ABSTRACT

ROGERS, ELYSE SUZANNE. Multipliers and Covers of Leibniz Algebras. (Under the direction of Dr. Kailash Misra and Dr. Ernest Stitzinger).

Leibniz algebras are a generalization of Lie algebras in which the anti-commutative property is no longer required. A Leibniz algebra is simply a vector space over a field in which all elements must satisfy the Leibniz identity. Imposing anti-commutativity on the Leibniz identity, gives the Jacobi identity, showing that a Lie algebra is a particular type of Leibniz algebra. A common theme in algebra research is extension theory. The dimension of a maximal central extension, called the cover, of a given algebra is found by looking at the dimension of the multiplier, the subset of the center that is added on to algebra in order to find the central extension. The covers and Schur multiplier have been studied in both group theory and Lie theory. In this research, we determine the structure and dimension of the multiplier for a Leibniz algebra. Although the algebras of matrices in the lower central series of the $n \times n$ strictly upper triangular matrices can be considered as Lie algebras, we have instead treated them as Leibniz algebras satisfying the Leibniz identity, hence we have considered the algebraic structure of both the cover and the multiplier of this algebra to also be Leibniz. We have determined the dimension of the multiplier for the $n \times n$ strictly upper triangular matrices, the $n \times n$ upper triangular matrices, and the general formula for the dimension of the multiplier of each algebra in the lower central series of the strictly upper triangular matrices. Finally the dimension of the multiplier of the Heisenberg algebra and each of the Leibniz variations of the Heisenberg algebra are found. From each of these, the dimension of the cover is found in each case. It is known in Lie theory that the covers of a particular algebra are isomorphic and the same has been determined in this research for Leibniz algebras. This is not the case for group theory.

Multiplies and Covers of Leibniz Algebras

by
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DEDICATION

To my mother and father for their constant love and support, and to my grandfather's memory; the man who always believed in me and became my first student when I was just a small child with big dreams of becoming a math teacher.

BIOGRAPHY

Elyse Suzanne Rogers was born in Epsom, England in 1993 to Royston and Linda Rogers. She grew up in London, England and attended The Holy Cross School in New Malden, graduating in 2011. Elyse studied at the University of Surrey where she earned an MMath in Mathematics in 2015. This included a 6 month study abroad program at North Carolina State University. Having attained her Master's degree from the University of Surrey, Elyse returned to North Carolina State University in the summer of 2015 to further her graduate studies. 2019 saw the completion of her doctoral degree in Mathematics under the advisement of Dr. Misra and Dr. Stitzinger. Elyse will now commence her career in academia in the fall of 2019 as an Assistant Professor of Mathematics at Taylor University.

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Chapter 1

Introduction

From [1] and [2], a Leibniz algebra, A, is a vector space over a field F equipped with the bilinear map

$$[\ ,\ ] : A \times A \to A$$

such that all $x, y, z \in A$ satisfy the Leibniz identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

For a Leibniz algebra, the left multiplication operator is defined by $L_x : A \to A$ where $L_x(y) = [x, y]$, $\forall\ y \in A$ and the right multiplication operator is defined by $R_x : A \to A$ where $R_x(y) = [y, x]$, $\forall\ y \in A$. A Leibniz algebra is a generalization of a Lie algebra. As the left and right multiplication by an element in a Leibniz algebra are not necessarily related, this gives many more brackets of elements to consider. However, if we were to allow $[x, y] = -[y, x]$ for all elements $x, y \in A$, then the Leibniz identity would become the Jacobi identity and A would be a Lie algebra. This shows that a Lie algebra is a particular type of Leibniz algebra.

For a finite dimensional Leibniz algebra A generated by a set, X, let $\alpha : X \to A$ be a mapping. Then there is a free Leibniz algebra F and a homomorphism $\pi : F \to A$ that is an extension of α. Taking $R = Ker(\pi)$, then by definition, R is an ideal of F and so we have the following free presentation of the Leibniz algebra A.

$$0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$$

The ideas established for multipliers and covers were first discovered by Schur in group theory, as seen in [3] and [4]. He said, that for a group G, there is a pair (H, A) such that

1. $G \cong H/A$
2. $A \subseteq Z(H) \cap [H, H]$

He found that the second members of the defining pairs for groups were bounded. Taking the H of maximal order for a group G gives a second member A that is isomorphic to a group that was defined as the multiplier of G. In group theory, he went on to show that all the second members of maximal defining pairs of the group G are isomorphic and are defined to be the second cohomology group.

This is also the case in Lie and Leibniz algebra as the second members of the maximal defining pairs of the algebra are abelian Lie algebras and therefore they are unique up to isomorphism. The first thing that is seen in this research is the fact that the first members of the maximal defining pairs of a Leibniz algebra are isomorphic. The same was found in the Lie algebra case by Peggy Batten in [8] and [9]. This is in fact not the case in group theory. This is one of several things that has been discovered to be different in group theory compared to Lie and Leibniz algebras.

Following this theory, we will study examples of covers and multipliers of different Lie algebras and non-Lie, Leibniz algebras. Taking A to be the particular algebra in question, we will denote K to be its cover and M to be its multiplier. From the above free presentation of a Leibniz algebra, we know there exists a homomorphism $\pi : K \to A$ and therefore a section μ can be used to take the basis elements of A back into the cover K. Using these basis elements of K, the Leibniz identity, and a particular Leibniz bracket structure, we will be able to find linear relations between the basis elements of M, hence finding its structure and dimension. From this, the dimension of K can be found.

In chapter 3, we will discuss the cover and multiplier of the $n \times n$ strictly upper triangular matrices. This is followed by chapter 4 where this idea is extended to establish a general formula for the dimension of the multiplier of each algebra in the lower central series of the $n \times n$ strictly upper triangular matrices. In [5] and [6], the dimension of the multiplier of each of these algebras was found when a Lie cover was used, however we will now look at the dimension of the Leibniz cover of each algebra. In chapter 5, the cover and multiplier of the $n \times n$ upper triangular matrices is studied.

Finally in chapter 6, we look at 3 different types of Heisenberg algebra, the first of which is the regular Lie Heisenberg algebra and the second and third types are Leibniz algebras that each take on a slightly different structure and hence are non-Lie Leibniz algebras. The covers and multipliers are studied for these algebras and as we will see, the dimension of each of the multipliers is the same in all 3 cases.

Chapter 2

The Uniqueness of the Cover and Multiplier

Definition 2.0.1. *A pair of algebras (K, M) is said to be a defining pair for the Leibniz algebra L, if:*

1. $L \cong K/M$
2. $M \subseteq Z(K) \cap K^2$

Lemma 2.0.2. *If $dim(K/Z(K)) = n$ then $dim([K, K]) \leq n^2$.*

Proof. Letting $\{x_1, ..., x_n\}$ be a basis for $K/Z(K)$, then the generating set of $[K, K]$ is $\{[x_i, x_j] : 1 \leq i, j \leq n\}$ as the elements within $Z(K)$ are central by definition and therefore $[K/Z(K), K/Z(K)] = [K, K]$. Hence, $dim([K, K]) \leq n^2$. □

Lemma 2.0.3. *If L is a finite dimensional Leibniz algebra, with dimension m, and K is the first term in the defining pair for L then $dim(K) \leq m(m + 1)$.*

Proof. As we know that $dim(K/Z(K)) \leq dim(K/M) = dim(L) = m$, then from the previous lemma, $dim(M) \leq dim([K, K]) \leq m^2$. Therefore, $dim(K) = dim(L) + dim(M) \leq m + m^2 = m(m + 1)$. □

This shows that for a finite dimensional algebra $L \cong K/M$, both members of the defining pair, K and M must be bounded. Therefore, the pair $(K.M)$ is a maximal defining pair if K is of maximal dimension, and satisfies the above conditions. In this case, K is called the cover of L and M is the multiplier of L. As M is central, it is an abelian algebra. Hence, it is a unique Lie algebra while K is a Leibniz algebra.

Just as in the case with Lie algebras, but unlike the case in group theory, the covers of L can be shown to be isomorphic.

The following definition is found in [8] and the same definition can be used for Leibniz algebras.

Definition 2.0.4. *For the following set*

$$C(L) = \{(J, \lambda) : \lambda \in Hom(J, L), \lambda \text{ surjective, } Ker(\lambda) \subseteq Z(J) \cap [J, J]\}$$

an element (H, π) is a universal element in the set if for every $(J, \lambda) \in C(L)$, there is a homomorphism $\beta \in Hom(H, J)$ such that $\lambda \circ \beta = \pi$.

This can be seen in the following diagram:

$$\begin{tikzcd} & J \arrow[dr, "\lambda"] & \\ H \arrow[ur, "\beta"] \arrow[rr, "\pi"] & & L \end{tikzcd}$$

Figure 2.1 Showing $\lambda \circ \beta = \pi$

It can be seen that defining pairs of L are directly related to the elements of the set $C(L)$. Taking an element $(J, \lambda) \in C(L)$, then from the definition of $C(L)$ that $Ker(\lambda) \subseteq J^2 \cap Z(J)$ and from the isomorphism theorem, $L \cong J/Ker(\lambda)$. This is the exact definition for $(J, Ker(\lambda))$ to be a defining pair for L. Then looking at it the other way around, if (J, M) is a defining pair for L, then by the isomorphism theorems, there is a surjective homomorphism $\lambda : J \to L$ where $M = Ker(\lambda) \subseteq Z(J) \cap J^2$. This is the exact definition of $(J, \lambda) \in C(L)$.

Using the idea that the elements of $C(L)$ and the defining pairs of L are directly related to each other, we will show that the element (K, λ) in $C(L)$ is a universal element of $C(L)$ if and only if K is actually a cover of L. This will be the strategy to show that all covers of L are isomorphic. The following lemmas have been adapted from the Lie case in [8] having been modified to satisfy Leibniz algebras.

Lemma 2.0.5. *If K is a finite dimensional Leibniz algebra, then $Z(K) \cap K^2$ is contained in every maximal subalgebra of K.*

Proof. If A is a maximal subalgebra of K, then $(Z(K) \cap K^2) + A$ is also a subalgebra of K. However, as A is maximal, then either $(Z(K) \cap K^2) + A = A$ or $(Z(K) \cap K^2) + A = K$. If $(Z(K) \cap K^2) + A = K$, then $[K, K] = [(Z(K) \cap K^2) + A, (Z(K) \cap K^2) + A] = [A, A] \subset A$. This says $[K, K] \subset A$ but this contradicts that $(Z(K) \cap K^2) + A \neq A$. Therefore, $(Z(K) \cap K^2) + A = A$ and $(Z(K) \cap K^2) \subset A$, so $(Z(K) \cap K^2)$ is contained in every maximal subalgebra of K. $\square$

Lemma 2.0.6. *Let $(J, \lambda) \in C(L)$ and let $\pi \in Hom(K, L)$, where $\pi(K) = L$. Suppose that $\beta \in Hom(K, J)$ such that $\lambda \circ \beta = \pi$. Then $\beta(K) = J$, so β is surjective.*

$$\begin{tikzcd} & J \arrow[dr, "\lambda"] & \\ K \arrow[ur, "\beta"] \arrow[rr, "\pi"] & & L \end{tikzcd}$$

Figure 2.2 Showing $\lambda \circ \beta = \pi$

Proof. As λ takes J to L and π takes K to L, we know that for some $j \in J$ and some $k \in K$, $\lambda(j) = \pi(k)$. Then $\pi(k) = \lambda \circ \beta(k) = \lambda(j)$ by definition. Therefore, $\beta(k) - j \in Ker(\lambda)$ and $J = Ker(\lambda) + Im(\beta)$ as $\beta(k)$ is in the image of β, and $j \in J$. Then by definition, $Ker(\lambda) \subseteq Z(J) \cap J^2$ and from the previous lemma, $Z(J) \cap J^2$ is contained in every maximal subalgebra of J. Suppose $Im(\beta) \neq J$, then as it is a subalgebra of J, it must be contained in some maximal subalgebra A of J where $A \neq J$. Therefore, $Im(\beta) + (Z(J) \cap J^2) \subseteq A$. But $Ker(\lambda) \subseteq Z(J) \cap J^2$, therefore this would mean $J = A$ which is a contradiction. Therefore $Im(\beta) = J$, meaning that β is surjective. $\square$

Bringing together these lemmas and definitions, if $(K, \pi) \in C(L)$ is such that for every $(J, \lambda) \in C(L)$ there is a homomorphism $\beta \in Hom(K, J)$ such that $\lambda \circ \beta = \pi$, then from the previous lemmas, β is a surjective homomorphism and $dim(J) \leq dim(K)$. Therefore, K has maximal dimension and hence it is a cover of L. All covers are therefore isomorphic to one another as each cover must have the same dimension and must be the homomorphic image of of K. So it has been shown that as long as a universal element (K, π) exists, it will lead to the covers, K of L, to be isomorphic to each other. Therefore, it is left to show the existence of a universal element in the Leibniz case.

We use the following free presentation for the Leibniz algebra L:

$$0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$$

where $R = Ker(\pi)$.

For the Leibniz algebra, we must be aware that when factoring out Leibniz brackets of algebras, both the left and right brackets must be factored out in order to not get a one sided ideal.

So let

$$B = R/([F, R] + [R, F])$$

$$C = F/([F, R] + [R, F])$$

$$D = (F^2 \cap R)/([F, R] + [R, F])$$

We can see that $[B, C] = 0$ as $[B, C] = [R, F]/([F, R] + [R, F]) = 0$ and in the same way $[C, B] = 0$ as $[C, B] = [F, R]/([F, R] + [R, F]) = 0$. Furthermore, D is a central ideal of C as it is contained within the center of C and is an ideal.

We will show that there exists a central ideal E of C where E is complementary to D in B such that $(C/E, \bar{\pi}) \in C(L)$ and then for $(K, \lambda) \in C(L)$, there is a $\bar{\sigma} \in Hom(C/E, K)$ such that $\bar{\pi} = \lambda \circ \bar{\sigma}$. In order to show this, we will show that the following diagrams will commute where $\bar{\pi}$ is induced from π_1 and π_1 is induced from π. Furthermore, $\bar{\sigma}$ is induced from σ_1 and σ_1 is induced from σ.

$$\begin{tikzcd} & K \arrow[dr, "\lambda"] & \\ F \arrow[ur, "\sigma"] \arrow[rr, "\pi"] & & L \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[dr, "\lambda"] & \\ C \arrow[ur, "\sigma_1"] \arrow[rr, "\pi_1"] & & L \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[dr, "\lambda"] & \\ C/E \arrow[ur, "\bar{\sigma}"] \arrow[rr, "\bar{\pi}"] & & L \end{tikzcd}$$

Figure 2.3 $\bar{\pi}$ is induced from π_1 and π_1 is induced from π
$\bar{\sigma}$ is induced from σ_1 and σ_1 is induced from σ

Lemma 2.0.7. *$(C/E, B/E)$ is a defining pair for L where E is a central ideal of C and a complementary subspace to D in B.*

Proof. In order to show that $(C/E, B/E)$ is a defining pair for L, both parts of the definition of a defining pair need to hold. Therefore, it needs to be shown that $L \cong (C/E)/(B/E)$ and $B/E \subseteq Z(C/E) \cap [C/E, C/E]$.

First, it is noted that $(C/E)/(B/E) \cong C/B \cong (F/([F, R] + [R, F]))/(R/([F, R] + [R, F])) \cong F/R \cong L$ by definition from the free presentation.

Next, it is noted that as $[B, C] = 0$ and $[C, B] = 0$, $B \subseteq Z(C)$ which implies that $B/E \subseteq Z(C/E)$. Also, $D = ([F, F] \cap R/([F, R] + [R, F])) \subseteq [F, F]/([F, R] + [R, F]) \cong [F/([F, R] + [R, F]), F/([F, R] + [R, F])] = [C, C]$ and therefore, using the fact that E is complementary to D in B, $B/E \cong (D \bigoplus E)/E \subseteq ([C, C] + E)/E \cong [C/E, C/E]$. So $B/E \subseteq Z(C/E) \cap [C/E, C/E]$. Therefore, $(C/E, B/E)$ is a defining pair for L. □

From [8], the following lemma for the Lie algebra case, also carries into the Leibniz case.

Lemma 2.0.8. *For an element $x \in F$, then $x \in R$ if and only if $\sigma(x) \in Ker(\lambda)$.*

Using this lemma, the following lemma can be proved.

Lemma 2.0.9. *$[R, F] + [F, R] \subseteq Ker(\sigma)$ and therefore, σ induces the surjective homomorphism $\sigma_1 \in Hom(C, M)$ where $\lambda \circ \sigma_1 = \pi_1$.*

Proof. Take an element $[r_1, f_1] + [f_2, r_2] \in [R, F] + [F, R]$, then $\sigma([r_1, f_1] + [f_2, r_2]) = [\sigma(r_1), \sigma(f_1)] + [\sigma(f_2), \sigma(r_2)] = 0$ as from the previous lemma, $\sigma(r_1)$, $\sigma(r_2) \in Ker(\lambda)$ and from the definition of $C(L)$, $Ker(\lambda) \subseteq Z(M) \cap [M, M]$ with $\sigma(f) \in M$. Therefore, σ induces $\sigma_1 \in Hom(C, M)$ where, for an element $c = f + [R, F] + [F, R] \in C$, $\lambda \circ \sigma_1(c) = \lambda \circ \sigma_1(f + [R, F] + [F, R]) = \lambda \circ \sigma(f) = \pi(f) = \pi_1(f + [R, F] + [F, R]) = \pi_1(c)$. So, $\lambda \circ \sigma_1 = \pi_1$ where σ_1 is surjective as σ is surjective. □

The following lemmas correspond to the Lie algebra case in [8] and [9].

Lemma 2.0.10. *$\sigma_1(B) = Ker(\lambda)$ and $\sigma_1(D) = Ker(\lambda)$. So $B = D + Ker(\sigma_1)$.*

Taking E to be a complementary subspace of D in B, such that, from the above lemma, $E \subseteq Ker(\sigma_1)$. Then $(C/E, B/E)$ is a defining pair for L and $\bar{\sigma}$ is induced by σ_1 such that $\lambda \circ \bar{\sigma} = \bar{\pi}$.

Lemma 2.0.11. *If K is a cover of L, then C/E is a cover of L and the multiplier, $M = D$.*

Proof. By definition of the cover of L, $dim(K) \geq dim(C/E)$, but also as K is the homomorphic image of C/E under a surjective mapping, $dim(C/E) \geq dim(K)$ and therefore, $dim(C/E) = dim(K)$ and C/E is a cover of L. Then as $L \cong C/B$ and by definition, $B/E \cong D$ with $(C/E, B/E)$ being a defining pair for L, the multiplier, M of L is $M = D$
$= ([F, F] \cap R)/[F, R] + [R, F]$. □

So for any $(J, \lambda) \in C(L)$ with the free presentation as given previously, there is an E complementary to D in B such that C/E is the cover of L and there is a homomorphism $\bar{\sigma}$ induced from σ_1 which is itself induced from σ. It has already been shown that $E \subseteq Ker(\sigma_1)$ and therefore, E depends on J. Using this information, a C/E needs to be found that satisfies the property of a universal element. This is done by showing that the algebras, C/E, are isomorphic to each other.

In order to show that the algebras C/E are isomorphic to one another, a specific form of F is found. Take the $dim(L) = n$. As L is the homomorphic image of F this means that F is also generated by n elements. Then as E is the complementary subspace to D in B and using the second isomorphism theorem, $E \cong B/D \cong R/([F, F] \cap R) \cong ([F, F] + R)/[F, F] \subseteq F/[F, F]$. This is abelian and generated by n elements.

Therefore E is a finite dimensional abelian algebra. Then in a similar way as seen for the Lie algebra case, take an element $(J, \lambda) \in C(L)$. If E_1 is a complementary subspace to D in B, then $\sigma_1 \in Home(C, J)$ can be induced from σ, with $E_1 \subseteq Ker(\sigma_1)$ and $\lambda \circ \sigma_1 = \pi_1$ such that $\pi_1 \in Hom(C, L)$.

Then using the fact that $E \subseteq F/[F,F]$, $F/[F,F] \cap [F,F]/([F,R]+[R,F]) = 0 \implies E \cap [C,C] = E_1 \cap [C,C] = 0$. Because of this fact, we can take an extension G of D whereby $C = E \bigoplus G$ and we can take an extension G_1 of D whereby $C = E \bigoplus G_1$. As E and E_1 have the same dimension and are both abelian, they must be isomorphic to each other. Just as in the Lie case found in [8], from the work above it can be seen that it is also the case in Leibniz algebras that $G \cong G_1 \implies C/E \cong C/E_1$.

In summary, for each $(J, \lambda) \in C(L)$, there is a surjective homomorphism that has been induced from σ, namely $\bar{\sigma} \in (C/E, J)$ such that $\lambda \circ \bar{\sigma} = \bar{\pi}$. Therefore, $(C/E, \bar{\pi})$ is a universal element in $C(L)$ where C/E is a cover of L and each cover is isomorphic to C/E as each one is a homomorphic image of C/E and has the same dimension. Also for any element $(J, \lambda) \in C(L)$ then there is a homomorphism $\beta \in Hom(K, J)$ where $\lambda \circ \beta = \pi$ where K is a cover of L and $(K, \pi) \in C(L)$. So (K, π) is a universal element if and only if K is a cover of L.

Theorem 2.0.12. *Let L be a finite dimensional algebra. For the following free presentation of L,*

$$0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$$

let $B = R/([F,R]+[R,F])$, $C = F/([F,R]+[R,F])$, and $D = ([F,F] \cap R)/([F,R]+[R,F])$. Then all covers of L are of the form C/E where E is the complement to D in B and all covers are isomorphic to each other. The multiplier, M of L is $D \cong B/E$. The universal elements in $C(L)$ are the elements (K, λ) where K is the cover of L.

Chapter 3

Strictly Upper Triangular Matrices k=0

The structure and dimension of the Lie cover and multiplier of the Lie algebra of strictly upper triangular matrices has already been found [8] and [9]. In this chapter, the Leibniz cover and multiplier of the algebra are found. By first finding the structure and hence the dimension of the multiplier of the algebra, the dimension of the cover can also be established. A formula for the dimension of the multiplier can be found in terms of n for general $n \times n$ strictly upper triangular matrices.

The elements in the Lie algebra of strictly upper triangular matrices L are each of the unit matrices $E_{i,j}$ where $1 \leq i < j \leq n$. Let (K, M) be a defining pair for L and let $F(s,t)$ be the image of $E_{s,t}$ under the section $l : L \to K$. The Leibniz bracket of elements of K can then be defined as:

$$[F(s,t), F(a,b)] = \begin{cases} F(s,b) + y(s,t,a,b) & \textit{when } t = a \\ -F(a,t) + y(s,t,a,b) & \textit{when } s = b \\ y(s,t,a,b) & \textit{otherwise} \end{cases}$$

Using this relationship between elements of the cover, a basis for the multiplier and it's dimension, can be found.

As the basis elements in the algebra of $n \times n$ strictly upper triangular matrices are of the form $E_{s,t}$ where $1 \leq s < t \leq n$, the elements of the multiplier must be of the form $y(s,t,a,b)$ where $s < t$ and $a < b$.

Theorem 3.0.1. *If $b \geq a+4$ or $t \geq s+4$, then $y(s,t,a,b) = 0$. So $y(s,t,a,b) \neq 0$ only has the potential to occur if both $b \leq a+3$ and $t \leq s+3$.*

Proof. To show that this is the case, take an element of the multiplier, $y(s,t,a,b)$. It is known that $t \geq s + (k+1) \implies t \geq s+1$ and similarly $b \geq a + (k+1) \implies b \geq a+1$. It can be seen that for some c such that $0 < c < t - s$, the following Leibniz identity holds:

$$\begin{aligned}[F(s,s+c),[F(s+c,t),F(a,b)]] &= [[F(s,s+c),F(s+c,t)],F(a,b)] \\ &\quad + [F(s+c,t),[F(s,s+c),F(a,b)]]\end{aligned}$$

Now assume that $t = s + 4$. As long as $a \neq s + 1$ and $b \neq s + 1$, we can take $c = 1$ for the above Leibniz identity to hold, otherwise, take $c = 2$. This will always make the element $y(s, t, a, b) = 0$ no matter what gap is between a and b. The same theory applies when taking $b = a + 4$ and letting the gap between s and t be whatever is desired. This can be extended to consider the cases where $t > s + 4$ and $b > a + 4$ and the same theory applies.

□

3.1 Elements of the multiplier that arise from [F(s,t),F(a,b)] where $t \neq a$ and $s \neq b$

As the non-zero elements of the basis of the multiplier must satisfy $t < s + 4$ and $b < a + 4$, this gives 9 remaining cases to consider for the elements $y(s, t, a, b)$:

1. (a) $t = s + 1$ and $b = a + 1$
   (b) $t = s + 1$ and $b = a + 2$
   (c) $t = s + 1$ and $b = a + 3$
2. (a) $t = s + 2$ and $b = a + 1$
   (b) $t = s + 2$ and $b = a + 2$
   (c) $t = s + 2$ and $b = a + 3$
3. (a) $t = s + 3$ and $b = a + 1$
   (b) $t = s + 3$ and $b = a + 2$
   (c) $t = s + 3$ and $b = a + 3$

We begin by going through each case, first where $t \neq a$ and $s \neq b$, to see which elements $y(s, t, a, b)$ of the multiplier can be eliminated depending on the placement of the values of s, t, a, b in relation to one another.

Theorem 3.1.1. *The only cases that will give non-zero elements of the multiplier are cases 1.(a), 1.(b), and 2.(a).*

Proof. This is proved in a similar manner to the previous theorem. It can be seen that the cases 2.(c), 3.(b), and 3.(c) have no elements to consider and will always give all 0 elements of the multiplier. In case 2.(c), the elements of the multiplier are of the form $y(s, s + 2, a, a + 3)$. By taking a value c such that $0 < c < 3$, the following Leibniz identity can be created:

$$\begin{aligned}[F(s, s + 2), [F(a, a + c), F(a + c, a + 3)]] = {} & [[F(s, s + 2), F(a, a + c)], F(a + c, a + 3)] \\ & + [F(a, a + c), [F(s, s + 2), F(a + c, a + 3)]]\end{aligned}$$

If $c = 1$ then this identity will equal 0 as long as $s \neq a + 1$ and $s \neq a - 1$. In each of these cases, let $c = 2$ and the identity will once again equal 0. The spacing of these values can be seen on the following 2 number lines:

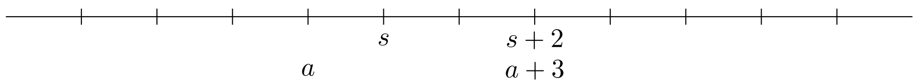

Figure 3.1 $y(s, s+2, a, a+3)$ where $a < s < t = b$

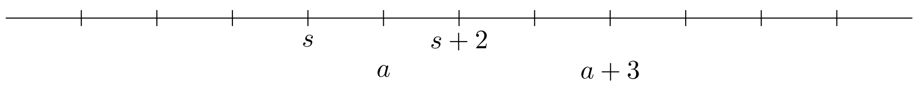

Figure 3.2 $y(s, s+2, a, a+3)$ where $s < a < t < b$

It therefore follows, no matter what the values of s and a are in the identity above, a value c can always be chosen that will make $y(s, s+2, a, a+3) = 0$.

In a very similar way, for the case 3.(b) the elements of the multiplier are of the form $y(s, s+3, a, a+2)$. By taking a value c such that $0 < c < 3$, the following Leibniz identity can be created:

$$\begin{aligned}[F(s, s+c), [F(s+c, s+3), F(a, a+2)]] &= [[F(s, s+c), F(s+c, s+3)], F(a, a+2)] \\ &\quad + [F(s+c, s+3), [F(s, s+c), F(a, a+2)]]\end{aligned}$$

If $c = 1$ then this identity will equal 0 as long as $s \neq a+1$ and $s \neq a-1$. In each of these cases, by letting $c = 2$, the identity will again equal 0. The spacing of these values can be seen on the following 2 number lines:

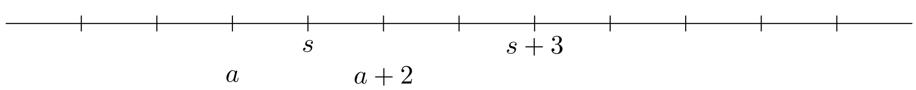

Figure 3.3 $y(s, s+3, a, a+2)$ where $a < s < b < t$

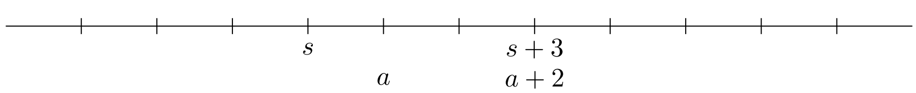

Figure 3.4 $y(s, s+3, a, a+2)$ where $s < a < b = t$

In the same way, for the case 3.(c) where elements of the multiplier are of the form $y(s, s+3, a, a+3)$, the following Leibniz identity can be created

$$\begin{aligned}[F(s, s+c), [F(s+c, s+3), F(a, a+3)]] &= [[F(s, s+c), F(s+c, s+3)], F(a, a+3)] \\ &\quad + [F(s+c, s+3), [F(s, s+c), F(a, a+3)]]\end{aligned}$$

Once more, we can again take $c = 1$ as long as $s \neq a+2$ and $s \neq a-1$, in which case we can take c=2 as can be seen from the spacing of the values on the following number lines:

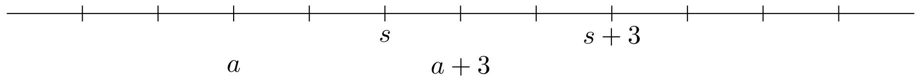

Figure 3.5 $y(s, s+3, a, a+3)$ where $a < s < b < t$

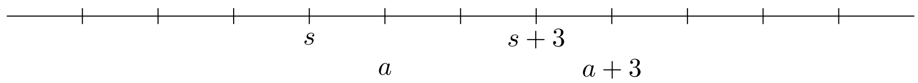

Figure 3.6 $y(s, s+3, a, a+3)$ where $s < a < t < b$

Therefore, as every multiplier element from these cases is equal to 0, this leaves 6 cases to consider.

For the case 1.(c), the elements of the multiplier are of the form $y(s, s+1, a, a+3)$. By taking a c such that $0 < c < 3$, we find:

$$\begin{aligned}[F(s,s+1),[F(a,a+c),F(a+c,a+3)]] &= [[F(s,s+1),F(a,a+c)],F(a+c,a+3)] \\ &\quad + [F(a,a+c),[F(s,s+1),F(a+c,a+3)]]\end{aligned}$$

As long as $s \neq a+1$, c can always be taken to be $c = 1$ or $c = 2$ in order for the above identity to reveal that $y(s, s+1, a, a+3) = 0$. We can see from the number line below that there would not be sufficient space for the value $a + c$ in order to reveal that elements of this form would equal 0.

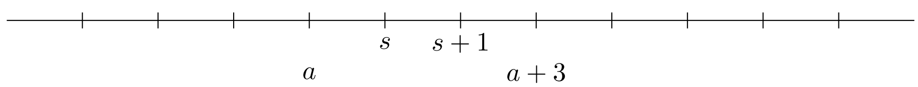

Figure 3.7 $y(s, s+1, a, a+3)$ where $a < s < t < b$

However, from taking the Leibniz identity on 2 different sets of elements from the cover, the following is found:

1.

$$\begin{aligned}[F(s,s+1),[F(s-1,s),F(s,s+2)]] &= [[F(s,s+1),F(s-1,s)],F(s,s+2)] \\ &\quad + [F(s-1,s),[F(s,s+1),F(s,s+2)]]\end{aligned}$$

$$\implies [F(s,s+1), F(s-1,s+2) + y(s-1,s,s,s+2)]$$
$$= [-F(s-1,s+1) + y(s,s+1,s-1,s), F(s,s+2)]$$
$$\implies [F(s,s+1), F(s-1,s+2)] = -[F(s-1,s+1), F(s,s+2)]$$
$$\implies y(s,s+1,s-1,s+2) = -y(s-1,s+1,s,s+2)$$

2.

$$[F(s,s+1),[F(s-1,s+1),F(s+1,s+2)]] = [[F(s,s+1),F(s-1,s+1)],F(s+1,s+2)] + [F(s-1,s+1),[F(s,s+1),F(s+1,s+2)]]$$

$$\implies [F(s,s+1),F(s-1,s+2)+y(s-1,s+1,s+1,s+2)]$$
$$= [F(s-1,s+1),F(s,s+2)+y(s,s+1,s+1,s+2)]$$
$$\implies [F(s,s+1),F(s-1,s+2)] = [F(s-1,s+1),F(s,s+2)]$$
$$\implies y(s,s+1,s-1,s+2) = y(s-1,s+1,s,s+2)$$

It can be seen that $y(s,s+1,a,a+3) = y(s,s+1,s-1,s+2)$ is equal to both $-y(s-1,s+1,s,s+2)$ and $y(s-1,s+1,s,s+2)$. Therefore, these elements must equal 0 and can also be eliminated from the basis of the multiplier.

In case 2.(b) the elements of the multiplier are of the form $y(s,s+2,a,a+2)$. Using the following Leibniz identity where $s \neq a+1$ and $s \neq a-1$, it can be seen that the elements of this form are equal to 0.

$$[F(s,s+2),[F(a,a+1),F(a+1,a+2)]] = [[F(s,s+2),F(a,a+1)],F(a+1,a+2)] + [F(a,a+1),[F(s,s+2),F(a+1,a+2)]]$$

$$\implies [F(s,s+2),F(a,a+2)+y(a,a+1,a+1,a+2)] = 0$$
$$\implies [F(s,s+2),F(a,a+2)] = 0$$
$$\implies y(s,s+2,a,a+2) = 0$$

Accordingly, the only types of elements that need to be considered in this case are those of the form $y(s,s+2,s+1,s+3)$ and $y(s,s+2,s-1,s+1)$. For elements of the form $y(s,s+2,s+1,s+3)$, the Leibniz identity of 2 different sets of elements of the cover will give 2 separate relations involving elements of this form:

1.

$$[F(s,s+1),[F(s+1,s+2),F(s+1,s+3)]] = [[F(s,s+1),F(s+1,s+2)],F(s+1,s+3)] + [F(s+1,s+2),[F(s,s+1),F(s+1,s+3)]]$$

$$\implies 0 = [F(s,s+2)+y(s,s+1,s+1,s+2),F(s+1,s+3)] + [F(s+1,s+2),F(s,s+3)+y(s,s+1,s+1,s+3)]$$

$$\implies 0 = [F(s,s+2),F(s+1,s+3)] + [F(s+1,s+2),F(s,s+3)]$$
$$\implies y(s,s+2,s+1,s+3) = -y(s+1,s+2,s,s+3)$$

2.

$$[F(s,s+2),[F(s+1,s+2),F(s+2,s+3)]] = [[F(s,s+2),F(s+1,s+2)],F(s+2,s+3)] + [F(s+1,s+2),[F(s,s+2),F(s+2,s+3)]]$$

$$\implies [F(s,s+2),F(s+1,s+3)+y(s+1,s+2,s+2,s+3)]$$

$$= [F(s+1,s+2), F(s,s+3) + y(s,s+2,s+2,s+3)]$$
$$\implies [F(s,s+2), F(s+1,s+3)] = [F(s+1,s+2), F(s,s+3)]$$
$$\implies y(s,s+2,s+1,s+3) = y(s+1,s+2,s,s+3)$$

So we see that the elements $y(s, s+2, s+1, s+3)$ are equal the positive and negative of $y(s+1, s+2, s, s+3)$ and therefore they must also equal 0 and as such there are no elements of the multiplier that arise from this case.

The same applies for the case $y(s, s+2, s-1, s+1)$. Using the same type of calculation, we find that these elements are equal to the elements $y(s, s+1, s-1, s+2)$ found in case 1.(c) and therefore can also be eliminated.

In case 3.(a) the elements are of the form $y(s, s+3, a, a+1)$. This will follow symmetrically from the case 1.(c). By taking c such that $0 < c < 3$, we find:

$$\begin{aligned}[F(s,s+c),[F(s+c,s+3),F(a,a+1)]] &= [[F(s,s+c),F(s+c,s+3)],F(a,a+1)] \\ &\quad + [F(s+c,s+3),[F(s,s+c),F(a,a+1)]]\end{aligned}$$

As long as $s \neq a-1$, c can always be taken to be $c = 1$ or $c = 2$ in order for the above identity to reveal that $y(s, s+3, a, a+1) = 0$. We can see from the number line below that there is insufficient space for the value $s+c$ in order to reveal that elements of this form would equal 0.

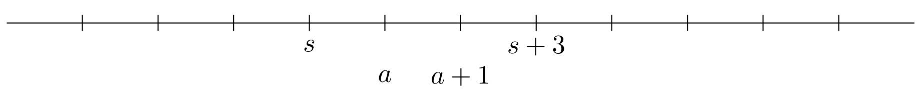

Figure 3.8 $y(s, s+3, a, a+1)$ where $s < a < b < t$

If we now consider the case where $s = a-1$, a Leibniz identity of 2 different sets of elements of the cover can be taken in order to get relations involving elements of the form $y(s, s+3, s+1, s+2)$. These are the following:

1.

$$\begin{aligned}[F(s,s+1),[F(s+1,s+3),F(s+1,s+2)]] &= [[F(s,s+1),F(s+1,s+3)],F(s+1,s+2)] \\ &\quad + [F(s+1,s+3),[F(s,s+1),F(s+1,s+2)]]\end{aligned}$$

$$\implies 0 = [F(s,s+3) + y(s,s+1,s+1,s+3), F(s+1,s+2)]$$
$$+[F(s+1,s+3), F(s,s+2) + y(s,s+1,s+1,s+2)]$$
$$\implies 0 = [F(s,s+3), F(s+1,s+2)] + [F(s+1,s+3), F(s,s+2)]$$
$$\implies y(s,s+3,s+1,s+2) = -y(s+1,s+3,s,s+2)$$

2.

$$\begin{aligned}[F(s,s+2),[F(s+2,s+3),F(s+1,s+2)]] &= [[F(s,s+2),F(s+2,s+3)],F(s+1,s+2)] \\ &\quad + [F(s+2,s+3),[F(s,s+2),F(s+1,s+2)]]\end{aligned}$$

$$\implies [F(s,s+2), -F(s+1,s+3) + y(s+2,s+3,s+1,s+2)]$$
$$= [F(s,s+3) + y(s,s+2,s+2,s+3), F(s+1,s+2)]$$
$$\implies -[F(s,s+2), F(s+1,s+3)] = [F(s,s+3), F(s+1,s+2)]$$
$$\implies -y(s,s+2,s+1,s+3) = y(s,s+3,s+1,s+2)$$

It was found in case 2.(b) that both $y(s+1,s+3,s,s+2)$ and $y(s,s+2,s+1,s+3)$ are 0. Therefore, $y(s,s+3,s+1,s+2) = 0$ and no elements for the basis of the multiplier are found in this case.

Therefore, by eliminating the possibility of any of these 6 cases contributing elements to the multiplier, it can be seen that the only possibilities are those in cases 1.(a), 1.(b), and 2.(a).

□

Now we must consider the remaining 3 cases to discover the elements that will remain in the basis of the multiplier.

Case 1. (a) t = s+1 and b = a+1

This corresponds to elements of the form $y(s,s+1,a,a+1)$. The Leibniz identity cannot be used to uncover these elements within the multiplier when $a > s+1$ or when $s > a+1$ as there is insufficient space between s and t or a and b in order to find a value c to create a Leibniz identity involving elements of the cover. Therefore, there is no way for these to be eliminated and these elements are included in the basis of the multiplier.

When $a > s+1$, there are $\frac{(n-2)(n-3)}{2}$ elements in the multiplier of the form $y(s,s+1,a,a+1)$. Similiarly, we can see that there are $\frac{(n-2)(n-3)}{2}$ elements in the multiplier of the form $y(s,s+1,a,a+1)$ when $s > a+1$. We know that this is the case for $a > s+1$ because the minimum value of s is 1 so the minimum value of a is therefore 3. Furthermore, the maximum value of a is $n-1$ as the maximum value of $a+1$ is n. Therefore a can range in value from 3 to $n-1$. This gives $n-3$ possible values for a when $s = 1$. When s increases by 1 to become $s = 2$, the value of a can then range in value from 4 to $n-1$. This gives $n-4$ possible values of a for this value of s. Continuing with this idea, for $s = 3$ there are $n-5$ possible values of a and so on. This carries on until there is only one value that each a and $a+1$ can take. Therefore there are $(n-3)+(n-4)+...+1$ different values of a for different values of s. This can be written as $\frac{(n-2)(n-3)}{2}$. The same applies to the case where $s > a+1$ and by following exactly the same steps, we find that there are also $\frac{(n-2)(n-3)}{2}$ elements in the multiplier of the form $y(s,s+1,a,a+1)$ under this restriction.

The only other form that we must look at in this case, such that $t \neq a$ and $s \neq b$, is $y(s,s+1,s,s+1)$. There are $n-1$ different values that s can take in this case, and we cannot take a Leibniz bracket of elements from the cover in order to eliminate elements of this form. Therefore, there are $n-1$ elements of this type in the basis of the multiplier.

Case 1. (b) t = s+1 and b = a+2

This corresponds to elements of the form $y(s,s+1,a,a+2)$. First we look at the cases where $a > s+1$ or $s > a+2$. Taking the following Leibniz bracket it can be shown that these elements are in fact 0 in both of these cases and therefore can be eliminated from the basis of the multiplier.

$$\begin{aligned}[F(s,s+1),[F(a,a+1),F(a+1,a+2)]] &= [[F(s,s+1),F(a,a+1)],F(a+1,a+2)] \\ &\quad + [F(a,a+1),[F(s,s+1),F(a+1,a+2)]]\end{aligned}$$

As we know that none of $s, s+1, a, a+1$ are equal to one another in the cases where $a > s+1$ and $s > a+2$, and $y(s, a+1, a+1, a+2) \in Z(K)$, the above reduces to the following brackets:

$$\begin{aligned}&[F(s,s+1),F(a,a+2)+y(a,a+1,a+1,a+2)] = 0 \\ &\implies [F(s,s+1),F(a,a+2)] = 0 \\ &\implies y(s,s+1,a,a+2) = 0\end{aligned}$$

Therefore, as $y(s, s+1, a, a+2) = 0$, elements of this form can be eliminated from the basis of the multiplier.

In order to make it easier to visualize the placement of values for the remaining elements in this case, the number lines are drawn below to show their positions. The first set of elements to consider in this case are those of the form $y(s, s+1, a, a+2) = y(s, s+1, s-1, s+1)$.

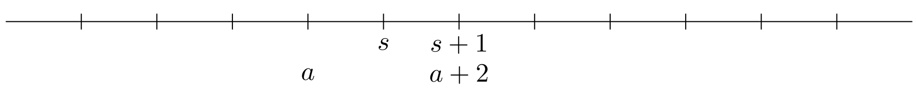

Figure 3.9 $y(s, s+1, a, a+2)$ where $a < s < b = t$

Taking a Leibniz identity involving elements in the cover, we are able to find a relationship between elements in the multiplier:

$$\begin{aligned}[F(s,s+1),[F(s-1,s),F(s,s+1)]] &= [[F(s,s+1),F(s-1,s)],F(s,s+1)] \\ &\quad + [F(s-1,s),[F(s,s+1),F(s,s+1)]]\end{aligned}$$

$$\begin{aligned}&\implies [F(s,s+1),F(s-1,s+1)+y(s-1,s,s,s+1)] \\ &= [-F(s-1,s+1)+y(s,s+1,s-1,s),F(s,s+1)] \\ &\implies [F(s,s+1),F(s-1,s+1)] = -[F(s-1,s+1),F(s,s+1)] \\ &\implies y(s,s+1,s-1,s+1) = -y(s-1,s+1,s,s+1)\end{aligned}$$

As such, the elements found in this case correspond to the elements that will arise in case 2.(a). The minimum value of $s-1$ is 1 and the maximum value of $s+1$ to be n. So the minimum value that s can take is 2 and the maximum value it can achieve is $n-1$. Hence, s can take $n-2$ different values, therefore giving $n-2$ different possibilities for this element of the multiplier.

The following number line corresponds to the elements $y(s, s+1, a, a+2) = y(s, s+1, s, s+2)$.

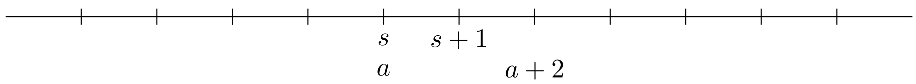

Figure 3.10 $y(s, s+1, a, a+2)$ where $a = s < t < b$

Taking a Leibniz identity involving elements in the cover, we are able to find a relation between elements in the multiplier:

$$\begin{aligned}[F(s,s+1),[F(s,s+1),F(s+1,s+2)]] &= [[F(s,s+1),F(s,s+1)],F(s+1,s+2)] \\ &\quad + [F(s,s+1),[F(s,s+1),F(s+1,s+2)]]\end{aligned}$$

$$\begin{aligned}&\implies [F(s,s+1),F(s,s+2)+y(s,s+1,s+1,s+2)] \\ &= [F(s,s+1),F(s,s+2)+y(s,s+1,s,s+2)] \\ &\implies [F(s,s+1),F(s,s+2)] = [F(s,s+1),F(s,s+2)]\end{aligned}$$

However, this gives us no new relation, but it does show that elements of the multiplier of this form cannot be eliminated and so they must be counted as elements of the basis of the multiplier. The minimum value that s can take is 1 and the maximum value that $s+2$ can take is n. Therefore the maximum value that s can take is $n-2$. So there are $n-2$ different values that s can take, hence there are $n-2$ elements of the multiplier of this type. Any other element in this case has either $s = b$ or $t = a$.

Case 2. (a) t = s+2 and b = a+1

This corresponds to elements of the form $y(s, s+2, a, a+1)$. First, we consider the case where $a > s+2$ or $s > a+1$. Taking the following Leibniz identity of elements of the cover, it can be shown that these elements are 0 in both of these cases and therefore can be eliminated from the basis of the multiplier.

$$\begin{aligned}[F(s,s+1),[F(s+1,s+2),F(a,a+1)]] &= [[F(s,s+1),F(s+1,s+2)],F(a,a+1)] \\ &\quad + [F(s+1,s+2),[F(s,s+1),F(a,a+1)]]\end{aligned}$$

$$\begin{aligned}&\implies [F(s,s+2)+y(s,s+1,s+1,s+2),F(a,a+1)] = 0 \\ &\implies [F(s,s+2),F(a,a+1)] = 0 \\ &\implies y(s,s+2,a,a+1) = 0\end{aligned}$$

The following number line corresponds to elements of the form $y(s, s+2, a, a+1) = y(s, s+2, s, s+1)$.

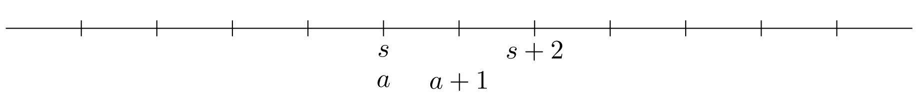

Figure 3.11 $y(s, s+2, a, a+1)$ where $a = s < b < t$

Taking the following Leibniz identity of elements from the cover, a relation involving elements of this form can be established:

$$[F(s,s+1),[F(s+1,s+2),F(s,s+1)]] = [[F(s,s+1),F(s+1,s+2)],F(s,s+1)] + [F(s+1,s+2),[F(s,s+1),F(s,s+1)]]$$

$$\implies [F(s,s+1),-F(s,s+2)+y(s+1,s+2,s,s+1)]$$
$$= [F(s,s+2)+y(s,s+1,s+1,s+2),F(s,s+1)]$$
$$\implies -[F(s,s+1),F(s,s+2)] = [F(s,s+2),F(s,s+1)]$$
$$\implies -y(s,s+1,s,s+2) = y(s,s+2,s,s+1)$$

The elements $y(s,s+1,s,s+2)$ were accounted for in case 1.(b), so there are no new elements that arise in this situation.

The following number line corresponds to the elements $y(s,s+2,a,a+1) = y(s,s+2,s+1,s+2)$.

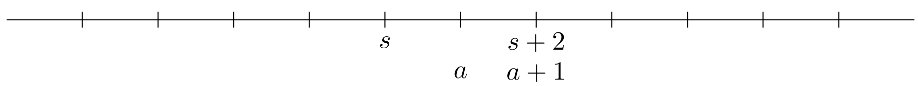

Figure 3.12 $y(s,s+2,a,a+1)$ where $s < a < b = t$

Taking the following Leibniz identity of elements from the cover, a relationship involving elements of this form can be established:

$$[F(s,s+1),[F(s+1,s+2),F(s+1,s+2)]] = [[F(s,s+1),F(s+1,s+2)],F(s+1,s+2)] + [F(s+1,s+2),[F(s,s+1),F(s+1,s+2)]]$$

$$\implies 0 = [F(s,s+2)+y(s,s+1,s+1,s+2),F(s+1,s+2)] + [F(s+1,s+2),F(s,s+2)+y(s,s+1,s+1,s+2)]$$

$$\implies 0 = [F(s,s+2),F(s+1,s+2)] + [F(s+1,s+2),F(s,s+2)]$$
$$\implies y(s,s+2,s+1,s+2) = -y(s+1,s+2,s,s+2)$$

The elements $y(s+1,s+2,s,s+2)$ were already accounted for in case 1.(b) and therefore, we have shown that in this case there are no new elements that appear in the basis of the multiplier.

3.2 Elements of the multiplier that arise from [F(s,t),F(a,b)] where $t = a$ or $s = b$

The only cases left to investigate are the elements of the cover such that $t = a$ or where $s = b$. First we will consider those where $t = a$. A change of basis within the cover can be used in order to eliminate elements in the basis of the multiplier. The following change of basis is used:

$$G(s,t) = F(s,t) + y(s,t-1,t-1,t)$$

As the multiplier is within the center of the cover, the elements $G(s,t)$ multiply in the same way as the elements $F(s,t)$. Using this change of basis and the Leibniz identity, we will find the elements that remain in the basis.

Theorem 3.2.1. *Every element in the multiplier of the form $y(s,s+c,s+c,b)$ where $c \geq 1$ and $s+c \leq b-1$ can be eliminated from the basis of the multiplier. Therefore there are no elements of the form $y(s,t,a,b)$ where $t=a$ in the basis of the multiplier.*

Proof. If $s+c=b-1$ this corresponds to elements within the basis of the form $y(s,b-1,b-1,b)$ and from the above change of basis, these elements can be eliminated. If $s+c<b-1$ then the following relationship can be found using the Leibniz identity:

$$\begin{aligned}[F(s,s+c),[F(s+c,b-1),F(b-1,b)]] &= [[F(s,s+c),F(s+c,b-1)],F(b-1,b)] \\ &\quad + [F(s+c,b-1),[F(s,s+c),F(b-1,b)]]\end{aligned}$$

$$\begin{gathered}\implies [F(s,s+c),F(s+c,b)+y(s+c,b-1,b-1,b)] = \\ [F(s,b-1)+y(s,s+c,s+c,b-1),F(b-1,b)] \\ \implies [F(s,s+c),F(s+c,b)] = [F(s,b-1),F(b-1,b)] \\ \implies F(s,b)+y(s,s+c,s+c,b) = F(s,b)+y(s,b-1,b-1,b) \\ \implies y(s,s+c,s+c,b) = y(s,b-1,b-1,b)\end{gathered}$$

It has already been found from the change of basis formula that the elements $y(s,b-1,b-1,b)$ can be eliminated from the basis of the multiplier and consequently, every element $y(s,s+c,s+c,b)$ where $s+c<b-1$ can be eliminated.

This means that every element of the form $y(s,s+c,s+c,b)$ can be eliminated from the basis of the multiplier and therefore there are no elements of the form $t=a$ in the basis.

□

The next case that needs to be considered are the elements where $s=b$ in $y(s,t,a,b)$. These elements can also be written in the form $y(s+c,s+d,s,s+c)$ where $d>c\geq 1$. If $c>1$ then the following elements can be placed into a Leibniz identity to uncover a relationship between the elements:

$$\begin{aligned}[F(s+c,s+d),[F(s,s+1),F(s+1,s+c)]] &= [[F(s+c,s+d),F(s,s+1)],F(s+1,s+c)] \\ &\quad + [F(s,s+1),[F(s+c,s+d),F(s+1,s+c)]]\end{aligned}$$

$$\begin{gathered}\implies [F(s+c,s+d),F(s,s+c)+y(s,s+1,s+1,s+c)] = \\ [F(s,s+1),-F(s+1,s+d)+y(s+c,s+d,s+1,s+c)] \\ \implies [F(s+c,s+d),F(s,s+c)] = [F(s,s+1),-F(s+1,s+d)] \\ \implies -F(s,s+d)+y(s+c,s+d,s,s+c) = -F(s,s+d)-y(s,s+1,s+1,s+d) \\ \implies y(s+c,s+d,s,s+c) = -y(s,s+1,s+1,s+d)\end{gathered}$$

It was found in the case where $t=a$ that the elements $y(s,s+1,s+1,s+d)$ can be eliminated from the basis of the multiplier. Therefore the elements $y(s+c,s+d,s,s+c)$ where $d>c>1$ can also be eliminated.

If $c=1$ and $d>2$ then the following elements can be placed into a Leibniz identity to uncover a relationship between the elements:

$$\begin{aligned}[F(s+1,s+2),[F(s+2,s+d),F(s,s+1)]] &= [[F(s+1,s+2),F(s+2,s+d)],F(s,s+1)] \\ &+ [F(s+2,s+d),[F(s+1,s+2),F(s,s+1)]]\end{aligned}$$

$$\implies 0 = [F(s+1,s+d) + y(s+1,s+2,s+2,s+d), F(s,s+1)]$$
$$+ [F(s+2,s+d), -F(s,s+2) + y(s+1,s+2,s,s+1)]$$

$$\implies 0 = [F(s+1,s+d), F(s,s+1)] + [F(s+2,s+d), -F(s,s+2)]$$

$$\implies 0 = -F(s,s+d) + y(s+1,s+d,s,s+1) + F(s,s+d) - y(s+2,s+d,s,s+2)$$

$$\implies y(s+1,s+d,s,s+1) = y(s+2,s+d,s,s+2)$$

It was found previously that the elements $y(s+2,s+d,s,s+2)$ can be eliminated from the basis of the multiplier and therefore the elements $y(s+1,s+d,s,s+1)$ where $d > 2$ can also be eliminated.

As can be seen from the above calculations, the only elements of the multiplier that cannot be eliminated from the basis are those of the form $y(s+1,s+2,s,s+1)$. There are $n-2$ possible values of s and therefore there are $n-2$ elements of the multiplier of this form.

From looking at the cases where $t \neq a$, $s \neq b$ and $t = a$ or $s = b$, the total number of elements in the basis of the Leibniz multiplier of the algebra of strictly upper triangular matrices is:

$$(n-2)(n-3) + (n-1) + 3(n-2) = n^2 - n - 1$$

where $n \geq 3$.

3.3 Comparison between the Lie and Leibniz multipliers of the strictly upper triangular matrices

It was found in [9] that the dimension of the Lie multiplier of the strictly upper triangular matrices is:

$$\frac{(n-2)(n+1)}{2}$$

Table 3.1 Counting multiplier elements for the strictly upper triangular matrices

n	Lie Multiplier	Lie Cover	Leibniz Multiplier	Leibniz Cover
3	2	5	5	8
4	5	11	11	17
5	9	19	19	29
10	44	89	89	134
20	189	379	379	569

This table shows the comparison of the Lie and Leibniz cases in the dimension of the cover and multiplier for the $n \times n$ strictly upper triangular matrices where $n \geq 3$. The size of the cover was determined by adding the dimension of the algebra, in this case $\frac{n(n-1)}{2}$, to the dimension of the multiplier. It can be seen from the values in the table that the Leibniz cover and multiplier are always greater than the Lie cover and multiplier for this algebra.

Chapter 4

The General Form

In this chapter, we will establish the general formula for the number of elements in the multiplier for each algebra in the lower central series. The lower central series is the series of algebras of matrices $L_0 \subseteq L_1 .... \subseteq L_{n-2}$ such that the Lie bracket $[L_0, L_k] = L_{k+1}$ for each $0 \leq k \leq n-2$. We take k to be the number of diagonals of zeros above the main diagonal in the matrices within the algebra. Therefore, the strictly upper triangular matrices correspond to $k = 0$, their derived algebra corresponds to $k = 1$ etc.

4.1 Finding the General Form

Before looking at the general form of the dimension formula for the multiplier, there are a few things that need to be noted. Firstly, we know $F(c, d)$ is the image of $E_{c,d}$ under the section $\mu : L \to K$. It can be seen that $c + (k+1) \leq d$ for each k due to the basis elements $E_{c,d}$ for each algebra L_k. Accordingly, the smallest space that there can be between s and t or a and b in each element $y(s, t, a, b)$ of the multiplier is $k + 1$. It can also be seen that $c + (2k + 2) \geq d$. Therefore, the largest space between s and t or a and b in each element $y(s, t, a, b)$ of the multiplier is $2k + 2$. This is useful when looking at elements in the general case as it eliminates many elements of the basis.

Theorem 4.1.1. *For elements of the multiplier of the form $y(s, t, a, b)$, the largest space between s and t and the largest space between a and b is $2k + 2$. If $t \geq s + (2k + 3)$ or $b \geq a + (2k + 3)$, then $y(s, t, a, b) = 0$.*

Proof. We know that the smallest space between s and t and the smallest space between a and b is $k + 1$. Taking a space of $2k + 3$ between a and b and some space that is greater than or equal to $k + 1$ between s and t, this gives a multiplier element of the form $y(s, t, a, a + (2k + 3))$ where $t \geq s + (k + 1)$. In order to eliminate elements of this form, the Leibniz identity of the following elements in performed:

$$\begin{aligned}[F(s,t), [F(a, a+x), F(a+x, a+(2k+3))]] &= [[F(s,t), F(a, a+x)], F(a+x, a+(2k+3))] \\ &+ [F(a, a+x), [F(s,t), F(a+x, a+(2k+3))]]\end{aligned}$$

If $t \neq a + (k + 1)$, then take $x = (k + 1)$. Otherwise, take $x = (k + 2)$. From the above Leibniz identity, this will then give $y(s, t, a, a + (2k + 3)) = 0$. Therefore, as long as $t \geq s + (k + 1)$, the multiplier elements $y(s, t, a, a + (2k + 3)) = 0$ and they may be eliminated. The same theory applies for any multiplier element of the form $y(s, t, a, a + x)$ where $x \geq 2k + 3$.

If $t = s+(2k+3)$ and $b \geq a+(k+1)$, this leads to elements of the multiplier of the form $y(s, s+(2k+3), a, b)$. In order to eliminate these elements, the Leibniz identity of the following elements is taken:

$$\begin{aligned}[F(s, s+x), [F(s+x, s+(2k+3)), F(a, b)]] &= [[F(s, s+x), F(s+x, s+(2k+3))], F(a, b)] \\ &+ [F(s+x, s+(2k+3)), [F(s, s+x), F(a, b)]]\end{aligned}$$

If $b \neq s+(k+1)$, then take $x = k+1$. If not, take $x = k+2$. From the above Leibniz identity, this will then give $y(s, s+(2k+3), a, b) = 0$. Consequently, as long as $b \geq a+(k+1)$, the multiplier elements $y(s, s+(2k+3), a, b) = 0$ and can be eliminated. The same theory applies for any multiplier element of the form $y(s, s+x, a, b)$ where $x \geq 2k+3$.

□

The cases where $t = s+(2k+2)$ or where $b = a+(2k+2)$ are looked at separately below.

Theorem 4.1.2. *When $t = s+(2k+2)$ or $b = a+(2k+2)$, the only elements of the multiplier that remain are those of the form $y(s, s+(2k+2), s, s+(k+1))$ and $y(s, s+(2k+2), s+(k+1), s+(2k+2))$ up to isomorphism.*

Proof. First consider the case where $t = s+(2k+2)$. This gives an element of the multiplier of the form $y(s, s+(2k+2), a, b)$. This element arises from taking the following Leibniz identity:

$$\begin{aligned}[F(s, x), [F(x, s+(2k+2)), F(a, b)]] &= [[F(s, x), F(x, s+(2k+2))], F(a, b)] \\ &+ [F(x, s+(2k+2)), [F(s, x), F(a, b)]]\end{aligned}$$

When $a \neq s+(k+1)$ and $b \neq s+(k+1)$, then take $x = s+(k+1)$ in the above identity and simplify to get $y(s, s+(2k+2), a, b) = 0$. So the only cases that need to be considered are those of the form $y(s, s+(2k+2), s+(k+1), b)$ and $y(s, s+(2k+2), a, s+(k+1))$.

First consider the elements $y(s, s+(2k+2), s+(k+1), b)$. If $b = s+(2k+2)$, then the element $y(s, s+(2k+2), s+(k+1), s+(2k+2))$ can be uncovered using:

$$\begin{aligned}&[F(s, s+(k+1)), [F(s+(k+1), s+(2k+2)), F(s+(k+1), s+(2k+2))]] \\ &= [[F(s, s+(k+1)), F(s+(k+1), s+(2k+2))], F(s+(k+1), s+(2k+2))] + \\ &\quad [F(s+(k+1), s+(2k+2)), [F(s, s+(k+1)), F(s+(k+1), s+(2k+2))]]\end{aligned}$$

This gives:

$$y(s, s+(2k+2), s+(k+1), s+(2k+2)) = -y(s+(k+1), s+(2k+2), s, s+(2k+2))$$

If $b > s+(2k+2)$, then from the Leibniz identity:

$$\begin{aligned}&[F(s, s+(k+1)), [F(s+(k+1), s+(2k+2)), F(s+(k+1), b)]] \\ &= [[F(s, s+(k+1)), F(s+(k+1), s+(2k+2))], F(s+(k+1), b)] + \\ &\quad [F(s+(k+1), s+(2k+2)), [F(s, s+(k+1)), F(s+(k+1), b)]] \\ &\implies y(s, s+(2k+2), s+(k+1), b) = -y(s+(k+1), s+(2k+2), s, b)\end{aligned}$$

However, we can see from the right hand side of the equation, $b \geq a + (2k+3)$, which from the previous theorem will eliminate these elements from the basis of the multiplier. The only elements that cannot be eliminated are those of the form $y(s, s+(2k+2), s+(k+1), s+(2k+2))$.

Now consider elements of the form $y(s, s+(2k+2), a, s+(k+1))$. If $a = s$ then the element $y(s, s+(2k+2), s, s+(k+1))$ can be uncovered using:

$$[F(s, s+(k+1)), [F(s+(k+1), s+(2k+2)), F(s, s+(k+1))]] =$$

$$[[F(s, s+(k+1)), F(s+(k+1), s+(2k+2))], F(s, s+(k+1))]+$$
$$[F(s+(k+1), s+(2k+2)), [F(s, s+(k+1)), F(s, s+(k+1))]]$$

This gives:

$$y(s, s+(2k+2), s, s+(k+1)) = -y(s, s+(k+1), s, s+(2k+2))$$

If $a < s$, then the Leibniz identity:

$$[F(s, s+(k+1)), [F(s+(k+1), s+(2k+2)), F(a, s+(k+1))]] =$$

$$[[F(s, s+(k+1)), F(s+(k+1), s+(2k+2))], F(a, s+(k+1))]+$$

$$[F(s+(k+1), s+(2k+2)), [F(s, s+(k+1)), F(a, s+(k+1))]]$$

$$\implies y(s, s+(2k+2), a, s+(k+1)) = -y(s, s+(k+1), a, s+(2k+2))$$

However, on the right hand side of the equation $a \leq s + (2k+3)$ and these elements are eliminated from the basis of the multiplier by the previous theorem. The only elements that cannot be eliminated from the basis of the multiplier are those of the form
$y(s, s+(2k+2), s, s+(k+1))$.

When $b = a + (2k+2)$, by the same theory as above and using the same Leibniz identity, it is found that:

$$y(s, s+(k+1), s, s+(2k+2)) = -y(s, s+(2k+2), s, s+(k+1))$$

and

$$y(s+(k+1), s+(2k+2), s, s+(2k+2)) = -y(s, s+(2k+2), s+(k+1), s+(2k+2))$$

Therefore, as these elements have already been accounted for, these are the only 2 forms of this type of element in the basis of the multiplier.

□

By counting the number of elements of the multiplier that have the form $y(s, s+(2k+2), s, s+(k+1))$ or $y(s, s+(2k+2), s+(k+1), s+(2k+2))$ it can be seen that there are $n-(2k+2)$ values of s for each type, hence there are $n-(2k+2)$ elements of the multiplier of each type, giving $2(n-(2k+2))$ elements in total.

Therefore, the elements $y(s, t, a, b)$ still remain to be examined where $s+(k+1) \leq t \leq s+(2k+1)$ and $a+(k+1) \leq b \leq a+(2k+1)$.

4.1.1 Elements of the multiplier $y(s,t,a,b)$ such that $s+(k+1) \leq t \leq s+(2k+1)$ and $a+(k+1) \leq b \leq a+(2k+1)$ and $s \neq a$ and $t \neq b$

We begin by looking at the case where $s < a < t < b$ and $t < s+(2k+2)$, $b < a+(2k+2)$ in $y(s,t,a,b)$. We need to look at the possible values of s,t,a, and b, starting with b. The smallest value of b is (smallest value of t) + 1. The smallest value of t is $s+(k+1)$, therefore, the smallest value of b is $s+(k+2)$. The largest value of b is (largest value of a) + (2k+1). The largest value of a in this situation is (largest value of t) - 1. The largest value of t is $s+(2k+1)$ and therefore the largest value of a is $s+2k$. So the largest value of b is $s+(4k+1)$. This gives $3k$ different values of b. Just as in the Lie case as found in [5], we look at the first k values of b, then the second k values of b, and subsequently the third k values of b.

The first k cases include $b = s+(k+2)$, up to $b = s+(2k+1)$. When $b =$(smallest value of t) + 1 = s + (k+2), this is where t takes on its minimum value, $t = s+(k+1)$. There are no other values that t can take that would satisfy the inequality $s < a < t < b$. The number of values that a can take when $b = s+(k+2)$ also needs to be found. It is known that $b \geq a+(k+1)$ as the smallest space between a and b is $k+1$. Therefore when $b = s+(k+2)$, $a \leq s+1$. But as $s < a$, this implies that the only value of a in this case is $a = s+1$. So when $s < a < t < b$, and $b = s+(k+2)$, there is only one possible type of multiplier element, $y(s, s+(k+1), s+1, s+(k+2))$.

Now we look at the case where $b =$ (smallest value of t) + 2 $= s+(k+3)$. In order to still satisfy the inequality $s < a < t < b$, and taking into account that the smallest difference between s and t is $k+1$, the possible values of t in this case are $t = s+(k+1)$ and $t = s+(k+2)$. When $t = s+(k+1)$ or $t = s+(k+2)$, and $b = s+(k+3)$, the possible values of a are $a = s+1$, and $a = s+2$ in order to still satisfy the inequality. With 2 possible values of a and 2 possible values of t, this gives $2^2 = 4$ possible types of multiplier element in this case, $y(s, s+(k+1), s+1, s+(k+3))$, $y(s, s+(k+1), s+2, s+(k+3))$, $y(s, s+(k+2), s+1, s+(k+3))$ and $y(s, s+(k+2), s+2, s+(k+3))$.

We then carry on in this fashion until the *k-th* value of b which is $b =$ (smallest value of t) + k = s + (2k + 1). For this value of b, the possible values of t are $t = s+(k+1)$, $t = s+(k+2)$, ..., $t = s+2k$. The possible values of a are $a = s+1$, $a = s+2$, ..., $a = s+k$. It is known these are the possible values of a because we have shown above that the first possible value of a is $a = s+1$. The largest value of a is $a = s+k$ as $b = s+(2k+1)$ and the smallest space between a and b is $(k+1)$. It should also be noted that $s < a < t < b$ is also satisfied here. As there are k different values for t and k different values for a, there are k^2 different types of multiplier element with this particular value of b in this case.

This means that when $b = s+(k+1)+i$ for $1 \leq i \leq k$, there are i^2 types of multiplier elements in each case. Then for each case of $b = s+(k+1)+i$, with $s < a < t < b$ where b takes on the highest value, there are $(n-((k+1)+i))$ different elements of the multiplier as s varies and when i is fixed. Therefore in total, when $b = s+(k+1)+i$ for $1 \leq i \leq k$ there are:

$$\sum_{i=1}^{k} i^2 \cdot (n-((k+1)+i))$$

elements of the multiplier.

Now looking at the second k values of b. The smallest value of b in this case is $b = s+(2k+2)$. The smallest value that t can take is $t = s+(k+1)$. The largest value of t is $t = s+(2k+1)$. This is the largest value of t as the largest space between s and t is $2k+1$ and this shows that in this case, t will always be smaller than b, which satisfies the inequality $s < a < t < b$. The possible values of a are $a = s+1$, $a = s+2$, ..., $a = s+(k+1)$ as the smallest space between a and b is $k+1$ and the largest space that we are considering is $2k+1$. However, if $a = s+(k+1)$, then $t = a$ if t equals its lowest possible value, so $t = s+(k+1)$. This does not satisfy the inequality $s < a < t < b$. Therefore, this possibility must be eliminated from this case. As such, there are $k+1$ possible values of t and $k+1$ possible values of a however the case where $t = a = s+(k+1)$ must be eliminated. So there are $(k+1)^2 - 1$ elements in the basis of the multiplier with $s < a < t < b$ with $b = s+(2k+2)$.

The next value of b to consider is $b = s+(2k+3)$. The possible values of t in this case are once again $t = s+(k+1)$, $t = s+(k+2)$, ..., $t = s+(2k+1)$. The possible values of a are $a = s+2$, $a = s+3$, ..., $a = s+(k+2)$ as $a+(k+1) \leq b \leq a+(2k+1)$. However, we must consider the inequality $s < a < t < b$. When $a = s+(k+1)$, the value $t = s+(k+1)$ cannot be used in order to satisfy the inequality. When $a = s+(k+2)$, the values $t = s+(k+1)$ or $t = s+(k+2)$ can also not be used. This takes out 3 possibilities of types of element in the multiplier. Therefore, there are $k+1$ possible values of t and $k+1$ possible values of a, and 3 possibilities that do not need to be considered here, leaving $(k+1)^2 - 3$ types of multiplier element in this case.

Carrying on in this fashion up to $b = s+(3k+1)$, the possible values of t in this case are $t = s+(k+1)$, $t = s+(k+2)$, ..., $t = s+(2k+1)$. It can be found that the possible values of a are $a = s+k$, $a = s+(k+1)$, ..., $a = s+2k$. This gives $k+1$ possible values of t and $k+1$ possible values of a. However, looking at the inequality $s < a < t < b$, if $a = s+(k+1)$, then $t = s+(k+1)$ which is not possible in this case. If $a = s+(k+2)$, the values of t that are not possible are $t = s+(k+1)$ or $t = s+(k+2)$. Continuing on until $a = s+2k$, for this value, we find that the cases that are not possible are $t = s+(k+1)$, $t+s+(k+2)$, ..., $t = s+2k$. This means $1+2+...+k$ types of element in the multiplier must be eliminated as they do not fit into this case. This gives $(k+1)^2 - (1+2+...+k) = (k+1)^2 - \frac{k(k+1)}{2}$ elements in this case.

Now adding up all types of elements in the multiplier that satisfy $b = s+(2k+1)+i$ for $1 \leq i \leq k$ with $s < a < t < b$, we have found that for each i there are $(k+1)^2 - \frac{i(i+1)}{2}$ types of multiplier elements. Then in each case, as i stays fixed and s varies and where b is the largest value with $s < a < t < b$, there are $n - ((2k+1)+i)$ elements of the multiplier of each form. Therefore in total, when $b = s+(2k+1)+i$ for $1 \leq i \leq k$ with $s < a < t < b$, there are:

$$\sum_{i=1}^{k} \left((k+1)^2 - \frac{i(i+1)}{2} \right) \cdot (n - ((2k+1)+i))$$

elements of the multiplier.

When considering the last k values of b that satisfy this inequality, the first value of b that needs to be considered is $b = s+(3k+2)$. The possible values of t are $t = s+(k+1)$, $t = s+(k+2)$, ..., $t = s+(2k+1)$. The possible values of a are $a = s+(k+1)$, $a = s+(k+2)$, ..., $a = s+(2k+1)$ as the smallest space between a and b is $k+1$ and the largest space is $2k+1$. However, the value $a = s+(2k+1)$ cannot be used as it is greater than or equal to every possible value of t. Therefore there are $k+1$ possible values of t and k possible values of a. When $a = s+(k+1)$, $t = s+(k+1)$ is not possible as $t = a$ in this case, however, the other k values of t are possible. When $a = s+(k+2)$, $t = s+(k+1)$ or $t = s+(k+2)$ are not possible, so there are 2 possibilities that have to be eliminated. Continuing on with this case in this fashion until $a = s + 2k$, it is found that the only possibility for t is $t = s + (2k + 1)$, which eliminates k values for t. Thus, the total number of types of multiplier elements in this case for this value of b is $k(k + 1) - \frac{k(k+1)}{2}$ $= \frac{k(k+1)}{2}$.

The next value of b to consider is $b = s+(3k+3)$. The possible values of t are $t = s+(k+1)$, $t = s+(k+2)$, ..., $t = s + (2k + 1)$. The possible values of a are $a = s + (k + 2)$, $a = s + (k + 2)$, ..., $a = s + (2k + 2)$. The values $a = s + (2k + 1)$ and $a = s + (2k + 2)$ cannot be used because these values of a are greater than or equal to all of the possible values of t. This gives $k + 1$ possible values for t and $k - 1$ possible values for a. When $a = s+(k+2)$, $t = s+(k+1)$ or $t = s+(k+2)$ these values are not possible as these do not satisfy the inequality $s < a < t < b$. This eliminates 2 possibilities for values of t with this particular a, but the other $k - 1$ values for t are still possible. When $a = s + (k + 3)$, $t = s + (k + 1)$, $t = s + (k + 2)$ or $t = s + (k + 3)$ again, these are not possible, which eliminates 3 possible values of t for this value of a. Carrying on in this way, when $a = s + 2k$, the only value of t that can be used is $t = s + (2k + 1)$, which eliminates k possible values of t for this a. Accordingly, for this particular value of b, there are $1 + 2 + ... + (k - 1) = \frac{(k-1)k}{2}$ possible elements of this form in the multiplier.

We carry on in this fashion until the final value of b which is $b = s + (4k + 1)$. The possible values of t are again $t = s + (k + 1)$, $t = s + (k + 2)$, ..., $t = s + (2k + 1)$. The possible values of a are $a = s + 2k$, $a = s + (2k + 1)$, ..., $a = s + 3k$. The only value of a that can be used in this case is $a = s + 2k$ as all other values of a are greater than or equal to the other values of t and therefore do not satisfy the inequality $s < a < t < b$. When $a = s + 2k$, the only possible value of t is $t = s + (2k + 1)$. This gives only 1 possible type of element of the multiplier.

The next task is to find the number of elements of this form. The elements satisfy $b = s + (3k + 1) + i$ for $1 \leq i \leq k$ with $s < a < t < b$. For each i there are $\frac{(k+1-i)(k+2-i)}{2}$ types of multiplier element. Then in each case, as i stays fixed and s varies, where b is the largest value with $s < a < t < b$, there are $n - ((3k + 1) + i)$ elements of the multiplier of each form. Therefore in total, when $b = s + (3k + 1) + i$ for $1 \leq i \leq k$ with $s < a < t < b$, there are:

$$\sum_{i=1}^{k} \frac{(k + 1 - i)(k + 2 - i)}{2} \cdot (n - ((3k + 1) + i))$$

elements of the multiplier.

Adding these together, it can be seen that the total number of elements of the multiplier of the form $y(s, t, a, b)$ with $s < a < t < b$ where b is of the form $b = s + (k + 1) + i$ where $1 \leq i \leq 3k$, is:

$$\sum_{i=1}^{k} \Bigg[i^2 \cdot (n - ((k + 1) + i)) + \left((k + 1)^2 - \frac{i(i + 1)}{2} \right) \cdot (n - ((2k + 1) + i)) + \frac{(k + 1 - i)(k + 2 - i)}{2} \cdot (n - ((3k + 1) + i)) \Bigg]$$

The case where $a < s < b < t$ such that $a + (k + 1) \leq b \leq a + (2k + 1)$ and $s + (k + 1) \leq t \leq s + (2k + 1)$ actually yields the same result as the above. This time, each of s, t, a and b will depend on a in each case instead of s this time, as a will have the smallest value. It can be seen by symmetry that this case will actually yield the same result as the above and because of this there are:

$$\sum_{i=1}^{k}\Bigg[i^2\cdot(n-((k+1)+i))+\left((k+1)^2-\frac{i(i+1)}{2}\right)\cdot(n-((2k+1)+i))$$
$$+\frac{(k+1-i)(k+2-i)}{2}\cdot(n-((3k+1)+i))\Bigg]$$

elements of the multiplier that satisfy this condition.

The next case for consideration is where $s < a < b < t$ for the multiplier elements $y(s,t,a,b)$. We have shown before that $a+(k+1) \leq b \leq a+(2k+1)$ and $s+(k+1) \leq t \leq s+(2k+1)$. If $s < a < b < t$, this means that t will have the largest value among s,t,a,b. We will let a,b,t depend on the smallest value of these, s, just as before.

We begin by looking at the case where t takes on its smallest value, $t = s+(k+1)$. Then let a take on its smallest possible value, in this case $a = s+1$. This means that the smallest value that b can take here is $b = s+(k+2)$ which does not satisfy $s < a < b < t$. Therefore, $t \neq s+(k+1)$ when $s < a < b < t$.

Next, look at the case where $t = s+(k+2)$. If $a = s+1$, then again the smallest value of b is $s+(k+2)$ which again does not satisfy $s < a < b < t$. Therefore $t \neq s+(k+2)$ for $s < a < b < t$.

When $t = s+(k+3)$, take a to be its smallest possible value $a = s+1$. Then the smallest value b can take is again $b = s+(k+2)$. This is the only possible value for b in order for the inequality $s < a < b < t$ to be satisfied.

When $t = s+(k+4)$, then if $a = s+1$, $b = s+(k+2)$ and $b = s+(k+3)$ are both possibilities that satisfy the inequality in this case. If $a = s+2$ then $b = s+(k+3)$ is another possibility. Therefore, there are 3 different types of multiplier elements in this case that satisfy the inequality $s < a < b < t$.

Continuing on in this way, it can be seen that the highest value that t can take in this case is $t = s+(2k+1)$. Because of the inequality, $s < a < b < t$, together with the relationships between s and t and a and b, the smallest possible value of a is $a = s+1$ while the largest value of a is $a = s+(k-1)$. The smallest value of b in this case is $b = s+(k+2)$ while the largest value of b is $b = s+2k$. When $a = s+1$, any of the above values of b are possible, therefore there are $k-1$ possible values of b. When $a = s+2$, the smallest possible value of b is $b = s+(k+3)$ and the largest value of b is $b = s+2k$, giving $k-2$ possible values for b. Continuing in this fashion, when $a = s+(k-1)$, the only possible value of b is $b = s+2k$, giving just 1 possibility. As such, there are $1+2+....+(k-1)$ types of multiplier that satisfy these values and inequalities.

Therefore, if $s < a < b < t$ and $t = s+(k+2)+i$ for $1 \leq i \leq k-1$, then there are $1+2+...+i$ types of multiplier for each i. This means there are:

$$\sum_{j=1}^{i} j = \frac{i(i+1)}{2}$$

types of multiplier elements for each $1 \leq i \leq k-1$. Now looking at how many of each type of multiplier element there are. If $t = s+(k+2)+i$ with t being the largest value in the inequality $s < a < b < t$, then there must be $n-((k+2)+i)$ elements in the multiplier for each value of t. Therefore, there are:

$$\sum_{i=1}^{k-1} \frac{i(i+1)}{2} \cdot (n-((k+2)+i))$$

different multiplier elements that satisfy the inequality $s < a < b < t$.

By symmetry, the above theory can be taken and applied to the case where $a < s < t < b$. The calculations above will be the same, except that this time each value will depend on the smallest value a instead of s and we would go through the possible values to identify the largest possible value of b. By doing this, the number of different multiplier elements that satisfy the inequality $a < s < t < b$ is:

$$\sum_{i=1}^{k-1} \frac{i(i+1)}{2} \cdot (n-((k+2)+i))$$

The next case for consideration are elements of the multiplier $y(s,t,a,b)$ where $s < t < a < b$. Again, we must remember that $a+(k+1) \leq b \leq a+(2k+1)$ and $s+(k+1) \leq t \leq s+(2k+1)$. Therefore, the smallest value of b is $a+(k+1)$ and the smallest value of t is $s+(k+1)$, so the multiplier element should take the form $y(s, s+(k+1)+i, a, a+(k+1)+j)$. The largest value of t is $s+(2k+1)$ and the largest value of b is $a+(2k+1)$, so $0 \leq i, j \leq k$. These elements of the multiplier can also be written as $y(s, s+x, a, a+y)$ where $k+1 \leq x \leq 2k+1$, $k+1 \leq y \leq 2k+1$, and $s+x < a$.

When $a+y = n$ and at the same time keeping a fixed, the largest value of $t = s+x$ is $n-y-1$, so the value of s can go from 1 to $n-x-y-1 = n-(x+y+1)$. This gives $n-(x+y+1)$ possible values of s in this case.

When $a+y = n-1$ and at the same time keeping a fixed, the largest value of $s+x$ is $n-y-2$. Therefore the values of s go from 1 to $n-x-y-2 = n-(x+y+2)$. This gives $n-(x+y+2)$ possible values of s in this case.

The smallest value of s is 1, so the smallest value of $b = a+y$ is $x+y+2$. When $a+y = x+y+2$, then the value of a is $x+2$. The only value of t in this case that satisfies $t < a$ is $x+1$, giving the value of s as 1.

Accordingly, the total number of different values of s in this case will give the total number of multiplier elements for a fixed x and y. This gives $1+2+...+n-(x+y+1) = \frac{(n-(x+y+1))(n-(x+y))}{2}$ total values of s. This can also be written as $1+2+...+n-((k+1)+i+(k+1)+j+1) = 1+2+...+n-(2k+3+i+j) = \frac{(n-(2k+3+i+j))(n-(2k+2+i+j))}{2}$ different multiplier elements for a fixed i and j.

Adding together the number of elements of the multiplier that satisfy $s < t < a < b$ as i and j vary, there are:

$$\sum_{j=0}^{k} \sum_{i=0}^{k} \frac{(n-(2k+3+i+j))(n-(2k+2+i+j))}{2}$$

different elements of the multiplier satisfying this inequality.

The case such that $a < b < s < t$ also needs to be considered. By symmetry, there will be the same number of elements in the multiplier that satisfy this inequality as those that satisfy $s < t < a < b$, so there are:

$$\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2k+3+i+j))(n-(2k+2+i+j))}{2}$$

that satisfy this inequality. So in total, there are:

$$\sum_{j=0}^{k}\sum_{i=0}^{k}(n-(2k+3+i+j))(n-(2k+2+i+j))$$

elements of the multiplier that satisfy either $s<t<a<b$ or $a<b<s<t$.

Ideally, we would like to write this as a single summation over i so that we can add all of the summations together for the total number of elements in the multiplier. For the summation:

$$\sum_{j=0}^{k}\sum_{i=0}^{k}(n-(2k+3+i+j))(n-(2k+2+i+j))$$

If we sum the elements form $i=0$ up to $i=k$ this would just give a summation over j with $k+1$ terms within the summation. This would give us:

$$\sum_{j=0}^{k}[(n-(2k+3+j))(n-(2k+2+j))+(n-(2k+4+j))(n-(2k+3+j))$$
$$+...+(n-(3k+3+j))(n-(3k+2+j))]$$

The next step is to sum each of these terms from $j=0$ up to $j=k$. This can be written in matrix format so that each of the elements in the matrix can be summed together at the end, with the first row corresponding to the first term being summed from $j=0$ up to $j=k$, the second row corresponding to the second term etc. This will give us a $(k+1)$x$(k+1)$ matrix.

$$\begin{bmatrix}(n-(2k+3))(n-(2k+2)) & (n-(2k+4))(n-(2k+3)) & \dots & (n-(3k+3))(n-(3k+2))\\ (n-(2k+4))(n-(2k+3)) & (n-(2k+5))(n-(2k+4)) & \dots & (n-(3k+4))(n-(3k+3))\\ (n-(2k+5))(n-(2k+4)) & (n-(2k+6))(n-(2k+5)) & \dots & (n-(3k+5))(n-(3k+4))\\ \vdots & \vdots & \ddots & \vdots\\ (n-(3k+3))(n-(3k+2)) & (n-(3k+4))(n-(3k+3)) & \dots & (n-(4k+3))(n-(4k+2))\end{bmatrix}$$

It can be seen that the terms on each of the diagonals starting from the top left corner, are the terms repeated 1, 2, 3, up to k+1 times. By looking at the pattern that has emerged here, the double summation from before can be written as the sum of 2 single summations over i in the following way:

$$\sum_{i=1}^{k+1}i\cdot(n-((2k+1)+i))(n-((2k+2)+i))+$$

$$\sum_{i=1}^{k}(k-(i-1))\cdot(n-((3k+2)+i))(n-((3k+3)+i))$$

Another set of elements that need to be examined are those where $s = a$ or $b = t$. When $s = a$ and $b \neq t$, there are 2 possibilities; either $s = a < b < t$ or $s = a < t < b$. Looking first at the case where $s = a < t < b$. The smallest possible value of t is $s + (k + 1)$ and therefore the smallest possible value of b is $s + (k + 2)$. The largest possible value of b is $s + (2k + 1)$ and the largest value of t is $s + 2k$.

By examining each of these cases, it can be seen that the summation of the number of terms of this form can be found. When $b = s + (k + 2)$, the only possible value of t is $t = s + (k + 1)$. When $b = s + (k + 3)$ there are 2 possible values of t, $t = s + (k + 1)$ and $t = s + (k + 2)$. Carrying on in this fashion up to $b = s + (2k + 1)$ this gives k possible values for t; $t = s + (k + 1)$, $t = s + (k + 3)$, ..., $t = s + 2k$. Therefore, for $b = s + (k + 1) + i$, where $1 \leq i \leq k$ there are i possible values of t and as such, there are i different types of multiplier element for each $b = s + (k + 1) + i$.

If $b = s + (k + 1) + i$, with b being the largest value in $s = a < t < b$, there are $(n - ((k + 1) + i))$ different values for s and hence there are $(n - ((k + 1) + i))$ different multiplier elements for each i. As such, the total number of multiplier elements $y(s, t, a, b)$ that satisfy $s = a < t < b$ is:

$$\sum_{i=1}^{k} i \cdot (n - ((k + 1) + i))$$

In a similar way, by symmetry, it can also be seen that in the second case where $s = a < b < t$, there are also:

$$\sum_{i=1}^{k} i \cdot (n - ((k + 1) + i))$$

multiplier elements, $y(s, t, a, b)$, that satisfy this inequality.

We must also look at the elements of the multiplier $y(s, t, a, b)$ such that $b = t$ and $s \neq a$. There are 2 possibilities; either $s < a < b = t$ or $a < s < b = t$. We will first consider $s < a < b = t$. The smallest value of a is $a = s + 1$ and therefore the smallest value of b and t is $b = t = s + (k + 2)$. The largest value of b and t is $b = t = s + (2k + 1)$, therefore the largest value of a is $a = s + k$. Going through each of these case, the summation of the number of terms of this form can be found. When $b = s + (k + 2)$, the only possible value of a is $s + 1$. When $b = s + (k + 3)$, there are 2 possible values of a, $a = s + 1$ and $a = s + 2$. Carrying on in this fashion up to $b = s + (2k + 1)$, it can be seen that there are k possible values of a; $a = s + 1$, ..., $a = s + k$. Therefore, for $b = s + (k + 1) = i$, for $1 \leq i \leq k$, there are i possible values of a and therefore there are i different types of multiplier element for each $b = s + (k + 1) = i$.

If $b = s + (k + 1) + i$, with b being the largest value in $s < a < b = t$, there are $(n - ((k + 1) + i))$ different values for s and hence there are $(n - ((k + 1) + i))$ different multiplier elements for each i. Hence, the total number of multiplier elements $y(s, t, a, b)$ that satisfy $s < a < b = t$ is:

$$\sum_{i=1}^{k} i \cdot (n - ((k + 1) + i))$$

In a similar way, by symmetry, we can also see that in the second case where $a < s < b = t$, there are also

$$\sum_{i=1}^{k} i \cdot (n - ((k + 1) + i))$$

multiplier elements, $y(s,t,a,b)$, that satisfy this inequality.

Finally, consider the case where $s = a < b = t$. This corresponds to elements of the form $y(s, s+x, s, s+x)$ where $k+1 \leq x \leq 2k+1$. For each value of x we get 1 type of element of the multiplier. Then when $t = b = s + (k+i)$ for $1 \leq i \leq k+1$ there are $n - (k+i)$ different elements of the multiplier of the form $y(s, s+x, s, s+x)$. Therefore, there are:

$$\sum_{i=1}^{k+1} (n - (k+i))$$

different elements of the multiplier that satisfy $s = a < b = t$.

4.1.2 Elements $y(s,t,a,b)$ such that $s + (k+1) \leq t \leq s + (2k+1)$ and $a + (k+1) \leq b \leq a + (2k+1)$ and $s = b$ or $t = a$

The last set of cases that need to considered are those where $s = b$ or $t = a$. First we will consider those with $t = a$. These will therefore be multiplier elements of the form $y(s,a,a,b)$.

We will split this into 2 separate cases. We will look at the elements of the multiplier $y(s,a,a,s+(2k+2+i))$ where $1 \leq i \leq k$ and then look at $y(s,a,a,s+(2k+2+i))$ where $i \geq k+1$. First looking at those elements where $1 \leq i \leq k$. It can been seen that we do not need to look at the case where $i = 0$ as the elements $y(s,a,a,s+(2k+2))$ can be eliminated from the multiplier. This is because the only value that a can take in this case is $s + (k+1)$ in order for the spaces between s and a and a and b to be sufficiently large enough. However in this case, this type of element can also be written as $y(s, b-(k+1), b-(k+1), b)$, which we have already stated is eliminated using a change of basis. Therefore we do not need to consider the case where $i = 0$. It can also seen that if $b < s + 2k + 2$ there is not a large enough space between s and b in order to have a value for a so this case does not need to be considered. Now considering $1 \leq i \leq k$ for $y(s,a,a,s+(2k+2+i))$.

When $i = 1$, there is a space of $2k+3$ between s and b. In order for the spaces between s and a and a and b to be sufficiently large enough, the only type of multiplier element in this case is $y(s, s+(k+1), s+(k+1), s+(2k+3))$. We cannot have $y(s, s+(k+2), s+(k+2), s+(2k+3))$ as this would give the multiplier element of the form $y(s, b-(k+1), b-(k+1), b)$ which can be eliminated by a change of basis. So when $i = 1$, there is 1 type of multiplier element.

When $i = 2$, there is a space of $2k+4$ between s and b. By the same reasoning as above, the only types of multiplier element in this case are $y(s, s+(k+1), s+(k+1), s+(2k+4))$ and $y(s, s+(k+2), s+(k+2), s+(2k+4))$. So when $i = 2$, there are 2 different types of multiplier element.

We carry on in this fashion until $i = k$ in which there is a space of $3k+2$ between s and b. In this case, there are k possible types of multiplier element, $y(s, s+(k+1), s+(k+1), s+(3k+2))$, $y(s, s+(k+2), s+(k+2), s+(3k+2))$, ..., $y(s, s+2k, s+2k, s+(3k+2))$.

Therefore, it can be seen that for each i such that $1 \leq i \leq k$, there are i different types of multiplier elements that are of the form $y(s,a,a,s+(2k+2+i))$. Keeping i fixed and now allowing s to vary, there are $(n-(2k+2+i))$ different elements in the multiplier for each i as b is the largest value among s, a and b in these cases. So there are:

$$\sum_{i=1}^{k} i \cdot (n - (2k+2+i))$$

different elements of the multiplier that satisfy $y(s,a,a,s+(2k+2+i))$ for $1 \leq i \leq k$.

We must now look at the elements of the multiplier $y(s,a,a,s+(2k+2+i))$ such that $i \geq k+1$. Another way of writing this is $y(s,a,a,s+(3k+2)+i)$ where $i \geq 1$. The largest space between s and a is $2k+1$ and the largest space between a and b is $2k+1$. The smallest gap between these elements is $k+1$. This means that the largest value of a is $a = s+(2k+1)$ and therefore the largest value of b is $b = s+(4k+2)$. Therefore $1 \leq i \leq k$.

When $i = 1$, the multiplier is of the form $y(s,a,a,s+(3k+3))$. The possible types of element for the multiplier in this case are $y(s,s+(k+2),s+(k+2),s+(3k+3))$, $y(s,s+(k+3),s+(k+3),s+(3k+3))$, ..., $y(s,s+(2k+1),s+(2k+1),s+(3k+3))$. The elements of the form $y(s,s+(k+1),s+(k+1),s+(3k+3))$ are not possible as this gives a space of $2k+2$ between a and b. Therefore there are k possibilities for types of multiplier element when $i = 1$.

When $i = 2$, the multiplier is of the form $y(s,a,a,s+(3k+4))$. The possible types of element for the multiplier in this case are $y(s,s+(k+3),s+(k+3),s+(3k+4))$, $y(s,s+(k+4),s+(k+4),s+(3k+4))$, ..., $y(s,s+(2k+1),s+(2k+1),s+(3k+4))$. Again, the elements $y(s,s+(k+1),s+(k+1),s+(3k+4))$ or $y(s,s+(k+2),s+(k+2),s+(3k+4))$ are not possible because there is too large a space between a and b. This gives $k-1$ possible types of element for the multiplier when $i = 2$.

We carry on in this fashion until $i = k$ where the multiplier is of the form $y(s,a,a,s+(4k+2))$. The only possible type of multiplier element in this case is $y(s,s+(2k+1),s+(2k+1),s+(4k+2))$. Therefore there is only 1 possible type of multiplier element when $i = k$.

Therefore we can see that for each i such that $1 \leq i \leq k$, there are $(k-(i-1))$ different types of multiplier element that are of the form $y(s,a,a,s+(3k+2)+i)$. Keeping i fixed and now allowing s to vary, we can see that there are $(n-(3k+2+i))$ different elements in the multiplier for each i as b is the largest value among s, a and b in these cases. We can therefore see that there are:

$$\sum_{i=1}^{k}(k-(i-1)) \cdot (n-(3k+2+i))$$

different elements of the multiplier that satisfy $y(s,a,a,s+(3k+2)+i)$ for $1 \leq i \leq k$. We can now put these 2 separate cases together. We have that there are:

$$\sum_{i=1}^{k} i \cdot (n-(2k+2+i)) + \sum_{i=1}^{k}(k-(i-1)) \cdot (n-(3k+2+i))$$

different elements of the multiplier when $t = a$ in $y(s,t,a,b)$.

Now we need to look at the case where $s = b$. This gives multiplier elements of the form $y(s,t,a,s)$. We can therefore see that $a < t$ but the relationship between a and t needs to be found. The smallest space between s and t is $k+1$ and the same goes for the space between a and $b = s$. The largest space between these elements is $2k+1$. Therefore, $t \geq s+(k+1)$, $t \leq s+(2k+1)$ and $s \geq a+(k+1)$, $s \leq a+(2k+1)$. Putting these together, $a+(2k+2) \leq t \leq a+(4k+2)$, $a+(k+1) \leq s \leq a+(2k+1)$ and therefore, $a \leq s \leq t$.

As a will have the smallest value, let s and t depend on a. We will look at the different values that t can have and for each value of t, look at the number of different types of multiplier element are possible.

First look at the elements of the multiplier $y(s, a+(2k+1)+i, a, s)$ for $1 \leq i \leq k+1$, then look at the elements of the form $y(s, a+(3k+2)+i, a, s)$ for $1 \leq i \leq k$. So for the elements $y(s, a+(2k+1)+i, a, s)$ for $1 \leq i \leq k+1$, look at each value of t and see how many different types of multiplier element are possible.

When $i = 1$, this means $t = a + (2k+2)$. The only possible value of s in this case that will satisfy the above inequalities is $s = a + (k+1)$. If s is any larger then the space between s and t would be too small. Therefore, when $i = 1$, there is only 1 possible type of multiplier element.

When $i = 2$, this means $t = a + (2k+3)$. By the same theory as above, the only possible values of s in this case are $s = a + (k+1)$ and $s = a + (k+2)$. Therefore, when $i = 2$ there are 2 possible types of element in the multiplier.

Carrying on in this fashion, when $i = k+1$, this means that $t = a + (3k+2)$. The possible values of s in this case are $s = a + (k+1)$, $s = a + (k+2)$, ..., $s = a + (2k+1)$. We cannot take any s larger than this or the space between s and a would be too large and the space between s and t would be too small. Therefore, when $i = k+1$ there are $k+1$ different types of multiplier element in this case.

So for multiplier elements of the form $y(s, a+(2k+1)+i, a, s)$ for $1 \leq i \leq k+1$, there are i different types of multiplier element for each i. Now keeping i fixed and letting a vary, there are $(n - ((2k+1)+i))$ different elements of the multiplier for each i as t is the largest value out of s, a and t. Putting this together we can see that there are:

$$\sum_{i=1}^{k+1} i \cdot (n - ((2k+1)+i))$$

elements of the multiplier of the form $y(s, a+(2k+1)+i, a, s)$ for $1 \leq i \leq k+1$.

Now looking at the second case where $s = b$ which looks at elements of the multiplier of the form $y(s, a+(3k+2)+i, a, s)$ for $1 \leq i \leq k$. When $i = 1$, so $t = a + (3k+3)$, the possible values of s are $s = a + (k+2)$, $s = a + (k+3)$, ..., $s = a + (2k+1)$. s cannot take ant smaller value than this as there would be too large a space between s and t and if s is any larger then the space between s and a would be too large and the space between s and t would be too small. Therefore, when $i = 1$, there are k different types of multiplier element in this case.

When $i = 2$, so $t = a + (3k+4)$, the possible values of s are $s = a + (k+3)$, $s = a + (k+4)$, ..., $s = a + (2k+1)$. Again if s is any smaller, the gap between s and t would be too large. Therefore, when $i = 2$, there are $k-1$ types of multiplier element in this case.

We carry on in this fashion up to $i = k$, so $t = a + (4k+2)$. The only possible value of s in this case is $s = a + (2k+1)$ and therefore when $i = k$ there is only 1 type of multiplier element.

So for multiplier elements of the form $y(s, a+(3k+2)+i, a, s)$ for $1 \leq i \leq k$, there are $(k-(i-1))$ different types of multiplier element for each i. Now keeping i fixed and letting a vary, there are $(n - ((3k+2)+i))$ different elements of the multiplier for each i as t is the largest value out of s, a and t. Putting this together, there are:

$$\sum_{i=1}^{k} (k - (i-1)) \cdot (n - ((3k+2)+i))$$

elements of the multiplier of the form $y(s, a+(3k+2)+i, a, s)$ for $1 \leq i \leq k$. Putting these 2 separate cases together, there are:

$$\sum_{i=1}^{k+1} i \cdot (n-((2k+1)+i)) + \sum_{i=1}^{k}(k-(i-1)) \cdot (n-((3k+2)+i))$$

different elements of the multiplier when $s = b$ in $y(s,t,a,b)$.

4.1.3 The sum of all elements of the multiplier

Adding together the total number of elements of the multiplier, we will end up with the following formula which can then be simplified:

$$\sum_{i=1}^{k} 2i^2 \cdot (n-((k+1)+i)) + \sum_{i=1}^{k}(2(k+1)^2 - i(i+1)) \cdot (n-((2k+1)+i))$$

$$+\sum_{i=1}^{k}(k-(i-1))(k-(i-2)) \cdot (n-((3k+1)+i)) + \sum_{i=1}^{k-1} i(i+1) \cdot (n-((k+2)+i))$$

$$+\sum_{i=1}^{k+1} i \cdot (n-((2k+1)+i))(n-((2k+1)+(i+1)))$$

$$+\sum_{i=1}^{k}(k-(i-1)) \cdot (n-((3k+2)+i))(n-((3k+2)+(i+1)))$$

$$+\sum_{i=1}^{k} 4i \cdot (n-((k+1)+i)) + \sum_{i=1}^{k+1}(n-(k+i))$$

$$+\sum_{i=1}^{k} i \cdot (n-((2k+2)+i)) + \sum_{i=1}^{k}(k-(i-1)) \cdot (n-((3k+2)+i))$$

$$+\sum_{i=1}^{k+1} i \cdot (n-((2k+1)+i)) + \sum_{i=1}^{k}(k-(i-1)) \cdot (n-((3k+2)+i))$$

$$+2(n-(2k+2))$$

The first step to simplifying this formula is to make any summation term a sum from $i = 1$ up to $i = k$. We can then combine them into one sum from $i = 1$ up to $i = k$ with some other added terms.

1.

$$\sum_{i=1}^{k-1} i(i+1) \cdot (n-((k+2)+i)) = \left[\sum_{i=1}^{k} i(i+1) \cdot (n-((k+2)+i))\right] - [k(k+1) \cdot (n-(2k+2))]$$

2.

$$\sum_{i=1}^{k+1} i \cdot (n-((2k+1)+i))(n-((2k+1)+(i+1)))$$

$$= \left[\sum_{i=1}^{k} i \cdot (n - ((2k+1)+i))(n - ((2k+1)+(i+1)))\right]$$

$$+ [(k+1) \cdot (n - (3k+2))(n - (3k+3))]$$

3.
$$\sum_{i=1}^{k+1} (n - (k+i)) = \left[\sum_{i=1}^{k} (n - (k+i))\right] + (n - (2k+1))$$

4.
$$\sum_{i=1}^{k+1} i \cdot (n - ((2k+1)+i)) = \left[\sum_{i=1}^{k} i \cdot (n - ((2k+1)+i))\right] + [(k+1) \cdot (n - (3k+2))]$$

Therefore, the above summation can be written as:

$$(n - (2k+1)) + [(2 - k(k+1)) \cdot (n - (2k+2))] + [(k+1) \cdot (n - (3k+2))]$$

$$+ [(k+1) \cdot (n - (3k+2))(n - (3k+3))] +$$

$$\sum_{i=1}^{k} [(n - (k+i)) + 2i(i+2) \cdot (n - ((k+1)+i)) + [i(i+1) \cdot (n - ((k+2)+i))]$$

$$+ \left[(2(k+1)^2 - i^2) \cdot (n - ((2k+1)+i))\right] + i(n - ((2k+2)+i))$$

$$+ [(k - (i-1))(k - (i-2)) \cdot (n - ((3k+1)+i))]$$

$$+ [2(k - (i-1)) \cdot (n - ((3k+2)+i))]$$

$$+ i(n - ((2k+1)+i))(n - ((2k+2)+i))$$

$$+ [(k - (i-1)) \cdot (n - ((3k+2)+i))(n - ((3k+3)+i))]]$$

All of the terms in the final 4 lines of this equation are summed over i from 1 to k. The highest power of i is i^3. By multiplying out each of the 9 terms that are within summations over i and using the following summation formulas:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

the summations can then be eliminated in this formula to get a formula for the general case in terms of just k and n.

Simplifying the formula term by term:

1. $\sum_{i=1}^{k}(n-(k+i))$

$= \sum_{i=1}^{k}(n-k) - \sum_{i=1}^{k} i$

$= k(n-k) - \frac{k(k+1)}{2}$

$= nk - k^2 - \frac{1}{2}k^2 - \frac{1}{2}k$

$= nk - \frac{1}{2}k - \frac{3}{2}k^2$

2. $\sum_{i=1}^{k} 2i(i+2) \cdot (n-((k+1)+i))$

$= \sum_{i=1}^{k}(2i^2+4i) \cdot ((n-k-1)-i)$

$= 2(n-k-1)\sum_{i=1}^{k} i^2 + 4(n-k-1)\sum_{i=1}^{k} i - 2\sum_{i=1}^{k} i^3 - 4\sum_{i=1}^{k} i^2$

$= 2(n-k-1)\frac{k(k+1)(2k+1)}{6} + 4(n-k-1)\frac{k(k+1)}{2} - \frac{k^2(k+1)^2}{2} - \frac{2k(k+1)(2k+1)}{3}$

$= \frac{1}{3}(n-k-1)(2k^3+3k^2+k) + 2(n-k-1)(k^2+k) - \frac{1}{2}(k^4+2k^3+k^2)$
$-\frac{2}{3}(2k^3+3k^3+k)$

$= \frac{1}{3}(2nk^3+3nk^2+nk-2k^3-3k^2-k-2k^4-3k^3-k^2) + (-\frac{1}{2}k^4-k^3-\frac{1}{2}k^2)$
$+2(nk^2+nk-k^3-k^2-k^2-k) + (-\frac{4}{3}k^3-2k^2-\frac{2}{3}k)$

$= n(\frac{2}{3}k^3+k^2+\frac{1}{3}k) - \frac{2}{3}k^4 - \frac{5}{3}k^3 - \frac{4}{3}k^2 - \frac{1}{3}k + n(2k^2+2k) - 2k^3 - 4k^2 - 2k$
$-\frac{1}{2}k^4 - k^3 - \frac{1}{2}k^2 - \frac{4}{3}k^3 - 2k^2 - \frac{2}{3}k$

$= n(\frac{2}{3}k^3+3k^2+\frac{7}{3}k) - \frac{7}{6}k^4 - 6k^3 - \frac{47}{6}k^2 - 3k$

3. $\sum_{i=1}^{k} i(i+1) \cdot (n-((k+2)+i))$

$= \sum_{i=1}^{k}(i^2+i) \cdot ((n-k-2)-i)$

$= (n-k-2)\sum_{i=1}^{k} i^2 + (n-k-2)\sum_{i=1}^{k} i - \sum_{i=1}^{k} i^3 - \sum_{i=1}^{k} i^2$

$= (n-k-2)\frac{k(k+1)(2k+1)}{6} + (n-k-2)\frac{k(k+1)}{2} - \frac{k^2(k+1)^2}{2} - \frac{k(k+1)(2k+1)}{6}$

$$= \tfrac{1}{6}(n-k-2)(2k^3+3k^2+k) + \tfrac{1}{2}(n(k^2+k) - k^3 - k^2 - 2k^2 - 2k)$$
$$- \tfrac{1}{4}(k^4+2k^3+k^2) - \tfrac{1}{6}(2k^3+3k^2+k)$$

$$= \tfrac{1}{6}(n(2k^3+3k^2+k) - 2k^4 - 3k^3 - k^2 - 4k^3 - 6k^2 - 2k) + n(\tfrac{1}{2}k^2 + \tfrac{1}{2}k)$$
$$- \tfrac{1}{2}k^3 - \tfrac{3}{2}k^2 - k - \tfrac{1}{4}k^4 - \tfrac{1}{2}k^3 - \tfrac{1}{4}k^2 - \tfrac{1}{3}k^3 - \tfrac{1}{2}k^2 - \tfrac{1}{6}k$$

$$= n(\tfrac{1}{3}k^3 + \tfrac{1}{2}k^2 + \tfrac{1}{6}k + \tfrac{1}{2}k^2 + \tfrac{1}{2}k) - \tfrac{1}{3}k^4 - \tfrac{7}{6}k^3 - \tfrac{7}{6}k^2 - \tfrac{1}{3}k - \tfrac{1}{2}k^3$$
$$- \tfrac{3}{2}k^2 - k - \tfrac{1}{4}k^4 - \tfrac{1}{2}k^3 - \tfrac{1}{4}k^2 - \tfrac{1}{3}k^3 - \tfrac{1}{2}k^2 - \tfrac{1}{6}k$$

$$= n(\tfrac{1}{3}k^3 + k^2 + \tfrac{2}{3}k) - \tfrac{7}{12}k^4 - \tfrac{5}{2}k^3 - \tfrac{41}{12}k^2 - \tfrac{3}{2}k$$

4. $\sum\limits_{i=1}^{k}(2(k+1)^2 - i^2)\cdot(n-((2k+1)+i))$

$$= \sum_{i=1}^{k}(2(k+1)^2 - i^2)((n-2k-1)-i)$$

$$= \left[\sum_{i=1}^{k} 2(k+1)^2(n-2k-1)\right] - \left[(n-2k-1)\sum_{i=1}^{k} i^2\right] - \left[2(k+1)^2\sum_{i=1}^{k} i\right] + \sum_{i=1}^{k} i^3$$

$$= 2k(k+1)^2(n-2k-1) - \tfrac{(n-2k-1)k(k+1)(2k+1)}{6} - k(k+1)^3 + \tfrac{k^2(k+1)^2}{4}$$

$$= (2k^3+4k^2+2k)(n-2k-1) - \tfrac{1}{6}(n-2k-1)(k^2+k)(2k+1)$$
$$- k(k^3+3k^2+3k+1) + \tfrac{1}{4}k^2(k^2+2k+1)$$

$$= n(2k^3+4k^2+2k) - 4k^4 - 8k^3 - 4k^2 - 2k^3 - 4k^2 - 2k$$
$$+ \tfrac{1}{6}(n-2k-1)(-2k^3-3k^2-k) - k^4 - 3k^3 - 3k^2 - k + \tfrac{1}{4}k^4 + \tfrac{1}{2}k^3 + \tfrac{1}{4}k^2$$

$$= n(2k^3+4k^2+4k) - 4k^4 - 10k^3 - 8k^2 - 2k - k^4 - 3k^3 - 3k^2 - k + \tfrac{1}{4}k^4 + \tfrac{1}{2}k^3 + \tfrac{1}{4}k^2$$
$$+ \tfrac{1}{6}(n(-2k^3-3k^2-k) + 4k^4 + 8k^3 + 5k^2 + k)$$

$$= n(2k^3+4k^2+2k - \tfrac{1}{3}k^3 - \tfrac{1}{2}k^2 - \tfrac{1}{6}k) - 4k^4 - 10k^3 - 8k^2 - 2k + \tfrac{2}{3}k^4$$
$$+ \tfrac{4}{3}k^3 + \tfrac{5}{6}k^2 + \tfrac{1}{6}k - k^4 - 3k^3 - 3k^2 - k + \tfrac{1}{4}k^4 + \tfrac{1}{2}k^3 + \tfrac{1}{4}k^2$$

$$= n(\tfrac{5}{3}k^3 + \tfrac{7}{2}k^2 + \tfrac{11}{6}k) - \tfrac{49}{12}k^4 - \tfrac{67}{6}k^3 - \tfrac{119}{12}k^2 - \tfrac{17}{6}k$$

5. $\sum\limits_{i=1}^{k} i\cdot(n-((2k+2)+i))$

$$= \sum_{i=1}^{k} i((n-2k-2)-i)$$

$$= (n-2k-2)\sum_{i=1}^{k} i - \sum_{i=1}^{k} i^2$$

$= (n-2k-2)\frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6}$

$= \frac{1}{2}(n-2k-2)(k^2+k) + \frac{1}{6}(-k^2-k)(2k+1)$

$= \frac{1}{2}(n(k^2+k) - 2k^3 - 2k^2 - 2k^2 - 2k) + \frac{1}{6}(-2k^3 - k^2 - 2k^2 - k)$

$= n(\frac{1}{2}k^2 + \frac{1}{2}k) - k^3 - 2k^2 - k - \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{1}{6}k$

$= n(\frac{1}{2}k^2 + \frac{1}{2}k) - \frac{4}{3}k^3 - \frac{5}{2}k^2 - \frac{7}{6}k$

6. $\sum_{i=1}^{k}(k-(i-1))(k-(i-2)) \cdot (n-((3k+1)+i))$

$= \sum_{i=1}^{k}((k+1)-i)((k+2)-i)((n-3k-1)-i)$

$= \sum_{i=1}^{k}((k+1)(k+2) - i(k+1) - i(k+2) + i^2)((n-3k-1)-i)$

$= \sum_{i=1}^{k}((k+1)(k+2) - 2ik - 3i + i^2)((n-3k-1)-i)$

$= \sum_{i=1}^{k}((k+1)(k+2) - i(2k+3) + i^2)((n-3k-1)-i)$

$$= \left[\sum_{i=1}^{k}(k+1)(k+2)(n-3k-1)\right] - \left[(2k+3)(n-3k-1)\sum_{i=1}^{k} i\right]$$
$$+ \left[(n-3k-1)\sum_{i=1}^{k} i^2\right] - \left[(k+1)(k+2)\sum_{i=1}^{k} i\right] + \left[(2k+3)\sum_{i=1}^{k} i^2\right] - \sum_{i=1}^{k} i^3$$

$$= k(k+1)(k+2)(n-3k-1) - (2k+3)(n-3k-1)\frac{k(k+1)}{2} + (n-3k-1)\frac{k(k+1)(2k+1)}{6}$$
$$-\frac{k(k+1)^2(k+2)}{2} + \frac{k(k+1)(2k+1)(2k+3)}{6} - \frac{k^2(k+1)^2}{4}$$

$$= (n-3k-1)(k^3+3k^2+2k) + \frac{1}{2}(-2k^3-5k^2-3k)(n-3k-1)$$
$$+(n-3k-1)(\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k) + \frac{1}{2}(-k^4-2k^3-k^2-2k^3-4k^2-2k)$$
$$+\frac{1}{6}(4k^4+8k^3+3k^2+4k^3+8k^2+3k) - \frac{1}{4}(k^4+2k^3+k^2)$$

$$= n(k^3+3k^2+2k) - 3k^4 - 9k^3 - 6k^2 - k^3 - 3k^2 - 2k + n(-k^3 - \frac{5}{2}k^2 - \frac{3}{2}k)$$
$$+\frac{1}{2}(6k^4+15k^3+9k^2+2k^3+5k^2+3k) + n(\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k)$$
$$-k^4 - \frac{3}{2}k^3 - \frac{1}{2}k^2 - \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{1}{6}k - \frac{1}{2}k^4 - 2k^3 - \frac{5}{2}k^2 - k$$
$$+\frac{2}{3}k^4 + 2k^3 + \frac{11}{6}k^2 + \frac{1}{2}k - \frac{1}{4}k^4 - \frac{1}{2}k^3 - \frac{1}{4}k^2$$

$$= n(k^3+3k^2+2k) - 3k^4 - 10k^3 - 9k^2 - 2k + n(-k^3 - \frac{5}{2}k^2 - \frac{3}{2}k) + 3k^4 + \frac{17}{2}k^3$$
$$+7k^2 + \frac{3}{2}k + n(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k) - k^4 - \frac{11}{6}k^3 - k^2 - \frac{1}{6}k - \frac{1}{2}k^4 - 2k^3 - \frac{5}{2}k^2 - k$$
$$+\frac{2}{3}k^4 + 2k^3 + \frac{11}{6}k^2 + \frac{1}{2}k - \frac{1}{4}k^4 - \frac{1}{2}k^3 - \frac{1}{4}k^2$$

$= n(\frac{1}{3}k^3 + k^2 + \frac{2}{3}k) - \frac{13}{12}k^4 - \frac{23}{6}k^3 - \frac{47}{12}k^2 - \frac{7}{6}k$

7. $\sum\limits_{i=1}^{k} 2(k-(i-1)) \cdot (n-((3k+2)+i))$

$= \sum\limits_{i=1}^{k} (2k-2i+2)(n-((3k+2)+i))$

$= \sum\limits_{i=1}^{k} ((2k+2)-2i)((n-3k-2)-i)$

$= \left[\sum\limits_{i=1}^{k} (2k+2)(n-3k-2)\right] - \left[(2k+2)\sum\limits_{i=1}^{k} i\right] - \left[2(n-3k-2)\sum\limits_{i=1}^{k} i\right] + 2\sum\limits_{i=1}^{k} i^2$

$= k(2k+2)(n-3k-2) - k(k+1)^2 - k(k+1)(n-3k-2) + \frac{k(k+1)(2k+1)}{3}$

$= (n-3k-2)(2k^2+2k) - k(k^2+2k+1) + (n-3k-2)(-k^2-k) + \frac{1}{3}(2k^3+3k^2+k)$

$= n(2k^2+2k) - 6k^3 - 10k^2 - 4k - k^3 - 2k^2 - k + n(-k^2-k) + 3k^3 + 5k^2 + 2k + \frac{2}{3}k^3 + k^2 + \frac{1}{3}k$

$= n(k^2+k) - \frac{10}{3}k^3 - 6k^2 - \frac{8}{3}k$

8. $\sum\limits_{i=1}^{k} i \cdot (n-((2k+1)+i))(n-((2k+2)+i))$

$= \sum\limits_{i=1}^{k} i \cdot ((n-2k-1)-i)((n-2k-2)-i)$

$= \sum\limits_{i=1}^{k} i(n-2k-1)(n-2k-2) - \sum\limits_{i=1}^{k} i^2(n-2k-1) - \sum\limits_{i=1}^{k} i^2(n-2k-2) + \sum\limits_{i=1}^{k} i^3$

$= (n-2k-1)(n-2k-2)\sum\limits_{i=1}^{k} i - (2n-4k-3)\sum\limits_{i=1}^{k} i^2 + \sum\limits_{i=1}^{k} i^3$

$= (n-2k-1)(n-2k-2)\frac{k(k+1)}{2} + \frac{k^2(k+1)^2}{4} - (2n-4k-3)\frac{k(k+1)(2k+1)}{6}$

$= \frac{1}{2}(k^2+k)(n^2+n(-2k-2)+n(-2k-1)+(-2k-1)(-2k-2)) + \frac{1}{4}k^2(k^2+2k+1)$
$+\frac{1}{6}(2n-4k-3)(-2k^3-3k^2-k)$

$= \frac{1}{2}(k^2+k)(n^2+n(-4k-3)+4k^2+6k+2) + \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$
$+\frac{1}{3}n(-2k^3-3k^2-k) + \frac{1}{6}(-4k-3)(-2k^3-3k^2-k)$

$= n^2(\frac{1}{2}k^2 + \frac{1}{2}k) + n(\frac{1}{2}(-4k^3-7k^2-3k)) + 2k^4 + 3k^3 + k^2 + 2k^3 + 3k^2 + k$
$+\frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2 + n(-\frac{2}{3}k^3 - k^2 - \frac{1}{3}k) + \frac{1}{6}(8k^4 + 18k^3 + 13k^2 + 3k)$

$= n^2(\frac{1}{2}k^2 + \frac{1}{2}k) + n(-2k^3 - \frac{7}{2}k^2 - \frac{3}{2}k - \frac{2}{3}k^3 - k^2 - \frac{1}{3}k)$
$+\frac{4}{3}k^4 + 3k^3 + \frac{13}{6}k^2 + \frac{1}{2}k + \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2 + 2k^4 + 5k^3 + 4k^2 + k$

$= n^2(\frac{1}{2}k^2 + \frac{1}{2}k) + n(-\frac{8}{3}k^3 - \frac{9}{2}k^2 - \frac{11}{6}k) + \frac{43}{12}k^4 + \frac{17}{2}k^3 + \frac{77}{12}k^2 + \frac{3}{2}k$

9. $\sum_{i=1}^{k}(k-(i-1)) \cdot (n-((3k+2)+i))(n-((3k+3)+i))$

$= \sum_{i=1}^{k}((k+1)-i) \cdot (n-((3k+2)+i))(n-((3k+3)+i))$

$= \sum_{i=1}^{k}((k+1)-i)((n-3k-2)(n-3k-3)+i^2-i(n-3k-2)-i(n-3k-3))$

$= \sum_{i=1}^{k}((k+1)-i)((n-3k-2)(n-3k-3)+i^2-i(2n-6k-5))$

$= \left[\sum_{i=1}^{k}(k+1)(n-3k-2)(n-3k-3)\right] + \left[(k+1)\sum_{i=1}^{k}i^2\right]$
$- \left[(k-1)(2n-6k-5)\sum_{i=1}^{k}i\right] - \left[(n-3k-2)(n-3k-3)\sum_{i=1}^{k}i\right]$
$- \sum_{i=1}^{k}i^3 + \left[(2n-6k-5)\sum_{i=1}^{k}i^2\right]$

$= k(k+1)(n-3k-2)(n-3k-3) + \frac{k(k+1)^2(2k+1)}{6} - (2n-6k-5)\frac{k(k+1)^2}{2}$
$-(n-3k-2)(n-3k-3)\frac{k(k+1)}{2} - \frac{k^2(k+1)^2}{4} + (2n-6k-5)\frac{k(k+1)(2k+1)}{6}$

$= (k^2+k)(n^2+n(-3k-2)+n(-3k-3)+(-3k-2)(-3k-3)) + \frac{1}{6}(2k^4+5k^3+4k^2+k)$
$+\frac{1}{2}(-k^3-2k^2-k)(2n-6k-5) + \frac{1}{2}(-k^2-k)(n^2+n(-3k-2)+n(-3k-3)$
$+(-3k-2)(-3k-3)) - \frac{1}{4}k^4 - \frac{1}{2}k^3 - \frac{1}{4}k^2 + \frac{1}{6}(2n-6k-5)(2k^3+3k^2+k)$

$= (k^2+k)(n^2+n(-6k-5)+9k^2+15k+6) + \frac{1}{3}k^4 + \frac{5}{6}k^3 + \frac{2}{3}k^2 + \frac{1}{6}k + n(-k^3-2k^2-k)$
$+\frac{1}{2}(6k^4+17k^3+16k^2+5k) + \frac{1}{2}(-k^2-k)(n^2+n(-6k-5)+9k^2+15k+6) - \frac{1}{4}k^4 - \frac{1}{2}k^3$
$-\frac{1}{4}k^2 + \frac{1}{3}n(2k^3+3k^2+k) + \frac{1}{6}(-6k-5)(2k^3+3k^2+k)$

$= n^2(k^2+k) + n(-6k^3-11k^2-5k) + 9k^4 + 24k^3 + 21k^2 + 6k + \frac{1}{3}k^4 + \frac{5}{6}k^3 + \frac{2}{3}k^2$
$+\frac{1}{6}k + n(-k^3-2k^2-k) + 3k^4 + \frac{17}{2}k^3 + 8k^2 + \frac{5}{2}k + n^2(-\frac{1}{2}k^2 - \frac{1}{2}k)$
$+n(3k^3 + \frac{11}{2}k^2 + \frac{5}{2}k) - \frac{9}{2}k^4 - 12k^3 - \frac{21}{2}k^2 - 3k - \frac{1}{4}k^4 - \frac{1}{2}k^3 - \frac{1}{4}k^2$
$+n(\frac{2}{3}k^3 + k^2 + \frac{1}{3}k) - 2k^4 - \frac{14}{3}k^3 - \frac{7}{2}k^2 - \frac{5}{6}k$

$= n^2(\frac{1}{2}k^2 + \frac{1}{2}k) + n(-\frac{10}{3}k^3 - \frac{13}{2}k^2 - \frac{19}{6}k) + \frac{67}{12}k^4 + \frac{97}{6}k^3 + \frac{185}{12}k^2 + \frac{29}{6}k$

Adding these 9 parts together, we will get:

$$n^2(k^2+k) + n(-3k^3 - k^2 + 3k) + \frac{9}{4}k^4 - \frac{7}{2}k^3 - \frac{53}{4}k^2 - \frac{13}{2}k$$

The part of the formula that was outside of the summation also needs to be simplified. Simplifying these terms:

$$(n-(2k+1))+(2-k(k+1))\cdot(n-(2k+2))+(k+1)\cdot(n-(3k+2))$$
$$+(k+1)\cdot(n-(3k+2))(n-(3k+3))$$

$$= n-2k-1+2n-4k-4-nk^2-nk+2k^3+4k^2+2k+nk+n-3k^2-5k-2$$
$$+(k+1)(n^2-n(6k+5)+9k^2+15k+6)$$

$$= n-2k-1+2n-4k-4-nk^2-nk+2k^3+4k^2+2k+nk+n-3k^2-5k-2+n^2k-n(6k^2+5k)$$
$$+9k^3+15k^2+6k+n^2-n(6k+5)+9k^2+15k+6$$

$$= n^2(k+1)+n(-7k^2-11k-1)+11k^3+25k^2+12k-1$$

Adding this formula to the simplified summation formula, we get the total number of elements of the multiplier in terms of n and k:

$$n^2(k^2+2k+1)+n(-3k^3-8k^2-8k-1)+\frac{9}{4}k^4+\frac{15}{2}k^3+\frac{47}{4}k^2+\frac{11}{2}k-1$$

This formula works for $n \geq 4k+3$.

4.2 Specific Cases

4.2.1 k=1

This case corresponds to the derived algebra of the $n \times n$ strictly upper triangular matrices. Substituting in $k=1$ into the general formula for the dimension of the multiplier in the Leibniz case, this gives:

$$4n^2-20n+26$$

This is the case for $n \geq 7$. However, it was found in [5] in the Lie case that the dimension of the multiplier for $k=1$ is:

$$2n^2-11n+16$$

Therefore, taking some values for n, we are able to compare the dimension of the multipliers in the Lie and Leibniz case.

Table 4.1 Counting multiplier elements for k=1

n	Leibniz Multiplier Dimension	Lie Multiplier Dimension
7	82	37
8	122	56
9	170	79
10	226	106
11	290	137

4.2.2 k=2

Substituting in $k = 2$ into the general formula for the dimension of the multiplier in the Leibniz case, this gives:

$$9n^2 - 73n + 160$$

This is the case for $n \geq 11$. However, it was found in the Lie case in [5] that the dimension of the multiplier for $k = 2$ is:

$$\frac{9}{2}n^2 - \frac{77}{2}n + 93$$

Therefore, taking some values for n, we are able to compare the dimension of the multipliers in the Lie and Leibniz case.

Table 4.2 Counting multiplier elements for k=2

n	Leibniz Multiplier Dimension	Lie Multiplier Dimension
11	446	214
12	580	279
13	732	353
14	902	436
15	1090	528

This again reveals that the Leibniz multiplier and hence the cover always has a greater dimension than the multiplier and cover found in the Lie case.

Chapter 5

Upper Triangular Matrices - The Solvable Case

The dimension of the Leibniz multiplier has been found for all algebras within the lower central series of the strictly upper triangular matrices. The next algebra that needs to be considered is the Lie algebra of upper triangular matrices. The elements in the basis of this algebra are the unit matrices $E_{s,t}$ for $s \leq t$. Let (K, M) be a defining pair for L and let $F(s,t)$ be the image of $E_{s,t}$ under the section $\mu : L \to K$. The Leibniz bracket of elements of K can then be defined as:

$$[F(s,t), F(a,b)] = \begin{cases} F(s,b) + y(s,t,a,b) & \textit{when } t = a \\ -F(a,t) + y(s,t,a,b) & \textit{when } s = b \\ y(s,t,a,b) & \textit{otherwise} \end{cases}$$

Using this relationship between elements of K, a basis for the multiplier and therefore it's dimension, can be found. The following theorem can be used to eliminate many elements of the multiplier.

Theorem 5.0.1. *If $b \geq a+3$ or $t \geq s+3$, then $y(s,t,a,b) = 0$. So $y(s,t,a,b) \neq 0$ only has the potential to occur if both $b \leq a+2$ and $t \leq s+2$.*

Proof. To show that this is the case, take an element of the multiplier, $y(s,t,a,b)$. It is known that $t \geq s$ and $b \geq a$. It can be seen that for some c such that $0 \leq c \leq t-s$, the following Leibniz identity holds:

$$\begin{aligned}[F(s,s+c), [F(s+c,t), F(a,b)]] &= [[F(s,s+c), F(s+c,t)], F(a,b)] \\ &\quad + [F(s+c,t), [F(s,s+c), F(a,b)]]\end{aligned}$$

Now assume that $t = s+3$. As long as $a \neq s+1$ and $b \neq s+1$, we can take $c = 1$ for the above Leibniz identity to hold, otherwise, take $c = 2$. This will always make the element $y(s,t,a,b) = 0$ no matter what gap is between a and b. The same theory applies when taking $b = a+3$ and letting the gap between s and t be whatever is desired. This can be extended to consider the cases where $t > s+3$ and $b > a+3$ and the same theory applies.

□

Now consider the elements $y(s,t,a,b)$ such that $t = a$. By using the following change of basis:

$$G(s,t) = F(s,t) + y(s,t,t,t)$$

the elements $G(s,t)$ multiply in the same way as the elements $F(s,t)$ as the multiplier is within the center of the cover. Using this change of basis and the Leibniz identity, we will find the elements that remain in the basis.

Theorem 5.0.2. *Every element in the multiplier of the form $y(s, s+c, s+c, b)$ where $c \geq 0$ and $s+c \leq b$ can be eliminated from the basis of the multiplier. Therefore there are no elements of the form $y(s,t,a,b)$ where $t = a$ in the basis of the multiplier.*

Proof. From the above change of basis, the elements $y(s,t,t,t)$ can be eliminated from the basis of the multiplier. Using the Leibniz identity on the following elements of the cover such that $c \geq 0$, $s+c \leq t$, but $t \neq s$, a relationship between elements of the cover is found:

$$\begin{aligned}[F(s,s+c),[F(s+c,t),F(t,t)]] &= [[F(s,s+c),F(s+c,t)],F(t,t)] \\ &\quad + [F(s+c,t),[F(s,s+c),F(t,t)]]\end{aligned}$$

$$\begin{aligned}\implies [F(s,s+c),F(s+c,t)+y(s+c,t,t,t)] &= [F(s,t)+y(s,s+c,s+c,t),F(t,t)] \\ &\quad + [F(s+c,t),y(s,s+c,t,t)]\end{aligned}$$

$$\implies [F(s,s+c),F(s+c,t)] = [F(s,t),F(t,t)]$$

$$\implies F(s,t)+y(s,s+c,s+c,t) = F(s,t)+y(s,t,t,t)$$

$$\implies y(s,s+c,s+c,t) = y(s,t,t,t)$$

Therefore all elements of the form $y(s,s+c,s+c,t)$ are equal to the elements $y(s,t,t,t)$ and so they can also be eliminated from the basis of the multiplier. The elements such that $s = t$ correspond to the elements $y(s,s,s,s)$ and can be eliminated using the change of basis. Accordingly, all elements of the multiplier $y(s,t,a,b)$ such that $t = a$ can be eliminated from the basis of the multiplier.

□

Next, consider the elements of the multiplier of the form $y(s,t,a,b)$ such that $s = b$. These elements can be rewritten in the form $y(s+c, s+d, s, s+c)$.

Theorem 5.0.3. *Every element in the multiplier of the form $y(s+c,s+d,s,s+c)$ such that $0 \leq c \leq d$ can be eliminated from the basis of the multiplier.*

Proof. For the elements $y(s+c,s+d,s,s+c)$ such that $0 < c \leq d$, the following Leibniz identity of elements from the cover will reveal a relationship between these elements and other elements from the multiplier:

$$\begin{aligned}[F(s+c,s+d),[F(s,s),F(s,s+c)]] &= [y(s+c,s+d,s,s),F(s,s+c)] \\ &\quad + [F(s,s),[F(s+c,s+d),F(s,s+c)]]\end{aligned}$$

$$\begin{aligned}\implies [F(s+c,s+d),F(s,s+c)+y(s,s,s,s+c)] &= [y(s+c,s+d,s,s),F(s,s+c)] \\ &\quad + [F(s,s),-F(s,s+d)+y(s+c,s+d,s,s+c)]\end{aligned}$$

$$\implies [F(s+c,s+d),F(s,s+c)] = -[F(s,s),F(s,s+d)]$$

$$\implies -F(s,s+d) + y(s+c,s+d,s,s+c) = -F(s,s+d) - y(s,s,s,s+d)$$
$$\implies y(s+c,s+d,s,s+c) = -y(s,s,s,s+d)$$

It was seen in the previous theorem that the elements $y(s,s,s,s+d)$ can be eliminated from the basis and therefore the elements $y(s+c,s+d,s,s+c)$ where $0 < c \leq d$ can also be eliminated.

Now consider the case where $c = 0$ and $d > 0$. This corresponds to the elements $y(s,s+d,s,s)$ where $d > 0$. These elements can be eliminated using the following Leibniz identity of elements from the cover:

$$\begin{aligned}[F(s,s),[F(s,s+d),F(s,s)]] &= [[F(s,s),F(s,s+d)],F(s,s)] \\ &\quad + [F(s,s+d),[F(s,s),F(s,s)]]\end{aligned}$$

$$\begin{aligned}\implies [F(s,s), -F(s,s+d) + y(s,s+d,s,s)] &= [F(s,s+d) + y(s,s,s,s+d), F(s,s)] \\ &\quad + [F(s,s+d), y(s,s,s,s)]\end{aligned}$$

$$\implies -[F(s,s),F(s,s+d)] = [F(s,s+d),F(s,s)]$$
$$\implies -F(s,s+d) - y(s,s,s,s+d) = -F(s,s+d) + y(s,s+d,s,s)$$
$$\implies -y(s,s,s,s+d) = y(s,s+d,s,s)$$

It was previously found in the case where $t = a$ that the elements $y(s,s,s,s+d)$ can be eliminated from the basis of the multiplier and therefore, the elements $y(s,s+d,s,s)$ can also be eliminated. In the case where $c = d = 0$, this corresponds to elements of the multiplier of the form $y(s,s,s,s)$ which have already been eliminated using the change of basis. Therefore there are no elements of the form $s = b$ in the basis of the multiplier.

□

As the non-zero elements of the multiplier must satisfy $t < s+3$ and $b < a+3$, this gives 9 remaining cases to consider for the elements $y(s,t,a,b)$:

1. t=s
   (a) b=a
   (b) b=a+1
   (c) b=a+2
2. t=s+1
   (a) b=a
   (b) b=a+1
   (c) b=a+2
3. t=s+2
   (a) b=a
   (b) b=a+1

(c) b=a+2

First, consider the elements such that $t = s + 2$. These will be elements of the form $y(s, s+2, a, a+c)$ where $c = 0, 1$, or 2. The elements from the cases 3.(a), 3.(b), and 3.(c) can be eliminated using the following Leibniz identity of elements from the cover:

$$[F(s, s+d), [F(s+d, s+2), F(a, a+c)]] = [[F(s, s+d), F(s+d, s+2)], F(a, a+c)] \\ + [F(s+d, s+2), [F(s, s+d), F(a, a+c)]]$$

where $0 \leq d \leq 2$ and $c \geq 0$.

It has already been found that the elements $y(s, t, a, b)$ such that $t = a$ or $s = b$ can be eliminated from the basis of the multiplier. Therefore, we need not consider the cases such that $a = s + 2$ or $a + c = s$ as these elements have already be eliminated. For the other cases, take $d = 1$. This will allow all elements of the form $y(s, s+2, a, a+c)$ to equal 0 except when $a = s + 1$ or when $a + c = s + 1$. In the above Leibniz identity, when $a = s + 1$, take $d = 0$ and when $a + c = s + 1$, take $d = 2$. These elements will then also equal 0 and can be eliminated from the basis of the multiplier. Therefore, there are no elements from 3.(a), 3.(b), or 3.(c) in the basis of the multiplier.

We can examine cases 1.(c) and 2.(c) in a similar way. These cases give rise to elements of the form $y(s, s+c, a, a+2)$ such that $c = 0, 1$. Using the following Leibniz identity of elements from the cover, these elements can be eliminated from the basis of the multiplier:

$$[F(s, s+c), [F(a, a+d), F(a+d, a+2)]] = [[F(s, s+c), F(a, a+d)], F(a+d, a+2)] \\ + [F(a, a+d), [F(s, s+c), F(a+d, a+2)]]$$

where $0 \leq d \leq 2$ and $c \geq 0$.

Again, it was previously found that the elements $y(s, t, a, b)$ such that $t = a$ or $s = b$ can be eliminated from the basis of the multiplier. Therefore, we need not consider the cases such that $s = a + 2$ or $s + c = a$ as these elements have already been eliminated. For all the other cases, take $d = 1$. This will allow elements of the form $y(s, s+c, a, a+2)$ to equal 0 except when $s = a + 1$ or $s + c = a + 1$. When $s = a + 1$, take $d = 0$ and when $s + c = a + 1$ take $d = 2$ in the above Leibniz identity. These elements will then equal 0 and can be eliminated from the basis of the multiplier. Therefore, there are no elements from 1.(c) or 2.(c) in the basis of the multiplier. This leaves 4 cases to consider.

Case 1.(a) t =s and b=a

This corresponds to elements of the multiplier of the form $y(s, s, a, a)$. A Leibniz identity of elements from the cover cannot be established as brackets of the form $[F(s, s), F(s, s)]$ or $[F(a, a), F(a, a)]$ will simply equal an element of the multiplier. Therefore, as no Leibniz identity can be created in order to generate this element of the multiplier, elements of the multiplier cannot be eliminated in this way. However, the elements $y(s, s, s, s)$ can be eliminated using the change of basis $G(s, s) = F(s, s) + y(s, s, s, s)$. As such, this case has $n^2 - n$ elements in the basis of the multiplier.

Case 1.(b) t=s and b=a+1

This case corresponds to elements of the form $y(s, s, a, a+1)$. If $a \leq s - 2$, then using the following

Leibniz identity, the elements of this form can be eliminated:

$$\begin{aligned}[F(s,s),[F(a,a),F(a,a+1)]] &= [[F(s,s),F(a,a)],F(a,a+1)] \\ &\quad + [F(a,a),[F(s,s),F(a,a+1)]]\end{aligned}$$

$$\implies [F(s,s),F(a,a+1)+y(a,a,a,a+1)] = 0$$

$$\implies y(s,s,a,a+1) = 0$$

where $a \leq s-2$.

Now looking at the case where $a = s-1$, this corresponds to elements of the form $y(s,s,s-1,s)$. These elements can be eliminated as $s = b$. If $a = s$, then this corresponds to the elements of the form $y(s,s,s,s+1)$ which can also be eliminated from the basis of the multiplier as these elements are of the form $t = a$.

Finally for this case, we consider the elements of the form $y(s,s,a,a+1)$ where $a \geq s+1$. Again, using the Leibniz identity:

$$\begin{aligned}[F(s,s),[F(a,a),F(a,a+1)]] &= [[F(s,s),F(a,a)],F(a,a+1)] \\ &\quad + [F(a,a),[F(s,s),F(a,a+1)]]\end{aligned}$$

$$\implies [F(s,s),F(a,a+1)+y(a,a,a,a+1)] = 0$$

$$\implies y(s,s,a,a+1) = 0$$

We can now see that when $a \geq s+1$, these elements can also be eliminated from the basis of the multiplier. Therefore, there are no elements of the form $y(s,s,a,a+1)$ in the basis of the multiplier.

Case 2.(a) t=s and b=a+1

This case corresponds to elements of the form $y(s,s+1,a,a)$. If $a \leq s-1$ then elements of this form can be eliminated using the following Leibniz identity of elements from the cover:

$$\begin{aligned}[F(s,s),[F(s,s+1),F(a,a)]] &= [[F(s,s),F(s,s+1)],F(a,a)] \\ &\quad + [F(s,s+1),[F(s,s),F(a,a)]]\end{aligned}$$

$$\implies 0 = [F(s,s+1)+y(s,s,s,s+1),F(a,a)]$$

$$\implies y(s,s+1,a,a) = 0$$

If $a = s$ this corresponds to the elements $y(s,s+1,s,s)$. These elements can be eliminated from the basis of the multiplier as these elements are of the form $s = b$. If $a = s+1$ this corresponds to the case where $t = a$ and therefore we know that this can be eliminated by the change of basis. If $a \geq s+2$ then using the following Leibniz bracket, all elements of the form $y(s,s+1,a,a)$ that satisfy this inequality can be eliminated from the basis of the multiplier:

$$\begin{aligned}[F(s,s),[F(s,s+1),F(a,a)]] &= [[F(s,s),F(s,s+1)],F(a,a)] \\ &\quad + [F(s,s+1),[F(s,s),F(a,a)]]\end{aligned}$$

$$\implies 0 = [F(s,s+1)+y(s,s,s,s+1),F(a,a)]$$

$$\implies y(s,s+1,a,a) = 0$$

Therefore there are no elements of the form $y(s, s+1, a, a)$ in the basis of the multiplier.

Case 2.(b) t=s+1 and b=a+1

Next we must consider the case where $t = s+1$ and $b = a+1$ which corresponds to elements of the multiplier of the form $y(s, s+1, a, a+1)$. If $a \leq s-2$ then using the following Leibniz identity, we can eliminate these elements from the basis of the multiplier:

$$\begin{aligned}[F(s,s),[F(s,s+1),F(a,a+1)]] &= [[F(s,s),F(s,s+1)],F(a,a+1)] \\ &\quad + [F(s,s+1),[F(s,s),F(a,a+1)]]\end{aligned}$$

$$\implies 0 = [F(s,s+1) + y(s,s,s,s+1), F(a,a+1)]$$
$$\implies y(s,s+1,a,a+1) = 0$$

If $a = s-1$, this corresponds to elements of the form $y(s, s+1, s-1, s)$. These elements can be eliminated as this is a case such that $s = b$.If $a = s$, this corresponds to elements of the form $y(s, s+1, s, s+1)$. Using the following Leibniz identity, these elements can be eliminated:

$$\begin{aligned}[F(s,s+1),[F(s,s),F(s,s+1)]] &= [[F(s,s+1),F(s,s)],F(s,s+1)] \\ &\quad + [F(s,s),[F(s,s+1),F(s,s+1)]]\end{aligned}$$

$$\implies [F(s,s+1), F(s,s+1) + y(s,s,s,s+1)] = [-F(s,s+1) + y(s,s+1,s,s), F(s,s+1)]$$
$$\implies y(s,s+1,s,s+1) = -y(s,s+1,s,s+1)$$
$$\implies y(s,s+1,s,s+1) = 0$$

If $a = s+1$, then this corresponds to elements of the form $y(s, s+1, s+1, s+2)$. In this case, $t = a$ and therefore these can be eliminated from the basis of the multiplier. If $a \geq s+2$, then the elements of the form $y(s, s+1, a, a+1)$ that satisfy this equality can also be eliminated from the basis of the multiplier by the following Leibniz identity:

$$\begin{aligned}[F(s,s),[F(s,s+1),F(a,a+1)]] &= [[[F(s,s),F(s,s+1)],F(a,a+1)] \\ &\quad + [F(s,s+1),[F(s,s),F(a,a+1)]]\end{aligned}$$

$$\implies 0 = [F(s,s+1) + y(s,s,s,s+1), F(a,a+1)]$$
$$\implies y(s,s+1,a,a+1) = 0$$

Therefore there are no elements of the form $y(s, s+1, a, a+1)$ in the basis of the multiplier.

Accordingly, the only elements of the multiplier are those of the form $y(s, s, a, a)$ such that $s \neq a$ which gives $n^2 - n = n(n-1)$ elements of the multiplier.

Below is a table showing the difference in dimension of the Lie multiplier and Leibniz multiplier of the algebra of the upper triangular matrices. The Lie case can be found in [8].

Table 5.1 Counting multiplier elements for the upper triangular matrices

n	Leibniz Multiplier Dimension	Lie Multiplier Dimension
2	2	1
3	6	3
4	12	6
5	20	10
6	30	15

As such, just as we found for the dimension of the multiplier of each algebra in the lower central series, the Leibniz multiplier has greater dimension than the Lie multiplier for each n.

Chapter 6

The Heisenberg Algebra

When looking at the Leibniz cover and multiplier of the Heisenberg algebra, there are 3 possible cases to consider:

1. Where the brackets for the cover satisfy the Lie brackets in the Heisenberg algebra.
2. Where right multiplication by an element is 0, giving an algebra that is Leibniz and not Lie.
3. Where there is a mix of some brackets equal to the negative of the right multiplication and some brackets where the right multiplication by an element is equal to 0.

In each case, the general form for the dimension of the multiplier will be found, as well as looking at specific examples.

6.1 The Heisenberg Lie Algebra

Taking the Heisenberg algebra L of dimension $2n+1 > 3$ to have the same bracket structure as the Heisenberg Lie algebra, then for an algebra with basis, v_1, v_2, ..., v_{2n}, v, the bracket structure is as follows:

$$[v_{2i-1}, v_{2i}] = v$$

$$[v_{2i}, v_{2i-1}] = -v$$

All other brackets are 0. Now letting (K, M) be the defining pair for the Lie algebra L, the task is to find the structure of the Leibniz multiplier M. Take μ to be a section that acts as $\mu(v_i) = x_i$ and $\mu(v) = x$. This is the same section that was used in [9] but now a basis for the Leibniz multiplier of this algebra will be found. Letting y_i, z_i, a_{ij}, b_{ij}, c_{ij}, d_{ij}, e, f_i, g_i, h_i, and k_i be elements of the multiplier, the brackets of the elements in the cover will be as follows:

$$[x_{2i-1}, x_{2i}] = x + y_i, \quad 1 \leq i \leq n$$

$$[x_{2i}, x_{2i-1}] = -x + z_i, \quad 1 \leq i \leq n$$

$$[x_{2i-1}, x_{2j}] = a_{ij}, \quad 1 \leq i \neq j \leq n$$

$$[x_{2i}, x_{2j-1}] = b_{ij}, \qquad 1 \leq i \neq j \leq n$$

$$[x_{2i-1}, x_{2j-1}] = c_{ij}, \qquad 1 \leq i, j \leq n$$

$$[x_{2i}, x_{2j}] = d_{ij}, \qquad 1 \leq i, j \leq n$$

$$[x, x] = e$$

$$[x_{2i-1}, x] = f_i, \qquad 1 \leq i \leq n$$

$$[x, x_{2i-1}] = g_i, \qquad 1 \leq i \leq n$$

$$[x_{2i}, x] = h_i, \qquad 1 \leq i \leq n$$

$$[x, x_{2i}] = k_i, \qquad 1 \leq i \leq n$$

By using the Leibniz identity on these basis elements $x_1, x_2, ..., x_{2n}, x$, it is possible to eliminate some of the elements of the multiplier in order to find the number of basis elements that are independent and non-zero, and hence we can find the dimension of this basis. From the Leibniz brackets below, the following elements can be eliminated:

$$[x, [x_{2i-1}, x_{2i}]] = [[x, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x, x_{2i}]]$$

$$\implies [x, x + y_i] = [g_i, x_{2i}] + [x_{2i-1}, k_i]$$

$$\implies [x, x] = 0$$

$$\implies e = 0$$

$$[x_{2i}, [x_{2i-1}, x_{2i}]] = [[x_{2i}, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x_{2i}, x_{2i}]]$$

$$\implies [x_{2i}, x + y_i] = [-x + z_i, x_{2i}] + [x_{2i-1}, d_{ii}]$$

$$\implies [x_{2i}, x] = -[x, x_{2i}]$$

$$\implies h_i = -k_i$$

$$[x_{2i-1}, [x_{2i}, x_{2i-1}]] = [[x_{2i-1}, x_{2i}], x_{2i-1}] + [x_{2i}, [x_{2i-1}, x_{2i-1}]]$$

$$\implies [x_{2i-1}, -x + z_i] = [x + y_i, x_{2i-1}] + [x_{2i}, c_{ii}]$$

$$\implies -[x_{2i-1}, x] = [x, x_{2i-1}]$$

$$\implies -f_i = g_i$$

For the following, assume that $j \neq i$:

$$[x_{2j}, [x_{2i-1}, x_{2i}]] = [[x_{2j}, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x_{2j}, x_{2i}]]$$

$$\implies [x_{2j}, x + y_i] = [b_{ji}, x_{2i}] + [x_{2i-1}, d_j i]$$

$$\implies [x_{2j}, x] = 0$$

$$\implies h_j = 0 \implies k_j = 0$$

$$[x_{2j-1}, [x_{2i-1}, x_{2i}]] = [[x_{2j-1}, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x_{2j-1}, x_{2i}]]$$

$$\implies [x_{2j-1}, x + y_i] = [c_{ji}, x_{2i}] + [x_{2i-1}, a_j i]$$

$$\implies [x_{2j-1}, x] = 0$$

$$\implies f_j = 0 \implies g_j = 0$$

Therefore, the elements e, f_i, g_i, h_i, and k_i can be eliminated from the basis of the multiplier for $1 \leq i \leq n$. A change of basis can be performed among the y_i terms in order to eliminate y_1. By letting $[x_1, x_2] = s = x + y_1$, this allows the element y_1 to be eliminated and then $\hat{y}_i = y_i - y_1$ for $1 < i \leq n$. This does not change the multiplication among the brackets and still allows each $\hat{y}_i$ to be central and an element of the multiplier.

As such, we can count the number of elements within the basis of the multiplier to find its dimension:

Table 6.1 Elements in the basis of the multiplier of the Heisenberg Lie algebra

Type of Multiplier Element	Number of Elements
y_i	$n-1$
z_i	n
a_{ij}	$n(n-1)$
b_{ij}	$n(n-1)$
c_{ij}	n^2
d_{ij}	n^2

In total, for a $2n+1 > 3$ dimensional Heisenberg Lie algebra, there are $4n^2 - 1$ elements in the basis of the multiplier. Therefore, the dimension of the cover is $(4n^2 - 1) + (2n+1) = 4n^2 + 2n = 2n(2n+1)$.

6.1.1 The Heisenberg Lie Algebra of Dimension 3

The Heisenberg algebra of dimension 3 has the basis v_1, v_2, v. Taking the section μ, this gives $\mu(v_1) = x_1$, $\mu(v_2) = x_2$, and $\mu(v) = x$. The following brackets of elements in the cover give the 9 elements, w_i, of the multiplier. Then by looking at the Leibniz identity, we will find which elements of the multiplier can be eliminated.

$$[x_1, x_2] = x + w_1$$

$$[x_2, x_1] = -x + w_2$$

$$[x_1, x] = w_3$$

$$[x, x_1] = w_4$$

$$[x_2, x] = w_5$$

$$[x, x_2] = w_6$$

$$[x_1, x_1] = w_7$$

$$[x_2, x_2] = w_8$$

$$[x, x] = w_9$$

Using the Leibniz identity, the following relationships are found:

$$[x, [x_1, x_2]] = [[x, x_1], x_2] + [x_1, [x, x_2]]$$

$$\implies [x, x + w_1] = [w_4, x_2] + [x_1, w_6]$$

$$\implies [x, x] = 0$$

$$\implies w_9 = 0$$

$$[x_1, [x_2, x_1]] = [[x_1, x_2], x_1] + [x_2, [x_1, x_1]]$$

$$\implies [x_1, -x + w_2] = [x + w_1, x_1] + [x_2, w_7]$$

$$\implies -[x_1, x] = [x, x_1]$$

$$\implies -w_3 = w_4$$

$$[x_2, [x_1, x_2]] = [[x_2, x_1], x_2] + [x_1, [x_2, x_2]]$$

$$\implies [x_2, x + w_1] = [-x + w_2, x_2] + [x_1, w_8]$$

$$\implies [x_2, x] = -[x, x_2]$$

$$\implies w_5 = -w_6$$

A change of basis can be performed by letting $[x_1, x_2] = y = x + w_1$. This then eliminates the element w_1 and $\hat{w}_i = w_i - w_1$. This will not change the multiplication of the brackets but will allow another element to be eliminated from the basis of the multiplier.

Therefore, by taking into account the dependence between elements of the multiplier and by eliminating the elements w_1 and w_9, we are left with 5 elements in the basis of the multiplier, giving a cover that has 8 elements.

It can be seen that this does not agree with the general formula (and this was to be expected) which is stated above. This is because for the 3 dimensional Heisenberg, we must take $2n + 1 = 3 \implies n = 1$. Therefore, an i and j that are different from one another cannot be established in order to get the multiplier elements a_{ij} and b_{ij}. Although this agrees with the fact that there are $n(n-1) = 0$ of each type, these elements of the multiplier need to exist in order to take Leibniz identities of elements of the cover so that other elements may be eliminated. This can be seen in the calculations that allow $h_j = 0$ and $f_j = 0$ in the general form calculation above. Accordingly, as these elements cannot be eliminated, the 3 dimensional Heisenberg case does not satisfy the general formula.

However, it must be noted that the 3 dimensional Heisenberg Lie algebra is isomorphic to the Lie algebra of 3x3 strictly upper triangular matrices. Therefore, their Leibniz multiplier should have equal dimension, as well as their covers. This is the case with both algebras having Leibniz multiplier of dimension 5 and cover of dimension 8.

6.1.2 The Heisenberg Lie Algebra of Dimension 5

The Heisenberg algebra of dimension 5 has the basis v_1, v_2, v_3, v_4, v. Taking the section μ, this gives $\mu(v_i) = x_i$ for $1 \leq i \leq 4$ and $\mu(v) = x$. Taking the brackets of the elements in the cover, we find that we can establish relationships with 25 possible elements, w_i in the basis of the multiplier.

$$[x_1, x_2] = x + w_1$$
$$[x_3, x_4] = x + w_2$$
$$[x_2, x_1] = -x + w_3$$
$$[x_4, x_3] = -x + w_4$$
$$[x_1, x_1] = w_5$$
$$[x_1, x_3] = w_6$$
$$[x_1, x_4] = w_7$$
$$[x_1, x] = w_8$$
$$[x_2, x_2] = w_9$$
$$[x_2, x_3] = w_{10}$$
$$[x_2, x_4] = w_{11}$$
$$[x_2, x] = w_{12}$$
$$[x_3, x_1] = w_{13}$$
$$[x_3, x_2] = w_{14}$$
$$[x_3, x_3] = w_{15}$$
$$[x_3, x] = w_{16}$$
$$[x_4, x_1] = w_{17}$$
$$[x_4, x_2] = w_{18}$$
$$[x_4, x_4] = w_{19}$$
$$[x_4, x] = w_{20}$$
$$[x, x_1] = w_{21}$$
$$[x, x_2] = w_{22}$$
$$[x, x_3] = w_{23}$$
$$[x, x_4] = w_{24}$$
$$[x, x] = w_{25}$$

By taking the following Leibniz brackets, the following elements of the multiplier can be eliminated:

$$[x, [x_1, x_2]] = [[x, x_1], x_2] + [x_1, [x, x_2]]$$
$$\implies [x, x + y_1] = [w_{21}, x_2] + [x_1, w_{22}]$$
$$\implies [x, x] = 0$$
$$\implies w_{25} = 0$$

$$[x_1, [x_2, x_1]] = [[x_1, x_2], x_1] + [x_2, [x_1, x_1]]$$
$$\implies [x_1, -x + w_3] = [x + w_1, x_1] + [x_2, w_5]$$
$$\implies -[x_1, x] = [x, x_1]$$
$$\implies -w_8 = w_{21}$$

$$[x_2,[x_1,x_2]] = [[x_2,x_1],x_2] + [x_1,[x_2,x_2]]$$

$$\implies [x_2, x + w_1] = [-x + w_3, x_2] + [x_1, w_9]$$

$$\implies [x_2, x] = -[x, x_2]$$

$$\implies w_{12} = -w_{22}$$

$$[x_3,[x_4,x_3]] = [[x_3,x_4],x_3] + [x_4,[x_3,x_3]]$$

$$\implies [x_3, -x + w_4] = [x + w_2, x_3] + [x_4, w_{15}]$$

$$\implies -[x_3, x] = [x, x_3]$$

$$\implies -w_{16} = w_{23}$$

$$[x_4,[x_3,x_4]] = [[x_4,x_3],x_4] + [x_3,[x_4,x_4]]$$

$$\implies [x_4, x + w_2] = [-x + w_4, x_4] + [x_3, w_{19}]$$

$$\implies [x_4, x] = -[x, x_4]$$

$$\implies w_{20} = -w_{24}$$

$$[x_1,[x_3,x_4]] = [[x_1,x_3],x_4] + [x_3,[x_1,x_4]]$$

$$\implies [x_1, x + w_2] = [w_6, x_4] + [x_3, w_7]$$

$$\implies [x_1, x] = 0$$

$$\implies w_8 = 0 \implies w_{21} = 0$$

$$[x_2,[x_3,x_4]] = [[x_2,x_3],x_4] + [x_3,[x_2,x_4]]$$

$$\implies [x_2, x + w_2] = [w_{10}, x_4] + [x_3, w_{11}]$$

$$\implies [x_2, x] = 0$$

$$\implies w_{12} = 0 \implies w_{22} = 0$$

$$[x_3,[x_1,x_2]] = [[x_3,x_1],x_2] + [x_1,[x_3,x_2]]$$

$$\implies [x_3, x + w_1] = [w_{13}, x_2] + [x_1, w_{14}]$$

$$\implies [x_3, x] = 0$$

$$\implies w_{16} = 0 \implies w_{23} = 0$$

$$[x_4, [x_1, x_2]] = [[x_4, x_1], x_2] + [x_1, [x_4, x_2]]$$

$$\implies [x_4, x + w_1] = [w_{17}, x_2] + [x_1, w_{18}]$$

$$\implies [x_4, x] = 0$$

$$\implies w_{20} = 0 \implies w_{24} = 0$$

In a similar way to the last case for the 3 dimensional Heisenberg algebra, a change of basis can be performed by letting $[x_1, x_2] = y = x + w_1$. This then allows w_1 to be eliminated and then $\hat{w}_i = w_i - w_1$. This will not change the brackets but will allow another element to be eliminated from the basis of the multiplier.

This leaves 15 elements in the basis of the Leibniz multiplier. Therefore, the cover will have dimension 20 for the 5 dimensional Heisenberg Lie algebra. The dimension of the Leibniz multiplier agrees with the formula found in the general case for a $2n + 1$ dimensional Heisenberg Lie algebra by using $n = 2$.

6.1.3 Comparison Between the Lie and Leibniz Multiplier for the Heisenberg Lie Algebra

For the following values of n, we can show the difference in dimension between the Lie multiplier, found in [9], and the Leibniz multiplier found above for the Heisenberg Lie algebra. For $n > 1$, the Lie multiplier has dimension $2n^2 - n - 1$, while when $n = 1$, the algebra is isomorphic to the 3x3 strictly upper triangular matrices, as such, its Lie multiplier has the same dimension as the Lie multiplier of the 3x3 strictly upper triangular matrices.

Table 6.2 Counting multiplier elements for the Heisenberg Lie algebra

n	Dimension of Algebra	Leibniz Multiplier Dimension	Lie Multiplier Dimension
1	3	5	2
2	5	15	5
3	7	35	14
4	9	63	27
5	11	99	44

6.2 The Heisenberg Leibniz Algebra - Case 1

The first case of a Heisenberg Leibniz algebra is when the bracket $[x_{2i-1}, x_{2i}] = x + y_i$ still remains the same but the right multiplication in the algebra is now 0, so $[x_{2i}, x_{2i-1}] = z_i$ for all $1 \leq i \leq n$. All other brackets remain the same from the Lie case. We will now investigate what elements of the multiplier of this algebra can be eliminated.

$$[x_{2i-1}, [x_{2i}, x_{2i-1}]] = [[x_{2i-1}, x_{2i}], x_{2i-1}] + [x_{2i}, [x_{2i-1}, x_{2i-1}]]$$

$$\implies [x_{2i-1}, z_i] = [x + y_i, x_{2i-1}] + [x_{2i}, c_{ii}]$$

$$\implies 0 = [x, x_{2i-1}]$$

$$\implies g_i = 0$$

$$[x, [x_{2i-1}, x_{2i}]] = [[x, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x, x_{2i}]]$$

$$\implies [x, x + y_i] = [g_i, x_{2i}] + [x_{2i-1}, k_i]$$

$$\implies [x, x] = 0$$

$$\implies e = 0$$

$$[x_{2i}, [x_{2i-1}, x_{2i}]] = [[x_{2i}, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x_{2i}, x_{2i}]]$$

$$\implies [x_{2i}, x + y_i] = [z_i, x_{2i}] + [x_{2i-1}, d_{ii}]$$

$$\implies [x_{2i}, x] = 0$$

$$\implies h_i = 0$$

$$[x_{2i-1}, [x_{2i}, x_{2i}]] = [[x_{2i-1}, x_{2i}], x_{2i}] + [x_{2i}, [x_{2i-1}, x_{2i}]]$$

$$\implies [x_{2i-1}, d_{ii}] = [x + y_i, x_{2i}] + [x_{2i}, x + y_i]$$

$$\implies 0 = [x, x_{2i}] + [x_{2i}, x]$$

$$\implies 0 = k_i + h_i$$

$$\implies k_i = -h_i \implies k_i = 0$$

Then assuming $j \neq i$:

$$[x_{2j-1}, [x_{2i-1}, x_{2i}]] = [[x_{2j-1}, x_{2i-1}], x_{2i}] + [x_{2i-1}, [x_{2j-1}, x_{2i}]]$$

$$\implies [x_{2j-1}, x + y_i] = [c_{ji}, x_{2i}] + [x_{2i-1}, a_{ji}]$$

$$\implies [x_{2j-1}, x] = 0$$

$$\implies f_j = 0$$

Therefore, the elements e, f_i, g_i, h_i and k_i for $1 \leq i \leq n$ can be eliminated from the basis of the multiplier. Once again, by performing a change of basis among the y_i elements in the multiplier, we are able to eliminate the element y_1. Letting $[x_1, x_2] = s = x + y_1$, we eliminate y_1 and let $\hat{y}_i = y_i - y_1$ for $1 < i \leq n$. This does not change the multiplication among the brackets and still allows the $\hat{y}_i$ elements to be central and elements of the multiplier.

As such, the same elements of the multiplier remain as in the case of the Heisenberg Lie algebra and so the dimension of the Leibniz multiplier for the Leibniz Heisenberg algebra is $4n^2 - 1$ and the cover will have dimension $4n^2 + 2n$.

6.2.1 The Heisenberg Leibniz Algebra of Dimension 3

The Heisenberg algebra of dimension 3 has the basis v_1, v_2, v. Taking the section μ, this gives $\mu(v_1) = x_1$, $\mu(v_2) = x_2$ and $\mu(v) = x$. Taking the brackets of the elements in the cover, we get the following relationships with 9 possible elements, w_i, in the basis of the multiplier. The task is to use the Leibniz identity to try to eliminate some of those elements.

$$[x_1, x_2] = x + w_1$$

$$[x_2, x_1] = w_2$$

$$[x_1, x] = w_3$$

$$[x, x_1] = w_4$$

$$[x_2, x] = w_5$$

$$[x, x_2] = w_6$$

$$[x_1, x_1] = w_7$$

$$[x_2, x_2] = w_8$$

$$[x, x] = w_9$$

Using the Leibniz identity, the following elements are eliminated:

$$[x, [x_1, x_2]] = [[x, x_1], x_2] + [x_1, [x, x_2]]$$

$$\implies [x, x + w_1] = [w_4, x_2] + [x_1, w_6]$$

$$\implies [x, x] = 0$$

$$\implies w_9 = 0$$

$$[x_2, [x_1, x_2]] = [[x_2, x_1], x_2] + [x_1, [x_2, x_2]]$$

$$\implies [x_2, x + w_1] = [w_2, x_2] + [x_1, w_8]$$

$$\implies [x_2, x] = 0$$

$$\implies w_5 = 0$$

$$[x_1, [x_2, x_1]] = [[x_1, x_2], x_1] + [x_2, [x_1, x_1]]$$

$$\implies [x_1, w_2] = [x + w_1, x_1] + [x_2, w_7]$$

$$\implies 0 = [x, x_1]$$

$$\implies w_4 = 0$$

$$[x_1, [x_2, x_2]] = [[x_1, x_2], x_2] + [x_2, [x_1, x_2]]$$

$$\implies [x_1, w_8] = [x + w_1, x_2] + [x_2, x + w_1]$$

$$\implies 0 = [x, x_2] + [x_2, x]$$

$$\implies w_6 = -w_5 \implies w_6 = 0$$

We are unable to eliminate the element w_3 in this case, however this element can be eliminated for a higher dimensional Leibniz Heisenberg algebra. We can eliminate w_1 using the change of basis. This shows that the 3 dimensional Heisenberg gives us a special case and the Leibniz multiplier of the Leibniz Heisenberg algebra has dimension 4, which is less than the dimension of the Leibniz multiplier of the Lie Heisenberg algebra, which has dimension 5. In all other higher dimensional Heisenberg algebras, the dimension of the multipliers is the same.

6.2.2 The Heisenberg Leibniz algebra of Dimension 5

The Heisenberg algebra of dimension 5 has the basis v_1, v_2, v_3, v_4, v. Taking the section μ, this gives $\mu(v_i) = x_i$ for $1 \leq i \leq 4$ and $\mu(v) = x$. Taking the brackets of the elements in the cover, this gives 25 possible elements, w_i in the basis of the multiplier. The only difference in the brackets compared to the brackets of elements in the cover of the 5 dimensional Lie Heisenberg algebra are the following:

$$[x_2, x_1] = w_3$$

$$[x_4, x_3] = w_4$$

This is because unlike in the Lie case, the multiplication of elements on the right must be 0, so when multiplying the corresponding elements in the cover, we should get an element of the multiplier.

When looking at Leibniz identity to see which elements of the multiplier can be eliminated, according to the general case, we should find that there are 15 elements in the basis of the multiplier.

$$[x, [x_1, x_2]] = [[x, x_1], x_2] + [x_1, [x, x_2]]$$

$$\implies [x, x + w_1] = [w_{21}, x_2] + [x, w_{22}]$$

$$\implies [x, x] = 0$$

$$\implies w_{25} = 0$$

$$[x_1, [x_2, x_1]] = [[x_1, x_2], x_1] + [x_2, [x_1, x_1]]$$

$$\implies [x_1, w_3] = [x + w_1, x_1] = [x_2, w_5]$$

$$\implies 0 = [x, x_1]$$

$$\implies w_{21} = 0$$

$$[x_2,[x_1,x_2]] = [[x_2,x_1],x_2] + [x_1,[x_2,x_2]]$$

$$\implies [x_2, x + w_1] = [w_3, x_2] + [x_1, w_9]$$

$$\implies [x_2, x] = 0$$

$$\implies w_{12} = 0$$

$$[x_1,[x_3,x_4]] = [[x_1,x_3],x_4] + [x_3,[x_1,x_4]]$$

$$\implies [x_1, x + w_2] = [w_6, x_4] + [x_3, w_7]$$

$$\implies [x_1, x] = 0$$

$$\implies w_8 = 0$$

$$[x_4,[x_3,x_4]] = [[x_4,x_3],x_4] + [x_3,[x_4,x_4]]$$

$$\implies [x_4, x + w_2] = [w_4, x_4] + [x_3, w_{19}]$$

$$\implies [x_4, x] = 0$$

$$\implies w_{20} = 0$$

$$[x_3,[x_1,x_2]] = [[x_3,x_1],x_2] + [x_1,[x_3,x_2]]$$

$$\implies [x_3, x + w_1] = [w_{13}, x_2] + [x_1, w_{14}]$$

$$\implies [x_3, x] = 0$$

$$\implies w_{16} = 0$$

$$[x_3,[x_4,x_3]] = [[x_3,x_4],x_3] + [x_4,[x_3,x_3]]$$

$$\implies [x_3, w_4] = [x + w_2, x_3] + [x_4, w_{15}]$$

$$\implies 0 = [x, x_3]$$

$$\implies w_{23} = 0$$

$$[x_1,[x_2,x_2]] = [[x_1,x_2],x_2] + [x_2,[x_1,x_2]]$$

$$\implies [x_1, w_9] = [x + w_1, x_2] + [x_2, x + w_1]$$

$$\implies 0 = [x, x_2] + [x_2, x]$$

$$\implies w_{22} = -w_{12} \implies w_{22} = 0$$

$$[x_3,[x_4,x_4]] = [[x_3,x_4],x_4] + [x_4,[x_3,x_4]]$$
$$\implies [x_3,w_{19}] = [x+w_2,x_4] + [x_4,x+w_2]$$
$$\implies 0 = [x,x_4] + [x_4,x]$$
$$\implies w_{20} = -w_{24} \implies w_{24} = 0$$

Again, a change of basis can be performed by letting $[x_1,x_2] = y = x + w_1$. This then allows w_1 to be eliminated and $\hat{w_i} = w_i - w_1$. This will not change the multiplication of the brackets but will allow another element to be eliminated from the basis of the multiplier. Accordingly, 10 elements of the multiplier can be eliminated, which gives a 15 dimensional multiplier. This is exactly the dimension of the multiplier that can be seen from the general case using $n = 2$.

6.3 The Heisenberg Leibniz Algebra - Case 2

The second case of a Heisenberg Leibniz algebra is where the bracket $[x_{2i-1},x_{2i}] = x + y_i$ still remains the same but the right multiplication by an element in the algebra can either be 0 or it can be negative of the left multiplication of the same element in the algebra. Thus, for elements in the cover, we have $[x_{2i},x_{2i-1}] = z_i$ for some values of i between 1 and n and for the other values of i between 1 and n, $[x_{2i},x_{2i-1}] = -x+z_i$. All other brackets remain the same from the Lie case. We must now determine what elements of the multiplier can be eliminated in this case.

Without loss of generality, we will assume that $[x_{2i},x_{2i-1}] = -x + z_i$ for $1 \leq i \leq m$ for some $m < n$ and $[x_{2i},x_{2i-1}] = z_i$ for $m < i \leq n$. Let us first consider which elements can be eliminated for the index $1 \leq i \leq m$.

$$[x_{2i},[x_{2i-1},x_{2i}]] = [[x_{2i},x_{2i-1}],x_{2i}] + [x_{2i-1},[x_{2i},x_{2i}]]$$
$$\implies [x_{2i},x+y_i] = [-x+z_i,x_{2i}] + [x_{2i-1},d_{ii}]$$
$$\implies [x_{2i},x] = -[x,x_{2i}]$$
$$\implies h_i = -k_i$$

$$[x_{2i-1},[x_{2i},x_{2i-1}]] = [[x_{2i-1},x_{2i}],x_{2i-1}] + [x_{2i},[x_{2i-1},x_{2i-1}]]$$
$$\implies [x_{2i-1},-x+z_i] = [x+y_i,x_{2i-1}] + [x_{2i},c_{ii}]$$
$$\implies -[x_{2i-1},x] = [x,x_{2i-1}]$$
$$\implies g_i = -f_i$$

$$[x,[x_{2i-1},x_{2i}]] = [[x,x_{2i-1}],x_{2i}] + [x_{2i-1},[x,x_{2i}]]$$
$$\implies [x,x+y_i] = [g_i,x_{2i}] + [x_{2i-1},k_i]$$

$$\implies [x,x]=0$$

$$\implies e=0$$

For $j \neq i$:

$$[x_{2i},[x_{2j-1},x_{2j}]] = [[x_{2i},x_{2j-1}],x_{2j}] + [x_{2j-1},[x_{2i},x_{2j}]]$$

$$\implies [x_{2i},x+y_j] = [b_{ij},x_{2j}] + [x_{2j-1},d_{ij}]$$

$$\implies [x_{2i},x]=0$$

$$\implies h_i=0 \implies k_i=0$$

$$[x_{2i-1},[x_{2j-1},x_{2j}]] = [[x_{2i-1},x_{2j-1}],x_{2j}] + [x_{2j-1},[x_{2i-1},x_{2j}]]$$

$$\implies [x_{2i-1},x+y_j] = [c_{ij},x_{2j}] + [x_{2j-1},a_{ij}]$$

$$\implies [x_{2i-1},x]=0$$

$$\implies f_i=0 \implies g_i=0$$

Now consider the elements that can be eliminated with the index $m < i \leq n$.

$$[x_{2i},[x_{2i-1},x_{2i}]] = [[x_{2i},x_{2i-1}],x_{2i}] + [x_{2i-1},[x_{2i},x_{2i}]]$$

$$\implies [x_{2i},x+y_i] = [z_i,x_{2i}] + [x_{2i-1},d_{ii}]$$

$$\implies [x_{2i},x]=0$$

$$\implies h_i=0$$

$$[x_{2i-1},[x_{2i},x_{2i-1}]] = [[x_{2i-1},x_{2i}],x_{2i-1}] + [x_{2i},[x_{2i-1},x_{2i-1}]]$$

$$\implies [x_{2i-1},z_i] = [x+y_i,x_{2i-1}] + [x_{2i},c_{ii}]$$

$$\implies 0=[x,x_{2i-1}]$$

$$\implies g_i=0$$

$$[x_{2i-1},[x_{2i},x_{2i}]] = [[x_{2i-1},x_{2i}],x_{2i}] + [x_{2i},[x_{2i-1},x_{2i}]]$$

$$\implies [x_{2i-1},d_{ii}] = [x+y_i,x_{2i}] + [x_{2i},x+y_i]$$

$$\implies 0=[x,x_{2i}]+[x_{2i},x]$$

$$\implies k_i=-h_i \implies k_i=0$$

For $j \neq i$:

$$[x_{2i-1}, [x_{2j-1}, x_{2j}]] = [[x_{2i-1}, x_{2j-1}], x_{2j}] + [x_{2j-1}, [x_{2i-1}, x_{2j}]]$$

$$\implies [x_{2i-1}, x + y_j] = [c_{ij}, x_{2j}] + [x_{2j-1}, a_{ij}]$$

$$\implies [x_{2i-1}, x] = 0$$

$$\implies f_i = 0$$

Therefore, for all $1 \leq i \leq n$, the elements e, f_i, g_i, h_i and k_i can be eliminated, including y_1 which is achieved by a change of basis. As we have seen previously with the Heisenberg Lie algebra and the first type of Heisenberg Leibniz algebra, the dimension of the multiplier is $4n^2 - 1$ and therefore the dimension of the cover in this case is $4n^2 + 2n$.

6.3.1 The Heisenberg Leibniz Algebra of Dimension 5

The Heisenberg algebra of dimension 5 has the basis v_1, v_2, v_3, v_4, v. Taking the section μ, this gives $\mu(v_i) = x_i$ for $1 \leq i \leq 4$ and $\mu(v) = x$. Taking the brackets of the elements in the cover, this gives 25 possible elements, w_i in the basis of the multiplier. Without loss of generality, we will assume the following:

$$[x_1, x_2] = x + w_1$$

$$[x_3, x_4] = x + w_2$$

$$[x_2, x_1] = -x + w_3$$

$$[x_4, x_3] = w_4$$

All other brackets remain the same as the brackets established for the cover of the 5-dimensional Lie case. The following elements of the multiplier can be eliminated:

$$[x, [x_1, x_2]] = [[x, x_1], x_2] + [x_1, [x, x_2]]$$

$$\implies [x, x + w_1] = [w_{21}, x_2] + [x_1, w_{22}]$$

$$\implies [x, x] = 0$$

$$\implies w_{25} = 0$$

$$[x_2, [x_1, x_2]] = [[x_2, x_1], x_2] + [x_1, [x_2, x_2]]$$

$$\implies [x_2, x + w_1] = [-x + w_3, x_2] + [x_1, w_9]$$

$$\implies [x_2, x] = [-x, x_2]$$

$$\implies w_{12} = -w_{22}$$

$$[x_3, [x_1, x_2]] = [[x_3, x_1], x_2] + [x_1, [x_3, x_2]]$$

$$\implies [x_3, x + w_1] = [w_{13}, x_2] + [x_1, w_{14}]$$

$$\implies [x_3, x] = 0$$

$$\implies w_{16} = 0$$

$$[x_4, [x_1, x_2]] = [[x_4, x_1], x_2] + [x_1, [x_4, x_2]]$$

$$\implies [x_4, x + w_1] = [w_{17}, x_2] + [x_1, w_{18}]$$

$$\implies [x_4, x] = 0$$

$$\implies w_{20} = 0$$

$$[x_1, [x_2, x_1]] = [[x_1, x_2], x_1] + [x_2, [x_1, x_1]]$$

$$\implies [x_1, -x + w_3] = [x + w_1, x_1] + [x_2, w_5]$$

$$\implies -[x_1, x] = [x, x_1]$$

$$\implies -w_8 = w_{21}$$

$$[x_3, [x_4, x_3]] = [[x_3, x_4], x_3] + [x_4, [x_3, x_3]]$$

$$\implies [x_3, w_4] = [x + w_2, x_3] + [x_4, w_{15}]$$

$$\implies 0 = [x, x_3]$$

$$\implies w_{23} = 0$$

$$[x_3, [x_4, x_4]] = [[x_3, x_4], x_4] + [x_4, [x_3, x_4]]$$

$$\implies [x_3, w_{19}] = [x + w_2, x_4] + [x_4, x + w_2]$$

$$\implies 0 = [x, x_4] + [x_4, x]$$

$$\implies w_{24} = -w_{20} \implies w_{24} = 0$$

$$[x_3, [x_4, x_1]] = [[x_3, x_4], x_1] + [x_4, [x_3, x_1]]$$

$$\implies [x_3, w_{17}] = [x + w_2, x_1] + [x_4, w_{13}]$$

$$\implies 0 = [x, x_1]$$

$$\implies w_{21} = 0 \implies w_8 = 0$$

$$[x_3, [x_4, x_2]] = [[x_3, x_4], x_2] + [x_4, [x_3, x_2]]$$

$$\implies [x_3, w_{18}] = [x + w_2, x_2] + [x_4, w_{14}]$$

$$\implies 0 = [x, x_2]$$

$$\implies w_{22} = 0 \implies w_{12} = 0$$

Again, as in the previous cases, a change of basis can be performed by letting $[x_1, x_2] = y = x + w_1$. This eliminates the element w_1 and gives $\hat{w_i} = w_i - w_1$. This will not change the brackets but will allow another element to be eliminated from the basis of the multiplier. Therefore, 10 elements of the multiplier can be eliminated, which gives a 15 dimensional multiplier. This is exactly the dimension of the multiplier that can be seen from the general case using $n = 2$.

Clearly it can be seen that a 3-dimensional Heisenberg Leibniz algebra cannot be found in Case 2.

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