} = \mathbf{0}$ in (3.2.13), we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk} e^{\xi(\mathbf{t}'-\mathbf{t}, z) - \frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t}, z)} w(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ & = \oint \frac{dz}{2\pi i} z^{-nm} e^{-\xi(\bar{\mathbf{t}}'-\bar{\mathbf{t}}, z^{-1}) + \frac{1}{2}\xi_m(\bar{\mathbf{t}}'-\bar{\mathbf{t}}, z^{-1})} \bar{w}(s', \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

According to Remark 3.2.4, we can replace z^{nk} in the left-hand side by $f(z^k)$, and z^{-nm} in the

right-hand side by $f(z^{-m})$, for any $f(z) \in \mathbb{C}[[z]]$. If we do it for

$$f(z^k) = e^{\frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t},z)} = \exp \frac{1}{2} \sum_{n=1}^{\infty} (t'_{nk} - t_{nk}) z^{nk} = \exp \frac{1}{2} \sum_{n=1}^{\infty} (\bar{t}'_{nm} - \bar{t}_{nm}) z^{nk},$$

then $f(z^{-m}) = e^{\frac{1}{2}\xi_m(\bar{\mathbf{t}}'-\bar{\mathbf{t}},z^{-1})}$, and we obtain

$$\begin{aligned} & \oint \frac{dz}{2\pi i} e^{\xi(\mathbf{t}'-\mathbf{t},z)} w(s', \mathbf{t}', \bar{\mathbf{t}}, \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} \exp\left(-\sum_{m \nmid i} (\bar{t}'_i - \bar{t}_i) z^{-i}\right) \bar{w}(s', \mathbf{t}', \bar{\mathbf{t}}, \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

Now setting $\bar{t}'_i = \bar{t}_i$ for $m \nmid i$, $\mathbf{t}' = \mathbf{t} + [u^{-1}]$ and using

$$\xi([u^{-1}], z) = \sum_{i=1}^{\infty} \frac{u^{-i}}{i} z^i = -\log\left(1 - \frac{z}{u}\right), \quad (3.2.14)$$

(cf. (1.2.1)), we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} \left(1 - \frac{z}{u}\right)^{-1} w(s, \mathbf{t} + [u^{-1}], \bar{\mathbf{t}}, \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} \bar{w}(s, \mathbf{t} + [u^{-1}], \bar{\mathbf{t}}, \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

Notice that \bar{w} and \bar{w}^* are formal power series of z , while $w - 1$ and $w^* - 1$ are formal power series of z^{-1} (see (3.1.9)). Hence, the right-hand side of this equation vanishes. For the left-hand side, we can use Cauchy's formula: for $f(z) = \sum_{i \in \mathbb{Z}} f_i z^i$,

$$\operatorname{Res}_z \frac{f(z)}{1 - zu^{-1}} = \operatorname{Res}_z \sum_{k \geq 0} \sum_{i \in \mathbb{Z}} f_i u^{-k} z^{k+i} = \sum_{k \geq 0} f_{-k-1} u^{-k} = u f(u)_-.$$

Applying this formula, we obtain

$$u \left(w(s, \mathbf{t} + [u^{-1}], \bar{\mathbf{t}}, \mathbf{x}, u) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, u) - 1 \right) = 0.$$

From this and (3.1.14), we can derive (3.2.11). Equation (3.2.12) is proved similarly; we start

by setting $s' = s - 1$, $\mathbf{a} = \mathbf{b} = 0$ in (3.2.13) and get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk-1} e^{\xi(\mathbf{t}'-\mathbf{t},z)-\frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t},z)} w(s-1, \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-nm-1} e^{-\xi(\bar{\mathbf{t}}'-\bar{\mathbf{t}},z^{-1})+\frac{1}{2}\xi_m(\bar{\mathbf{t}}'-\bar{\mathbf{t}},z^{-1})} \bar{w}(s-1, \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

We then replace z^{nk} on the left hand side with

$$f(z^k) = e^{-\frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t},z)} = \exp -\frac{1}{2} \sum_{n=1}^{\infty} (t'_{nk} - t_{nk}) z^{nk} = \exp -\frac{1}{2} \sum_{n=1}^{\infty} (\bar{t}'_{nm} - \bar{t}_{nm}) z^{nk},$$

then $f(z^{-m}) = e^{-\frac{1}{2}\xi_m(\bar{\mathbf{t}}'-\bar{\mathbf{t}},z^{-1})}$, so we get

$$\begin{aligned} & \oint \frac{dz}{2\pi i} \exp \left(\sum_{k \nmid i} (t'_i - t_i) z^i \right) z^{-1} w(s-1, \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} e^{-\xi(\bar{\mathbf{t}}'-\bar{\mathbf{t}},z^{-1})} z^{-1} \bar{w}(s-1, \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

Finally, setting $t'_i = t_i$ for $k \nmid i$, and $\bar{\mathbf{t}}' = \bar{\mathbf{t}} - [u]$ gives

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{-1} w(s-1, \mathbf{t}, \bar{\mathbf{t}} - [u], \mathbf{x}, z) w^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) \\ &= \oint \frac{dz}{2\pi i} z^{-1} \left(\frac{1}{1-uz^{-1}} \right) \bar{w}(s-1, \mathbf{t}, \bar{\mathbf{t}} - [u], \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z). \end{aligned}$$

The residue to of the left-hand side is 1. We apply a second version of Cauchy's formula for $f(z) = \sum_{i \in \mathbb{Z}} f_i z^i$ to compute the residue of the right-hand side:

$$\text{Res}_z \frac{f(z)}{z-u} = f(u)_+.$$

Using this, we get

$$\bar{w}(s-1, \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x}, z) \bar{w}^*(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = 1,$$

from which (3.2.12) follows. □

Theorem 3.2.7. *A function τ is a tau-function of the EBTH if and only if it satisfies the*

following bilinear equation:

$$\begin{aligned}
& \oint \frac{dz}{2\pi i} z^{nk+s'-s} e^{\xi(\mathbf{t}'-\mathbf{t},z)-\frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t},z)} \\
& \quad \times \tau(s' - \xi(\mathbf{a}, z^k), \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}) \\
& \quad \times \tau(s - \xi(\mathbf{b}, z^k), \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}) \\
& = \oint \frac{dz}{2\pi i} z^{-nm+s'-s} e^{\xi(\bar{\mathbf{t}}-\bar{\mathbf{t}}',z^{-1})-\frac{1}{2}\xi_m(\bar{\mathbf{t}}-\bar{\mathbf{t}}',z^{-1})} \\
& \quad \times \tau(s' + 1 - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}' + [z], \mathbf{x} + \mathbf{a}) \\
& \quad \times \tau(s - 1 - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}} - [z], \mathbf{x} + \mathbf{b}),
\end{aligned} \tag{3.2.15}$$

for all $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_1, b_2, \dots)$, $n \in \mathbb{Z}_{\geq 0}$ and $s - s' \in \mathbb{Z}$.

Proof. First, we plug in (3.2.6) the expressions for ψ , $\bar{\psi}$, ψ^* , $\bar{\psi}^*$ in terms of τ (see (3.1.14), (3.1.15), (3.2.11), (3.2.12)). Then, by Remark 3.2.4, we can replace z^{nk} on the left-hand side of (3.2.6) by

$$z^{nk} \tau(s' - \xi(\mathbf{a}, z^k), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}) \tau(s - \xi(\mathbf{b}, z^k), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}),$$

and z^{-nm} on the right-hand side by

$$z^{-nm} \tau(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}) \tau(s - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}).$$

Therefore, (3.2.6) is equivalent to (3.2.15). \square

If we apply the change of variables from Remark 3.1.3, we get the bilinear equation from [18] (see (85)–(87) there) as a special case of (3.2.15) after setting $\mathbf{x} = \mathbf{0}$, $\mathbf{a} = \mathbf{x}'$, $\mathbf{b} = \mathbf{x}''$ in (3.2.15). Conversely, we can obtain (3.2.15) from the bilinear equation of [18] by observing that if $\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})$ is a tau-function for the EBTH, then so is $\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x} + \mathbf{c})$ for any constant \mathbf{c} .

3.3 Two difference Fay identities for the EBTH

From Theorem 3.2.7, we can derive the following difference Fay identities for the EBTH (cf. [53]). We will again use the shift convention of Remark 3.1.4.

Theorem 3.3.1. *If τ is a tau-function of the EBTH, then for any $\lambda, \mu \in \mathbb{C}^*$, we have*

$$\begin{aligned}
& (\lambda - \mu) \tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) \tau(s - 1, \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \\
& = \lambda \tau(s, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \tau(s - 1, \mathbf{t} - [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \\
& - \mu \tau(s, \mathbf{t} - [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \tau(s - 1, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}}, \mathbf{x})
\end{aligned} \tag{3.3.1}$$

and

$$\begin{aligned}
& (\lambda - \mu)\tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [\lambda] + [\mu], \mathbf{x})\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) \\
& = \lambda\tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [\lambda], \mathbf{x})\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu], \mathbf{x}) \\
& \quad - \mu\tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [\mu], \mathbf{x})\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\lambda], \mathbf{x}).
\end{aligned} \tag{3.3.2}$$

Proof. Using the same trick as in the proof of Proposition 3.2.6, we can rewrite the bilinear equation (3.2.15) as

$$\begin{aligned}
& \oint \frac{dz}{2\pi i} z^{nk+s'-s} e^{\xi(\mathbf{t}'-\mathbf{t}, z)} \\
& \quad \tau(s' - \xi(\mathbf{a}, z^k), \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a})\tau(s - \xi(\mathbf{b}, z^k), \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}) \\
& = \oint \frac{dz}{2\pi i} z^{-nm+s'-s} \exp\left(-\sum_{m \nmid i} (\bar{t}'_i - \bar{t}_i) z^{-i}\right) \\
& \quad \times \tau(s' + 1 - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}' + [z], \mathbf{x} + \mathbf{a})\tau(s - 1 - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}} - [z], \mathbf{x} + \mathbf{b}).
\end{aligned}$$

Then setting

$$n = 0, \quad s' - s = 1, \quad \mathbf{a} = \mathbf{b} = \mathbf{0}, \quad \mathbf{t}' = \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}], \quad \bar{t}'_i = \bar{t}_i,$$

for $m \nmid i$ gives

$$\begin{aligned}
& \oint \frac{dz}{2\pi i} \frac{z}{(1 - z\lambda^{-1})(1 - z\mu^{-1})} \\
& \quad \times \tau(s + 1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})\tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \\
& = \oint \frac{dz}{2\pi i} z \tau(s + 2, \bar{\mathbf{t}} + [\lambda^{-1}] + [\mu^{-1}] + [z], \bar{\mathbf{t}}, \mathbf{x})\tau(s - 1, \mathbf{t} - [z], \bar{\mathbf{t}}, \mathbf{x}) \\
& = 0.
\end{aligned}$$

To compute the residue in the left side, we use

$$\begin{aligned}
\frac{z}{(1 - z\lambda^{-1})(1 - z\mu^{-1})} & = \frac{(1 - \mu z^{-1}) - (1 - \lambda z^{-1})}{(1 - \lambda z^{-1})(1 - \mu z^{-1})} \\
& = \frac{z^{-1}(\lambda - \mu)}{(1 - \lambda z^{-1})(1 - \mu z^{-1})} \\
& = \frac{1}{\lambda^{-1} - \mu^{-1}} \left(\frac{1}{1 - z\lambda^{-1}} - \frac{1}{1 - z\mu^{-1}} \right)
\end{aligned} \tag{3.3.3}$$

and the property that if

$$f(z) = \sum_{i=0}^{\infty} f_i z^{-i},$$

then

$$\begin{aligned} \operatorname{Res}_z \frac{f(z)}{1 - z\lambda^{-1}} &= \operatorname{Res}_z \sum_{i,j=0}^{\infty} z^j \lambda^{-i} f_i z^{-1} \\ &= \sum_{i=0}^{\infty} f_{i+1} \lambda^{-i} \\ &= \lambda \sum_{i=0}^{\infty} f_{i+1} \lambda^{-i-1} \\ &= \lambda(f(\lambda) - f_0). \end{aligned} \tag{3.3.4}$$

We obtain

$$\begin{aligned} &(\lambda - \mu)\tau(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x})\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) \\ &\quad - \lambda\tau(s+1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}, \mathbf{x})\tau(s, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \\ &\quad + \mu\tau(s+1, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}}, \mathbf{x})\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}, \mathbf{x}) = 0, \end{aligned}$$

which gives (3.3.1) after the shift $s \mapsto s-1$, $\mathbf{t} \mapsto \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}]$.

Similarly, to prove (3.3.2), we begin by multiplying the left side of the bilinear equation (3.2.15) by $f(z^k) = -\frac{1}{2}\xi_k(\mathbf{t}' - \mathbf{t}, z)$ and equivalently, the right side by $f(z^{-m}) = \frac{1}{2}\xi_m(\bar{\mathbf{t}} - \bar{\mathbf{t}}', z^{-1})$ and rewrite it as

$$\begin{aligned} &\oint \frac{dz}{2\pi i} z^{nk+s'-s} \exp\left(\sum_{k \nmid i} (t'_i - t_i) z^i\right) \\ &\quad \tau(s' - \xi(\mathbf{a}, z^k), \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a})\tau(s - \xi(\mathbf{b}, z^k), \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}) \\ &= \oint \frac{dz}{2\pi i} z^{-nm+s'-s} e^{\xi(\bar{\mathbf{t}} - \bar{\mathbf{t}}', z^{-1})} \\ &\quad \times \tau(s' + 1 - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}}' + [z], \mathbf{x} + \mathbf{a})\tau(s - 1 - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}} - [z], \mathbf{x} + \mathbf{b}). \end{aligned}$$

Next we set

$$n = 0, \quad s - s' = 1, \quad \mathbf{a} = \mathbf{b} = \mathbf{0}, \quad \bar{\mathbf{t}}' = \bar{\mathbf{t}} - [\lambda] - [\mu] \quad t'_i = t_i$$

for $k \nmid i$ in (3.2.15) to obtain

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{-1} \tau(s-1, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}} - [\lambda] - [\mu], x) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}, x) \\ &= \oint \frac{dz}{2\pi i} z^{-1} \frac{1}{(1 - \lambda z^{-1})(1 - \mu z^{-1})} \tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\lambda] - [\mu] + [z], x) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [z], x). \end{aligned}$$

The residue on the left hand side is computed using

$$\operatorname{Res}_z z^{-1} f(z) = f_0, \quad \text{if } f(z) = \sum_{i=0}^{\infty} f_i z^{-i}$$

and the residue on the right side is computed by applying (3.3.3) and using

$$\operatorname{Res}_z \frac{g(z)}{(1 - \lambda z^{-1})} = \lambda g(\lambda), \quad \text{if } g(z) = \sum_{i \geq 0} g_i z^i$$

(cf. (3.3.4)) to obtain

$$\begin{aligned} & \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [\lambda] - [\mu], x) \tau(s, \mathbf{t}, \bar{\mathbf{t}}, x) = \\ & \frac{\lambda}{\lambda - \mu} \tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\mu], x) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [\lambda], x) \\ & - \frac{\mu}{\lambda - \mu} \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [\lambda], x) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} - [\mu], x). \end{aligned}$$

This gives (3.3.2) if we make the shift $s \mapsto s+1, \bar{\mathbf{t}} \mapsto \bar{\mathbf{t}} + [\lambda] + [\mu]$.

□

CHAPTER 4

DARBOUX TRANSFORMATIONS OF THE EBTH

In this chapter, we begin by reviewing results from [13], which can be applied to factoring difference operators, and will be useful for understanding Darboux transformations of the EBTH. Next, we will review results from [15, 42] which state explicitly how to do Darboux transformations for a Lax operator and wave function of the EBTH. We will use this to obtain our first main result, which states that the action of a Darboux transformation on a wave function for the EBTH amounts to acting on the corresponding tau function with a vertex operator.

4.1 Darboux Transformations of the EBTH

4.1.1 Factoring difference operators

Let \mathcal{R} be a unital ring, and σ an endomorphism of \mathcal{R} . Following [13], we define the skew polynomial ring $\mathcal{R}[x; \sigma]$ as the set of polynomials

$$f(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, \dots, a_n \in \mathcal{R}$. We use the usual definition of addition, and define multiplication by

$$xa = \sigma(a)x$$

for all $a \in \mathcal{R}$.

Definition 4.1.1. *If $f(x)$ is a skew-polynomial, then its σ -evaluation is defined by*

$$f(t; \sigma) = a_n(t\sigma(t) \cdots \sigma^{n-1}(t)) + a_{n-1}(t\sigma(t) \cdots \sigma^{n-2}(t)) + \cdots + a_2(t\sigma(t)) + a_1t + a_0$$

for $t \in \mathcal{R}$.

The following theorem from [13] provides a way to factor skew polynomials.

Theorem 4.1.2. *Let \mathcal{R} be a ring and $f(x) \in \mathcal{R}[x; \sigma]$. Then for some $q(x)$, $f(x) = q(x)(x - t)$ if and only if $f(t; \sigma) = 0$.*

The theorem is originally stated for a $K[x; \sigma]$ where K is a field, for later results. However, because the proof relies on the division algorithm to write $f(x) = q(x)(x - t) + r(x)$, for our purposes, we do not require \mathcal{R} to be a field as we would only need to divide the leading coefficient of $f(x)$ by 1.

Let \mathcal{R} be the ring of functions in the variable s , and $\sigma = \Lambda = e^{\partial_s}$, then any $A \in \mathcal{A}_{\text{fin}} \cap \mathcal{A}_+$ is a skew polynomial over \mathcal{R} . The following corollary is a special case of Theorem 4.1.2 (cf. Lemma 1.3.1).

Corollary 4.1.3. *Let $a = a(s) \in \mathcal{R}$ and $R = (\Lambda - aI)$. Suppose ϕ is some function in \mathcal{R} such that $\phi \in \ker(R)$. A difference operator $A = \sum_{i=0}^n a_i \Lambda^i$ can be factored as $A = QR$ if and only if $\phi \in \ker(A)$.*

Proof. If $\phi \in \ker R$ and $A = QR$, then $A\phi = Q(R\phi) = 0$. On the other hand, suppose that $\phi \in \ker(R)$ and $\phi \in \ker(A)$. Then since

$$R\phi = (\Lambda - aI)\phi = 0,$$

we can write $a = \frac{(\Lambda\phi)}{\phi}$. We first evaluate, for any $n \in \mathbb{Z}^{\geq 0}$,

$$\begin{aligned} a(\Lambda a) \cdots (\Lambda^{n-1} a) &= \frac{(\Lambda\phi)}{\phi} \frac{(\Lambda^2\phi)}{(\Lambda\phi)} \cdots \frac{(\Lambda^{n-1}\phi)}{(\Lambda^{n-2}\phi)} \frac{(\Lambda^n\phi)}{(\Lambda^{n-1}\phi)} \\ &= \frac{(\Lambda^n\phi)}{\phi}. \end{aligned}$$

Next, letting $\sigma = \Lambda$ and $f(\Lambda) = A = \sum_i^n a_i \Lambda^i$, we see that

$$\begin{aligned}
f(a; \Lambda) &= \sum_{i=1}^n a_i (a(\Lambda a) \cdots (\Lambda^{i-1} a)) \\
&= \sum_{i=0}^n a_i \frac{(\Lambda^i \phi)}{\phi} \\
&= \frac{1}{\phi} \sum_{i=0}^n a_i (\Lambda^i \phi) \\
&= \frac{1}{\phi} A \phi \\
&= 0
\end{aligned}$$

since we assumed $\phi \in \ker(A)$. Hence, by Theorem 4.1.2 we can conclude that $A = Q(\Lambda - aI) = QR$. \square

Remark 4.1.4. *If $A \in \mathcal{A}_{\text{fin}}$ is of the form $A = a_1 \Lambda^{-n} + a_2 \Lambda^{-n+1} + \cdots + a_{n+m} \Lambda^m$, the result still holds. Observe that $\Lambda^n A \in \mathcal{A}_+$ and for $\phi \in \ker(A)$,*

$$(\Lambda^n A)\phi = \Lambda^n(A\phi) = 0.$$

Thus, by Corollary 4.1.3, if $R = (\Lambda - aI)$ with $\phi \in \ker(R)$, we can write $\Lambda^n A = QR$, and $A = (\Lambda^{-n}Q)R$. By similar reasoning, we can use an operator R of the form $R = (1 - \tilde{a}\Lambda^{-1})$.

4.1.2 Darboux transformations of L and ψ

Darboux transformations for the ETH were first considered by G. Carlet in [14], and a generalization to the EBTH was given in [42]. The following theorem is equivalent to Theorem 3.4 from [42] and gives a formula for the wave function, $\psi^{[N]}$, and Lax operator, $L^{[N]}$, after N iterations of the Darboux transformation. In order to state the theorem, we need to introduce some notation.

We will suppose that $\mathcal{U} \subset \mathbb{C}$ is an open set such that the wave function $\psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z)$ is defined for $z \in \mathcal{U}$, i.e., the formal power series $w(z)$ from (3.1.9) is convergent for $z \in \mathcal{U}$. Then for $z_i \in \mathcal{U}$, we will denote $\psi_i = \psi|_{z=z_i}$. We define the discrete Wronskian of functions $f_i = f_i(s)$

by

$$\text{Wr}_\Lambda(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \Lambda^{-1}(f_1) & \Lambda^{-1}(f_2) & \cdots & \Lambda^{-1}(f_n) \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda^{-n+1}(f_1) & \Lambda^{-n+1}(f_2) & \cdots & \Lambda^{-n+1}(f_n) \end{vmatrix}.$$

We would like to establish a few facts about the discrete Wronskian for later use.

Lemma 4.1.5. *The functions f_1, f_2, \dots, f_N (where $f_i = f_i(s)$) are linearly independent if and only if $\text{Wr}_\Lambda(f_1, f_2, \dots, f_N) \neq 0$.*

Proof. Suppose f_1, f_2, \dots, f_N are linearly independent. Then for every s , the equation

$$c_1 f_1(s) + c_2 f_2(s) + \cdots + c_N f_N(s) = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_N = 0$. In particular, the system

$$\begin{bmatrix} f_1(s) & f_2(s) & \cdots & f_N(s) \\ f_1(s-1) & f_2(s-1) & \cdots & f_N(s-1) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(s-N+1) & f_2(s-N+1) & \cdots & f_N(s-N+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4.1.1)$$

has only the trivial solution. Since the $N \times N$ matrix in (4.1.1) is the discrete Wronskian matrix, we have shown that its determinant, $\text{Wr}_\Lambda(f_1, f_2, \dots, f_N)$, is nonzero. On the other hand, we suppose that $\text{Wr}_\Lambda(f_1, f_2, \dots, f_N) \neq 0$ for every s , but the equation

$$c_1 f_1(s) + c_2 f_2(s) + \cdots + c_N f_N(s) = 0$$

holds for every s . Then the system (4.1.1) has a solution other than the trivial solution, which contradicts our assumption. \square

The following theorem provides an explicit formulation for Darboux transformations on any wave function ψ and Lax operator L of the EBTH.

Theorem 4.1.6 ([42]). *Let ψ be a wave function for the EBTH and L its corresponding Lax operator. For fixed $N \geq 1$ and $z_1, \dots, z_N \in \mathcal{U}$, consider the difference operator $R^{[N]}$ defined by*

$$R^{[N]} f = (-1)^N \frac{\text{Wr}_\Lambda(\psi_1, \dots, \psi_N, f)}{\text{Wr}_\Lambda(\Lambda^{-1}(\psi_1), \dots, \Lambda^{-1}(\psi_N))},$$

where $\psi_i = \psi|_{z=z_i}$. Then

$$L^{[N]} = R^{[N]}L(R^{[N]})^{-1}, \quad \psi^{[N]} = R^{[N]}\psi$$

are a Lax operator and wave function for the EBTH, which are obtained from L and ψ after N Darboux transformations.

To illustrate the theorem, consider the case of a single Darboux transformation. Then

$$\begin{aligned} \psi^{[1]} &= -\frac{\text{Wr}_\Lambda(\psi_1, \psi)}{\text{Wr}_\Lambda(\Lambda^{-1}(\psi_1))} = -\frac{1}{\Lambda^{-1}(\psi_1)} \begin{vmatrix} \psi_1 & \psi \\ \Lambda^{-1}(\psi_1) & \Lambda^{-1}(\psi) \end{vmatrix} \\ &= \psi - \frac{\psi_1}{\Lambda^{-1}(\psi_1)} \Lambda^{-1}(\psi), \quad \text{where } \psi_1 = \psi|_{z=z_1}. \end{aligned} \quad (4.1.2)$$

Hence,

$$R^{[1]} = I - \frac{\psi_1}{\Lambda^{-1}(\psi_1)} \Lambda^{-1},$$

where I denotes the identity operator (cf. [14]). Notice that $L\psi_1 = z_1\psi_1$ and $R^{[1]}\psi_1 = 0$. Hence, by Corollary 4.1.3, the difference operator $L - z_1I$ factors as

$$L - z_1I = Q^{[1]}R^{[1]}$$

for some difference operator $Q^{[1]}$. Then the new Lax operator $L^{[1]}$ is obtained from the Darboux transformation

$$L^{[1]} - z_1I = R^{[1]}Q^{[1]},$$

and will have a wave function $\psi^{[1]}$. The next Darboux transformation is done the same way, by starting from $L^{[1]}$, $\psi^{[1]}$ and z_2 in place of L , ψ and z_1 , respectively. The significance of Theorem 4.1.6 is that, after N steps, the Lax operator $L^{[N]}$ and wave function $\psi^{[N]}$ can be expressed only in terms of the initial L and ψ .

Lemma 4.1.7. *If L and ψ are solutions to the EBTH, then $L^{[1]}$ and $\psi^{[1]}$ defined as in Theorem 4.1.6 are also solutions to the EBTH.*

Proof. We prove this lemma using Corollary 4.1.3 and the product rule from calculus. Since ψ is assumed to be a solution to the EBTH, it must satisfy the equations (3.1.11). To show that $\psi^{[1]} = R^{[1]}\psi$ is also a solution to the EBTH, as was shown in the proof of Theorem 3.2.6 it suffices to show that $\psi^{[1]}$ also satisfies these equations for some difference operators. For $n \in \mathbb{Z}_{\geq 1} \setminus k\mathbb{Z}$, we have

$$\begin{aligned}
\frac{\partial \psi^{[1]}}{\partial t_n} &= \partial_{t_n}(R^{[1]}\psi) \\
&= \frac{\partial R^{[1]}}{\partial t_n}\psi + R^{[1]}\frac{\partial \psi}{\partial t_n} \\
&= \frac{\partial R^{[1]}}{\partial t_n}\psi + R^{[1]}(L^{n/k})_+\psi \\
&= \left(\frac{\partial R^{[1]}}{\partial t_n} + R^{[1]}(L^{n/k})_+ \right) \psi.
\end{aligned} \tag{4.1.3}$$

Since the operator being applied to ψ in (4.1.3) is an element of \mathcal{A}_{fin} , we also check that

$$\left(\frac{\partial R^{[1]}}{\partial t_n} + R^{[1]}(L^{n/k})_+ \right) \psi_1 = \partial_{t_n}(R^{[1]}\psi_1) = 0.$$

By Corollary 4.1.3 we can conclude that (4.1.3) can be rewritten as

$$B^{[1]}R^{[1]}\psi$$

for some difference operator $B^{[1]} \in \mathcal{A}_{\text{fin}}$. Specifically, we have shown that

$$\frac{\partial \psi^{[1]}}{\partial t_n} = B^{[1]}\psi^{[1]}.$$

Similarly, we have that

$$\begin{aligned}
\frac{\partial \psi^{[1]}}{\partial t_{nk}} &= R^{[1]}\frac{\partial \psi}{\partial t_{nk}} + \frac{\partial R^{[1]}}{\partial t_{nk}}\psi \\
&= \left(R^{[1]}A_n + \frac{\partial R^{[1]}}{\partial t_{nk}} \right) \psi \\
&= A_n^{[1]}R^{[1]}\psi \\
&= A_n^{[1]}\psi^{[1]}
\end{aligned}$$

for some difference operator $A_n^{[1]}$, using the same argument as above. Finally, we check that the derivative with respect to the x_n variables amounts to acting on $\psi^{[1]}$ with a differential-difference

operator (cf. (3.1.11)):

$$\begin{aligned}
\frac{\partial \psi^{[1]}}{\partial x_n} &= R^{[1]} \frac{\partial \psi}{\partial x_n} + \frac{\partial R^{[1]}}{\partial x_n} \psi \\
&= R^{[1]} (L^n \partial_s + P_n) \psi + \frac{\partial R^{[1]}}{\partial x_n} \psi \\
&= R^{[1]} L^n (R^{[1]})^{-1} R^{[1]} \partial_s \psi + R^{[1]} P_n \psi + \frac{\partial R^{[1]}}{\partial x_n} \psi \\
&= R^{[1]} L^n (R^{[1]})^{-1} \left(\partial_s (R^{[1]} \psi) - \frac{\partial R^{[1]}}{\partial s} \psi \right) + R^{[1]} P_n \psi + \frac{\partial R^{[1]}}{\partial x_n} \psi \\
&= (L^{[1]})^n \partial_s (R^{[1]} \psi) - R^{[1]} L^n (R^{[1]})^{-1} \frac{\partial R^{[1]}}{\partial s} \psi + R^{[1]} P_n \psi + \frac{\partial R^{[1]}}{\partial x_n} \psi.
\end{aligned}$$

If we let $\psi = \psi_1$ above, we get from the the last line that

$$(L^{[1]})^n \partial_s (R^{[1]} \psi_1) - R^{[1]} L^n (R^{[1]})^{-1} \frac{\partial R^{[1]}}{\partial s} \psi_1 + R^{[1]} P_n \psi_1 + \frac{\partial R^{[1]}}{\partial x_n} \psi_1 = 0.$$

Clearly, $(L^{[1]})^n \partial_s (R^{[1]} \psi_1) = 0$, so that

$$\left(-R^{[1]} L^n (R^{[1]})^{-1} \frac{\partial R^{[1]}}{\partial s} + R^{[1]} P_n + \frac{\partial R^{[1]}}{\partial x_n} \right) \psi_1 = 0.$$

Again, by Corollary 4.1.3 we can conclude that the operator

$$-R^{[1]} L^n (R^{[1]})^{-1} \frac{\partial R^{[1]}}{\partial s} + R^{[1]} P_n + \frac{\partial R^{[1]}}{\partial x_n}$$

can be factored as $P_n^{[1]} R^{[1]}$ for some difference operator $P_n^{[1]}$, giving us the desired result,

$$\frac{\partial \psi^{[1]}}{\partial x_n} = \left((L^{[1]})^n \partial_s + P_n^{[1]} \right) \psi^{[1]}.$$

This completes the proof of the lemma. □

Before we proceed with our proof of Theorem 4.1.6, we introduce the following lemmas.

Lemma 4.1.8. *Let $\psi_1, \psi_2, \dots, \psi_N$ be such that $L\psi_i = z_i\psi_i$ for $i = 1, 2, \dots, N$. Let $h(L) = (L - z_1 I)(L - z_2 I) \cdots (L - z_N I)$. Then $h(L) = QR$ for some $R = I + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \cdots + a_N \Lambda^{-N}$.*

Proof. Construct a difference operator $R_1 = I + \tilde{a}_1 \Lambda^{-1}$ with $\psi_1 \in \ker R_1$. Then $h(L) = Q_1 R_1$ by Corollary 4.1.3. Similarly, construct an operator $R_2 = I + \tilde{a}_2 \Lambda^{-1}$ such that $R_1 \psi_2 \in \ker(R_2)$. Then since $h(L)\psi_2 = Q_1(R_1 \psi_2) = 0$, using Corollary 4.1.3, we can write $Q_1 = Q_2 R_2$ for some

other difference operator Q_2 . It follows that $h(L) = Q_2 R_2 R_1$. Repeating this process for each remaining ψ_i , we get that $h(L) = Q_N R_N \cdots R_2 R_1 = QR$ where

$$R = \prod_{i=1}^N (I + \tilde{a}_i \Lambda^{-1}) = I + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \cdots + a_N \Lambda^{-N}.$$

□

Lemma 4.1.9. *The operator R from Lemma 4.1.8 is unique and is given by*

$$Rf = (-1)^N \frac{\text{Wr}_\Lambda(\psi_1, \psi_2, \dots, \psi_N, f)}{\text{Wr}_\Lambda((\Lambda^{-1}\psi_1), (\Lambda^{-1}\psi_2), \dots, (\Lambda^{-1}\psi_N))}. \quad (4.1.4)$$

Proof. Using the fact that $R\psi_i = 0$ for each i , we have N equations of the form

$$\psi_i + a_1 \Lambda^{-1} \psi_i + \cdots + a_N \Lambda^{-N} \psi_i = 0$$

yielding the matrix equation

$$\begin{bmatrix} \Lambda^{-1}\psi_1 & \Lambda^{-2}\psi_1 & \cdots & \Lambda^{-N}\psi_1 \\ \Lambda^{-1}\psi_2 & \Lambda^{-2}\psi_2 & \cdots & \Lambda^{-N}\psi_2 \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda^{-1}\psi_N & \Lambda^{-2}\psi_N & \cdots & \Lambda^{-N}\psi_N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -\psi_1 \\ -\psi_2 \\ \vdots \\ -\psi_N \end{bmatrix}. \quad (4.1.5)$$

Since the ψ_i were eigenfunctions of L corresponding to distinct eigenvalues, they are linearly independent. Hence by Lemma 4.1.5, the $N \times N$ matrix in (4.1.5) is invertible, and we can find unique a_i satisfying the system. Observe that the operator in (4.1.4) is also an operator of the form $1 + b_1 \Lambda^{-1} + b_2 \Lambda^{-2} + \cdots + b_N \Lambda^{-N}$ with kernel $\text{span}(\psi_1, \psi_2, \dots, \psi_N)$, so by uniqueness, it is equal to R . □

Now we are in a position to prove Theorem 4.1.6. Our proof differs from the one originally presented in [42] and resembles what is done in [8].

Proof of Theorem 4.1.6. We will use induction on N . For $N = 1$, let $R_1 = (I + a\Lambda^{-1})$ be such that $R_1\psi_1 = 0$. Then

$$R_1 = I - \frac{\psi_1}{(\Lambda^{-1}\psi_1)} \Lambda^{-1} = R^{[1]}.$$

Applying the Darboux transformation once gives

$$\begin{aligned} L^{[1]} &= R_1 Q_1 = R^{[1]} Q_1 = R^{[1]} L (R^{[1]})^{-1} \\ \psi^{[1]} &= R^{[1]} \psi. \end{aligned}$$

We assume the statement is true for the first $N \geq 1$ iterations of the Darboux transformation. Then

$$\psi^{[N]} = R^{[N]}\psi^{[N-1]} = R_N R_{N-1} \cdots R_2 R_1 \psi$$

where $R_i = I - \frac{\psi_{i+1}^{[i]}}{(\Lambda^{-1}\psi_{i+1}^{[i]})}\Lambda^{-1}$ and

$$L^{[N]} = R^{[N]}L(R^{[N]})^{-1}.$$

Observe that $R_i\psi^{[i]}|_{z=z_{i+1}} = 0$. To complete the $(N+1)$ -th step of the Darboux transformation, we use Corollary 4.1.3 and the property that $L^{[N]}\psi^{[N]} = z\psi^{[N]}$ to write

$$L^{[N]} - z_{N+1}I = Q_{N+1}R_{N+1}$$

where

$$R_{N+1} = I - \frac{\psi_{N+1}^{[N]}}{(\Lambda^{-1}\psi_{N+1}^{[N]})}\Lambda^{-1}.$$

Then we obtain

$$\begin{aligned} L^{[N+1]} - z_{N+1}I &= R_{N+1}Q_{N+1} \\ &= R_{N+1}(L^{[N]} - z_{N+1}I)(R_{N+1})^{-1} \\ &= R_{N+1}R_N \cdots R_2 R_1 L R_1^{-1} R_2^{-1} \cdots R_N^{-1} R_{N+1}^{-1} - z_{N+1}I \end{aligned}$$

By the inductive hypothesis, we have that $\text{span}\{\psi_1, \psi_2, \dots, \psi_N\} \subset \ker(R_{N+1}R_N \cdots R_2 R_1)$. We also see that by construction,

$$\begin{aligned} 0 &= R_{N+1}\psi^{[N]}|_{z=z_{N+1}} \\ &= R_{N+1}R_N \cdots R_2 R_1 \psi|_{z=z_{N+1}} \\ &= R_{N+1}R_N \cdots R_2 R_1 \psi_{N+1} \end{aligned}$$

so that $\psi_{N+1} \in \ker(R_{N+1}R_N \cdots R_2 R_1)$. By Lemma 4.1.9, $R^{[N+1]}$ is the unique operator whose kernel is $\text{span}\{\psi_1, \psi_2, \dots, \psi_{N+1}\}$. \square

The authors of [42] also provide the following similar theorem corresponding to Darboux transformations on $\bar{\psi}$. This time, we will suppose that $\mathcal{U} \subset \mathbb{C}$ is an open set such that the wave function $\bar{\psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z)$ is defined for $z \in \mathcal{U}$, i.e., the formal power series $\bar{w}(z)$ from (3.1.9) is convergent for $z \in \mathcal{U}$. Then for $z_i \in \mathcal{U}$, we will denote $\bar{\psi}_i = \bar{\psi}|_{z=z_i}$. We also need to define a

different analog of the Wronskian of functions $f_i = f_i(s)$ by

$$\text{Wr}_\Lambda^+(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \Lambda(f_1) & \Lambda(f_2) & \cdots & \Lambda(f_n) \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda^{n-1}(f_1) & \Lambda^{n-1}(f_2) & \cdots & \Lambda^{n-1}(f_n) \end{vmatrix}.$$

Theorem 4.1.10 ([42]). *Let $\bar{\psi}$ be a wave function for the EBTH and L its corresponding Lax operator. For fixed $N \geq 1$ and $z_1, \dots, z_N \in \mathcal{U}$, consider the difference operator $\bar{R}^{[N]}$ defined by*

$$\bar{R}^{[N]} f = (-1)^N \frac{\text{Wr}_\Lambda^+(\bar{\psi}_1, \dots, \bar{\psi}_N, f)}{\text{Wr}_\Lambda^+(\bar{\psi}_1, \dots, \bar{\psi}_N)},$$

where $\bar{\psi}_i = \bar{\psi}|_{z=z_i}$. Then

$$L^{[N]} = \bar{R}^{[N]} L (\bar{R}^{[N]})^{-1}, \quad \bar{\psi}^{[N]} = \bar{R}^{[N]} \bar{\psi}$$

are a Lax operator and wave function for the EBTH, which are obtained from L and $\bar{\psi}$ after N Darboux transformations.

In this case, we have

$$\bar{R}^{[1]} = \left(\frac{\Lambda \bar{\psi}_1}{\bar{\psi}_1} - \Lambda \right).$$

Since

$$L - z_1^{-1} \bar{\psi} = 0, \quad \bar{R}^{[1]} \bar{\psi}_1 = 0,$$

by Corollary 4.1.3

$$L - z_1^{-1} I = \bar{Q}^{[1]} \bar{R}^{[1]}$$

for some difference operator $\bar{Q}^{[1]}$, and

$$L^{[1]} - z_1^{-1} I = \bar{R}^{[1]} \bar{Q}^{[1]}, \quad \bar{\psi}^{[1]} = \bar{R}^{[1]} \bar{\psi}$$

are a new Lax operator and wave function for the EBTH. This can be verified by a process similar to what was done in Theorem 4.1.6 for ψ .

4.1.3 Action of Darboux transformations on τ

We can now state the main results of this thesis. Using Theorem 4.1.6, Theorem 4.1.10, and the Fay identity (3.3.1), we will prove that the actions of a Darboux transformation corresponding

to both ψ and $\bar{\psi}$ on the tau-function is given by the *vertex operators*

$$\Gamma_+(z) = e^{-\partial_s} e^{\xi(\mathbf{t}, z)} \exp\left(-\sum_{n=1}^{\infty} \frac{\partial_{t_n}}{n} z^{-n}\right), \quad (4.1.6)$$

and

$$\Gamma_-(z) = z^s e^{\partial_s} e^{-\xi(\bar{\mathbf{t}}, z^{-1})} \exp\left(\sum_{n=1}^{\infty} \frac{\partial_{\bar{t}_n}}{n} z^n\right), \quad (4.1.7)$$

respectively. Note that $\exp(-\sum_{n=1}^{\infty} \frac{\partial_{t_n}}{n} z^{-n})$ acts as the shift operator $\mathbf{t} \mapsto \mathbf{t} - [z^{-1}]$, while $e^{-\partial_s} = \Lambda^{-1}$ acts as the shift $s \mapsto s - 1$.

Theorem 4.1.11. *Let ψ be a wave function for the EBTH, and $\psi^{[1]}$ be the wave function after one Darboux transformation on ψ (see (4.1.2)). Let τ and $\tau^{[1]}$ be their corresponding tau-functions. Then $\tau^{[1]} = \Gamma_+(z_1)\tau$, i.e.,*

$$\tau^{[1]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) = e^{\xi(\mathbf{t}, z_1)} \tau(s-1, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x}). \quad (4.1.8)$$

Proof. Using (4.1.2), (3.1.14) and $\Lambda^{-1}(\chi) = z^{-1}\chi$, we express $\psi^{[1]}$ in terms of τ as follows:

$$\begin{aligned} \psi^{[1]} &= \frac{\chi}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) \tau(s-1, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x})} \\ &\times \left(\tau(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \tau(s-1, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \right. \\ &\quad \left. - z^{-1} z_1 \tau(s, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \tau(s-1, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x}) \right). \end{aligned} \quad (4.1.9)$$

On the other hand, again by (3.1.14),

$$\psi^{[1]} = \frac{\tau^{[1]}(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau^{[1]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \chi.$$

Substituting $\tau^{[1]} = \Gamma_+(z_1)\tau$ into the right side of this equation gives

$$\frac{(1 - z^{-1} z_1) \tau(s-1, \mathbf{t} - [z^{-1}] - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s-1, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x})} \chi, \quad (4.1.10)$$

where we used that, by (3.2.14),

$$e^{\xi(\mathbf{t} - [z^{-1}], z_1)} = e^{\xi(\mathbf{t}, z_1)} e^{-\xi([z^{-1}], z_1)} = e^{\xi(\mathbf{t}, z_1)} (1 - z^{-1} z_1). \quad (4.1.11)$$

If we set $\lambda = z$, $\mu = z_1$ in the Fay identity (3.3.1), we see that the above two expressions (4.1.9) and (4.1.10) are equal. Therefore, $\tau^{[1]} = \Gamma_+(z_1)\tau$. \square

If we do N Darboux transformations of τ , we can apply Theorem 4.1.11 repeatedly to obtain the tau-function

$$\tau^{[N]} = \Gamma_+(z_N) \cdots \Gamma_+(z_2) \Gamma_+(z_1) \tau, \quad (4.1.12)$$

which corresponds to the Lax operator $L^{[N]}$ and wave function $\psi^{[N]}$ from Theorem 4.1.6. Multiplying these vertex operators, it follows that

$$\tau^{[N]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) = V_N e^{\sum_{i=1}^N \xi(\mathbf{t}, z_i)} \tau(s - N, \mathbf{t} - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\mathbf{t}}, \mathbf{x}), \quad (4.1.13)$$

where

$$V_N = \prod_{1 \leq i < j \leq N} \left(1 - \frac{z_i}{z_j}\right). \quad (4.1.14)$$

We can verify directly that, for any tau-function τ of the EBTH, the function $\tau^{[N]}$ given by (4.1.13) satisfies the bilinear equations (3.2.15) and hence is a tau-function of the EBTH as well. Here, we illustrate this for the case when $N = 1$. Given a solution τ to the EBTH, we have

$$\tau^{[1]} = \Gamma_+(z_1) \tau = \exp\left(\sum_{n=1}^{\infty} t_n z_1^n\right) \tau(s - 1, \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}}, \mathbf{x}).$$

If we plug $\tau^{[1]}$ into the bilinear equation, the left side becomes

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk+s'-s} e^{\xi(\mathbf{t}'-\mathbf{t}, z) - \frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t}, z)} \\ & \times \exp\left(\sum_{n=1}^{\infty} \left(t'_n - \frac{z^{-n}}{n}\right) z_1^n\right) \tau(s' - 1 - \xi(\mathbf{a}, z^k), \mathbf{t}' - [z_1^{-1}] - [z^{-1}], \bar{\mathbf{t}}', \mathbf{x} + \mathbf{a}) \\ & \times \exp\left(\sum_{n=1}^{\infty} \left(t_n + \frac{z^{-n}}{n}\right) z_1^n\right) \tau(s - 1 - \xi(\mathbf{b}, z^k), \mathbf{t} - [z_1^{-1}] + [z^{-1}], \bar{\mathbf{t}}, \mathbf{x} + \mathbf{b}) \end{aligned}$$

and the right side becomes

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{-nm+s'-s} e^{\xi(\bar{\mathbf{t}}-\bar{\mathbf{t}}', z^{-1}) - \frac{1}{2}\xi_m(\bar{\mathbf{t}}-\bar{\mathbf{t}}', z^{-1})} \\ & \times \exp\left(\sum_{n=1}^{\infty} t'_n z_1^n\right) \tau(s' - \xi(\mathbf{a}, z^{-m}), \mathbf{t}' - [z_1^{-1}], \bar{\mathbf{t}}' + [z], \mathbf{x} + \mathbf{a}) \\ & \times \exp\left(\sum_{n=1}^{\infty} t_n z_1^n\right) \tau(s - 2 - \xi(\mathbf{b}, z^{-m}), \mathbf{t} - [z_1^{-1}], \bar{\mathbf{t}} - [z], \mathbf{x} + \mathbf{b}). \end{aligned}$$

We can see that these are equal if we start with the fact that τ already satisfies (3.2.15), multiply

both sides by $\exp\left(\sum_{n=1}^{\infty} t_n z_1^n\right) \exp\left(\sum_{n=1}^{\infty} t'_n z_1^n\right)$, and then make the shifts $s' \mapsto s' - 1$, $s \mapsto s - 1$, $\mathbf{t} \mapsto \mathbf{t} - [z_1^{-1}]$, $\mathbf{t}' \mapsto \mathbf{t}' - [z_1^{-1}]$. Thus $\tau^{[1]}$ satisfies the bilinear equation and is a solution to the EBTH.

By a similar procedure, we obtain the following result for Darboux transformations done on a wave function $\bar{\psi}$.

Theorem 4.1.12. *Let $\bar{\psi}$ be a wave function for the EBTH and $\bar{\psi}^{[1]}$ the wave function after one Darboux transformation (see Theorem 4.1.10). Let τ and $\tau^{[1]}$ be their corresponding tau-functions. Then $\tau^{[1]} = \Gamma_-(z_1)\tau$ where*

$$\Gamma_-(z) = z^s e^{\partial_s} e^{-\xi(\bar{\mathbf{t}}, z^{-1})} \exp\left(\sum_{n=1}^{\infty} \frac{\partial_{\bar{t}_n}}{n} z^n\right). \quad (4.1.15)$$

Proof. The proof is similar to that of Theorem 4.1.6. We use Theorem 4.1.10, (3.1.15) and $\Lambda(\bar{\chi}) = z\bar{\chi}$, to write the wave function after a single Darboux transformation as

$$\begin{aligned} \bar{\psi}^{[1]} &= \frac{\bar{\chi}}{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x})} \\ &\times \left(z_1 \tau(s+2, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x})\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x}) \right. \\ &\quad \left. - z \tau(s+2, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x})\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x}) \right). \end{aligned} \quad (4.1.16)$$

On the other hand, again by (3.1.14),

$$\bar{\psi}^{[1]} = \frac{\tau^{[1]}(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x})}{\tau^{[1]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \bar{\chi}.$$

Substituting $\tau^{[1]} = \Gamma_-(z_1)\tau$ into the right side of this equation gives

$$\frac{(z_1 - z)\tau(s+2, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + [z], \mathbf{x})}{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x})}. \quad (4.1.17)$$

If we set $\lambda = z_1$, $\mu = z$ in the Fay identity (3.3.2), we see that the above two expressions (4.1.16) and (4.1.17) are equal. Therefore, $\tau^{[1]} = \Gamma_-(z_1)\tau$. □

We can again use the bilinear equation (3.2.15) to show directly that if τ is a tau-function for the EBTH, then

$$\tau^{[N]} = (z_N \cdots z_1)^s \prod_{1 \leq i < j \leq N} (z_i - z_j) e^{\sum_{i=1}^N -\xi(\bar{\mathbf{t}}, z_i^{-1})} \tau(s+N, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + \cdots + [z_N], \mathbf{x}) \quad (4.1.18)$$

is also a tau-function. We will illustrate it here in the case of $\tau^{[1]}$. Plugging

$$\Gamma_{-}(z_1)\tau = z_1^s \exp\left(-\sum_{n=1}^{\infty} \bar{t}_n z_1^{-n}\right) \tau(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x})$$

in to the left side of (3.2.15), gives

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{nk+s'-s} e^{\xi(\mathbf{t}'-\mathbf{t}, z) - \frac{1}{2}\xi_k(\mathbf{t}'-\mathbf{t}, z)} \\ & \times z_1^{s'-\xi(\mathbf{a}, z^k)} \exp\left(-\sum_{n=1}^{\infty} \bar{t}'_n z_1^{-n}\right) \tau(s'+1 - \xi(\mathbf{a}, z^k), \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}' + [z_1], \mathbf{x} + \mathbf{a}) \\ & \times z_1^{s-\xi(\mathbf{b}, z^k)} \exp\left(-\sum_{n=1}^{\infty} \bar{t}_n z_1^{-n}\right) \tau(s+1 - \xi(\mathbf{b}, z^k), \mathbf{t} + [z^{-1}], \bar{\mathbf{t}} + [z_1], \mathbf{x} + \mathbf{b}). \end{aligned} \quad (4.1.19)$$

Plugging into the right side gives

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{-nm+s'-s} e^{\xi(\bar{\mathbf{t}}-\bar{\mathbf{t}}', z^{-1}) - \frac{1}{2}\xi_m(\bar{\mathbf{t}}-\bar{\mathbf{t}}', z^{-1})} \\ & \times z_1^{s'+1-\xi(\mathbf{a}, z^{-m})} \exp\left(-\sum_{n=1}^{\infty} \left(\bar{t}'_n + \frac{z^n}{n}\right) z_1^{-n}\right) \\ & \times \tau(s'+2 - \xi(\mathbf{a}, z^{-m}), \mathbf{t}', \bar{\mathbf{t}} + [z] + [z_1], \mathbf{x} + \mathbf{a}) \\ & \times z_1^{s-1-\xi(\mathbf{b}, z^{-m})} \exp\left(-\sum_{n=1}^{\infty} \left(\bar{t}_n - \frac{z^n}{n}\right) z_1^{-n}\right) \\ & \times \tau(s - \xi(\mathbf{b}, z^{-m}), \mathbf{t}, \bar{\mathbf{t}} - [z] + [z_1], \mathbf{x} + \mathbf{b}). \end{aligned} \quad (4.1.20)$$

Starting with the fact that τ already satisfies (3.2.15), we can use Remark 3.2.4 to multiply the left side by $f(z^k) = z_1^{-\xi(\mathbf{a}, z^k) - \xi(\mathbf{b}, z^k)}$ and the right side by $f(z^{-m}) = z_1^{-\xi(\mathbf{a}, z^{-m}) - \xi(\mathbf{b}, z^{-m})}$.

We also multiply both sides by $\exp\left(-\sum_{n=1}^{\infty} \bar{t}'_n z_1^{-n}\right) \exp\left(-\sum_{n=1}^{\infty} \bar{t}_n z_1^{-n}\right)$ and make the shifts $s \mapsto s+1$, $s' \mapsto s'+1$, $\bar{\mathbf{t}} \mapsto \bar{\mathbf{t}} + [z_1]$, $\bar{\mathbf{t}}' \mapsto \bar{\mathbf{t}}' + [z_1]$, we see that (4.1.19) and (4.1.20) are equal. Therefore $\tau^{[1]}$ satisfies the bilinear equation and is a solution to the EBTH.

Combining the above two results, we can conclude that

$$\Gamma_{\epsilon_N}(z_N) \cdots \Gamma_{\epsilon_1}(z_1) \tau$$

is a tau-function for the EBTH for any choice of signs $\epsilon_i = \pm$ (cf. [34, Chapter 14]).

4.2 Generalized Fay identities

In this section, as an application of Theorems 4.1.6 and 4.1.11, we derive generalized difference Fay identities for the EBTH similar to what was done in [2] for the case of KP hierarchy.

Theorem 4.2.1. *Let ψ be a wave function for the EBTH with a corresponding tau-function τ , and let $\psi_i = \psi|_{z=z_i}$. Then*

$$\begin{aligned} \text{Wr}_\Lambda(\psi_1, \dots, \psi_N) &= \chi_1 \cdots \chi_N \prod_{1 \leq i < j \leq N} (z_j^{-1} - z_i^{-1}) \\ &\times \frac{\tau(s - N + 1, \mathbf{t} - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s - N + 1, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})}, \end{aligned} \quad (4.2.1)$$

where $\chi_i = \chi|_{z=z_i}$.

In this theorem, z_1, \dots, z_N are complex numbers in a certain domain $\mathcal{U} \subset \mathbb{C}$, in which ψ is defined. Alternatively, equation (4.2.1) makes sense as an identity of formal power series in $z_1^{-1}, \dots, z_N^{-1}$, if we write $\psi = w\chi$ for a formal power series w in z^{-1} (see (3.1.9)), while the exponentials in χ are not expanded.

Proof of Theorem 4.2.1. We will prove the claim by induction on N . The case $N = 1$ reduces to (3.1.14) for $z = z_1$, since $\text{Wr}_\Lambda(\psi_1) = \psi_1$. Now suppose that (4.2.1) holds for some $N \geq 1$. By Theorem 4.1.6, we have

$$\psi^{[N]} = (-1)^N \frac{\text{Wr}_\Lambda(\psi_1, \dots, \psi_N, \psi)}{\text{Wr}_\Lambda(\Lambda^{-1}(\psi_1), \dots, \Lambda^{-1}(\psi_N))}.$$

After setting $z = z_{N+1}$, we obtain

$$\psi^{[N]}|_{z=z_{N+1}} = (-1)^N \frac{\text{Wr}_\Lambda(\psi_1, \dots, \psi_N, \psi_{N+1})}{\text{Wr}_\Lambda(\Lambda^{-1}(\psi_1), \dots, \Lambda^{-1}(\psi_N))}.$$

By the inductive assumption, the denominator is given by (4.2.1) after shifting $s \mapsto s - 1$:

$$\begin{aligned} \text{Wr}_\Lambda(\Lambda^{-1}(\psi_1), \dots, \Lambda^{-1}(\psi_N)) &= z_1^{-1} \cdots z_N^{-1} \chi_1 \cdots \chi_N \prod_{1 \leq i < j \leq N} (z_j^{-1} - z_i^{-1}) \\ &\times \frac{\tau(s - N, \mathbf{t} - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s - N, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})}. \end{aligned}$$

On the other hand, again by (3.1.14),

$$\psi^{[N]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau^{[N]}(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau^{[N]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \chi.$$

Let us plug here the formula (4.1.13) for $\tau^{[N]}$ and set $z = z_{N+1}$. Using (4.1.11) as before, we see that

$$\begin{aligned} \tau^{[N]}(s, \mathbf{t} - [z_{N+1}^{-1}], \bar{\mathbf{t}}, \mathbf{x}) &= V_N \prod_{i=1}^N (1 - z_i z_{N+1}^{-1}) e^{\sum_{i=1}^N \xi(\mathbf{t}, z_i)} \\ &\times \tau(s - N, \mathbf{t} - [z_1^{-1}] - \cdots - [z_{N+1}^{-1}], \bar{\mathbf{t}}, \mathbf{x}). \end{aligned}$$

Hence,

$$\psi^{[N]}|_{z=z_{N+1}} = \chi_{N+1} \prod_{i=1}^N (1 - z_i z_{N+1}^{-1}) \frac{\tau(s - N, \mathbf{t} - [z_1^{-1}] - \cdots - [z_{N+1}^{-1}], \bar{\mathbf{t}}, \mathbf{x})}{\tau(s - N, \mathbf{t} - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\mathbf{t}}, \mathbf{x})}.$$

Comparing the above two expressions for $\psi^{[N]}|_{z=z_{N+1}}$, we obtain (4.2.1) with $N + 1$ in place of N . This completes the proof of the theorem. \square

Furthermore, from Theorems 4.1.10 and 4.1.12, we obtain generalized Fay identities with respect to $\bar{\mathbf{t}}$.

Theorem 4.2.2. *Let $\bar{\psi}$ be a wave function for the EBTH with a corresponding tau-function τ , and let $\bar{\psi}_i = \bar{\psi}|_{z=z_i}$. Then*

$$\begin{aligned} \text{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N) &= \bar{\chi}_1 \cdots \bar{\chi}_N \prod_{1 \leq i < j \leq N} (z_j - z_i) \\ &\times \frac{\tau(s + N, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + \cdots + [z_N], \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})}, \end{aligned} \tag{4.2.2}$$

where $\bar{\chi}_i = \bar{\chi}|_{z=z_i}$.

Proof. We will prove this claim by induction on N as well. When $N = 1$, we have

$$\text{Wr}_{\Lambda}^+(\bar{\psi}_1) = \bar{\psi}_1 = \frac{\tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [z_1], \mathbf{x})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \bar{\chi}_1$$

by (3.1.15). Next, we suppose that (4.2.2) holds for some $N \geq 1$. By Theorem 4.1.10, we have

$$\bar{\psi}^{[N]} = (-1)^N \frac{\text{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N, \bar{\psi})}{\text{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N)}.$$

After setting $z = z_{N+1}$, we obtain

$$\bar{\psi}^{[N]}|_{z=z_{N+1}} = (-1)^N \frac{\text{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N, \bar{\psi}_{N+1})}{\text{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N)}.$$

By the inductive assumption, the denominator is given by (4.2.2). On the other hand, again by (3.1.15),

$$\bar{\psi}^{[N]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, z) = \frac{\tau^{[N]}(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x})}{\tau^{[N]}(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x})} \bar{\chi}.$$

We will now plug in the formula (4.1.18) for $\tau^{[N]}$ and set $z = z_{N+1}$. Using (4.1.11) as before, we see that

$$\begin{aligned} \tau^{[N]}(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z_{N+1}], \mathbf{x}) &= (z_{N+1} \cdots z_1)^{s+1} \prod_{1 \leq i < j \leq N} (z_i - z_j) \prod_{i=1}^{N+1} (1 - z_{N+1} z_i^{-1}) \\ &\times e^{\sum_{i=1}^N -\xi(\bar{\mathbf{t}}, z_i^{-1})} \tau(s+N+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + \cdots + [z_N]). \end{aligned}$$

Hence,

$$\bar{\psi}^{[N]}|_{z=z_{N+1}} = \bar{\chi}_{N+1} \prod_{i=1}^N (z_i - z_{N+1}) \frac{\tau(s+N+1, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + \cdots + [z_{N+1}], \mathbf{x})}{\tau(s+N, \mathbf{t}, \bar{\mathbf{t}} + [z_1] + \cdots + [z_N], \mathbf{x})}.$$

Comparing these two expressions and writing

$$\mathrm{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N, \bar{\psi}_{N+1}) = (-1)^N \mathrm{Wr}_{\Lambda}^+(\bar{\psi}_1, \dots, \bar{\psi}_N) \bar{\psi}^{[N]}|_{z=z_{N+1}},$$

we obtain (4.2.2) with $N+1$ in place of N . Thus, we have completed the proof of the theorem. \square

CHAPTER 5

CONCLUSION

5.1 Conclusion

In this thesis, we proved a bilinear equation for the extended bigraded Toda hierarchy (EBTH), which is equivalent to the bilinear equation of Carlet and van de Leur [18] after a change of variables but uses Takasaki's more convenient notation from [54]. Our proof is also shorter than the one originally presented in [18]. From the bilinear equation, we derived difference Fay identities for the EBTH and showed that the action of the Darboux transformations on the wave functions ψ , $\bar{\psi}$ corresponds to acting on the tau-function by certain vertex operators Γ_+ , Γ_- . As an application, we obtained generalized Fay identities for the EBTH.

5.2 Future Directions

In this section we provide an overview of possible projects to which the results presented in this thesis can be applied.

5.2.1 Tau function for the EBTH

A natural question is to determine explicitly the initial tau-function corresponding to the trivial Lax operator $L = \Lambda^k + \Lambda^{-m}$, from which we can generate other solutions of the EBTH with Darboux transformations. Wave functions for this Lax operator were given in [14, 42] in the cases $k = m = 1$ and $k = m = 2$, but they correspond to a wave function ϕ satisfying $L\phi = (z^k + z^{-m})\phi$, not $L\psi = z^k\psi$. We would like to determine the initial tau-function for the version of the EBTH presented here.

5.2.2 Equivalence of Fay identities to ETH and EBTH

In [56] it was shown that if a function satisfies certain Fay identities for the KP hierarchy, then it is a tau function for the hierarchy; in other words, the Fay identities are equivalent to the entire system. Analogous results for the 2D Toda hierarchy were obtained in [58]. These results are quite useful, because, in general, showing that a function satisfies the Fay identities is simpler than showing that it satisfies the bilinear equation. An interesting follow up to this thesis would be to obtain similar results for the ETH and EBTH.

5.2.3 \mathcal{W} -Algebras

Another interesting question is whether one can generate a \mathcal{W} -algebra from the vertex operators Γ_+ and Γ_- , as was done for the KP hierarchy in [5, 6, 21]. One can construct a Virasoro algebra based on [12, 24], but it would be interesting to try to construct a more general \mathcal{W} -algebra of symmetries by modifying the vertex operators Γ_+ and Γ_- (cf. [7, 11, 46]).

5.2.4 Bispectral problem

Recall the bispectral problem discussed in Subsection 1.3.3. An algebra A of differential operators is called *bispectral* if there exists an eigenfunction $\Psi(x, z)$ for which (1.3.9) holds for every $L \in A$. In [8, 9, 10], the bispectral problem was considered for the eigenfunctions $\Psi(x, z)$ corresponding to wave functions of the KP hierarchy. Bispectral operators were obtained by applying Darboux transformations to specific wave functions known as the *Bessel* and *Airy wave functions*. We would like to use our results about Darboux transformations to find solutions to the bispectral problem [25] for the EBTH (cf. [8, 9, 10]). The bispectral problem was first extended to difference operators in the case of the discrete KP hierarchy in [37] and then expanded upon in [32].

5.2.5 Cluster Algebras

Cluster algebras were introduced in 2002 by Fomin and Zelevinsky as a method of understanding dual canonical bases and total positivity in semisimple groups [26, 27, 28]. They have connections to many fields, including, but not limited to, Poisson geometry [29], combinatorics [17], string theory [1], and algebraic geometry [36]. In [29], Gekhtman, Shapiro, and Vainshtein construct Bäcklund-Darboux transformations of the 2D Toda hierarchy in terms of cluster algebras using standard facts about the Poisson-Lie structure of GL_n . A possible future direction would be to extend these results to the ETH and EBTH.

5.2.6 Random Matrices

A *random matrix* is an $N \times N$ matrix whose entries are random variables. They most commonly arise in probability, statistics, and physics, but due to Jimbo, Miwa, Mori and Sato [39, 40] have been shown to be connected to the theory of integrable hierarchies. For example, in [4], Adler, Shiota and van Moerbeke use the vertex operator corresponding to Darboux transformations for the KP hierarchy and some combinatorics to develop techniques in random matrix theory. Another possible future direction would be to establish connections between the ETH and EBTH and random matrices using some of the results presented in this thesis.

REFERENCES

- [1] M. Ali et al, *BPS quivers and spectra of complete $\mathcal{N} = 2$ quantum field theories*. Comm. Math. Phys. **323** (2013), no. 3, 1185–1227.
- [2] M. Adler, T. Shiota, and P. van Moerbeke, *From the w_∞ algebra to its central extension: a τ -function approach*. Phys. Lett. A **194** (1994), 33–43.
- [3] M. Adler, T. Shiota, and P. van Moerbeke, *A Lax representation for the vertex operator and the central extension*. Comm. Math. Phys. **171** (1995), 547–588.
- [4] M. Adler, T. Shiota, and P. van Moerbeke, *Random matrices, vertex operators and the Virasoro algebra*. Phys. Lett. A **208** (1995), 67–78.
- [5] M. Adler and P. van Moerbeke, *A matrix integral solution to two-dimensional W_p -gravity*. Comm. Math. Phys. **147** (1992), 25–56.
- [6] M. Adler and P. van Moerbeke, *Birkhoff strata, Bäcklund transformations, and regularization of isospectral operators*. Adv. Math. **108** (1994), 140–204.
- [7] B. Bakalov and T. Milanov, *\mathcal{W} -constraints for the total descendant potential of a simple singularity*. Compositio Math. **149** (2013), 840–888.
- [8] B. Bakalov, E. Horozov, and M. Yakimov, *Bäcklund–Darboux transformations in Sato’s Grassmannian*. Serdica Math. J. **22** (1996), 571–586.
- [9] B. Bakalov, E. Horozov, and M. Yakimov, *General methods for constructing bispectral operators*. Phys. Lett. A. **222** (1996), 59–66.
- [10] B. Bakalov, E. Horozov, and M. Yakimov *Bispectral algebras of commuting ordinary differential operators*. Comm. Math. Phys. **190** (1997), 331–373.
- [11] B. Bakalov and M. Sullivan, *Twisted logarithmic modules of lattice vertex algebras*. Trans. Amer. Math. Soc. (2018), published online, 33 pp.; <https://doi-org.prox.lib.ncsu.edu/10.1090/tran/7703>.

- [12] B. Bakalov and W. Wheeler, *Additional symmetries of the extended bigraded Toda hierarchy*. J. Phys. A **49** (2016), 055201, 25 pp.
- [13] J. Bergen, M. Giesbrecht, P.N. Shivakumar, and Y. Zhang, *Factorizations for difference operators*. Adv. Difference Equ. **2015**, 2015:57, 6 pp.
- [14] G. Carlet, *Extended Toda hierarchy and its Hamiltonian structure*. Ph.D. thesis (2003), SISSA; <http://gcarlet.droppages.com/pdf/phdth.pdf>.
- [15] G. Carlet, *The extended bigraded Toda hierarchy*. J. Phys. A **39** (2006), 9411–9435.
- [16] G. Carlet, B. Dubrovin, and Y. Zhang, *The extended Toda hierarchy*. Mosc. Math. J. **4** (2004), 313–332.
- [17] F. Chapoton, S. Fomin, and A. Zelevinsky, *Polytopal realizations of generalized associahedra*. Canad. Math. Bull. **45** (2002), 537–566.
- [18] G. Carlet and J. van de Leur, *Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{C}P^1$ orbifolds*. J. Phys. A **46** (2013), no. 40, 405205, 16 pp.
- [19] R.V. Churchill and J.W. Ward, *Complex variables and applications*. Fourth edition. McGraw-Hill Book Co., New York, 1984. x+339pp.
- [20] G. Darboux, *Leçons sur la théorie générale des surfaces*, 2ème partie. Paris: Gauthiers-Villars, 1889.
- [21] L.A. Dickey, *On additional symmetries of the KP hierarchy and Sato’s Bäcklund transformation*. Comm. Math. Phys. **167** (1995), 227–233.
- [22] L.A. Dickey, *Soliton equations and Hamiltonian systems*. Second edition. Adv. Ser. in Math. Phys., vol. 26. World Scientific, River Edge, NJ, 2003.
- [23] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*. Compositio Math. **111** (1998), 167–219.

- [24] B. Dubrovin and Y. Zhang, *Virasoro symmetries of the extended Toda hierarchy*. Comm. Math. Phys. **250** (2004), 161–193.
- [25] J.J. Duistermaat and F.A. Grünbaum, *Differential equations in the spectral parameter*. Comm. Math. Phys. **103** (1986), 177–240.
- [26] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*. J. Amer. Math Soc. **15** (2002), no. 2, 497–529.
- [27] S. Fomin and A. Zelevinsky, *Cluster algebras. II. Finite type classification*. Invent. Math. **154** (2003), no. 1 63–121.
- [28] S. Fomin and A. Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*. Duke Math. J. **126** (2005), no. 1, 1–52.
- [29] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster Algebras and Poisson Geometry*. Mathematical Surveys and Monograph, **167**, Amer. Math. Soc., Providence, RI, 2010
- [30] E. Getzler, *The Toda conjecture*. In: “Symplectic geometry and mirror symmetry” (Seoul, 2000), 51–79, World Sci., River Edge, NJ, 2001.
- [31] F.A. Grünbaum, *The limited angle reconstruction problem in computer tomography*. Proc. Symp. Appl. Math. **27**, AMS, L. Shepp (ed.) (1982)
- [32] F.A. Grünbaum and M. Yakimov, *Discrete bispectral Darboux transformations from Jacobi operators*. Pacific J. Math. **204** (2002), no. 2, 395–431.
- [33] C. Gu, H. Hu, and Z. Zhou, *Darboux transformations in integrable systems*. Mathematical Physics Studies, 26. Springer, Dordecht, 2005.
- [34] V.G. Kac, *Infinite dimensional Lie algebras*, third edition. Cambridge University Press, Cambridge, 1990.
- [35] V.G. Kac, *Vertex algebras for beginners*. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997. viii+141 pp.

- [36] M. Kontsevich and Y. Soibelman, *Stability structures, Donaldson-Thomas invariants and cluster transformations*. Preprint (2008), arXiv:0811.2435.
- [37] L. Haine and P. Iliev, *Commutative rings of difference operators and an adelic flag manifold*. Internat. Math. Res. Notices **2000**, no. 6, 261–323.
- [38] E.L. Ince, *Ordinary Differential Equations*. Dover Publications, New York, 1944. viii+558 pp.
- [39] M. Jimbo, T. Miwa, Y. Môri, M. Sato, *Studies on holonomic quantum fields. XVI. Density matrix of impenetrable Bose gas*. Proc. Japan. Acad. Ser. A Math. Sci. **55** (1979), 317–322
- [40] M. Jimbo, T. Miwa, Y. Môri, M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*. Phys. D. **1** (1980), 80–158.
- [41] C. Li, J. He, K. Wu, and Y. Cheng, *Tau function and Hirota bilinear equations for the extended bigraded Toda hierarchy*. J. Math. Phys. **51** (2010), no. 4, 043515, 32 pp.
- [42] C. Li and T. Song, *Multifold Darboux transformations of the extended bigraded Toda hierarchy*. Z. Naturforschung A. **71** (2016), no. 4, 357–371.
- [43] F. Magri and J.P. Zubelli, *Bi-Hamiltonian formalism and the Darboux–Crum method. I. From the KP to mKP hierarchy*. Inverse Problems **13** (1997), 755–780.
- [44] V.B. Matveev and M.A. Salle, *Darboux Transformations and solitons*. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.
- [45] T. Milanov, *Gromov–Witten theory of $\mathbb{C}P^1$ and integrable hierarchies*. Preprint (2006); <https://arxiv.org/pdf/math-ph/0605001.pdf>.
- [46] T. Milanov, *Hirota quadratic equations for the extended Toda hierarchy*. Duke Math. J. **138** (2007), 161–178.
- [47] T. Milanov and H.H. Tseng, *The spaces of Laurent polynomials, Gromov–Witten theory of \mathbb{P}^1 -orbifolds, and integrable hierarchies*. J. Reine Angew. Math. **622** (2008), 189–235.

- [48] T. Miwa, M. Jimbo, and E. Date *Solitons*. Differential equations, symmetries, and infinite dimensional algebras. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, 200. x+108 pp.
- [49] Y. Ogawa, *On the (2+1)-dimensional extension of 1-dimensional Toda lattice hierarchy*. J. Nonlinear Math. Phys. **15** (2008), no. 1, 48–65.
- [50] A. Okounkov and R. Pandharipande, *The equivariant Gromov–Witten theory of \mathbb{P}^1* . Ann. of Math. (2) **163** (2006), 561–605.
- [51] M. Sato, *Soliton equations and the universal Grassmann manifold (by Noumi in Japanese)*. Math. Lect. Note Ser. **18** (1984), Sophia University, Tokyo.
- [52] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*. Invent. Math. **83** (1986), no. 2, 333–382.
- [53] K. Takasaki, *Differential Fay identities and auxiliary linear problem of integrable hierarchies*. Exploring new structures and natural constructions in mathematical physics, 387–441, Adv. Stud. Pure Math., **61**, Math. Soc. Japan, Tokyo, 2011.
- [54] K. Takasaki, *Two extensions of 1D Toda hierarchy*. J. Phys. A **43** (2010), no. 43, 434032, 15 pp.
- [55] K. Takasaki, *Toda hierarchies and their applications*. J. Phys. A **51** (2018), 203001, 35 pp.
- [56] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*. Rev. Math. Phys. **7** (1995), no. 5, 743–808.
- [57] A.-P.-E. ten Kroode and M.-J. Bergvelt, *The homogeneous realization of the basic representation of $A_1^{(1)}$ and the Toda lattice*. Lett. Math. Phys. **12** (1986), no. 2, 139–147.
- [58] L.P. Teo, *Fay-like identities of the Toda lattice hierarchy and its dispersionless limit*. Rev. Math. Phys. **18**, no. 10, 1055–1073.

- [59] M. Toda, *Wave Propagation in Anharmonic Lattices*. J. Phys. Soc. Japan **23** (1967), 501–506.
- [60] K. Ueno and K. Takasaki, *Toda lattice hierarchy*. In: “Group representations and systems of differential equations” (Tokyo, 1982), 1–95, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.
- [61] P. van Moerbeke, *Integrable foundations of string theory*. In: “Lectures on integrable systems” (Sophia-Antipolis, 1991), 163–267, World Sci. Publ., River Edge, NJ, 1994.
- [62] W.B. Wheeler, *Additional symmetries of the extended Toda hierarchy*. Ph.D. thesis (2015), NC State University; <http://www.lib.ncsu.edu/resolver/1840.16/10461>.
- [63] Y. Zhang, *On the CP^1 topological sigma model and the Toda lattice hierarchy*. J. Geom. Phys. **40** (2002), 215–232.