## ABSTRACT

WESELCOUCH, MICHAEL. The Uniqueness and Irreducibility of $P$-partition Generating Functions. (Under the direction of Ricky Liu).

The $(P,\omega)$*-partition generating function* of a labeled poset $(P,\omega)$ is a quasisymmetric function enumerating certain order-preserving maps from $P$ to $\mathbb{Z}^+$. We aim to characterize when two labeled posets have the same partition generating function.

We simplify the question by restricting our attention to the naturally labeled case, that is, when $\omega$ is order preserving. In this case, a decreasing run in a linear extension corresponds to an antichain in the poset. Using the Hopf algebras of posets and quasisymmetric functions, we give necessary conditions for two posets to have the same generating function. In particular, we show that they must have the same number of antichains of each size and give an explicit formula for counting this number. This fact is a consequence of the existence of a family of linear functions that determine the number of maximal and minimal elements in a naturally labeled poset.

The shape of a poset is a partition that is determined by the sizes of unions of chains in $P$. We show that the shape of a poset is determined by the support of its partition generating function in the fundamental quasisymmetric function basis. We also discuss which shapes guarantee uniqueness of the $P$-partition generating function. In the case that it is not uniquely determined, we give a method of constructing pairs of non-isomorphic posets with the same generating function.

Next, we approach our question from a new direction by expanding the $(P,\omega)$-partition generating function in terms of the type 1 quasisymmetric power sum basis $\{\psi_\alpha\}$. Using this expansion, we show that connected, naturally labeled posets have irreducible $P$-partition generating functions. We also show that series-parallel posets are uniquely determined by their partition generating function. We conclude by giving a combinatorial interpretation for the coefficients of the $\psi_\alpha$-expansion of the $(P,\omega)$-partition generating function akin to the Murnaghan-Nakayama rule.

The Uniqueness and Irreducibility of $P$-partition Generating Functions

by
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# DEDICATION

To my family.

## BIOGRAPHY

Michael Weselcouch grew up in East Haddam, CT. He studied mathematics at Assumption College in Worcester, MA. While at Assumption, he ran cross country and was a practice player on the women's basketball team. He graduated with a B.A. in mathematics in 2014. The following semester, he began his graduate studies in mathematics at North Carolina State University. In his spare time, Mike enjoys running and puzzle solving.

## ACKNOWLEDGEMENTS

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# Introduction

For a finite poset $P = (P, \prec)$ with a bijective labeling $\omega\colon P \to [n]$, the $(P,\omega)$-*partition generating function* $K_{(P,\omega)}(\mathbf{x})$ is a quasisymmetric function enumerating certain order-preserving maps from $P$ to $\mathbb{Z}^+$. This generating function is of interest because many families of symmetric and quasisymmetric functions can be expressed nicely in terms of the $(P,\omega)$-partition generating functions. For example, every elementary symmetric function, complete homogenous symmetric function, and skew Schur function can be expressed as the partition generating function of a certain labeled poset. Additionally, the chromatic quasisymmetric function of Shareshian and Wachs [22] and the quasisymmetric functions associated to matroids of Billera, Jia, and Reiner [4] can be expressed as a nonnegative sum of some $K_{(P,\omega)}(\mathbf{x})$.

In this chapter we will motivate some of the questions that are answered in this thesis and give an overview of what we will cover. We then discuss some preliminaries about the main objects used in this thesis. These objects are compositions and partitions, posets, $(P,\omega)$-partitions, and quasisymmetric functions.

## 1.1 Overview

The question of when two distinct posets can have the same $(P,\omega)$-partition generating function has been studied extensively in the case of skew Schur functions [5, 17, 19], by McNamara and Ward [18] for general labeled posets, and by Hasebe and Tsujie [11] for rooted trees. The initial goal of this thesis is to consider the naturally labeled case, that is, to give necessary and sufficient conditions for when two naturally labeled posets have the same $(P,\omega)$-partition generating function. We say that $P$ is *naturally labeled* if $x \preceq y$ implies $x \leq y$ as integers. We will write $P$ instead of $(P,\omega)$ whenever $P$ is naturally labeled.

In general, it is not true that a poset can be distinguished by its $P$-partition generating function. The smallest case in which two distinct naturally labeled posets have the same partition generating function is the two 7-element posets shown below. We will explore this example further in Section 3.3, where we give a general construction for non-isomorphic posets with the same generating function.

In the early chapters of this thesis, we will rely heavily on the expansion of $K_P(\mathbf{x})$ in the fundamental quasisymmetric function basis. It is a famous result in the study of $(P, \omega)$-partitions that the expansion $K_P(\mathbf{x})$ in the fundamental quasisymmetric function basis depends only on the linear extensions of $P$. A linear extension is a permutation that agrees with the relations of $P$, that is, if $x \prec y$ in $P$, then $x$ must appear before $y$ in the permutation. In fact, $K_P(\mathbf{x})$ is uniquely determined by the multiset of descent sets of the linear extensions of $P$.

The set of finite posets form a basis for a combinatorial Hopf algebra. Informally, a combinatorial Hopf algebra is a graded Hopf algebra with a basis consisting of combinatorial objects. The map that sends a poset to its partition generating function is a Hopf algebra morphism which implies that it commutes with each of the comultiplication maps. In Chapter 2, we will use this fact and tools from the combinatorial Hopf algebra structure on posets due to Schmitt [21] (see also [1]) to prove that if $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then for all triples $(k, i, j)$, $P$ and $Q$ must have the same number of $k$-element order ideals that have $i$ maximal elements and whose complement has $j$ minimal elements. In particular, they must have the same number of antichains of each size, proving a conjecture of McNamara and Ward [18]. As a result of our proof, one can compute certain coefficients in the fundamental quasisymmetric function expansion of $K_P(\mathbf{x})$ explicitly in terms of the number of such ideals.

In Chapter 3, we will show that if $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $P$ and $Q$ must have the same shape. Here, the *shape* of a finite poset, denoted $\mathrm{sh}(P)$, is the partition $\lambda$ whose conjugate partition $\lambda'$ satisfies

$$\lambda'_1 + \lambda'_2 + \cdots + \lambda'_i = a_i,$$

where $a_i$ is the largest number of elements in a union of $i$ antichains of $P$. In fact, we will prove a stronger statement, namely that if the support of $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$ in the fundamental quasisymmetric function basis is the same, then $P$ and $Q$ must have the same shape. The proof of this result depends on the fact that descents in the linear extensions of $P$ correspond to antichains in $P$. Knowing that the shape of a poset is determined by its partition generation function, we ask the following question: for which partitions $\lambda$ does $\mathrm{sh}(P) = \lambda$ guarantee that $P$ is uniquely determined by $K_P(\mathbf{x})$?

We show that if $\mathrm{sh}(P)$ has at most two parts, is a hook shape, or has the form $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$, then $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ implies $P \cong Q$. Conversely, we show that if $\mathrm{sh}(P)$ contains $(3, 3, 1)$ or $(2, 2, 2, 2)$, then $K_P(\mathbf{x}) = K_Q(\mathbf{x})$ does not necessarily imply $P \cong Q$ by constructing two distinct posets of this shape with the same generating function. It remains to be answered what happens when $\mathrm{sh}(P) = (\lambda_1, 2, 2, 1, \ldots, 1)$.

In Chapter 4, we study the expansion of $K_{(P,\omega)}(\mathbf{x})$ in the type 1 quasisymmetric power sum basis $\{\psi_\alpha\}$ introduced by Ballantine, Daugherty, Hicks, Mason, and Niese in [3]. Such expansions have previously been considered in the naturally labeled case by Alexandersson and Sulzgruber in [2].

It is well known that the $(P, \omega)$-partition generating function of a disconnected poset is reducible since it can be expressed as a product over the connected components of $P$. McNamara and Ward [18] asked whether the converse is true, namely whether the $(P, \omega)$-partition generating function of a connected poset is always irreducible in QSym. If said question is true, then one can determine the number of connected components of $P$ from $K_{(P,\omega)}(\mathbf{x})$. We answer this question affirmatively in the naturally labeled case. In particular, we relate part of the $\psi_\alpha$-expansion of $K_P(\mathbf{x})$ to certain *zigzag labelings* of $P$, which exist only for connected posets. We then use this to deduce that connected, naturally labeled posets have irreducible $P$-partition generating functions in Section 4.4. Unfortunately, our proof does not extend to all connected labeled posets $(P, \omega)$. It remains open whether all connected labeled posets $(P, \omega)$ have irreducible $(P, \omega)$-partition generating functions. (See [13] for some discussion of irreducibility in QSym.)

An important object in this thesis is a pair of linear functionals $\eta$ and $\tilde{\eta}$ on QSym (called $\min_1$ and $\max_1$ in Chapter 2) that can be used to determine if a poset has exactly 1 minimal or 1 maximal element. These functionals also have the property that they send any reducible element of QSym to 0. In fact, they send any element in the span of the reducible elements of QSym to 0. In Section 4.3, we express $\eta$ and $\tilde{\eta}$ in terms of the basis $\{\psi_\alpha\}$ and use this to describe the action of various involutions of QSym on this basis.

Hasebe and Tsujie showed in [11] that all rooted trees are uniquely determined by their partition generating function. Their proof relies on their result that rooted trees have irreducible partition generating functions. They then asked whether series-parallel posets can be distinguished by their partition generating functions. (A poset is *series-parallel* if it can be built from one-element posets using ordinal sum and disjoint union operations.) This is a natural question to ask since every rooted tree is series-parallel. We use the previously stated result on irreducibility to give a complete, affirmative answer to this question in Section 4.5.

Alexandersson and Sulzgruber [2] show that when $(P, \omega)$ is naturally labeled, $K_P(\mathbf{x}) = \sum_\alpha c_\alpha \psi_\alpha$ is $\psi$-positive, and they give a combinatorial interpretation for the coefficients $c_\alpha$. The interpretation they gave depends on certain $P$-partitions that we call *pointed $P$-partitions.* In

Section 4.6, we extend this work and give a (signed) combinatorial interpretation for the coefficients in the $\psi_\alpha$-expansion of $K_{(P,\omega)}(\mathbf{x})$ for any labeled poset. This interpretation generalizes the Murnaghan-Nakayama rule for computing the expansion of a skew Schur function in terms of power sum symmetric functions.

In summary, in Chapter 1 we will discuss some background information and preliminaries; in Chapter 2 we will use tools from the combinatorial Hopf algebra structure on posets to count various statistics about the poset; in Chapter 3 we will study posets of a fixed shape and see whether there exists distinct posets of that shape with the same partition generating function; in Chapter 4 we study the combinatorics of the $(P, \omega)$-partition generating function when expanded in the type 1 quasisymmetric power sum basis.

## 1.2 Preliminaries

We begin with some preliminaries about compositions and partitions, posets, $(P, \omega)$-partitions, and quasisymmetric functions.

### 1.2.1 Compositions and partitions

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of $n$ is a finite sequence of positive integers summing to $n$. (When it is clear from context, we will remove the parentheses and commas when writing a composition.) A *weak composition* of $n$ is a finite sequence of nonnegative integers summing to $n$. A *partition* of $n$ is a composition of $n$ whose parts are in weakly decreasing order. Given a composition $\alpha$ and partition $\lambda$, we write $\alpha \sim \lambda$ if $\lambda$ is formed by rearranging the parts of $\alpha$ in weakly decreasing order. We use the notation $\alpha \vDash n$ if $\alpha$ is a composition of $n$ and $\lambda \vdash n$ if $\lambda$ is a partition of $n$.

We will use the shorthand $1^n$ to denote the composition $(\underbrace{1, 1, \ldots, 1}_{n})$. The *reverse of* $\alpha$, denoted $\alpha^r$, is the composition formed by reversing the order of $\alpha$. The *length of* $\alpha$, denoted $l(\alpha)$, is the number of parts of $\alpha$.

The compositions of $n$ are in bijection with the subsets of $[n-1]$ in the following way: for any composition $\alpha$, define

$$D(\alpha) = \{\alpha_1, \quad \alpha_1 + \alpha_2, \quad \ldots, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \subseteq [n-1].$$

Likewise, for any subset $S = \{s_1, s_2, \ldots, s_{k-1}\} \subseteq [n-1]$ with $s_1 < s_2 < \cdots < s_{k-1}$, we can define the composition

$$\mathrm{co}(S) = (s_1, \quad s_2 - s_1, \quad s_3 - s_2, \quad \ldots, \quad s_{k-1} - s_{k-2}, \quad n - s_{k-1}).$$

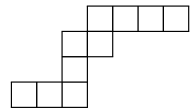


Figure 1.1: The ribbon representation of $\alpha = (3, 1, 2, 4)$.

Given two nonempty compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$, their *concatenation* is

$$\alpha \cdot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l),$$

and their *near-concatenation* is

$$\alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m).$$

Observe that if $\alpha \vDash n$ and $\beta \vDash m$, then both $\alpha \cdot \beta \vDash (n + m)$ and $\alpha \odot \beta \vDash (n + m)$.

The *ribbon representation* for a composition $\alpha$ is the diagram having rows of sizes $(\alpha_1, \ldots, \alpha_k)$ read from bottom to top with exactly one column of overlap between adjacent rows. For example, Figure 1.1 depicts the ribbon representation of $\alpha = (3, 1, 2, 4)$. Each composition can be written as the near-concatenation of compositions of all 1s. The composition $\alpha$ can be expressed as $\alpha = 1^{a_1} \odot 1^{a_2} \odot \cdots \odot 1^{a_l}$, where $a_i$ is the number of boxes in the $i$th column of $\alpha$'s ribbon representation. We will refer to the expansion $\alpha = 1^{a_1} \odot 1^{a_2} \odot \cdots \odot 1^{a_l}$ as the *near-concatenation decomposition* of $\alpha$.

If $\alpha$ and $\beta$ are both compositions of $n$, then we say that $\alpha$ *refines* $\beta$ (equivalently, $\beta$ *coarsens* $\alpha$), denoted $\alpha \preceq \beta$, if

$$\beta = (\alpha_1 + \cdots + \alpha_{i_1}, \quad \alpha_{i_1+1} + \cdots + \alpha_{i_1+i_2}, \quad \ldots, \quad \alpha_{i_1+\cdots+i_{k-1}+1} + \cdots + \alpha_{i_1+\cdots+i_k}),$$

for some $i_1, i_2, \ldots, i_k$ summing to $l(\alpha)$. Equivalently, $\alpha \preceq \beta$ if and only if $D(\beta) \subseteq D(\alpha)$.

For example, if $\alpha = (3, 1, 2, 4, 2)$ and $\beta = (4, 2, 6)$, then $\alpha$ refines $\beta$.

### 1.2.2 Posets

We will now define "poset" as well as define some basic terminology related to posets.

**Definition 1.2.1.** A *poset* $P = (P, \preceq)$ is a set $P$ with a binary relation $\preceq$ such that for all $x$, $y$, and $z$ in $P$, we have:

1. $x \preceq x$,
2. if $x \preceq y$ and $y \preceq x$, then $x = y$, and

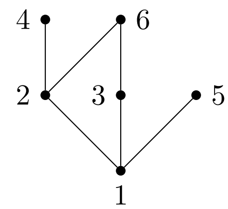



Figure 1.2: The following is the Hasse diagram for the poset on the set $\{1, 2, 3, 4, 5, 6\}$ with the order relation $x \preceq y$ if and only if $x$ divides $y$. Even though $1 \prec 4$, there is not an edge drawn from 1 to 4 because 4 does not cover 1.

3. if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

We say that $x$ is *related* to $y$ if either $x \preceq y$ or $y \preceq x$. We write $x \prec y$ if $x \preceq y$, but $x \neq y$. Note that the definition of a poset allows us to have elements that are not related to each other. In this case we say that those elements are *incomparable*.

**Definition 1.2.2.** An element $x$ is said to be *covered* by $y$ if $x \prec y$ and there is no element $z$ such that $x \prec z \prec y$.

In a finite poset, all relations can be determined from the covering relations.

**Definition 1.2.3.** We say that an element $x$ is *maximal* if there is no element $y$ such that $x \prec y$. We say that an element $x$ is *minimal* if there is no element $y$ such that $y \prec x$.

We will discuss the concept of maximal and minimal elements further in Chapter 2.

**Definition 1.2.4.** A poset $P$ is said to be *chain* if any two elements in $P$ are comparable. Similarly, we say that a subset $C \subseteq P$ is a *chain* if when regarded as a subposet of $P$, $C$ is a chain. The chain $C$ of $P$ is called *maximal* if it is not contained in a larger chain of $P$.

An example of a chain is the set of integers with the standard $\leq$ ordering. In this thesis, we restrict our attention to finite posets.

### Hasse diagrams and ranked posets

If $P = (P, \prec)$ is a finite poset, then the *Hasse diagram* of $P$ is the graph in the plane whose vertices are the elements of $P$ and an edge is drawn upwards from $x$ to $y$ whenever $y$ covers $x$ as shown in Figure 1.2. The Hasse diagram is a tool for visualizing posets.

The set of minimal elements can be seen in the Hasse diagram as they are the only elements that are not at the top of an edge. Similarly, the set of maximal elements are the elements that are not at the bottom of an edge. In Figure 1.2, the element 1 is the only minimal element and $\{4, 5, 6\}$ are the maximal elements.

**Definition 1.2.5.** A finite poset $P$ is said to be *ranked* (or *graded*) if every maximal chain of $P$ has the same length. In this case there is a rank function $\rho\colon P \to \mathbb{N}$ that satisfies the following:

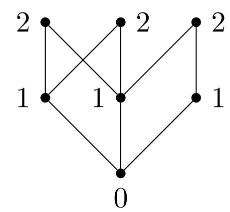
(i) if $x$ is minimal, then $\rho(x) = 0$, and

(ii) if $y$ covers $x$, then $\rho(y) = \rho(x) + 1$.

We say that if $\rho(x) = i$, then $x$ has *rank* $i$.

We will now consider an example of a ranked poset.

**Example 1.2.6.** The following is the Hasse diagram of a ranked poset with the vertices labeled with their rank.




### Distributive lattice

If $x$ and $y$ are in a poset $P$, then we say that $z$ is an *upper bound* of $x$ and $y$ if $x \preceq z$ and $y \preceq z$. The *join* (or *least upper bound*) of $x$ and $y$ is an upper bound $z$ of $x$ and $y$ such that if $w$ is an upper bound of $x$ and $y$, then $z \preceq w$. Note that in a poset, there need not be a join of every pair of elements. We denote the join of $x$ and $y$ by $x \vee y$.

Similarly, we say that $z$ is a *lower bound* of $x$ and $y$ if $x \succeq z$ and $y \succeq z$. The *meet* (or *greatest lower bound*) of $x$ and $y$ is a lower bound $z$ of $x$ and $y$ such that if $w$ is a lower bound of $x$ and $y$, then $z \succeq w$. Again, there need not be a meet of every pair of elements. We denote the meet of $x$ and $y$ by $x \wedge y$.

**Definition 1.2.7.** A *lattice* is a poset $L$ for which every pair of elements has a join and a meet. We say that a lattice $L$ is a *distributive lattice* if for all $x$, $y$, and $z$ in $L$ we have:

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

We will see in Section 2.1.2 that every finite poset $P$ can be associated with a unique distributive lattice $J(P)$.

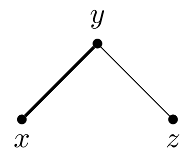



Figure 1.3: The following labeled poset has $\omega(x) > \omega(y) > \omega(z)$.

### 1.2.3 $(P, \omega)$-partitions

Let $P = (P, \prec)$ be a finite poset of size $n$. A *labeling* of $P$ is a bijection $\omega \colon P \to \{1, 2, \ldots, n\}$.

**Definition 1.2.8.** For a labeled poset $(P, \omega)$, a $(P, \omega)$*-partition* is a map $\theta \colon P \to \mathbb{Z}^+$ that satisfies the following:

(a) If $x \preceq y$, then $\theta(x) \leq \theta(y)$.

(b) If $x \preceq y$ and $\omega(x) > \omega(y)$, then $\theta(x) < \theta(y)$.

We say that the *weight* of a $(P, \omega)$-partition $\theta$ is the weak composition

$$\mathrm{wt}(\theta) = (|\theta^{-1}(1)|, |\theta^{-1}(2)|, \ldots).$$

**Definition 1.2.9.** The $(P, \omega)$*-partition generating function* $K_{(P,\omega)}(x_1, x_2, \ldots)$ for a labeled poset $(P, \omega)$ is given by

$$K_{(P,\omega)}(x_1, x_2, \ldots) = \sum_{(P,\omega)\text{-partition } \theta} x_1^{|\theta^{-1}(1)|} x_2^{|\theta^{-1}(2)|} \cdots,$$

where the sum ranges over all $(P, \omega)$-partitions $\theta$.

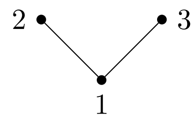
If $(P, \omega)$ is disconnected, then $K_{(P,\omega)}(\mathbf{x})$ is the product of the $(P, \omega)$-partition generating functions of the connected components of $P$.

Note that $K_{(P,\omega)}(x_1, x_2, \ldots)$ depends only on the relative order of $\omega(x)$ and $\omega(y)$ when $y$ covers $x$. In the Hasse diagram of $(P, \omega)$, we will use a bold edge (or *strict edge*) to represent when $x \prec y$ but $\omega(x) > \omega(y)$, while we will use a plain edge (or *natural edge*) when $x \prec y$ and $\omega(x) < \omega(y)$ (see Figure 1.3).

If $\omega$ is order-preserving, then $K_{(P,\omega)}(x_1, x_2, \ldots)$ depends only on the structure of $P$. In this case, we call $P$ *naturally labeled* and write $K_P(\mathbf{x}) = K_{(P,\omega)}(\mathbf{x})$.

We will now give an example to show that $K_{(P,\omega)}(\mathbf{x})$ does in fact depend on our choice of labeling.

**Example 1.2.10.** Let $(P, \omega_1)$ be the following naturally labeled poset.

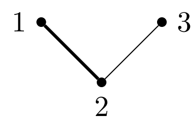



It follows that if $\theta$ is a $(P, \omega_1)$-partition, then $\theta(1) \leq \theta(2)$ and $\theta(1) \leq \theta(3)$. One can compute $K_{(P,\omega_1)}(\mathbf{x})$ as

$$\begin{aligned} K_{(P,\omega)}(\mathbf{x}) = & \, x_1^3 + x_2^3 + x_3^3 + x_4^3 \cdots \\ & + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 \cdots \\ & + 2x_1^2 x_2 + 2x_1^2 x_3 + 2x_2^2 x_3 + 2x_1^2 x_4 \cdots \\ & + 2x_1 x_2 x_3 + 2x_1 x_2 x_4 \cdots . \end{aligned}$$

For clarity we will now state where each term in the expansion came from. The terms in the top row of $K_{(P,\omega_1)}(\mathbf{x})$ correspond to when $\theta(1) = \theta(2) = \theta(3)$, the terms in the second row correspond to when $\theta(1) < \theta(2) = \theta(3)$, the terms in the third row correspond to when $\theta(1) = \theta(2) < \theta(3)$ or $\theta(1) = \theta(3) < \theta(2)$, and the terms in the bottom row correspond to when $\theta(1) < \theta(2) < \theta(3)$ or $\theta(1) < \theta(3) < \theta(2)$.

Now let $(P, \omega_2)$ be the following labeled poset.




It follows that if $\theta$ is a $(P, \omega_2)$-partition, then $\theta(2) < \theta(1)$ and $\theta(2) \leq \theta(3)$. One can compute $K_{(P,\omega_2)}(\mathbf{x})$ as

$$\begin{aligned} K_{(P,\omega_2)}(\mathbf{x}) = & \, x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 \cdots \\ & + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 \cdots \\ & + 2x_1 x_2 x_3 + 2x_1 x_2 x_4 \cdots . \end{aligned}$$

The terms in the top row of $K_{(P,\omega_2)}(\mathbf{x})$ correspond to when $\theta(2) < \theta(1) = \theta(3)$, the terms in the second row correspond to when $\theta(2) = \theta(3) < \theta(1)$, and the terms in the bottom row correspond to when $\theta(2) < \theta(1) < \theta(3)$ or $\theta(2) < \theta(3) < \theta(1)$.

We see that even though the underlying structure of $(P, \omega_1)$ and $(P, \omega_2)$ are equivalent, $K_{(P,\omega_1)}(\mathbf{x}) \neq K_{(P,\omega_2)}(\mathbf{x})$ because of our choice of labeling.

In this thesis, we will usually restrict our attention to the case when $(P, \omega)$ is *naturally labeled*, that is, when $\omega$ is an order-preserving map. In this case, $K_P(\mathbf{x})$ does not depend on our choice of natural labeling but only on the underlying structure of $P$.

**Linear extensions**

A *linear extension* of a labeled poset $(P, \omega)$ with ground set $[n]$ is a permutation $\pi$ of $[n]$ that respects the relations in $P$, that is, if $x \preceq y$, then $\pi^{-1}(x) \leq \pi^{-1}(y)$. The set of all linear extensions of $P$ is denoted $\mathcal{L}(P, \omega)$. Note that $|\mathcal{L}(P, \omega)|$ is the coefficient of $x_1 x_2 \cdots x_n$ in $K_P(\mathbf{x})$. For example, the set of linear extensions of $(P, \omega_2)$ from Example 1.2.10 is $\{213, 231\}$.

### 1.2.4 Quasisymmetric functions

A *quasisymmetric function* in the variables $x_1, x_2, \ldots$ (with coefficients in $\mathbb{C}$) is a formal power series $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ of bounded degree such that, for any composition $\alpha$, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ equals the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ whenever $i_1 < i_2 < \cdots < i_k$. We denote the algebra of quasisymmetric functions by $\mathrm{QSym} = \bigoplus_{n \geq 0} \mathrm{QSym}_n$, graded by degree.

We will begin by considering two natural bases for QSym, the monomial basis and the fundamental basis. In Chapter 4, we will consider a third basis, the type 1 quasisymmetric power sum basis. The *monomial quasisymmetric function basis* $\{M_\alpha\}$, indexed by compositions $\alpha$, is given by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

For example, $M_{(2,1)} = \sum_{i<j} x_i^2 x_j$.

The *fundamental quasisymmetric function basis* $\{L_\alpha\}$ is also indexed by compositions $\alpha$ and is given by

$$L_\alpha = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_s < i_{s+1} \text{ if } s \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

In terms of the monomial basis,

$$L_\alpha = \sum_{\beta \preceq \alpha} M_\beta,$$

where the sum runs over all refinements $\beta$ of $\alpha$. By Möbius inversion, this implies that

$$M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{l(\beta) - l(\alpha)} L_\beta.$$

For any labeled poset $(P, \omega)$ (on the ground set $[n]$), $K_{(P,\omega)}(\mathbf{x})$ is a quasisymmetric function, and we can express it in terms of the fundamental basis $\{L_\alpha\}$ using the linear extensions of $(P, \omega)$. For any linear extension $\pi \in \mathcal{L}(P, \omega)$, define the *descent set* of $\pi$ to be $\mathrm{des}(\pi) = \{i \mid \pi(i) > \pi(i+1)\}$. Abbreviating $\mathrm{co}(\mathrm{des}(\pi))$ by $\mathrm{co}(\pi)$, we then have the following result.

**Theorem 1.2.11** ([24]). *Let $P$ be a (labeled) poset on $[n]$. Then*

$$K_{(P,\omega)}(\mathbf{x}) = \sum_{\pi \in \mathcal{L}(P,\omega)} L_{\mathrm{co}(\pi)}.$$

This result is Corollary 7.19.5 from [24]; we sketch a proof here for completeness.

*Proof.* Let $\pi \in \mathcal{L}(P,\omega)$ and let $\mathrm{des}(\pi)$ be the descent set of $\pi$. We say a $(P,\omega)$-partition $\theta$ is *$\pi$-compatible* if

$$\theta(\pi_1) \leq \theta(\pi_2) \leq \cdots \leq \theta(\pi_n), \text{ and}$$

$$\theta(\pi_i) < \theta(\pi_{i+1}) \text{ if } i \in \mathrm{des}(\pi).$$

It follows that the set of $\pi$-compatible $(P,\omega)$-partitions contribute to the $L_{\mathrm{co}(\pi)}$ term in the fundamental basis expansion of $K_{(P,\omega)}(\mathbf{x})$. In fact, every $(P,\omega)$-partition is $\pi$-compatible for exactly one $\pi \in \mathcal{L}(P,\omega)$. To find said $\pi$ from $\theta$, read off the entries set to 1 by $\theta$ in order, then entries sent to 2, etc. □

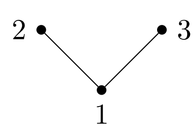
In other words, the descent sets of the linear extensions of $(P,\omega)$ determine its $(P,\omega)$-partition generating function. We will use this fact later when showing that two naturally labeled posets $P$ and $Q$ have different $P$-partition generating functions. In some cases we can just show that a linear extension of $P$ has a descent set that no linear extension of $Q$ has, thus implying that $K_P(\mathbf{x}) \neq K_Q(\mathbf{x})$. For completeness, we will now express $K_{(P,\omega)}(\mathbf{x})$ in terms of the monomial basis $\{M_\alpha\}$.

**Theorem 1.2.12.** *Let $P$ be a (labeled) poset on $[n]$. Then*

$$K_{(P,\omega)}(\mathbf{x}) = \sum_{\theta} M_{\mathrm{wt}(\theta)},$$

*where the sum runs over all $(P,\omega)$-partitions $\theta$ that are surjective onto some $[k]$.*

**Example 1.2.13.** We will reconsider the naturally labeled poset $(P,\omega_1)$ from Example 1.2.10.

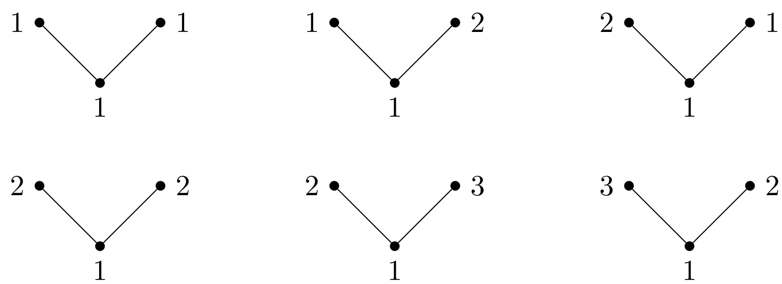



We computed $K_{(P,\omega_1)}(\mathbf{x})$ as

$$\begin{aligned} K_{(P,\omega_1)}(\mathbf{x}) = {} & x_1^3 + x_2^3 + x_3^3 + x_4^3 \cdots \\ & + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_1x_4^2 \cdots \\ & + 2x_1^2x_2 + 2x_1^2x_3 + 2x_2^2x_3 + 2x_1^2x_4 \cdots \end{aligned}$$

$$+ 2x_1x_2x_3 + 2x_1x_2x_4 \cdots .$$

The following are the $P$-partitions that contribute to the monomial basis expansion of $K_{(P,\omega_1)}(\mathbf{x})$.



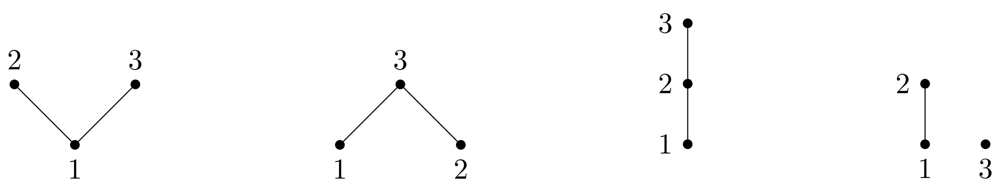
In terms of the monomial basis, $K_{(P,\omega_1)}(\mathbf{x}) = M_3 + M_{12} + 2M_{21} + 2M_{111}$. The set of linear extension of $(P, \omega_1)$ is $\mathcal{L}(P, \omega_1) = \{123, 132\}$. Therefore by Theorem 1.2.11

$$K_{(P,\omega_1)}(\mathbf{x}) = L_3 + L_{21}.$$

This also follows from the fact that $L_3 = M_3 + M_{12} + M_{21} + M_{111}$ and $L_{21} = M_{21} + M_{111}$.

Note that the set of partition generating functions for naturally labeled posets is not a linearly independent set. We can see this in the following example.

**Example 1.2.14.** Let $P_1$, $P_2$, $Q_1$, and $Q_2$ be the posets show below.



$P_1$ $P_2$ $Q_1$ $Q_2$

The partition generating functions for the four posets above are:

$$\begin{aligned}
K_{P_1}(\mathbf{x}) &= L_3 + L_{21}, \\
K_{P_2}(\mathbf{x}) &= L_3 + L_{12}, \\
K_{Q_1}(\mathbf{x}) &= L_3, \\
K_{Q_2}(\mathbf{x}) &= L_3 + L_{21} + L_{12}.
\end{aligned}$$

It follows that $K_{P_1}(\mathbf{x}) + K_{P_2}(\mathbf{x}) = K_{Q_1}(\mathbf{x}) + K_{Q_2}(\mathbf{x})$.

In the next chapter, we will show the existence of linear functions on QSym that can determine information about a poset from its partition generating function. If the set of partition generating functions for naturally labeled posets was a linearly independent set, then the existence of these linear functions would be trivial. This set was studied recently by Féray [7], where he gives a combinatorial description of the linear dependence relations of the set. The description given relies on a combinatorial operation called *cyclic inclusion-exclusion*. We will use an operation similar to this in Section 3.3, when we construct two posets with the same partition generating function.

# Chapter 2

# Antichains and combinatorial Hopf algebras

We begin this chapter by giving some background on posets and combinatorial Hopf algebras. We then use the connection between the combinatorial Hopf algebra structure of posets and quasisymmetric functions to count certain properties of a poset from its partition generating function. In particular, for all triples $(k, i, j)$, we will count the number of $k$-element order ideals that have $i$ maximal elements and whose complement has $j$ minimal elements. This is to say that two posets can have the same partition generating function only if they have the same number of such ideals for all values of $k$, $i$, and $j$. Summing over all values of $k$, $i$, and $j$, we count the number of antichains in $P$.

We conclude the chapter by stating results that will be used in Chapter 3. These results will later be used to show that certain posets are uniquely determined by their partition generating function.

Throughout this chapter, we take $P$ to be a naturally labeled poset with ground set $[n]$. The results in this chapter can also be found in [14].

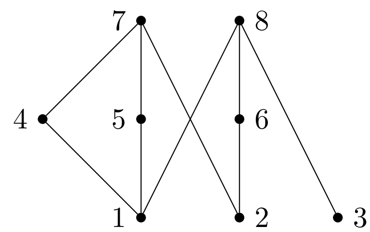
## 2.1 Poset background

In this section we will discuss background on antichains, the poset of order ideals, and the flag $f$-vector and flag $h$-vector.

### 2.1.1 Antichains

An *antichain* is a subset $A$ of a poset $P$ such that any two elements of $A$ are incomparable. The antichain structure of a naturally labeled poset $P$ plays an important role in determining which sets can appear as descent sets for linear extensions of $P$. Since $P$ is naturally labeled, elements $i < j$ form an antichain in $P$ if and only if there exists a linear extension of $P$ in which $j$ appears immediately before $i$. This means that every descent in a linear extension of $P$ is formed by a 2-element antichain. Similarly, if there is a linear extension of $P$ that has $i$ consecutive descents, then these elements form an $(i+1)$-element antichain in $P$. This shows that the sizes of some of the antichains of $P$ can be obtained from $K_P(\mathbf{x})$.

**Example 2.1.1.** Let $P$ be the following poset.




Note that $A = \{3, 6, 7\}$ is an antichain in $P$ and $\pi = 12457638$ is a linear extension of $P$. The elements of $A$ appear consecutively and in decreasing order in $\pi$.

### 2.1.2 Poset of order ideals

An *order ideal*, or *ideal* for short, is a subset $I \subseteq P$ such that if $x \in I$ and $y \prec x$, then $y \in I$. There is a one-to-one correspondence between ideals and antichains, namely, the maximal elements of an ideal form an antichain. A *principal order ideal* is an ideal with a unique maximal element. The dual notion of an order ideal is a *filter*: it is a subset $J \subseteq P$ such that if $x \in J$ and $y \succ x$, then $y \in J$.

The set of all order ideals of $P$, ordered by inclusion, forms a poset that we will denote $J(P)$. In fact, $J(P)$ is a finite (graded) distributive lattice. The rank of an element of $J(P)$ is the number of elements in the corresponding ideal of $P$. See Figure 2.1 for an example.

If $J(P)$ has a unique element of some rank $k$, then $P$ can be expressed as an *ordinal sum* $P = Q \oplus R$ with $|Q| = k$. Here, the ordinal sum $Q \oplus R$ is the poset on the disjoint union of $Q$ and $R$ with relations $x \preceq y$ if and only if $x \preceq_Q y$, $x \preceq_R y$, or $x \in Q$ and $y \in R$. For an example of an ordinal sum, see Figure 2.2.

**Definition 2.1.2.** A finite poset $P$ is *irreducible* if $P = Q \oplus R$ implies that either $Q = \varnothing$ or $R = \varnothing$.

Each poset has a unique *ordinal sum decomposition* $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ with $P_i$ irreducible. If $|P_i| = n_i$, then $J(P)$ has exactly one element in ranks 0, $n_1$, $n_1 + n_2$, …, $n_1 + n_2 + \cdots + n_k$.

We will now show that the partition generating functions of the irreducible components of $P$ can be determined from $K_P(\mathbf{x})$.

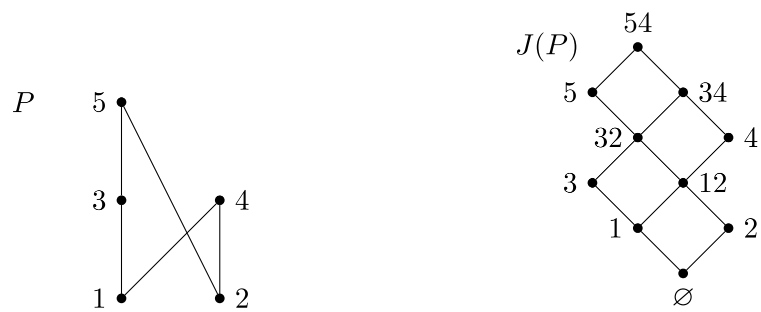



Figure 2.1: A poset $P$ and its poset of order ideals $J(P)$. The labels of the elements in $J(P)$ correspond the the maximal elements of the corresponding order ideal in $P$.

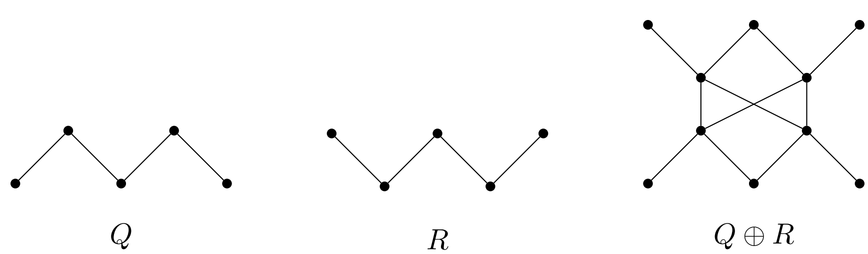



Figure 2.2: The posets $Q$ and $R$ can be combined to make a new poset $Q \oplus R$. The only order ideal of $Q \oplus R$ with five elements is the order ideal containing only the elements of $Q$.

**Lemma 2.1.3.** *Suppose $P$ and $Q$ have ordinal sum decompositions $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ and $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_j$. If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $k = j$, and $K_{P_i}(\mathbf{x}) = K_{Q_i}(\mathbf{x})$ for $i = 1, \ldots, k$.*

*Proof.* Since $P$ is naturally labeled, if elements $a$ and $b$ form a descent in a linear extension of $P$, then $a$ and $b$ must both lie in the same $P_i$ for some $i$. This means that no linear extension of $P$ has a descent in the locations $n_1$, $n_1 + n_2$, $\ldots$, $n_1 + n_2 + \cdots + n_{k-1}$. For any other possible descent location $r$, let $I$ be an ideal of $P$ of size $r$ containing an element whose label $x$ is as large as possible. Since $I$ is not an ordinal summand of $P$, some minimal element $y$ of $P \setminus I$ is not greater than $x$ in $P$. Then $(I \cup \{y\}) \setminus \{x\}$ is also an ideal of $P$ of size $r$, so we must have $y < x$ by our choice of $I$. Hence there is a linear extension of $P$ with a descent in location $r$ (that begins with the elements of $I$ ending with $x$, followed by $y$). Since we can determine all possible descent sets of linear extensions of $P$ from the expansion of $K_P(\mathbf{x})$ in the fundamental basis by Theorem 1.2.11, we can thereby determine $k$ and all $n_i = |P_i|$ from $K_P(\mathbf{x})$.

To get $K_{P_i}(\mathbf{x})$ from $K_P(\mathbf{x})$, note that linear extensions of $P$ can be broken up into $k$ parts: the first $n_1$ elements form a linear extension of $P_1$, the next $n_2$ elements form a linear extension of $P_2$, and so on. Then define $\pi_i \colon \mathrm{QSym}_n \to \mathrm{QSym}_{n_i}$ by

$$\pi_i(L_\alpha) = \begin{cases} L_\beta & \text{if } \alpha = (n_1 + \cdots + n_{i-1}) \odot \beta \odot (n_{i+1} + \cdots + n_k), \\ 0 & \text{otherwise.} \end{cases}$$

extended linearly. It follows that $K_{P_i}(\mathbf{x}) = \pi_i(K_P(\mathbf{x}))$. □

### 2.1.3 Flag $f$-vector and flag $h$-vector

For any $S \subseteq [n]$, define the subposet $J(P)_S = \{I \in J(P) \mid |I| \in S\}$. Let $f(S)$ denote the number of maximal chains in $J(P)_S$. The function $f \colon 2^{[n]} \to \mathbb{Z}$ is called the *flag $f$-vector* of $J(P)$. Also define $h(S)$ by

$$h(S) = \sum_{T \subseteq S} (-1)^{|S-T|} f(T).$$

This function $h$ is called the *flag $h$-vector* of $J(P)$.

In the case when $P$ is naturally labeled, the flag $f$-vector and flag $h$-vector appear in the expansion of $K_P(\mathbf{x})$ as follows:

$$K_P(\mathbf{x}) = \sum_\alpha f_{D(\alpha)} M_\alpha = \sum_\alpha h_{D(\alpha)} L_\alpha.$$

Here $f_S$ and $h_S$ are the flag $f$-vector and flag $h$-vector of $J(P)$, respectively.

The *$L$-support of $K_P(\mathbf{x})$* is defined by

$$\mathrm{supp}_L(K_P(\mathbf{x})) = \{\alpha \mid h_{D(\alpha)} \neq 0\}.$$

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\nabla \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \nabla"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{K} \arrow[d, "\mathrm{id} \otimes 1_A"'] & A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{K} \otimes A \arrow[d, "1_A \otimes \mathrm{id}"] \\
A \otimes A \arrow[r, "\nabla"] & A & A \otimes A \arrow[l, "\nabla"']
\end{tikzcd}
$$

Figure 2.3: The commutative diagrams for Definition 2.2.1.

$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{K} \arrow[r] & A & \mathbb{K} \otimes A \arrow[l] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \epsilon"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "\mathrm{id}"'] & A \otimes A \arrow[u, "\epsilon \otimes \mathrm{id}"']
\end{tikzcd}
$$

Figure 2.4: The commutative diagrams for Definition 2.2.2.

## 2.2 Combinatorial Hopf algebra

The ring of quasisymmetric functions is well known to be a combinatorial Hopf algebra. Additionally, the set of isomorphism classes of finite posets forms the basis of a combinatorial Hopf algebra. In order to discuss the connection between isomorphism classes of finite posets and the ring of quasisymmetric functions fully, we must first give a formal definition of what it means to be a combinatorial Hopf algebra. This will require a few preliminary definitions. Throughout these definitions, we take $\mathbb{K}$ to be a field and $1_{\mathbb{K}}$ to be multiplicative identity in $\mathbb{K}$.

**Definition 2.2.1.** A *unital associative algebra* $(A, \nabla, 1_A)$ is a vector space $A$ over a field $\mathbb{K}$ with linear maps (multiplication) $\nabla \colon A \otimes A \to A$ and (unit) $1_A \colon \mathbb{K} \to A$ which satisfy the commutative diagrams in Figure 2.3. In the diagrams, the map id denotes the identity map on $A$, the map $A \to A \otimes \mathbb{K}$ sends $a \mapsto a \otimes 1_{\mathbb{K}}$, and the map $A \to \mathbb{K} \otimes A$ sends $a \mapsto 1_{\mathbb{K}} \otimes a$.

**Definition 2.2.2.** A *counital coassociative algebra* $(A, \Delta, \epsilon)$ is a vector space $A$ over a field $\mathbb{K}$ with linear maps (comultiplication) $\Delta \colon A \to A \otimes A$ and (counit) $\epsilon \colon A \to \mathbb{K}$ which satisfy the commutative diagrams in Figure 2.4. In these diagrams, the map $A \otimes \mathbb{K} \to A$ sends $a \otimes 1_{\mathbb{K}} \mapsto a$ and the map $\mathbb{K} \otimes A \to A$ sends $1_{\mathbb{K}} \otimes a \mapsto a$.

We see in the left diagram of Figure 2.3 that the multiplication is associative, that is, $\nabla \circ (\nabla \otimes \mathrm{id}) = \nabla \circ (\mathrm{id} \otimes \nabla)$. The left diagram of Figure 2.4 shows a similar property for the comultiplication.

**Definition 2.2.3.** A *morphism of coalgebras* $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ is a linear map $\phi \colon A \to B$ such that the diagrams in Figure 2.5 commute.

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "\phi"'] & A \otimes A \arrow[d, "\phi \otimes \phi"] \\ B \arrow[r, "\Delta_B"] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\epsilon_A"'] & & B \arrow[dl, "\epsilon_B"] \\ & \mathbb{K} & \end{tikzcd}$$

Figure 2.5: The commutative diagrams for Definition 2.2.3.

The diagram on the left of Figure 2.5 shows that a morphism of coalgebras commutes with the coproduct, whereas the diagram on the right shows that a morphism of coalgebras commutes with the counit as well.

The following definition relies heavily on the coalgebra structure of $\mathbb{K}$ and $A \otimes A$ where $A$ is a coalgebra. Before stating the next definition, we will first show that $\mathbb{K}$ and $A \otimes A$ are indeed coalgebras. We take $\Delta_{\mathbb{K}}$ to be the canonical isomorphism from $\mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$ and $\epsilon_{\mathbb{K}}$ to be the identity $\mathrm{id}\colon \mathbb{K} \to \mathbb{K}$. It follows that $(\mathbb{K}, \Delta_{\mathbb{K}}, \epsilon_{\mathbb{K}})$ is a coalgebra. Let the function $\tau\colon A \otimes B \to B \otimes A$ be the linear function defined by $\tau(x \otimes y) = y \otimes x$ for all $x$ in $A$ and $y$ in $B$. For all coalgebras $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$, we can construct a new coalgebra $(A \otimes B, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ using the linear function $\tau$. We define the comultiplication $\Delta_{A \otimes B}$ by first applying $\Delta_A \otimes \Delta_B$ to an element of $A \otimes B$. This results in an element in $A \otimes A \otimes B \otimes B$. We then apply the map $\mathrm{id} \otimes \tau \otimes \mathrm{id}$ to get an element in $A \otimes B \otimes A \otimes B$ as desired. The counit $\epsilon_{A \otimes B}$ is composition of $\epsilon_A \otimes \epsilon_B$ with the canonical isomorphism from $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$. That is, first apply $\epsilon_A \otimes \epsilon_B$ resulting in an element in $\mathbb{K} \otimes \mathbb{K}$, and then apply the canonical isomorphism $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$. It follows that $A \otimes B$ is a coalgebra for all coalgbras $A$ and $B$.

**Definition 2.2.4.** A *bialgebra* $(A, \nabla, 1_A, \Delta, \epsilon)$ over a field $\mathbb{K}$ satisfies the following:

- $A$ is a vector space over $\mathbb{K}$,
- $(A, \nabla, 1_A)$ is a unital associative algebra,
- $(A, \Delta, \epsilon)$ is a counital coassociative algebra; and
- the compatibility conditions expressed by the commutative diagrams in Figure 2.6.

We now have the required background to state the definition of a Hopf algebra.

**Definition 2.2.5.** A *Hopf algebra* $\mathcal{H} = (A, \nabla, 1_A, \Delta, \epsilon, S)$ is a bialgebra $(A, \nabla, 1_A, \Delta, \epsilon)$ with a $\mathbb{K}$-linear map $S\colon A \to A$, called the *antipode*, such that the diagram in Figure 2.7 commutes.

A vector space $A$ is a *graded vector space* if $A = \bigoplus_{n \geq 0} A_n$. If $A$ is a graded vector space, then we can obtain a natural grading on $A \otimes A$ by taking each element of $A_i \otimes A_j$ to be an

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] & A \arrow[r, "\Delta"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & & A \otimes A \otimes A \otimes A \arrow[u, "\nabla \otimes \nabla"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[rr, "\nabla"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & A \arrow[dl, "\epsilon"] \\
& \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} \arrow[dl, "1_A \otimes 1_A"'] \arrow[dr, "1_A"] & \\
A \otimes A & & A \arrow[ll, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[rr, "\mathrm{id}"] \arrow[dr, "1_A"'] & & \mathbb{K} \\
& A \arrow[ur, "\epsilon"'] &
\end{tikzcd}
$$

Figure 2.6: The commutative diagrams for Definition 2.2.4.

$$
\begin{tikzcd}
& A \otimes A \arrow[r, "S \otimes \mathrm{id}"] & A \otimes A \arrow[dr, "\nabla"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\epsilon"] & \mathbb{K} \arrow[r, "1_A"] & & A \\
& A \otimes A \arrow[r, "\mathrm{id} \otimes S"] & A \otimes A \arrow[ur, "\nabla"'] &
\end{tikzcd}
$$

Figure 2.7: The commutative diagram for Definition 2.2.5.

element of the $i+j$ graded component of $A \otimes A$. We say a graded vector space $A = \bigoplus_{n \geq 0} A_n$ over a field $\mathbb{K}$ is *connected* if $A_0 \cong \mathbb{K}$. It should be noted that every graded, connected bialgebra is a Hopf algebra [10].

**Definition 2.2.6.** A *combinatorial Hopf algebra* $\mathcal{H}$ is a graded connected Hopf algebra over a field $\mathbb{K}$ equipped with a *character* (multiplicative linear function) $\zeta \colon \mathcal{H} \to \mathbb{K}$.

See [1] for more details on the formal definition of a combinatorial Hopf algebra. Informally, a combinatorial Hopf algebra is a Hopf algebra whose basis elements are combinatorial objects. The multiplication gives a way to combine these objects, and the comultiplication gives a way to decompose these objects.

### 2.2.1 Reduced incidence Hopf algebra

Let $\mathcal{J}$ denote the set of all finite distributive lattices up to isomorphism. The free $\mathbb{C}$-module $\mathbb{C}[\mathcal{J}]$, whose basis consists of isomorphism classes of distributive lattices $[J] \in \mathcal{J}$, can be given a Hopf algebra structure known as the *reduced incidence Hopf algebra*. The multiplication, unit, comultiplication, and counit are defined as follows:

$$\begin{aligned}
\nabla([J_1] \otimes [J_2]) &:= [J_1 \times J_2], \\
1_{\mathbb{C}[\mathcal{J}]} &:= [o], \\
\Delta([J]) &:= \sum_{x \in J} [\hat{0}, x] \otimes [x, \hat{1}], \\
\epsilon([J]) &:= \begin{cases} 1 & \text{if } |J| = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Here $[o]$ is the isomorphism class of the one-element lattice, and $\hat{0}$ and $\hat{1}$ are the minimum and maximum elements of a lattice.

In fact, the reduced incidence Hopf algebra can be made into a combinatorial Hopf algebra after choosing an appropriate character. We define the character of the reduced incidence Hopf algebra to be the map $\zeta \colon \mathbb{C}[\mathcal{J}] \to \mathbb{C}$ defined on basis elements by $\zeta([J]) = 1$ for all $J$ and extended linearly.

These functions can likewise be defined on the free $\mathbb{C}$-module $\mathbb{C}[\mathcal{P}]$ whose basis consists of isomorphism classes of finite posets. Explicitly:

$$\begin{aligned}
\nabla([P_1] \otimes [P_2]) &:= [P_1 \sqcup P_2], \\
1_{\mathbb{C}[\mathcal{P}]} &:= \varnothing, \\
\Delta([P]) &:= \sum_{\text{ideal } I \subseteq P} [I] \otimes [P \setminus I],
\end{aligned}$$

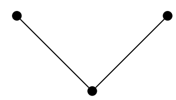
$$\epsilon([P]) := \begin{cases} 1 & \text{if } |P| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding character of $\mathbb{C}[\mathcal{P}]$ is $\zeta_{\mathcal{P}} \colon \mathbb{C}[\mathcal{P}] \to \mathbb{C}$ defined by $\zeta([P]) = 1$ for all $P$, extended linearly. These functions are all compatible with the map $J$ that sends $[P]$ to $[J(P)]$, so $J$ is a Hopf isomorphism between $\mathbb{C}[\mathcal{P}]$ and $\mathbb{C}[\mathcal{J}]$.

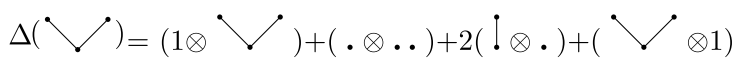
We define the *graded comultiplication* $\Delta_{k,n-k}([P])$ to be the part of $\Delta([P])$ of bidegree $(k, n-k)$, that is,

$$\Delta_{k,n-k}([P]) := \sum_{\substack{I \subseteq P \\ |I|=k}} [I] \otimes [P \setminus I].$$

**Example 2.2.7.** Let $P$ be the poset shown below.



It follows that $\Delta([P])$ is



The comultiplication allows us to work with the order ideals of a poset.

### 2.2.2 Hopf algebra of quasisymmetric functions

The ring of quasisymmetric functions QSym is also a Hopf algebra. The Hopf algebra structure of QSym has been studied extensively in [10]. For the purposes of this chapter, we will only need to consider the comultiplication. The comultiplication is defined on the fundamental quasisymmetric function basis by

$$\Delta(L_\alpha) := \sum_{\substack{(\beta,\gamma) \\ \alpha=\beta\cdot\gamma \text{ or } \beta\odot\gamma}} L_\beta \otimes L_\gamma.$$

The *graded comultiplication* $\Delta_{k,n-k}(L_\alpha)$ is given by

$$\Delta_{k,n-k}(L_\alpha) := \sum_{\substack{(\beta,\gamma) \\ \alpha=\beta\cdot\gamma \text{ or } \beta\odot\gamma \\ |\beta|=k}} L_\beta \otimes L_\gamma.$$

We omit the multiplication, unit, and counit for now. We will discuss multiplication in QSym in Section 4.4.

$$
\begin{tikzcd}
P \arrow[r, "\Delta"] \arrow[d, "K"'] & \sum I \otimes P\backslash I \arrow[d, "K \otimes K"] \\
K_P(\mathbf{x}) \arrow[r, "\Delta"'] & \sum K_I(\mathbf{x}) \otimes K_{P\backslash I}(\mathbf{x})
\end{tikzcd}
$$

Figure 2.8: A commutative diagram connecting $\mathcal{P}$ with QSym. The sums run over all order ideals $I \subseteq P$.

The map $K : \mathcal{P} \to \mathrm{QSym}$ that sends $P$ to the $P$-partition generating function $K_P(\mathbf{x})$ is the unique Hopf morphism that satisfies $\zeta_{\mathcal{P}} = \zeta_{\mathcal{Q}} \circ K$, where the character $\zeta_{\mathcal{Q}}$ for QSym is the linear function that sends $L_{(n)}$ to 1 for all $n$ and all other $L_\alpha$ to 0. It follows that the diagram in Figure 2.8 is a commutative diagram.

## 2.3 Necessary conditions

In this section, we will describe various necessary conditions for two naturally labeled posets to have the same partition generating function.

### 2.3.1 Order ideals and antichains

Let $\mathrm{anti}_{k,i,j}$ be the function that sends a poset $P$ to the number of $k$-element ideals $I$ of $P$ such that $I$ has $i$ maximal elements and $P \setminus I$ has $j$ minimal elements. This is equal to the number of rank $k$ elements of $J(P)$ that cover $i$ elements and are covered by $j$ elements. We will show that if $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ for all $k$, $i$, and $j$.

First we will need the following lemmas.

**Lemma 2.3.1.** *Let $P$ be a naturally labeled finite poset.*

*(a) If $P$ has exactly $j$ maximal elements, then there are $\binom{j-1}{k}$ linear extensions of $P$ whose descent set is $\{n-k, n-k+1, \ldots, n-1\}$.*

*(b) If $P$ has exactly $j$ minimal elements, then there are $\binom{j-1}{k}$ linear extensions of $P$ whose descent set is $\{1, 2, \ldots, k\}$.*

*Proof.* Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a linear extension of $P$, and suppose $\mathrm{des}(\sigma) = \{n-k, n-k+1, \ldots, n-1\}$. It follows that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{n-k} > \sigma_{n-k+1} > \cdots > \sigma_n.$$

This implies that $\sigma_{n-k} = n$ and $\{\sigma_{n-k+1}, \ldots, \sigma_n\}$ must be maximal elements, for if $\sigma_i$ is not maximal, then there is some $\sigma_j \succ \sigma_i$ in $P$ (with $j > i > n-k$ since $\sigma$ is a linear extension). But since $P$ is naturally labeled, this would imply $\sigma_i < \sigma_j$, which is impossible. Therefore $\{\sigma_{n-k+1}, \ldots, \sigma_n\}$ is a $k$-element subset of the maximal elements of $P$ other than $n$. There are $\binom{j-1}{k}$ such subsets, and each corresponds to a linear extension with the desired descent set.

The proof for (b) follows similarly. □

We will now show the existence of a family of linear functions that can be used to count the number of maximal or minimal elements in a poset.

**Lemma 2.3.2.** *(a) There exists a linear function* $\max_i \colon \mathrm{QSym} \to \mathbb{C}$ *satisfying*

$$\max_i(K_P(\mathbf{x})) = \begin{cases} 1 & \text{if } P \text{ has exactly } i \text{ maximal elements,} \\ 0 & \text{otherwise.} \end{cases}$$

*(b) There exists a linear function* $\min_i \colon \mathrm{QSym} \to \mathbb{C}$ *satisfying*

$$\min_i(K_P(\mathbf{x})) = \begin{cases} 1 & \text{if } P \text{ has exactly } i \text{ minimal elements,} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We claim that the following function defined on the basis $\{L_\alpha\}$ of $\mathrm{QSym}_n$, extended linearly, satisfies this condition.

$$\max_i(L_\alpha) = \begin{cases} (-1)^{(k-i+1)}\binom{k}{i-1} & \text{if } \alpha = \alpha(k) := (n-k-1) \odot 1^{k+1} \text{ for } i-1 \le k < n, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.2.11, $K_P(\mathbf{x}) = \sum_\alpha c_\alpha L_\alpha$, where $c_\alpha$ is the number of linear extensions of $P$ with descent set $D(\alpha)$. Evaluating $\max_i$ on $K_P(\mathbf{x})$, we have

$$\max_i(K_P(\mathbf{x})) = \max_i\left(\sum_\alpha c_\alpha L_\alpha\right) = \sum_\alpha c_\alpha \max_i(L_\alpha) = \sum_{k=0}^{n} c_{\alpha(k)}(-1)^{(k-i+1)}\binom{k}{i-1}.$$

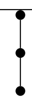
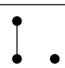
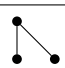
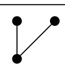
Suppose $P$ has exactly $j$ maximal elements. By Lemma 2.3.1, $c_{\alpha(k)} = \binom{j-1}{k}$ because $D(\alpha(k)) = \{n-k, n-k+1, \ldots, n-1\}$. Substituting this equality into the summation we have:

$$\max_i(K_P(\mathbf{x})) = \sum_{k=0}^{n}(-1)^{(k-i+1)}\binom{j-1}{k}\binom{k}{i-1}$$

$$
\begin{aligned}
&= \binom{j-1}{i-1} \sum_{k=0}^{n} (-1)^{(k-i+1)} \binom{j-i}{k-i+1} \\
&= \binom{j-1}{i-1} \delta_{i,j} \\
&= \delta_{i,j}.
\end{aligned}
$$

The proof of (b) follows similarly. □

**Example 2.3.3.** We now compute $\max_2$ on the partition generating functions of all 3-element posets. From Lemma 2.3.2, we have $\max_2(L_3) = 0$, $\max_2(L_{21}) = 1$, $\max_2(L_{12}) = 0$, and $\max_2(L_{111}) = -2$.

| $P$ | $K_P(\mathbf{x})$ | $\max_2(K_P(\mathbf{x}))$ |
|---|---|---|
|  | $L_3$ | $\max_2(K_P(\mathbf{x})) = 0$ |
|  | $L_3 + L_{21} + L_{12}$ | $\max_2(K_P(\mathbf{x})) = 0 + 1 + 0 = 1$ |
|  | $L_3 + L_{12}$ | $\max_2(K_P(\mathbf{x})) = 0 + 0 = 0$ |
|  | $L_3 + L_{21}$ | $\max_2(K_P(\mathbf{x})) = 0 + 1 = 1$ |
|  | $L_3 + 2L_{21} + 2L_{12} + L_{111}$ | $\max_2(K_P(\mathbf{x})) = 0 + 2 + 0 - 2 = 0$ |

We get 1 whenever $P$ has exactly 2 maximal elements and 0 otherwise.

As a simple application of Lemma 2.3.2, we can apply $\max_i$ along with the coproduct to see the following result, which will be used later as a tool to show that two posets do not have the same partition generating function.

**Corollary 2.3.4.** *Suppose that for some $k$ and $i$, $P$ has a unique ideal $I$ of size $k$ with $i$ maximal elements. Then $K_{P\backslash I}(\mathbf{x})$ can be determined from $K_P(\mathbf{x})$.*

*Proof.* The partition generating function for $P\backslash I$ is

$$K_{P\backslash I}(\mathbf{x}) = (\max_i \otimes id)\Delta_{k,n-k} K_P(\mathbf{x}).$$

□

A similar result can be stated when there is a unique filter $I$ such that $|I| = k$ and $I$ has $i$ minimal elements.

It follows from Lemma 2.3.2 that $\max_i(K_P(\mathbf{x}))$ and $\min_i(K_P(\mathbf{x}))$ can be expressed as a linear combination of the coefficients of the fundamental basis expansion of $K_P(\mathbf{x})$. Observe

$$\begin{tikzcd}
P \arrow[r, "\Delta_{k,n-k}"] \arrow[d, "K"'] & \sum P_{(1)} \otimes P_{(2)} \arrow[d, "K \otimes K"'] \arrow[dr, "\max_i^* \otimes \min_j^*"] & \\
K_P(\mathbf{x}) \arrow[r, "\Delta_{k,n-k}"] & \sum K_{P_{(1)}}(\mathbf{x}) \otimes K_{P_{(2)}}(\mathbf{x}) \arrow[r, "\max_i \otimes \min_j"'] & \mathrm{anti}_{k,i,j}(P)
\end{tikzcd}$$

Figure 2.9: The commutative diagram used in computing $\mathrm{anti}_{k,i,j}(P)$.

that if we order the compositions in lexicographic order, then the leading term in the expansion of $\max_i(K_P(\mathbf{x}))$ is $(n-i)\odot 1^i$, and the leading term in the expansion of $\min_i(K_P(\mathbf{x}))$ is $1^i \odot (n-i)$. We will now use these linear functions along with the coproduct to express $\mathrm{anti}_{k,i,j}(P)$ as a linear combination of the coefficients of the fundamental basis expansion of $K_P(\mathbf{x})$.

**Theorem 2.3.5.** *If* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$*, then* $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ *for all triples* $(k, i, j)$*.*

*Proof.* We will prove this result by finding a linear function that takes $K_P(\mathbf{x})$ to $\mathrm{anti}_{k,i,j}(P)$.

Recall that there is a Hopf morphism $K : \mathcal{P} \to \mathrm{QSym}$ that sends $P$ to $K_P(\mathbf{x})$. It follows that $K$ is compatible with comultiplication, $(K \otimes K) \circ \Delta = \Delta \circ K$, and graded comultiplication, $(K \otimes K) \circ \Delta_{k,n-k} = \Delta_{k,n-k} \circ K$.

Define $\max_i^* : \mathbb{C}[\mathcal{P}] \to \mathbb{C}$ by $\max_i^* = \max_i \circ K$. Thus $\max_i^*(P) = 1$ if $P$ has exactly $i$ maximal elements, otherwise $\max_i^*(P) = 0$. Similarly define $\min_i^* = \min_i \circ K$.

Consider the commutative diagram in Figure 2.9.

We can compute $\mathrm{anti}_{k,i,j}(P)$ by evaluating the composition of the top row of functions on $P$ as

$$\mathrm{anti}_{k,i,j}(P) = ((\max_i^* \otimes \min_j^*) \circ \Delta_{k,n-k})(P),$$

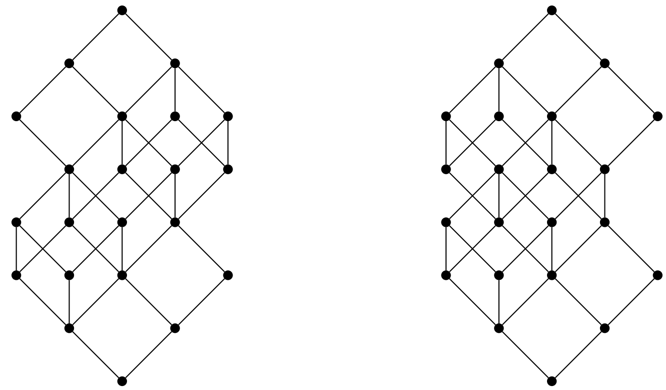
or equivalently we can compute $\mathrm{anti}_{k,i,j}(P)$ by evaluating the composition of the bottom row of functions to $K_P(\mathbf{x})$ as

$$\mathrm{anti}_{k,i,j}(P) = ((\max_i \otimes \min_j) \circ \Delta_{k,n-k})(K_P(\mathbf{x})).$$

This shows that $\mathrm{anti}_{k,i,j}(P)$ only depends on $K_P(\mathbf{x})$. Therefore if two posets, $P$ and $Q$, have the same partition generating function, then $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$. $\square$

In particular, by summing over $k$ and $j$, we arrive at the following corollary, conjectured by McNamara and Ward [18].

**Corollary 2.3.6.** *If* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$*, then* $P$ *and* $Q$ *have the same number of antichains of each size.*



$J(P)$ $\qquad$ $J(Q)$

Figure 2.10: Above are the poset of order ideals for two posets $P$ and $Q$ with the same partition generating function. Since $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$, $J(P)$ and $J(Q)$ have same number of rank $k$ elements that cover $i$ elements and are covered by $j$ elements.

The number $\mathrm{anti}_{k,i,j}(P)$ can be interpreted in terms of the poset of order ideals as $\mathrm{anti}_{k,i,j}(P)$ counts the number of rank $k$ elements of $J(P)$ that cover $i$ elements and are covered by $j$ elements. Figure 2.10 shows the poset of order ideals of distinct posets with the same partition generating function.

We just showed that $\mathrm{anti}_{k,i,j}(P)$ can be expressed as a linear combination of certain coefficients of the fundamental basis expansion of $K_P(\mathbf{x})$. In fact, $((\max_i \otimes \min_j) \circ \Delta_{k,n-k})(L_\alpha) = 0$ unless $\alpha$ is of the form $\alpha = (a) \odot 1^b \odot 1^c \odot (n-a-b-c)$, so $\mathrm{anti}_{k,i,j}(P)$ only depends on the coefficients for these compositions in $K_P(\mathbf{x})$. If we order the compositions in lexicographic order, then the leading coefficient of $\mathrm{anti}_{k,i,j}(P)$ is $c_{\alpha(k,i,j)}(P)$, where

$$\alpha(k,i,j) = (k-i) \odot 1^i \odot 1^j \odot (n-k-j).$$

One can then deduce the following result.

**Corollary 2.3.7.** *Let $c_\alpha(P)$ and $c_\alpha(Q)$ denote the coefficent of $L_\alpha$ in $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$, respectively. If* $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ *for all $k, i, j$, then $c_\alpha(P) = c_\alpha(Q)$ for all compositions $\alpha$ of the form $\alpha = (a) \odot 1^b \odot 1^c \odot (n-a-b-c)$.*

*Proof.* Let $C = \{\alpha \mid \alpha = (a) \odot 1^b \odot 1^c \odot (n-a-b-c)\}$. We showed in Theorem 2.3.5 that for all triples $(k,i,j)$, $\mathrm{anti}_{k,i,j}(P)$ can be expressed as a linear combination of $c_\alpha$ for $\alpha \in C$. Each of these $c_\alpha$ appear as the leading coefficient in the expansion of some $\mathrm{anti}_{k,i,j}(P)$. In particular, $c_{\alpha(a,b,c)}$ is the leading coefficient of the expansion for $\mathrm{anti}_{a+b,b,c}(P)$. Therefore the matrix that

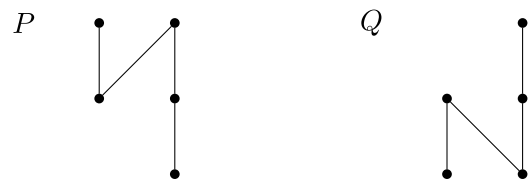



Figure 2.11: The jump sequences of $P$ and $Q$ are equal, $\mathrm{jump}(P) = \mathrm{jump}(Q) = (2, 2, 1)$, however $K_P(\mathbf{x}) \neq K_Q(\mathbf{x})$.

expresses $\mathrm{anti}_{k,i,j}(P)$ as a linear combination of $c_{\alpha(a,b,c)}$ has full rank, so the coefficient of $L_\alpha$ in $K_P(\mathbf{x})$ is determined by the values of $\mathrm{anti}_{k,i,j}(P)$ for all $\alpha \in C$. $\square$

In other words, some easily counted statistics on $J(P)$ determine a number of the coefficients in the fundamental basis expansion of $K_P(\mathbf{x})$.

### 2.3.2 Jump

Let the *jump* of an element be the maximum number of relations in a saturated chain from the element down to a minimal element. We define the *jump sequence* to be $\mathrm{jump}(P) = (j_0, \ldots, j_k)$, where $j_i$ equals the number of elements with jump $i$, and $k$ is the maximum jump of an element.

McNamara and Ward prove in [18] that if two posets have the same $P$-partition generating function, then they must have the same jump sequence. The converse, however, is not true (see Figure 2.11). The jump sequence of a naturally labeled poset can be interpreted in terms of minimal elements. Let $P_i$ denote the subposet of $P$ that consists of elements of $P$ with jump greater than or equal to $i$. Then $j_i$ is equal to the number of minimal elements of $P_i$, and $P_{i+1}$ is obtained from $P_i$ by removing its minimal elements. McNamara and Ward prove the following result.

**Lemma 2.3.8** ([18], Corollary 5.3)**.** *If $P$ and $Q$ have the same partition generating function, then so do $P_i$ and $Q_i$, the induced subposets consisting of elements of jump at least $i$.*

We prove a similar result on the $L$-support of $K_P(\mathbf{x})$.

**Lemma 2.3.9.** *If $K_P(\mathbf{x})$ and $K_Q(\mathbf{x})$ have the same $L$-support, then so do $K_{P_i}(\mathbf{x})$ and $K_{Q_i}(\mathbf{x})$, the partition generating functions for the induced subposets consisting of elements of jump at least $i$.*

*Proof.* Any linear extension of $P$ that begins with $j_0 - 1$ descents must start with the minimal elements of $P$ in descending order followed by a linear extension of $P_1$, and no linear extension can start with more descents. Thus $\alpha \in \mathrm{supp}_L(K_{P_1}(\mathbf{x}))$ if and only if $1^{j_0} \odot \alpha \in \mathrm{supp}_L(K_P(\mathbf{x}))$, where $j_0$ is the maximum value for which some such $\alpha$ exists. We can repeat this $i$ times to see that $\beta \in \mathrm{supp}_L(K_{P_i}(\mathbf{x}))$ if and only if $(1^{j_0} \odot 1^{j_1} \odot \cdots \odot 1^{j_{i-1}} \odot \beta) \in \mathrm{supp}_L(K_P(\mathbf{x}))$. $\square$

A similar proof can be used to give an alternate argument for Lemma 2.3.8.

We define the *upward jump* of an element to be the maximum number of relations in a saturated chain from the element up to a maximal element. We define the *upward jump sequence* to be up-jump$(P) = (j'_0, \ldots, j'_k)$, where $j'_i$ equals the number of elements with upward jump $i$, and $k$ is the maximum up-jump of an element. We then let the *jump pair* of an element $x$ be jumppair$(x) = (\mathrm{jump}(x), \text{up-jump}(x))$.

**Lemma 2.3.10.** *If $\mathrm{supp}_L(K_P(\mathbf{x})) = \mathrm{supp}_L(K_Q(\mathbf{x}))$, then $P$ and $Q$ have the same number of elements with jump pair $(i, j)$ for all $i$ and $j$.*

*Proof.* Let $P_{i,j}$ be the induced subposet of $P$ consisting of all elements with jump at least $i$ and up-jump at least $j$. By the previous lemma and its dual, $\mathrm{supp}_L(P_{i,j})$ is determined by $K_P(\mathbf{x})$, hence so is $|P_{i,j}|$. This implies the result since the number of elements with jump pair $(i, j)$ is $|P_{i,j}| - |P_{i+1,j}| - |P_{i,j+1}| + |P_{i+1,j+1}|$. $\square$

Another similar result is given in the following lemma.

**Lemma 2.3.11.** *If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $P$ and $Q$ have the same number of elements with principal order ideal size $i$ and up-jump $j$.*

*Proof.* Let $P_{0,j}$ be the induced subposet consisting of elements with up-jump at least $j$. From the dual of Lemma 2.3.9, the generating function for $P_{0,j}$ is determined by $K_P(\mathbf{x})$. The number of elements with principal order ideal size $i$ and up-jump $j$ in $P$ is the same as the number of maximal elements with principal order ideal size $i$ in $P_{0,j}$. The function $(\max_1 \otimes \zeta)\Delta_{i,|P_{0,j}|-i}$ evaluated on $K_{P_{0,j}}(\mathbf{x})$ gives us the number of elements in $P_{0,j}$ whose principal order ideal has $i$ elements. We can count the number of these that are maximal by evaluating $(\max_1 \otimes \zeta)\Delta_{i,|P_{0,j+1}|-i}$ on $K_{P_{0,j+1}}(\mathbf{x})$ and taking the difference. $\square$

We will use the previous four lemmas to prove the main results in Chapter 3.

## 2.4 Discussion and open questions

In this chapter, we showed that much of the order ideal structure of a poset can be determined by its partition generating function. This helps when doing a computer search to find posets with the same generating function. It is much quicker to count $\mathrm{anti}_{k,i,j}(P)$ than it is to compute the entire generating function.

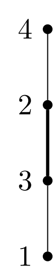
Another statistic on the structure of a poset that is of interest is the number of connected components. It is natural to ask if the number of connected components of $P$ is determined by $K_P(\mathbf{x})$. At this point in the thesis, we still do not have the tools to count the number of connected components. We will discuss this further in Chapter 4. One question to consider is stated below.

**Question 2.4.1.** Do the functions $\max_i$ and $\min_i$ tell us anything about the structure of a general labeled poset?

In Section 4.3.1, we investigate this question when $i = 1$.

In a naturally labeled poset, the minimal elements are the elements with jump 0. McNamara and Ward [18] extended the definition of jump to define the jump of an element in a labeled poset. With their definition, an element in a labeled poset can have jump 0 and not be minimal. However, every minimal element has jump 0 in the new definition. They showed that the number of elements with jump 0 can be determined from the monomial basis support of $K_{(P,\omega)}(\mathbf{x})$. Unlike in the naturally labeled case, we know that there is no linear function that can determine this number. If said function existed, it would have to be $\min_i$ for some $i$. This is because the partition generating functions of naturally labeled posets span QSym and only the minimal elements in a naturally labeled poset have jump 0. The following example shows that these functions cannot determine the number of jump 0 elements in a labeled poset.

**Example 2.4.2.** Let $(P, \omega)$ be the following chain. This labeled poset has 1 element with jump 0.




It follows that $K_{(P,\omega)}(\mathbf{x}) = L_{22}$ and $\min_i(K_{(P,\omega)}(\mathbf{x})) = 0$ for all $i$.

# Chapter 3

# Shape

In this chapter, we will assign to each poset a partition that is determined by the antichain structure of the poset. This partition is known as the shape of a poset and was introduced by Greene [9] in 1976. We will show that the shape of a poset $P$ is uniquely determined by the support of $K_P(\mathbf{x})$ in the fundamental quasisymmetric function basis. We then consider naturally labeled posets of a fixed shape and ask whether they are uniquely determined by their partition generating function. We show that every poset whose shape is two rows, hook shaped, or nearly hook shaped is uniquely determined by its partition generating function. The proofs of these results rely heavily on work done in Chapter 2.

We then show that for every partition $\lambda$ that contains either $(3,3,1)$ or $(2,2,2,2)$, there exist distinct posets of shape $\lambda$ that have the same partition generating function. We first show that the result holds when $\lambda = (3,3,1)$ or $\lambda = (2,2,2,2)$ by giving an example of a pair of posets with shape $\lambda$ that have the same partition generating function. We build off of these examples to prove the result.

It is not clear whether or not the shape of a not naturally labeled poset $(P,\omega)$ can be determined from its partition generating function, but we do know that the results in Section 3.2 do not hold for all labeled posets.

Throughout this chapter, we take $P$ to be a naturally labeled poset with ground set $[n]$, unless stated otherwise. The results from this chapter can also be found in [14].

## 3.1 Background

A famous result in the theory of posets is the theorem of Dilworth [6] which states that the maximal size of an antichain in a poset $P$ is equal to the minimal number of chains into which $P$ can be partitioned. This theorem also holds when "chain" and "antichain" are interchanged. Greene [9] gave the following generalization of Dilworth's theorem which considers the maximal size of a union of antichains in $P$.

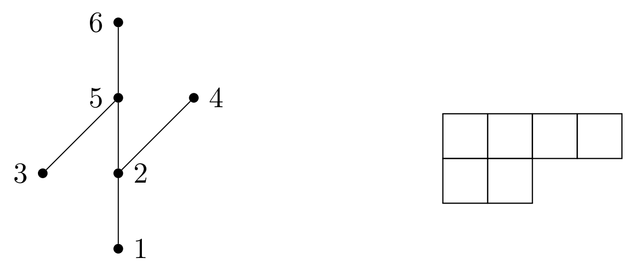
For $k = 0, 1, \ldots$, let $a_k$ (resp. $c_k$) denote the maximum cardinality of a union of $k$ antichains (resp. chains) in $P$. Let $\lambda_k = c_k - c_{k-1}$ and $\lambda'_k = a_k - a_{k-1}$ for all $k \geq 1$.

**Theorem 3.1.1** (The Duality Theorem for Finite Partially Ordered Sets, [9])**.** *For any finite poset $P$, the sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ are weakly decreasing and form conjugate partitions of the number $n = |P|$.*

**Definition 3.1.2.** The *shape* of a finite poset, denoted $\mathrm{sh}(P)$, is the partition $\lambda$ that satisfies $\lambda'_1 + \cdots + \lambda'_i = a_i$ for all $i$, where $a_i$ is the largest number of elements in a union of $i$ antichains of $P$.

To illustrate, consider the following example.

**Example 3.1.3.** The following is a poset $P$ and its shape.




We have $c_1 = 4$, $c_2 = c_3 = \cdots = 6$, implying that $\lambda = (4, 2)$. Similarly, $a_1 = 2$, $a_2 = 4$, $a_3 = 5$, $a_4 = a_5 = \cdots = 6$, implying that $\lambda' = (2, 2, 1, 1)$. These partitions are conjugates.

The *width* of a poset $P$ is the length of its longest antichain. If $\mathrm{sh}(P) = \lambda$, then the width of $P$ is $\lambda'_1$.

In the previous chapter we showed that the number of antichains of a fixed size is determined by $K_P(\mathbf{x})$. It is natural to ask if the size of a union of antichains can also be determined. We will now show that the shape of the poset $P$ is determined by $K_P(\mathbf{x})$, or more specifically, by its support in the fundamental basis.

**Theorem 3.1.4.** *If* $\mathrm{supp}_L(K_P(\mathbf{x})) = \mathrm{supp}_L(K_Q(\mathbf{x}))$*, then* $\mathrm{sh}(P) = \mathrm{sh}(Q)$*.*

*Proof.* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of $n$, and let $\mathrm{sh}(P) = \lambda$. Define $B(\alpha) = \#\{a \mid a \in D(\alpha) \text{ and } a - 1 \notin D(\alpha)\}$, that is, $B(\alpha)$ is the number of decreasing runs (with at least two elements) in a permutation with descent set $D(\alpha)$. We also define $L_i(\alpha) = i + |D(\alpha)|$.

We will prove that the shape of $P$ is determined by its support by showing that, for $i \leq \lambda_1$,

$$\lambda'_1 + \cdots + \lambda'_i = \max\{L_i(\alpha) \mid \alpha \in \mathrm{supp}_L(K_P(\mathbf{x})) \text{ and } B(\alpha) \leq i\}.$$

First, let $\beta$ be the composition for which this maximum occurs. There is a linear extension of $P$ with descent set $D(\beta)$ that has at most $i$ decreasing runs. These decreasing runs (together with possibly some single elements) correspond to $i$ antichains of $P$, and the total number of elements in the union of these antichains is $L_i(\beta)$. Since $\lambda'_1 + \cdots + \lambda'_i$ is by definition the largest number of elements in a union of $i$ antichains of $P$,

$$\max\{L_i(\alpha) \mid \alpha \in \mathrm{supp}_L(K_P(\mathbf{x})) \text{ and } B(\alpha) \leq i\} \leq \lambda'_1 + \cdots + \lambda'_i.$$

Conversely, let $A_1, A_2, \ldots, A_i$ be antichains such that $|A_1|+|A_2|+\cdots+|A_i| = \lambda'_1+\cdots+\lambda'_i$. Without loss of generality, we can take $A_j \leq A_{j+1}$ for $j = 1, \ldots i-1$, meaning that for each $y \in A_{j+1}$, there exists $x \in A_j$ with $x \preceq y$. We can do this because in the subposet $A_1 \cup A_2 \cup \cdots \cup A_i$, the longest chain has at most $i$ elements, so we can redefine $A_1$ to be the elements with jump 0 in this subposet, $A_2$ to be the elements with jump 1, and so on.

For $j = 1, \ldots, i$, let $I_j$ denote the order ideal generated by $A_1, \ldots, A_j$. Then let $B_j = I_j \backslash (A_j \cup I_{j-1})$, and let $B_{i+1} = P \setminus I_i$. There is a linear extension $\pi$ of $P$ of the form $\pi = B_1 A_1 B_2 A_2 \ldots A_i B_{i+1}$, where the entries in each $B_i$ appear in increasing order and the entries in each $A_i$ appear in decreasing order. It follows that

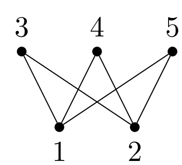
$$L_i(\mathrm{co}(\pi)) - i = |\mathrm{des}(\pi)| \geq \sum_{j=1}^{i}(|A_j| - 1) = \lambda'_1 + \cdots + \lambda'_i - i.$$

Therefore $L_i(\mathrm{co}(\pi)) \geq \lambda'_1 + \cdots + \lambda'_i$, which implies that

$$\lambda'_1 + \cdots + \lambda'_i \leq \max\{L_i(\alpha) \mid \alpha \in \mathrm{supp}_L(K_P(\mathbf{x})) \text{ and } B(\alpha) \leq i\}. \qquad \square$$

Therefore the shape of a poset $P$ is determined by the compositions that appear with a nonzero coefficient in the fundamental quasisymmetric function expansion of $K_P(\mathbf{x})$.

**Example 3.1.5.** Let $P$ be the following poset.




The partition generating function for $P$ is:

$$K_P(\mathbf{x}) = L_5 + 2L_{41} + 2L_{32} + L_{311} + L_{14} + 2L_{131} + 2L_{122} + L_{1211}.$$

It follows that the support of $K_P(\mathbf{x})$ in the fundamental basis is:

$$\operatorname{supp}_L(K_P(\mathbf{x})) = \{5, 41, 32, 311, 14, 131, 122, 1211\}.$$

From the composition 311 we see that $\lambda'_1 = 3$, and from the composition 1211 we see that $\lambda'_1 + \lambda'_2 = 5$. Therefore $\operatorname{sh}(P) = (2, 2, 1)$.

**Corollary 3.1.6.** *If* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$*, then* $\operatorname{sh}(P) = \operatorname{sh}(Q)$.

*Proof.* This result follows directly from the previous theorem. $\square$

We should note that Theorem 3.1.4 and Corollary 3.1.6 rely on the fact that we have naturally labeled posets. However, the shape of a poset is independent of the labeling. In Figure 3.1, we have four examples of labeled posets that have the same partition generating function. Each pair also has the same shape. This gives some evidence that the shape of a not naturally labeled poset may be determined by its partition generating function.

## 3.2 Uniqueness from shape

Since Theorem 3.1.4 shows that posets with the same generating function must have the same shape, one can ask for which shapes is a naturally labeled poset of that shape uniquely determined by its generating function. In other words, for which $\lambda$ do all nonisomorphic posets of shape $\lambda$ have distinct partition generating functions?

We will prove that this holds for three cases below: width two posets, hook shaped posets, and nearly hook shaped posets.

### 3.2.1 Width two posets

In this section we consider posets whose shape has at most two parts, that is to say, the width of the poset is at most two. Dilworth's theorem [6] states that if the width of $P$ is 2, then $P$ can be partitioned into 2 chains, $C_1$ and $C_2$. We will use the notation

$$P = C_1 \uplus C_2$$

to denote our choice of partition. In the case when $P$ is irreducible, the minimal elements of $C_1$ and $C_2$ are the minimal elements of $P$. We can embed $J(P)$ into $\mathbb{N}^2$ by mapping an ideal $I$ to the point $(a_1, a_2)$ where $a_i = |I \cap C_i|$. Hence when referring to $J(P)$ we will treat it as a sublattice of $\mathbb{N}^2$.

**Example 3.2.1.** The following is a width 2 poset along with its poset of order ideals embedded in $\mathbb{N}^2$.

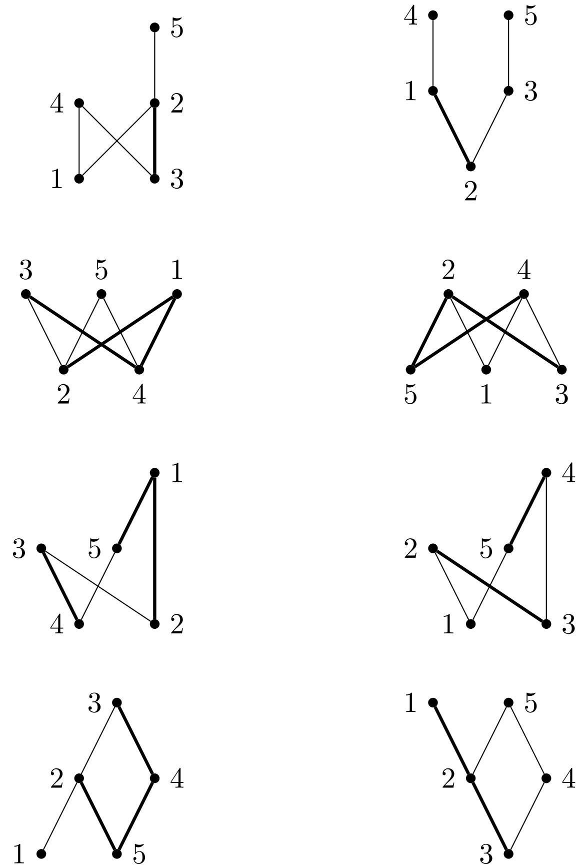



Figure 3.1: Each pair of labeled posets has the same partition generating function and shape.

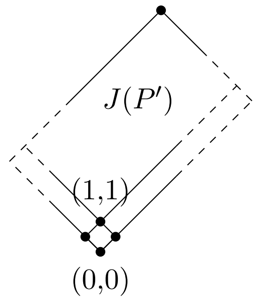
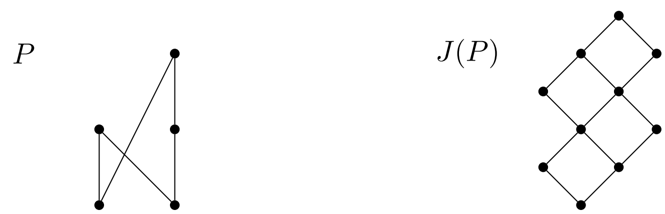



Figure 3.2: The subposet of elements greater than or equal to $(1,1)$ is isomorphic to $J(P')$.




We will show that any poset of width two is uniquely determined by its partition generating function. We will first need several useful lemmas about the structure of $P$.

**Lemma 3.2.2.** *Let $P'$ be the induced subposet of $P$ consisting of all elements that are not minimal. The generating function for $P'$ is determined by $K_P(\mathbf{x})$.*

*Proof.* This follows immediately from Lemma 2.3.8 when $i = 1$. $\square$

In the case when $P$ has width 2 and is irreducible (and hence has two minimal elements), we can explicitly find the partition generating function for $P'$ as

$$K_{P'}(\mathbf{x}) = (\max_2 \otimes id) \circ \Delta_{2,n-2} K_P(\mathbf{x}).$$

In terms of $J(P)$, the subposet of elements greater than or equal to $(1,1)$ is isomorphic to $J(P')$, see Figure 3.2.

It is not true that $P'$ must be irreducible if $P$ is irreducible, but there are some restrictions for what the ordinal sum decomposition of $P'$, or indeed of any filter of $P$, can be.

**Lemma 3.2.3.** *If $P = C_1 \uplus C_2$ is irreducible, then for all filters $F \subseteq P$, $F$ can be expressed as $F = C \oplus R$, where $C$ is a (possibly empty) chain satisfying $C \subseteq C_1$ or $C \subseteq C_2$, and $R$ is irreducible.*

*Proof.* Let $F$ be a filter of $P$. We can express $F$ as $F = C \oplus R$, where $R$ is irreducible. It remains to be shown that $C$ is a chain contained in either $C_1$ or $C_2$.

Suppose that $C$ is not a chain. This means that the width of $C$ is 2. Every element in $P \backslash F$ must be less than an element of $C$ or else the width of $P$ would be at least 3. Therefore every element of $R$ is greater than every element of $P \backslash R$ implying that $P$ is reducible. Therefore $C$ is a chain.

We conclude the proof by showing that either $C \subseteq C_1$ or $C \subseteq C_2$. Suppose the minimal element of $C$ is an element of $C_1$. If $C$ contained an element $c_2 \in C_2$, then $c_2$ would be related to all of the elements in $P$, which cannnot happen if $P$ is irreducible. $\square$

We say that an ideal $I$ of $P$ is a *chain ideal* if $I$ is a chain. If $P$ is irreducible, let $a$ and $b$ (assume $a \leq b$) be the sizes of the two maximal chain ideals of $P$. One of these chain ideals will be contained in $C_1$ and the other in $C_2$. If, say, the largest chain ideal in $C_1$ has $a$ elements, then the $(a+1)$th element of $C_1$ is the smallest element of $C_1$ greater than the minimum element of $C_2$. The value of $b$ can be described similarly.

**Lemma 3.2.4.** *Let $P = C_1 \uplus C_2$ be irreducible. Then the values of $a$ and $b$ are determined by $K_P(\mathbf{x})$, and if $a \neq b$, then there exists exactly one $(a+1)$-element chain ideal. If $I_{a+1}$ is this chain ideal and $P'' = P \setminus I_{a+1}$, then $K_{P''}(\mathbf{x})$ is determined by $K_P(\mathbf{x})$.*

*Proof.* When the width of $P$ is at most 2, $P$ has at most 2 chain ideals of any given size. Also, since $P$ is irreducible, the only way for an ideal to have exactly one minimal element is if it is a chain ideal. Since $a \leq b$, the value of $a$ is the largest number such that $P$ has two $a$-element chain ideals, and $b$ is the smallest number such that $P$ has no $(b+1)$-element chain ideals.

The number of $k$-element ideals in $P$ is counted by $\text{rank}_k(K_P(\mathbf{x})) = \sum_{i,j} \text{anti}_{k,i,j}(K_P(\mathbf{x}))$. The $k$-element chain ideals are exactly the $k$-element ideals in $P$ that do not contain both minimal elements. We can count the number of $k$-elements ideals of $P$ that contain both minimal elements by counting the number of $(k-2)$-element ideals in $P'$. This is counted by $\text{rank}_{k-2}(K_{P'}(\mathbf{x}))$. Therefore the number of $k$-element chain ideals in $P$ is counted by $\text{rank}_k(K_P(\mathbf{x})) - \text{rank}_{k-2}(K_{P'}(\mathbf{x}))$. Thus the value of $a$ is the largest number such that

$$\text{rank}_a(K_P(\mathbf{x})) - \text{rank}_{a-2}(K_{P'}(\mathbf{x})) = 2,$$

while the value of $b$ is smallest number such that

$$\text{rank}_{b+1}(K_P(\mathbf{x})) - \text{rank}_{b-1}(K_{P'}(\mathbf{x})) = 0.$$

We will now show that if $a \neq b$, then $K_{P''}(\mathbf{x})$ is determined by $K_P(\mathbf{x})$. Since there is a unique $(a+1)$-element chain ideal and $P$ is irreducible, there is a unique $(a+1)$-element ideal with exactly one minimal element. Therefore the result follows as in Corollary 2.3.4:

$$K_{P''}(\mathbf{x}) = (\min_1 \otimes id)\Delta_{a+1,n-a-1} K_P(\mathbf{x}). \qquad \square$$

Note that if the width of $P$ is at most two, then the width of any induced subposet is also at most two. In particular, the widths of $P'$ and $P''$ are both at most two.

We are now ready to prove the main result of this section.

**Theorem 3.2.5.** *If the width of $P$ is at most 2, then $P$ is uniquely determined by $K_P(\mathbf{x})$. If $P$ is irreducible, then $P$ has a unique decomposition $P = C_1 \uplus C_2$ (up to reordering).*

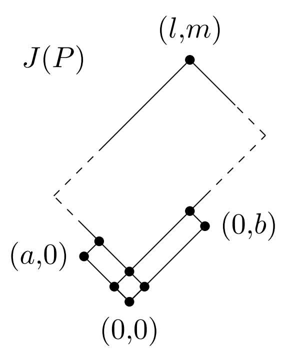
*Proof.* We prove this by induction on the size of $P$. The case when $P$ has one element is trivial. By Lemma 2.1.3, we can assume that $P$ is irreducible.

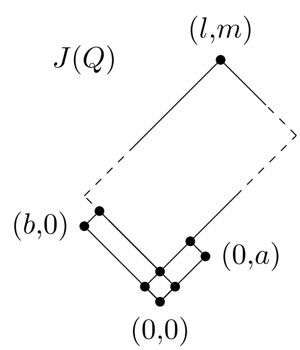
Now suppose that $P$ is an irreducible width two poset, and assume that the theorem holds for all smaller width two posets. By Lemma 3.2.2, we can determine the generating function for $P'$, and by induction, $K_{P'}(\mathbf{x})$ uniquely determines $P'$. Therefore, if $Q$ is a poset such that $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $P' \cong Q'$.

Case 1: $P'$ is irreducible.

By induction, there is a unique decomposition $P' = C_1' \uplus C_2'$ into two chains (up to reordering). Let $|C_1'| = l-1$ and $|C_2'| = m-1$. If $P = C_1 \uplus C_2$, then $C_1'$ and $C_2'$ must be obtained from $C_1$ and $C_2$ by removing their minimal elements. Since $K_P(\mathbf{x})$ determines $a$ and $b$ by Lemma 3.2.4, there are at most two possibilities for how these minimal elements can compare to the elements in the other chain, depending on whether the maximal chain ideal in $C_1$ has $a$ elements or $b$ elements. Let $P$ and $Q$ be the two posets obtained in this way, and suppose $K_P(\mathbf{x}) = K_Q(\mathbf{x})$. In terms of $J(P), J(Q) \subset \mathbb{N}^2$, the principal filter generated by $(1,1)$ in either is $J(P') \cong J(Q')$, and

$$J(P) = J(P') \cup [(0,0),(a,0)] \cup [(0,0),(0,b)],$$
$$J(Q) = J(P') \cup [(0,0),(b,0)] \cup [(0,0),(0,a)].$$



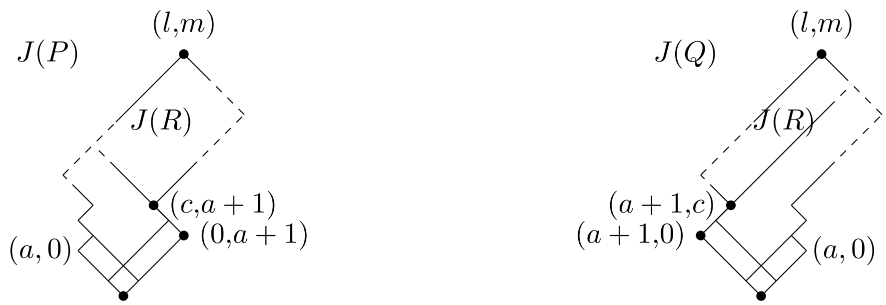




If $a = b$, then clearly $P \cong Q$. Otherwise, Lemma 3.2.4 states that we can determine the generating functions for $P''$ and $Q''$ from $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, so by induction $P'' \cong Q''$. In terms of $J(P)$ and $J(Q)$, we have that

$$J(P) \supset [(0, a+1), (l, m)] \cong [(a+1, 0), (l, m)] \subset J(Q).$$

By Lemma 3.2.3, these subposets have the form $P'' \cong Q'' = C \oplus R$ where $C$ is a (possibly empty) chain contained in one of the two chains of $P$ or $Q$, and $R$ is an irreducible width 2 poset. Let $c = |C|$. (Note $c = 0$ unless $b = a + 1$.) Since $R$ is a subposet of both $P$ and $Q$,

$$J(R) \cong [(c, a+1), (l, m)] \subset J(P),$$
$$J(R) \cong [(a+1, c), (l, m)] \subset J(Q).$$




Both of these embeddings of $J(R)$ correspond to a partition of $R$ into two chains. By induction, since $R$ is irreducible, the partition of $R$ into two chains is unique up to reordering, which corresponds to a reflection of $J(R)$.

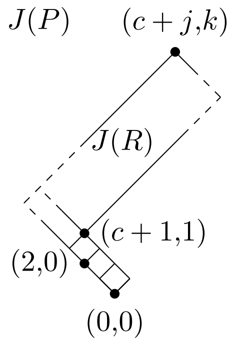
If $J(R)$ is embedded in the same way in both $J(P)$ and $J(Q)$, then $c = a + 1$. But then in $J(P') \cong J(Q')$, $(a+1, a+1)$ is the only element in its rank, contradicting irreducibility.

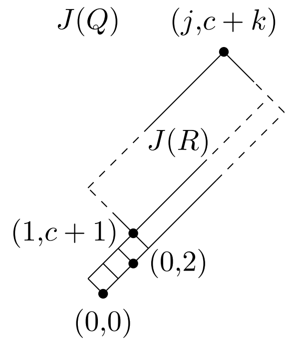
Otherwise the embeddings of $J(R)$ in $J(P)$ and $J(Q)$ are reflections of one another, that is, the isomorphism between $[(c, a+1), (l, m)] \subset J(P)$ and $[(a+1, c), (l, m)] \subset J(Q)$ must be $(x, y) \leftrightarrow (y, x)$. But this implies that $J(P')$ is symmetric, so we can extend this isomorphism to get $J(P) \cong J(Q)$. Hence $P \cong Q$, and the isomorphism corresponds to a reordering of the two chains.

Case 2: $P'$ is reducible.

By Lemma 3.2.3, $P' = C \oplus R$ where $C$ is a nonempty chain and $R$ is irreducible. Since $R$ is irreducible it can be partitioned uniquely into two chains $A$ and $B$. Suppose $|A| = j - 1$, $|B| = k - 1$, and $|C| = c \geq 1$. If $P = C_1 \uplus C_2$, with $A \subset C_1$ and $B \subset C_2$, then by Lemma 3.2.3, either $C \subset C_1$ or $C \subset C_2$. Again, by Lemma 3.2.4, $a$ and $b$ are determined. In fact, we must have $a = 1$ (the maximal chain ideal in the chain not containing $C$ can only have size 1) and $b > 1$.

There are again two possibilities, so let $P$ be the poset where $C \subset C_1$, and let $Q$ be the poset where $C \subset C_2$. The subposet $J(R)$ must be isomorphic to both of the intervals $[(c+1,1),(c+j,k)] \subset J(P)$ and $[(1,c+1),(j,c+k)] \subset J(Q)$.




By Lemma 3.2.4, we have that $J(P'') \cong J(Q'')$, that is to say,

$$J(P) \supset [(2,0),(c+j,k)] \cong [(0,2),(j,c+k)] \subset J(Q).$$

Also $P''$ and $Q''$ must be irreducible. (If they were reducible, then there would be only one rank 3 element of $J(P)$ and $J(Q)$, namely, $(2,1)$ and $(1,2)$, respectively.) This can only happen if the isomorphism is a translation or if it is a reflection.

If the isomorphism is a translation, then $c = 2$, and the translation is $(x,y) \leftrightarrow (x-2,y+2)$. However, $J(P'')$ and $J(Q'')$ are not translations of each other. We know this because $(3,1) \in J(P)$ and $P$ is irreducible, so we must also have the rank 4 element $(4,0) \in J(P'')$, but $(2,2) \notin J(Q'')$. Therefore this possibility cannot happen.

If the isomorphism is a reflection, then $j = k$, and the isomorphism is $(x,y) \leftrightarrow (y,x)$. Since

$$\begin{aligned} J(P) &= J(P'') \cup \{(0,0),(1,0),(0,1),(1,1)\}, \\ J(Q) &= J(Q'') \cup \{(0,0),(1,0),(0,1),(1,1)\}, \end{aligned}$$

the isomorphism between $J(P'')$ and $J(Q'')$ can be extended to $J(P)$ and $J(Q)$. This isomorphism corresponds to a reordering of the chains $C_1$ and $C_2$. $\square$

Theorem 3.2.5 tells us that any poset $P$ whose shape $\lambda$ has at most two parts has a unique $P$-partition generating function.

### 3.2.2 Hook shaped posets

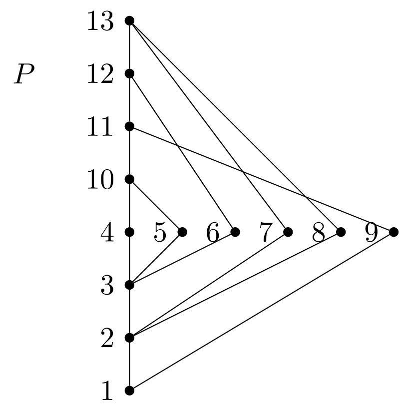
A partition $\lambda$ is said to be *hook shaped* if $\lambda_2 \leq 1$. Hook shaped partitions are therefore of the form $\lambda = (\lambda_1, 1, 1, \ldots, 1)$. In this section, we will show that a poset whose shape is a hook is determined not just by $K_P(\mathbf{x})$ but by $\mathrm{supp}_L(K_P(\mathbf{x}))$.

**Theorem 3.2.6.** *If* $\mathrm{sh}(P)$ *is hook shaped, then* $P$ *is determined by* $\mathrm{supp}_L(K_P(\mathbf{x}))$*, that is, if* $\mathrm{supp}_L(K_P(\mathbf{x})) = \mathrm{supp}_L(K_Q(\mathbf{x}))$*, then* $P \cong Q$*.*

*Proof.* If $\mathrm{sh}(P)$ is hook shaped, then $P$ can be expressed as the union of a chain $C$ and an antichain $A$ where $|C \cap A| = 1$. The jump pair of each element in $C$ is determined by its position in the chain. Each element of $A$ can cover at most one element of $C$ and is covered by at most one element of $C$. For each $a \in A$, jumppair$(a)$ is determined by the element that $a$ covers and the element that covers $a$. This implies that hook shaped posets are determined by the jump pairs of their elements. Since the multiset of jumppair$(a)$ for all $a$ is determined by $\mathrm{supp}_L(K_P(\mathbf{x}))$ by Lemma 2.3.10, it follows that if $\mathrm{sh}(P)$ is hook shaped and $\mathrm{supp}_L(K_P(\mathbf{x})) = \mathrm{supp}_L(K_Q(\mathbf{x}))$, then $P \cong Q$. ☐

**Corollary 3.2.7.** *If* $\mathrm{sh}(P)$ *is hook shaped and* $K_P(\mathbf{x}) = K_Q(\mathbf{x})$*, then* $P \cong Q$*.*

**Example 3.2.8.** Consider the following two hook shaped posets.




The partition generating functions for these posets do not have the same $L$-support because the element $9 \in P$ has jump 1 and up-jump 3, but no element in $Q$ has this jump pair.

### 3.2.3 Nearly hook shaped posets

In this section, we will show that if the shape of a poset $P$ is *nearly hook shaped*, that is, if $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$, then $P$ is uniquely determined by $K_P(\mathbf{x})$.

**Lemma 3.2.9.** *Any finite poset* $P$ *has a unique antichain* $A$ *of maximum size such that any other antichain of maximum size is contained in the order ideal* $I(A)$ *generated by* $A$*.*

*Proof.* By Dilworth's theorem, the minimum number of chains into which $P$ can be partitioned is the maximum size of an antichain of $P$. Hence in any such partition, each chain must contain one element from every antichain of maximum size. Then let $A$ consist of the largest element in each chain that is contained in some antichain of maximum size. ☐

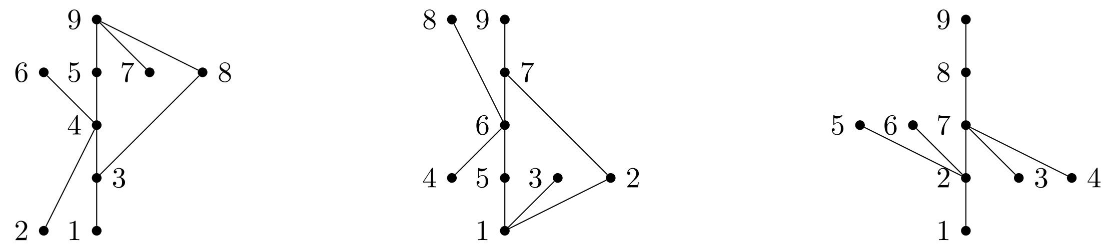



Figure 3.3: The three posets above each have shape $(5, 2, 1, 1)$. The poset on left is Type 1 with maximal antichain $\{5, 6, 7, 8\}$. The poset in the center is Type 2 with maximal antichain $\{2, 3, 4, 5\}$. The poset on the right is Type 3 with maximal antichain $\{3, 4, 5, 6\}$.

Let $A$ be the unique maximum antichain of $P$ as described above. By Theorem 2.3.5, both $|A| = \lambda_1'$ and $|I(A)| = m$ are determined by $K_P(\mathbf{x})$. Let $P^-$ be the subposet of $P$ consisting of elements less than $A$ in $P$, so that $P^- = I(A) \backslash A$. The partition generating function for $P^-$ is

$$K_{P^-}(\mathbf{x}) = (id \otimes \min_{\lambda_1'})(\Delta_{m-\lambda_1', n-(m-\lambda_1')}(K_P(\mathbf{x}))).$$

When $\lambda = \mathrm{sh}(P)$ is nearly hook shaped, $P^-$ must either be a chain, or it can be partitioned into a chain and a single element $x$. Since the width of $P^-$ is less than or equal to 2, $P^-$ is determined by $K_{P^-}(\mathbf{x})$ and hence by $K_P(\mathbf{x})$.

Similarly, let $P^+$ be the subposet of $P$ consisting of elements greater than an element of $A$ in $P$. As with $P^-$, the width of $P^+$ is less than or equal to 2, so $P^+$ is determined by $K_{P^+}(\mathbf{x})$, which is also determined by $K_P(\mathbf{x})$ by Corollary 2.3.4.

Since $\lambda$ is nearly hook shaped, it cannot be the case that both $P^-$ and $P^+$ have width two. We will say that:

(i) $P$ is Type 1 if $\mathrm{width}(P^-) = 2$ and $\mathrm{width}(P^+) \leq 1$,

(ii) $P$ is Type 2 if $\mathrm{width}(P^-) \leq 1$ and $\mathrm{width}(P^+) = 2$,

(iii) $P$ is Type 3 if $\mathrm{width}(P^-) \leq 1$ and $\mathrm{width}(P^+) \leq 1$.

Since we can determine the widths of $P^-$ and $P^+$ from $K_{P^-}(\mathbf{x})$ and $K_{P^+}(\mathbf{x})$, the type of $P$ is determined by $K_P(\mathbf{x})$. Note that the dual of a Type 2 poset is Type 1, so if we can show that Type 1 posets are determined by their $P$-partition generating functions, then Type 2 posets will be as well. (To get the generating function for the dual of a poset $P$, reverse each composition in the expansion of $K_P(\mathbf{x})$ in the $\{M_\alpha\}$-basis.) See Figure 3.3 for examples of Type 1, Type 2, and Type 3 posets.

**Lemma 3.2.10.** *If $P$ is a Type 1 poset, then $I(A)$ is determined by $K_P(\mathbf{x})$ up to isomorphism.*

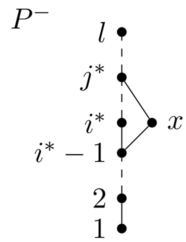



Figure 3.4: The element $x$ covers $i^*-1$ and is covered by $j^*$ in $P^-$.

*Moreover, if $x \in P^-$ does not lie in a maximum length chain of $P$, then the number of elements in $A$ that only cover $x$ is determined by $K_P(\mathbf{x})$.*

*Proof.* Suppose that $P$ is Type 1 and that the result holds for all Type 1 posets with fewer elements that $P$. Since $P$ is Type 1, a maximum length chain in $P$ intersects $P^-$ in an $l$-element chain $C$, with a single element $x \in P^-$ remaining. Label the elements of $C$ by $1, 2, \ldots, l$ from bottom to top. Suppose that $x$ covers $i^*-1$ and $x$ is covered by $j^*$. If $x$ is minimal, then we let $i^* = 1$, while if $x$ is maximal in $P^-$, then we let $j^* = l+1$. See Figure 3.4.

For each antichain $S \subseteq P^-$, define $B(S)$ to be the set of elements in $A$ that cover the elements of $S$ and no other elements. (By convention, $B(\{0\}) = B(\varnothing)$.) Determining $I(A)$ is equivalent to finding the values $b(S) = |B(S)|$ for all $S$.

Lemma 2.3.10 and Theorem 2.3.5 state that the following statistics on $P$ are determined by $K_P(\mathbf{x})$: (i) the number of elements of $P$ whose jump is $i$ for all $i$; and (ii) the number of elements of $P$ whose principal order ideal has $i+1$ elements. These statistics can be counted in the following way for $i \leq l$:

(i)

$$\#\{p \in P \mid \mathrm{jump}(p) = i\} = \begin{cases} b(\{i\}) + 1 & \text{if } i < i^* - 1, \\ b(\{i^*-1\}) + 2 & \text{if } i = i^* - 1, \\ b(\{i^*\}) + b(\{i^*, x\}) + b(\{x\}) + 1 & \text{if } i = i^*, \\ b(\{i\}) + b(\{i, x\}) + 1 & \text{if } i^* < i < j^*, \\ b(\{i\}) + 1 & \text{if } j^* \leq i \leq l. \end{cases}$$

(ii)

$$\sum_j \text{anti}_{i+1,1,j}(P) = \begin{cases} b(\{i\})+1 & \text{if } i < i^*-1, \\ b(\{i^*-1\})+2 & \text{if } i = i^*-1, \\ b(\{i^*\})+b(\{x\})+1 & \text{if } i = i^*, \\ b(\{i\})+b(\{i-1,x\})+1 & \text{if } i^* < i < j^*, \\ b(\{i-1\})+1 & \text{if } j^* \leq i \leq l. \end{cases}$$

Since these statistics are all determined by $K_P(\mathbf{x})$, we can use these to find the values of $b(\{i\})$ for all $i \neq i^*$, $b(\{i,x\})$ for all $i$, and the value of $b(\{i^*\})+b(\{x\})$. It remains to be shown that $b(\{i^*\})$ and $b(\{x\})$ can be determined by $K_P(\mathbf{x})$.

Case 1: $j^* > i^*+1$.

Let $\hat{P}$ be the poset formed by removing all elements with jump less than $i^*-1$ from $P$. By Lemma 2.3.8, $K_{\hat{P}}(\mathbf{x})$ is determined by $K_P(\mathbf{x})$. The set of minimal elements of $\hat{P}$ is $\{i^*,x\}\cup B(\{i^*-1\})$. By Theorem 2.3.5, we can determine the values of $j$ such that $\text{anti}_{1,1,j}(\hat{P}) \neq 0$. Such $j$ (counted with multiplicity $\text{anti}_{1,1,j}(\hat{P})$) are the number of minimal elements remaining when one of the minimal elements of $\hat{P}$ is removed. These values are $b(\{i^*-1\})+1$ for removing an element of $B(\{i^*-1\})$, $b(\{i^*-1\})+b(\{x\})+1$ for removing $x$, and $b(\{i^*-1\})+b(\{i^*\})+2$ for removing $i^*$. Since we have already determined the value of $b(\{i^*-1\})$, we can determine the set $\{b(\{x\}), b(\{i^*\})+1\}$. If these values are equal, then we can determine $b(\{x\})$ and $b(\{i^*\})$. We will now assume that these values are not equal.

Let $M = \max\{b(\{x\}), b(\{i^*\})+1\}$, and consider the ideal $I \subseteq \hat{P}$ that has $b(\{i^*-1\})+M+2$ elements, $b(\{i^*-1\})+M+1$ of which are maximal. It is either the case that the maximal elements of $I$ are $B(\{i^*-1\})\cup B(\{x\})\cup\{i^*\}$, or $B(\{i^*-1\})\cup B(\{i^*\})\cup\{i^*+1,x\}$. Since there is no other ideal with the same cardinality and number of maximal elements as $I$, Corollary 2.3.4 says that $K_{\hat{P}\backslash I}(\mathbf{x})$ is determined by $K_{\hat{P}}(\mathbf{x})$. Observe that $\hat{P}\backslash I$ is hook shaped, so by Theorem 3.2.6 it is uniquely determined by $K_{\hat{P}\backslash I}(\mathbf{x})$.

If the maximal elements of $I$ are $B(\{i^*-1\})\cup B(\{x\})\cup\{i^*\}$, then the length of the longest chain in $\hat{P}\backslash I$ is $\lambda_1 - i^*$. Similarly, if the maximal elements of $I$ are $B(\{i^*-1\})\cup B(\{i^*\})\cup\{i^*+1,x\}$, then the length of the longest chain in $\hat{P}\backslash I$ is $\lambda_1 - i^* - 1$. Since $\hat{P}\backslash I$ is determined, we can find the length of its longest chain, which allows us to distinguish $b(\{x\})$ and $b(\{i^*\})+1$.

Case 2: $j^* = i^*+1$.

This case follows similarly to Case 1, but the set we can determine is $\{b(\{x\}), b(\{i^*\})\}$. Since there is an automorphism of $P^-$ that switches $x$ and $i^*$, this is enough to determine $I(A)$ up to isomorphism. However, if $i^* = l$, then $x$ and $i^*$ are both maximal in $P^-$, so $x$ may not lie in a maximum length chain of $P$. In this case, we need to determine $b(\{x\})$, so assume $b(\{x\}) \neq b(\{i^*\})$, and let $M = \max\{b(\{x\}), b(\{i^*\})\}$.

Again, as in Case 1, let $\hat{P}$ be the poset of elements of $P$ of jump at least $i^*-1$. Then there is a unique ideal $I \subseteq \hat{P}$ with $b(\{i^*-1\})+M+2$ elements, $b(\{i^*-1\})+M+1$ of which are maximal: either the maximal elements are $B(\{i^*-1\}) \cup B(\{x\}) \cup \{i^*\}$ or $B(\{i^*-1\}) \cup B(\{i^*\}) \cup \{x\}$. As in Case 1, $K_{\hat{P}\setminus I}(\mathbf{x})$ is determined by $K_P(\mathbf{x})$, so we can determine the length of the longest chain in $\hat{P} \setminus I$. When $x$ does not lie in a maximum length chain of $P$, we must have $b(\{x\}) = M$ if the longest chain in $\hat{P} \setminus I$ has $\lambda_1' - i^*$ elements, while $b(\{i^*\}) = M$ if the longest chain in $\hat{P} \setminus I$ has $\lambda_1' - i^* - 1$ elements. $\square$

Note that in Lemma 3.2.10, we chose the element $i^*$ to lie in the longest chain of $P$, so up-jump$(i^*) = \lambda_1 - i^* \geq$ up-jump$(x)$ in $P$. While $x$ must be smaller than some element of $A$ (by maximality of $A$), it may be maximal in $P^-$. In this case, we need to determine the smallest element of the chain $P^+$ that is greater than $x$ (if there is one).

We use the notation $|V_a|$ to denote the cardinality of the principal filter whose minimum element is $a$.

**Lemma 3.2.11.** *If $P$ is a Type 1 poset, and $x \in P^-$ is not contained in a maximum length chain of $P$, then the smallest element of $P^+$ that is greater than $x$ is determined by $K_P(\mathbf{x})$.*

*Proof.* We will prove this by induction on the size of $P$. Suppose the statement holds for all Type 1 posets with fewer elements than $P$. If $x$ is not maximal in $P^-$, then the statement is trivial, so we will assume that $x$ is maximal in $P^-$.

Recall from Lemma 2.3.10 and Lemma 2.3.11 that the multiset of up-jump values of the elements with a fixed jump is determined by $K_P(\mathbf{x})$, as is the multiset of $|V_a|$ of the elements $a$ with a fixed jump. Using the notation of Lemma 3.2.10, this implies that we can determine from $K_P(\mathbf{x})$ the multisets

$$S_1 = \{\text{up-jump}(a) + 1 \mid a \in B(\{i^*-1\})\} \cup \{\text{up-jump}(i^*) + 1, \text{up-jump}(x) + 1\},$$
$$S_2 = \{|V_a| \mid a \in B(\{i^*-1\})\} \cup \{|V_{i^*}|, |V_x|\}.$$

We know up-jump$(i^*)+1 = \lambda_1 - i^* + 1$, and $|V_{i^*}|$ is determined by Lemma 3.2.10. Moreover, for all elements $a \in B(\{i^*-1\})$, $|V_a| =$ up-jump$(a) + 1$. Therefore, if we compare $S_1$ and $S_2$, we will be able to determine $|V_x|$ (and therefore the number of elements of $P^+$ that are greater than $x$) in all cases except when $|V_x| =$ up-jump$(x) + 1$.

In this exceptional case, the principal filter of $\{x\}$ is a chain. Then we can determine up-jump$(x)$ by considering the poset $\hat{P}$ formed by removing the maximal elements from $P$. We can determine $K_{\hat{P}}(\mathbf{x})$ from $K_P(\mathbf{x})$ (by the dual of Lemma 3.2.2), and the shape of $\hat{P}$ is either hook shaped or it is nearly hook shaped.

If sh$(\hat{P})$ is hook shaped, then $x$ is covered by exactly one element in $A$ and that element is maximal in $P$. This implies that $x$ is not related to any element in $P^+$.

If $\mathrm{sh}(\hat{P})$ is nearly hook shaped, then by induction we know the smallest element in $\hat{P}^+$ that is greater than $x$. This is the same element that $x$ is less than in $P^+$. $\square$

If every element of $P^-$ is contained in a maximum length chain of $P$, then there is an automorphism of $P^-$ switching $i^*$ and $x$. In this case, we will need a way of distinguishing the elements $i^*$ and $x$ in Lemma 3.2.10 if $b(\{i^*\}) \neq b(\{x\})$.

**Lemma 3.2.12.** *Suppose there is an automorphism of $P^-$ that switches $i^*$ and $x$, and $b(\{i^*\}) > b(\{x\})$. Then the multiset $\{\text{up-jump}(a) \mid a \in B(\{x\}) \cup B(\{i^*, x\})\}$ is determined by $K_P(\mathbf{x})$.*

*Proof.* The ideal $I$ whose maximal elements are $B(\varnothing) \cup B(\{1\}) \cup \cdots \cup B(\{i^*\}) \cup \{x\}$ has $\sum_{i=0}^{i^*} b(\{i\}) + i^* + 1$ elements, all but $i^*$ of which are maximal. Suppose there were another ideal in $P$ whose set of maximal elements $S$ had the same size. It must be the case that $S$ contains an element greater than $i^*$ and an element greater than $x$, but this would imply that the ideal has at least $i^* + 1$ elements that are not maximal.

Now from Corollary 2.3.4, $K_{P \setminus I}(\mathbf{x})$ is determined by $K_P(\mathbf{x})$. In particular, the up-jumps of the minimal elements of $P \setminus I$ are determined. Since the minimal elements are $B(\{x\}) \cup B(\{i^*, x\})$ and $i^* + 1$ if it exists (which has up-jump $\lambda_1 - i^* - 1$), the result follows. $\square$

We are now ready to prove the main theorem of this section.

**Theorem 3.2.13.** *If $\mathrm{sh}(P) = \lambda = (\lambda_1, 2, 1, 1, \ldots, 1)$ is nearly hook shaped, then $P$ is uniquely determined by $K_P(\mathbf{x})$.*

*Proof.* First we will assume that $P$ is Type 1. We will induct on the size of $P^+$. If $|P^+| = 0$, then Lemma 3.2.10 implies the result.

Now suppose the statement holds for all Type 1 posets with smaller $P^+$. Let $\hat{P}$ be $P$ with its maximal elements removed, which we can determine from $K_P(\mathbf{x})$. We will show that there is a unique way to recover $P$ from $\hat{P}$ given $K_P(\mathbf{x})$. In order to show this, we need to consider the case when $\mathrm{sh}(\hat{P})$ is hook shaped and when it is nearly hook shaped.

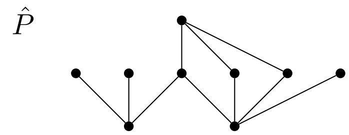
If $\mathrm{sh}(\hat{P})$ is hook shaped, then every element that covers $x$ in $A$ must be maximal in $P$. Given $P^-$ and Lemma 3.2.11, we know which element of the chain must cover and be covered by $x$ in $P$, so we can find an element in $\hat{P}$ that corresponds to $x$ in $P$. Since we know $P^-$, and Lemma 3.2.10 tells us the number of elements in $P$ that cover any ideal in $P^-$, there is a unique way to add the missing elements of $A$ to $\hat{P}$. We also add a new maximal element to the top of the chain that covers all the maximal elements of $\hat{P}$ (except possibly $x$). The only other relation that can occur in $P$ is that $x$ may also be covered by this final maximal element at the top of $P^+$, which we can again determine from Lemma 3.2.11.

If $\mathrm{sh}(\hat{P})$ is nearly hook shaped, then to get $P$ from $\hat{P}$, we must add a maximal element that covers all of the maximal elements of $\hat{P}$, then add elements to the longest antichain of $\hat{P}$ until

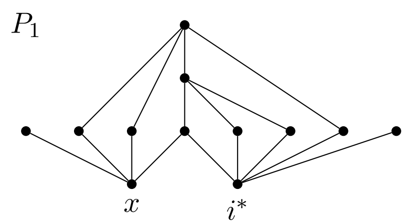
Lemma 3.2.10 is satisfied. However, there may be some ambiguity if there is an automorphism of $P^-$ that switches $i^*$ and $x$ that does not extend to an automorphism of $\hat{P}$, and $b(\{i^*\}) \neq b(\{x\})$ in $P$. In this case, the multiset of up-jump values of elements of $B(\{i^*\})$ must differ from that of $B(\{x\})$, so Lemma 3.2.12 is enough to distinguish $x$ from $i^*$.

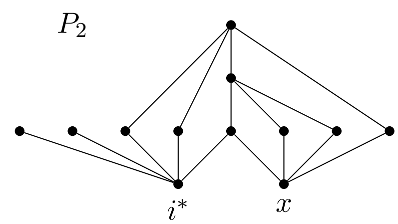
Therefore, the result holds when $P$ is Type 1, as well as for Type 2 since the dual of a Type 2 poset is Type 1. Finally, if $P$ is Type 3, then $P$ can be expressed as a union of a chain and an antichain $A$ (which do not intersect). As in the proof of Theorem 3.2.6, $P$ is then determined by the jump pairs of its elements, which can be determined from $K_P(\mathbf{x})$ by Theorem 2.3.10. □

**Example 3.2.14.** Suppose $b(\{i^*\}) = 4$ and $b(\{x\}) = 3$ and the poset $\hat{P}$ is shown below.




Note that there is an automorphism of $P^-$ that switches the two minimal elements. In order to determine $P$ from $\hat{P}$, we need to determine which of the minimal elements is $x$ and which one is $i^*$. The following two posets are both formed by adding maximal elements to $\hat{P}$, and they both satisfy $b(\{i^*\}) = 4$, $b(\{x\}) = 3$.




However, in $P_1$, the up-jump values for elements of $B(\{x\})$ are $\{0, 1, 1\}$, while in $P_2$, they are $\{1, 2, 2\}$. Thus we can distinguish these two cases by Lemma 3.2.12.

In summary, we have shown that if $\mathrm{sh}(P) = (\lambda_1, \lambda_2)$, $\mathrm{sh}(P) = (\lambda_1, 1, \ldots, 1)$ or $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$, then $P$ is uniquely determined by its $P$-partition generating function.

For most of the remaining shapes, we present a negative result in the next section.

## 3.3 Posets with the same $P$-partition generating function

In this section, we give a method for constructing distinct posets with the same partition generating function.

**Definition 3.3.1.** Suppose that $P$ and $Q$ are finite posets. If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then we say that $P$ and $Q$ are $K$-*equivalent*.

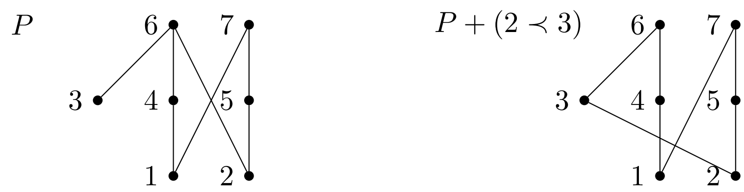



Figure 3.5: The posets $P$ (left) and $P + (2 \prec 3)$ (right).

Given a poset $P$ and a pair of incomparable elements $(x, y)$, write $P + (x \prec y)$ for the poset obtained by adding the relation $x \prec y$ to $P$ (and taking the transitive closure). See Figure 3.5 for an example.

**Lemma 3.3.2.** *Suppose $R$ is a finite poset and $\phi\colon R \to R$ is an automorphism. Let $e = (e_1, e_2)$ and $f = (f_1, f_2)$ be two pairs of incomparable elements in $R$ such that in $R + (f_2 \prec f_1)$, both $e_1 \prec e_2$ and $\phi^{-1}(e_1) \prec \phi^{-1}(e_2)$. If $m > 0$ is the smallest positive integer such that $\phi^{m+1}(e) = e$, then*

$$\begin{aligned}
P &= R + (f_1 \prec f_2) + (e_1 \prec e_2) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m-1}(e_1) \prec \phi^{m-1}(e_2)),\\
Q &= R + (f_1 \prec f_2) + (\phi(e_1) \prec \phi(e_2)) + (\phi^2(e_1) \prec \phi^2(e_2)) + \cdots + (\phi^{m}(e_1) \prec \phi^{m}(e_2)).
\end{aligned}$$

*are $K$-equivalent (assuming both are naturally labeled).*

*Proof.* Let

$$S = R + (e_1 \prec e_2) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m-1}(e_1) \prec \phi^{m-1}(e_2)).$$

Every partition of $S$ is either a partition of $P$ or a partition of $S + (f_2 \prec f_1)$ (which is not naturally labeled), so $K_S(\mathbf{x}) = K_P(\mathbf{x}) + K_{S+(f_2 \prec f_1)}(\mathbf{x})$. Solving for $K_P(\mathbf{x})$ gives

$$K_P(\mathbf{x}) = K_S(\mathbf{x}) - K_{S+(f_2 \prec f_1)}(\mathbf{x}).$$

Similarly, let

$$S' = R + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m}(e_1) \prec \phi^{m}(e_2)).$$

The partitions of $Q$ are the partitions of $S'$ with the partitions of $S' + (f_2 \prec f_1)$ removed, so

$$K_Q(\mathbf{x}) = K_{S'}(\mathbf{x}) - K_{S'+(f_2 \prec f_1)}(\mathbf{x}).$$

Observe that $S \cong S'$ since $S' = \phi(S)$, so $S$ and $S'$ are trivially $K$-equivalent.

By assumption, $(e_1 \prec e_2)$ and $(\phi^m(e_1) \prec \phi^m(e_2))$ in $R + (f_2 \prec f_1)$. It follows that

$$\begin{aligned} S + (f_2 \prec f_1) &= R + (f_2 \prec f_1) + (e_1 \prec e_2) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m-1}(e_1) \prec \phi^{m-1}(e_2)) \\ &= R + (f_2 \prec f_1) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m-1}(e_1) \prec \phi^{m-1}(e_2)) \end{aligned}$$

and

$$\begin{aligned} S' + (f_2 \prec f_1) &= R + (f_2 \prec f_1) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m}(e_1) \prec \phi^{m}(e_2)) \\ &= R + (f_2 \prec f_1) + (\phi(e_1) \prec \phi(e_2)) + \cdots + (\phi^{m-1}(e_1) \prec \phi^{m-1}(e_2)). \end{aligned}$$

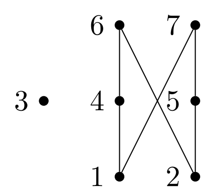
These are the same poset, so $K_{S+(f_2 \prec f_1)}(\mathbf{x}) = K_{S'+(f_2 \prec f_1)}(\mathbf{x})$. Therefore $P$ and $Q$ are $K$-equivalent. $\square$

We will now give some examples of posets that can be shown to be $K$-equivalent by using the previous lemma.

**Example 3.3.3.** Consider the following 7-element posets. These posets are not isomorphic but they are $K$-equivalent.

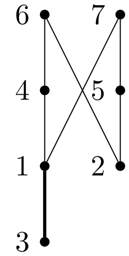

$P$ $Q$

We can express $P$ and $Q$ in terms of a subposet $R$ with a nontrivial automorphism along with some additional covering relations.




$R$

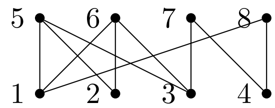
The automorphism $\phi$ is the map that fixes 3 and swaps the two chains. Let $e = (3, 6)$, $\phi(e) = (3, 7)$, and $f = (1, 3)$. Below we have the poset $R + (3 \prec 1)$.



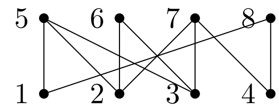

$R + (3 \prec 1)$

Since both $3 \prec 6$ and $3 \prec 7$ in $R{+}(3 \prec 1)$, it follows from Lemma 3.3.2 that $K_P(\mathbf{x}) = K_Q(\mathbf{x})$.

**Example 3.3.4.** Consider the following nonisomorphic 8-element posets.

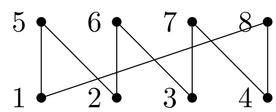



$P$



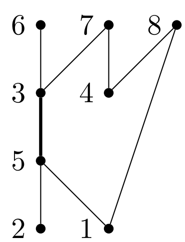

$Q$

The poset $R$ shown below has an automorphism $\phi$ given by the permutation $(1234)(5678)$.




$R$

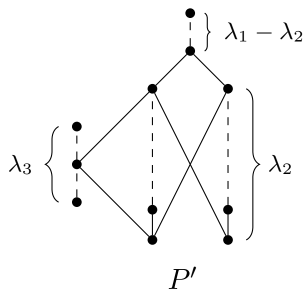
Let $e = (1, 6)$, $\phi(e) = (2, 7)$, and $f = (3, 5)$. Below we have the poset $R + (5 \prec 3)$.




$R + (5 \prec 3)$

Since both $1 \prec 6$ and $2 \prec 7$ in $R{+}(5 \prec 3)$, it follows from Lemma 3.3.2 that $K_P(\mathbf{x}) = K_Q(\mathbf{x})$.

Observe that the posets in Example 3.3.3 have shape $(3, 3, 1)$ and the posets in Example 3.3.4 have shape $(2, 2, 2, 2)$. We can generalize these examples to construct pairs of posets of any larger shape that are $K$-equivalent.



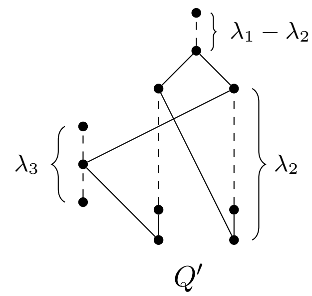




Figure 3.6: The posets $P'$ and $Q'$ from the proof of Theorem 3.3.5.

**Theorem 3.3.5.** *For all partitions $\lambda$ with $\lambda \supset (3,3,1)$ or $\lambda \supset (2,2,2,2)$, there exist posets $P$ and $Q$ such that $P \not\cong Q$, $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$, and $K_P(\mathbf{x}) = K_Q(\mathbf{x})$.*

*Proof.* We will prove this result by building off of the posets from Example 3.3.3 and Example 3.3.4. Observe that if $\mathrm{sh}(P) = \mu = (\mu_1, \dots, \mu_k)$ and $\mathrm{sh}(Q) = \nu = (\nu_1, \dots, \nu_l)$, then

$$\mathrm{sh}(P \oplus Q) = \mu + \nu = (\mu_1 + \nu_1, \mu_2 + \nu_2, \dots).$$

Also observe that

$$\mathrm{sh}(P \sqcup Q) = \mu \cup \nu = (\mu_1' + \nu_1', \mu_2' + \nu_2', \dots)'.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition that contains either $(3,3,1)$ or $(2,2,2,2)$. If $\lambda$ contains $(3,3,1)$ then we will first form two $K$-equivalent posets that have shape $(\lambda_1, \lambda_2, \lambda_3)$ and then take the disjoint union with the poset of disjoint chains of sizes $\lambda_4, \lambda_5, \dots, \lambda_k$.

Consider the following posets $P'$ and $Q'$ depicted in Figure 3.6 of shape $(\lambda_1, \lambda_2, \lambda_3)$.

Since $\lambda_2 \geq 3$, $P' \not\cong Q'$. As in Example 3.3.3, it follows from Lemma 3.3.2 that $P'$ and $Q'$ are $K$-equivalent. Now let $R$ be the poset of disjoint chains of sizes $\lambda_4, \lambda_5, \dots, \lambda_k$, and let $P = P' \sqcup R$ and $Q = Q' \sqcup R$. These posets have the desired shape $\lambda$, and since $K_{P'}(\mathbf{x}) = K_{Q'}(\mathbf{x})$, it follows that $K_P(\mathbf{x}) = K_Q(\mathbf{x})$.

Now suppose $\lambda \supset (2,2,2,2)$ but it does not contain $(3,3,1)$, so $\lambda$ has the form $\lambda = (\lambda_1, 2^j, 1^l)$. Let $C$ be a $(\lambda_1 - 2)$-element chain, and let $R$ be the poset with $j-3$ disjoint 2-element chains and $l$ disjoint single elements. Let $P_8$ and $Q_8$ be the 8-element posets from Example 3.3.4. If we let $P = (P_8 \oplus C) \sqcup R$ and $Q = (Q_8 \oplus C) \sqcup R$, then $P$ and $Q$ have the desired shape. Since $P_8$ and $Q_8$ are $K$-equivalent, $K_P(\mathbf{x}) = K_Q(\mathbf{x})$. $\square$

The only remaining shapes for which it is not known whether there exists non-isomorphic $K$-equivalent posets are those of the form $(\lambda_1, 2, 2, 1, 1, \dots, 1)$.

## 3.4 Discussion and open questions

In this chapter we asked for which partitions $\lambda$ is a naturally labeled poset of that shape uniquely determined by its generating function. We showed that if $\mathrm{sh}(P) = (\lambda_1, \lambda_2)$, $\mathrm{sh}(P) = (\lambda_1, 1, \ldots, 1)$, or if $\mathrm{sh}(P) = (\lambda_1, 2, 1, \ldots, 1)$, then $P$ is uniquely determined by $K_P(\mathbf{x})$. We then showed that for all partitions $\lambda$ that contain $(3, 3, 1)$ or $(2, 2, 2, 2)$, there exists distinct posets of shape $\lambda$ that are $K$-equivalent.

**Question 3.4.1.** Are all posets of shape $(\lambda_1, 2, 2, 1, 1, \ldots, 1)$ uniquely determined by their partition generating function?

The techniques used in the proof of Theorem 3.2.13 may be useful if there is a positive answer to Question 3.4.1. The following question considers posets like those in Example 3.3.3.

**Question 3.4.2.** What can be said about $K$-equivalent posets of shape $(\lambda_1, \lambda_2, 1)$?

If $P$ and $Q$ are $K$-equivalent posets of shape $(\lambda_1, \lambda_2, 1)$, we expect that $P$ and $Q$ must be posets that are related by Lemma 3.3.2. That is, $P$ and $Q$ must contain a subposet $R$ that has a nontrivial automorphism, and there must be a cover relation in both $P$ and $Q$ that behaves nicely with this automorphism. We believe this to be the case, because there are not many nontrivial automorphisms of posets of shape $(\lambda_1, \lambda_2, 1)$.

A final question to consider follows.

**Question 3.4.3.** Can the shape of a labeled poset $(P, \omega)$ be determined from $K_{(P,\omega)}(\mathbf{x})$?

It may be difficult to answer this question using the fundamental basis expansion of $K_{(P,\omega)}(\mathbf{x})$ because a linear extension of $(P, \omega)$ can have a descent formed by elements in a chain. It is possible that the cyclic inclusion-exclusion operation of Féray [7] will help answer this question; however, we have not considered this approach yet.

# Chapter 4

# Quasisymmetric power sums

In this chapter, we study the expansion of $K_{(P,\omega)}(\mathbf{x})$ in the type 1 quasisymmetric power sum basis $\{\psi_\alpha\}$ introduced in [3]. We begin by studying the Hopf algebra structure of QSym in this basis. We then see that the $\min_1$ function plays an important role in the expansion of $K_{(P,\omega)}(\mathbf{x})$ in the $\psi_\alpha$-basis. In Section 4.3, we express the functions $\min_1$ and $\max_1$ in terms of the basis $\{\psi_\alpha\}$ and use this to describe the action of various involutions of QSym on this basis.

The $\psi_\alpha$-basis has the advantage that multiplication can be expressed easily in this basis. We aim to answer whether one can determine the number of connected components of $P$ from $K_{(P,\omega)}(\mathbf{x})$. Recall that $K_{(P,\omega)}(\mathbf{x})$ factors into a product of the partition generating functions of the connected components of $P$. If we can show that $K_{(P,\omega)}(\mathbf{x})$ is irreducible whenever $P$ is connected, then this will imply that we can count the number of connected components, as the number of irreducible factors of $K_{(P,\omega)}(\mathbf{x})$ would equal the number of connected components. We answer this question in the naturally labeled case in Section 4.4.

We relate part of the $\psi_\alpha$-expansion of $K_P(\mathbf{x})$ to certain zigzag labelings of $P$, which exist only for connected posets. We use this to show that connected, naturally labeled posets have irreducible $P$-partition generating functions. We will see, however, that our proof does not extend to all connected labeled posets.

A poset is series-parallel if it can be built from one-element posets using ordinal sum and disjoint union operations. In Section 4.5, we use the result on irreducibility to show that naturally labeled, series-parallel posets are uniquely determined by their partition generating function. This answers the question asked by Hasebe and Tsujie in [11]. They asked this question as a way to generalize their results on rooted trees since every rooted tree is series-parallel. They showed that all naturally labeled, rooted trees are uniquely determined by their partition generating function.

We conclude by giving a (signed) combinatorial interpretation for the coefficients in the $\psi_\alpha$-expansion of $K_{(P,\omega)}(\mathbf{x})$ for any labeled poset. Our interpretation generalizes the interpretation of the naturally labeled case given by Alexandersson and Sulzgruber [2]. We will discuss this in detail in Section 4.6, but in short the interpretation they gave depends on so-called pointed $P$-partitions. In fact, our interpretation also generalizes the Murnaghan-Nakayama rule for computing the expansion of a skew Schur function in terms of the power sum symmetric functions.

The results in this chapter can also be found in [15].

## 4.1 Composition functions

The following function will be important in a number of combinatorial formulas.

**Definition 4.1.1.** Given a refinement $\alpha$ of $\beta$, let $\alpha^{(i)}$ be the composition consisting of the parts of $\alpha$ that combine to form $\beta_i$, so $\alpha^{(i)} \vDash \beta_i$. We then define

$$\pi(\alpha) = \prod_{i=1}^{l(\alpha)} \sum_{j=1}^{i} \alpha_j \qquad \text{and} \qquad \pi(\alpha, \beta) = \prod_{i=1}^{l(\beta)} \pi(\alpha^{(i)}).$$

Observe that $\pi(\alpha) = \pi(\alpha, (n))$.

**Example 4.1.2.** If $\alpha = (2, 1, 4, 2, 1)$ and $\beta = (3, 7)$, then $\alpha \preceq \beta$, and

$$\pi(\alpha, \beta) = (2 \cdot (2 + 1))(4 \cdot (4 + 2) \cdot (4 + 2 + 1)) = 1008.$$

This function has the following combinatorial interpretation. Given compositions $\alpha, \beta \vDash n$ with $\alpha \preceq \beta$, let

$$\begin{aligned} D(\alpha) &= \{s_1, s_2, \ldots, s_{l-1}\}, & \qquad 0 = s_0 < s_1 < s_2 < \cdots < s_{l-1} < s_l = n, \\ D(\beta) &= \{s_{i_1}, s_{i_2}, \ldots, s_{i_{k-1}}\}, & \qquad 0 = i_0 < i_1 < i_2 < \cdots < i_{k-1} < i_k = l. \end{aligned}$$

Let $\Pi(\alpha, \beta)$ (denoted $\mathrm{Cons}_{\alpha \preceq \beta}$ in [3]) be the set of permutations $\sigma \in S_n$ such that $\sigma_j \leq \sigma_{s_i}$ for all $s_{i_{m-1}} < j \leq s_i \leq s_{i_m}$.

**Example 4.1.3.** Let $\alpha = (1, 1, 4, 2, 1)$ and $\beta = (2, 7)$. It follows that $D(\alpha) = \{1, 2, 6, 8\}$ and $D(\beta) = \{2\}$. Then

$$\sigma = 236417589 \in \Pi(\alpha, \beta),$$

but

$$\sigma = 142738569 \notin \Pi(\alpha, \beta)$$

because $\sigma_6 > \sigma_8$.

The following result is Lemma 3.7 from [3]; we sketch a proof here for completeness.

**Lemma 4.1.4.** *For compositions $\alpha \preceq \beta$,*

$$|\Pi(\alpha, \beta)| = \frac{n!}{\pi(\alpha, \beta)}.$$

*Proof.* Choose $i_{m-1} < i \leq i_m$. For a random permutation $\sigma \in S_n$, the probability that $\sigma_{s_i} = \max_{j \in (s_{i_{m-1}}, s_i]} \sigma_j$ is

$$\frac{1}{s_i - s_{i_{m-1}}} = \frac{1}{\alpha_1^{(m)} + \cdots + \alpha_{i-i_{m-1}}^{(m)}}.$$

It is easy to check that these probabilities are independent, and taking the product over all $i$ gives $\frac{1}{\pi(\alpha,\beta)}$. $\square$

Using Lemma 4.1.4, one can prove the following identity, which we will need later.

**Lemma 4.1.5.** *Let $\alpha$ be a composition of $n$. Then*

$$\frac{n!}{\pi(\alpha^r)} = \sum_{\beta \succeq \alpha} (-1)^{l(\alpha)-l(\beta)} \frac{n!}{\pi(\alpha, \beta)}.$$

*Proof.* The left hand side counts the number of permutations $\sigma$ such that for all $i = 1, \ldots, l$, $\max\{\sigma_{s_{i-1}+1}, \ldots, \sigma_n\} = \sigma_{s_i}$. In other words, out of the last $\alpha_i + \cdots + \alpha_l$ values of $\sigma$, the largest of those values is in position $s_i$. Observe that these are the permutations in $\Pi(\alpha, \alpha)$ such that $\sigma_{s_1} > \sigma_{s_2} > \cdots > \sigma_{s_l}$.

The right hand side counts pairs $(\sigma, \beta)$ such that $\sigma \in \Pi(\alpha, \beta) \subseteq \Pi(\alpha, \alpha)$ with a sign depending on the length of $\beta$. We will describe a sign-reversing involution whose fixed points are the permutations that are counted by the left hand side. Let $j$ be the smallest positive number such that $\sigma_{s_j} < \sigma_{s_{j+1}}$. Then $\sigma \in \Pi(\alpha, \beta')$, where

$$D(\beta') = \begin{cases} D(\beta) \cup \{s_j\} & \text{if } s_j \notin D(\beta), \\ D(\beta) \setminus \{s_j\} & \text{if } s_j \in D(\beta). \end{cases}$$

In either case, $l(\beta') = l(\beta) \pm 1$, so $(\sigma, \beta) \mapsto (\sigma, \beta')$ is a sign-reversing involution. The fixed points are exactly the permutations in $\Pi(\alpha, \alpha)$ where $\sigma_{s_1} > \sigma_{s_2} > \cdots > \sigma_{s_l}$, which are counted by the left hand side. $\square$

For a composition $\alpha$, define $z_\alpha = 1^{m_1} m_1! \cdot 2^{m_2} m_2! \cdots$, with $m_i$ being the multiplicity of $i$ in $\alpha$. This number is the size of the centralizer of a group element $g \in S_n$ whose cycle type is $\alpha$.

### 4.1.1 Shuffles

An important notion when working with compositions and quasisymmetric functions is that of shuffles.

**Definition 4.1.6.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$. The *multiset of shuffles* of $\alpha$ and $\beta$ is defined by

$$\alpha \sqcup\!\sqcup \beta = \{(\gamma_{\sigma_1}, \gamma_{\sigma_2}, \ldots, \gamma_{\sigma_{k+l}}) : \sigma \in \mathrm{Sh}_{k,l}\}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{k+l})$ is the concatenation $\alpha \cdot \beta$, and $\mathrm{Sh}_{k,l}$ is the subset of permutations

$$\mathrm{Sh}_{k,l} = \{\sigma \in S_{k+l} : \sigma_1^{-1} < \sigma_2^{-1} < \cdots < \sigma_k^{-1};\ \sigma_{k+1}^{-1} < \sigma_{k+2}^{-1} < \cdots < \sigma_{k+l}^{-1}\}.$$

**Example 4.1.7.** Let $\alpha = (4, 2, 1)$ and $\beta = (3, 1)$, then the multiset of shuffles of $\alpha$ and $\beta$ is:

$$(4,2,1,3,1), (4,2,3,1,1), (4,2,3,1,1), (4,3,2,1,1), (4,3,2,1,1),$$

$$(4,3,1,2,1), (3,4,2,1,1), (3,4,2,1,1), (3,4,1,2,1), (3,1,4,2,1).$$

The shuffle operator $\sqcup\!\sqcup$ is commutative, meaning $\alpha \sqcup\!\sqcup \beta = \beta \sqcup\!\sqcup \alpha$ as multisets.

## 4.2 Type 1 Quasisymmetric power sum background

In this section, we will describe a third basis for the ring of quasisymmetric functions, the type 1 quasisymmetric power sum basis. We will first give the formal definition of the basis, as well as a way to expand the basis elements in terms of the monomial basis. We will then see that we can use this basis to refine QSym. We conclude the section by discussing the Hopf algebra structure of QSym in terms of the type 1 quasisymmetric power sum basis.

### 4.2.1 Quasisymmetric power sums

Formally, the *type 1 quasisymmetric power sum basis*, as defined in [3], is the basis $\{\Psi_\alpha\}$ of QSym that satisfies $\langle \Psi_\alpha, \boldsymbol{\Psi}_\beta \rangle = z_\alpha \delta_{\alpha,\beta}$, where $\boldsymbol{\Psi}_\beta$ is the noncommutative power sum of the first kind (introduced in [8]).

The type 1 quasisymmetric power sum basis refines the power sum symmetric functions (see Section 4.6.3) as

$$p_\lambda = \sum_{\alpha \sim \lambda} \Psi_\alpha,$$

where the sum runs over all compositions $\alpha$ that rearrange to the partition $\lambda$. We will consider the unnormalized version of the type 1 quasisymmetric power sum basis with basis elements

$\{\psi_\alpha\}$, given by $\psi_\alpha = \frac{\Psi_\alpha}{z_\alpha}$. From now on, when we refer to the type 1 quasisymmetric power sum basis, we are referring to the unnormalized version $\psi_\alpha$ unless stated otherwise.

We can express $\psi_\alpha$ in terms of the monomial basis:

$$\psi_\alpha = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha, \beta)} M_\beta,$$

where the sum runs over all coarsenings $\beta$ of $\alpha$. This was proven in [3], but for the purposes of this thesis, this can be taken as the definition of $\psi_\alpha$.

If $f = \sum_\alpha c_\alpha \psi_\alpha$, then we say that the *$\psi$-support of $f$* is the set of compostions $\alpha$ such that $c_\alpha \neq 0$.

For more information on the type 1 quasisymmetric power sum basis, see [2, 3].

### 4.2.2 Length

The type 1 quasisymmetric power sum basis has the following multiplicative property as shown in [3]:

$$\psi_\alpha \psi_\beta = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} \psi_\gamma.$$

Every composition $\gamma$ in the $\psi$-support of $\psi_\alpha \psi_\beta$ satisfies $l(\gamma) = l(\alpha) + l(\beta)$. We can then use the $\psi$-basis to refine $\mathrm{QSym}_n$:

$$\mathrm{QSym}_n = \bigoplus_{\lambda \vdash n} \mathrm{QSym}_\lambda,$$

where $\mathrm{QSym}_\lambda$ is spanned by $\{\psi_\alpha \colon \alpha \sim \lambda\}$. Observe that if $f \in \mathrm{QSym}_\lambda$ and $g \in \mathrm{QSym}_\mu$, then $f \cdot g \in \mathrm{QSym}_\nu$, where $\nu$ is the partition formed by combining and rearranging the parts of $\lambda$ and $\mu$.

We define $\mathrm{QSym}_{n,m}$ to be

$$\mathrm{QSym}_{n,m} := \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) = m}} \mathrm{QSym}_\lambda.$$

If $f \in \mathrm{QSym}_{n,m}$, then we say the *length* of $f$ is $m$. This gives a grading for $\mathrm{QSym}_n$,

$$\mathrm{QSym}_n = \bigoplus_{m \geq 0} \mathrm{QSym}_{n,m}.$$

If $f \in \mathrm{QSym}_{n_1, m_1}$ and $g \in \mathrm{QSym}_{n_2, m_2}$, then $f \cdot g \in \mathrm{QSym}_{n_1+n_2, m_1+m_2}$.

### 4.2.3 Hopf Algebra

Recall that the ring of quasisymmetric functions is a Hopf algebra; in particular, it is equipped with a coproduct. The coproduct on the type 1 quasisymmetric power sum basis is defined as follows:

$$\Delta(\psi_\alpha) = \sum_{\beta \cdot \gamma = \alpha} \psi_\beta \otimes \psi_\gamma.$$

This follows from the results of [3].

We define the *graded comultiplication* $\Delta_\alpha(\psi_\beta)$ to be

$$\Delta_\alpha(\psi_\beta) := \sum_{\substack{\gamma^{(1)} \cdots \gamma^{(l)} = \beta \\ \gamma^{(i)} \vDash \alpha_i}} \psi_{\gamma^{(1)}} \otimes \cdots \otimes \psi_{\gamma^{(l)}},$$

where each $\gamma^{(i)}$ is a composition of $\alpha_i$. (Therefore $\Delta_\alpha(\psi_\beta) = 0$ unless $\alpha \succeq \beta$.) In other words, the graded coproduct gives one graded component of the iterated coproduct.

Applying the comultiplication to a $(P, \omega)$-partition generating function gives

$$\Delta_\alpha(K_{(P,\omega)}(\mathbf{x})) = \sum K_{(P_1,\omega)}(\mathbf{x}) \otimes \cdots \otimes K_{(P_l,\omega)}(\mathbf{x}),$$

where for each $i$, $|P_i| = \alpha_i$; $P_1, P_2, \ldots, P_l$ partition $P$; and $P_1 \cup \cdots \cup P_i$ is an order ideal of $P$ (see, for instance [10]). Note that we abuse notation slightly by writing $K_{(P_i,\omega)}(\mathbf{x})$ since the $\omega$ that appears in $K_{(P_i,\omega)}(\mathbf{x})$ is actually the restriction of $\omega$ to $P_i$.

## 4.3 Operations on QSym

In this section, we will express some useful linear functionals in terms of the type 1 quasisymmetric power sum basis. We will then see how some well known automorphisms of QSym act on this basis.

### 4.3.1 The $\min_1$ and $\max_1$ functionals on QSym

In Section 2.3, we showed the existence of linear functionals $\min_1$ and $\max_1$ on QSym that satisfy

$$\min_1(K_P(\mathbf{x})) = \begin{cases} 1 & \text{if } P \text{ has exactly 1 minimal elements,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\max_1(K_P(\mathbf{x})) = \begin{cases} 1 & \text{if } P \text{ has exactly 1 maximal elements,} \\ 0 & \text{otherwise.} \end{cases}$$

whenever $P$ is a naturally labeled poset. In this chapter, to simplify notation, we denote $\min_1$ by $\eta$ and $\max_1$ by $\tilde{\eta}$. On the fundamental quasisymmetric function basis, these functions act as follows:

$$\eta(L_\alpha) = \begin{cases} (-1)^k & \text{if } \alpha = (1^k, n-k) \text{ for } 0 \leq k < n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\eta}(L_\alpha) = \begin{cases} (-1)^k & \text{if } \alpha = (n-k, 1^k) \text{ for } 0 \leq k < n, \\ 0 & \text{otherwise.} \end{cases}$$

These functions can do more than just determine if a poset has exactly one minimal element or one maximal element: they can be used to test if a quasisymmetric function is irreducible.

**Lemma 4.3.1.** *For all non-constant homogeneous $f, g \in \mathrm{QSym}$, $\eta(f \cdot g) = \tilde{\eta}(f \cdot g) = 0$.*

*Proof.* The $P$-partition generating functions of naturally labeled posets span QSym [25]. Therefore we can express $f$ and $g$ as a linear combination of these partition generating functions. The product of the partition generating functions of any two naturally labeled posets gets sent to 0 by $\eta$ since no disconnected poset has exactly 1 minimal element.

A similar proof shows that $\tilde{\eta}(f \cdot g) = 0$.

(One can also easily prove this result using the fundamental basis.) $\square$

Note that if $f$ is a (homogeneous, non-constant) quasisymmetric function and $\tilde{\eta}(f) \neq 0$ (or similarly if $\eta(f) \neq 0$), then Lemma 4.3.1 tells us that $f$ is irreducible (and in fact does not lie in the span of homogeneous reducible elements of QSym).

It is straightforward to evaluate $\eta$ and $\tilde{\eta}$ on the monomial basis.

**Lemma 4.3.2.** *On the monomial basis $\{M_\alpha\}$,*

$$\tilde{\eta}(M_\alpha) = (-1)^{l(\alpha)-1}\alpha_1 \quad \text{and} \quad \eta(M_\alpha) = (-1)^{l(\alpha)-1}\alpha_{l(\alpha)}.$$

*Proof.* Expanding $M_\alpha$ in the $\{L_\alpha\}$ basis and applying $\tilde{\eta}$ gives

$$\tilde{\eta}(M_\alpha) = \sum_{\beta \preceq \alpha} (-1)^{l(\beta)-l(\alpha)}\, \tilde{\eta}(L_\beta).$$

Since $\tilde{\eta}(L_\beta) = 0$ unless $\beta = (n-k, 1^k)$ we have

$$\tilde{\eta}(M_\alpha) = \sum_{i=0}^{\alpha_1 - 1} (-1)^{n-\alpha_1+i+1-l(\alpha)}\, \tilde{\eta}(L_{(\alpha_1 - i, 1^{n-\alpha_1+i})}).$$

But $\tilde{\eta}(L_{(\alpha_1-i,1^{n-\alpha_1+i})}) = (-1)^{n-\alpha_1+i}$, therefore $\tilde{\eta}(M_\alpha) = (-1)^{l(\alpha)-1}\alpha_1$ as desired.

A similar argument shows that $\eta(M_\alpha) = (-1)^{l(\alpha)-1}\alpha_{l(\alpha)}$. $\square$

We can now evaluate the functions $\tilde{\eta}$ and $\eta$ on the type 1 quasisymmetric power sum basis.

**Lemma 4.3.3.** *On the type 1 quasisymmetric power sum basis $\{\psi_\alpha\}$,*

$$\tilde{\eta}(\psi_\alpha) = \sum_{i=1}^{l(\alpha)} \frac{(-1)^{l(\alpha)-i}}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi((\alpha_{i+1} \cdots \alpha_{l(\alpha)})^r)},$$

*and*

$$\eta(\psi_\alpha) = \begin{cases} 1 & \text{if } \alpha = (n), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We first calculate $\tilde{\eta}(\psi_\alpha)$. Expanding $\psi_\alpha$ in the monomial basis and applying $\tilde{\eta}$ using Lemma 4.3.2 gives

$$\tilde{\eta}(\psi_\alpha) = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha,\beta)}\, \tilde{\eta}(M_\beta) = \sum_{\beta \succeq \alpha} \frac{(-1)^{l(\beta)-1}\beta_1}{\pi(\alpha,\beta)}.$$

Let $l(\alpha) = l$. For each $\beta$ that coarsens $\alpha$, $\beta_1 = \alpha_1 + \cdots + \alpha_i$ for some value of $i$, and the composition $\gamma = (\beta_2, \ldots, \beta_{l(\beta)})$ coarsens $\alpha^{(i)} := (\alpha_{i+1}, \ldots, \alpha_l)$. We can then group together the compositions $\beta$ by their first component:

$$\begin{aligned} \tilde{\eta}(\psi_\alpha) &= \sum_{i=1}^{l} \sum_{\gamma \succeq \alpha^{(i)}} \frac{(-1)^{l(\gamma)}(\alpha_1 + \cdots + \alpha_i)}{\pi(\alpha, (\alpha_1 + \cdots + \alpha_i) \cdot \gamma)} \\ &= \sum_{i=1}^{l} \sum_{\gamma \succeq \alpha^{(i)}} \frac{(-1)^{l(\gamma)}}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi(\alpha^{(i)}, \gamma)} \end{aligned}$$

By Lemma 4.1.5, we have that, for all $i$,

$$\begin{aligned} \sum_{\gamma \succeq \alpha^{(i)}} \frac{(-1)^{l(\gamma)}}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi(\alpha^{(i)}, \gamma)} &= \frac{(-1)^{l(\alpha^{(i)})}}{\pi(\alpha_1 \cdots \alpha_{i-1})} \sum_{\gamma \succeq \alpha^{(i)}} \frac{(-1)^{l(\alpha^{(i)})-l(\gamma)}}{\pi(\alpha^{(i)}, \gamma)} \\ &= \frac{(-1)^{l-i}}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi((\alpha_{i+1} \cdots \alpha_l)^r)} \end{aligned}$$

as desired.

We will now show that $\eta(\psi_\alpha) = 0$ unless $\alpha = (n)$, in which case $\eta(\psi_{(n)}) = 1$. As before, we will express $\psi_\alpha$ in terms of the monomial basis, and evaluate $\eta$ on both sides. This gives

$$\eta(\psi_\alpha) = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha,\beta)}\, \eta(M_\beta) = \sum_{\beta \succeq \alpha} \frac{(-1)^{l(\beta)-1}\beta_{l(\beta)}}{\pi(\alpha,\beta)}.$$

When $l(\alpha) > 1$, for each $\gamma$ that coarsens $(\alpha_1, \dots, \alpha_{l-1})$, both $\beta = \gamma \cdot (\alpha_l)$ and $\beta' = \gamma \odot (\alpha_l)$ are coarsenings of $\alpha$. Therefore

$$\sum_{\beta \succeq \alpha} \frac{(-1)^{l(\beta)-1} \beta_{l(\beta)}}{\pi(\alpha, \beta)} = \sum_{\gamma \succeq (\alpha_1, \dots, \alpha_{l-1})} \left( \frac{(-1)^{l(\gamma)} \alpha_l}{\pi(\alpha, \gamma \cdot (\alpha_l))} + \frac{(-1)^{l(\gamma)-1}(\gamma_{l(\gamma)} + \alpha_l)}{\pi(\alpha, \gamma \odot (\alpha_l))} \right).$$

But for all $\gamma$,

$$\pi(\alpha, \gamma \cdot (\alpha_l)) = \pi((\alpha_1, \dots, \alpha_{l-1}), \gamma) \cdot \alpha_l,$$

and

$$\pi(\alpha, \gamma \odot (\alpha_l)) = \pi((\alpha_1, \dots \alpha_{l-1}), \gamma) \cdot (\gamma_{l(\gamma)} + \alpha_l).$$

Therefore each term in the sum vanishes, so $\eta(\psi_\alpha) = 0$ when $l(\alpha) > 1$. When $\alpha = (n)$, we have

$$\eta(\psi_{(n)}) = \frac{n}{\pi((n), (n))} = 1. \qquad \square$$

**Example 4.3.4.** Let $\alpha = 2314$. We have

$$\tilde{\eta}(\psi_{2314}) = \frac{(-1)^3}{(4 \cdot 5 \cdot 8)} + \frac{(-1)^2}{(2)(4 \cdot 5)} + \frac{(-1)^1}{(2 \cdot 5)(4)} + \frac{(-1)^0}{(2 \cdot 5 \cdot 6)} = \frac{1}{96}.$$

Due to the simplicity of the behavior of $\eta$ on $\psi_\alpha$, we can now compute the coefficients in the $\psi_\alpha$-expansion of any quasisymmetric function.

**Theorem 4.3.5.** *Suppose* $f \in \mathrm{QSym}$ *and* $f = \sum_\alpha c_\alpha \psi_\alpha$. *Then*

$$c_\alpha = \eta^{\otimes l(\alpha)}(\Delta_\alpha f).$$

*Proof.* Since $\eta$ is linear, we can prove this by showing that $\eta^{\otimes l(\alpha)}(\Delta_\alpha \psi_\beta) = \delta_{\alpha,\beta}$. Recall that

$$\Delta_\alpha(\psi_\beta) = \sum_{\substack{\gamma^{(1)} \dots \gamma^{(l)} = \beta \\ |\gamma^{(i)}| = \alpha_i}} \psi_{\gamma^{(1)}} \otimes \dots \otimes \psi_{\gamma^{(l)}}$$

and $\eta(\psi_\alpha) = \delta_{l(\alpha),1}$. The only way compositions with length 1 can concatenate to $\beta$ is if the compositions are $(\beta_1)$, $(\beta_2), \dots$. Since $\beta_i = \alpha_i$ for all $i$, it follows that $\eta^{\otimes l(\alpha)}(\Delta_\alpha \psi_\beta) = \delta_{\alpha,\beta}$. $\square$

It should be noted that Theorem 4.3.5 follows immediately from the combinatorial description of the coefficient of $\psi_\alpha$ in $K_P(\mathbf{x})$ given by Alexandersson and Sulzgruber [2]. Indeed, consider the following definition of a pointed $P$-partition.

**Definition 4.3.6.** Let $P$ be a naturally labeled poset. A $P$-partition $\theta$ is *pointed* if $\theta$ is surjective onto $[k]$ for some $k$, and $\theta^{-1}(i)$ has a unique minimal element for all $i \in [k]$.

It is shown in [2] that the coefficient of $\psi_\alpha$ in $K_P(\mathbf{x})$ is the number of pointed $P$-partitions with weight $\alpha$. This is the same as evaluating $\eta$ on each factor in the graded coproduct of $K_P(\mathbf{x})$. We will state this as a corollary to Theorem 4.3.5.

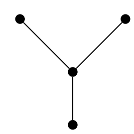
**Corollary 4.3.7** ([2], Theorem 5.4)**.** *Let $P$ be a naturally labeled poset. Then*

$$K_P(\mathbf{x}) = \sum_{\theta} \psi_{\mathrm{wt}(\theta)},$$

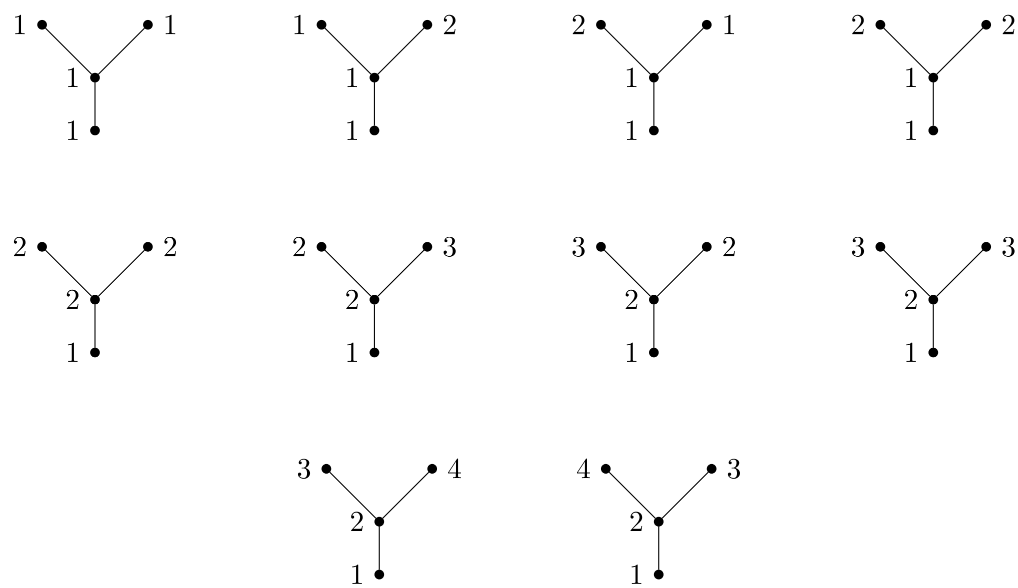
*where the sum runs over all pointed $P$-partitions $\theta$.*

*Proof.* Suppose $K_P(\mathbf{x}) = \sum_\alpha c_\alpha \psi_\alpha$. By Theorem 4.3.5, $c_\alpha = \eta^{\otimes l(\alpha)}(\Delta_\alpha K_P(\mathbf{x}))$. Recall that $\eta(K_P(\mathbf{x})) = 1$ if $P$ has exactly one minimal element and $\eta(K_P(\mathbf{x})) = 0$ otherwise. This means that $\eta^{\otimes l(\alpha)}(\Delta_\alpha K_P(\mathbf{x}))$ is the number of ways to partition $P$ into $P_1, P_2, \ldots, P_{l(\alpha)}$ where for all $i$, $|P_i| = \alpha_i$, $P_1 \cup \cdots \cup P_i$ is an order ideal of $P$, and $P_i$ has exactly one minimal element. This is exactly the number of pointed $P$-partitions with weight $\alpha$. $\square$

**Example 4.3.8.** Let $P$ be the following poset.



The $P$-partitions that are surjective onto some $[k]$ are:



It follows that the monomial basis expansion of $K_P(\mathbf{x})$ is

$$K_P(\mathbf{x}) = M_4 + 2M_{31} + M_{22} + M_{13} + 2M_{121} + M_{112} + 2M_{1111}.$$

However, the rightmost $P$-partitions in the first and second row are not pointed $P$-partitions, so they do not contribute to the type 1 quasisymmetric power sum basis expansion of $K_P(\mathbf{x})$. Therefore the expansion of $K_P(\mathbf{x})$ in the $\psi_\alpha$-basis is

$$K_P(\mathbf{x}) = \psi_4 + 2\psi_{31} + \psi_{13} + 2\psi_{121} + 2\psi_{1111}.$$

In Section 4.6 we will extend this result to all labeled posets.

### 4.3.2 Automorphisms

For this section, suppose that $(P, w)$ is a labeled poset. Let $P^*$ be the poset formed by reversing the relations of $P$, that is $x \preceq_P y \iff y \preceq_{P^*} x$. Let $w^*$ be the labeling of $P$ or $P^*$ defined by $w^*(x) = n - w(x) + 1$.

We will consider three well known automorphisms of QSym: $\omega$, $\rho$, and $\psi$. These automorphisms act as follows on the fundamental basis $\{L_\alpha\}$:

$$\begin{aligned} \omega(L_\alpha) &= L_{(\alpha^c)^r}, \\ \rho(L_\alpha) &= L_{\alpha^r}, \\ \psi(L_\alpha) &= L_{\alpha^c}. \end{aligned}$$

Here $\alpha^c$ is the composition such that $D(\alpha)$ and $D(\alpha^c)$ are complementary subsets of $[n-1]$.

These automorphisms perform the following actions on $K_{(P,\omega)}(\mathbf{x})$:

$$\begin{aligned} \omega(K_{(P,w)}(\mathbf{x})) &= K_{(P^*,w)}(\mathbf{x}), \\ \rho(K_{(P,w)}(\mathbf{x})) &= K_{(P^*,w^*)}(\mathbf{x}), \\ \psi(K_{(P,w)}(\mathbf{x})) &= K_{(P,w^*)}(\mathbf{x}). \end{aligned}$$

Informally, in terms of the Hasse diagram of $P$, $\rho$ flips $P$ upside down, $\psi$ switches natural edges with strict edges, and $\omega$ does both. This is shown in Figure 4.1. Each of these automorphism can be expressed as the composition of the other two. Since we are using the letter $\psi$ for basis elements, to avoid confusion, we will use the notation $\omega\rho$ for $\psi$. For more on these automorphisms, see Section 3.6 in [16].

The authors of [3] show that $\omega(\psi_\alpha) = (-1)^{|\alpha|-l(\alpha)}\psi_{\alpha^r}$ (though they do not give an expansion for the other two automorphisms). This result is easy to deduce given the earlier results in this

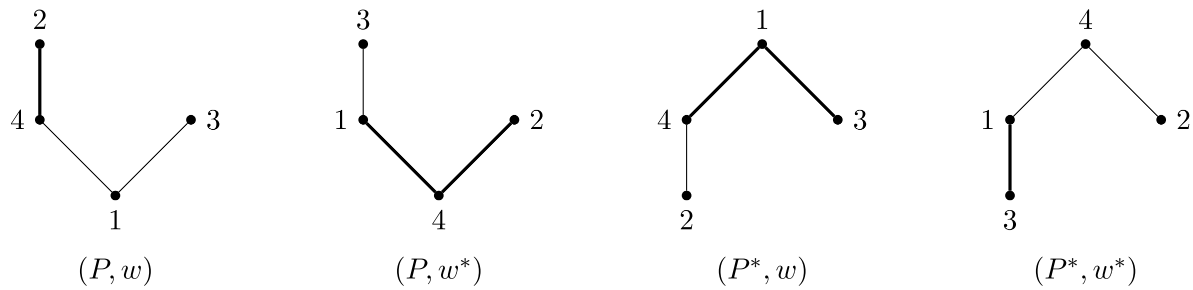



Figure 4.1: The Hasse diagrams for $(P, w)$, $(P, w^*)$, $(P^*, w)$, and $(P^*, w^*)$. Their partition generating functions are related as follows: $\omega(K_{(P,w)}(\mathbf{x})) = K_{(P^*,w)}(\mathbf{x})$, $\rho(K_{(P,w)}(\mathbf{x})) = K_{(P^*,w^*)}(\mathbf{x})$, $\psi(K_{(P,w)}(\mathbf{x})) = K_{(P,w^*)}(\mathbf{x})$.

section.

**Theorem 4.3.9** ([3])**.** *For any composition $\alpha$, $\omega(\psi_\alpha) = (-1)^{|\alpha|-l(\alpha)}\psi_{\alpha^r}$.*

*Proof.* If $\alpha = (1^k, n-k)$, then $(\alpha^c)^r = (1^{n-k-1}, k+1)$. Hence, for all compositions $\alpha$, $\eta\circ\omega(L_\alpha) = (-1)^{|\alpha|-1}\,\eta(L_\alpha)$. In other words, $\eta\circ\omega$ acts like $(-1)^{n-1}\,\eta$ on $\mathrm{QSym}_n$. By Theorem 4.3.5, the coefficient of $\psi_\beta$ in $\omega(\psi_\alpha)$ is

$$\eta^{\otimes l(\beta)}\,\Delta_\beta(\omega(\psi_\alpha)) = (\eta\circ\omega)^{\otimes l(\beta)}\Delta_{\beta^r}(\psi_\alpha).$$

This is only nonzero if $\beta = \alpha^r$, in which case it equals $\prod_i (-1)^{|\alpha_i|-1} = (-1)^{|\alpha|-l(\alpha)}$. □

Since $\eta(L_\alpha) = \tilde{\eta}(L_{\alpha^r})$, it follows that $\eta\circ\rho = \tilde{\eta}$. This allows us to give an expansion for $\omega\rho(\psi_\alpha)$ in the type 1 quasisymmetric power sum basis.

**Theorem 4.3.10.** *Let $\rho(\psi_\alpha) = \sum_\beta c_\beta\psi_\beta$. Then*

$$c_\beta = \tilde{\eta}^{\otimes l(\beta)}\,\Delta_{\beta^r}(\psi_\alpha).$$

*Proof.* Using the fact that $\eta\circ\rho = \tilde{\eta}$, we have from Theorem 4.3.5 that

$$c_\beta = \eta^{\otimes l(\beta)}\,\Delta_\beta(\rho(\psi_\alpha)) = (\eta\circ\rho)^{\otimes l(\beta)}\Delta_{\beta^r}(\psi_\alpha) = \tilde{\eta}^{\otimes l(\beta)}\,\Delta_{\beta^r}(\psi_\alpha).$$ □

Combining these results immediately gives the following theorem.

**Theorem 4.3.11.** *Let $\omega\rho(\psi_\alpha) = \sum_\beta c_\beta\psi_\beta$. Then*

$$c_\beta = (-1)^{|\beta|-l(\beta)}\,\tilde{\eta}^{\otimes l(\beta)}\,\Delta_\beta(\psi_\alpha).$$

In particular, $\psi_\beta$ can only appear in the expansion of $\rho(\psi_\alpha)$ if $\beta^r$ coarsens $\alpha$, and likewise $\psi_\beta$ can only appear in the expansion of $\omega\rho(\psi_\alpha)$ if $\beta$ coarsens $\alpha$. Of course, these coefficients can be computed explicitly using Lemma 4.3.3.

**Example 4.3.12.** By the previous theorem, the expansion of $\omega\rho(\psi_{3421})$ in the $\psi_\alpha$ basis is

$$\omega\rho(\psi_{3421}) = \psi_{3421} + \frac{1}{2}\psi_{343} + \frac{1}{4}\psi_{361} + \frac{1}{8}\psi_{37} - \frac{1}{12}\psi_{721} - \frac{1}{24}\psi_{73} - \frac{1}{28}\psi_{91} - \frac{4}{189}\psi_{10}.$$

## 4.4 Irreducibility of $K_P(\mathbf{x})$

In this section, we will restrict our attention to the case when $P$ is naturally labeled. We will show that in this case, $K_P(\mathbf{x})$ is irreducible whenever $P$ is connected, which partially answers a question from [18].

As a remark, it was shown in [13] that a homogeneous quasisymmetric function is reducible in QSym if and only if it is reducible in the ring $\mathbb{C}[[x_1, x_2, \dots]]$ of formal power series of bounded degree.

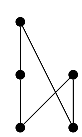
### 4.4.1 Minimal length

Let $\tau_m \colon \mathrm{QSym}_n \to \mathrm{QSym}_{n,m}$ be the projection map onto the length $m$ part; in other words,

$$\tau_m(\psi_\alpha) = \begin{cases} \psi_\alpha & \text{if } l(\alpha) = m, \\ 0 & \text{otherwise.} \end{cases}$$

Given a poset $P$, let $\tilde{K}_P(\mathbf{x}) := \tau_m(K_P(\mathbf{x}))$, where $m$ is the minimum length composition in the $\psi_\alpha$-expansion of $K_P(\mathbf{x})$. In other words, $\tilde{K}_P(\mathbf{x})$ consists of the terms in the expansion of $K_P(\mathbf{x})$ in the $\psi_\alpha$-basis of minimal length. As we will see shortly, $m$ will equal the number of minimal elements of $P$.

**Example 4.4.1.** Let $P$ be the following 5-element poset.



Then

$$\begin{aligned} K_P(\mathbf{x}) = \psi_{23} + \psi_{14} + 2\psi_{221} + 2\psi_{131} + 2\psi_{122} + \psi_{113} \\ + 2\psi_{2111} + 4\psi_{1211} + 4\psi_{1121} + 2\psi_{1112} + 8\psi_{11111}, \end{aligned}$$

$$\tilde{K}_P(\mathbf{x}) = \psi_{23} + \psi_{14}.$$

Just as with the usual notion of degree, we can check for irreducibility of a quasisymmetric function by checking irreducibility on the part of minimal length.

**Lemma 4.4.2.** *If $K_P(\mathbf{x})$ is reducible, then $\tilde{K}_P(\mathbf{x})$ is also reducible.*

*Proof.* Recall that multiplication in terms of the type 1 quasisymmetric power sum basis is the shuffle product, so length gives a grading on QSym. If $K_P(\mathbf{x})$ is reducible, then $K_P(\mathbf{x})$ can be expressed as $K_P(\mathbf{x}) = f \cdot g$ for some nonconstant homogeneous $f, g \in \mathrm{QSym}$. If $\tilde{f}$ and $\tilde{g}$ are the terms of shortest length in $f$ and $g$, respectively, then $\tilde{K}_P(\mathbf{x}) = \tilde{f} \cdot \tilde{g}$. $\square$

It follows that if $\tilde{K}_P(\mathbf{x})$ is irreducible, then $K_P(\mathbf{x})$ must also be irreducible.

We now give a combinatorial interpretation for $\tilde{K}_P(\mathbf{x})$ in terms of certain special pointed $P$-partitions. Let $\{z_1, \ldots, z_m\}$ be the set of minimal elements of $P$. For any subset $\{i_1, \ldots, i_l\} \subseteq [m]$, we will denote by $V(z_{i_1}, \ldots, z_{i_l})$ the filter of $P$ whose minimal elements are $\{z_{i_1}, \ldots, z_{i_l}\}$.

**Definition 4.4.3.** Suppose $P$ has $m$ minimal elements and $\sigma \in S_m$. The *$\sigma$-partition* $\theta_\sigma$ is the pointed $P$-partition that sends any element $x \in P$ to the largest number $j$ such that $z_{\sigma_j} \preceq x$. In other words, $\theta_\sigma^{-1}(i) = V(z_{\sigma_i}) \setminus V(z_{\sigma_{i+1}}, \ldots, z_{\sigma_m})$ for $i = 1, \ldots, m$.

Denote by $\alpha(\sigma)$ the weight of $\theta_\sigma$. Explicitly,

$$\begin{aligned}\alpha(\sigma)_i &= |V(z_{\sigma_i}) \setminus V(z_{\sigma_{i+1}}, \ldots, z_{\sigma_m})| \\ &= |V(z_{\sigma_i}, \ldots, z_{\sigma_m})| - |V(z_{\sigma_{i+1}}, \ldots, z_{\sigma_m})|.\end{aligned}$$

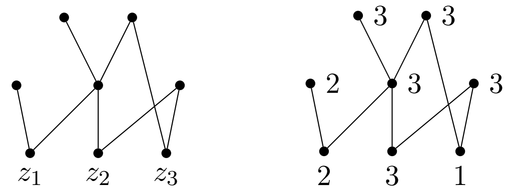
**Lemma 4.4.4.** *Let $P$ be a poset with $m$ minimal elements. Then*

$$\tilde{K}_P(\mathbf{x}) = \sum_{\sigma \in S_m} \psi_{\alpha(\sigma)}.$$

*Proof.* This follows from Corollary 4.3.7. Note that in any pointed $P$-partition, the minimal elements of $P$ must be sent to different values. Hence the minimum length part of $\tilde{K}_P(\mathbf{x})$ has length (at least) $m$.

If $\theta \colon P \to [m]$ is a pointed $P$-partition, then there exists some permutation $\sigma$ such that $\theta(z_{\sigma_i}) = i$. Since $\theta^{-1}(m)$ must be a principal filter of $P$, it must be $V(z_{\sigma_m})$. Similarly, $\theta^{-1}(m-1)$ must be a principal filter of $P \setminus V(z_{\sigma_m})$, so it must be $V(z_{\sigma_{m-1}}) \setminus V(z_{\sigma_m})$. Continuing in this manner, we see that we must have $\theta = \theta_\sigma$, and the result follows. $\square$

**Example 4.4.5.** Let $P$ be the poset shown below (on the left). On the right is the $\sigma$-partition of $P$ when $\sigma = 312$ with weight $\alpha(\sigma) = (1, 2, 5)$.




Similarly, one can compute for all $\sigma \in S_3$:

$$\alpha(123) = 233, \qquad \alpha(132) = 215, \qquad \alpha(213) = 143,$$
$$\alpha(231) = 125, \qquad \alpha(312) = 125, \qquad \alpha(321) = 125.$$

It follows from Lemma 4.4.4 that

$$\tilde{K}_P(\mathbf{x}) = \psi_{233} + \psi_{215} + \psi_{143} + 3\psi_{125}.$$

### 4.4.2 Evaluating functions in $\psi_\alpha$-basis

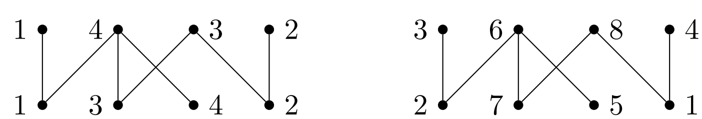
We will now give a combinatorial interpretation for the value of $\tilde{\eta}$ when applied to some term $\psi_{\alpha(\sigma)}$ appearing in $\tilde{K}_P(\mathbf{x})$. To do this, we will need the following technical notion.

**Definition 4.4.6.** Let $\pi, \sigma \in S_m$. We say that a bijection $\phi\colon P \to [n]$ is a $(\pi, \sigma)$-*labeling* if the following are true:

1. $\phi(z_{\pi_1}) > \phi(z_{\pi_2}) > \cdots > \phi(z_{\pi_m})$; and
2. for all $y \in P$, $\phi(y) \geq \phi(z)$, where $z$ is the minimal element such that $\theta_\sigma(z) = \theta_\sigma(y)$.

Denote by $T_P(\pi, \sigma)$ the set of all $(\pi, \sigma)$-labelings of $P$.

**Example 4.4.7.** Shown below are the 1423-partition $\theta$ of a poset $P$ (left), and an example of a $(2314, 1423)$-labeling $\phi$ (right). Here $z_1, z_2, z_3, z_4$ are taken to be the minimal elements from left to right.




It is clear that condition (1) holds since $\phi(z_2) > \phi(z_3) > \phi(z_1) > \phi(z_4)$.

For condition (2), consider, for instance, the element $x$ that covers $z_2$ and $z_4$. Since $\theta(x) = \theta(z_2) = 3$, we must have $\phi(x) \geq \phi(z_2)$.

Consider the operations $f_i, g_i \colon S_m \to S_m$ for $i = 1, \ldots, m-1$ defined by

$$\begin{aligned} f_i(\pi) &= \pi_1 \cdots \widehat{\pi_i} \cdots \pi_m \pi_i, \\ g_i(\pi) &= \pi_1 \cdots \pi_{i-1} \pi_m \pi_i \cdots \pi_{m-1}. \end{aligned}$$

Observe that $f_i$ and $g_i$ are inverses of each other. For any subset $S = \{s_1, \ldots, s_l\} \subseteq [m-1]$ with $s_1 < \cdots < s_l$, define

$$\begin{aligned} f_S &= f_{s_1} \circ f_{s_2} \circ \cdots \circ f_{s_l}, \\ g_S &= g_{s_l} \circ g_{s_{l-1}} \circ \cdots \circ g_{s_1}. \end{aligned}$$

Explicitly,

$$f_S(\pi) = \pi_1 \cdots \widehat{\pi_{s_1}} \cdots \widehat{\pi_{s_2}} \cdots \widehat{\pi_{s_l}} \cdots \pi_m \pi_{s_l} \pi_{s_{l-1}} \cdots \pi_{s_1}.$$

We can now give the following description of $\tilde{\eta}(\psi_{\alpha(\sigma)})$.

**Lemma 4.4.8.** *Let $P$ be a poset of size $n$ with $m$ minimal elements, and let $\sigma \in S_m$. Then*

$$(n-1)!\, \tilde{\eta}(\psi_{\alpha(\sigma)}) = \sum_{S \subseteq [m-1]} (-1)^{|S|} |T_P(g_S(\sigma), \sigma)|.$$

*Proof.* Note that for fixed $\sigma$, the permutations $g_S(\sigma)$ are distinct for distinct sets $S$, so the sets $T_P(g_S(\sigma), \sigma)$ are all disjoint.

Fix a value of $i$ with $1 \leq i \leq m$. Every labeling $\phi$ of $P$ in the union

$$\bigcup_{\substack{S \subseteq [m-1] \\ |S| = m-i}} T_P(g_S(\sigma), \sigma)$$

must satisfy the following two conditions:

1. $\phi(z_{\sigma_1}) > \phi(z_{\sigma_2}) > \cdots > \phi(z_{\sigma_{i-1}}) > \phi(z_{\sigma_i}) < \phi(z_{\sigma_{i+1}}) < \cdots < \phi(z_{\sigma_{m-1}}) < \phi(z_{\sigma_m})$; and
2. for all $y \in P$, $\phi(y) \geq \phi(z)$, where $z$ is the minimal element such that $\theta_\sigma(z) = \theta_\sigma(y)$.

These conditions are equivalent to the following:

- $\phi(z_{\sigma_i}) = 1$ (which of course is the smallest label among all $n$ elements of $P$);
- for $j = i-1, i-2, \ldots, 1$, $z_{\sigma_j}$ has the smallest label among $P \setminus V(\sigma_{z_{j+1}}, \ldots, \sigma_{z_m})$, and
- for $j = i+1, i+2, \ldots, m$, $z_{\sigma_j}$ has the smallest label among $V(\sigma_{z_j}, \ldots, \sigma_{z_m})$.

It is easy to check that for a random bijection from $P$ to $[n]$, these events are all independent of one another. If we let $\alpha = \alpha(\sigma)$, then

$$\begin{aligned} |P \setminus V(\sigma_{z_{j+1}}, \ldots, \sigma_{z_m})| &= \alpha_1 + \cdots + \alpha_j, \\ |V(\sigma_{z_j}, \ldots, \sigma_{z_m})| &= \alpha_j + \cdots + \alpha_m. \end{aligned}$$

Hence the number of such labelings $\phi$ is

$$\frac{n!}{n \cdot \prod_{j=1}^{i-1}(\alpha_1 + \cdots + \alpha_j) \cdot \prod_{j=i+1}^{m}(\alpha_j + \cdots + \alpha_m)} = \frac{(n-1)!}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi((\alpha_{i+1} \cdots \alpha_m)^r)}.$$

When we include the $(-1)^{|S|}$ and sum over all values of $i$, we find that the right hand side equals

$$\sum_{i=1}^{m} \frac{(-1)^{m-i}(n-1)!}{\pi(\alpha_1 \cdots \alpha_{i-1}) \cdot \pi((\alpha_{i+1} \cdots \alpha_m)^r)}.$$

This value is exactly $(n-1)!\,\tilde{\eta}(\psi_{\alpha(\sigma)})$ by Lemma 4.3.3, as desired. $\square$

The following two lemmas about $T_P(\pi, \sigma)$ will be helpful for giving a combinatorial interpretation for $\tilde{\eta}(\tilde{K}_P(\mathbf{x}))$. This is because many of the terms on the right hand side of Lemma 4.4.8 for different $\sigma$ will cancel out due to the principle of inclusion-exclusion.

**Lemma 4.4.9.** *Let $P$ be a poset with $m$ minimal elements. For all $S \subseteq [m-1]$ and all $\pi \in S_m$,*

$$T_P(\pi, f_S(\pi)) = \bigcap_{s \in S} T_P(\pi, f_s(\pi)).$$

*Proof.* We will show that the two sets above are equivalent by showing that the inequalities that $\phi$ must satisfy are the same in both sets. Clearly, a labeling $\phi$ must satisfy $\phi(z_{\pi_1}) > \phi(z_{\pi_2}) > \cdots > \phi(z_{\pi_m})$ to be in either of the sets. Since this determines the inequalities that must be satisfied among minimal elements of $P$, we will turn our attention to some non-minimal element $y \in P$. Define $a = \theta_\pi(y)$, where $\theta_\pi$ is the $\pi$-partition of $P$.

For the left hand side, if $\phi \in T_P(\pi, f_S(\pi))$, then we must have that $\phi(y) > \phi(z_{\pi_{s'}})$, where $s'$ is the image of $y$ in the $f_S(\pi)$-partition of $P$. Let $S_y = \{s \in S \colon y \succ z_{\pi_s}\}$. If $S_y$ is nonempty, then $s'$ is the minimum element of $S_y$, while if $S_y$ is empty, then $s' = a$. Note that by the definition of $\theta_\pi$, $a \geq s$ for all $s \in S_y$. Thus $s'$ is the minimum element of $S_y \cup \{a\}$.

For the right hand side, $\phi \in T_P(\pi, f_s(\pi))$ implies $\phi(y) > \phi(z_{\pi_s})$ if $s \in S_y$; otherwise $\phi(y) > \phi(z_{\pi_a})$. Thus $\phi$ will lie in the intersection of all of these sets if $\phi(y)$ is larger than the maximum of $\phi(z_{\pi_s})$ for all $s \in S_y \cup \{a\}$. But by condition (1) on $\phi$, this maximum is attained at $\phi(z_{\pi_{s'}})$. $\square$

We will now see that for all $i$, any $(\pi, f_i(\pi))$-labeling is also a $(\pi, \pi)$-labeling.

**Lemma 4.4.10.** *Let $P$ be a poset with $m$ minimal elements. For all $\pi \in S_m$,*

$$\bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi)) \subseteq T_P(\pi, \pi).$$

*Proof.* Let $\phi \in \bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi))$. This means that $\phi$ is a $(\pi, f_i(\pi))$-labeling for some value of $i$. No matter the value of $i$, $\phi$ must satisfy $\phi(z_{\pi_1}) > \phi(z_{\pi_2}) > \cdots > \phi(z_{\pi_m})$, which also must be satisfied to be in $T_P(\pi, \pi)$.

It remains to show that for all $y \in P$, $\phi(y) \geq \phi(z_{\pi_a})$, where $a = \theta_\pi(y)$, and $\theta_\pi$ is the $\pi$-partition of $P$. If $y \not\succeq z_{\pi_i}$, then $\phi(y) \geq \phi(z_{\pi_a})$. If instead $y \succeq z_{\pi_i}$, then $\phi(y) \geq \phi(z_{\pi_i})$, but $\phi(z_{\pi_i}) \geq \phi(z_{\pi_a})$ since $i \leq a$ by the definition of $\theta_\pi$. Therefore $\phi \in T_P(\pi, \pi)$. $\square$

Note that a $(\pi, \pi)$-labeling $\phi$ appears on the right hand side of Lemma 4.4.10 but not the left hand side if and only if, for all $i = 1, \ldots, m-1$, there exists $y \succ z_{\pi_i}$ with $\phi(y) < \phi(z_{\pi_i})$. We will describe such labelings more thoroughly in the next section.

### 4.4.3 Zigzag labelings

In this section, we introduce the notion of a zigzag labeling, which will allow us to give a combinatorial interpretation for $\tilde{\eta}(\tilde{K}_P(\mathbf{x}))$.

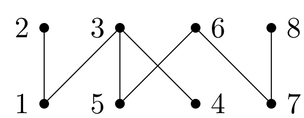
**Definition 4.4.11.** A bijection $\phi\colon P \to [n]$ is a *zigzag labeling* if the following conditions hold:

1. if $x$ is minimal and $\phi(x) \neq 1$, then there exists $y \succ x$ with $\phi(y) < \phi(x)$; and

2. if $x$ is not minimal, then there exists a minimal element $y \prec x$ with $\phi(y) < \phi(x)$.

We say that a zigzag labeling $\phi$ has *type* $\pi$ if $\phi(z_{\pi_1}) > \phi(z_{\pi_2}) > \cdots > \phi(z_{\pi_m})$.

**Example 4.4.12.** The following is an example of a zigzag labeling of $P$ of type 4231. (We take $z_1, z_2, z_3, z_4$ to be the minimal elements drawn from left to right.)




Zigzag labelings can be used to determine whether or not a poset is connected.

**Lemma 4.4.13.** *A poset $P$ has a zigzag labeling if and only if it is connected.*

*Proof.* Suppose $P$ is connected and has $m$ minimal elements. Since $P$ is connected, there exists a permutation $\pi' \in S_m$ such that $V(z_{\pi'_1}, \ldots, z_{\pi'_{i-1}}) \cap V(z_{\pi'_i}) \neq \varnothing$ for all $i$. Indeed, if it were not possible to find $\pi'_i$ given $\pi'_1, \dots, \pi'_{i-1}$, then $V(z_{\pi'_1}, \dots, z_{\pi'_{i-1}})$ would be a connected component of $P$, which contradicts the fact that $P$ is connected.

An example of a zigzag labeling $\phi$ of $P$ is as follows: label $z_{\pi'_1}$ with 1, and label the rest of $V(z_{\pi'_1})$ with $2, 3, \dots$. Then label $z_{\pi'_2}$ with the lowest number available, and label the elements of $V(z_{\pi'_1}, z_{\pi'_2}) \setminus V(z_{\pi'_1})$ with the next lowest remaining labels. Continue this process until all elements of $P$ are labeled. For all $i > 1$, there exists an element $x \in V(z_{\pi'_1}, \dots, z_{\pi'_{i-1}}) \cap V(z_{\pi'_i})$, and this element satisfies $x \succ z_{\pi'_i}$ and $\phi(x) < \phi(z_{\pi'_i})$. Therefore $\phi$ is a zigzag labeling. (One can check that $\phi$ has type $\pi$, where $\pi$ is the reverse of $\pi'$.)

Now suppose $P$ is disconnected, and let $\phi\colon P \to [n]$ be any bijection. Let $x$ be the element with the smallest label that is not in the same connected component as the element labeled 1. Then $x$ is not comparable to any element $y$ with a smaller label, so $\phi$ cannot be a zigzag labeling. $\square$

We will now use the lemmas in the previous section to show that we can count zigzag labelings of a fixed type by an alternating sum.

**Lemma 4.4.14.** *Let $P$ be a finite poset with $m$ minimal elements, and let $\pi \in S_m$. The set of zigzag labelings of $P$ of type $\pi$ is*

$$T_P(\pi, \pi) \setminus \bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi)),$$

*and the number of such labelings is*

$$\sum_{S \subseteq [m-1]} (-1)^{|S|} |T_P(\pi, f_s(\pi))|.$$

*Proof.* First note that for a labeling $\phi$ of type $\pi$, condition (2) for being a $(\pi, \pi)$-labeling is equivalent to condition (2) for being a zigzag labeling. Indeed, for any non-minimal $x \in P$, if there exists a minimal element $z_{\pi_i} \prec x$ with $\phi(z_{\pi_i}) < \phi(x)$, then certainly we can take $i$ to be the maximum value such that $z_{\pi_i} \prec x$ since that will only decrease the value of $\phi(z_{\pi_i})$.

A labeling $\phi \in T_P(\pi, \pi)$ does not lie in $T_P(\pi, f_i(\pi))$ if there exists some element $y \succ z_{\pi_i}$ with $\phi(y) < \phi(z_{\pi_i})$. Hence this occurs for all $i < m$ when $\phi$ satisfies condition (1) of being a zigzag labeling, which completes the proof of the first claim.

From the principle of inclusion-exclusion and Lemma 4.4.9,

$$\sum_{S \subseteq [m-1]} (-1)^{|S|} |T_P(\pi, f_s(\pi))| = |T_P(\pi, \pi)| - \left| \bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi)) \right|.$$

By Lemma 4.4.10, we have that

$$|T_P(\pi,\pi)| - \left|\bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi))\right| = \left|T_P(\pi,\pi) \setminus \bigcup_{i=1}^{m-1} T_P(\pi, f_i(\pi))\right|.$$

This is exactly the number of zigzag labelings of type $\pi$. $\square$

We now have the required tools to prove that the number of zigzag labelings of $P$ can be determined from $K_P(\mathbf{x})$.

**Theorem 4.4.15.** *The number of zigzag labelings of $P$ is counted by $(n-1)!\,\tilde{\eta}(\tilde{K}_P(\mathbf{x}))$.*

*Proof.* By Lemmas 4.4.4 and 4.4.8,

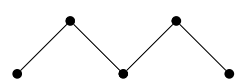
$$\begin{aligned}(n-1)!\,\tilde{\eta}(\tilde{K}_P(\mathbf{x})) &= (n-1)! \sum_{\sigma\in S_m} \tilde{\eta}(\psi_{\alpha(\sigma)}) \\ &= \sum_{\sigma\in S_m} \sum_{S\subseteq[m-1]} (-1)^{|S|} |T_P(g_S(\sigma),\sigma)|\end{aligned}$$

For all $\pi \in S_m$, we can group together the terms for which $g_S(\sigma) = \pi$, meaning $\sigma = f_S(\pi)$ since $g_S$ and $f_S$ are inverses. It follows that

$$\sum_{\sigma\in S_m} \sum_{S\subseteq[m-1]} (-1)^{|S|} |T_P(g_S(\sigma),\sigma)| = \sum_{\pi\in S_m} \sum_{S\subseteq[m-1]} (-1)^{|S|} |T_P(\pi, f_s(\pi))|.$$

By Lemma 4.4.14, this counts the number of zigzag labelings of $P$ (of any type). $\square$

**Example 4.4.16.** Let $P$ be the following naturally labeled poset.



One can compute

$$K_P(\mathbf{x}) = 4\psi_{122} + 2\psi_{113} + 4\psi_{1211} + 8\psi_{1121} + 4\psi_{1112} + 16\psi_{11111}.$$

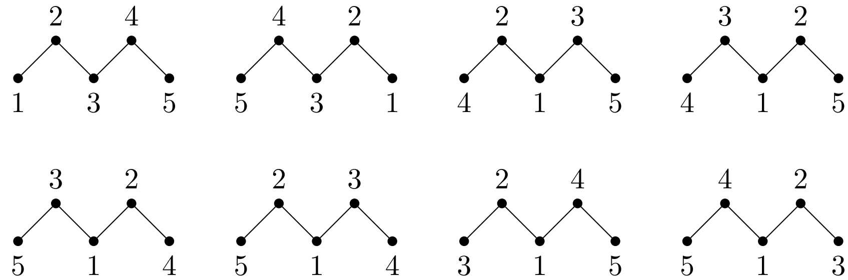
It follows that

$$\tilde{K}_P(\mathbf{x}) = 4\psi_{122} + 2\psi_{113}.$$

Applying $\tilde{\eta}$ and multiplying by 4!, we have

$$4! \cdot \tilde{\eta}(\tilde{K}_P(\mathbf{x})) = 24(-\tfrac{1}{6} + \tfrac{1}{2}) = 8.$$

The following are the 8 zigzag labelings of $P$:




Now that we have this combinatorial interpretation, it is easy to show our main result about the irreduciblity of $K_P(\mathbf{x})$.

**Theorem 4.4.17.** *A poset $P$ is connected if and only if $K_P(\mathbf{x})$ is irreducible over* QSym.

*Proof.* This is an immediate consequence of Theorem 4.4.15 and Lemma 4.4.13. If $P$ is connected, then $(n-1)!\,\tilde{\eta}(\tilde{K}_P(\mathbf{x})) > 0$, so it follows from Lemma 4.3.1 that $\tilde{K}_P(\mathbf{x})$ is irreducible. Therefore by Lemma 4.4.2, $K_P(\mathbf{x})$ is irreducible. $\square$

In fact, this result implies that if $P$ is connected, then $K_P(\mathbf{x})$ does not even lie in the span of the homogeneous, reducible elements of QSym.

This result also tells us how $K_P(\mathbf{x})$ factors as a product of irreducible partition generating functions. It is well known that QSym is isomorphic to a polynomial ring and hence is a unique factorization domain (see [12]).

**Corollary 4.4.18.** *Let $P$ be a naturally labeled poset. Then the irreducible factorization of $K_P(\mathbf{x})$ is given by $K_P(\mathbf{x}) = \prod_i K_{P_i}(\mathbf{x})$, where $P_i$ are the connected components of $P$.*

*Proof.* This follows from the fact that $K_P(\mathbf{x})$ factors into a product of the partition generation functions of its connected components, and each of these is irreducible by Theorem 4.4.17. $\square$

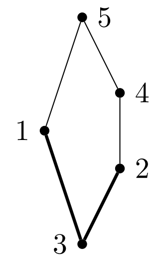
This result also gives a condition on when two posets can have the same partition generating function based on their connected components.

**Corollary 4.4.19.** *Let $P$ and $Q$ be naturally labeled posets. Let $P_1, \ldots, P_k$ be the connected components of $P$, and $Q_1, \ldots, Q_l$ the connected components of $Q$. If $K_P(\mathbf{x}) = K_Q(\mathbf{x})$, then $k = l$, and there exists a permutation $\pi \in S_k$ such that $K_{P_i}(\mathbf{x}) = K_{Q_{\pi_i}}(\mathbf{x})$ for all $i$.*

*Proof.* This follows immediately from Corollary 4.4.18 and the fact that QSym is a unique factorization domain. (There are no scalar factors since Theorem 1.2.11 implies that the coefficient of $L_n$ in the expansion of any $K_{P_i}(\mathbf{x})$ or $K_{Q_j}(\mathbf{x})$ in the $L_\alpha$-basis is 1.) $\square$

It is still open whether there exists a connected labeled poset $(P,\omega)$ whose partition generating function $K_{(P,\omega)}(\mathbf{x})$ is reducible. Our approach will not work in this more general setting: for a general labeled poset $(P,\omega)$, it is possible for $K_{(P,\omega)}(\mathbf{x})$ to be irreducible even when $\widetilde{K}_{(P,\omega)}(\mathbf{x})$ is reducible. (Moreover, $K_{(P,\omega)}(\mathbf{x})$ may be irreducible yet lie in the span of homogeneous, reducible elements of QSym.)

**Example 4.4.20.** Let $(P,\omega)$ be the the following labeled poset with the generating function shown.




$$\begin{aligned} K_{(P,\omega)}(\mathbf{x}) = &- \psi_{32} - \psi_{23} - \psi_{311} - \psi_{221} - 3\psi_{212} + \psi_{122} \\ &+ \psi_{113} + 3\psi_{2111} + \psi_{1211} + \psi_{1121} + 3\psi_{1112} + 3\psi_{11111}. \end{aligned}$$

If we define $\widetilde{K}_{(P,\omega)}(\mathbf{x}) = \tau_2(K_{(P,\omega)}(\mathbf{x}))$, then we have

$$\begin{aligned} \widetilde{K}_{(P,\omega)}(\mathbf{x}) &= -\psi_{32} - \psi_{23} \\ &= -\psi_3 \cdot \psi_2. \end{aligned}$$

Since $\widetilde{K}_{(P,\omega)}(\mathbf{x})$ is reducible, it follows that $\tilde{\eta}(\widetilde{K}_{(P,\omega)}(\mathbf{x})) = 0$ even though $P$ is connected. However, one can check that $K_{(P,\omega)}$ is itself irreducible.

## 4.5 Series-parallel posets

We will now turn our attention to a collection of naturally labeled posets known as *series-parallel posets*. We will use the results of the previous section to show that distinct series-parallel posets have distinct partition generating functions.

**Definition 4.5.1.** The class $\mathcal{SP}$ of *series-parallel posets* is the smallest collection of posets that satisfies the following:

- the one-element poset $1_{\mathcal{P}}$ lies in $\mathcal{SP}$;
- if $P \in \mathcal{SP}$ and $Q \in \mathcal{SP}$, then $P \sqcup Q \in \mathcal{SP}$; and

- if $P \in \mathcal{SP}$ and $Q \in \mathcal{SP}$, then $P \oplus Q \in \mathcal{SP}$.

Recall that the ordinal sum $P \oplus Q$ is the poset on the disjoint union of $P$ and $Q$ with relations $x \preceq y$ if and only if $x \preceq_P y$, $x \preceq_Q y$, or $x \in P$ and $y \in Q$.

It is well known that $P \in \mathcal{SP}$ if and only if $P$ is $N$-avoiding [26]. (A poset is $N$-*avoiding* if there does not exist an induced poset on four elements $\{a, b, c, d\} \subseteq P$ with relations $a \prec b \succ c \prec d$.)

Note that if a series-parallel poset is disconnected, then it can be expressed as the disjoint union of series-parallel posets. If it is connected (and has more than one element), then it can be expressed as an ordinal sum of series-parallel posets.

In [11], Hasebe and Tsujie show that if a poset $P$ is a finite rooted tree, then it can be distinguished by its $P$-partition generating function. A finite rooted tree is a special type of a series-parallel poset. They then ask if the same can be said about series-parallel posets. We will now give an answer to this question.

**Theorem 4.5.2.** *Let $P$ be a series-parallel poset. Then $P$ is uniquely determined (up to isomorphism) by $K_P(\mathbf{x})$.*

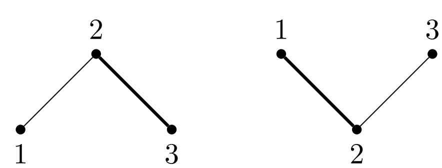
*Proof.* We will prove this by induction on the size of $P$. If $P$ has a single element, then the result holds trivially since there is a unique single-element poset. Now suppose that the results holds for all series-parallel posets with fewer than $|P|$ elements.

If $P$ is disconnected, then $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_k$, where each $P_i$ is a connected series-parallel poset. By Lemma 4.4.18, the irreducible factors of $K_P(\mathbf{x})$ are the partition generating functions of the connected components of $P$. By induction, we can determine the connected components from their partition generating functions.

We will now assume that $P$ is connected and series-parallel. We can express $P$ as $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$, where each $P_i$ is series-parallel. By Lemma 2.1.3, we can determine $K_{P_i}(\mathbf{x})$ from $K_P(\mathbf{x})$ for all $i$. Since each $P_i$ is series-parallel, it follows by induction that it is uniquely determined by $K_{P_i}(\mathbf{x})$. $\square$

It should be noted that Theorem 4.5.2 does not hold when $(P, \omega)$ is not naturally labeled.

**Example 4.5.3.** The following labeled series-parallel posets have the same partition generating functions.




Although the labeled posets in Example 4.5.3 are not isomorphic, they are both series-parallel.

## 4.6 A combinatorial description of the coefficients

In this section, we will give a combinatorial interpretation for the coefficients in the $\psi_\alpha$-expansion of $K_{(P,\omega)}(\mathbf{x})$ akin to the Murnaghan-Nakayama rule.

### 4.6.1 Generalized ribbons

We will begin by considering the following question: for which labeled posets $(P,\omega)$ does $\eta(K_{(P,\omega)}(\mathbf{x})) \neq 0$? By Lemma 4.3.3, this is equivalent to asking when the composition $(n)$ lies in the $\psi$-support of $K_{(P,\omega)}(\mathbf{x})$.

In the fundamental quasisymmetric function basis $\{L_\alpha\}$,

$$\eta(L_\alpha) = \begin{cases} (-1)^k & \text{if } \alpha = (1^k, n-k), \\ 0 & \text{otherwise.} \end{cases}$$

This means that we must only consider linear extensions of $(P,\omega)$ whose descent set is $\{1,\ldots,k\}$ for some value of $k$ when computing $\eta(K_{(P,\omega)}(\mathbf{x}))$.

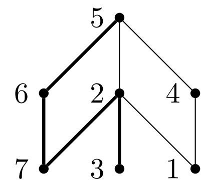
**Lemma 4.6.1.** *Suppose $(P,\omega)$ is a labeled poset. If there exists a chain $a \prec b \prec c$ such that $\omega(a) < \omega(b) > \omega(c)$, then $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$.*

*Proof.* Every linear extension of $(P,\omega)$ has an ascent at some point between the appearance of $\omega(a)$ and $\omega(b)$, as well as a descent between the appearance of $\omega(b)$ and $\omega(c)$. Therefore $K_{(P,\omega)}(\mathbf{x})$ has no compositions of the form $\alpha = (1^k, n-k)$ in its $L$-support, so $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$. □

We say that a labeled poset $(P,\omega)$ is a *generalized ribbon* if $(P,\omega)$ does not contain a chain $a \prec b \prec c$ such that $\omega(a) < \omega(b) > \omega(c)$. Note that any generalized ribbon has a maximum order ideal $I$ containing only strict edges, in which case $P \setminus I$ is naturally labeled (i.e., contains only natural edges).

Denote by $J$ the set of elements in $I$ such that $(P \setminus I) \cup J$ is naturally labeled and $I \setminus J$ is an order ideal. In other words, $J$ is the set of elements $j$ such that the principal order ideal generated by $j$ contains only strict edges, while the principal filter generated by $j$ contains only natural edges. Then $J$ must be a subset of the maximal elements of $I$. Note that for all $j \in J$ and $x \in P$, if $j$ and $x$ share an edge in the Hasse diagram of $(P,\omega)$, then $\omega(x) > \omega(j)$. Observe that $J$ is nonempty since the element labeled 1 (which we denote by 1) must be a maximal element in $I$, and $(P \setminus I) \cup \{1\}$ is naturally labeled, so $1 \in J$.

**Example 4.6.2.** Suppose that $(P,\omega)$ is the following generalized ribbon.




The ideal $I$ has maximal elements $\{1,3,6\}$, while $J=\{1\}$.

This set $J$ can be used to compute the value of $\eta(K_{(P,\omega)}(\mathbf{x}))$ for generalized ribbons.

**Lemma 4.6.3.** *Suppose $(P,\omega)$ is a generalized ribbon, and let $I$ and $J$ be defined as above. Then*

$$\eta(K_{(P,\omega)}(\mathbf{x})) = \begin{cases} (-1)^{|I|-1} & \text{if } |J|=1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We only need to consider the linear extensions of $(P,\omega)$ whose descent set is $\{1,\ldots,k\}$ for some value of $k$. The only linear extensions of $(P,\omega)$ with a descent set of this form must begin with, for some subset $A \subseteq J\setminus\{1\}$, the elements of $I\setminus A$ in decreasing order and end with the elements of $(P\setminus I)\cup A$ in increasing order. Therefore we can express $\eta(K_{(P,\omega)}(\mathbf{x}))$ as

$$\eta(K_{(P,\omega)}(\mathbf{x})) = \sum_{A\subseteq J\setminus\{1\}} (-1)^{|I|-|A|-1}.$$

This value is 0 unless $|J|=1$, in which case $\eta(K_{(P,\omega)}(\mathbf{x})) = (-1)^{|I|-1}$. □

Observe that if $P$ is naturally labeled, then $\eta(K_P(\mathbf{x}))=1$ if and only if $P$ has a unique minimal element. (In this case, $I$ and $J$ are both just the set of minimal elements of $P$.)

We say that $(P,\omega)$ is *rooted* if $\eta(K_{(P,\omega)}(\mathbf{x}))\neq 0$. In other words, $(P,\omega)$ is rooted if it is a generalized ribbon and $|J|=1$, that is, 1 is the unique element whose principal ideal contains only strict edges and whose principal filter contains only natural edges.

**Corollary 4.6.4.** *If $(P,\omega)$ is rooted, then it is connected.*

*Proof.* If $(P,\omega)$ is disconnected, then $K_{(P,\omega)}(\mathbf{x})$ is the product of the $(P,\omega)$-partition generating functions of the connected components of $P$. By Lemma 4.3.1, $\eta(K_{(P,\omega)}(\mathbf{x}))=0$.

Alternatively, the elements with the minimum label in each connected component of $P$ lie in the set $J$. □

### 4.6.2 Enriched $(P,\omega)$-partitions

We will now give an alternate combinatorial description of rooted posets in terms of enriched $(P,\omega)$-partitions. We will then use this to give a combinatorial formula (with signs) for the coefficient of $\psi_\alpha$ in $K_{(P,\omega)}(\mathbf{x})$.

Let $\mathbf{P'}$ be the nonzero integers with the total order

$$-1 < 1 < -2 < 2 < \cdots .$$

For $k \in \mathbf{P'}$, we define the absolute value of $k$ by $|k| = i \in \mathbb{Z}^+$ if $k \in \{-i, i\}$, and we say $k < 0$ if $k = -|k|$.

The following definition is due to Stembridge [27].

**Definition 4.6.5.** Let $(P,\omega)$ be a labeled poset. An *enriched* $(P,\omega)$*-partition* is a map $f\colon P \to \mathbf{P'}$ such that $|f|$ is surjective onto $[k]$ for some $k$, and if $x$ is covered by $y$, then we have

(i) $f(x) \leq f(y)$,

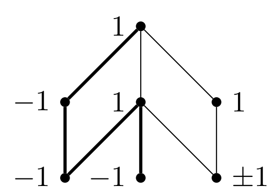
(ii) if $|f(x)| = |f(y)|$ and $\omega(x) < \omega(y)$, then $f(y) > 0$,

(iii) if $|f(x)| = |f(y)|$ and $\omega(x) > \omega(y)$, then $f(x) < 0$.

It should be noted that if $x \prec y \prec z$ is a chain with $\omega(x) < \omega(y) > \omega(z)$, then for all enriched $(P,\omega)$-partitions $f$, $|f(x)| < |f(z)|$. It follows that for each $i$, the subposet $P_i = \{x \in P \mid |f(x)| = i\}$ is a generalized ribbon.

Suppose that $f$ is an enriched $(P,\omega)$-partition and we know the value of $|f(x)|$ for all $x$. Is this enough information to determine the value of $f(x)$ for all $x$? The answer to this question is no.

**Example 4.6.6.** Let $(P,\omega)$ be the following poset. Suppose $f$ is an enriched $(P,\omega)$-partition, and $|f(x)| = 1$ for all $x$.




The element in the bottom right can either be sent to $-1$ or $1$ by $f$.

For each $x \in P$ and enriched $(P,\omega)$-partition $f$, define the map $f_x\colon P \to \mathbf{P'}$ by

$$f_x(y) = \begin{cases} -f(x) & \text{if } x = y, \\ f(y) & \text{otherwise.} \end{cases}$$

We say that an element $x \in P$ is *ambiguous with respect to* $f$ if $f_x$ is still an enriched $(P, \omega)$-partition.

Let $\mathbf{P}^* = \mathbf{P}' \cup \{1^*, 2^*, \dots\}$, where each $i^*$ satisfies $-i < i^* < i$ and $|i^*| = i$. For an enriched $(P, \omega)$-partition $f$, consider the map $f^* \colon P \to \mathbf{P}^*$ defined by

$$f^*(x) = \begin{cases} |f(x)|^* & \text{if } x \text{ is ambiguous with respect to } f, \\ f(x) & \text{otherwise.} \end{cases}$$

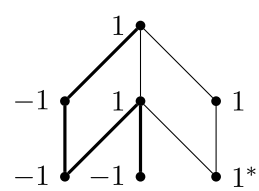
The map $f^*$ is called a *starred* $(P, \omega)$*-partition*. Note that $f^*$ is uniquely determined by the generalized ribbons $P_i$.

The *ambiguity of* $f^*$ is the sequence $\mathrm{amb}(f^*) = (a_1, a_2, \dots, a_k)$, where $a_i = |(f^*)^{-1}(i^*)|$, the number of elements labeled $i^*$ by $f^*$, The *sign of* $f^*$ is

$$\mathrm{sign}(f^*) = (-1)^{|\{x | f^*(x) < 0\}|},$$

and the *weight of* $f^*$ is $\mathrm{comp}(f^*) = (b_1, b_2, \dots, b_k)$, where $b_i$ is the total number of elements labeled $-i$, $i^*$, or $i$ by $f$.

**Example 4.6.7.** Let $f$ be the enriched $(P, \omega)$-partition from Example 4.6.6. Then the starred $(P, \omega)$-partition $f^*$ is shown below.




Here, $\mathrm{amb}(f^*) = (1)$, $\mathrm{sign}(f^*) = (-1)^3$, and $\mathrm{comp}(f^*) = (7)$.

We say that a starred $(P, \omega)$-partition $f^*$ is a *pointed* $(P, \omega)$*-partition* if $\mathrm{amb}(f^*) = (1, 1, \dots, 1)$. For instance, if $f \colon P \to [n]$ is any linear extension of $(P, \omega)$, then $f^*$ is a pointed $(P, \omega)$-partition.

The following lemma is equivalent to Lemma 4.6.3.

**Lemma 4.6.8.** *Let $(P, \omega)$ be a labeled poset. Then there exists a pointed $(P, \omega)$-partition $f^*$ with* $\mathrm{comp}(f^*) = (n)$ *if and only if $(P, \omega)$ is rooted. In this case,* $\eta(K_{(P,\omega)}(\mathbf{x})) = \mathrm{sign}(f^*)$*; otherwise* $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$.

*Proof.* We know from Lemma 4.6.1 that in order for $\eta(K_{(P,\omega)}(\mathbf{x})) \neq 0$, there cannot be a chain $a \prec b \prec c$ with $\omega(a) < \omega(b) > \omega(c)$. If there are no such chains, then there exists a starred $(P, \omega)$-partition $f^*$ with $\mathrm{comp}(f^*) = (n)$. Explicitly, $f^*$ labels the elements of $P$ is as follows:

if an element lies at the bottom of a strict edge, then it gets sent to $-1$; if it lies at the top of a natural edge, then it gets sent to 1; otherwise, it gets sent to $1^*$. There are no elements that lie both at the bottom of a strict edge and at the top of a natural edge.

The elements that get sent to $1^*$ are what we called $J$ in Lemma 4.6.3, and the elements that get sent to $-1$ or $1^*$ are what we called $I$. Therefore the result holds by Lemma 4.6.3. $\square$

A pointed $(P, \omega)$-partition $f^*$ can be interpreted as a partitioning of $(P, \omega)$ such that each part is rooted. Specifically, each subposet $P_i = \{x \in P \mid |f^*(x)| = i\}$ is a rooted poset, and the weight of $f^*$ is the composition $(|P_1|, |P_2|, \dots)$. This allows us to give a combinatorial description of the coefficient of $\psi_\alpha$ in $K_{(P,\omega)}(\mathbf{x})$.

**Theorem 4.6.9.** *Let $(P, \omega)$ be a labeled poset. Then*

$$K_{(P,\omega)}(\mathbf{x}) = \sum_{f^*} \operatorname{sign}(f^*)\psi_{\operatorname{comp}(f^*)},$$

*where the sum ranges over all pointed $(P, \omega)$-partitions $f^*$.*

*Proof.* Recall that Theorem 4.3.5 states that the coefficient of $\psi_\alpha$ in $K_{(P,\omega)}(\mathbf{x})$ can be computed by $\eta^{\otimes l}(\Delta_\alpha K_{(P,\omega)}(\mathbf{x}))$, and that

$$\Delta_\alpha K_{(P,\omega)}(\mathbf{x}) = \sum K_{(P_1,\omega)}(\mathbf{x}) \otimes \cdots \otimes K_{(P_l,\omega)}(\mathbf{x}),$$

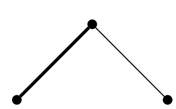
where $|P_i| = \alpha_i$, $P_1 \cup \cdots \cup P_i$ is an order ideal of $P$, and $P_1, P_2, \dots, P_{l(\alpha)}$ partition $P$.

Therefore the coefficient of $\psi_\alpha$ in $K_{(P,\omega)}(\mathbf{x})$ is given by

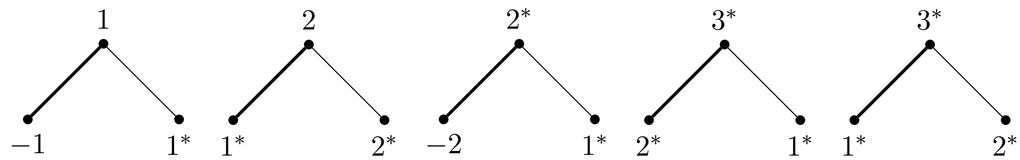
$$\eta^{\otimes l}(\Delta_\alpha K_{(P,\omega)}(\mathbf{x})) = \sum \eta(K_{(P_1,\omega)}(\mathbf{x})) \otimes \cdots \otimes \eta(K_{(P_l,\omega)}(\mathbf{x})).$$

By Lemma 4.6.8, a term in this summation is 0 unless each $(P_i, \omega)$ is rooted, meaning there exists a pointed $(P_i, \omega)$-partition $f_i^*$ with weight $(\alpha_i)$. These can be combined to form a unique pointed $(P, \omega)$-partition $f^*$ with $\operatorname{comp}(f^*) = \alpha$. Since $\eta(K_{(P_i,\omega)}(\mathbf{x})) = \operatorname{sign}(f_i^*)$, it follows that $\eta(K_{P_1}(\mathbf{x})) \otimes \cdots \otimes \eta(K_{P_l}(\mathbf{x})) = \operatorname{sign}(f^*)$. $\square$

**Example 4.6.10.** Let $(P, \omega)$ be the following poset.



The following are the pointed $(P, \omega)$-partitions.




Therefore $K_{(P,\omega)}(\mathbf{x}) = -\psi_3 + \psi_{12} - \psi_{12} + 2\psi_{111} = -\psi_3 + 2\psi_{111}$.

Observe that although the labeled poset in the previous example has a pointed $(P, \omega)$-partition with weight $(1, 2)$, the composition $(1, 2)$ is not in the $\psi$-support of $K_{(P,\omega)}(\mathbf{x})$.

One should note that the combinatorial interpretation given in Theorem 4.6.9 is consistent with the description given in [2] (or Corollary 4.3.7 above) when $(P, \omega)$ in naturally labeled. The only pointed $(P, \omega)$-partitions in the naturally labeled case are when each $P_i$ has a unique minimal element.

### 4.6.3 Murnaghan-Nakayama rule

In this section, we will compare Theorem 4.6.9 to the Murnaghan-Nakayama rule, which expresses a Schur symmetric function in terms of power sum symmetric functions.

We begin with some background on symmetric functions and tableaux. (For more information, see [20, 24].) A *symmetric function* in the variables $x_1, x_2, \dots$ (with coefficients in $\mathbb{C}$) is a formal power series $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ of bounded degree such that, for any composition $\alpha$, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ equals the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ whenever $i_1, i_2, \dots, i_k$ are distinct. We denote the algebra of symmetric functions by $\Lambda$. Clearly every symmetric function is also quasisymmetric.

The *power sum symmetric function basis* $\{p_\lambda\}$, indexed by partitions $\lambda$, is given by $p_n = x_1^n + x_2^n + \cdots$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. In [3], it was shown that

$$\frac{p_\lambda}{z_\lambda} = \sum_{\alpha \sim \lambda} \psi_\alpha,$$

where the sum runs over all compositions $\alpha$ that rearrange to the partition $\lambda$.

For any partition $\lambda$, the *Young diagram* of shape $\lambda$ is a collection of boxes arranged in left-justified rows such that row $i$ has $\lambda_i$ boxes. If $\lambda$ and $\mu$ are partitions such that $\mu_i \leq \lambda_i$ for all $i$, then the Young diagram of the (skew) shape $\lambda/\mu$ is the set-theoretic difference between the Young diagram of shape $\lambda$ and the Young diagram of shape $\mu$.

A *semistandard (Young) tableau* (SSYT) of (skew) shape $\lambda/\mu$ is a labeling of the boxes of the Young diagram of shape $\lambda/\mu$ such that the entries in the rows are weakly increasing from

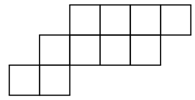


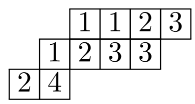
Figure 4.2: The Young diagram of shape 652/21.

left to right, and the entries in the columns are strictly increasing from top to bottom. If $T$ is an SSYT of shape $\lambda/\mu$, then we write $\lambda/\mu = \mathrm{sh}(T)$. We say that $T$ has *type* $\alpha = (\alpha_1, \alpha_2, \dots)$, denoted $\alpha = \mathrm{type}(T)$, if $T$ has $\alpha_i = \alpha_i(T)$ parts equal to $i$. For any SSYT $T$ of type $\alpha$, we write $x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots$. The *skew Schur function* $s_{\lambda/\mu}$ is the formal power series

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_T x^T,$$

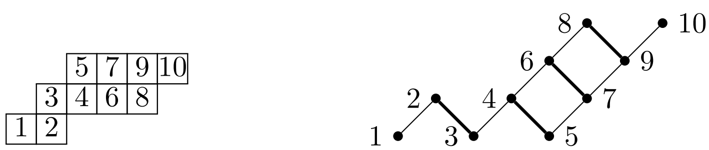
where the sum runs over all SSYT $T$ of shape $\lambda/\mu$. The Schur functions $\{s_\lambda\}$ for partitions $\lambda$ form a basis for $\Lambda$.

**Example 4.6.11.** The following is a SSYT of shape 652/21 and type $(3, 3, 3, 1)$.




Define $P_{\lambda/\mu}$ to be the poset whose elements are the squares $(i, j)$ of $\lambda/\mu$, partially ordered componentwise. Define a labeling $\omega_{\lambda/\mu} \colon P_{\lambda/\mu} \to [n]$ as follows: the bottom square of the first column of $P_{\lambda/\mu}$ is labeled 1. The labeling then proceeds in order up the first column, then up the second column, and so forth.

**Example 4.6.12.** The following is $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ when $\lambda = 652$ and $\mu = 21$.




It follows immediately from the definition of $K_{(P,\omega)}(\mathbf{x})$ that

$$K_{(P_{\lambda/\mu}, \omega_{\lambda/\mu})}(\mathbf{x}) = \sum_T x^T = s_{\lambda/\mu}(\mathbf{x}),$$

where the sum runs over all SSYT $T$ of shape $\lambda/\mu$.

We can express $s_{\lambda/\mu}$ in terms of the power sum symmetric function basis using the following combinatorial rule, known as the *Murnaghan-Nakayama rule*.

A *border strip* is a connected skew shape with no $2 \times 2$ square. Define the *height* $\mathrm{ht}(B)$ of a border strip $B$ to be one less than its number of rows.

A *border-strip tableau* of shape $\lambda/\mu$ and type $\alpha = (\alpha_1, \alpha_2, \dots)$ is an assignment of positive integers to the squares of $\lambda/\mu$ such that

- every row and column is weakly increasing,
- the integer $i$ appears $\alpha_i$ times, and
- the set of squares occupied by $i$ forms a border strip.

The *height* of a border-strip tableau $T$, denoted $\mathrm{ht}(T)$, is the sum of the heights of the border strips that make up $T$.

**Theorem 4.6.13** (Murnaghan-Nakayama rule)**.** *For partitions $\lambda$, $\mu$, and $\nu$,*

$$s_{\lambda/\mu} = \sum_{\nu} \chi^{\lambda/\mu}(\nu) \frac{p_\nu}{z_\nu}.$$

*Here $\chi^{\lambda/\mu}(\nu) = \sum_T (-1)^{\mathrm{ht}(T)}$, where the sum ranges over all border-strip tableaux of shape $\lambda/\mu$ and type $\nu$.*

When $\mu = \varnothing$, Theorem 4.6.13 gives a change of basis formula for expressing Schur functions in terms of the power sum symmetric functions.

Since the quasisymmetric power sums refine the symmetric power sums, the Murnaghan-Nakayama rule also gives a description of the $\psi_\alpha$-expansion of $s_{\lambda/\mu}$. Since $\chi^{\lambda/\mu}(\nu)$ also does not depend on the order of the parts of $\nu$, we find that the Murnaghan-Nakayama rule is equivalent to

$$s_{\lambda/\mu} = \sum_{\alpha} \chi^{\lambda/\mu}(\alpha) \psi_\alpha,$$

where the sum ranges over all compositions $\alpha$.

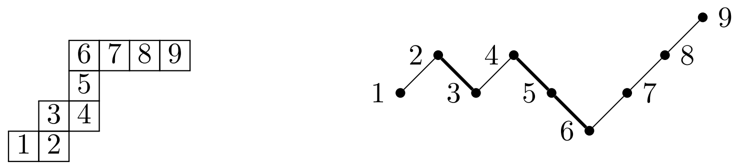
In fact, this description agrees with the one obtained by Theorem 4.6.9 above.

**Proposition 4.6.14.** *When $(P, \omega) = (P_{\lambda/\mu}, \omega_{\lambda/\mu})$, $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$ unless $\lambda/\mu$ is a border strip, in which case $\eta(K_{(P,\omega)}(\mathbf{x})) = (-1)^{\mathrm{ht}(\lambda/\mu)}$.*

*Proof.* If $\lambda/\mu$ contains a $2 \times 2$ square, then $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ contains a chain $a \prec b \prec c$ with $\omega(a) < \omega(b) > \omega(c)$. It follows from Lemma 4.6.1 that $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$. Similarly, if $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ is disconnected, then by Corollary 4.6.4, we have $\eta(K_{(P,\omega)}(\mathbf{x})) = 0$. Therefore the only way to have $\eta(K_{(P,\omega)}(\mathbf{x})) \neq 0$ is if $\lambda/\mu$ is connected and contains no $2 \times 2$ square, that is, if it is a border strip.

Using the terminology of Lemma 4.6.3, when $\lambda/\mu$ is a border strip, the elements of $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ that lie in $I$ are the leftmost boxes in the rows of the Young diagram with shape $\lambda/\mu$. Therefore $|I|$ is equal to the number of rows of $\lambda/\mu$. The only element of $J$ is the box in the southwest corner. By Lemma 4.6.3, we have that $\eta(K_{(P,\omega)}(\mathbf{x})) = (-1)^{\mathrm{ht}(\lambda/\mu)}$. $\square$

**Example 4.6.15.** The following is $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$ when $\lambda = 6332$ and $\mu = 221$.




Since $\lambda/\mu$ is a border strip, it follows that $\eta(K_{(P_{\lambda/\mu},\omega_{\lambda/\mu})}(\mathbf{x})) = (-1)^3$. We see that $I = \{1, 3, 5, 6\}$ and $J = \{1\}$.

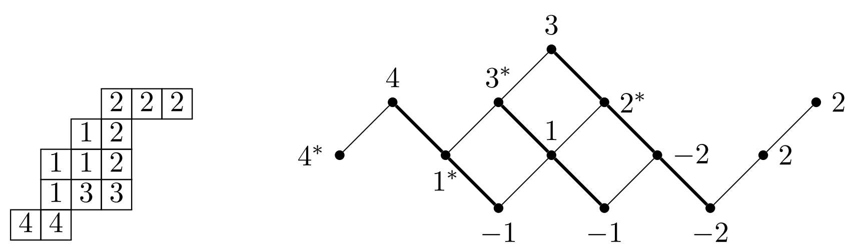
**Corollary 4.6.16.** *For partitions $\lambda$ and $\mu$ and compositions $\alpha$,*

$$s_{\lambda/\mu} = \sum_{\alpha} \chi^{\lambda/\mu}(\alpha)\psi_\alpha.$$

*Here $\chi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{\mathrm{ht}(T)}$, where the sum ranges over all border-strip tableaux of shape $\lambda/\mu$ and type $\alpha$.*

*Proof.* By Proposition 4.6.14, a pointed $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$-partition $f^*$ is equivalent to a border-strip tableau $T$ of shape $\lambda/\mu$. In this case, $\mathrm{wt}(f^*) = \mathrm{type}(T)$ and $\mathrm{sign}(f^*) = (-1)^{\mathrm{ht}(T)}$, so the result follows immediately from Theorem 4.6.9. $\square$

**Example 4.6.17.** Let $\lambda = (6, 4, 4, 4, 2)$ and $\mu = (3, 2, 1, 1)$. The following is a border strip tableau of shape $\lambda/\mu$ and the pointed $(P_{\lambda/\mu}, \omega_{\lambda/\mu})$-partition $f^*$ that corresponds to it.




## 4.7 Discussion and open questions

In this chapter, we showed that connected, naturally labeled posets have irreducible partition generating functions. We did this by finding a linear function that sends all reducible elements of QSym to 0. The kernel of this function also contains some irreducible elements of QSym so we were not able to use it to show that all connected labeled posets have irreducible partition generating functions. This question is still open.

**Question 4.7.1.** If $P$ is a connected, labeled poset, is $K_{(P,\omega)}(\mathbf{x})$ always irreducible over QSym?

At this point, we are far from giving a complete answer to this question. We may want to simplify the question by considering the case when $(P, \omega)$ is a generalized ribbon. We see in Example 4.4.20 that the techniques used in the naturally labeled case do not extend to generalized ribbons.

In Section 4.5, we showed that naturally labeled, series-parallel posets are uniquely determined by their partition generating functions. We also gave an example of two distinct labeled series-parallel posets with the same partition generating function.

**Question 4.7.2.** Let $(P, \omega_1)$ and $(Q, \omega_2)$ be labeled posets, and suppose that $P$ is series-parallel. If $K_{(P,\omega_1)}(\mathbf{x}) = K_{(Q,\omega_2)}(\mathbf{x})$, does this imply that $Q$ is series-parallel?

As with Question 3.4.3, the cyclic inclusion-exclusion operation of Féray [7] may be helpful in answering this question.

Another question to consider follows.

**Question 4.7.3.** What is the $\psi$-support of $K_{(P,\omega)}(\mathbf{x})$?

This question may be difficult, but even a partial answer can be useful in determining whether or not $K_{(P,\omega)}(\mathbf{x})$ is a symmetric function. Suppose $\alpha$ and $\beta$ are both compositions that rearrange to the partition $\lambda$. If one can show that $\alpha$ is in the $\psi$-support of $K_{(P,\omega)}(\mathbf{x})$, but $\beta$ is not, then that is enough to show that $K_{(P,\omega)}(\mathbf{x})$ is not symmetric.

This leads us to the question asked by Stanley in [23].

**Question 4.7.4.** For what labeled posets $(P, \omega)$ is $K_{(P,\omega)}(\mathbf{x})$ symmetric?

Stanley conjectures that $K_{(P,\omega)}(\mathbf{x})$ is symmetric only when $(P, \omega)$ corresponds to some skew Young diagram and proves this conjecture in the naturally labeled case. The only way a naturally labeled poset can correspond to a skew Young diagram is if it is a disjoint union of chains. The combinatorial interpretation given in Theorem 4.6.9 may be useful in answering Question 4.7.4. A natural direction to begin with is when $(P, \omega)$ is a generalized ribbon. In this case, if $(P, \omega)$ corresponds to a skew Young diagram, then the skew shape must be a disjoint union of ribbons.

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