ABSTRACT

CHANDLER, ALEX. On Thin Posets and Categorification. (Under the direction of Radmila Sazdanović).

Motivated by the cube construction of Khovanov homology, we discuss thin posets and their capacity to support homology theories. We outline a general technique to construct homology theories which categorify polynomials that admit formulas as rank alternating sums over thin posets. We present two main applications of this general setup. The first application is a broken circuit model for chromatic homology, categorifying Whitney's broken circuit theorem for the chromatic polynomial of graphs. The second is a categorification of certain generalized Vandermonde determinants. Finally, we give a theorem which characterizes torsion in locally thin regions of Khovanov homology. As an application of this theorem, we give explicit computations of the integral Khovanov homology for an infinite class of 3-braids strictly containing the 3-strand torus links.

On Thin Posets and Categorification

by
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To Amy.

BIOGRAPHY

Alex Chandler was born in Commerce Township, Michigan on August 9th, 1988, and grew up in South Lyon, Michigan. Much of his childhood and young adulthood was spent playing music and retro video games, and studying various martial arts. While at Washtenaw Community College, his love of music led him to learn the physics of sound and wave motions, and in turn this led him to the study of mathematics. At Michigan State University, he double majored in Mathematics and Physics. During his first summer of graduate school, he participated in an REG program, where he studied homology theories of links and graphs with Radmila Sazdanović, whom would soon after become his PhD advisor. On March 5th 2019, he married Amy Yang, whom he met during his first year of graduate school at NC State.

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INTRODUCTION

Categorification can be thought of as something like a mathematician's version of Plato's allegory of the cave. In Plato's allegory, a group of prisoners are confined in a dark cave facing a blank wall. Outside the cave, there is a fire which casts 2D shadows on the cave wall as 3D objects (birds, deer, etc...) pass by. The prisoners have no understanding of 3D objects and are aware only of the motions of the shadows on the wall. Plato's idea of a philosopher is "one who escapes the cave" and learns the true nature of these 3D objects. With this allegory in mind, one might venture to imagine that many of the familiar mathematical objects in our experience are really "shadows" of some more structured objects hidden from our view. This is the idea of categorification. Inspired by the ideas of Crane and Frenkel [CF94], we now attempt to "escape the cave".

There is no precise all-encompassing definition of categorification, but one might describe it as the process of finding category theoretic analogues of set theoretic or algebraic structures, or more generally, as replacing n-categories by $(n+1)$-categories (see for example [BD98]). Following Baez and Dolan [BD98] we give an analogy between set theory and category theory in Figure 1.1. To categorify a set S one should find a category $\mathscr{C}$ and a function (ideally a bijection) $\mathrm{Decat}: \{\text{isom. classes of } \mathscr{C}\} \to S$. If S has some extra structure (for example a group or ring structure), then $\mathscr{C}$ should have natural isomorphisms between objects (known as *coherence conditions*) which descend under the function Decat to the appropriate structural equations between elements in S. In this context, we say that $\mathscr{C}$ *categorifies* S, the object x in $\mathscr{C}$) *categorifies* the element $\mathrm{Decat}([x])$ of S, and $\mathrm{Decat}([x])$ is the *decategorification* of x. In general, there may be many ways to categorify a categorification $\longrightarrow$

sets	categories
elements	objects
equations between elements	isomorphisms between objects
functions	functors
equations between functions	natural transformations between functors

$\longleftarrow$ decategorification

Figure 1.1 An analogy between set theory and category theory taken from [BD98]. We can think of a set as a 0-category and a category as a 1-category. More generally, categorification should replace an n category by an $(n+1)$ category.

given object, but some ways may be more useful/natural than others. The “right" categorification should be one that not only lifts structures present at the decategorified level, but also introduces interesting new structures. We now provide some standard examples of categorification. Examples 1.0.1 and 1.0.2 can be thought of as the basic building blocks for more interesting categorifications such as Examples 1.0.3 and 1.0.4. Here, we keep the discussion somewhat informal, but these ideas can be made precise with the concept of the Grothendieck group (see Theorem 2.6.1, Example 2.6.2 and Example 2.6.3).

Example 1.0.1. The natural numbers $(\mathbb{N},+,\cdot)$ form the structure of a *rig*, that is, a ring without necessarily having additive inverses. The rig $(\mathbb{N},+,\cdot)$ is categorified by the category $\Bbbk$-**Vect** of finite dimensional vector spaces over a field $\Bbbk$. Decategorification is done by taking the dimension of the vector space: $\mathrm{Decat}(V)=\dim(V)$. Direct sums and tensor products of vector spaces behave nicely in this regard: $\dim(V\oplus W)=\dim(V)+\dim(W)$ and $\dim(V\otimes W)=\dim(V)\dim(W)$. More precisely, $(\mathbb{N},+,\cdot)$ can be recovered from $(\Bbbk\text{-}\mathbf{Vect},\oplus,\otimes)$ as the collection of isomorphism classes of objects in $\Bbbk$-**Vect** with operations $[V]+[W]=[V\oplus W]$ and $[V]\cdot[W]=[V\otimes W]$. The category $\Bbbk$-**Vect** satisfies all of the appropriate coherence conditions needed to categorify $\mathbb{N}$ as a rig, and for this reason is called a *rig category*. Not only are the usual properties of the natural numbers lifted to the category of vector spaces, but upon categorification, we introduce a variety of useful new tools and structures unavailable at the decategorified level (that is, all of linear algebra!).

An alternative way to categorify $(\mathbb{N},+,\cdot)$ is with $(\mathbf{FinSet},\amalg,\times)$ where **FinSet** is the category of finite sets (whose morphisms are functions), $\amalg$ denotes disjoint union, and $\times$ denotes cartesian product. Decategorification is done by taking cardinality. Again, we satisfy the desired properties $|S\amalg T|=|S|+|T|$ and $|S\times T|=|S|\cdot|T|$. This approach is widely used in enumerative combinatorics:

to show two natural numbers $n, m \in \mathbb{N}$ are equal, combinatorialists like to find sets N and M with cardinalities n and m respectively, and show that N and M are isomorphic (in this category, isomorphic means "in bijection with").

Example 1.0.2. The ring $(\mathbb{Z}, +, \cdot)$ of integers is categorified by the category $\mathscr{C}^b(\Bbbk\text{-}\mathbf{Vect})$ of bounded chain complexes of $\Bbbk$-vector spaces. Integers are categorified by chain complexes V_* where decategorification is done by taking the Euler characteristic of a chain complex $\mathrm{Decat}(V_*) = \chi(V_*) = \sum_{n\in\mathbb{Z}}(-1)^n \dim(V_i) \in \mathbb{Z}$. Again, direct sums and tensor products behave as desired: $\chi(V_* \oplus W_*) = \chi(V_*) + \chi(W_*)$ and $\chi(V_* \otimes W_*) = \chi(V_*)\chi(W_*)$. The category $\mathscr{C}^b(\Bbbk\text{-}\mathbf{Vect})$ satisfies all of the appropriate coherence conditions needed to categorify $\mathbb{Z}$ as a ring. Again, upon categorification, we have introduced some very useful additional structure not available at the decategorified level (that is, all of homological algebra!). In the same manner, we can think of integers as being categorified by cochain complexes V^* (as opposed to chain complexes V_*). Note that in this example and the previous one, $\Bbbk$-**Vect** can be replaced with the category R-**mod** of finitely generated modules over a commutative ring R (in this case dim should be replaced by rank).

In the previous two examples, we see categorifications of $\mathbb{N}$ as a rig and of $\mathbb{Z}$ as a ring. As of yet, there is no known categorification of $\mathbb{Q}$ or $\mathbb{R}$ as rings. As a step in this direction, Khovanov and Tian give a categorification of $\mathbb{Z}[\frac{1}{2}]$ in [KT17]. Next, we give two classical examples of novel categorifications, both providing a rich variety of new tools and structures in their respective fields.

Example 1.0.3. Let **Sim** denote the category of simplicial complexes and simplicial maps, and let $\Delta \in \mathrm{Ob}(\mathbf{Sim})$. Let $f_n(\Delta)$ denote the number of n dimensional faces in Δ. The Euler characteristic $\chi(\Delta) = \sum_{n\geq 0}(-1)^n f_n(\Delta)$ is categorified by the simplicial homology $H_*(\Delta)$ of Δ in the sense that

$$\sum_{n\geq 0}(-1)^n \dim H_n(\Delta) = \chi(\Delta). \tag{1.1}$$

For this reason, given a chain complex C_*, the quantity $\sum_{n\in\mathbb{Z}}(-1)^n \dim C_n = \sum_{n\in\mathbb{Z}}(-1)^n \dim H_n(C_*)$ is called the Euler characteristic of C_*. Simplicial homology contains the information of the Euler characteristic, but also has much more information about the space. For example, each homology group $H_n(\Delta)$ is a topological invariant, and the rank of the homology group $H_n(\Delta)$ indicates the number of 'holes' of dimension n in Δ (see for example [Hat02, Chapter 2]).

The Euler characteristic of simplicial complexes is just a function $\chi : \mathrm{Ob}(\mathbf{Sim}) \to \mathbb{Z}$ whereas homology is a functor $H_* : \mathbf{Sim} \to \mathbb{Z}\text{-}\mathbf{gmod}$, where $\mathbb{Z}$-**gmod** denotes the category of graded $\mathbb{Z}$-modules. Thus, not only do we get a stronger invariant, but for each simplicial map $\Delta \to \Delta'$ we get an induced map $H_*(\Delta) \to H_*(\Delta')$. This functoriality is what gives simplicial homology its true power as compared to the Euler characteristic. For instance, functoriality of simplicial homology provides an easy proof of the Brouwer fixed-point theorem [Hat02, Corollary 2.15].

Example 1.0.4. A more recent example of categorification is the Khovanov homology of a knot/link. Khovanov homology is a bigraded Abelian group which categorifies the Jones polynomial of a link in the sense that one may recover the Jones polynomial by taking the (graded) Euler characteristic of the Khovanov homology. One can find details of this construction in Chapter 5 of this thesis. This theory was developed by Khovanov in [Kho00], with a more topological interpretation featured in the work of Bar-Natan in [BN05], and a more combinatorial construction featured by Viro in [Vir04]. The Jones polynomial is a powerful link invariant, and the Khovanov homology is a strictly stronger link invariant. Moreover, Khovanov homology can be shown to be functorial, meaning that a cobordism between links induces maps between the Khovanov homologies of those links. In particular this functoriality has proven useful in producing a lower bound for the slice genus of a knot (see [Ras10]). Rasmussen used this fact in [Ras10] to give a combinatorial proof of the Milnor conjecture, computing the slice genus of torus links.

The success of Khovanov's categorification of the Jones polynomial has inspired many other categorifications. For example, Ozsváth and Szabó's categorification of the Alexander polynomial in [OS04] done independently by Rasmussen in [Ras03]. In fact, there is a whole family $\{P_n\}_{n\in\mathbb{N}}$ of polynomial link invariants (including the Jones polynomial P_2, and the Alexander polynomial P_0, as special cases) which are categorified by bigraded homology theories [KR04]. Khovanov and Rozansky's categorification of the HOMFLY-PT polynomial (developed in [KR08] and [Kho07]) requires a triply-graded link homology theory. Similar in spirit to these theories, are Helme-Guizon and Rong's categorification of the chromatic polynomial in [HGR05], Hepworth's categorification of the magnitude of a graph in [HW15], Sazdanović and Yip's categorification of the Stanley chromatic symmetric function [SY18], Stǒsić's categorification of the dichromatic polynomial [Sto08], and many more. Chapter 2 of this thesis is dedicated to this approach to categorification and outlining its use in the most general setting possible.

The theme of this thesis is categorification. In particular, categorification using the technique outlined by Khovanov in his categorification of the Jones polynomial. In the construction of Khovanov homology, Khovanov lifts a state sum formula for the Kauffman bracket (a rank-alternating sum over the Boolean lattice) to a functor on the Boolean lattice (viewing the Boolean lattice as a category). From this functor, he obtains a chain complex, and thus a homology theory. This homology theory categorifies the Kauffman bracket (and hence the Jones polynomial) in the sense that we recover the Kauffman bracket by taking its graded Euler characteristic. We generalize his approach by replacing the Boolean lattice by a broader class of posets under which this technique works. Thus we equip ourselves with the ability to categorify a broader class of invariants, those which admit rank alternating sums over thin posets. This broader class of posets is known as the class of *thin posets*.

This thesis is organized as follows. In Chapter 2, we discuss thin posets, and their capacity to support homology theories, outlining how these can be used to categorify ring elements (usually polynomials) of a certain form (rank-alternating sums over thin posets). In Chapter 3, we use a result from Chapter 2 based on algebraic Morse theory to categorify Whitney's broken circuit theorem for chromatic polynomials and chromatic symmetric functions. As an application, we give a simplified model for computation, as well as restrictions on the homological support of the graph homology theories of Helme-Guizon and Rong [HGR05], and Sazdanović and Yip [SY18]. In Chapter 4, we give a categorification of the Vandermonde determinant using the techniques from Chapter 2. In Chapter 5, we prove that locally thin regions of Khovanov homology (under some minor restrictions) can have only $\mathbb{Z}_2$ torsion. As an application, we give explicit computations of the integral Khovanov homology for an infinite class of 3-braids strictly containing the 3-strand torus links.

CHAPTER

2

THIN POSETS, CW POSETS, AND CATEGORIFICATION

In 2000, Khovanov [Kho00] categorified the Jones polynomial by lifting Kauffman's 'state sum' formula (a sum over a Boolean lattice) to a homology theory built from a functor on the Boolean lattice. This construction was further explained and popularized soon after by Bar-Natan [BN02] and Viro [Vir04]. Since Khovanov's categorification, many authors have used similar techniques to categorify other polynomials which admit state sum formulas over Boolean lattices (see for example [HGR05; SY18; DL15; HW15; Sto08]). In this chapter, we set out a general framework for a larger class of posets, called thin posets (containing Boolean lattices as a special case), in which this categorification technique can be applied.

In Section 2.2 we introduce the class of diamond transitive thin posets. Diamond transitive thin posets P have the property that there are far fewer conditions needed to check that a functor on P is well defined. In particular, for diamond transitive thin posets, one need only show that compositions of morphisms agree along any two coinitial coterminal directed paths of length 2 in the Hasse diagram of P, whereas in general one must check this condition for chains of all lengths. The most important example of thin posets appearing in this chapter are the CW posets, that is, the face posets of regular CW complexes. The first main result of this chapter is given in Theorem 2.4.12, where we show that CW posets are diamond transitive. We surmise a partial converse to this result in Conjecture 2.8.1. The technique outlined in Section 2.5 for obtaining homology theories from functors on thin posets P requires the use of a so-called balanced coloring on P, that is, a $\{1,-1\}$ coloring of the cover relations in P which plays nicely with the 'thin structure' in P (see Definition 2.3.5). However, balanced colorings on thin posets are not unique. The second main result in this chapter is Theorem 2.5.4, where we show that for diamond transitive thin posets, the homology theory obtained from a given functor is independent of the choice of balanced coloring. The proof of Theorem 2.5.4 depends on a topological interpretation of the property of diamond transitivity. In particular, we define a regular CW complex $X(P)$ whose 1-skeleton is the Hasse diagram $H(P)$, and show in Theorem 2.3.4 that if P is diamond transitive, then $X(P)$ is simply connected.

In this thesis, we concern ourselves with topological/combinatorial invariants of spaces X which are related to the incidence ordering in the collection of cells $\mathscr{F}(X)$ in some cell decomposition of those spaces, namely, those which admit formulas of the type $\sum_{e\in\mathscr{F}(X)}(-1)^{\dim(e)}f(e)$ where $f:\mathscr{F}(X)\to R$ is a function to some ring R. For example, Euler characteristics, h-polynomials, Tutte polynomials, and Jones polynomials all arise in this form. Notice these formulas (which we shall call rank alternators) do not depend on the partial order on $\mathscr{F}(X)$ in any way. In this context, categorification can be thought of as the process of upgrading these invariants in such a way that the partial order is taken into account. The next main result of this chapter is Theorem 2.6.6, where we outline how to categorify such invariants using functors on thin posets. The last main result of this chapter is Theorem 2.7.7, where we use algebraic Morse theory to simplify the calculation of the homology theory coming from a functor on a thin poset, given that the Hasse diagram of the poset has an acyclic matching for which the functor in question assigns isomorphisms to each edge in the matching.

We begin with a review of some needed terminology and notations from the theory of partially ordered sets. Nearly all of the material in Section 2.1 is standard and can be found in [Sta98]. For the more obscure material, citations will be provided along the way.

2.1 Background on Partially Ordered Sets

A *partially ordered set* (or *poset*) is a set P together with a reflexive, antisymmetric, transitive relation, denoted $\leq_P$ (or just $\leq$ if P is understood). All posets in this thesis are assumed to be finite. A poset P is *totally ordered* if for all $x, y \in P$, $x \leq y$ or $y \leq x$. A *linear extension* of a poset $(P, \leq)$ is a total order $(P, \leq')$ on P with the property that $x \leq y$ implies $x \leq' y$ for all $x, y \in P$. If $x \leq y$ and $x \neq y$ we write $x < y$. If P has a unique minimal (respectively maximal) element, we denote this element by $\hat{0}$ (respectively $\hat{1}$). A *lower order ideal* in P is a subset L of P with the property that if $y \in L$ and $x \leq y$ then $x \in L$. A map $f : P \to Q$ between posets is *order preserving* if $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for all $x, y \in P$, and is called an *order embedding* if the converse of this implication also holds. Notice that an order embedding is necessarily injective. An order embedding $f : P \to Q$ is an *isomorphism* if f is also surjective. A *subposet* of P is a poset P' together with an injective order preserving map

$f : P' \to P$. A subposet P' is *induced* if $f : P' \to P$ is an order embedding.

A *cover relation* in a poset is a pair $(x, y) \in P \times P$ with $x \leq y$ such that there is no $z \in P$ with $x < z < y$, and in this case we write $x \lessdot y$. Let $C(P)$ denote the set of all cover relations in P. The *Hasse diagram* of a finite poset is the directed graph $(P, C(P))$ with nodes P and a directed edge drawn from left to right from x to y if and only if $x \lessdot y$ (Hasse diagrams are normally drawn bottom up, but in this thesis we draw them from left to right as this convention will be more useful for the construction of homology theories). We will call a poset *connected* if its Hasse diagram is connected. A *chain* in a poset P is a totally ordered subposet $C \subseteq P$. We say that a chain C is *from x to y* if x (resp. y) is the smallest (resp. largest) element of C. A chain in P is *saturated* if every cover relation in C is a cover relation in P. A chain is *maximal* if it is not properly contained in any other chain. Equivalently, a maximal chain is a saturated chain from a minimal element of P to a maximal element of P.

An *abstract simplicial complex* is a collection Δ of subsets of a fixed finite set V (called the *vertex set*) such that if $F \in \Delta$ and $E \subseteq F$, then $E \in \Delta$. Each element of Δ is called a *face* of Δ. The *geometric realization* of Δ is gotten by embedding the vertex set as an affinely independent set in $\mathbb{R}^n$ for large enough n, and taking the union of the convex hulls of each $F \in \Delta$ (this is well defined up to homeomorphism). Given a poset P, the *order complex*, $\Delta(P)$, is the abstract simplicial complex whose faces are the chains in P. That is, $\Delta(P) = \{F \subseteq P \mid F \text{ is totally ordered}\}$. One should note that we always have $\varnothing \in \Delta(P)$.

A poset P is *graded* if there is a function $\mathrm{rk} : P \to \mathbb{N}$ with the property that $x \lessdot y$ implies $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. Given a graded poset P, the *rank* of P, denoted $\mathrm{rk}(P)$, is the maximum length of any chain in P, where the length of a chain C is defined as $|C| - 1$. If P has $\hat{0}$, we will always assume that $\mathrm{rk}(\hat{0}) = 0$, and so in this case, we will have $\mathrm{rk}(P) = \max\{\mathrm{rk}(x) \mid x \in P\}$. If P is a graded poset, and $x \leq y$ in P, the *length* of the interval $[x, y] = \{z \in P \mid x \leq z \leq y\}$ is $\mathrm{rk}(y) - \mathrm{rk}(x)$. Equivalently, the length of $[x, y]$ is the length of any saturated chain from x to y.

Definition 2.1.1 ([Bjö84, Section 4])**.** A graded poset P is *thin* if every closed nonempty interval of length 2 has exactly 4 elements. In a thin poset, a closed nonempty interval of length 2 is called a *diamond*.

Remark 2.1.2. There is another class of posets referred to as thin posets in [BM11]. The above definition of thin posets is taken from [Bjö84] and has nothing to do with the one given in [BM11].

We now give a class of examples which we will make heavy use of throughout this thesis. In every example involving the concept of a 'face poset' we will always assume the existence of the 'empty face' which is contained in every other face, thus such posets will always have unique minimal elements $\hat{0}$. If P is a poset with $\hat{0}$, we will often write $\bar{P} = P \setminus \{\hat{0}\}$.

Example 2.1.3. Let S be a set. The *Boolean lattice*, 2^S, is the collection of subsets of S partially ordered by inclusion. The Boolean lattice is graded by cardinality, and is thin since any interval of length 2 is of the form $\{T, T\cup\{x\}, T\cup\{y\}, T\cup\{x,y\}\}$. See Figure 2.1 for an example of the Hasse diagram of a Boolean lattice.

Example 2.1.4. Let Δ be an *abstract simplicial complex*. The *face poset* of Δ is the poset $\mathscr{F}(\Delta)$, with underlying set Δ, partially ordered by inclusion. Every interval in $\mathscr{F}(\Delta)$ is isomorphic to a Boolean lattice, and therefore face posets of abstract simplicial complexes are also thin posets. Since the face poset of a simplex is itself a Boolean lattice, Example 2.1.3 is a special case of this example. The reader is referred to [Koz07, Section 2.1] for background information on abstract simplicial complexes and face posets if needed. Simplicial complexes give an effective way to study topological spaces using combinatorial methods due to the following fact: Given a topological space X, suppose there is a simpicial complex Δ such that $||\Delta||$ is homeomorphic to X. In this case, Δ is referred to as a *triangulation* of X. Then the homeomorphism type of X is determined by the poset $\mathscr{F}(\Delta)$ in the sense that $X \cong \Delta(\mathscr{F}(\Delta))$. In fact, the order complex $\Delta(\mathscr{F}(\Delta))$ is the barycentric subdivision of Δ (see for example [Wac06, Section 1]).

Example 2.1.5. Let Γ be a *polyhedral complex*, that is, a collection Γ of polyhedra in $\mathbb{R}^n$ for some n such that:

1. if $F \in \Gamma$ and E is a face of F, then $E \in \Gamma$,

2. the intersection of any two polyhedra in Γ is a face of each.

See [Koz07, Section 2.2] for background information on polyhedral complexes if needed. The *face poset* of Γ is the poset $\mathscr{F}(\Gamma)$ with underlying set Γ, partially ordered by inclusion. Notice that the face poset of a polyhedral complex consisting of a single simplex (and all of its faces) is a Boolean lattice. Thus, Example 2.1.4 is a special case of this example. See Figure 2.1 for an example of the Hasse diagram of a polytopal complex (in this case, just a single polygon).

Example 2.1.6. For any Coxeter group W, the Bruhat order $\mathrm{Br}(W)$ is a graded poset, with rank function given by the length of a reduced word. See [BB06, Section 2] for background information on the Bruhat order if needed. Björner noted in [BB06, Lemma 2.7.3] that $\mathrm{Br}(W)$ is a thin poset.

Example 2.1.7. Recall a *ball* in a topological space X is a subspace B of X such that B is homeomorphic to the subspace $\{x \in \mathbb{R}^d \,:\, |x| \le 1\}$ of $\mathbb{R}^d$ for some d. The *interior* of a ball is the image of $\{x \in \mathbb{R}^d \,:\, |x| < 1\}$ under such a homeomorphism. Let X be a Haussdorff space with a *regular CW decomposition*, Γ. That is, Γ is a collection of closed balls in X such that the interiors of balls in Γ partition X, and for each ball $\sigma \in \Gamma$ of positive dimension, the boundary $\partial\sigma$ is a union of balls in Γ. In this case, Γ is called a *regular CW complex*, and one will often write $||\Gamma|| = \cup_{\sigma\in\Gamma}\sigma$ so in particular $||\Gamma|| = X$. The space $||\Gamma||$ is called the *geometric realization* of Γ. There is a more general way to decompose a topological space, using so-called CW complexes [Hat02, Chapter 0], which are defined by specifying attaching maps to glue balls of different dimensions together. One may alternatively define regular CW complexes as those CW complexes whose attaching maps are all homeomorphisms.

Given a regular CW complex Γ, its *face poset*, $\mathscr{F}(\Gamma)$, is the poset with underlying set Γ, partially ordered by containment. See [BB06, Appendix A2.5] or [Bjö84] for background information on regular CW complexes and their face posets. It was noted by Björner that face posets of regular CW complexes are thin [Bjö84, Proposition 3.1 and Figure 1(a)]. This is not true in general for face posets of CW complexes. Following the terminology in [Bjö84], we will refer to face posets of regular CW complexes as *CW posets*. See Definition 2.1.8 for a poset theoretic definition and Theorem 2.1.9 to see that the two definitions are equivalent. As shown in [Bjö84, Section 2], Examples 2.1.3, 2.1.4, 2.1.5 and 2.1.6 are all special cases of face posets of regular CW complexes. From the perspective of a combinatorialist interested in topology, regular CW complexes Γ are important because the combinatorics of $\mathscr{F}(\Gamma)$ determines the topology of $||\Gamma||$ in the sense that $||\Gamma|| \cong \Delta(\mathscr{F}(\Gamma))$ [BB06, Fact A.2.5.2]. This is not true in general for face posets of CW complexes. Thus regular CW complexes can be thought of as those CW complexes which can be completely understood combinatorially.

Definition 2.1.8 ([Bjö84, Definition 2.1])**.** A *CW poset* is a poset P with $\hat{0}$, and at least one other element, for which any open interval of the form $(\hat{0}, x)$ is homeomorphic to a sphere (that is, the order complex $\Delta(\hat{0}, x)$ is homeomorphic to a sphere).

The following properties of CW posets are proven in [Bjö84] Proposition 2.6, Figure 1(a), and Sections 2.3-2.5.

Theorem 2.1.9 ([Bjö84])**.** *Let P be a finite poset.*

1. *P is a CW poset if and only if P is a face poset of a regular CW complex.*
2. *If P is a CW poset then P is thin.*
3. *If P is a CW poset and L is a lower order ideal in P, then L is a CW poset.*
4. *If P is a CW poset and $x \in P$, then $\{y \in P \mid y \geq x\}$ is a CW poset.*
5. *Face posets of simplicial complexes, Bruhat orders on Coxeter groups, and face posets of polytopal complexes are all CW posets.*

Example 2.1.10. An *Eulerian poset* is a poset in which every nontrivial interval has the same number of elements in even rank as in odd rank. This condition immediately implies that intervals of length two are diamonds, so all Eulerian posets are thin. In fact, Stanley showed in [Sta94, Section 1], that CW posets are Eulerian, so this example includes all of the previous examples as special cases.

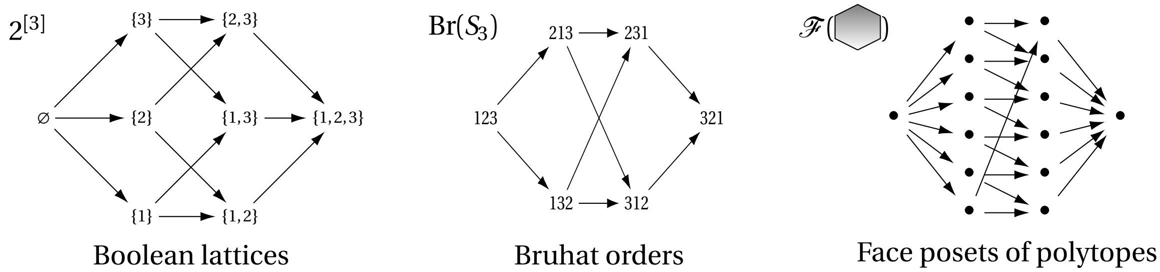


Figure 2.1 In this figure we see the Boolean lattice $2^{[3]}$ consisting of all subsets of $[3] = \{1,2,3\}$ (or equivalently the face poset of the 2-simplex), the Bruhat order on the symmetric group S_3, and the face poset of a hexagon.

2.2 The Diamond Group and Diamond Transitivity

Given a thin poset P, let S denote the set of diamonds in P (see Definition 2.1.1). Let $W(P)$ denote the group with presentation $\langle S \mid R\rangle$, where $R = \{d^2 \mid d \in S\}$. Given $x, y \in P$ with $x \le y$, let $\mathscr{C}_{x,y}$ denote the set of all maximal chains in the interval $[x, y]$ and let $\mathscr{C}_P$ (or just $\mathscr{C}$ when P is understood) denote the disjoint union of all of the sets $\mathscr{C}_{x,y}$ for each pair (x, y) where $x \le y$ in P. In other words, $\mathscr{C} = \mathscr{C}_P$ is the set of all saturated chains in P.

Definition 2.2.1. Given a diamond $d = \{x, a, b, y\} \in S$ (where $x \lessdot a \lessdot y$ and $x \lessdot b \lessdot y$) and a saturated chain $C \in \mathscr{C}$, define $dC \in \mathscr{C}$ as follows:

- If $x \lessdot a \lessdot y$ is a subchain of C, define $dC = (C \setminus \{a\}) \cup \{b\}$,
- If $x \lessdot b \lessdot y$ is a subchain of C, define $dC = (C \setminus \{b\}) \cup \{a\}$,
- Otherwise, define $dC = C$.

We will refer to the process of passing from C to dC as performing a *diamond move* on C.

Intuitively, if C contains one side of the diamond d, we form dC by rerouting C along the other side of the diamond d. Notice that for any $d \in S$, and any $C \in \mathscr{C}$, $d(dC) = C$, and therefore this defines an action of $W(P)$ on $\mathscr{C}$ via $(d_1 \dots d_{k-1} d_k)(C) = d_1(\dots d_{k-1}(d_k(C))\dots)$. Let $N(P)$ denote the subgroup of $W(P)$ generated by all words which act trivially on $\mathscr{C}$. The following two facts are special cases of a more general well known result, but we provide proofs here for convenience.

Lemma 2.2.2. *For any thin poset P with diamond set S, $N(P)$ is a normal subgroup of $W(P)$.*

Proof. Let $w \in N(P)$, $y \in W(P)$, and $C \in \mathscr{C}$. Then

$$y^{-1} w y(C) = y^{-1}(w(y(C)) = y^{-1}(y(C)) = (y y^{-1})(C) = eC = C.$$

Therefore $y^{-1}wy \in N$ and we conclude that $N(P)$ is normal. □

Definition 2.2.3. Given a thin poset P, the *diamond group*, $D(P)$ is defined as the quotient of $W(P)$ by the normal subgroup $N(P)$:

$$D(P) = W(P)/N(P).$$

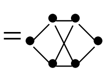
Recall, a group action is called *faithful* if the only element of G which fixes every element of X is the identity element.

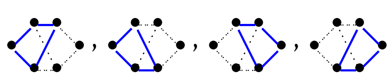
Lemma 2.2.4. *Given a thin poset P, $D(P)$ acts faithfully on the set $\mathscr{C}$ of saturated chains.*

Proof. Given $w \in W(P)$ let us denote the class of w in $D(P)$ by $[w]$. The action of $W(P)$ on $\mathscr{C}$ decends to an action of $D(P)$ on $\mathscr{C}$ via $[w]C = wC$. This is well defined since if $[w] = [v]$, then $w^{-1}v \in N(P)$ and thus $w^{-1}v$ acts trivially on all elements of $\mathscr{C}$. Therefore, for all $C \in \mathscr{C}$, $wC = w(w^{-1}v)(C) = vC$. This is a group action since $[v]([w]C) = v(wC) = (vw)(C) = ([vw])C = ([v][w])C$ for all $v, w \in W(P)$ and $C \in \mathscr{C}$. Suppose $[w] \in D(P)$ satisfies $[w]C = C$ for all $C \in \mathscr{C}$. Then $wC = C$ for any $C \in \mathscr{C}$ and therefore $w \in N(P)$ and therefore $[w]$ is the identity element in $D(P)$. □

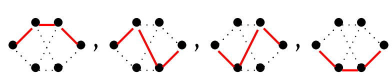
Recall that if a group G acts on a set X, and $Y \subseteq X$ is a subset which is closed under the action of G, then G also acts on Y by restriction. In our case, notice that for each $x \le y$, $\mathscr{C}_{x,y} \subseteq \mathscr{C}$ is closed under the action of $D(P)$. An action of G on X is called *transitive* if for any $x, y \in X$ there exists $g \in G$ such that $gx = y$. In other words, if we let $\mathscr{O}_x = \{gx \mid g \in G\}$ denote the *orbit* of x under the action of G, then G acts transitively on X if $X = \mathscr{O}_x$ for some (equivalently for all) $x \in X$.

Definition 2.2.5. A thin poset P is *diamond transitive* if $D(P)$ acts transitively on $\mathscr{C}_{x,y}$ for each pair (x, y) such that $x \le y$ in P.

Example 2.2.6. Let $P = \mathrm{Br}(S_3) =$ ,

$S = \{$  $\}$

and

$\mathscr{C}_{\hat{0},\hat{1}} = \{$  $\}$.

Then $D(P)$ acts on $\mathscr{C}_{\hat{0},\hat{1}}$ according to the table in Figure 2.2.

We now give a characterization of small intervals in diamond transitive thin posets.

Theorem 2.2.7. *Intervals of length 2 and 3 in a finite diamond transitive thin poset P are of the following prescribed combinatorial type:*

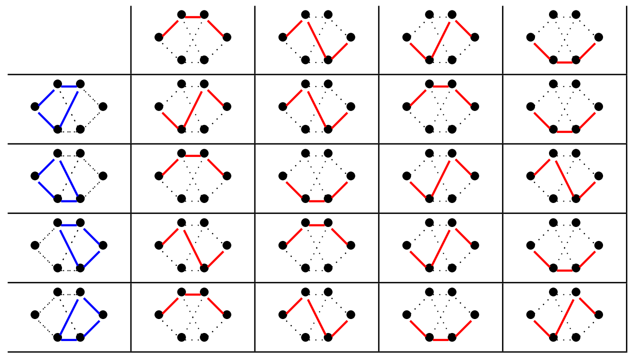
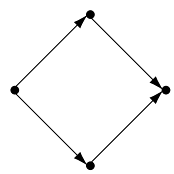
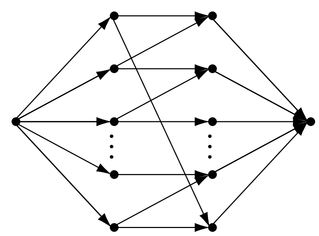


Figure 2.2 The top row shows the set of maximal chains (in red), and the leftmost column shows the set of diamonds (in blue) for the Bruhat order on S_3. The maximal chain shown in row d and column C is the chain dC defined by the action of $D(P)$ on $\mathscr{C}_{\hat{0},\hat{1}}$.



Intervals of length 2



Intervals of length 3

Proof. The result for intervals of length 2 follows from thinness. Consider now an interval $[x, y]$ of length 3 in a diamond transitive thin poset. Take a chain of the form $x \lessdot a_0 \lessdot b_0 \lessdot y$ in $[x, y]$. The interval $[x, b_0]$ must be a diamond, so there must be $a_1 \neq a_0 \in P$ with $x \lessdot a_1 \lessdot b_0$. The interval $[a_1, y]$ must be a diamond, so there must be an element $b_1 \neq b_0 \in P$ with $a_1 \lessdot b_1 \lessdot y$. The interval $[a_0, y]$ must be a diamond so either $a_0 \lessdot b_1$ or we continue the process by adding elements a_2, b_2 with $a_2 \lessdot b_1$ and $a_2 \lessdot b_2$. Since P is finite, one must eventually stop adding such elements a_i, b_i. Now, the interval $[x, b_0]$ must be a diamond, so eventually we must connect a_0 to b_k for some k.

Once we reach k such that $a_0 \lessdot b_k$, we claim there can be no more elements in $[x, y]$ besides those already considered. Suppose there were some element a_{k+1} in $[x, y]$ with $x \lessdot a_{k+1}$. If $a_{k+1} \lessdot b_i$ for some $i \leq k$ then thinness is contradicted so there must be some b_{k+1} with $a_{k+1} \lessdot b_{k+1} \lessdot y$. If $a_i \lessdot b_{k+1}$ for any $i \leq k$ then again thinness is contradicted. However now, there is no way to get to the chain $x \lessdot a_{k+1} \lessdot b_{k+1} \lessdot y$ from the chain $x \lessdot a_k \lessdot b_k \lessdot y$ by diamond moves, thus contradicting diamond transitivity. Thus every interval of length 3 is of the combinatorial type shown in the statement above. $\square$

Remark 2.2.8. Every CW poset has intervals of length 2 and 3 isomorphic to those shown in Theorem 2.2.7, as noted by Björner in [Bjö84, Figure 1]. Thus one may wonder the relationship between diamond transitive thin posets and CW posets. As a partial answer to this, Theorem 2.4.12 tells us that all CW posets are diamond transitive. Conjecture 2.8.1 surmises a partial converse to this.

The following theorem, while less general than the forthcoming Theorem 2.4.12, provides an opportunity to present a direct proof of diamond transitivity.

Theorem 2.2.9. *Face posets of simplicial complexes are diamond transitive.*

Proof. Let P be the face poset of a simplicial complex. Recall intervals in face posets of simplicial complexes are Boolean lattices. Consider an interval $[x, y]$ in P. Then by the above remark, $[x, y] \cong 2^{[n]}$ for some n. The set of elements in $2^{[n]}$ of a given rank (that is, all subsets of $[n]$ of a given size) can be totally ordered lexicographically, and this induces a lexicographic on maximal chains. Let C_0 denote the lexicographically largest chain $\varnothing \subseteq \{n\} \subseteq \{n-1, n\} \subseteq \cdots \subseteq [n]$.

We now show, by induction on the distance $d(C, C_0)$ in lexicographic order between C and C_0 that there exists $w \in D(P)$ such that $wC = C_0$. The base case is trivial since for distance 0 we have $C = C_0$. Otherwise, since C is not lexicographically largest, there is a subchain $S \subseteq S+i \subseteq S+i+j$ of C with $i < j$. Now consider the (lexicographically larger) chain dC where $d = \{S, S+i, S+j, S+i+j\}$. By induction we have $w \in D(P)$ such that $w(dC) = C_0 = (wd)C$. □

2.3 The Diamond Space and Balanced Colorability

In this section, we give a topological interpretation of diamond transitivity. This interpretation turns out to be useful in regard to understanding certain $\{1, -1\}$ colorings (called balanced colorings) of the cover relations of P, which we use in Section 2.5 to define homology theories.

Definition 2.3.1. Given a thin poset P, we construct a regular CW complex $X(P)$ as follows:

1. The 0-skeleton, $X^0(P)$ of $X(P)$ is the set P.

2. The 1-skeleton, $X^1(P)$ of $X(P)$ is the (undirected) Hasse diagram of P, that is, we glue in a 1-cell connecting pairs (x, y) whenever $x \lessdot y$.

3. The 2-skeleton $X^2(P) = X(P)$ of $X(P)$ is gotten by gluing 2-cells into each diamond in P. Since we will add no higher dimensional cells, $X(P)$ is equal to the 2-skeleton.

Call the resulting regular CW complex $X(P)$ the *diamond space* of P.

Let P be a thin poset. Fix a continuous map $\phi : X^1(P) \to \mathbb{R}$, so that for each $a \in X^0(P)$, $\phi(a) = \mathrm{rk}(a)$, and on each closed 1-cell e in $X^1(P)$, corresponding to an edge in $H(P)$ from a to b, $\phi|_e$ is a linear isomorphism onto the interval $[\mathrm{rk}(a), \mathrm{rk}(b)] \subseteq \mathbb{R}$.

Definition 2.3.2. Let $r_1, r_2 \in \mathbb{R}$. A continuous map $\gamma : [r_1, r_2] \to X^1(P)$ is *monotonically increasing* (resp. *monotonically decreasing*) if for all $s, t \in [r_1, r_2]$, $s < t$ implies $\phi(\gamma(s)) < \phi(\gamma(t))$ (resp. $\phi(\gamma(s)) > \phi(\gamma(t))$).

Lemma 2.3.3. *Let P be a diamond transitive thin poset with $\hat{0}$. Any continuous map $\gamma : [0,1] \to X(P)$ with $\gamma(0) = \gamma(1) = \hat{0}$ is homotopic to a piecewise linear map $\gamma' : [0,1] \to X^1(P)$ with $\gamma'(0) = \gamma'(1) = \hat{0}$ such that there exists $r \in (0,1)$ so that $\gamma'|_{[0,r]}$ is monotonically increasing and $\gamma'|_{[r,1]}$ is monotonically decreasing.*

Proof. Let $\gamma : [0,1] \to X(P)$. Using the cellular approximation theorem [Hat02, Theorem 4.8], we see that γ is homotopic to a map $\bar{\gamma}$ whose image is contained in $X^1(P)$. Then, by applying the simplicial approximation theorem [Hat02, Theorem 2C.1] to $\bar{\gamma}$, we see that γ is homotopic to a map $\tilde{\gamma}$ that is simplicial with respect to some iterated barycentric subdivision of $[0,1]$. In other words, there is a partition $0 = r_0 < r_1 < \cdots < r_{2k} = 1$ of $[0,1]$ such that

1. $\tilde{\gamma}(r_0) = \tilde{\gamma}(r_{2k}) = \hat{0}$
2. $\tilde{\gamma}(r_i) \in X^0(P)$ for each i
3. For $0 \le i \le 2k-1$, $\tilde{\gamma}(r_i) \lessdot \tilde{\gamma}(r_{i+1})$ or $\tilde{\gamma}(r_{i+1}) \lessdot \tilde{\gamma}(r_i)$
4. If $r \in [0,1] \setminus \{r_0, \ldots, r_{2k}\}$ then $\tilde{\gamma}(r) \in X^1(P) \setminus X^0(P)$
5. For $0 \le i \le 2k-1$, $\tilde{\gamma}$ is linear (in particular either monotonically increasing or decreasing) on $[r_i, r_{i+1}]$.

Choose $i_1 < \cdots < i_{2\ell}$ where $\{i_1, \ldots, i_{2\ell}\} \subseteq \{1, \ldots, 2k\}$ such that $\tilde{\gamma}$ is monotonically increasing on $[0, r_{i_1}]$, monotonically decreasing on $[r_{i_1}, r_{i_2}]$, monotonically increasing on $[r_{i_1}, r_{i_3}]$ and so on. We will show by induction on ℓ (that is, half the number of subintervals in such a partition) that $\tilde{\gamma}$ is homotopic to a map γ' as desired in the statement of this lemma. If $\ell = 1$, then $\tilde{\gamma}$ monotonically increases on $[0, r_{i_1}]$, and monotonically decreases on $[r_{i_1}, 1]$, so we are done. If $\ell > 1$ we proceed as follows. Recall that paths α, β can be multiplied to form the product path $\alpha\beta$ which traverses first α and then β. Also recall that, given a path $\alpha : [a,b] \to X$ in a topological space X, we let α^{-1} denote the path defined by $\alpha^{-1} : [-b, -a] \to X$ where $\alpha^{-1}(t) = \alpha(-t)$, so α^{-1} traverses the same points as α but in the opposite direction (see [Hat02, Section 1.1]). In the Hasse diagram $H(P)$, $\tilde{\gamma}|_{[0, r_{i_1}]}$ corresponds to a directed path from $\hat{0}$ to some element $x_1 \in P$. Now, $(\tilde{\gamma}|_{[r_{i_1}, r_{i_2}]})^{-1}$ corresponds to a directed path in $H(P)$ starting at some element x_2 and ending at x_1. Let $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_n = x_2$ be a directed path in $H(P)$, and let $\beta : [0,1] \to X^1(P)$ be a monotonically increasing piecewise linear map corresponding to this directed path. Then $\beta(\tilde{\gamma}|_{[r_{i_1}, r_{i_2}]})^{-1}$ is piecewise linear and monotonically increasing from $\hat{0}$ to x_1. Since P is diamond transitive, there is a sequence of diamond moves taking the path $\tilde{\gamma}|_{[0, r_{i_1}]}$ to the path $\beta(\tilde{\gamma}|_{[r_{i_1}, r_{i_2}]})^{-1}$. Each diamond move defines a homotopy in $X(P)$ taking a path along one side of the diamond to the path along the other side. Thus, $\tilde{\gamma} = \tilde{\gamma}|_{[0,r_{i_1}]}\tilde{\gamma}|_{[r_{i_1},1]}$ is homotopic to the path $\beta(\tilde{\gamma}|_{[r_{i_1},r_{i_2}]})^{-1}\tilde{\gamma}|_{[r_{i_1},1]}$ which is homotopic to $\gamma' = \beta(\tilde{\gamma}|_{[r_{i_2},1]})$. Notice that γ' has a partition of its domain $0 < r_{i_3} < \cdots < r_{i_{2\ell}}$ with $2(\ell-1)$ subintervals so that γ' is monotonic on each subinterval. Thus we are done by induction. □

Theorem 2.3.4. *Let P be a diamond transitive thin poset with $\hat{0}$. Then $\pi_1(X(P),\hat{0})$ is trivial.*

Proof. Suppose that P is diamond transitive. By Lemma 2.3.3, any loop $\gamma : [0,1] \to X(P)$ based at $\hat{0}$ can be homotoped to a loop $\gamma' : [0,1] \to X^1(P)$ such that there exists $r \in (0,1)$ for which $\gamma'|_{[0,r]}$ increases monotonically to some element $x \in P$ and $\gamma'|_{[r,1]}$ decreases monotonically from x back to $\hat{0}$. Let $C_1 = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = x)$ be the saturated chain in P corresponding to $\gamma'|_{[0,r]}$ and let $C_2 = (\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k = x)$ be the saturated chain in P corresponding to $(\gamma'|_{[r,1]})^{-1}$. Since P is diamond transitive, there is a sequence of diamond moves that takes C_1 to C_2, and this defines a homotopy from $\gamma'|_{[0,r]}$ to $(\gamma'|_{[r,1]})^{-1}$. Thus γ' is homotopic to $\gamma'|_{[0,r]}(\gamma'|_{[0,r]})^{-1}$ which is homotopic to the constant map at $\hat{0}$. Thus we have shown that any loop in $X(P)$ based at $\hat{0}$ is nullhomotopic. □

Definition 2.3.5 ([BB06, Section 2.7])**.** A *balanced coloring* on a thin poset P is a function $c : C(P) \to \{1,-1\}$ if each diamond in P is assigned an odd number of -1 colored edges. A thin poset P is *balanced colorable* if it admits a balanced coloring.

We have now seen a relationship between the combinatorial property of diamond transitivity in P to the topology of the space $X(P)$. Next, we show that the property of balanced colorability can also be thought of in terms of the diamond space $X(P)$. Balanced colorings of P are $\{1,-1\}$ colorings of the edges in the Hasse diagram $H(P)$. Thus balanced colorings can be viewed as elements of the cochain group $C^1(X(P),\mathbb{Z}_2) = \mathrm{Hom}(C_1(X(P),\mathbb{Z}_2)$. Warning: the coefficient groups for cohomology are usually written with additive notation, however because of our situation, we will use multiplicative notation, identifying $(\{0,1\},+)$ with $(\{1,-1\},\cdot)$. Consider the function $\phi_0 : C_2(X(P)) \to \mathbb{Z}_2 = \{1,-1\}$ sending every 2-cell to $-1 \in \mathbb{Z}_2$. Recall, 2-cells correspond to diamonds in P, so we will denote 2-cells by d and each 2-cell has boundary consisting of four edges $\{e_1^d, e_2^d, e_3^d, e_4^d\}$. Given $\psi : C_1(X(P)) \to \mathbb{Z}_2$, $\delta\psi : C_2(X(P)) \to \mathbb{Z}_2$ is defined on 2-cells d with boundary $\{e_1^d, e_2^d, e_3^d, e_4^d\}$ by $\delta\psi(d) = \psi(e_1^d)\psi(e_2^d)\psi(e_3^d)\psi(e_4^d)$. If $\delta\psi = \phi_0$, then $\psi(e_1^d)\psi(e_2^d)\psi(e_3^d)\psi(e_4^d) = -1$ for each diamond d, or in other words, ψ is a balanced coloring. Since $X(P)$ has no 3-cells, ϕ_0 is guaranteed to be a cocycle. Since we are working over the field $\mathbb{Z}_2$, we have $H^k(X(P),\mathbb{Z}_2) \cong \mathrm{Hom}(H_k(X(P),\mathbb{Z}_2),\mathbb{Z}_2)$ for all k. Thus we have proven the following.

Proposition 2.3.6. *Let $\phi_0 : C_2(X(P)) \to \mathbb{Z}_2 = \{1,-1\}$ denote the cocycle sending every 2-cell to $-1 \in \mathbb{Z}_2$. Then P is balanced colorable if and only if $[\phi_0]$ is trivial in $H^2(X(P),\mathbb{Z}_2)$.*

In particular, this says that if $H^2(X(P),\mathbb{Z}_2)$ is trivial, then P is balanced colorable. However, after working out any number of simple examples, the reader will notice that this is very rarely going to be the case. The following result will be applicable in a wider variety of situations.

Proposition 2.3.7. *Let P be a thin poset and suppose that $X(P)$ forms the 2-skeleton of a regular CW complex $Z(P)$ in which each 3-cell has an even number of 2-dimensional faces, and suppose that $H^2(Z(P),\mathbb{Z}_2)=0$. Then P is balanced colorable.*

Proof. Since $X(P)$ and $Z(P)$ have the same 2-skeleton, we have that $C^2(X(P),\mathbb{Z}_2) = C^2(Z(P),\mathbb{Z}_2)$ so we can identify ϕ_0 as a 2-cochain in $Z(P)$. Let F be a 3-cell in $Z(P)$. By assumption, F has an even number of 2-dimensional faces, $d_1,\dots,d_{2n}$. In our multiplicative notation ($\mathbb{Z}_2=\{1,-1\}$) we have $\delta\phi_0(F)=\prod_{i=1}^{2n}\phi_0(d_i)=(-1)^{2n}=1$. Therefore $\delta\phi_0$ is the identity element in $C^3(Z(P),\mathbb{Z}_2)$, so $\phi_0\in C^2(Z(P),\mathbb{Z}_2)$ is a cocycle. Since $H^2(Z(P),\mathbb{Z}_2)=0$, ϕ_0 is a coboundary. Thus $\phi_0=\delta\psi$ for some $\psi: C_1(Z(P))\to\mathbb{Z}_2$. Since $X(P)$ and $Z(P)$ have the same 1-skeleton, we have $C_1(X(P))=C_1(Z(P))$ so we can identify $\psi: C_1(X(P))\to\mathbb{Z}_2$ as a 1-cochain in $X(P)$. Therefore we have that $[\phi_0]$ is trivial in $H^2(X(P),\mathbb{Z}_2)$ and so by Propsitition 2.3.6, P is balanced colorable. □

Example 2.3.8. As an easy application of Proposition 2.3.7, we see that the Boolean lattice B_n is balanced colorable since $X(B_n)$ is the 2-skeleton of the n-cube. Note that every 3-cell in an n-cube is a 3-cube, and thus has an even number of faces. Khovanov makes use of such a balanced coloring in [Kho00, Section 3.3], en route to constructing a homology theory which categorifies the Jones polynomial. More generally, this argument shows that any poset whose Hasse diagram is the 1-skeleton of a *cubical polytope* is balanced colorable.

Definition 2.3.9. Given a thin poset P, a *central coloring* of P is a map $d: C(P)\to\{1,-1\}$ such that every diamond has an even number of -1's.

Let P be a thin poset, and let Φ denote the group of all edge colorings $f: C(P)\to\mathbb{Z}_2=\{1,-1\}$ of the Hasse diagram, with group operation defined by pointwise multiplication of functions. Let $\Phi^b\subseteq\Phi$ denote the set of balanced colorings and let $\Phi^c\subseteq\Phi$ denote the set of central colorings.

Lemma 2.3.10. *If P is diamond transitive, then for any $d\in\Phi^c$ there exists $f: P\to\{1,-1\}$ such that $\delta f=d$.*

Proof. Since P is diamond transitive, it follows from Theorem 2.3.4 that we have $H^1(X(P),\mathbb{Z}_2)=0$. Since $c\in\Phi^c$, we have $c\in\ker\delta^1=\mathrm{im}\delta^0$. □

Lemma 2.3.11. *For any $c_1,c_2\in\Phi^b$, $c_1c_2\in\Phi^c$.*

Proof. Consider a diamond with edges e,f,g,h. Then

$$c_1(e)c_1(f)c_1(g)c_1(h)=-1$$

and

$$c_2(e)c_2(f)c_2(g)c_2(h)=-1$$

so

$$(c_1c_2)(e)(c_1c_2)(f)(c_1c_2)(g)(c_1c_2)(h) = 1. \qquad \square$$

Lemma 2.3.12. *For any $c_1, c_2 \in \Phi^b$ there exists $d \in \Phi^c$ such that $c_1 d = c_2$.*

Proof. Notice that for any $c \in \Phi$, $c^2 = \mathrm{id}_\Phi$. Let $d = c_1c_2$ (an element of Φ^c by Lemma 2.3.11). Then $c_1 d = c_1c_1c_2 = \mathrm{id}_\Phi c_2 = c_2$. $\square$

This homological interpretation of balanced colorings is utilized in Section 2.5, where we use balanced colorings to construct homology theories from thin posets. To conclude this section, we state a result of Björner, which also finds its utility in Section 2.5.

Theorem 2.3.13 ([BB06, Corollary 2.7.14])**.** *CW posets are balanced colorable.*

Proof. This follows from [BB06, Corollary 2.7.14] by noticing the proof is not specific to the Bruhat order, but holds for all CW posets. $\square$

2.4 CW posets and Obstructions to Diamond Transitivity

In this section, we begin by giving a way to construct thin posets which are not diamond transitive. We then show that this construction provides the only possible type of obstruction to diamond transitivity. This gives us a classification of diamond transitive posets in terms of obstructions, and this classification allows us to prove that all CW posets are diamond transitive.

Definition 2.4.1. Given two thin posets P (with $\hat{0}_P$ and $\hat{1}_P$) and Q (with $\hat{0}_Q$ and $\hat{1}_Q$) of the same rank $n \geq 3$, define the *pinch product* $P \ominus Q$ by

$$P \ominus Q = (P - \{\hat{0}_P, \hat{1}_P\}) \amalg (Q - \{\hat{0}_Q, \hat{1}_Q\}) \amalg \{\hat{0}, \hat{1}\}$$

where $\amalg$ denotes disjoint union. A partial order on $P \ominus Q$ is defined by setting $x \leq y$ if any of the following conditions hold:

1. $x = \hat{0}$ or $y = \hat{1}$,
2. $x, y \in P - \{\hat{0}_P, \hat{1}_P\}$ and $x \leq_P y$,
3. $x, y \in Q - \{\hat{0}_Q, \hat{1}_Q\}$ and $x \leq_Q y$.

In other words, the pinch product is formed by removing the $\hat{0}$ and $\hat{1}$ from each poset, taking the disjoint union, and then adding a new $\hat{0}$ and $\hat{1}$ to the result.

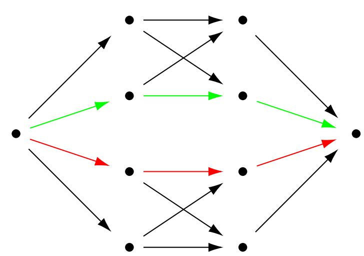
Lemma 2.4.2. *Given thin posets P and Q of the same rank $n \geq 3$, $P \ominus Q$ is thin.*

Proof. The pinch product $P \ominus Q$ is graded with rank function

$$\operatorname{rk}(x) = \begin{cases} 0 & \text{if } x = \hat{0} \\ n & \text{if } x = \hat{1} \\ \operatorname{rk}_P(x) & \text{if } x \in P - \{\hat{0}_P, \hat{1}_P\} \\ \operatorname{rk}_Q(x) & \text{if } x \in Q - \{\hat{0}_Q, \hat{1}_Q\} \end{cases}$$

Any interval of length two can be identified as either an interval of length two in P or an interval of length two in Q and is thus a diamond. $\square$

Example 2.4.3. Let $P = \operatorname{Br}(S_3) \ominus \operatorname{Br}(S_3)$. The Hasse diagram of P is shown below:



Notice the green chain and the red chain are in different orbits of $D(P)$, so P is not diamond transitive. In fact, it is not hard to see that P is the smallest possible non-diamond transitive thin poset. This poset also appears in [Sta94, Figure 2] as an example of an Eulerian poset which is not a CW poset. It is not hard to see that P is also the smallest possible thin poset with $\hat{0}$ which is not a CW poset.

Lemma 2.4.4. *For any two thin posets P, Q, with $\hat{0}$ and $\hat{1}$, of equal rank $n \geq 3$, there is a group isomorphism $f : D(P) \times D(Q) \to D(P \ominus Q)$ such that the following diagram commutes:*

$$\begin{tikzcd} D(P) \times D(Q) \arrow[r, "f"] \arrow[d, hook] & D(P \ominus Q) \arrow[d, hook] \\ S_{\mathscr{C}_P} \times S_{\mathscr{C}_Q} \arrow[r, hook] & S_{\mathscr{C}} \end{tikzcd}$$

Proof. Let S (resp. S_P, S_Q) be the set of diamonds in $P \ominus Q$ (resp. P, Q), and let $\mathscr{C}$ (resp. $\mathscr{C}_P, \mathscr{C}_Q$) be the set of saturated chains in $P \ominus Q$ (resp. P, Q). Then $S = S_P \amalg S_Q$ and $\mathscr{C} = \mathscr{C}_P \amalg \mathscr{C}_Q$. Furthermore notice that if $d \in S_P$ and $e \in S_Q$ that $de = ed$ in $D(P \ominus Q)$. Therefore we can factor $D(P \ominus Q) = D(P) \times D(Q)$ by writing any word $d_1 \dots d_k \in D(P)$ in the form $(e_1 \dots e_r)(f_{r+1} \dots f_k)$ where $e_i \in S_P$ and $f_j \in S_Q$ for all i, j. We define the map f by sending $(e_1 \dots e_r f_1 \dots f_s) \in D(P) \times D(Q)$ to $e_1 \dots e_r f_1 \dots f_s \in D(P \ominus Q)$. Commutativity of the above diagram follows from the fact that rearranging the diamonds in a word as prescribed by the map f does not change the action of the word on chains, (since diamonds in P commute with diamonds in Q). $\square$

Lemma 2.4.5. *For any two thin posets P, Q, with $\hat{0}$ and $\hat{1}$, of equal rank $n \geq 3$, $P \ominus Q$ is not diamond transitive.*

Proof. Given a maximal chain C in P, a maximal chain C' in Q, and $w \in D(P \ominus Q)$, the action of $f^{-1}(w)$ on C gives another maximal chain in P, so by the commutative diagram in Lemma 2.4.4, $wC \neq C'$ for any $w \in D(P \ominus Q)$. □

In what follows, given a subset S of a poset $(P, \leq)$, we will automatically identify S with the induced subposet of P with underlying set S and partial order $\leq'$ defined for all $x, y \in S$ by $x \leq' y$ if and only if $x \leq y$. Given any interval $[x, y]$ in P of length at least 2, the diamond group $D([x, y])$ acts on the set of maximal chains in $[x, y]$. Given a maximal chain C in $[x, y]$ (that is, a saturated chain from x to y), let $\mathcal{O}_C$ denote the orbit of C with respect to this action. Given any saturated chain C, define P_C to be the induced subposet of P with underlying set $\cup_{C' \in \mathcal{O}_C} C'$.

Definition 2.4.6. An induced subposet S of P is *saturated* if every saturated chain in S is also saturated in P.

Definition 2.4.7. A saturated subposet S of P is *diamond-complete* if every for every diamond d in P, and every saturated chain C in S, $dC \subseteq S$.

Lemma 2.4.8. *For any saturated chain C in a thin poset P, P_C is a diamond-complete thin subposet.*

Proof. Every saturated chain in P_C is a saturated chain in P by construction. Given a diamond d in P and a saturated chain C' in P_C, $dC' \in \mathcal{O}_C$ and therefore $dC' \subseteq P_C$ by definition, so P_C is diamond complete. To see that P_C is thin, suppose that $x < y$ in P_C and $\text{rk}(y) = \text{rk}(x) + 2$. Since $x < y$ is not saturated in P, it must not be saturated in P_C. Therefore there is an element $a \in P_C$ with $x < a < y$. Extend this to a maximal chain C' in P_C. Since P is thin, there exists another element $b \in P$ with $x < b < y$, and consider the diamond $d = \{x, a, b, y\}$. By definition of P_C, all elements of dC' are in P_C and therefore $b \in P_C$, so the interval $[x, y]$ in P_C consists of four elements $\{x, a, b, y\}$. □

Lemma 2.4.9. *Given $x < y$ in P with $\text{rk}(y) \geq \text{rk}(x) + 3$, suppose C and D are saturated chains from x to y such that $\mathcal{O}_C \neq \mathcal{O}_D$ and $P_C \cap P_D = \{x, y\}$. Then $P_C \cup P_D$ is a diamond-complete thin subposet of P isomorphic to $P_C \ominus P_D$.*

Proof. First we show that $P_C \cup P_D$ is induced by induction on the length ℓ of C. The base case is for $\ell = 3$. In this case, $C = x \lessdot c_1 \lessdot c_2 \lessdot y$ and $D = x \lessdot d_1 \lessdot d_2 \lessdot y$. Suppose towards a contradiction that $P_C \cup P_D$ is not induced. Then without loss of generality we must have $c_1 \lessdot d_2$. However in this case, (c_1, y) consists of at least 3 elements, contradicting thinness of P. Suppose now that $\ell > 3$, and again suppose towards a contradiction that $P_C \cup P_D$ is not induced. In this case, $C = x \lessdot c_1 \lessdot \cdots \lessdot c_{\ell-1} \lessdot y$ and $D = x \lessdot d_1 \lessdot \cdots \lessdot d_{\ell-1} \lessdot y$. Then without loss of generality there is a cover relation $c_i \lessdot d_{i+1}$ for some $1 \leq i \leq \ell - 2$. Let C' denote the chain $x \lessdot c_1 \lessdot \cdots \lessdot c_i \lessdot d_{i+1}$ and let D' denote the chain

$x \lessdot d_1 \lessdot \cdots \lessdot d_i \lessdot d_{i+1}$. Then $P_{C'}$ is an induced subposet of P_C and $P_{D'}$ is an induced subposet of P_D. Let C'' denote the chain $c_i \lessdot c_{i+1} \lessdot \cdots \lessdot \lessdot y$ and let D'' denote the chain $c_i \lessdot d_{i+1} \lessdot \cdots \lessdot y$. Then $P_{C''}$ is an induced subposet of P_C and $P_{D''}$ is an induced subposet of P_D. By induction, $P_{C'} \cup P_{D'}$ and $P_{C''} \cup P_{D''}$ are induced and meaning that there are no other cover relations between C and D besides $c_i \lessdot d_{i+1}$ and possibly $d_i \lessdot c_{i+1}$. This contradicts thinness of P since then we have $(c_i, d_{i+2}) = \{d_{i+1}\}$.

We have that $P_C \cup P_D$ is saturated since any saturated chain in $P_C \cup P_D$ is either a saturated chain in P_C or a saturated chain in P_D, and so by Lemma 2.4.8, is saturated in P. Furthermore, $P_C \cup P_D$ is diamond-complete since any saturated chain in $P_C \cup P_D$ lies entirely in P_C or P_D respectively, and thus by Lemma 2.4.9, any diamond move on such a chain gives another chain in P_C or P_D respectively. Lastly, $P_C \cup P_D$ is thin since any interval of length two in $P_C \cup P_D$ is either an interval of length two in P_C or in P_D and is therefore a diamond again by Lemma 2.4.8. □

Theorem 2.4.10. *Let P be a thin poset. Then P is diamond transitive if and only if P contains no diamond-complete subposet isomorphic to a pinch product of two thin posets.*

Proof. If P is not diamond transitive, one can find saturated chains C and D from x to y in P for some $x, y \in P$ such that $\mathcal{O}_C \neq \mathcal{O}_D$ and $P_C \cap P_D = \{x, y\}$. Lemma 2.4.9 then provides a diamond-complete thin subposet isomorphic to a pinch product. Conversely, suppose P contains a diamond-complete subposet isomorphic to a pinch product, say of the form $Q = Q_1 \ominus Q_2$ with some unique minimal element $\hat{0}_Q$ and maximal element $\hat{1}_Q$. Let $C_1 \subseteq Q_1$ and $C_2 \subseteq Q_2$ be a saturated chains from $\hat{0}_Q$ to $\hat{1}_Q$ in P. Note that Q_1 and Q_2 are diamond-complete by Lemma 2.4.8 since $Q_1 = P_{C_1}$ and $Q_2 = P_{C_2}$. For any sequence of diamonds $d_1 \dots d_r$ in P, $(d_1 \dots d_r)C_1 \subseteq Q_1$, so C_1 and C_2 are in different orbits of the action of $D(P)$ and therefore P is not diamond transitive. □

In Remark 2.2.8 and Example 2.4.3, we noted that CW posets share certain similarities with diamond transitive thin posets. We now further explore this connection.

Proposition 2.4.11. *If P can be written in the form $P = P_1 \ominus P_2$, where P_1 and P_2 are thin posets of equal rank, then P is not a CW poset.*

Proof. Since P has $\hat{1}$, if P is a face poset of a CW complex Y, then Y must be connected. Suppose to the contrary that $P = P_1 \ominus P_2$ is the face poset of a regular CW complex. Removing the top element, $P = P_1 \ominus P_2 \setminus \hat{1}$ is also the face poset of a regular CW complex $X = X_1 \amalg X_2$ where $X_1 = P_1 \setminus \hat{1}$ and $X_2 = P_2 \setminus \hat{1}$. But there is no way to glue an n-cell to an $n-1$ dimensional skeleton which is not connected, and obtain a connected space (i.e. the image of a continuous map $f : e^n \to X_1 \amalg X_2$ must be contained in either X_1 or X_2). □

Theorem 2.4.12. *CW posets are diamond transitive.*

Proof. Let P be a CW poset. We proceed by induction on $|P|$. If $|P| = 0$ or 1 the result is obvious. By Theorem 2.4.10, it suffices to show that P contains no diamond-complete subposets isomorphic to pinch products. Suppose towards a contradiction that P contains a diamond-complete subposet L such that L is isomorphic to a pinch product. If $L = P$ then this contradicts Proposition 2.4.11 so we can assume L is a proper subset of P. Now, L has $\hat{0}_L$ and $\hat{1}_L$ since it is isomorphic to a pinch product. By Theorem 2.1.9, we see that intervals of the form $[\hat{0}, x]$ and $[y, \hat{1}]$ in CW posets are CW posets. Suppose that P has $\hat{1}_P$ and $\hat{1}_L = \hat{1}_P$. Then the interval $[\hat{0}_L, \hat{1}_L]$ in P is a CW poset.

If $\hat{0}_L \neq \hat{0}_P$ then $[\hat{0}_L, \hat{1}_L]$ is a proper subset of P, and therefore by induction $[\hat{0}_L, \hat{1}_L]$ is diamond transitive. But L is a diamond-complete subposet in $[\hat{0}_L, \hat{1}_L]$ isomorphic to a pinch product, so this contradicts Theorem 2.4.11.

Suppose that $\hat{0}_L = \hat{0}_P$, and let $L = L_1 \ominus L_2$. Let $x \in P \setminus L$ and $y \in L \setminus \{\hat{0}, \hat{1}\}$. If $x \leq y$ then there must be some $x' \in P \setminus L$ and $y' \in L \setminus \{\hat{0}, \hat{1}\}$ such that $x' \lessdot y'$. The lower interval $[\hat{0}, y']$ is a CW poset and is therefore diamond transitive by induction. Therefore, a saturated chain from $\hat{0}$ to y' in L can be taken to a saturated chain from $\hat{0}$ which goes through x' to y'. But then we have $x' \in L$ by diamond-completeness. Similarly we can show that it is impossible to have $y \leq x$ with Let $x \in P \setminus L$ and $y \in L \setminus \{\hat{0}, \hat{1}\}$. Thus, setting $L' = P \setminus L \cup \{\hat{0}, \hat{1}\}$, we conclude that $L \cap L' = \{\hat{0}, \hat{1}\}$ and therefore we have $P = L \ominus L'$, contradicting Proposition 2.4.11.

If P does not have $\hat{1}_P$ or if P does have $\hat{1}_P$ but $\hat{1}_L \neq \hat{1}_P$, then the lower interval $[\hat{0}_P, \hat{1}_L]$ is a lower order ideal properly contained in P and so $[\hat{0}_P, \hat{1}_L]$ is a CW poset. Therefore, by induction, $[\hat{0}_P, \hat{1}_L]$ is diamond transitive. However, again L is a diamond-complete subposet in $[\hat{0}_P, \hat{1}_L]$ isomorphic to a pinch product, so this is a contradiction. □

Remark 2.4.13. Since CW posets are a special case of Eulerian posets, one might ask whether all Eulerian posets are diamond transitive. However, the poset in Example 2.4.3 provides an example of a non-diamond transitive Eulerian poset.

2.5 Homology of Functors on Thin Posets

A poset P can be thought of as a category with objects P and a unique morphism, denoted $x \leq y$, from x to y if and only if $x \leq y$ in P. A functor on a poset then can be thought of as a labeling of the nodes and edges of the Hasse diagram so that for any $x \leq y$, compositions along any two directed paths between x and y agree.

Let P be a poset (regarded as a category), let $\{F_x\}_{x \in P}$ be a labeling of the vertices of the Hasse diagram by objects of a category $\mathscr{C}$, and let $\{F_{y,x}\}_{x \lessdot y \in C(P)}$ be a labeling of the edges $C(P)$ of the Hasse diagram by morphims in $\mathscr{C}$, where $F_{y,x} : F_x \to F_y$. Suppose that for any $x \leq y$, and any two directed paths $x = x_0, x_1, \ldots, x_n = y$, $x = x'_0, x'_1, \ldots, x'_k = y$, from x to y, we have

$$F_{x_n,x_{n-1}} \circ \cdots \circ F_{x_1,x_0} = F_{x'_k,x'_{k-1}} \circ \cdots \circ F_{x'_1,x'_0}.$$

Then the following defines a functor $F : P \to \mathscr{C}$:

$$F(x) = F_x \quad \text{for all } x \in P \tag{2.1}$$

$$F(x \le y) = F_{x_n,x_{n-1}} \circ \cdots \circ F_{x_1,x_0} \tag{2.2}$$

where $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{n-1} \lessdot x_n = y$ is any directed path in the Hasse diagram from x to y.

Theorem 2.5.1. *Let P be a diamond transitive thin poset, let $\{F_x\}_{x\in P}$ be a labeling of P by objects of a category $\mathscr{C}$, and let $\{F_{y,x}\}_{x\lessdot y\in C(P)}$ be a labeling of $C(P)$ by morphims in $\mathscr{C}$, where $F_{y,x} : F_x \to F_y$. Then equations 2.1 and 2.2 determine a functor if and only if this labeling commutes on diamonds.*

Proof. Let C and C' denote two maximal chains from x to y where $x \le y$ in P. We proceed by induction on the minimal number of diamond moves needed to take C to C'. For the base case, if no diamond moves are needed, then $C = C'$ and we are done. If a positive number of diamond moves are needed, take a minimal word $w = w_1 \ldots w_k \in D(P)$ such that $wC = C'$. Set $C'' = w_k C$ and note $(w_1 \ldots w_{k-1})C'' = C$. Compositions along C and along $w_k C = C''$ agree since we have commutativity on diamonds. The minimum number of diamond moves needed to take C'' to C' is less than k so by induction compositions along C'' and C' agree. □

Definition 2.5.2. Let $\mathscr{A}$ be an abelian category, P a finite thin poset with balanced coloring c, and $F : P \to \mathscr{A}$ a covariant functor. Define a cochain complex, denoted $C^*(P,F,c)$ with differential δ given by the formulas

$$C^k(P,F,c) = \bigoplus_{\mathrm{rk}(x)=k} F(x) \qquad \delta^k = \sum_{\substack{x\lessdot y\\ \mathrm{rk}(x)=k}} c(x \lessdot y)F(x \lessdot y). \tag{2.3}$$

In the case that F is contravariant, we define a chain complex, denoted $C_*(P,F,c)$ with differential ∂ given by the formulas

$$C_k(P,F,c) = \bigoplus_{\mathrm{rk}(x)=k} F(x) \qquad \partial_k = \sum_{\substack{y\lessdot x\\ \mathrm{rk}(x)=k}} c(y \lessdot x)F(y \lessdot x). \tag{2.4}$$

Lemma 2.5.3 shows δ^2 and ∂^2 are 0 so these complexes are well defined. Denote the cohomology of $C^*(P,F,c)$ by $H^*(P,F,c)$ and homology of $C_*(P,F,c)$ by $H_*(P,F,c)$. Theorem 2.5.4 shows that this homology/cohomology does not depend on the balanced colorings, so we may instead write $H^*(P,F)$

or $H_*(P,F)$. If we wish to not distinguish between F being covariant/contravariant, or when it is clear from context, we may write expressions like $C(P,F,c)$ and $H(P,F,c)$ when we wish to make statements about both cases. To avoid the issue of balanced colorings, one may choose to work in a $\mathbb{Z}_2$-linear category where $1=-1$.

Lemma 2.5.3. *Let P be a thin poset with balanced coloring c, and let $F: P \to \mathscr{A}$ be a functor, where $\mathscr{A}$ is an abelian category. Then $C^*(P,F,c)$ is a cochain complex. In the case that F is contravariant, $C_*(P,F,c)$ is a chain complex.*

Proof. Suppose F is covariant. Given $A \in F(x)$, with $\mathrm{rk}(x)=k$, consider the component $\delta^2_{x,z}(A)$ of $\delta^2(A)$ in $F(z)$ where $\mathrm{rk}(z)=\mathrm{rk}(x)+2$. If $\delta^2_{x,z}(A)$ were nonzero, then by construction we must have $z \geq x$, in which case the interval $[x,z]=\{x,a,b,z\}$ is a diamond. Thus, we have

$$\begin{aligned}
\delta^2_{x,z} &= c(a \lessdot z)F(a \lessdot z)\circ c(x \lessdot a)F(x \lessdot a) + c(b \lessdot z)F(b \lessdot z)\circ c(x \lessdot b)F(x \lessdot b)\\
&= c(a \lessdot z)c(x \lessdot a)\big[F(a \lessdot z)\circ F(x \lessdot a)\big] + c(b \lessdot z)c(x \lessdot b)\big[F(b \lessdot z)\circ F(x \lessdot b)\big]\\
&= [c(a \lessdot z)c(x \lessdot a) + c(b \lessdot z)c(x \lessdot b)]\big[F(a \lessdot z)\circ F(x \lessdot a)\big]\\
&= 0
\end{aligned}$$

The computation for the case when F is contravariant can be done similarly, the only difference being that we would have $\mathrm{rk}(z)=\mathrm{rk}(x)-2$. □

Theorem 2.5.4. *Let P be a diamond transitive thin poset with balanced colorings $c_1, c_2 \in \Phi^b$. Let $\mathscr{A}$ be an abelian category and $F: P \to \mathscr{A}$ a functor. Then there is an isomorphism of complexes $C^*(P,F,c_1) \to C^*(P,F,c_2)$.*

Proof. Consider the cochain complex $(C^*(X(P),Z_2),\delta)$. Notice that $\ker\delta^1 = \Phi^c$ under the identification of $\{1,-1\}$ with $\mathbb{Z}_2 = \{0,1\}$. If P is diamond transitive, then $H_1(X(P)) = 0$ by Proposition 2.3.4 and thus $H^1(X(P),\mathbb{Z}_2) = \mathrm{Hom}(H_1(X(P),\mathbb{Z}_2),\mathbb{Z}_2) = 0$. Therefore any $c \in \Phi^c$ is a coboundary. Let $c_1, c_2 \in \Phi^b$. By Lemma 2.3.12 there is some $d \in \Phi^c$ such that $c_1 d = c_2$. Let $f: P \to \mathbb{Z}_2$ such that $d = \delta f$. Note: under the identification of $\mathbb{Z}_2$ with $\{1,-1\}$ this means that $d(x \lessdot y) = f(x)f(y)$. Define a chain map from $C^*(P,F,c_1) \to C^*(P,F,c_2)$ with components $f(x)id_{F(x)}: F(x) \to F(x)$ for each $x \in P$. In order to check this is a chain map, we must have, for any $x \lessdot y$, $c_2(x \lessdot y)F(x \lessdot y)f(x)\mathrm{id}_{F(x)} = f(y)\mathrm{id}_{F(y)}c_1(x \lessdot y)F(x \lessdot y)$, or equivalently, $c_2(x \lessdot y)f(x) = f(y)c_1(x \lessdot y)$. But $c_2 = d\,c_1$, so

$$\begin{aligned}
c_2(x \lessdot y)f(x) &= d(x \lessdot y)c_1(x \lessdot y)f(x)\\
&= f(x)f(y)c_1(x \lessdot y)f(x)\\
&= f(x)^2 f(y)c_1(x \lessdot y)\\
&= f(y)c_1(x \lessdot y).
\end{aligned}$$

□

Theorem 2.3.13, Proposition 2.4.12 and Theorem 2.5.4 together indicate that CW posets are the ideal setting for the construction of these homology theories.

Suppose $F : P \to \mathscr{A}$ is a covariant functor from a diamond transitive thin poset P to an abelian category $\mathscr{A}$. Suppose that P has a balanced coloring $c : C(P) \to \{1, -1\}$. If I is an upper or lower order ideal, then I is a thin poset, the restriction $c|_{C(I)}$ is a balanced coloring, and the restriction $F|_I : I \to \mathscr{A}$ is a functor. If I is an upper order ideal, then $C^*(I, F|_I, c|_I)$ is a subcomplex of $C^*(P, F, c)$, or equivalently, the inclusion $C^*(I, F|_I, c|_I) \hookrightarrow C^*(P, F, c)$ is a chain map. If F is contravariant, then restriction to a lower ideal gives a subcomplex.

Theorem 2.5.5. *Suppose P is a diamond transitive balanced colorable thin poset, and $F : P \to \mathscr{A}$ is a functor to an abelian category $\mathscr{A}$. Given an upper order ideal I in P, there is a long exact sequence*

$$\ldots H^*(I, F) \to H^*(P, F) \to H^*(P \setminus I, F) \xrightarrow{\partial} H^*(I, F) \ldots$$

where the map ∂ has degree 1. The analogous statement holds for lower order ideals in the case that F is contravariant, and in this case, ∂ has degree -1.

Proof. Let c be a balanced coloring of P. Since I is an upper order ideal, $P \setminus I$ is a lower order ideal, and $C^*(P, F, c)$ decomposes as a direct sum $C^*(P, F, c) = C^*(I, F|_I, c|_I) \oplus C^*(P \setminus I, F|_{P\setminus I}, c|_{P\setminus I})$ where $C^*(I, F|_I, c|_I)$ is a subcomplex of $C^*(P, F, c)$. Thus we get a short exact sequence of complexes

$$0 \to C^*(I, F|_I, c|_I) \hookrightarrow C^*(P, F, c) \twoheadrightarrow C^*(P \setminus I, F|_{P\setminus I}, c|_{P\setminus I}) \to 0$$

leading to the desired long exact sequence on cohomology. □

2.6 Categorification via Functors on Thin Posets

This section begins with some background information on Grothendieck groups, which can be found in [Wei13]. We will state these results without proof and refer the reader to the literature for details. We then show how to categorify certain ring elements, called rank alternators, with the homology theory obtained from a certain functor on a thin poset.

Let $\mathscr{A}$ be an abelian category, and let $\mathscr{C}^b(\mathscr{A})$ denote the category of bounded cochain complexes C in $\mathscr{A}$, that is, such that $C^i = 0$ for all but finitely many i. Note that $\mathscr{C}^b(\mathscr{A})$ is also an abelian category. Let $K_0(\mathscr{A})$ denote the *Grothendieck group* of $\mathscr{A}$, that is, the free abelian group generated by the set of isomorphism classes $\{[X] \mid X \in \mathrm{Ob}(\mathscr{C})\}$ modulo the relations $[X] = [X'] + [X'']$ for each short exact sequence $0 \to X' \to X \to X'' \to 0$. If $\mathscr{A}$ additionally has a monoidal structure $\otimes$, then $K_0(\mathscr{A})$ inherits the structure of a ring, with product $[X] \cdot [Y] = [X \otimes Y]$. The following result will be used extensively in later sections of this thesis.

Theorem 2.6.1 ([Wei13, Theorem 9.2.2])**.** *Let $\mathscr{A}$ be a monoidal abelian category. The Euler characteristic*

$$\chi : K_0(\mathscr{C}^b(\mathscr{A})) \to K_0(\mathscr{A}) \qquad [C] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [H^i(C)]$$

defines an isomorphism of rings. Furthermore, we have

$$\sum_{i\in\mathbb{Z}} (-1)^i [H^i(C)] = \sum_{i\in\mathbb{Z}} (-1)^i [C^i] \tag{2.5}$$

From now on, we identify $K_0(\mathscr{C}^b(\mathscr{A}))$ with $K_0(\mathscr{A})$ via the isomorphism in Theorem 2.6.1.

Example 2.6.2. Let R be a principal ideal domain. Let R-**mod** denote the (abelian) category of finitely generated (left) R modules. There is an isomorphism of rings:

$$K_0(R\text{-}\mathbf{mod}) \cong \mathbb{Z} \qquad [M] \mapsto \mathrm{rk} M$$

(see for example [Wei13, Section 2]). In light of Theorem 2.6.1 we have the following identification of rings:

$$K_0(\mathscr{C}^b(R\text{-}\mathbf{mod})) \cong \mathbb{Z} \qquad [C] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{rk} C^i.$$

Example 2.6.3. Let R be a principal ideal domain. Let R**-gmod** denote the (abelian) category of finitely generated graded (left) R modules. Given a graded module $M = \oplus_{i\in\mathbb{Z}} M^i \in R$**-gmod**, the *graded rank* of M is $q\mathrm{rk} M = \sum_{i\in\mathbb{Z}} q^i \mathrm{rk} M^i$. Let $\mathbb{Z}[q, q^{-1}]$ denote the ring of Laurent polynomials in the variable q. There is an isomorphism of rings:

$$K_0(R\text{-}\mathbf{gmod}) \cong \mathbb{Z}[q, q^{-1}] \qquad [M] \mapsto q\mathrm{rk} M = \sum_{i\in\mathbb{Z}} q^i \mathrm{rk} M^i$$

(see [Wei13, Example 7.14]). In light of Theorem 2.6.1 we have the following identification of rings:

$$K_0(\mathscr{C}^b(R\text{-}\mathbf{gmod})) \cong \mathbb{Z}[q, q^{-1}] \qquad [C] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i q\mathrm{rk} C^i = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \mathrm{rk} C^{i,j}.$$

The map $[C] \mapsto \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \mathrm{rk} C^{i,j}$ is usually referred to as the *graded Euler characteristic* (see for example [Kho00] or [BN02]).

Theorem 2.6.4. *Let $\mathscr{A}$ be a monoidal abelian category, P a balanced colorable thin poset, and $F : P \to \mathscr{A}$ a functor. Then in $K_0(\mathscr{C}^b(\mathscr{A}))$ we have the following equality:*

$$[H(P, F)] = \sum_{x\in P} (-1)^{\mathrm{rk}(x)} [F(x)].$$

Proof. We identify $H(P, F)$ in $\mathscr{C}^b(\mathscr{A})$ as a chain complex with all differentials being zero maps.

Then by Theorem 2.6.1, we have $[H(P,F)] = [C(P,F,c)]$ where c is a balanced coloring of P. Since $C^i(P,F,c) = \oplus_{\mathrm{rk}(x)=i} F(x)$, the desired equality follows from Theorem 2.6.1. $\square$

Theorem 2.6.4 provides the following categorification technique. In general, to categorify an element g of a ring R, one should find a monoidal abelian category $\mathscr{A}$ with an isomorphism $\phi : K_0(\mathscr{A}) \to R$, and an object $G \in \mathscr{C}^b(\mathscr{A})$ such that $[G] = g$ under the identification $\phi \circ \chi$ of $K_0(\mathscr{C}^b(\mathscr{A}))$ with R. In this case we say that G categorifies g.

Definition 2.6.5. Suppose P is a thin poset, R is a ring, and consider a function $f : P \to R$. The *rank alternator* for f is

$$\mathrm{RA}(P,f) = \sum_{x \in P} (-1)^{\mathrm{rk}(x)} f(x).$$

Theorem 2.6.6. *Let P be a balanced colorable thin poset, R a ring, and $f : P \to R$. Suppose $\mathscr{A}$ is a monoidal abelian category equipped with an isomorphism of rings $K_0(\mathscr{A}) \cong R$. Suppose $F : P \to \mathscr{A}$ is a functor with the property that for each $x \in P$, $[F(x)] = f(x)$. Then $H(P,F)$ categorifies* $\mathrm{RA}(P,f)$ *in the sense that* $[H(P,F)] = \mathrm{RA}(P,f)$.

Proof. This follows immediately from Theorem 2.6.4. $\square$

Remark 2.6.7. Theorem 2.6.6 gives the blueprint for all of the categorifications considered in this thesis.

We now look at some examples of interesting algebraic objects which arise as rank alternators over thin posets. Some of these examples have existing categorifications via the process outlined in Theorem 2.6.6 and others do not. We begin with the examples whose categorifications are known.

Example 2.6.8. Let X be a regular CW complex, and let $f_i(X)$ denote the number of cells in X of dimension i. Let $\bar{\mathscr{F}}(X)$ denote the face poset of X, with the empty face deleted. The Euler characteristic $\chi(X)$ of X is:

$$\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i f_i(X) = \sum_{F \in \bar{\mathscr{F}}(X)} (-1)^{|F|}.$$

Let $1_{R\text{-}\mathbf{mod}} : \bar{\mathscr{F}}(X) \to R\text{-}\mathbf{mod}$ denote the contravariant functor sending each $F \in \bar{\mathscr{F}}(X)$ to R and each cover relation to the identity map. Let c be a balanced coloring of $\bar{\mathscr{F}}(X)$ (we know one exists by Theorem 2.3.13). Then we can form the complex $C_*(\bar{\mathscr{F}}(X), 1_{R\text{-}\mathbf{mod}}, c)$ with homology $H_*(\bar{\mathscr{F}}(X), 1_{R\text{-}\mathbf{mod}})$ and by Theorem 2.6.6 we have $[H_*(\bar{\mathscr{F}}(X), 1_{R\text{-}\mathbf{mod}})] = \chi(X)$. Of course, this construction just recovers the definition of the cellular homology $H_*^{\mathrm{cell}}(X,R)$ of X with coefficients in R, so we have $H_*(\bar{\mathscr{F}}(X), 1_{R\text{-}\mathbf{mod}}) \cong H_*^{\mathrm{cell}}(X,R)$, and it is not much harder to see that $H_*(\mathscr{F}(X), 1_{R\text{-}\mathbf{mod}}) \cong \tilde{H}_*^{\mathrm{cell}}(X,R)$ where the tilde indicates reduced homology. Since the order complex $\Delta(\bar{\mathscr{F}}(X))$ is homeomorphic to X (see for example [BB06, Fact A2.5.2]), we find $H_*(\bar{\mathscr{F}}(X), 1_{R\text{-}\mathbf{mod}}) \cong H_*^{\mathrm{simp}}(\Delta(\bar{\mathscr{F}}(X)), R)$, where H^{simp} denotes the simplicial homology, and similarly one can see $H_*(\mathscr{F}(X), 1_{R\text{-}\mathbf{mod}}) \cong \tilde{H}_*^{\mathrm{simp}}(\Delta(\bar{\mathscr{F}}(X)), R)$.

Example 2.6.9. The Kauffman bracket of a link L is a rank alternating sum over a Boolean lattice (the collection of subsets of the set of crossings X of a diagram for L):

$$\langle L\rangle = \sum_{I\in 2^X} (-1)^{|I|} q^{|I|}(q+q^{-1})^{|D(I)|}$$

where $|D(I)|$ is the number of connected components in the Kauffman state corresponding to I. Khovanov homology (see Chapter 5 for details and references) is constructed by categorifying this formula as per the process outlined in Theorem 2.6.6 and then applying some grading shifts to obtain a homology theory which decategorifies to the Jones polynomial (which is a rescaling of the Kauffman bracket). This example was the original motivation for the generalization to Theorem 2.6.6. Ozsváth and Szabó's categorification of the Alexander polynomial [OS04] is of a similar nature (done independently by Rasmussen in [Ras03]).

Example 2.6.10. Let $G = (V, E)$ be a graph. The chromatic polynomial $\chi_G(x)$ of G counts the number of proper colorings $c : V \to [x]$ for $x \in \mathbb{N}$ (see Section 3.2 for details and background references). The chromatic polynomial admits a so-called "state sum" formula:

$$\chi_G(x) = \sum_{S\in 2^E} (-1)^{|S|} x^{k(S)}$$

where $k(S)$ denote the number of connected components in the spanning subgraph S. The chromatic homology of Helme-Guizon and Rong [HGR05] categorifies $\chi_G(x)$.

Similar in spirit is the chromatic symmetric function $X_G(\mathbf{x})$ defined by Stanley [Sta95] as a symmetric function generalization of the chromatic polynomial (see Section 3.3 for details and references). The Stanley chromatic symmetric function also admits a state sum formula:

$$X_G(\mathbf{x}) = \sum_{S\in 2^E} (-1)^{|S|} p_{\lambda(S)}$$

where $\lambda(S)$ is the integer partition encoded by the sizes of connected components in S and $p_{\lambda(S)}(\mathbf{x}) = p_{\lambda_1}(\mathbf{x})\dots p_{\lambda_k}(\mathbf{x})$ and $p_r(\mathbf{x}) = \sum_{i=0}^{\infty} x_i^r$ is the power sum symmetric function. Sazdanović and Yip categorify $X_G(\mathbf{x})$ in [SY18].

Both of these categorifications proceed using a method of the type outlined in Theorem 2.6.6, using the Boolean lattice. In Chapter 3 we explore an application of the forthcoming Theorem 2.7.7 to both of these homology theories.

Now that we have revisited some familiar homology theories, restated in terms of Theorem 2.6.6, we now explore some potential new ideas.

Example 2.6.11. The characteristic polynomial $f_P(t)$ of a ranked poset P is defined as

$$f_P(t) = \sum_{x\in P} \mu(x) t^{\mathrm{rk}(P)-\mathrm{rk}(x)}$$

where μ is the Möbius function on P (see for example [Sag99] for details on characteristic polynomials and Möbius functions). For Eulerian posets, $\mu(x) = (-1)^{\mathrm{rk}(x)}$ and therefore if P is Eulerian,

$$f_P(t) = \sum_{x\in P} (-1)^{\mathrm{rk}(x)} t^{\mathrm{rk}(P)-\mathrm{rk}(x)}.$$

From Example 2.6.8, we see that $[H_*(P, 1_{\mathscr{A}})] = \sum_{x\in P}(-1)^{\mathrm{rk}(x)}$ is exactly the characteristic polynomial for P evaluated at 1 (for Eulerian posets). In a similar fashion, one might imagine categorifying the characteristic polynomial itself (for Eulerian posets) by choosing a functor that sends $x \mapsto T^{\otimes \mathrm{rk}(P)-\mathrm{rk}(x)}$ for some fixed $T \in \mathscr{A}$. If such a functor F_T is constructed, then in $K_0(\mathscr{A})$, we have $[H_*(P, F_T)] = \sum_{x\in P}(-1)^{\mathrm{rk}(x)}[T]^{\mathrm{rk}(P)-\mathrm{rk}(x)} = f_P([T])$.

Example 2.6.12. This example is inspired by the discussion in [Sta86] pages 275-276. Let L be a lattice and let X be a subset of L such that $\hat{1} \notin X$ and if $s \in L$ and $s \neq \hat{1}$ then $s \leq t$ for some $t \in X$. We can take $X = L - \{\hat{0}, \hat{1}\}$ for example. By The Crosscut Theorem, [Sta98, Corollary 3.9.4],

$$\mu(\hat{0}, \hat{1}) = \sum_{k\in\mathbb{Z}} (-1)^k N_k$$

where N_k is the number of k-subsets of X whose meet is $\hat{0}$ and μ is the Möbius function. Define $X(L)$ to be the set of all subsets of X whose meet is not $\hat{0}$. Then $X(L)$ is a simplicial complex and the above equation can be rewritten as

$$\mu(\hat{0}, \hat{1}) = \sum_{F\in X(L)} (-1)^{|F|}.$$

Consider the constant functor $1_{\mathbb{Z}\text{-}\mathbf{mod}} : X(L) \to \mathbb{Z}\text{-}\mathbf{mod}$. By the discussion in Example 2.6.8, in the Grothendieck group we have $[H_*(X(L), 1_{\mathbb{Z}\text{-}\mathbf{mod}})] = \sum_{F\in X(L)}(-1)^{|F|} = \mu(\hat{0}, \hat{1})$. [Sta98, Proposition 3.8.6] states that $\mu(\hat{0}, \hat{1})$ is the (reduced) Euler characteristic of the order complex $\Delta(L')$ where $L' = L - \{\hat{0}, \hat{1}\}$ so we also have $[H_*(\Delta(L'), 1_{\mathbb{Z}\text{-}\mathbf{mod}})] = \mu(\hat{0}, \hat{1})$ in the Grothendieck group. Since they both categorify $\mu(\hat{0}, \hat{1})$, one may ask if these two homology theories are isomorphic. It turns out that $X(L)$ is in fact homotopy equivalent to $\Delta(L')$ (this is stated in [Sta98, page 276]) so the answer is yes, both being isomorphic to the common simplicial homology of the two homotopy equivalent spaces $X(L)$ and $\Delta(L')$ as per Example 2.6.11.

Example 2.6.13. Any determinant can be written by definition as a rank alternating sum over the

the symmetric group S_n:

$$\det(x_{i,j})_{i,j=1}^n = \sum_{\pi\in S_n}(-1)^{\mathrm{inv}(\pi)}x_{1,\pi(1)}\dots x_{n,\pi(n)}.$$

Recall that S_n has a partial order (the Bruhat order) $\mathrm{Br}(S_n)$ which is thin (in fact, it is a CW poset). In Chapter 4, we give a categorification of the Vandermonde determinant using a functor $F:\mathrm{Br}(S_n)\to\mathscr{A}$ from the Bruhat order on S_n to a symmetric monoidal abelian category $\mathscr{A}$.

Example 2.6.14. Let Δ be a simplicial complex. In combinatorics, one often encounters the f-polynomial $f(q)=\sum_{i=0}^m f_{i-1}q^i$ and the h-polynomial $h(q)=\sum_{i=0}^m h_iq^i$ where f_i is the number of faces of dimension i (subsets in Δ with $i+1$ vertices), $m-1$ is the maximum dimension of any face in Δ, and $h(q)=(1-q)^m f\left(\frac{q}{1-q}\right)$ (see for example [Zie12, Section 8.3]). In l^2-topology, it is $h(-q)$ that is of interest (see for example [Bor10, Section 1]). We can express $h(-q)$ in a way suitable for categorification:

$$\begin{aligned}
h(-q) &= (1+q)^m f\left(\frac{-q}{1+q}\right)\\
&= \sum_{i=0}^m f_{i-1}(-q)^i(1+q)^{m-i}\\
&= (1+q)^{(m-n)}\sum_{F\in\Delta}(-1)^{|F|}q^{|F|}(1+q)^{|V\setminus F|}.
\end{aligned}$$

To categorify $h(-q)$ using this formula, for example, one could construct a functor $H:\Delta\to R$**-gmod** such that $H(F)=A^{\otimes|V\setminus F|}\{|F|\}$ for some fixed $A\in R$**-gmod** where $\{|F|\}$ denotes a degree shift by $|F|$.

2.7 An Application from Algebraic Morse Theory

In 1998 Forman [For02] defined a version of Morse theory (coined “discrete Morse theory”) for simplicial complexes, yielding tools to greatly simplify homology calculations. In 2006, Sköldberg [Skö06] described a version of discrete Morse theory (coined “algebraic Morse theory”) for arbitrary chain complexes. Similar theories were developed independently by Jonsson in [Jon03] and by Jöllenbeck and Welker in [JW05]. In this section we review some results from [Skö06] and state a version of his main theorem in the setting of thin poset homology theories. Here we state everything in terms of cohomology (i.e. using differentials which increase homological grading), but one can also write everything in this section in terms of homology.

Definition 2.7.1 ([Skö06, Section 2])**.** A *based complex* of R-modules is a cochain complex (K,d) of R-modules with direct sum decompositions $K^n=\oplus_{\alpha\in I_n}K_\alpha^n$, for each $n\in\mathbb{Z}$, where the I_n are mutually disjoint index sets.

For $\alpha \in I_m$ and $\beta \in I_{m+1}$, let $d_{\beta,\alpha}$ denote the component of d from K^m_α to K^{m+1}_β:

$$d_{\beta,\alpha} = K^m_\alpha \hookrightarrow K^m \xrightarrow{d} K^{m+1} \twoheadrightarrow K^{m+1}_\beta.$$

Given a based complex K, construct a directed graph G_K with vertex set $V = \cup_n I_n$ and a directed edge from α to β whenever $d_{\alpha,\beta} \neq 0$.

Definition 2.7.2 ([Sköö06, Section 2])**.** A *matching* M on a directed graph $D = (V, E)$ is a collection of disjoint edges in E. A matching M is called *complete* if every vertex in D is incident with some edge in M. Given a directed graph D and a matching M on D, construct a directed graph D^M from D by reversing the direction of all directed edges in M. The matching M is called *acyclic* if there are no directed cycles in D^M. A vertex v in D^M is called M*-critical* if v is not incident with any of the edges in M. Let M^0 denote the set of M-critical vertices in D.

Definition 2.7.3 ([Sköö06, Section 2])**.** Let (K, d) be a based complex of R modules. A *Morse matching* on G_K is an acyclic matching such if there is an edge from α to β in M, then $d_{\beta,\alpha}$ is an isomorphism.

Theorem 2.7.4 ([Sköö06, Corollary 2])**.** *Let M be a Morse matching on the based complex K. If $\oplus_{\alpha\in M^0} K_\alpha$ forms a subcomplex of K, then K is homotopy equivalent to $\oplus_{\alpha\in M^0} K_\alpha$.*

Two easy corollaries of the previous theorem follow.

Corollary 2.7.5. *Let M be a complete Morse matching on the based complex K. Then K is homotopy equivalent to the zero complex.*

Proof. Since there are no critical cells, this is immediate from Theorem 2.7.4. □

Corollary 2.7.6. *Let M be a complete Morse matching on a subcomplex L of the based complex K. Then $H(K) \cong H(K/L)$.*

Proof. We begin with the short exact sequence $0 \to L \to K \to K/L \to 0$ and obtain a long exact sequence

$$\cdots \to H^r(L) \to H^r(K) \to H^r(K/L) \to H^{r+1}(L).$$

By Corollary 2.7.5, $H^r(L) = 0$ for all r so by exactness we have $H^r(K) \cong H^r(K/L)$ for all r. □

We now translate this to a result in the setting of thin poset cohomology. Let P be a balanced colorable thin poset with balanced coloring c, and a functor $F : P \to \mathscr{A}$, where $\mathscr{A}$ is an abelian category. Notice by definition that $K = C^*(P, F, c)$ is a based complex with index set $I_n = \{x \in P \mid \mathrm{rk}(x) = n\}$. For $x \in P$ with $\mathrm{rk}(x) = n$, we have $K^n_x = F(x)$, and for $x, y \in P$,

$$d_{y,x} = \begin{cases} c(x \lessdot y)F(x \lessdot y) & \text{if } x \lessdot y \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.7.7. *Let P be a thin poset, $\mathscr{A}$ an abelian category, and consider a functor $F : P \to \mathscr{A}$. Suppose that I is an upper or lower order ideal in P and there is a complete Morse matching M on I for which $F(e)$ is an isomorphism for each $e \in M$. Then*

$$H^*(P, F) \cong H^*(P \setminus I, F|_{P \setminus I}).$$

Proof. In the case that I is an upper order ideal, it follows that $C^*(I, F|_I, c|_I)$ is a subcomplex of $C^*(P, F, c)$ and Corollary 2.7.6 gives the desired result. In the case the I is a lower order ideal, then the $M^0 = P \setminus I$. Since $P \setminus I$ is an upper order ideal, it follows that $\bigoplus_{x \in M^0} F(x)$ is the subcomplex $C^*(P \setminus I, F|_{P \setminus I}, c|_{P \setminus I})$ and thus Theorem 2.7.4 tells us that $C^*(P, F, c)$ is homotopy equivalent to $C^*(P \setminus I, F|_{P \setminus I}, c|_{P \setminus I})$, and so in particular, $H^*(P, F) = H^*(P \setminus I, F|_{P \setminus I})$. □

2.8 Future Directions

We conclude this chapter with the following questions/ideas for future work with regard to thin posets and homology theories.

1. With Theorem 2.3.4, Theorem 2.4.12, Example 2.4.3, and Remark 2.2.8 in mind, we make the following conjecture.

   Conjecture 2.8.1. *Let P be a finite poset with $\hat{0}$. Then the following are equivalent*

   (a) P is a CW poset

   (b) P is a diamond transitive thin poset

   (c) $\pi_1(X(P)) = 0$

   Theorems 2.3.4 and 2.4.12 tell us $(a) \implies (b) \implies (c)$, so it remains to show that if P is a finite thin poset with $\hat{0}$ such that $\pi_1(X(P)) = 0$, then P is a CW poset.

2. There is another way to obtain homology theories from functors on posets. In [ET14], Everett and Turner show how to go back and forth between functors on a poset and presheaves on the Alexandrov topology of the poset (where open sets are lower order ideals). Using this, they are able to identify the Khovanov homology of a link with the derived functors of the inverse limit of the Khovanov functor F_{Kh} (i.e. the labeling of the n-cube in the cube construction of Khovanov homology). It should be possible to repeat this construction to describe all (or at least a large class of) thin poset homology theories $H(P, F)$ in terms of derived functors of the inverse limit of F (thinking of F as a presheaf), and thus give a homotopy theoretic interpretation of thin poset homology theories.

3. This question is a continuation of Example 2.6.8. Recall, given a CW poset P, and letting $\bar{P}$ denote P with $\hat{0}$ removed, we found that $H_*(P, 1_{R\textbf{-mod}}) \cong \tilde{H}_*^{\text{simp}}(\Delta(\bar{P}), R)$, where $\Delta(P)$ denotes the order complex of P, and $\tilde{H}^{\text{simp}}$ denotes the reduced simplicial homology. How does $H_*(P, 1_{R\textbf{-mod}})$ compare with $\tilde{H}_*^{\text{simp}}(\Delta(\bar{P}), R)$ for arbitrary thin posets?

4. The construction of cellular homology gives a way to balanced color CW posets (see Theorem 2.3.13), however the problem in the general case of thin posets remains open. Additionally, even for CW posets, there are only combinatorial formulas for balanced colorings in very few special cases (for example Boolean lattices). Is every Eulerian poset balanced colorable? Is every thin poset balanced colorable? Given a CW poset P, can one find a combinatorially motivated function $f : C(P) \to \mathbb{N}$ for which the map $C(P) \to \{\pm 1\}$, $c \mapsto (-1)^{f(c)}$ is balanced?

5. The h-polynomial of simplicial complexes has been of great interest in simplicial topology, occupying an entire chapter in Stanley's book [Sta07]. It is categorified by the Stanley-Reisner ring of the complex, which recovers the h-polynomial as its Hilbert series. For simplicial polytopes, the coefficients of the h-polynomial arise as dimensions of even intersection cohomology groups of an associated projective toric variety, thus satisfying a Poincaré duality known as the Dehn-Sommerville equations. The h-polynomial is also a rank-statistic generating function for a thin poset (the face poset), and should have a categorification of the type outlined in Theorem 2.6.6 (see Example 2.6.14). Variations of the h-polynomial, including the flag h-vector and the toric h-vector have a similar story. For a simplicial complex Δ, what is the relationship between the Stanley-Reisner ring, the homology theory $H(P, F)$ suggested in Example 2.6.14, and the intersection cohomology of the associated projective toric variety (all of which categorify the h-polynomial)? What about for the flag h-vector and toric h-vector?

6. In [LS14], Lipshitz, and Sarkar construct a spectrum whose singular cohomology is the Khovanov homology of a given link. They show that the stable homotopy type of the spectrum is a stronger link invariant than Khovanov homology. Such a construction endows the cohomology with extra struture, for example, Steenrod squares, and has led to a refined version of Rasmussen's s-invariant. This procedure has also been carried out for odd Khovanov homology [Sar18]. Given a functor $F : P \to \mathscr{A}$ from a thin poset to an abelian category, is there a way to prescribe a spectrum which recovers $H(P, F)$ as its singular homology?

CHAPTER

3

BROKEN CIRCUITS AND HOMOLOGICAL GRAPH INVARIANTS

In graph theory, one of the basic problems is the task of distinguishing graphs. This is often accomplished using *graph invariants*, that is, functions $i : \{\text{graphs}\} \to S$ where S is a set, such that if G and G' are isomorphic graphs, then $i(G) = i(G')$. Such functions provide a tool for distinguishing graphs: if $i(G) \neq i(G')$ then G and G' are not isomorphic.

In this chapter, we consider two homological graph invariants arising from functors on the Boolean lattice (as per the construction in Chapter 2): Helme-Guizon and Rong's chromatic homology [HGR05] which categorifies the chromatic polynomial, and Sazdanović and Yip's chromatic symmetric homology [SY18], which categorifies the Stanley chromatic symmetric function. The first main result of this chapter is Lemma 3.1.3, where we establish an acyclic matching on the Hasse diagram of the Boolean lattice of spanning subgraphs of a given graph. An application of Theorem 2.7.7, together with the Lemma 3.1.3, gives our next main results in this chapter: a categorification of Whitney's Broken Circuit Theorem in either context (Corollary 3.2.10 for chromatic homology or Corollary 3.3.5 for chromatic symmetric homology), thus yielding a significant simplification in the calculation of either homology theory. The categorification of Whitney's Broken Circuit Theorem in the case of chromatic homology was known to Sazdanović and Yip in 2015 in an unpublished joint work, where they used an inductive approach. The proof presented here is more general and allows for extension to other theories such as chromatic symmetric homology. As a result, we are able to obtain bounds on homological support for these homology theories in Corollaries 3.2.11 and 3.3.6.

3.1 An Acyclic Matching on Spanning Subgraphs

Let $G = (V, E)$ be a finite graph. That is, V is a finite set and $E \subseteq \binom{V}{2}$ is a collection of subsets of size two. A *cycle* (of length k) in G is a sequence $x_1, x_2, \ldots, x_k, x_1$ of k distinct vertices in G such that $\{x_i, x_{i+1}\}$ is an edge of G for $1 \leq i \leq k$ where subscripts are taken modulo k. The *edge space* of G is the $\mathbb{Z}_2$-vector space $\mathscr{E}(G)$ with basis E, and the *cycle space* of G is the subspace $\mathscr{C}(G)$ of $\mathscr{E}(G)$ spanned by cycles. We will identify each subset $S \subseteq E$ with the element $S = \sum_{e \in S} e \in \mathscr{E}(G)$, and also with the spanning subgraph of G with vertex set V and edge set S. Given $S, T \in \mathscr{E}(G)$, $S + T$ is thus identified with the symmetric difference $S \Delta T = S \cup T - S \cap T$. The well-known fact that a connected graph has an Eulerian circuit if and only if every vertex has even degree (this goes all the way back to Euler's solution to the bridges of Königsberg problem) takes the following form in the present context.

Lemma 3.1.1. *Given $S \subseteq E$, we have $S \in \mathscr{C}(G)$ if and only if S is a union of edge-disjoint cycles.*

Proof. The result in [Die97, Proposition 1.9.2] gives that $S \in \mathscr{C}(G)$ if and only if each vertex in S has even degree. Thus we must show that S is a union of edge-disjoint cycles if an only if each vertex in S has even degree. The forward direction is immediate since given a vertex v, each cycle containing v contributes 2 to the degree of v. We show the backwards direction by induction on the number of edges. If $|S| = 0$ and each vertex has even degree, then S is the empty union of cycles. For $|S| > 0$, there must exist a vertex v of degree ≥ 2. Consider a walk in S starting at v. Pick an edge $e_0 \in S$ containing $v = v_0$ and walk along e_0 to reach the vertex v_1. Since v_1 has positive even degree, there must be an edge $e_1 \neq e_0$ containing v_1. Walk along this edge to arrive at a vertex v_2. Since S is finite, this process must eventually reach a vertex v_k for which $v_{k+1} = v_i$ for some $i < k-1$. Thus S contains a cycle $C = e_i + e_{i+1} + \cdots + e_k \in \mathscr{C}(G)$. Consider the subgraph $S' = S + C$. Consider the degree of a vertex v in S'. If v is incident with an edge in C, then the degree of v in S' is two less than it is in S, and thus the degree of v in S' is even. If v is not incident with an edge in C, then the degree of v in S' is the same as the degree of v in S. We conclude that every vertex has even degree in S'. By induction, $S' = C_1 + \cdots + C_k$ where each C_i is a cycle and $C_i \cap C_j = \varnothing$ for $i \neq j$. Therefore $S = C + C_1 + \cdots + C_k$ is an edge disjoint union of cycles. □

Definition 3.1.2 ([Whi32])**.** Let $G = (V, E)$ be a graph, and let $\mathscr{O}$ be a fixed ordering $\mathscr{O} = (e_1, \ldots, e_m)$ of the edge set E. A *broken circuit* in G is $C + e \in \mathscr{E}(G)$ (or $C \setminus \{e\}$ as sets) where $C \subseteq E$ is a cycle in G and e is the edge in C with the largest label. Define $\mathrm{NBC}_{G,\mathscr{O}}$ (or just NBC when G and $\mathscr{O}$ are clear from context) to be the set of all $S \subset E$ such that S contains no broken circuits, and let $\mathrm{BC}_{G,\mathscr{O}}$ (or just BC) be its complement, that is, $\mathrm{BC} = 2^E \setminus \mathrm{NBC}$.

In Sections 3.2 and 3.3, we consider homology theories arising from functors on 2^E. Recall from Section 2.7, that Morse matchings (see Definition 2.7.3) on Hasse diagrams of posets give simplifications in computations of homology theories built from functors on those posets. Also recall from Definition 2.7.2 that M^0 denotes the set of M-critical vertices of a matching M on a directed graph D. Given $S \subseteq E$, let $k(S)$ denote the number of connected components in the subgraph S, and let $\lambda(S)$ denote the set partition of V into the connected components of S.

Lemma 3.1.3. *Let $G = (V, E)$ be a graph with a fixed ordering of its edges. The Hasse diagram of the Boolean lattice, 2^E, has an acyclic matching M_{BC} in which each matched pair $S \lessdot S'$ satisfies $k(S) = k(S')$, $\lambda(S) = \lambda(S')$, and $\mathrm{M}^0_{\mathrm{BC}} = \mathrm{NBC}$.*

Proof. Given a function $f : S \to S$ on a set S, recall the *orbit* of $x \in S$ is the set $\{f^n(x) \mid n \geq 0\}$. We construct a matching on the Hasse diagram of 2^E as follows. First we construct an involution $i : \mathrm{BC} \to \mathrm{BC}$ (that is, a function $i : \mathrm{BC} \to \mathrm{BC}$ such that i^2 is the identity function). Since i is an involution, the orbits of i form a collection of disjoint 2-element sets which covers BC. We define i in such a way that for any $S \in \mathrm{BC}$, $i(S)$ either covers S or is covered by S in BC. Therefore the orbits of i form a matching on the Hasse diagram of 2^E, which we denote M_{BC}, for which $M^0_{\mathrm{BC}} = 2^E \setminus \mathrm{BC} = \mathrm{NBC}$. We now define the involution i.

Given $S \in \mathrm{BC}$, pick $e \in E$ maximally among all broken circuits of the form $C + e$ in S, where $e \in C$. Define $i(S) = S + e$ (recall $S + e$ is either $S \cup \{e\}$ or $S \setminus \{e\}$ depending whether $e \in S$). Let $C' + f \subseteq i(S)$ be a broken circuit with f chosen maximally among all such broken circuits. Suppose that $e \neq f$. Then $f > e$ since $C + e$ is a broken circuit in $i(S)$, and thus $f \notin C$ (else we contradict maximality of e). Similarly,we have $e \in C'$ since otherwise $C' + f$ is a broken circuit in S again contradicting maximality of e. Lemma 3.1.1 tells us that $C + C'$ is a union of cycles, and we know that $e \notin C + C'$ and $f \in C + C'$ so $C + C' + f \subseteq S$ contains a broken circuit missing the edge f, thus contradicting maximality of e among broken circuits in S.

Let M_{BC} denote the matching consisting of the collection of orbits of i. To see that M_{BC} is acyclic, it suffices to give a linear extension of BC in which T follows S for each pair $(S \lessdot T)$ in M_{BC} (this is [Koz07, Theorem 11.2]). For each $e = (S \lessdot T) \in M$ let $u(e) = T$ and $d(e) = S$ (u for 'up' and d for 'down'). Let $u(M) = \{u(e) \mid e \in M\}$ and $d(M) = \{d(e) \mid e \in M\}$. Fix a total ordering on $d(M)$ by setting $U \leq V$ whenever $|U| < |V|$ or $|U| = |V|$ and U is lexicographically larger than V (that is, order ranks by reverse lexicographic order). Let $\mathrm{M}_{\mathrm{BC}} = \{(S_1, T_1), \ldots (S_n, T_n)\}$ with the S_i ordered as indicated above and $T_i = S_i + e_i$ where S_i has a distinguished broken circuit $C_i + e_i$. We will now show that $S_1, T_1, S_2, T_2, \ldots, S_n, T_n$ is such a linear extension of the inclusion ordering on BC (see for example Figure 3.1). Suppose that $U, V \in \mathrm{BC}$ and $U \subset V$. We now show by cases that U comes before V in the linear ordering $S_1, T_1, S_2, T_2, \ldots, S_n, T_n$.

Case 1. $U, V \in d(M)$. Then U comes before V in the linear ordering because $|U| < |V|$.

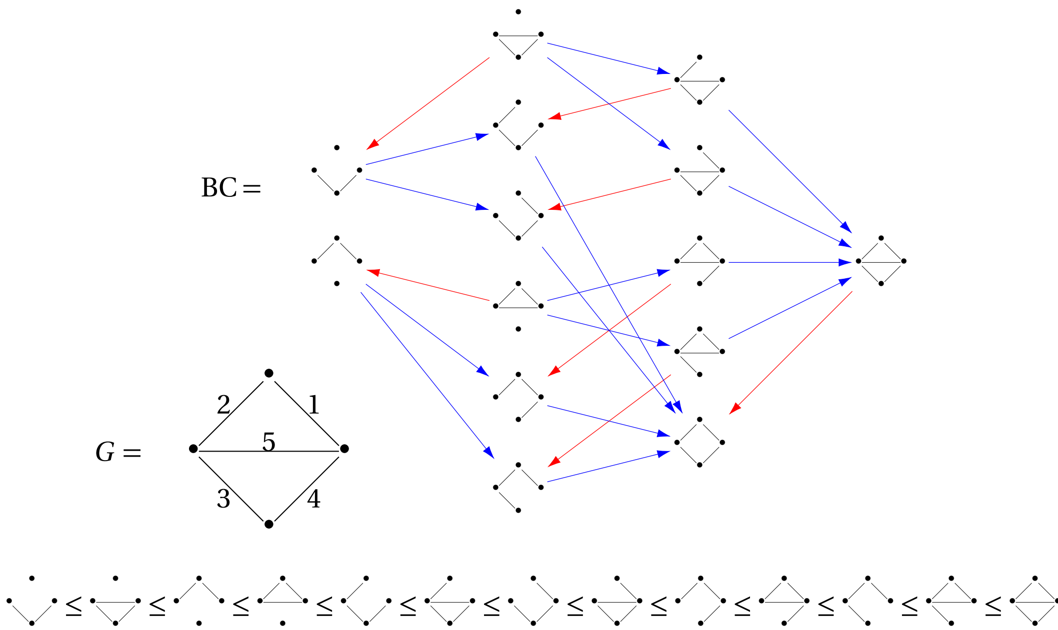


Figure 3.1 An example of the acyclic matching on BC for the graph G shown above, with matched edges shown in red. Ranks are ordered reverse lexicographically from top to bottom. The resulting linear extension is shown below.

Case 2. $U \in u(M), V \in d(M)$. Then $U = T_i$ and $V = S_j$ for some $i, j \in [n]$. Therefore $S_i \subsetneq T_i \subseteq S_j$ and thus $|S_i| < |S_j|$ so S_i comes before S_j in the linear extension by definition. Since T_i comes directly after S_i, we conclude $U = T_i$ comes before $V = S_j$.

Case 3. $U, V \in u(M)$. Let $U = T_i$ and $V = T_j$ for some $i, j \in [n]$. Then $|S_i| = |T_i| - 1$ and $|S_j| = |T_j| - 1$ and so $|S_i| < |S_j|$. We conclude that S_i comes before S_j in the linear extension and therefore T_i comes before T_j.

Case 4. $U \in d(M), V \in u(M)$. Let $U = S_i$ and $V = T_j$ for some $i, j \in [n]$. If $S_i \subseteq S_j$ then S_i comes weakly before S_j which comes before T_j in the linear extension, so we are done. Similarly, if $|S_i| < |S_j|$ we are done so let us assume $|S_i| = |S_j|$. Thus there exists $x \in S_j$ with $x \notin S_i$, $T_j = S_i + x = S_j + e_j$. Notice also that x, e_i, and e_j are mutually distinct edges. See Figure 3.2 for a schematic diagram of S_i, S_j, and T_j. It suffices to show that S_i is lexicographically larger than S_j, or equivalently, that $e_j > x$ in our fixed edge ordering.

Suppose towards a contradiction that $e_j < x$. Then $x \notin C_j$ (else we contradict maximality of e_j in C_j) and therefore $C_j \subseteq S_i$ (since $e_j \in S_i$) so we can conclude that $e_i > e_j$ by maximality of e_i in S_i. Thus we find that $e_i \notin C_j$ and furthermore $e_j \in C_i$ since otherwise $C_i + e_i \subseteq S_j$ (which would contradict maximality of e_j in S_j). Lemma 3.1.1 tells us that $C_i + C_j$ is an edge-disjoint union of cycles, and we have $e_i \in C_i + C_j$ and $e_j \notin C_i + C_j$ and therefore $C_i + C_j + e_i \subseteq S_j$ contains a broken

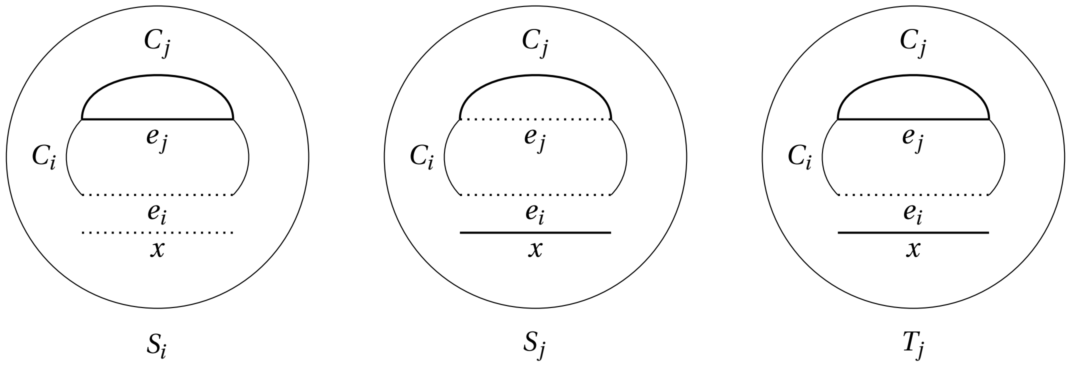


Figure 3.2 A schematic for visualizing the subgraphs S_i, S_j, and T_j.

circuit missing the edge e_i and therefore $e_j > e_i$ (contradicting our earlier finding that $e_i > e_j$). Therefore we conclude that $e_j > x$ and therefore S_i is lexicographically larger than S_j. □

3.2 Broken Circuits in Chromatic Homology

In 1852, Francis Guthrie noticed that only four colors were needed to color the map of counties of England in such a way that no two adjacent counties get the same color, and went on to ask if this is a general phenomena of maps. This question became known as the "four-color problem". Due to the simplicity of its statement, and the fact that it remained unsolved for so long, this became one of the most famous open problems in mathematics. The four-color problem can be stated graph theoretically as follows. Recall, a graph is *planar* if it can be drawn in the plane in such a way that no two edges cross.

Definition 3.2.1. A *proper coloring* of a graph $G = (V, E)$ is a function $c : V \to [n]$ where $[n] = \{1, 2, \ldots, n\}$ such that if there is an edge from x to y in G then $c(x) \neq c(y)$. A graph is *n-colorable* if there exists a proper coloring $c : V \to [n]$.

Problem 3.2.2 (Four-Color Problem). *Is every planar graph four-colorable?*

The four-color problem was answered in the affirmative in 1976 by Appel and Haken [AH77] by using a computer to check that no possible counterexample can exist. A direct proof is still highly sought after. A possible new approach to the four-color problem is given by Kronheimer and Mrowka [KM15] using instanton homology. In 1912, Birkhoff [Bir12] introduced the chromatic polynomial in an attempt to obtain an algebraic/analytic solution to the four-color problem. Although this attempt was unsuccessful, the idea contributed largely to the development of the field of algebraic graph theory.

Definition 3.2.3 ([Bir12])**.** Given a graph G, the *chromatic polynomial* $\chi_G : \mathbb{N} \to \mathbb{N}$ is defined by

$$\chi_G(x) = \#\{c : V \to [x] \mid c \text{ is proper}\}.$$

It is not clear from the definition that χ_G is in fact a polynomial. The following result establishes this fact and gives a very convenient way to calculate χ_G. Recall, given a set E, 2^E denotes the collection of all subsets of E, partially ordered by containment.

Lemma 3.2.4 ([Bir12])**.** *For any graph G,*

$$\chi_G(x) = \sum_{S \in 2^E} (-1)^{|S|} x^{k(S)}$$

where $k(S)$ is the number of connected components in S.

In 1932, Whitney found the following connection between the chromatic polynomial and broken circuits [Whi32], resulting in a significant simplification in the computation of chromatic polynomials.

Theorem 3.2.5 ([Whi32])**.** *For any graph G, and any fixed ordering of its edge set,*

$$\sum_{S \in \mathrm{BC}} (-1)^{|S|} x^{k(S)} = 0$$

and therefore

$$\chi_G(x) = \sum_{S \in \mathrm{NBC}} (-1)^{|S|} x^{k(S)}.$$

Proof. This follows directly from Lemma 3.1.3 by noticing that every contribution of the form $(-1)^{|S|} x^{k(S)}$ to $\chi_G(x)$ where $S \in \mathrm{BC}$ is matched with another contribution $(-1)^{|S'|} x^{k(S')}$ where $S' \in \mathrm{BC}$, $|S'| = |S| \pm 1$ and $k(S) = k(S')$. Thus all contributions from BC cancel and we are left with only the contributions from NBC. □

In 2005, Helme-Guizon and Rong [HGR05] constructed a homology theory which categorifies $\chi_G(x)$ in the sense of Theorem 2.6.6. Here, we describe their homology theory in terms of the notation from Chapter 2. Recall that R-**gmod** denotes the category of graded R-modules.

Definition 3.2.6. Given a graph $G = (V, E)$, a commutative ring R, and a graded R-algebra A with multiplication $m : A \otimes A \to A$, the (G, A)-*chromatic functor*, $F^{\mathrm{Ch}}_{G,A} : 2^E \to R$-**gmod** (or just F^{Ch} when G and A are understood) is defined as follows:

1. $F^{\mathrm{Ch}}(S) = A^{\otimes k(S)}$ for $S \in 2^E$, where $k(S)$ denotes the number of connected components in S.

2. Given a cover relation $S \lessdot S+e$ in 2^E, adding the edge e to S either joins two distinct connected components, or completes a cycle. In the former case, $F^{\mathrm{Ch}}(S \lessdot S+e)$ is defined by multiplying tensor factors of A corresponding to the joining connected components. In the latter case, $F^{\mathrm{Ch}}(S \lessdot S+e)$ is the identity map.

Lemma 3.2.7. *For any graph G, and any R-algebra A, Definition 3.2.6 determines a functor*

$$F^{\mathrm{Ch}}_{G,A} : 2^E \to R\text{-}\boldsymbol{gmod}.$$

Proof. We begin by remarking that 2^E is a CW poset, and is therefore diamond transitive. Thus by Theorem 2.5.1 it suffices to check that the "per edge maps" defined in Definition 3.2.6 part 2 commute on diamonds. Given $S \in 2^E$, consider $e, f \in E \setminus S$ and the corresponding diamond $\{S, S+e, S+f, S+e+f\}$.

Case 1 ($k(S+e+f) = k(S)$)**.** In this case, all maps are identity maps, and the result is trivial.

Case 2 ($k(S+e+f) = k(S)+1$)**.** In this case, both $F^{\mathrm{Ch}}(S+e \lessdot S+e+f) \circ F^{\mathrm{Ch}}(S \lessdot S+e)$ and $F^{\mathrm{Ch}}(S+f \lessdot S+e+f) \circ F^{\mathrm{Ch}}(S \lessdot S+f)$ consist of multiplying the two tensor factors corresponding to the two components of S which are joined in $S+e+f$.

Case 3 ($k(S+e+f) = k(S)+2$)**.** In this case, both $F^{\mathrm{Ch}}(S+e \lessdot S+e+f) \circ F^{\mathrm{Ch}}(S \lessdot S+e)$ and $F^{\mathrm{Ch}}(S+f \lessdot S+e+f) \circ F^{\mathrm{Ch}}(S \lessdot S+f)$ consist of multiplying the three tensor factors corresponding to the three components of S which are joined in $S+e+f$. These maps are equal by associativity in A. □

Given a graph $G = (V, E)$ where E has a fixed ordering $E = \{e_1, \dots, e_m\}$, recall the Boolean lattice 2^E has a balanced coloring $c : C(2^E) \to \{1, -1\}$ defined by $c(S \lessdot S+e_i) = (-1)^{\#\{j \in S \mid j<i\}}$. As per the construction in Definition 2.5.2, this data $(2^E, F^{\mathrm{Ch}}_{G,A}, c)$ determines a chain complex $C^*(2^E, F^{\mathrm{Ch}}_{G,A}, c)$ with homology $H^*(2^E, F^{\mathrm{Ch}}_{G,A})$ which we denote by $H^{\mathrm{Ch}}_A(G)$.

Theorem 3.2.8 ([HGR05])**.** *For any graph G, and any graded R-algebra A, under the identification*

$$K_0(\mathscr{C}^b(R\text{-}\boldsymbol{gmod})) \cong \mathbb{Z}[q, q^{-1}], \qquad [C] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i q\mathrm{rk} C^i,$$

we have

$$[H^{\mathrm{Ch}}_A(G)] = \chi_G(q\mathrm{rk} A).$$

Proof. This follows immediately by construction, as per Theorem 2.6.6. □

Remark 3.2.9. Helme-Guizon and Rong [HGR05], in their original construction of chromatic homology, mainly studied this construction with $A_2 = \mathbb{Z}[x]/(x^2)$ so that $q\mathrm{rk} A_2 = 1+q$, and one can recover the chromatic polynomial from $[H^{\mathrm{Ch}}_{A_2}(G)]$ with the substitution $q = x-1$. It turns out that $H^{\mathrm{Ch}}_{A_2}(G)$ is not a stronger graph invariant than $\chi_G(x)$, thus chromatic homology over other algebras is often studied. For example, see the work of Scofield and Sazdanovič [SS18] where they study chromatic homology over $A_n = \mathbb{Z}[x]/(x^n)$ for different values of n. In particular, they show that $H^{\mathrm{Ch}}_{A_3}(G)$ is a stronger graph invariant than $\chi_G(x)$.

Note that $\mathrm{NBC} \subseteq 2^E$ is a lower order ideal in 2^E and is thus also a CW poset by Theorem 2.1.9. Hence one can consider the restriction of F^{Ch} to NBC and the restriction of c to NBC, yielding a chain complex $C^*(\mathrm{NBC}, F^{\mathrm{Ch}}|_{\mathrm{NBC}}, c|_{\mathrm{NBC}})$ with homology $H^*(\mathrm{NBC}, F^{\mathrm{Ch}}|_{\mathrm{NBC}})$.

Corollary 3.2.10. *For any graph $G = (V,E)$, any graded R-algebra A, and any fixed ordering of the edge set E,*

$$H^{\mathrm{Ch}}_A(G) \cong H^*(\mathrm{NBC}, F^{\mathrm{Ch}}|_{\mathrm{NBC}}).$$

Proof. Recall that for a cover relation $S \lessdot S+e$ for which e completes a cycle in S, $F^{\mathrm{Ch}}(S \lessdot S+e)$ is an isomorphism. Therefore the result follows from Theorem 2.7.7 and Lemma 3.1.3. □

Taking the (graded) Euler characteristic of both sides (as per Example 2.6.3) of the above isomorphism yields Theorem 3.2.5, and therefore Corollary 3.2.10 can be viewed as a categorification of Whitney's Broken Circuit Theorem.

Corollary 3.2.11. *For any graph $G = (V,E)$ with n vertices and c connected components, $H^{\mathrm{Ch}}_A(G)$ is supported in bigradings (i, j) which satisfy $0 \le i \le n-c$.*

Proof. This follows from Corollary 3.2.10 along with the following argument. Suppose $S \subseteq E$ has $|S| \ge n-c+1$. Suppose towards a contradiction that S has no cycles. Then any connected component of S is a tree. Let S have r connected components $S_1, \ldots, S_r$ with $|S_i| = s_i$. Then S_i has $s_i - 1$ edges and so S has $\sum_{i=1}^r (s_i - 1) = n-r$ edges. But $r \ge c$, so $n-r < n-c+1 = |S|$, thus we have found a contradiction. Thus there are no generators in homological degrees larger than $n-c$. □

3.3 A Broken Circuit Model for Chromatic Symmetric Homology

In 1995, Stanley [Sta95] introduced the *c*hromatic symmetric function $X_G(x_1, x_2, \ldots)$, a multi-variable generalization of the chromatic polyomial $\chi_G(x)$ in the sense that $X_G(1, \ldots, 1, 0, \ldots) = \chi_G(k)$, where we have substituted 1 for the first k coordinates $x_1, \ldots x_k$, and 0 for the rest. The chromatic symmetric function of $G = (V,E)$ is defined as

$$X_G(\mathbf{x}) = \sum_{\kappa} x_{c(v_1)} \ldots x_{c(v_n)},$$

where the sum is over all proper colorings $c : V(G) \to \mathbb{N}$ of G. The chromatic symmetric function generalizes many properties of the chromatic polynomial, but in general, $X_G(\mathbf{x})$ is a stronger invariant than $\chi_G(x)$. Recall the power sum symmetric functions $p_k(\mathbf{x}) = \sum_{i=0}^{\infty} x_i^k$. Given any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we define $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_n}$. Any edge subset $T \subseteq E$ determines a partition $\lambda(T)$ of n, defined by the number of vertices in each connected component of S. The chromatic symmetric function, like the chromatic polynomial has a state sum formula:

$$X_G = \sum_{T \in 2^E} (-1)^{|T|} p_{\lambda(T)}. \tag{3.1}$$

Similar to the case for the chromatic polynomial, the chromatic symmetric function also has a significant simplification in terms of broken circuits.

Theorem 3.3.1 ([Sta95])**.** *For any graph G, and any fixed ordering of the edge set,*

$$\sum_{T \in \mathrm{BC}} (-1)^{|T|} p_{\lambda(T)} = 0,$$

and therefore

$$X_G(\mathbf{x}) = \sum_{T \in \mathrm{NBC}} (-1)^{|T|} p_{\lambda(T)}.$$

Proof. This follows directly from Lemma 3.1.3. □

With equation 3.1 in mind, Sazdanović and Yip [SY18] categorify X_G in an analogous way to Helme-Guizon and Rong's categorification of $\chi_G(x)$. We now recall their construction, but state it in terms of the thin poset homology theories of Chapter 2. Let $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$ and let $\alpha_i = e_i - e_{i-1}$. Let S^λ denote the irreducible $\mathbb{C}[S_n]$-modules indexed by $\lambda \vdash n$. In particular, $S^{(n-1,1)}$ is the standard $n-1$ dimensional $\mathbb{C}[S_n]$-module with basis $\alpha_1, \ldots, \alpha_{n-1}$. For $n \in \mathbb{N}$, let $\mathscr{L}_n$ denote the graded $\mathbb{C}[S_n]$-module

$$\mathscr{L}_n = \wedge^* S^{(n-1,1)} = \bigoplus_{k=0}^{n-1} \wedge^k S^{(n-1,1)} = \bigoplus_{k=0}^{n-1} S^{(n-k,1^k)}\{k\} \tag{3.2}$$

where $\{k\}$ denotes a grading shift shift by k. Let $T \subseteq E(G)$ be a spanning subgraph of G with r connected components, whose vertex sets we denote by $B_1, \ldots, B_r$. Let b_i denote the number of vertices in B_i. The sets $B_1, \ldots, B_r$ of vertices partition $V(G)$ so $S_{B_1} \times \cdots \times S_{B_r}$ is a Young subgroup of $S_{V(G)} \cong S_n$. Define the graded $\mathbb{C}[S_n]$-module

$$\mathscr{M}_T = \mathrm{Ind}_{S_{B_1} \times \cdots \times S_{B_r}}^{S_{V(G)}} (\mathscr{L}_{b_1} \otimes \cdots \otimes \mathscr{L}_{b_r}) \tag{3.3}$$

For each cover relation $T \lessdot T + e$ in the Boolean lattice 2^E, Sazdanović and Yip define a map $d_{T \lessdot T+e} : \mathscr{M}_T \to \mathscr{M}_{T+e}$. If adding e completes a cycle when added to T, $d_{T \lessdot T+e}$ is the identity map. In the case that e does not complete a cycle, we refer the reader to [SY18] for details. For the purposes of this section, the details of the per-edge maps in general are not needed.

Definition 3.3.2. Given a graph $G = (V, E)$ with $|V| = n$, define the *chromatic symmetric functor* $F_G^{\mathrm{CS}} : 2^E \to \mathbb{C}[S_n]$-**gmod** by

1. Given $T \subseteq E$, $F^{\mathrm{CS}}(T) = \mathscr{M}_T$,
2. Given a cover relation $T \lessdot T + e$, define $F^{\mathrm{CS}}(T \lessdot T + e) = d_{T \lessdot T+e}$.

As per the construction in Chapter 2, for any graph $G = (V, E)$, the data $(2^E, F_G^{\mathrm{CS}}, c)$ defines a cochain complex $C^*(2^E, F_G^{\mathrm{CS}}, c)$ in $\mathscr{C}^b(\mathbb{C}[S_n]$-**gmod**), with homology $H^*(2^E, F_G^{\mathrm{CS}})$. We will refer to $H^*(2^E, F_G^{\mathrm{CS}})$ as the *chromatic symmetric homology* of G, and denote it by $H^{\mathrm{CS}}(G)$. Let R_n denote the Grothendieck group of the category $\mathbb{C}[S_n]$-**gmod**. That is, R_n is the free abelian group on the isomorphism classes $[S^\lambda]$ of Specht modules indexed by all partitions λ of n. Define the graded ring $R = \oplus_{n\geq 0} R_n$ with multiplication defined as follows. Given $N \in \mathbb{C}[S_n]$-**gmod** and $M \in \mathbb{C}[S_m]$-**gmod** define $[M] \cdot [N] = [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} M \otimes N]$. One can also think of R as the Grothendieck ring of the category of all symmetric group representations. Let $\Lambda_{\mathbb{C}}$ denote the ring of symmetric functions over $\mathbb{C}$.

Theorem 3.3.3 ([Ful97, Section 7.3, Theorem 1])**.** *The homomorphism* $\mathrm{ch} : R \to \Lambda_{\mathbb{C}}$ *of graded rings defined by sending the Specht modules to the Schur functions* $[S^\lambda] \mapsto s_\lambda$ *is an isomorphism.*

From now on, we identify R with $\Lambda_{\mathbb{C}}$ via the isomorphism ch from Theorem 3.3.3.

Theorem 3.3.4 ([SY18])**.** *For any graph G, $H^{\mathrm{CS}}(G)$ categorifies the chromatic symmetric function in the sense of Theorem 2.6.6. That is, in $K_0(\mathscr{C}(\mathbb{C}[S_n]$-**gmod**$))$,*

$$[H^{\mathrm{CS}}(G)] = X_G(\mathbf{x}).$$

Proof. By the identification allowed by Theorem 3.3.3 $p_n = \sum_{i=0}^{n-1} (-1)^i [S^{(n-i,1^i)}]$ and therefore, for each $T \subseteq E$, we have $[\mathscr{M}_T] = p_{\lambda(T)}$, and the rest of the proof follows immediately by construction, considering equation 3.1. □

Again recall that NBC $\subseteq 2^E$ is a lower order ideal in 2^E and is thus also a CW poset by Theorem 2.1.9. Hence one can consider the restriction of F^{CS} to NBC and the restriction of c to NBC, yielding a chain complex $C^*(\mathrm{NBC}, F^{\mathrm{CS}}|_{\mathrm{NBC}}, c|_{\mathrm{NBC}})$ with homology $H^*(\mathrm{NBC}, F^{\mathrm{CS}}|_{\mathrm{NBC}})$.

Corollary 3.3.5. *For any graph $G = (V, E)$, and any fixed ordering of the edge set E,*

$$H^{\mathrm{CS}}(G) \cong H^*(\mathrm{NBC}, F^{\mathrm{CS}}|_{\mathrm{NBC}}).$$

Proof. Recall that for a cover relation $T \lessdot T + e$ for which e completes a cycle in T, $F^{\mathrm{CS}}(T \lessdot T + e)$ is an isomorphism. Therefore the result follows from Theorem 2.7.7 and Lemma 3.1.3. □

Taking the (graded) Euler characteristic of both sides (as per Example 2.6.3) of the above isomorphism yields Theorem 3.2.5, and therefore Corollary 3.3.5 can be viewed as a categorification of Whitney's Broken Circuit Theorem for the chromatic symmetric function.

Corollary 3.3.6. *For any graph G with n vertices and c connected components, $H^{\mathrm{CS}}(G)$ is supported in bigradings (i, j) which satisfy $0 \leq i \leq n-c$ and $0 \leq j \leq i$.*

Proof. For the bounds on i, the argument is the same as in the proof of Corollary 3.2.11. The bounds on j follow immediately from the definition of $F^{\mathrm{CS}}(T)$. □

3.4 Future Directions

We end this section with a list of question/ideas for future work regarding graph homology and broken circuits.

1. In 1980, Orlik and Solomon [OS80] showed that the characteristic polynomial of a complex hyperplane arrangement coincides with the graded Poincaré polynomial of the cohomology algebra (now called the *Orlik-Solomon algebra*) of the complement. Orlik and Terao [OT13] showed that the Orlik-Solomon algebras have a "no broken circuit" basis, i.e. one indexed by collections of hyperplanes which contain no broken circuits (broken circuits are defined for any matroid). Dancso and Licata [DL15] gave an alternative categorification of the characteristic polynomial of an arrangement by constructing a chain complex analogous to the odd Khovanov chain complex. What are the relationships between chromatic homology, Orlik-Solomon algebras, and Dancso-Licata's odd Khovanov homology for hyperplane arrangements? Are there maps/spectral sequences between them? Do they all have "no broken circuit" bases?

2. In [EH07], Eastwood and Hugget give another categorification of the chromatic polynomial as the Euler characteristic of the graph configuration space M_G obtained by deleting diagonals $\Delta_e = \{(x_1, \ldots, x_n) \in M^{\times n} \mid x_i = x_j\}$ from the n-fold product $M^{\times n}$ for each edge $e \in E(G)$. Baranovsky and Sazdanović show in [BS12] that there is a spectral sequence beginning at chromatic homology of G and converging to the homology of M_G. It seems plausible that this theory should also have a broken circuit model. In the case of the complete graph K_n, the homology of M_{K_n} admits the so-called Lambrechts-Stanley model [LS08] which appears to be equivalent to a broken circuit model. Can the Lambrechts-Stanley model be generalized to arbitrary graphs?

3. It is conjectured that the Stanley chromatic symmetric function $X_G(\mathbf{x})$ distinguishes trees, and this has been verified computationally on trees with up to 29 vertices. With the extra structure

endowed by categorification, is it more viable to show that Sazdanović and Yip's chromatic symmetric homology $H^{\mathrm{CS}}(G)$ distinguishes trees?

CHAPTER

4

A CATEGORIFICATION OF THE VANDERMONDE DETERMINANT

The categorification presented in this chapter is motivated by M. Khovanov's encouragement to categorify special types of determinants, as a step towards the categorification of linear algebra. In this chapter, we make use of the fact that any determinant can be expressed as a rank alternation function over the symmetric group, which has a thin partial order given by the Bruhat order $\mathrm{Br}(S_n)$. Given a link diagram D, we construct a functor (up to "stabilization") V_D from $\mathrm{Br}(S_n)$ to a category of colored cobordisms. Post composing with a certain topological quantum field theory (which is invariant up to this so-called stabilization), F, we obtain a functor $F \circ V_D$ to an abelian category $\mathscr{A}$. In Theorem 4.3.6, we show that the homology $H(\mathrm{Br}(S_n), F \circ V_D)$ categorifies a certain generalized Vandermonde determinant, and in the case that D is a diagram of the $(2, n)$ torus link, we show in Theorem 4.3.8, that $H(\mathrm{Br}(S_n), F \circ V_D)$ categorifies (a rescaling of) the usual Vandermonde determinant, $\det(x_i^j)_{i,j=1}^n$.

4.1 Background

4.1.1 The Vandermonde Determinant

In this chapter, we consider the *Vandermonde determinant*, usually defined as $\widetilde{V}(\vec{x}) = \det(x_i^{j-1})_{i,j=1}^n$ where $\vec{x} = (x_1, \ldots, x_n)$ is a list of variables. The Vandermonde determinant and its properties are useful in several areas of mathematics. The nonvanishing of $\widetilde{V}(\vec{x})$ for distinct values of its inputs shows that the polynomial interpolation problem is uniquely solvable. It is a standard result that $\widetilde{V}(\vec{x}) = \prod_{i<j}(x_j - x_i)$, and thus any alternating polynomial in variables $x_1, \ldots x_n$ is divisible by $\widetilde{V}(\vec{x})$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ one can define the generalized Vandermonde determinant $\widetilde{V}_\lambda(\vec{x}) = \det(x_i^{j+\lambda_{n-j+1}})_{i,j=1}^n$, an alternating polynomial. The quotient $\widetilde{V}_\lambda(\vec{x})/\widetilde{V}(\vec{x})$ arises in the Frobenius character formula which can be used to compute characters of representations of the symmetric group. The Vandermonde determinant also appears in the theory of BCH code, Reed-Solomon error correction codes [KlØ99], and can be used to define the discrete Fourier transform [Mas98]. For purposes of categorification, we find it more convenient to consider the following rescaling:

$$V(\vec{x}) = \begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \det(x_i^j)_{i,j=1}^n. \tag{4.1}$$

Linearity of the determinant yields the relation $V(\vec{x}) = x_1 \ldots x_n \widetilde{V}(\vec{x})$. More generally, for any sequence $\vec{s} = (s_1, \ldots s_n)$ of positive integers, the corresponding *generalized Vandermonde determinant* is

$$V_{\vec{s}}(\vec{x}) = \det(x_i^{s_j})_{i,j=1}^n. \tag{4.2}$$

Notice that $V_{(1,2,\ldots,n)}(\vec{x}) = V(\vec{x})$. The following expressions for the Vandermonde determinant will prove to be useful for the purposes of categorfication:

$$V_{\vec{s}}(\vec{x}) = \sum_{\pi \in S_n} (-1)^{\operatorname{sgn}(\pi)} x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \ldots x_n^{s_{\pi(n)}} \tag{4.3}$$

where S_n is the symmetric group on n letters. With the above expression, and with Theorem 2.6.6 in mind, it makes sense to view the Vandermonde determinant as the Euler characteristic of some cohomology theory. Thus the goal of this chapter is to use the construction in Chapter 2 of cohomology theories from functors on thin posets to accomplish this. First we recall some basic notations and concepts regarding the symmetric group S_n. All of the needed background information regarding the symmetric group can be found in [BB06].

4.1.2 The Bruhat Order

The *one-line notation* for $\pi \in S_n$ is $\pi = \pi_1\pi_2 ... \pi_n$ where $\pi_i = \pi(i)$. Let $(a_1, a_2, ..., a_k)$ denote the *cycle* sending a_i to a_{i+1} for $1 \le i \le k-1$, sending a_k to a_1, and fixing all elements of $[n]$ which do not appear in the list $a_1, ..., a_k$. A *transposition* is a cycle of the form (a_1, a_2). Any $\pi \in S_n$ can be written as a product of transpositions $\pi = \tau_1 \dots \tau_k$ and the quantity $\mathrm{sgn}(\pi) = (-1)^k$ is well defined. An *inversion* in π is a pair $(i, j) \in [n] \times [n]$ with $i < j$ and $\pi(i) > \pi(j)$. Recall, the *inversion number* $\mathrm{inv}(\pi)$ of a permutation $\pi \in S_n$ is the number of pairs (i, j) with $i < j$ such that $\pi(i) > \pi(j)$. It turns out that $\mathrm{sgn}(\pi) = (-1)^{\mathrm{inv}(\pi)}$ where $\mathrm{inv}(\pi)$ denotes the number of inversions in π. We now review a useful presentation of the Bruhat order on S_n as given in [BB06, Section 2.1]. As noted in [Sta98, Section 3.1], if P is a finite poset, then knowing the cover relations in P is enough to know all order relations (in this case $x < y$ if and only if there is a chain of cover relations $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y$). The *Bruhat order* on the symmetric group S_n, denoted $\mathrm{Br}(S_n)$ (or just S_n if the partial order is clear from context) is the poset determined by the following cover relations: given $\pi, \sigma \in S_n$, define $\pi \lessdot \sigma$ if and only if $\sigma = \pi\tau$ where τ is a transposition and $\mathrm{inv}(\sigma) = \mathrm{inv}(\pi) + 1$.

The cover relations in the Bruhat order can be described conveniently by looking at π and σ in one-line notation and τ in cycle notation. We observe that multiplying on the right by the transposition (i, j) swaps the i^{th} and j^{th} positions in one-line notation: $\pi_1 ... \pi_i ... \pi_j ... \pi_n \cdot (i, j) = \pi_1 ... \pi_j ... \pi_i ... \pi_n$. Thus $\pi \lessdot \sigma$ in S_n if and only if σ can be produced by finding and interchanging two entries π_i, π_j of π in one-line notation with $i < j$ and $\pi_i < \pi_j$ such that none of the numbers $\pi_{i+1}, ..., \pi_{j-1}$ are in the $\mathbb{Z}$-interval (π_i, π_j). See Figure 2.1 for an example of the Hasse diagram of the Bruhat order on S_3. The following well-known fact will be essential in our categorification.

Lemma 4.1.1 ([Bjö84, Section 2.5])**.** *The Bruhat order,* $\mathrm{Br}(S_n)$*, on the symmetric group* S_n*, is a CW poset with rank function* $\pi \mapsto \mathrm{inv}(\pi)$.

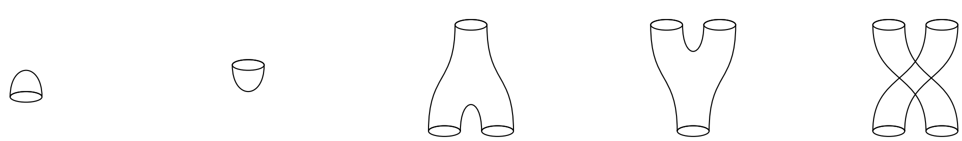
4.1.3 Cobordisms, TQFTs, and Frobenius Algebras

Bar-Natan's description [BN05] of Khovanov homology is of the type $C^*(P, F, c)$ presented in Chapter 2, where $P = 2^X$ (with X being the crossing set of a knot diagram), and F is gotten by composing a functor $2^X \to \mathbf{Cob}_2$ from 2^X to the category of 2-dimensional cobordisms, with a functor $\mathbf{Cob}_2 \to R$**-gmod** known as a TQFT (topological quantum field theory). Our categorification of $V(\vec{x})$ is defined similarly but will instead be gotten by constructing a functor which factors not through $\mathbf{Cob}_2$, but through the category $\mathbf{Cob}_2^{\vec{x}}$ of 2-dimensional colored cobordisms (see Definition 4.2.6). In this section we review cobordisms, TQFTs, and the equivalence between 2-dimensional TQFTs and commutative Frobenius algebras. All of the definitions and results in this section can be found in [Koc04].

Definition 4.1.2 ([Koc04, Definition 1.3.20])**.** Let $\mathbf{Cob}_n$ denote the the following category: objects are closed oriented $(n-1)$-dimensional manifolds, morphisms between objects M and N are (orientation preserving) diffeomorphism classes of oriented n-dimensional manifolds W with $\partial W = -M \amalg N$, where $-M$ denotes M with the opposite orientation and $\amalg$ denotes disjoint union. In this case we write $\partial_i W = M$ and $\partial_o W = N$ and say that M is the *in-boundary* and N is the *out-boundary* of W. Morphisms in $\mathbf{Cob}_n$ are called *oriented cobordisms*. Given $A \in \mathrm{Hom}(M, N)$ and $B \in \mathrm{Hom}(N, L)$ define the composition $B \circ A$ by gluing B to A along the identity map on N. See [Koc04] for details on the well-definedness of this construction.

Given two $(n-1)$-manifolds, M and N, one can form the disjoint union $M \amalg N$ which is again an $(n-1)$ manifold. Similarly, one can form disjoint unions of cobordisms. Disjoint unions endow the category $\mathbf{Cob}_n$ with the structure of a monoidal category. Given a diffeomorphism $\phi : M \to N$, one can form a cobordism from M to N by first forming the cylinder $[0,1] \times M$ and gluing N to $\{1\} \times M$ via the diffeomorphism ϕ. Notice that $M \amalg N$ is diffeomorphic to $N \amalg M$ via the map $\tau_{M,N}$ which interchanges factors. The cobordism corresponding to $\tau_{M,N}$ from $M \amalg N$ to $N \amalg M$ is called a *twist cobordism*. Twist cobordisms act as a symmetric braiding, and thus endow $\mathbf{Cob}_n$ with the structure of a symmetric monoidal category. In the case that M and N are both circles, the corresponding twist cobordism will be denoted as shown in Figure 4.1e.

In this chapter, we are concerned only with $\mathbf{Cob}_2$ where cobordisms are easy to visualize and classify. The objects of $\mathbf{Cob}_2$ are closed oriented 1-manifolds (disjoint unions of oriented circles). By the classification of surfaces, morphisms in $\mathbf{Cob}_2$ are oriented surfaces (that is, spheres and connect sums of tori with some number of disks removed) with oriented boundary. Thus in dimension 2, connected cobordisms are classified by three quantities: genus, number of in-boundary components, and number of out-boundary components. In our pictures, we will draw the in-boundary on the bottom of the cobordism and the out-boundary on the top (so pictures go from the bottom up). A standard result of Morse theory is that any cobordism in dimension 2 can be built by gluing and taking disjoint unions of the basic building blocks shown in Figure 4.1.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 4.1 (a) The *cap* cobordism from ○ to ∅. (b) The *cup* cobordism from ∅ to ○. (c) The *pants* cobordism from ○○ to ○. (d) The *copants* cobordism from ○ to ○○. (e) The twist cobordism from ○○ to ○○.

In Section 4.3, we categorify the Vandermonde determinant by first constructing a diagram in a category of cobordisms, and then using this diagram to construct a cochain complex which contains $V(\vec{x})$ as its Euler characteristic. To do this, we need a way to pass from the category of cobordisms to a category of modules.

Definition 4.1.3 ([Koc04, Definition 1.3.32])**.** An n-dimensional *topological quantum field theory (TQFT)* is a symmetric monoidal functor from $\mathbf{Cob}_n$ to a symmetric monoidal abelian category $\mathscr{A}$.

We will often restrict our attention to the case $\mathscr{A} = R$-**mod** for some commutative ring R. TQFTs over a ring R form a category, denoted $n\mathbf{TQFT}_R$, where morphisms are monoidal natural transformations.

$$
\begin{tikzcd}
& A\otimes A \arrow[dr, "\mu"] & \\
A\otimes A\otimes A \arrow[ur, "1\otimes\mu"] \arrow[dr, "\mu\otimes 1"'] & & A \\
& A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes A \arrow[dr, "\mu"] & \\
R\otimes A\cong A\cong A\otimes R \arrow[ur, "1\otimes\eta"] \arrow[rr, "1"] \arrow[dr, "\eta\otimes 1"'] & & A \\
& A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A\otimes A \arrow[dl, "1\otimes\Delta"'] & \\
A\otimes A\otimes A & & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
& A\otimes A \arrow[ul, "\Delta\otimes 1"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes A \arrow[dl, "1\otimes\epsilon"'] & \\
R\otimes A\cong A\cong A\otimes R & & A \arrow[ul, "\Delta"'] \arrow[ll, "1"'] \arrow[dl, "\Delta"] \\
& A\otimes A \arrow[ul, "\epsilon\otimes 1"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A\otimes A\otimes A \arrow[dr, "\mu\otimes 1"] & & \\
A\otimes A \arrow[ur, "1\otimes\Delta"] \arrow[r, "\mu"] \arrow[dr, "\Delta\otimes 1"'] & A \arrow[r, "\Delta"] & A\otimes A \\
& A\otimes A\otimes A \arrow[ur, "1\otimes\mu"'] &
\end{tikzcd}
$$

Figure 4.2 Axioms of a Frobenius algebra. The diagram on the right illustrates the Frobenius relation. The other four diagrams illustrate the associativity, unit, coassociativity, and counit relations.

Definition 4.1.4 ([Koc04, Definition 2.2.1 and Proposition 2.3.24])**.** A *Frobenius algebra* is a tuple $(A,\mu,\eta,\Delta,\varepsilon)$ such that (A,μ,η) is a unital associative algebra over a ring R and (A,Δ,ε) is a counital coassociative coalgebra over R for which the Frobenius relation holds:

$$(\mu\otimes 1)\circ(1\otimes\Delta)=\Delta\circ\mu=(1\otimes\mu)\circ(\Delta\otimes 1).$$

Commutative Frobenius algebras over a ring R form a category, denoted $\mathbf{cFA}_R$, whose morphisms are homomorphisms of Frobenius algebras, that is, algebra homomorphisms which are also coalgebra homomorphisms.

These two definitions, 4.1.3 and 4.1.4, look quite different at face value, however 2-dimensional TQFTs and (commutative) Frobenius algebras actually encode the same information. Upon comparison of the relations in $\mathbf{Cob}_2$ shown in Figure 4.3 and the axioms of commutative Frobenius algebras in Figure 4.2, we see that we can go back and forth between TQFTs and commutative Frobenius

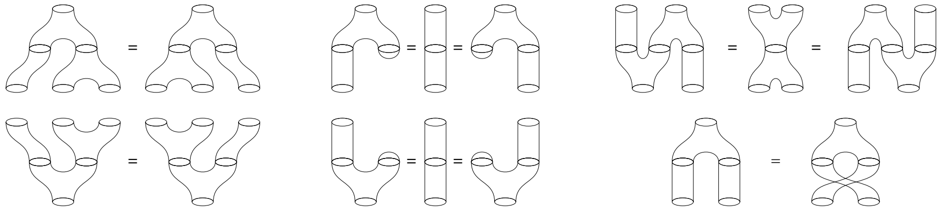


Figure 4.3 Equalities of cobordisms which correspond to axioms of Frobenius algebras (compare to Figure 4.2). The equality at the top right corresponds to the Frobenius relation, and the equality at the bottom right guarantees that the corresponding Frobenius algebra is commutative. The other four equalities correspond to the associativity, unit, coassociativity, and counit relations.

algebras via

$$F \longleftrightarrow \Big(F(\bigcirc), F(\;), F(\;), F(\;), F(\;) \Big). \tag{4.4}$$

Thus we have a bijection between 2-dimensional TQFTs and commutative Frobenius algebras. In fact, this bijection extends to an equivalence of categories.

Theorem 4.1.5 ([Koc04, Theorem 3.3.2])**.** *There is a canonical equivalence of categories*

$$2\mathbf{TQFT}_R \simeq \mathbf{cFA}_R.$$

4.2 Special TQFTs and Colored TQFTs

The Frobenius algebras/TQFTs in our construction need to satisfy the following additional property in order to define a functor $F^{\mathrm{Vand}} : \mathrm{Br}(S_n) \to \mathscr{A}$, from which we build a homology theory which categorifies $V(\vec{x})$.

Definition 4.2.1 ([Run05, Definition 3.5])**.** A Frobenius algebra $(A, \mu, \eta, \Delta, \varepsilon)$ is called *special* if

$$\mu \circ \Delta = 1_A.$$

Figure 4.4 provides the topological interpretation of this condition. Special Frobenius algebras are also sometimes referred to as *strongly separable* algebras. Aguiar uses this terminology in [Agu00] where he characterizes and provides many nice examples of these algebras.

Example 4.2.2. Define A_n to be the n-fold product

$$A_n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2.$$

Then A_n is a $\mathbb{Z}_2$-algebra of dimension n with addition and multiplication defined pointwise. Consider the basis $\{\vec{e}_i \mid 1 \le i \le n\}$ where $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$ has all entries 0 except for a single 1 in position i.

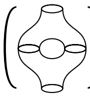
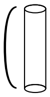
$$F\Big($$  $$\Big) = F\Big($$  $$\Big) \quad \Longleftrightarrow \quad \mu\circ\Delta = 1_A$$

Figure 4.4 The defining relation of a special Frobenius algebra (Definition 4.2.1), expressed in terms of the corresponding TQFT. One may interpret this as the fact that special 2D TQFTs (those corresponding to special Frobenius algebras) do not detect genus (see Lemma 4.2.4).

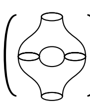
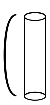
Define the counit $\varepsilon : A_n \to \mathbb{Z}_2$ by sending $\vec{a} \mapsto \sum_{i=1}^n a_i$ where $\vec{a} = (a_1,...,a_n)$. This determines the comultiplication $\Delta : A_n \to A_n \otimes A_n$ as given by the formula $\vec{a} \mapsto \sum_{i=1}^n a_i \vec{e}_i \otimes \vec{e}_i$. These maps endow A_n with the structure of a Frobenius algebra. Actually, since

$$(\mu\circ\Delta)(\vec{a}) = \mu\left(\sum_{i=1}^n a_i \vec{e}_i \otimes \vec{e}_i\right) = \sum_{i=1}^n a_i \vec{e}_i = \vec{a},$$

A_n is a special Frobenius algebra. Thus we have a family $\{A_n\}_{n\in\mathbb{Z}_+}$ of special Frobenius algebras over $\mathbb{Z}_2$, one for each positive integer n.

Instead of restricting to categories of modules, we will work in the more general setting of abelian categories.

Definition 4.2.3. Let $\mathscr{A}$ be a symmetric monoidal abelian category. A topological quantum field theory $F : \mathbf{Cob}_2 \to \mathscr{A}$ is *special* if

$$F\Big($$  $$\Big) = F\Big($$  $$\Big).$$

Lemma 4.2.4. *Let $F : \mathbf{Cob}_2 \to \mathscr{A}$ be a special 2-dimensional TQFT. Then for any two connected cobordisms $B, C \in \mathrm{Hom}(M,N)$ with $M,N \in \mathbf{Cob}_2$, we have $F(B) = F(C)$.*

Proof. Given a commutative special Frobenius algebra A, let F be the corresponding TQFT. In Figure 4.4 we see the defining relation $\mu\circ\Delta = 1$ expressed topologically. It follows that applying F to any connected cobordism with one incoming and one outgoing boundary also results in the identity map, because any such cobordism can be expressed by stacking the cobordism  on itself an appropriate number of times (just enough to get the correct genus).

Now, take objects $M,N \in \mathbf{Cob}_2$, connected cobordisms $C,D \in \mathrm{Hom}(M,N)$ and suppose M (respectively N) consists of k (respectively ℓ) disjoint circles. Then C and D both have k incoming boundary components and l outgoing boundary components. Since C and D are connected cobordisms in $\mathbf{Cob}_2$, they can be written in *normal form*. That is, write $C = C_1 \circ C_2 \circ C_3$ where C_3 has k incoming boundary components, one outgoing boundary component and genus 0, C_2 has one incoming boundary component, the same genus as C and one outgoing boundary component, and C_1 has one incoming boundary component, l outgoing boundary component, and genus 0.

Similarly, we can write $D = D_1 \circ D_2 \circ D_3$ where $D_1 = C_1$ and $D_3 = C_3$ and D_2 has one incoming and one outgoing boundary component and has the same genus as D. It follows from the previous paragraph that $F(C_2) = F(D_2) = 1$ so

$$F(C) = F(C_1) \circ F(C_3) = F(D_1) \circ F(D_3) = F(D). \qquad \square$$

Remark 4.2.5. In other words, Theorem 4.2.4 says that special 2-dimensional TQFTs do not detect genus.

We now introduce colors into the cobordism category, with different colors corresponding to the different variables present in $V(\vec{x})$.

Definition 4.2.6. Let S be a set, endowed with the discrete topology. Define $\mathbf{Cob}_k^S$ to be the category whose objects are pairs (M, ϕ_M) where M is a $(k-1)$-dimensional closed oriented manifold and ϕ_M is a continuous map from M to S. Morphisms between (M, ϕ_M) and (N, ϕ_N) are pairs $([A], \phi_A)$ where $[A]$ is an oriented cobordism class from M to N ($[\cdot]$ denotes the diffeomorphism class of A) and ϕ_A is a continuous map from A to S such that $\phi_A|_{\partial_i A} = \phi_M$ and $\phi_A|_{\partial_o A} = \phi_N$, where ∂_i and ∂_o denote the in-boundary and out-boundary respectively.

Intuitively, the objects of $\mathbf{Cob}_k^S$ are closed $(k-1)$-manifolds whose connected components are colored by elements of S and morphisms are (diffeomorphism classes of) k-manifolds with boundary and with each connected component given a color from S.

Convention 4.2.7. Let $\vec{x} = (x_1, \ldots, x_n)$. We introduce the slight abuse of notation $\mathbf{Cob}_k^{\vec{x}} = \mathbf{Cob}_k^{\{x_1,\ldots,x_n\}}$.

For convenience, instead of labeling connected components of manifolds with elements of the set, we display them in different colors. Our running example (starting with Example 4.2.8) is in the case $S = \{x_1, x_2, x_3\} \subseteq \mathbb{Z}_+$ with the convention $x_1 = \text{red}$, $x_2 = \text{green}$, $x_3 = \text{blue}$.

Example 4.2.8. Let $\vec{x} = (x_1, x_2, x_3)$ have distinct entries and consider the category $\mathbf{Cob}_2^{\vec{x}}$. Let $M =$ ○○○○○○ and $N =$ ○○○○○○. Figure 4.5a shows a colored cobordism from M to N and in Figure 4.5b we see an example of a cobordism for which no coloring gives a morphism in $\mathbf{Cob}_2^{\vec{x}}$.

(a) **(b)**

Figure 4.5 A colored cobordism in (a) and a non-example in (b).

$\mathbf{Cob}_k^S$ is a monoidal category under the operation of taking disjoint unions. Let $\mathbf{Cob}_k^S(x)$ be the subcategory of $\mathbf{Cob}_k^S$ consisting of all objects and morphisms labeled x, where $x \in S$. Then

$$\mathbf{Cob}_k^S \cong \prod_{x \in S} \mathbf{Cob}_k^S(x)$$

where each $\mathbf{Cob}_k^S(x)$ is isomorphic to $\mathbf{Cob}_k$. Given a morphism $C \in \mathbf{Cob}_k^S$ and $x \in S$, let C_x denote the component of C in $\mathbf{Cob}_k^S(x)$. In the case $S = [n]$, we will simply write $\mathbf{Cob}_k^n$.

Definition 4.2.9. Let $\mathscr{A}$ be a symmetric monoidal abelian category. A *colored topological quantum field theory* is a monoidal functor $F : \mathbf{Cob}_k^S \to \mathscr{A}$ for some finite set S such that the restriction F_x of F to the subcategory $\mathbf{Cob}_k^S(x)$ is a TQFT for each $x \in S$. In the case that $k = 2$, call a colored TQFT *special* if for each $x \in S$, F_x is a special TQFT. For brevity we may use the abbreviation *SC-TQFT* to refer to a special colored TQFT.

4.3 Categorifying Generalized Vandermonde Determinants

The goal of this section is to define a functor $F : \mathrm{Br}(S_n) \to \mathscr{A}$ to an abelian category so that the homology $H(\mathrm{Br}(S_n), F)$ categorifies the Vandermonde determinant in the sense of Theorem 2.6.6. We accomplish this in two steps, similar to Bar-Natan's approach to Khovanov homology in [BN05]. First we pass to a category of colored cobordisms, and then we apply a TQFT to land in an abelian category.

Definition 4.3.1. Let P be a poset, and S a finite set. A *functor up to stabilization*, F, from a diamond transitive thin poset P to $\mathbf{Cob}_2^S$ is a collection $\{F(x)\}_{x \in P}$ of objects in $\mathbf{Cob}_2^S$ (one for each element of P), and a collection of morphisms $\{F(x \lessdot y)\}_{x \lessdot y \in C(P)}$ (one for each cover relation in P) such that for any diamond $\{x, a, b, y\}$ in P, (where $x \lessdot a \lessdot y$ and $x \lessdot b \lessdot y$), for each $x \in S$ there exist closed oriented surfaces C, C' in $\mathbf{Cob}_2^S(x)$ such that

$$[F(a \lessdot y) \circ F(x \lessdot a)]_x \# C \cong [F(b \lessdot y) \circ F(x \lessdot b)]_x \# C',$$

where # denote the connect sum.

Lemma 4.3.2. *Let S be a finite set, P a diamond transitive thin poset, $\mathscr{A}$ an abelian category, $F : P \to \mathbf{Cob}_2^S$ a functor up to stabilization, and $G : \mathbf{Cob}_2^S \to \mathscr{A}$ a special colored TQFT. Then $G \circ F$ is a functor.*

Proof. This follows directly from Lemma 4.2.4, and Theorem 2.5.1. □

Recall from Theorem 2.1.9 that $\mathrm{Br}(S_n)$ is a CW poset, and therefore by Theorems 2.4.12 and 2.5.1, to define a functor from $\mathrm{Br}(S_n)$ it suffices to define per-edge maps which commute on diamonds. In the next step of our construction, we use knot/link diagrams and their smoothings as a way to pass from the Bruhat order on S_n to $\mathbf{Cob}_2^S$, as a step towards constructing a homology theory. For general knot theory background, the reader is referred to [Lic12]. Recall that crossings in knot diagrams have two types of smoothings, the *0-smoothing* and the *1-smoothing* (see Figure 4.6).

Figure 4.6 The 0 and 1 smoothings of a crossing in a knot diagram.

Definition 4.3.3. Let D be a link diagram, and let $X = \{c_1, \dots, c_n\}$ denote the set of crossings. We define a labeling $V_D : \mathrm{Br}(S_n) \to \mathbf{Cob}_2^n$ of the vertices and edges of the Hasse diagram $H(\mathrm{Br}(S_n))$ by objects and morphisms of $\mathbf{Cob}_2^n$ as follows:

1. For each $\pi \in S_n$, $V_D(\pi) = D_1^\pi \amalg \cdots \amalg D_n^\pi$ where $D_i^\pi \in \mathbf{Cob}_2^n(i)$ is the resolution of D gotten by giving the crossings $c_1, \dots, c_{\pi(i)}$ 1-smoothings, and all other crossings 0-smoothings. Call $V_D(\pi)$ the π*-smoothing* of D.

2. Suppose $\pi \lessdot \sigma \in \mathrm{Br}(S_n)$. Define

$$V_D(\pi \lessdot \sigma) = C_1^{\pi,\sigma} \amalg \dots \amalg C_n^{\pi,\sigma}$$

   where $C_i^{\pi,\sigma} \in \mathbf{Cob}_2^n(i)$ is the identity cobordism from D_i^π to D_i^σ if $\pi(i) = \sigma(i)$, or the unique connected genus zero cobordism from D_i^π to D_i^σ if $\pi(i) \neq \sigma(i)$.

Lemma 4.3.4. *For any link diagram D, V_D is a functor up to stabilization.*

Proof. Consider a diamond in the Bruhat order (shown in Figure 4.7). We proceed by analyzing the following cases:

$$\begin{tikzcd} & \sigma \arrow[dr] & \\ \pi \arrow[ur] \arrow[dr] & & \gamma \\ & \eta \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & D^\sigma \arrow[dr, "C^{\sigma,\gamma}"] & \\ D^\pi \arrow[ur, "C^{\pi,\sigma}"] \arrow[dr, "C^{\pi,\eta}"'] & & D^\gamma \\ & D^\eta \arrow[ur, "C^{\eta,\gamma}"'] & \end{tikzcd}$$

(a) **(b)**

Figure 4.7 In (a), a diamond in the Bruhat order and in (b) the corresponding labelings of V_D.

Case 1. *π and γ differ in 3 positions, i, j and k (in one-line notation).* Without loss of generality we can write $\sigma = \pi(i,j)$ and $\gamma = \sigma(i,k)$. Now, $(i,j)(i,k) = (i,k,j)$ can be written in exactly two other ways as a product of transpositions, $(i,k)(j,k) = (i,k,j) = (j,k)(i,j)$ so our diamond is given by either Figure 4.8a or 4.8b.

$$\begin{tikzcd}
& \sigma \arrow[dr, "{(i,k)}"] & \\
\pi \arrow[ur, "{(i,j)}"] \arrow[dr, "{(i,k)}"'] & & \gamma \\
& \eta \arrow[ur, "{(j,k)}"'] &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
& \sigma \arrow[dr, "{(i,k)}"] & \\
\pi \arrow[ur, "{(i,j)}"] \arrow[dr, "{(j,k)}"'] & & \gamma \\
& \eta \arrow[ur, "{(i,j)}"'] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
& \sigma \arrow[dr, "{(k,l)}"] & \\
\pi \arrow[ur, "{(i,j)}"] \arrow[dr, "{(k,l)}"'] & & \gamma \\
& \eta \arrow[ur, "{(i,j)}"'] &
\end{tikzcd}$$

(c)

Figure 4.8 In (a) and (b) we see the two possible diamonds which can appear in case 1. In (c) we see the only possible type of diamond which can appear in case 2.

First consider the diamond in Figure 4.8a. By construction, $C_i^{\pi,\sigma}$ is a connected genus zero cobordism from D_i^{π} to D_i^{σ} and $C_i^{\sigma,\gamma}$ is a connected genus zero cobordism from D_i^{σ} to D_i^{γ}. Thus $C_i^{\sigma,\gamma} \circ C_i^{\pi,\sigma}$ is a connected cobordism from D_i^{π} to D_i^{γ}. Going the other way, $C_i^{\pi,\eta}$ is a connected genus zero cobordism from D_i^{π} to D_i^{η} and $C_i^{\eta,\gamma}$ is the identity cobordism from D_i^{η} to D_i^{γ}. Thus $C_i^{\eta,\gamma} \circ C_i^{\pi,\eta}$ is also a connected cobordism from D_i^{π} to D_i^{γ}. Thus $C_i^{\eta,\gamma} \circ C_i^{\pi,\eta}$ and $C_i^{\sigma,\gamma} \circ C_i^{\pi,\sigma}$ differ only (possibly) in their genus. The same argument works for the pieces colored j and k. For $l \notin \{i,j,k\}$ things are even easier, as $C_l^{\sigma,\gamma} \circ C_l^{\pi,\sigma} = C_l^{\eta,\gamma} \circ C_l^{\pi,\eta}$ is by definition just the identity cobordism. A similar argument applies to the diamond in Figure 4.8b.

Case 2. *π and γ differ in 4 positions i, j, k and l (in one-line notation).* Then our diamond looks like one shown in Figure 4.8c. A similar argument works here as in case 1. $\square$

Convention 4.3.5. To avoid having to draw 2-dimensional pictures of cobordisms, we can use the following shortcut. To denote the connected genus zero cobordism from a smoothing D_i^{π} to another smoothing D_i^{σ}, simply circle which crossings change in the picture of D_i^{π}. See Figure 4.9 for an example. Warning: since we require the cobordisms of each color to be connected, the circles around crossings do not indicate local saddle cobordisms (like one might expect in comparison to Bar-Natan's notation in [BN02]). See Figure 4.9 for an example in the case $D = T_{2,3}$, and Figure 4.10 for a depiction of the functor V_D.

Let $F : \mathbf{Cob}_2^n \to \mathscr{A}$ be a special colored TQFT. Then by Lemma 4.3.2, $F \circ V_D : \mathrm{Br}(S_n) \to \mathscr{A}$ is a functor. Let c be a balanced coloring of $\mathrm{Br}(S_n)$. Then by the construction in Chapter 2, the data $(\mathrm{Br}(S_n), F \circ V_D, c)$ determines a chain complex $C^*(\mathrm{Br}(S_n), F \circ V_D, c)$ with homology $H^*(\mathrm{Br}(S_n), F \circ V_D)$ which we will denote by $H_F^{\mathrm{Vand}}(D)$ for brevity.

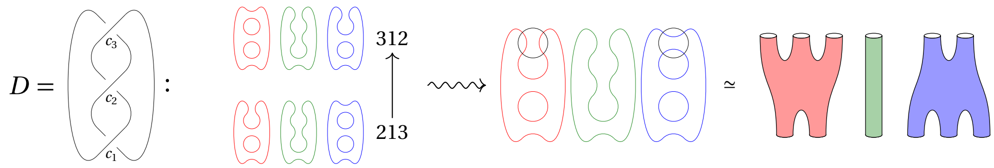


Figure 4.9 A shortcut notation for colored cobordisms, as explained in 4.3.5. The above picture denotes the cobordism $C^{213,312}$ from D^{213} to D^{312} where $D = T_{2,3}$.

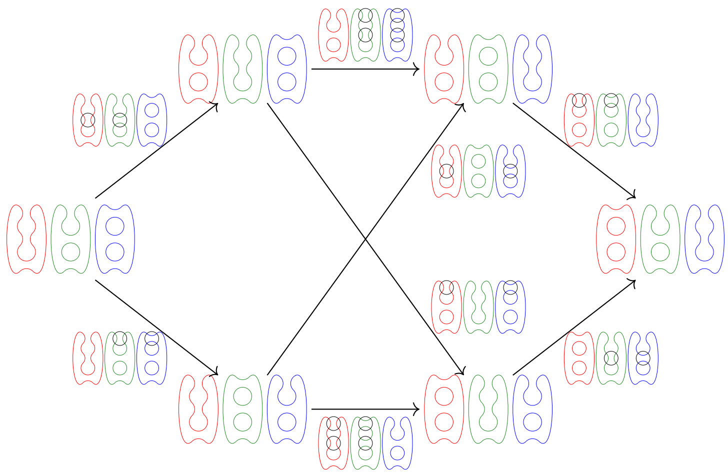


Figure 4.10 Shown above is the diagram for V_D in the case that D is a diagram of the torus knot $T_{2,3}$ with crossings ordered from bottom to top, and $S = \{\text{red}, \text{green}, \text{blue}\}$.

Theorem 4.3.6. *Let D be a link diagram with crossing set $X = \{c_1, \dots c_n\}$, $\mathscr{A}$ a symmetric monoidal abelian category, and $F : \mathbf{Cob}_2^n \to \mathscr{A}$ a special colored TQFT. Let $\vec{s} = (s_1, \dots s_n)$ where s_i is the number of circles in the resolution of D which gives $c_1, \dots c_i$ 1-smoothings and $c_{i+1}, \dots, c_n$ 0-smoothings. Let $x_i = [F(\bigcirc_i)] \in K_0(\mathscr{A})$, and let $\vec{x} = (x_1, \dots x_n)$. Then in $K_0(\mathscr{C}^b(\mathscr{A}))$, we have*

$$[H_F^{\text{Vand}}(D)] = V_{\vec{s}}(\vec{x}).$$

Proof. The smoothing D_i^π consists of $s_{\pi(i)}$ circles of color i by construction and thus $[(F \circ V_D)(\pi)] = [x_1]^{s_{\pi(1)}}[x_2]^{s_{\pi(2)}} \dots [x_n]^{s_{\pi(n)}}$, which is exactly the contribution from π to $V_{\vec{s}}(\vec{x})$ by equation 4.3. Thus the result follows from Theorem 2.6.6. $\square$

We now specialize our construction to the 2-torus link diagrams $T_{2,n}$. Recall $T_{2,n}$ denotes the closure of the diagram σ_1^n in the braid group B_2 where $\sigma_1 = \times$ (see figure 4.11a). It turns out (by Lemma 4.3.7 and Theorem 4.3.8) that smoothings of the diagram $T_{2,n}$ have the appropriate combinatorics needed to categorify the Vandermonde determinant $V(\vec{x})$.

Lemma 4.3.7. *Let $c_1, c_2, \ldots, c_n$ be a total ordering of the crossings of $T_{2,n}$ and consider $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$. Let $h(\vec{\alpha}) = \sum_i \alpha_i$ denote the height of $\vec{\alpha}$. Then if $h(\vec{\alpha}) > 0$, the smoothing of $T_{2,n}$ corresponding to $\vec{\alpha}$ consists of $h(\vec{\alpha})$ disjoint circles.*

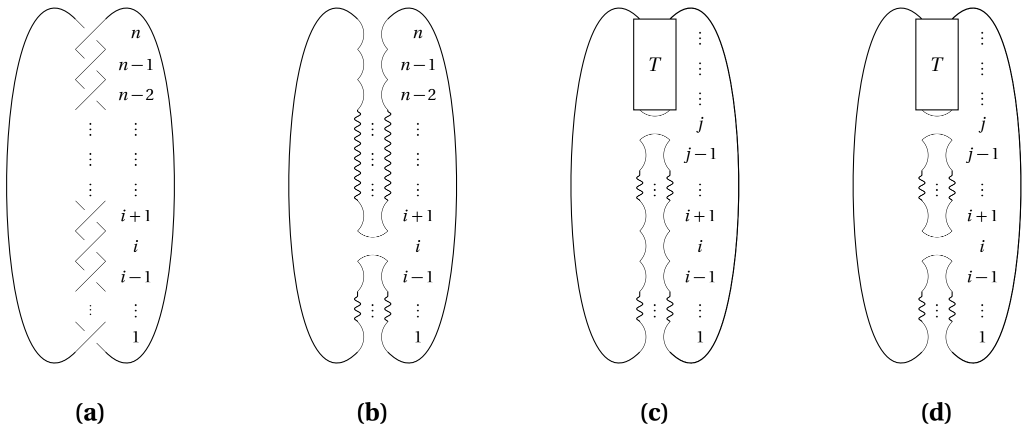


Figure 4.11 (a) The diagram $T_{2,n}$. (b) A smoothing of $T_{2,n}$ with height 1. (c) The smoothing $\vec{\alpha}'$ gotten by changing the first 1 in $\vec{\alpha}$ to 0. (d) The smoothing $\vec{\alpha}$ from the proof of Lemma 4.3.7.

Proof. We proceed by induction on $h(\vec{\alpha})$. If $h(\vec{\alpha}) = 1$, then $\alpha_i = 1$ for some $i \in [n]$ and $\alpha_j = 0$ for each $j \neq i$. The smoothing corresponding to $\vec{\alpha}$ is shown in Figure 4.11a and consists of one circle.

Now suppose that $h(\vec{\alpha}) > 1$, and choose i minimally such that $\alpha_i = 1$ and j minimally such that $j > i$ and $\alpha_j = 1$. Consider the tuple $\vec{\alpha}'$ for which $\alpha'_j = \alpha_j$ for $j \neq i$ and $\alpha'_i = 0$. The smoothings above the j^{th} crossing form a trivial tangle which we will denote by T, and this tangle is the same in both the smoothing corresponding to $\vec{\alpha}$ and the smoothing corresponding to $\vec{\alpha}'$. By induction, the smoothing corresponding to $\vec{\alpha}'$ consists of $h(\vec{\alpha}') = h(\vec{\alpha}) - 1$ disjoint circles. As shown in Figure 4.11c and 4.11d, changing the i^{th} smoothing from 0 to 1 splits one circle into two. Thus the smoothing corresponding to $\vec{\alpha}$ consists of $h(\vec{\alpha})$ disjoint circles. $\square$

Theorem 4.3.8. *Let $\mathscr{A}$ be a symmetric monoidal abelian category, and $F : \mathbf{Cob}_2^n \to \mathscr{A}$ a special colored TQFT. Let $x_i = [F(\bigcirc_i)] \in K_0(\mathscr{A})$, and let $\vec{x} = (x_1, \ldots x_n)$. Then in $K_0(\mathscr{C}^b(\mathscr{A}))$, we have*

$$[H_F^{\text{Vand}}(T_{2,n})] = V(\vec{x}).$$

Proof. By Lemma 4.3.7, the smoothing $(T_{2,n})_i^\pi$ consists of $\pi(i)$ circles of color i by construction and thus $[(F \circ V_D)(\pi)] = [x_1]^{\pi(1)}[x_2]^{\pi(2)} \ldots [x_n]^{\pi(n)}$, which is exactly the contribution from π to $V(\vec{x})$ by equation 4.3. Thus the result follows from Theorem 2.6.6. $\square$

4.4 Future Directions

This chapter presents the first example of a cohomology theory categorifying a determinant using the categorification technique outlined in Theorem 2.6.6. Hopefully the technique presented in this chapter will lead to categorifications of other interesting determinants. We end with a list of questions related to this categorification of the Vandermonde determinant.

1. Is it possible to choose the SC-TQFT F so as to obtain a complex $H_F^{\mathrm{Vand}}(D)$ whose cohomology is a link invariant?

2. Applying a cofactor expansion along the last column in $V(\vec{x}) = \det(x_i^j)_{i,j=1}^n$ yields

$$V(\vec{x}) = \sum_{k=1}^{n} (-1)^{n+k} x_k^n \; V\big(\vec{x}(i)\big) \tag{4.5}$$

   where $V\big(\vec{x}(i)\big) = \det(x_i^j)_{(i,j)\in([n]\setminus\{k\})\times[n-1]}$ is the Vandermonde determinant in variables $\vec{x}(i) = (x_1, \dots, \widehat{x_i}, \dots x_n)$. Is there a resolution of $H_F^{\mathrm{Vand}}(T_{2,n})$ which categorifies the relation (4.5)?

3. The first property one typically learns about the Vandermonde determinant is the product formula:

$$V(\vec{x}) = x_1 \dots x_n \prod_{i<j} (x_j - x_i). \tag{4.6}$$

   One could consider categorifying the Vandermonde determinant using equation (4.6) instead of (4.3). This could be accomplished with a family of objects $\{X_1, \dots X_n\}$ in an abelian category $\mathscr{A}$ by categorifying the difference $x_j - x_i$ to some morphism $X_j \to X_i$ where $[X_i] = x_i$ in $K_0(\mathscr{A})$ for all i. Then the complex $x_1 \otimes \dots \otimes x_n \bigotimes_{i<j} (x_j \to x_i)$, has Euler characteristic $V(\vec{x})$. One may ask if there is a choice of a SC-TQFT so that this complex is equivalent to $H_F^{\mathrm{Vand}}(T_{2,n})$.

4. In the representation theory of S_n, quotients of generalized Vandermonde determinants by the Vandermonde determinant $V(\vec{x})$ in the same variables can be expressed as a sum of symmetric functions with coefficients related to characters of certain representations of S_n. Can this relation be formulated on the categorified level in terms of the complexes $H_F^{\mathrm{Vand}}(D)$?

CHAPTER

5

KHOVANOV HOMOLOGY AND TORSION

In 1984, Jones discovered a powerful polynomial link invariant, now known as the Jones polynomial [Jon85]. In 1999, Khovanov [Kho00] categorified the Jones polynomial $J(L)$ to a bigraded homology theory $H(L)$ called Khovanov homology. Khovanov homology categorifies the Jones polynomial in the sense that the Jones polynomial of a link can be recovered as the graded Euler characteristic of its Khovanov homology. Each $H^{i,j}(L)$ is a finitely generated abelian group, and the Jones polynomial gets a contribution only from the free part of $H(L)$. Thus torsion in Khovanov homology is a new phenomena in knot theory which does not appear in the theory of Jones polynomials. Throughout this chapter we will use the term '$\mathbb{Z}_{p^r}$ torsion', p a prime, to mean a direct summand of isomorphism class $\mathbb{Z}_{p^r}$ in the primary decomposition of the integral Khovanov homology of a link.

Khovanov homology is equipped with two gradings: the homological grading i and the polynomial grading j. A link is *homologically thin* if its Khovanov homology is supported in bigradings $2i-j=s\pm1$ for some integer s. Non-split alternating links and quasi-alternating links are homologically thin [Lee08; MO08]. In [Shu14], Shumakovitch showed that homologically thin links have only $\mathbb{Z}_{2^r}$ torsion in their Khovanov homology, and in [Shu18] he used a relationship between the Turner and Bockstein differentials on $\mathbb{Z}_2$-Khovanov homology to show in fact there is only $\mathbb{Z}_2$ torsion. In this chapter, we prove a version of this result when the Khovanov homology of a link is thin over a restricted range of homological gradings.

Let $i_1, i_2 \in \mathbb{Z}$ with $i_1 < i_2$. We say that $H(L)$ is *thin over* $[i_1, i_2]$ if there is an $s \in \mathbb{Z}$ such that $H^{*,*}(L;\mathbb{Z}_p)$ is supported only in bigradings satisfying $2i-j=s\pm1$ for all i with $i_1 \le i \le i_2$ and for all primes p. If i is an integer with $i_1 \leq i \leq i_2$, then we say $i \in [i_1, i_2]$; we similarly define $i \in (i_1, i_2], (i_1, i_2)$, or $[i_1, i_2)$.

Theorem 5.3.4. Suppose that a link L satisfies:

1. $H(L)$ is thin over $[i_1, i_2]$ for integers i_1 and i_2,
2. $\dim_{\mathbb{Q}} H^{i_1,*}(L;\mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1,*}(L;\mathbb{Z}_p)$ for each odd prime p, and
3. $H^{i_1,*}(L)$ is torsion-free.

Then all torsion in $H^{i,*}(L)$ is $\mathbb{Z}_2$ torsion for $i \in [i_1, i_2]$.

We use Theorem 5.3.4 to show that certain families of closed 3-braids have only $\mathbb{Z}_2$ torsion in their Khovanov homology. Various techniques have been used to show that some other families of links only have $\mathbb{Z}_2$ torsion in their Khovanov homology or only have $\mathbb{Z}_2$ torsion in certain gradings. In [HG06], Helme-Guizon, Przytycki, and Rong established a connection between the Khovanov homology of a link and the chromatic graph homology of graphs associated to diagrams of the link. In [LS17], Lowrance and Sazdanović used this connection to show that in a range of homological gradings, Khovanov homology contains only $\mathbb{Z}_2$ torsion. This result can now be seen as a corollary to Theorem 5.3.4.

In [PS14], Przytycki and Sazdanović obtained explicit formulae for some torsion and proved that the Khovanov homology of semi-adequate links contains $\mathbb{Z}_2$ torsion if the corresponding Tait-type graph has a cycle of length at least 3. In [PS14], Przytycki and Sazdanović conjectured the following, connecting torsion in Khovanov homology to braid index.

Conjecture 5.0.1 ([PS14, Conjecture 6.1])**.**

1. *The Khovanov homology of a closed 3-braid can have only $\mathbb{Z}_2$ torsion.*
2. *The Khovanov homology of a closed 4-braid cannot have $\mathbb{Z}_{p^r}$ torsion for $p \neq 2$.*
3. *The Khovanov homology of a closed 4-braid can have only $\mathbb{Z}_2$ and $\mathbb{Z}_4$ torsion.*
4. *The Khovanov homology of a closed n-braid cannot have $\mathbb{Z}_{p^r}$ torsion for $p > n$ (p prime).*
5. *The Khovanov homology of a closed n-braid cannot have $\mathbb{Z}_{p^r}$ torsion for $p^r > n$.*

Counterexamples to parts 2, 3 and 5 are given in [Muk17], and a counterexample to part 4 has recently been constructed by Mukherjee [Muk]. However, part 1 remains open, and computations suggest that part 1 is indeed true. One goal of this (ongoing) project is to prove this.

Consider the braid group B_s on s-strands whose generators are shown in Figure 5.1. The *half twist* $\Delta \in B_s$ is defined as $\Delta = (\sigma_1\sigma_2 \ldots \sigma_{p-1})(\sigma_1\sigma_2 \ldots \sigma_{p-2}) \ldots (\sigma_1\sigma_2)(\sigma_1)$, and the *full twist* is Δ^2.

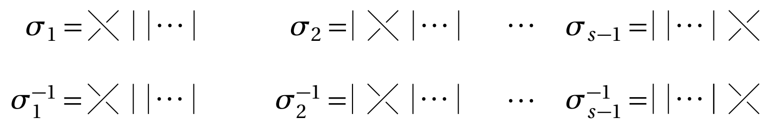


Figure 5.1 Generators and their inverses for the braid group B_s.

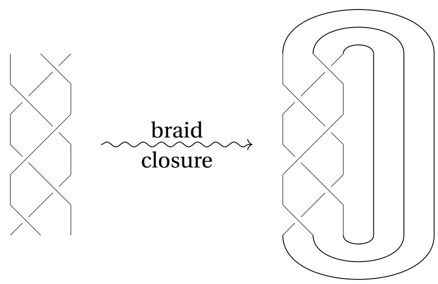


Figure 5.2 A braid diagram and its braid closure.

The closure of a braid diagram is a diagram of a link (see Figure 5.2), and a famous result of Alexander states that every link can be represented by the closure of a braid. For convenience, throughout this chapter, a braid word will be used to refer to either an element of the braid group or its braid closure depending on the context in which it appears.

If two elements of a braid group are conjugate, then the corresponding braid closures are isotopic as links. Therefore, it would be convenient to have a classification of elements of the braid group B_s up to conjugacy. For $s = 2$, of course, the classification is trivial. For $s \geq 4$, no classification is known. For $s = 3$, Murasugi provides the following.

Theorem 5.0.2 ([Mur74])**.** *Every element of the braid group B_3 is conjugate to a unique element of one of the following disjoint sets:*

$$\begin{aligned}
\Omega_0 &= \{\Delta^{2n} \mid n \in \mathbb{Z}\},\\
\Omega_1 &= \{\Delta^{2n}\sigma_1\sigma_2 \mid n \in \mathbb{Z}\},\\
\Omega_2 &= \{\Delta^{2n}(\sigma_1\sigma_2)^2 \mid n \in \mathbb{Z}\},\\
\Omega_3 &= \{\Delta^{2n+1} \mid n \in \mathbb{Z}\},\\
\Omega_4 &= \{\Delta^{2n}\sigma_1^{-p} \mid n \in \mathbb{Z}, p \in \mathbb{Z}_{>0}\},\\
\Omega_5 &= \{\Delta^{2n}\sigma_2^{q} \mid n \in \mathbb{Z}, q \in \mathbb{Z}_{>0}\},\ \textit{or}\\
\Omega_6 &= \{\Delta^{2n}\sigma_1^{-p_1}\sigma_2^{q_1}\ldots\sigma_1^{-p_r}\sigma_2^{q_r} \mid n \in \mathbb{Z}, p_i, q_i \in \mathbb{Z}_{>0} \text{ for } i = 1,\ldots,r\},
\end{aligned}$$

where $\mathbb{Z}_{>0}$ denotes the set of positive integers.

Along with Murasugi's classification, we use Theorem 5.3.4 to show that certain classes of 3-braids have only $\mathbb{Z}_2$ torsion in Khovanov homology (Theorem 5.4.6), taking a significant step in the direction of proving part 1 of the PS Braid Conjecture (Conjecture 5.0.1).

Theorem 5.4.6. *All torsion in the Khovanov homology of a closed 3-braid of type* $\Omega_0, \Omega_1, \Omega_2$*, or* Ω_3 *is* $\mathbb{Z}_2$ *torsion.*

In Section 5.4, we give an application of the main result, showing that all torsion in the Khovanov homology of links in $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 is $\mathbb{Z}_2$ torsion, and we give explicit calculations of the integral Khovanov homology of links in $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 (See Figures 5.3a, 5.3b, 5.4a, and 5.4b). We end with an explanation of why Theorem 5.3.4 does not apply to all closed 3-braids.

5.1 A Construction of Khovanov Homology

Khovanov homology is an invariant of oriented links $L \subset S^3$ with values in the category of bigraded modules over a commutative ring R with identity. We begin with a construction of Khovanov homology. Our conventions for positive and negative crossings, and for zero- and one-smoothings, are as given in Figure 5.5.

A bigraded R-module is an R-module M with a direct sum decomposition of the form $M = \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}$. The submodule $M^{i,j}$ is said to have *bigrading* (i,j). For our purposes we will refer to i as the *homological grading* and to j as the *polynomial grading*. Given two bigraded R-modules $M = \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}$ and $N = \bigoplus_{i,j\in\mathbb{Z}} N^{i,j}$, we define the direct sum $M \oplus_R N$ and tensor product $M \otimes_R N$ to be the bigraded R-modules with components $(M\oplus N)^{i,j} = M^{i,j} \oplus N^{i,j}$ and $(M \otimes N)^{i,j} = \bigoplus_{k+m=i, l+n=j} M^{k,l} \otimes N^{m,n}$. We also define homological and polynomial shift operators, denoted $[\cdot]$ and $\{\cdot\}$, respectively, by $(M[r])^{i,j} = M^{i-r,j}$ and $(M\{s\})^{i,j} = M^{i,j-s}$.

Consider the directed graph whose vertex set $\mathcal{V}(n) = \{0,1\}^n$ comprises n-tuples of 0's and 1's and whose edge set $\mathcal{E}(n)$ contains a directed edge from $I \in \mathcal{V}(n)$ to $J \in \mathcal{V}(n)$ if and only if all entries of I and J are equal except for one, where I is 0 and J is 1. One can think of the underlying graph as the 1-skeleton of the n-dimensional cube. For example, if $n = 4$ there are two outward edges starting from $(0,1,0,1)$, one ending at $(0,1,1,1)$ and the other ending at $(1,1,0,1)$. The height of a vertex $I = (k_1, k_2, \dots, k_n)$ is $h(I) = k_1 + k_2 + \cdots + k_n$. In other words, $h(I)$ is the number of 1's in I. If ε is an edge from I to J and they differ in the r-th entry, the height of ε is $|\varepsilon| := h(I)$. The sign of ε is $(-1)^{\varepsilon} := (-1)^{\sum_{i=1}^{r-1} k_i}$. In other words, the sign of ε is -1 if the number of 1's before the r-th entry is odd, and is $+1$ if even.

Let D be a diagram of a link L with n crossings $c_1, c_2, \dots, c_n$. To each vertex $I \in \mathcal{V}(n)$ we associate a collection of circles $D(I)$, called a Kauffman state, obtained by 0-resolving those crossings c_i for which $k_i = 0$ and 1-resolving those crossings c_j for which $k_j = 1$. The 0- and 1-resolution conventions are given in Figure 5.5. To each Kauffman state we associate a bigraded R-module $C(D(I))$ as follows.

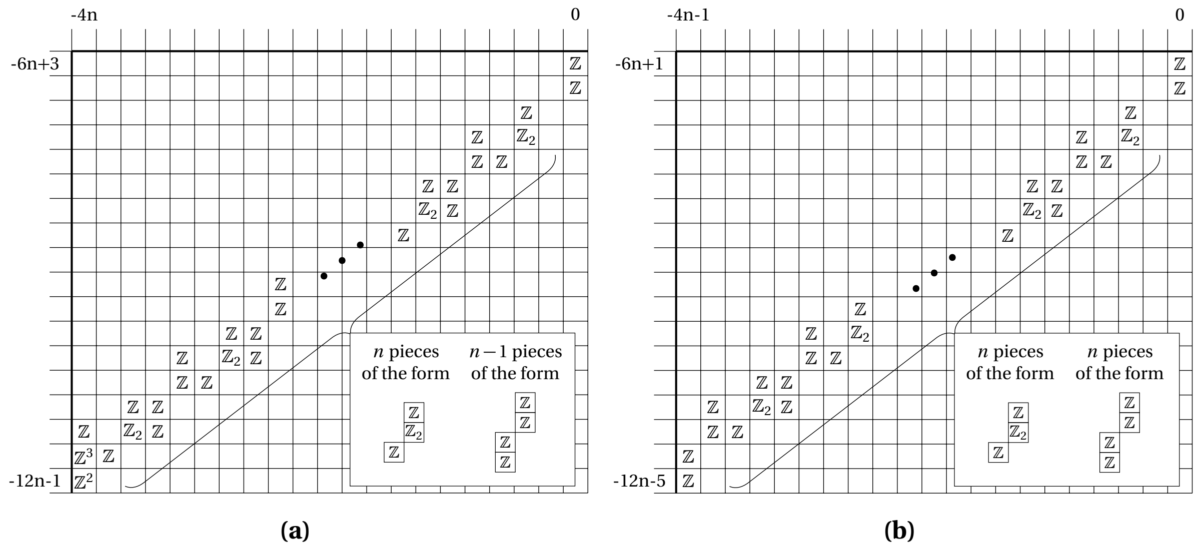


Figure 5.3 In (a) we have the integral Khovanov homology of $T(3,3n)$. In (b) we have the integral Khovanov homology of $T(3,3n+1)$.

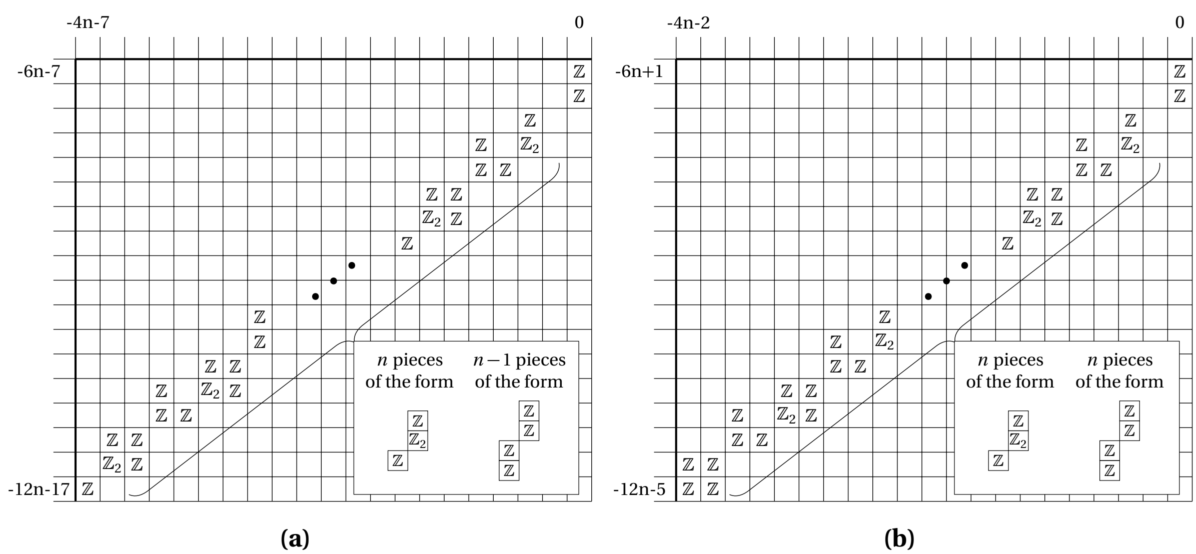


Figure 5.4 In (a) we have the integral Khovanov homology of $T(3,3n+2)$. In (b) we have the integral Khovanov homology of the braid closure of Δ^{2n+1}.

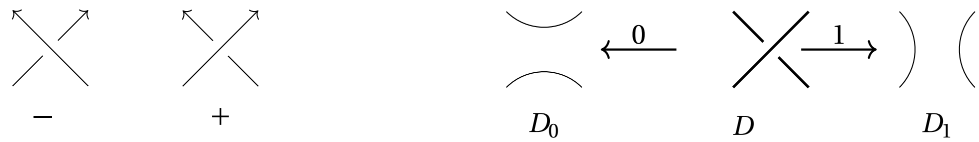


Figure 5.5 Conventions for crossings and smoothings in link diagrams.

Let $\mathscr{A}_2 = R[X]/X^2 \cong R1 \oplus RX$ be the bigraded module with generator 1 in bigrading $(0,1)$ and generator X in bigrading $(0,-1)$. Denoting the number of circles in $D(I)$ by $|D(I)|$, we define

$$C(D(I)) := \mathscr{A}_2^{\otimes|D(I)|}[h(I)]\{h(I)\}.$$

Here, each tensor factor of $\mathscr{A}_2$ is understood to be associated to a particular circle of $D(I)$. Having associated modules to vertices, we now associate maps to directed edges. Define R-module homomorphisms $m : \mathscr{A}_2 \otimes \mathscr{A}_2 \to \mathscr{A}_2$ and $\Delta : \mathscr{A}_2 \to \mathscr{A}_2 \otimes \mathscr{A}_2$ (called multiplication and comultiplication, respectively) by

$$m(1\otimes 1) = 1, \quad m(1 \otimes X) = X, \quad m(X \otimes 1) = X, \quad m(X \otimes X) = 0,$$

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

To an edge ε from I to J we associate the map $d_\varepsilon : C(D(I)) \to C(D(J))$ defined as follows.

(1) If $|D(J)| = |D(I)| - 1$, then $D(J)$ is obtained by merging two circles of $D(I)$ into one: d_ε acts as multiplication m on the tensor factors associated to the circles being merged, and acts as the identity map on the remaining tensor factors.

(2) If $|D(J)| = |D(I)| + 1$ then $D(J)$ is obtained by splitting one circle of $D(I)$ into two: d_ε acts as comultiplication Δ on the tensor factor associated to the circle being split, and acts as the identity map on the remaining tensor factors.

Suppose D has n_+ positive crossings and n_- negative crossings. Define the bigraded module

$$C^{*,*}(D) = \bigoplus_{I \in \mathscr{V}(n)} C(D(I))[-n_-]\{n_+ - 2n_-\}. \tag{5.1}$$

Define maps $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$ by $d^i = \sum_{|\varepsilon|=i} (-1)^\varepsilon d_\varepsilon$. In [Kho00], Khovanov shows that $(C(D), d)$ is a (co)chain complex, and since the maps d^i are polynomial degree-preserving, we get a bigraded homology R-module

$$H(L;R) = \bigoplus_{i,j\in\mathbb{Z}} H^{i,j}(L;R)$$

called the Khovanov homology of the link L with coefficients in R, or, more compactly, the R-Khovanov homology of L. Proofs that d is a differential and that these homology groups are independent of the diagram D and the ordering of the crossings can be found in [Kho00; Vir04; BN05]. If $R=\mathbb{Z}$ we simply write $H(L)=H(L;\mathbb{Z})$. In this article we will focus on the rings $R=\mathbb{Z},\mathbb{Q}$ and $\mathbb{Z}_p$, where p is a prime.

The (unnormalized) Jones polynomial is recovered as the graded Euler characteristic of Khovanov homology:

$$\hat{J}_L(q)=\sum_{i,j\in\mathbb{Z}}(-1)^i q^j\cdot \mathrm{rk}(H^{i,j}(L)).$$

5.2 Computational Tools

5.2.1 A Long Exact Sequence in Khovanov Homology

Given an oriented link diagram D, there is a long exact sequence relating D to the zero and one smoothings D_0 and D_1 at a given crossing of D [Kho00]. If the crossing is negative, we set $c=n_-(D_0)-n_-(D)$ to be the number of negative crossings in D_0 minus the number in D. For each j there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(D_1)\longrightarrow H^{i,j}(D)\longrightarrow H^{i-c,j-3c-1}(D_0)\xrightarrow{\delta_*} H^{i+1,j+1}(D_1)\longrightarrow \tag{5.2}$$

Note that in this case D_1 inherits an orientation, and D_0 must be given one. Similarly, if the crossing is positive, we set $c=n_-(D_1)-n_-(D)$ to be the number of negative crossings in D_1 minus the number in D. For each j there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i-c-1,j-3c-2}(D_1)\longrightarrow H^{i,j}(D)\longrightarrow H^{i,j-1}(D_0)\xrightarrow{\delta_*} H^{i-c,j-3c-2}(D_1)\longrightarrow \tag{5.3}$$

and note that in this case D_0 inherits an orientation, and D_1 must be given one.

5.2.2 The Bockstein Spectral Sequence

The Bockstein spectral sequence arises as the spectral sequence associated to an exact couple, and is a powerful tool for analyzing torsion in homology theories.

Definition 5.2.1 ([McC01, Section 2.2])**.** Let D_0 and E_0 be R-modules, and let $i_0:D_0\to D_0$, $j_0:D_0\to E_0$ and $k_0:E_0\to D_0$ be R-module homomorphisms satisfying $\ker(i_0)=\mathrm{im}(k_0)$, $\ker(j_0)=\mathrm{im}(i_0)$ and $\ker(k_0)=\mathrm{im}(j_0)$. The pentuple (D_0,E_0,i_0,j_0,k_0) is called an exact couple. Succinctly, an exact couple is an exact diagram of R-modules and homomorphisms of the following form.

$$\begin{tikzcd} D_0 \arrow[rr, "i_0"] & & D_0 \arrow[dl, "j_0"] \\ & E_0 \arrow[ul, "k_0"] & \end{tikzcd}$$

The map $d_0 : E_0 \to E_0$ defined by $d_0 := j_0 \circ k_0$ satisfies $d_0 \circ d_0 = (j_0 \circ k_0) \circ (j_0 \circ k_0) = j_0 \circ (k_0 \circ j_0) \circ k_0 = 0$ and so defines a differential on E_0. Define $E_1 = H(E_0, d_0)$ to be the homology of the pair (E_0, d_0), and define $D_1 = \operatorname{im}(i_0 : D_0 \to D_0)$. Also, define maps $i_1 : D_1 \to D_1$, $j_1 : D_1 \to E_1$ and $k_1 : E_1 \to D_1$ by $i_1 = i_0|_{D_1}$, $j_1(i_0(a)) = j_0(a) + d_0 E_0$ and $k_1(e + d_0 E_0) = k_0(e)$. The resulting pentuple $(D_1, E_1, i_1, j_1, k_1)$ is another exact couple by [McC01, Proposition 2.7]. Iterating this process produces a sequence $(E_r, d_r)_{r \geq 0}$ called the *spectral sequence associated to the exact couple* $(D_0, E_0, i_0, j_0, k_0)$.

Given a chain complex (C, d), let us denote its integral homology by $H(C)$ and its homology over coefficients in $\mathbb{Z}_p$ by $H(C; \mathbb{Z}_p)$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\text{red } p} \mathbb{Z}_p \longrightarrow 0 \tag{5.4}$$

where $\times p$ is multiplication by p, and red p is reduction modulo p. Tensoring a chain complex (C, d) with (5.4) yields a short exact sequence of chain complexes

$$0 \longrightarrow C \xrightarrow{\times p} C \xrightarrow{\text{red } p} C \otimes \mathbb{Z}_p \longrightarrow 0.$$

The associated long exact sequence in homology can be viewed as an exact couple

$$\begin{tikzcd} H(C) \arrow[rr, "\times p"] & & H(C) \arrow[dl, "\text{red } p"] \\ & H(C;\mathbb{Z}_p) \arrow[ul, "\partial"] & \end{tikzcd} \tag{5.5}$$

where ∂ is the connecting homomorphism.

Definition 5.2.2 ([McC01, Chapter 10])**.** Let p be prime. The spectral sequence $(E_B^r, d_B^r)_{r \geq 1}$ associated to the exact couple (5.5) is called the $\mathbb{Z}_p$ Bockstein spectral sequence.

The following properties of the Bockstein spectral sequence will be of great importance for our purposes, the proofs of which can be found in [McC01].

(B1) The first page is $(E_B^1, d_B^1) \cong (H(C; \mathbb{Z}_p), \beta)$ where $\beta = (\text{red } p) \circ \partial$.

(B2) The infinity page is $E_B^\infty \cong (H(C)/\text{Torsion}) \otimes \mathbb{Z}_p$.

(B3) If the Bockstein spectral sequence collapses on the E^k page for a particular bigrading (i,j), that is, if $(E_B^k)^{i,j} = (E_B^\infty)^{i,j}$, then $H^{i,j}(C)$ contains no $\mathbb{Z}_{p^r}$ torsion for $r \geq k$.

(B4) If $C = C^{*,*}$ is a bigraded complex with degree preserving differential, then the differentials d_B^r have bidegree $(1,0)$.

To show that all $\mathbb{Z}_{2^r}$ torsion in the homology of a chain complex is in fact $\mathbb{Z}_2$ torsion, it suffices by property $(B3)$ to show that the $\mathbb{Z}_2$ Bockstein spectral sequence collapses on the E_B^2 page. To this end, we will consider interactions between the Bockstein spectral sequence and the Turner spectral sequence, discussed below.

5.2.3 Bar-Natan Homology and the Turner Spectral Sequence

In [BN05] Bar-Natan gives a variant of $\mathbb{Z}_2$-Khovanov homology, now referred to as Bar-Natan homology. The construction of Bar-Natan homology follows that of Section 5.1 but with differential defined via multiplication and comultiplication given by

$$m_B(1\otimes 1) = 1, \quad m_B(1\otimes X) = X, \quad m_B(X\otimes 1) = X, \quad m_B(X\otimes X) = X,$$

$$\Delta_B(1) = 1\otimes X + X\otimes 1 + 1\otimes 1, \quad \Delta_B(X) = X\otimes X.$$

The resulting singly graded homology theory is a link invariant [BN05] denoted $BN^*(L)$.

Lemma 5.2.3 ([Tur06, Theorem 3.1])**.** *Let L be an oriented link with k components. Then,* $\dim_{\mathbb{Z}_2}(BN^*(L)) = 2^k$*. Specifically, if the components of L are $L_1, L_2, \ldots, L_k$, then*

$$\dim_{\mathbb{Z}_2}(BN^i(L)) = \#\left\{ E \subset \{1,2,\ldots,k\} \;\middle|\; 2\times \sum_{\ell\in E,\, m\in E^c} lk(L_\ell, L_m) = i \right\}$$

where $lk(L_\ell, L_m)$ is the linking number between L_ℓ and L_m.

Example 5.2.4. Let $T(3,q)$ be the $(3,q)$-torus link. Specifically, $T(3,3n)$ is the closure of the braid Δ^{2n}, $T(3,3n+1)$ is the closure of the braid $\Delta^{2n}\sigma_1\sigma_2$ and $T(3,3n+2)$ is the closure of $\Delta^{2n}(\sigma_1\sigma_2)^2$, each oriented so that crossings in Δ are negative crossings.

(1) The torus link $T(3,3n)$ has three components and $BN^0(T(3,3n)) \cong \mathbb{Z}_2^2$, $BN^{-4n}(T(3,3n)) \cong \mathbb{Z}_2^6$ and $BN^i(T(3,3n)) \cong 0$ for $i \neq 0$ or $-4n$.

(2) The torus knot $T(3,3n+1)$ satisfies $BN^0(T(3,3n+1)) \cong \mathbb{Z}_2^2$ and $BN^i(T(3,3n+1)) \cong 0$ for $i \neq 0$.

(3) The torus knot $T(3,3n-1)$ satisfies $BN^0(T(3,3n-1)) \cong \mathbb{Z}_2^2$ and $BN^i(T(3,3n-1)) \cong 0$ for $i \neq 0$.

In [Tur06] Turner defines a map d_T on the $\mathbb{Z}_2$-Khovanov complex in the same manner as the Khovanov differential d, but with multiplication and comultiplication given by

$$m_T(1\otimes 1)=0, \quad m_T(1\otimes X)=0, \quad m_T(X\otimes 1)=0, \quad m_T(X\otimes X)=X,$$

$$\Delta_T(1)=1\otimes 1, \quad \Delta_T(X)=0,$$

and the map d_T satisfies $d_T^2=0$ and $d\circ d_T+d_T\circ d=0$.

Definition 5.2.5 ([Tur06, Section 2])**.** Let D be a diagram of a link L and $(C(D;\mathbb{Z}_2),d)$ be the $\mathbb{Z}_2$-Khovanov complex. The spectral sequence $(E_T^r,d_T^r)_{r\geq 1}$ associated to the double complex $(C(D;\mathbb{Z}_2),d,d_T)$ is called the Turner spectral sequence.

The Turner spectral sequence satisfies the following properties.

(T1) The first page is $(E_T^1,d_T^1)\cong(H(L;\mathbb{Z}_2),d_T^*)$ where $d_T^*: H(L;\mathbb{Z}_2)\to H(L;\mathbb{Z}_2)$ is the induced map on homology.

(T2) Each map d_T^r is a differential of bidegree $(1,2r)$.

(T3) The Turner spectral sequence converges to Bar-Natan homology: $E_T^r \implies BN^*(L)$. In light of (T2), this simply means that taking the direct sum along the i-th column of the infinity page yields $BN^i(L)$.

The next ingredient in the proof of Theorem 5.3.4 is a 'vertical' differential on the $\mathbb{Z}_2$-Khovanov complex due to Khovanov [Kho00]. Let D be a diagram of a link L. A differential $\nu: C(D;\mathbb{Z}_2)\to C(D;\mathbb{Z}_2)$ is defined as follows. Recall that for a Kauffman state $D(I)$, the algebra $C(D(I))$ has $2^{|D(I)|}$ generators of the form $a_1\otimes a_2\otimes\cdots\otimes a_{|D(I)|}$ where $a_i\in\{1,X\}$ and $|D(I)|$ is the number of circles in $D(I)$. For each Kauffman state we define a map $\nu_{D(I)}: C(D(I))\to C(D(I))$ by sending a generator to the sum of all possible generators obtained by replacing a single X with a 1. For example, $X\otimes X\otimes X\mapsto 1\otimes X\otimes X+X\otimes 1\otimes X+X\otimes X\otimes 1$. We then extend $\nu_{D(I)}$ linearly to all of $C(D(I))$, and then to a map $\nu: C(D;\mathbb{Z}_2)\to C(D;\mathbb{Z}_2)$. The properties of this map relevant to our purposes are given below, the proofs of which can be found in Shumakovitch [Shu18].

(V1) The map ν is a differential of bidegree $(0,2)$.

(V2) The map ν commutes with the Khovanov differential d, and so induces a map (differential) $\nu^*: H(L;\mathbb{Z}_2)\to H(L;\mathbb{Z}_2)$ on homology.

(V3) The complex $(H(L;\mathbb{Z}_2),\nu^*)$ is acyclic, that is, it has trivial homology.

The following lemma of Shumakovitch is the key to proving Theorem 5.3.4. It relates the first $\mathbb{Z}_2$ Bockstein map with the Turner and vertical differentials. We use the behavior of the Turner spectral sequence and the acyclic homology induced by ν to determine when the $\mathbb{Z}_2$ Bockstein spectral sequence collapses.

Lemma 5.2.6 ([Shu18, Lemma 3.2.A])**.** *Let L be a link. The Bockstein, Turner and vertical differentials on the $\mathbb{Z}_2$-Khovanov homology $H(L;\mathbb{Z}_2)$ of L are related by $d_T^* = d_B^1 \circ \nu^* + \nu^* \circ d_B^1$.*

5.2.4 The Lee Spectral Sequence

Lee [Lee05] defined an endormorphism on Khovanov homology that Rasmussen [Ras10] used to define the s concordance invariant. Let R be either $\mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime. The Lee differential $d_L : C(D;R) \to C(D;R)$ is defined in the same way as the Khovanov differential d, but with multiplication and comultiplication given by

$$m_L(1\otimes 1) = 0, \quad m_L(1\otimes X) = 0, \quad m_L(X\otimes 1) = 0, \quad m_L(X\otimes X) = 1,$$

$$\Delta_L(X) = 0, \quad \Delta_L(X) = 1\otimes 1.$$

The map d_L satisfies $d_L^2 = 0$ and $d\circ d_L + d_L \circ d = 0$. The resulting singly graded homology theory is a link invariant, denoted Lee*$(L;R)$, and it behaves as follows.

Lemma 5.2.7 ([Lee05, Proposition 4.3])**.** *Let L be an oriented link with k components. Then,* $\dim_R(\mathrm{Lee}^*(L;R)) = 2^k$*. Specifically, if the components of L are $L_1, L_2, \ldots, L_k$, then*

$$\dim_R(\mathrm{Lee}^i(L;R)) = \#\left\{ E\subset\{1,2,\ldots,k\} \;\middle|\; 2\times \sum_{\ell\in E, m\in E^c} lk(L_\ell, L_m) = i \right\}$$

where $lk(L_\ell, L_m)$ is the linking number between L_ℓ and L_m.

Definition 5.2.8 ([Lee05, Section 4])**.** Let D be a diagram of a link L and $(C(D;R), d)$ be the R-Khovanov complex. The spectral sequence $(E_L^r, d_L^r)_{r\geq 1}$ associated to the double complex $(C(D;R), d, d_L)$ is called the R Lee spectral sequence.

The R-Lee spectral sequence satisfies the following properties.

(L1) The first page is $(E_L^1, d_L^1) \cong (H(L;R), d_L^*)$ where $d_L^* : H(L;R) \to H(L;R)$ is the induced map on homology.

(L2) Each map d_L^r is a differential of bidegree $(1, 4r)$.

(L3) The Lee spectral sequence converges to Lee homology: $E_L^r \implies$ Lee*$(L;R)$. In light of (L2), this simply means that taking the direct sum along the i-th column of the infinity page yields Lee$^i(L)$.

5.3 The Main Result

We now prove several lemmas that will lead to the proof of Theorem 5.3.4. Throughout the proofs, we take advantage of the properties of the $\mathbb{Z}_2$-Bockstein spectral sequence, Turner spectral sequence, Lee spectral sequence and vertical differential ν^*, described in Section 5.2.

Suppose that $H(L)$ is thin over the interval $[i_1, i_2]$, and let $H^{[i_1,i_2]}(L;R)$ denote the direct sum

$$H^{[i_1,i_2]}(L;R) = \bigoplus_{i=i_1}^{i_2} H^{i,*}(L;R).$$

Our first lemma states that all torsion in homological gradings $(i_1, i_2]$ must be supported on the lower diagonal.

Lemma 5.3.1. *If $H(L)$ is thin over $[i_1, i_2]$ where $H^{[i_1,i_2]}(L;\mathbb{Z})$ is supported in bigradings (i, j) with $2i - j = s \pm 1$ for some $s \in \mathbb{Z}$, then any torsion summand of $H^{[i_1,i_2]}(L;\mathbb{Z})$ with homological gradings $i > i_1$ is supported on the lower diagonal in bigrading $(i, 2i - s - 1)$.*

Proof. If $H^{i,2i-s+1}(L)$ has a nontrivial torsion summand for some $i \in (i_1, i_2]$, then the universal coefficient theorem implies that $H^{i-1,2i-s+1}(L;\mathbb{Z}_p)$ is nontrivial for some p, contradicting the fact that $H(L)$ is thin over $[i_1, i_2]$. Therefore all torsion in homological gradings $(i_1, i_2]$ appears in bigradings of the form $(i, 2i - s - 1)$. □

Our next lemma gives a condition to ensure that $H^{[i_1,i_2]}(L;\mathbb{Z})$ has no odd torsion.

Lemma 5.3.2. *Suppose that $H(L)$ is thin over $[i_1, i_2]$ and that*

$$\dim_{\mathbb{Q}} H^{i_1,*}(L;\mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1,*}(L;\mathbb{Z}_p)$$

for each odd prime p. Then $H^{[i_1,i_2]}(L;\mathbb{Z})$ contains no torsion of odd order.

Proof. We show that if $i \in [i_1, i_2]$, then there cannot be any torsion in $H^{i,*}(L)$ of the form $\mathbb{Z}_{p^r}$ for an odd prime p. By way of contradiction suppose that for some $i \in [i_1, i_2]$, the group $H^{i,2i-s-1}(L)$ contains a torsion summand of the form $\mathbb{Z}_{p^r}$. The universal coefficient theorem implies that

$$\dim_{\mathbb{Q}} H^{i,2i-s-1}(L;\mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i,2i-s-1}(L;\mathbb{Z}_p) \text{ and}$$
$$\dim_{\mathbb{Q}} H^{i-1,2i-s-1}(L;\mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i-1,2i-s-1}(L;\mathbb{Z}_p).$$

Since $H(L)$ is thin over $[i_1, i_2]$, Lemma 5.2.7 implies the infinity pages of the $\mathbb{Q}$ and $\mathbb{Z}_p$ Lee spectral sequences have the same dimension in each bigrading $(i, 2i - s \pm 1)$ for $i \in [i_1, i_2]$. Because the Lee map is of bidegree $(1,4)$, if $i_1 < i - 1$,

$$\dim_{\mathbb{Q}} H^{i-2,2i-s-5}(L;\mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i-2,2i-s-5}(L;\mathbb{Z}_p).$$

Since $H(L)$ is thin over $[i_1, i_2]$, it follows that $H^{i-1,2i-s-5}(L) = 0$, and in particular, there is no $\mathbb{Z}_{p^r}$ torsion summand in bigrading $(i-1, 2i-s-5)$. Thus the universal coefficient theorem implies there is a $\mathbb{Z}_{p^{r'}}$ torsion summand in bigrading $(i-2, 2i-s-5)$. A $\mathbb{Z}_{p^r}$ torsion summand in homological grading i induces a dimension inequality in homological gradings $i, i-1$, and $i-2$, and it induces another $\mathbb{Z}_{p^r}$ summand in homological grading $i-2$. Since $i \in [i_1, i_2]$ is arbitrary, repeating this argument for each new $\mathbb{Z}_{p^r}$ summand implies that

$$\dim_{\mathbb{Q}} H^{i_1,*}(L;\mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i_1,*}(L;\mathbb{Z}_p),$$

which is a contradiction. □

Assume that $H^{[i_1,i_2]}(L;\mathbb{Z})$ contains no torsion of odd order, and $H(L)$ is thin over $[i_1, i_2]$. If $R = \mathbb{Q}$ or $\mathbb{Z}_2$, then $H^{[i_1,i_2]}(L;R)$ can be decomposed as

$$H^{[i_1,i_2]}(L;R) = H^{[i_1,i_2]}_{<\infty}(L;R) \oplus H^{[i_1,i_2]}_{\infty}(L;R)$$

where $H^{[i_1,i_2]}_{\infty}(L;R)$ is the submodule that survives to the infinity page of the Lee or Turner spectral sequence when $R = \mathbb{Q}$ or $\mathbb{Z}_2$ respectively. Lemmas 5.2.3 and 5.2.7 imply that for any $i \in [i_1, i_2]$,

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^{i,2i-s-1}_{\infty}(L;\mathbb{Z}_2) &= \dim_{\mathbb{Q}} H^{i,2i-s-1}_{\infty}(L;\mathbb{Q}) \text{ and} \\ \dim_{\mathbb{Z}_2} H^{i,2i-s+1}_{\infty}(L;\mathbb{Z}_2) &= \dim_{\mathbb{Q}} H^{i,2i-s+1}_{\infty}(L;\mathbb{Q}). \end{aligned} \tag{5.6}$$

Each $\mathbb{Q}$ summand in $H(L;\mathbb{Q})$ corresponds to a $\mathbb{Z}$ summand in $H(L)$. Define $H^{[i_1,i_2]}_{\infty}(L)$ to be the direct sum of the $\mathbb{Z}$ summands corresponding to the $\mathbb{Q}$ summands in $H^{[i_1,i_2]}_{\infty}(L;\mathbb{Q})$. Also, define $H^{[i_1,i_2]}_{<\infty}(L)$ to be the direct sum of all torsion in $H^{[i_1,i_2]}(L)$ with the $\mathbb{Z}$ summands corresponding to the $\mathbb{Q}$ summands in $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Q})$. Then

$$H^{[i_1,i_2]}(L) = H^{[i_1,i_2]}_{<\infty}(L) \oplus H^{[i_1,i_2]}_{\infty}(L).$$

Suppose that $H^{[i_1,i_2]}(L)$ is supported in bigradings (i, j) where $2i - j = s \pm 1$ for some integer s. The Lee spectral sequence implies that if $i < i_2$, then a $\mathbb{Q}$ summand in bigrading $(i, 2i-s-1)$ of $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Q})$ is paired with a $\mathbb{Q}$ summand in bigrading $(i+1, 2i-s+3)$ of $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Q})$ (and vice versa). We call such pairs *knight move pairs*. The vertical differential implies that for all $i \in [i_1, i_2]$, a $\mathbb{Z}_2$ summand in bigrading $(i, 2i-s-1)$ of $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Z}_2)$ is paired with a $\mathbb{Z}_2$ summand in bigrading $(i, 2i-s+1)$ of $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Z}_2)$ (and vice versa). We call such pairs *pawn move pairs*.

A submodule S of $H^{[i_1,i_2]}_{<\infty}(L)$ is *closed under the Lee and vertical differentials* (or LV-closed) if it satisfies the following.

- If $i \in [i_1, i_2)$, then $\operatorname{rk} S^{i,2i-s-1} = \operatorname{rk} S^{i+1,2i-s+3}$.

- If $i \in [i_1, i_2)$, then $\dim_{\mathbb{Z}_2} S^{i,2i-s-1} \otimes \mathbb{Z}_2 = \dim_{\mathbb{Z}_2} S^{i+1,2i-s+1} \otimes \mathbb{Z}_2$.

Informally, a submodule S is LV-closed if it induces $\mathbb{Q}$ summands in $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Q})$ that come in knight move pairs (or partial knight move pairs in the cases that $i = i_1$ or i_2) and if it induces $\mathbb{Z}_2$ summands in $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Z}_2)$ that comes in pawn move pairs. A submodule of $H^{[i_1,i_2]}_{<\infty}(L)$ is *LV-indecomposable* if it is LV-closed and is not the direct sum of two proper LV-closed submodules.

A *knight move* submodule consists of two $\mathbb{Z}$ summands in bigradings $(i,2i-s-1)$ and $(i+1,2i-s+3)$ and a $\mathbb{Z}_{2^r}$ summand in bigrading $(i+1,2i-s+1)$ where $i < i_2$. An i_1*-knight move* submodule consists of a $\mathbb{Z}$ summand in bigrading $(i_1,2i_1-s+1)$ and a $\mathbb{Z}_{2^r}$ summand in bigrading $(i_1,2i_1-s-1)$. An i_2*-knight move* submodule consists of a $\mathbb{Z}$ summand in bigrading $(i_2,2i_2-s-1)$. Of course, an i_1-knight move is just the summands of the knight move occurring in the larger homological grading, but supported in homological grading i_1. Similarly, an i_2-knight move is just the summand of the knight move occurring in the smaller homological grading, but supported in homological grading i_2. A $\mathbb{Z}$*-chain* submodule consists of $\mathbb{Z}$ summands in every bigrading $(i,2i-s\pm 1)$ except in bigrading $(i_1,2i_1-s+1)$ where it contains either a $\mathbb{Z}$ or a $\mathbb{Z}_{2^r}$ summand. A $\mathbb{Z}_2$*-chain* submodule consists of $\mathbb{Z}_{2^{r_i}}$ summands in bigradings $(i,2i-s-1)$ for $i \in [i_1,i_2]$. Knight move, i_1-knight move, i_2-knight move, $\mathbb{Z}$-chain, and $\mathbb{Z}_2$-chain submodules are all LV-indecomposable. See Figure 5.6 for their depictions.

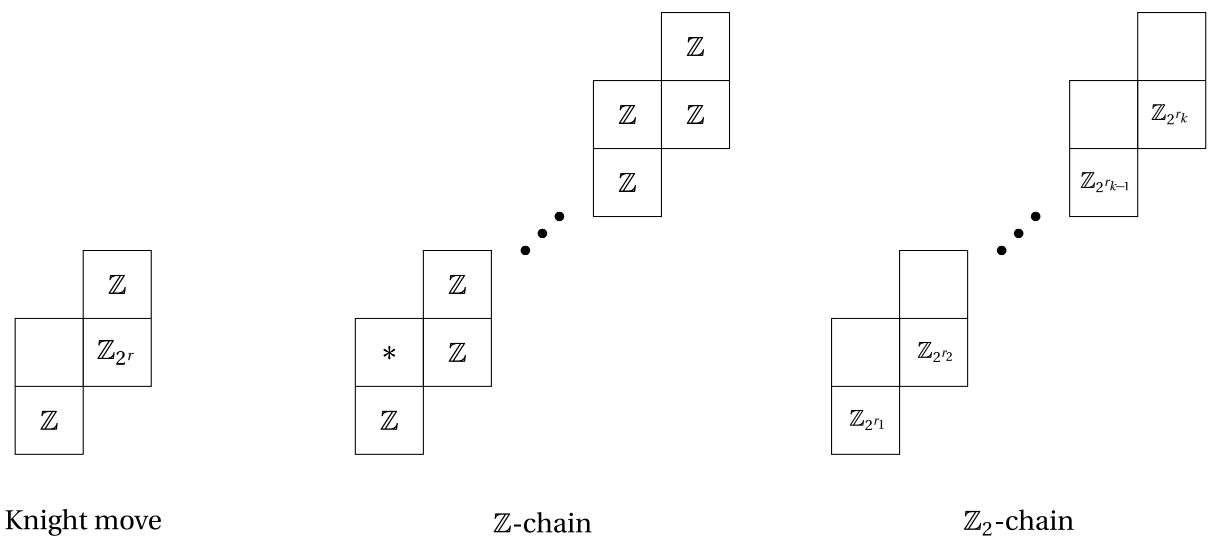


Figure 5.6 LV-indecomposable submodules of $H^{[i,j]}_{<\infty}(L)$.

Lemma 5.3.3. *If $H(L)$ is thin over $[i_1,i_2]$ and $H^{[i_1,i_2]}(L)$ contains no torsion of odd order, then $H^{[i_1,i_2]}_{<\infty}(L)$ is a direct sum of knight move, i_1-knight move, i_2-knight move, $\mathbb{Z}$-chain, and $\mathbb{Z}_2$-chain submodules.*

Proof. By construction, $H^{[i_1,i_2]}_{<\infty}(L)$ is a direct sum of LV-indecomposable submodules. Thus it remains to show that knight move, i_1-knight move, i_2-knight move, $\mathbb{Z}$-chain, and Z_2-chain submodules are the only LV-indecomposable submodules.

Let M be an LV-indecomposable submodule that contains a $\mathbb{Z}$ summand on the upper diagonal in bigrading $(i, 2i-s+1)$ for some $i \in [i_1, i_2]$. If $i > i_1$, the Lee spectral sequence then implies that M contains a $\mathbb{Z}$ summand in bigrading $(i-1, 2i-s-3)$. The vertical differential implies that M must contain either a $\mathbb{Z}$ summand or $\mathbb{Z}_{2^r}$ summand in bigrading $(i, 2i-s-1)$. If M contains a $\mathbb{Z}_{2^r}$ summand in bigrading $(i, 2i-s-1)$, then these three summands make it LV-closed, and hence it is the knight move submodule. If $i = i_1$, then the $\mathbb{Z}$ and $\mathbb{Z}_{2^r}$ summands in bigradings $(i_1, 2i_1-s\pm 1)$ form an i_1-knight move. Suppose, on the other hand, M contains a $\mathbb{Z}$ summand in bigrading $(i, 2i-s-1)$. Now the vertical differential implies that the $\mathbb{Z}$ summand in bigrading $(i-1, 2i-s-3)$ needs to pair with something in bigrading $(i-1, 2i-s-1)$. Since bigrading $(i-1, 2i-s-1)$ is on the upper diagonal, it cannot be a torsion summand unless $i-1 = i_1$. The Lee differential also implies that there is a $\mathbb{Z}$ summand on the upper diagonal in bigrading $(i+1, 2i-s+3)$. Each of these $\mathbb{Z}$ summands on the upper diagonal must induce a summand immediately beneath it on the lower diagonal due to the vertical differential. None of these lower diagonal summands can be $\mathbb{Z}_{2^r}$ summands because then M would have a knight move submodule as a proper submodule, making M LV-decomposable. The resulting submodule structure is thus necessarily a $\mathbb{Z}$-chain.

Now let M be an LV-indecomposable submodule that contains a $\mathbb{Z}_{2^r}$ summand on the lower diagonal in bigrading $(i, 2i-s-1)$ for some $i \in [i_1, i_2]$. The vertical differential implies that there is a $\mathbb{Z}$ summand in bigrading $(i, 2i-s+1)$ or a $\mathbb{Z}_{2^{r'}}$ summand in bigrading $(i+1, 2i-s+1)$. If there is a $\mathbb{Z}$ summand in bigrading $(i, 2i-s+1)$, then the Lee spectral sequence implies there is a $\mathbb{Z}$ summand in bigrading $(i-1, 2i-s-3)$, resulting in a knight move submodule. If there is a $\mathbb{Z}_{2^{r'}}$ summand in bigrading $(i+1, 2i-s+1)$, then we repeat this argument. No $\mathbb{Z}$ summand can be produced on the upper diagonal in this process because it would induce a knight move submodule of M contradicting that M is LV-indecomposable. The resulting submoudle structure is thus necessarily a $\mathbb{Z}_2$-chain.

Finally, if M contains a $\mathbb{Z}$ summand in bigrading $(i_2, 2i_2-s-1)$ and nothing in bigrading $(i_2, 2i_2-s+1)$, then M is LV-indecomposable and is an i_2-knight move. $\square$

We can now use Lemmas 5.3.1, 5.3.2, and 5.3.3 to prove our main theorem.

Theorem 5.3.4. *Suppose that a link L satisfies:*

1. *$H(L)$ is thin over $[i_1, i_2]$ for integers i_1 and i_2,*
2. *$\dim_{\mathbb{Q}} H^{i_1,*}(L;\mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1,*}(L;\mathbb{Z}_p)$ for each odd prime p, and*
3. *$H^{i_1,*}(L)$ is torsion-free.*

Then all torsion in $H^{i,*}(L)$ is $\mathbb{Z}_2$ torsion for $i \in [i_1, i_2]$.

Proof. Since $H(L)$ is thin over $[i_1, i_2]$, there is an integer s such that $H^{[i_1,i_2]}(L)$ is supported in bigradings (i, j) satisfying $2i - j = s \pm 1$. Lemma 5.3.1 implies that all torsion in $H^{[i_1,i_2]}(L)$ occurs on the lower diagonal, i.e. in bigradings $(i, 2i - s - 1)$. Lemma 5.3.2 implies that $H^{[i_1,i_2]}(L)$ does not contain any torsion summands of the form $\mathbb{Z}_{p^r}$ for any odd prime p. Therefore $H^{[i_1,i_2]}(L)$ consists of $\mathbb{Z}$ and $\mathbb{Z}_{2^r}$ summands, and can be decomposed as

$$H^{[i_1,i_2]}(L) = H^{[i_1,i_2]}_{<\infty}(L) \oplus H^{[i_1,i_2]}_{\infty}(L)$$

where $H^{[i_1,i_2]}_{\infty}(L)$ is torsion-free.

Lemma 5.3.3 implies that $H^{[i_1,i_2]}_{<\infty}(L)$ can be decomposed into a direct sum of its LV-indecomposable submodules. Those submodules are knight move submodules, i_1/i_2-knight move submodules, $\mathbb{Z}$-chain submodules, or $\mathbb{Z}_2$-chain submodules. The assumption that $H^{i_1,*}(L)$ is torsion-free implies that no i_1-knight move or $\mathbb{Z}_2$-chains occur. Suppose that $H^{[i_1,i_2]}_{<\infty}(L)$ is the direct sum of N_1 knight move submodules, N_2 $\mathbb{Z}$-chain submodules, and N_3 i_2-knight move submodules.

Let $d_T^{[i_1,i_2)}$ denote the sum of the induced Turner maps in homological gradings i_1 to $i_2 - 1$, that is

$$d_T^{[i_1,i_2)} = \sum_{i=i_1}^{i_2-1} (d_T^*)^{i,2i-s-1} + (d_T^*)^{i,2i-s+1}.$$

Since the i_2-knight move is entirely supported in homological grading i_2, it does not have a summand in the domain of $d_T^{[i_1,i_2)}$ and will not contribute to the rank of $d_T^{[i_1,i_2)}$. All of $H^{[i_1,i_2]}_{<\infty}(L;\mathbb{Z}_2)$ dies at the E_T^2 page of the Turner spectral sequence. The summands in the i_1 and i_2 homological gradings of a $\mathbb{Z}$-chain can either be killed as part of $d_T^{[i_1,i_2)}$ or from contributions of $H(L)$ outside of homological gradings $[i_1, i_2]$. Therefore, the rank of $d_T^{[i_1,i_2)}$ when restricted to a $\mathbb{Z}$ chain will be at least $2\left\lfloor \frac{i_2-i_1}{2} \right\rfloor$ and at most $2\left\lceil \frac{i_2-i_1}{2} \right\rceil$. The rank of $d_T^{[i_1,i_2)}$ when restricted to a knight move submodule is two. Putting all of this together yields the inequality

$$2N_1 + 2N_2\left\lfloor \frac{i_2-i_1}{2} \right\rfloor \le \mathrm{rk} d_T^{[i_1,i_2)} \le 2N_1 + 2N_2\left\lceil \frac{i_2-i_1}{2} \right\rceil.$$

Let $d_B^{[i_1,i_2)}$ denote the sum of the Bockstein maps in homological gradings i_1 to $i_2 - 1$, that is

$$d_B^{[i_1,i_2)} = \sum_{i=i_1}^{i_2-1} (d_B^1)^{i,2i-s+1}.$$

Since the only torsion summands in $H^{[i_1,i_2]}(L)$ come from knight move submodules, it follows that $\mathrm{rk} d_B^{[i_1,i_2)} \le N_1$. Finally, Lemma 5.2.6 and the fact that ν^* is an isomorphism imply that $\mathrm{rk} d_T^{[i_1,i_2)} =$

$2\mathrm{rk} d_B^{[i_1,i_2)}$. Therefore

$$N_1 + N_2 \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor \le \frac{1}{2}\,\mathrm{rk} d_T^{[i_1,-i_2)} = \mathrm{rk} d_B^{[i_1,i_2)} \le N_1,$$

which implies that $\mathrm{rk} d_B^{[i_1,i_2)} = N_1$. Hence the Bockstein spectral sequence in gradings i_1+1 to i_2 collapses at the second page, that is $(E_B^2)^i \cong (E_B^\infty)^i$ for $i_1 < i \le i_2$, and thus the only torsion in $H^{i,*}(L)$ is of the form $\mathbb{Z}_2$ for $i \in [i_1, i_2]$. □

The following immediate corollary will be useful for our 3-braid computations in the next section.

Corollary 5.3.5. *Let L be a link satisfying*

1. $\dim_{\mathbb{Q}} H(L;\mathbb{Q}) = \dim_{\mathbb{Z}_p} H(L;\mathbb{Z}_p)$ *for all odd primes p,*
2. *all torsion summands occur in a homologically thin region, and*
3. *the first homological grading of each homologically thin region is torsion-free.*

Then all torsion in $H(L)$ is $\mathbb{Z}_2$ torsion.

5.4 An Application to 3-Braids

There are a number of results about the Khovanov homology of closed 3-braids, but a full computation of the Khovanov homology of closed 3-braids remains open. Turner [Tur08] computed the Khovanov homology of the $(3,q)$ torus links $T(3,q)$ over coefficients in $\mathbb{Q}$ or $\mathbb{Z}_p$ for an odd prime p (see also [Sto09]). Benheddi [Ben17] computed the reduced Khovanov homology of $T(3,q)$ with coefficients in $\mathbb{Z}_2$. Both Turner and Benheddi's computations play a crucial role in our proofs.

The literature on the Khovanov homology of non-torus closed 3-braids is considerably more sparse. Baldwin [Bal08] proved that a closed 3-braid is quasi-alternating if and only if its Khovanov homology is homologically thin. Abe and Kishimoto [AK10] used the Rasmussen s-invariant to compute the alternation number and dealternating number of many closed 3-braids. Lowrance [Low11] computed the homological width of the Khovanov homology of all closed 3-braids.

Over the next few sections, we prove Theorem 5.4.6, showing that all torsion in the Khovanov homology of a closed braid in $\Omega_0, \Omega_1, \Omega_2$, or Ω_3 is $\mathbb{Z}_2$ torsion. First, we argue that it suffices to prove Theorem 5.4.6 when the exponent n in Δ^n in the braid word is non-negative. Next, we use the long exact sequences 5.2 and 5.3 to compute the Khovanov homology of these closed braids over $\mathbb{Q}$ and $\mathbb{Z}_p$ for an odd prime p. Then, we use Benheddi's results [Ben17] to assist our computations of the Khovanov homology of these closed 3-braids over $\mathbb{Z}_2$. Finally, we use Theorem 5.3.4 and Corollary 5.3.5 to complete the proof.

Remark 5.4.1. In this section, we use the same convention for torus links as Turner in [Tur08], that the torus link $T(3,q)$ has all negative crossings.

5.4.1 Reducing to the Case $n \geq 0$

Each braid in $\Omega_0, \Omega_1, \Omega_2$, or Ω_3 has a braid word of the form $\Delta^n \beta$ for some $\beta \in B_3$. The following observations imply that we may assume $n \geq 0$.

1. The mirror image $m(D)$ of a link diagram D is the diagram obtained by changing all crossings. On the level of braid words, $m: B_3 \to B_3$ is a group homomorphism satisfying $m(\sigma_i) = \sigma_i^{-1}$ and $m(\Delta) = \Delta^{-1}$. Recall that the torsion in Khovanov homology of a link diagram and the torsion of its mirror image differ only by a homological shift ([Kho00, Corollary 11]). So the Khovanov homology of L has $\mathbb{Z}_{p^r}$ torsion if and only if the Khovanov homology of its mirror $m(L)$ has $\mathbb{Z}_{p^r}$ torsion.

2. Consider the group homomorphism $\phi: B_3 \to B_3$ defined on generators by $\phi(\sigma_1) = \sigma_2$ and $\phi(\sigma_2) = \sigma_1$. If the braid word ω is a projection of a link L embedded in $\{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1\}$ to the plane $z = 0$, then the projection of L to the plane $z = 1$ is $\phi(\omega)$. Thus the map ϕ preserves the isotopy type of the braid word. Therefore the Khovanov homology of the closure of ω has $\mathbb{Z}_{p^r}$ torsion if and only if the Khovanov homology of the closure of $\phi(\omega)$ has $\mathbb{Z}_{p^r}$ torsion. Note that the homomorphism ϕ satisfies $\phi(\Delta) = \Delta$. See Figure 5.7 for an example of the action of ϕ on a braid diagram.

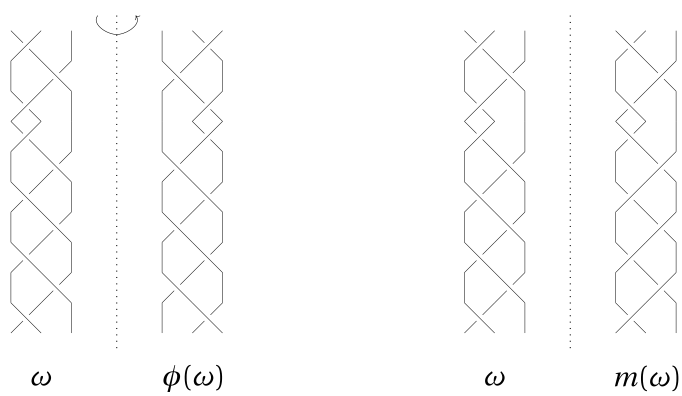


Figure 5.7 Left: The braid word $\omega = \Delta^2\sigma_1^2\sigma_2\sigma_1 \in \Omega_6$ and the corresponding diagram $\phi(\omega) = \Delta^2\sigma_2^2\sigma_1\sigma_2$. Think of $\phi(D)$ as D rotated about the dotted line. Right: The braid word ω and its mirror image $m(\omega)$.

The following equalities together with the above two arguments show that in all cases it suffices to determine torsion for $n \geq 0$:

$$m(\Delta^{-2n}) = \Delta^{2n} \tag{5.7}$$

$$m(\Delta^{-2n}\sigma_1\sigma_2) = (\sigma_1\sigma_2)^{3n-1} \tag{5.8}$$
$$m(\Delta^{-2n}(\sigma_1\sigma_2)^2) = (\sigma_1\sigma_2)^{3n-2} \tag{5.9}$$
$$m(\Delta^{-2n-1}) = \Delta^{2n+1} \tag{5.10}$$
$$m\phi(\Delta^{-2n}\sigma_1^{-p}) = \Delta^{2n}\sigma_2^{p} \tag{5.11}$$
$$m\phi(\Delta^{-2n}\sigma_2^{q}) = \Delta^{2n}\sigma_1^{-q} \tag{5.12}$$
$$m\phi(\Delta^{-2n}\sigma_1^{-p_1}\sigma_2^{q_1}\dots\sigma_1^{-p_r}\sigma_2^{q_r}) = \Delta^{2n}\sigma_2^{p_1}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r}. \tag{5.13}$$

5.4.2 Odd Torsion in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$

We begin with a theorem, shown by Turner in [Tur08], that will be useful in conjunction with Murasugi's classification of 3-braids and the long exact sequence of Section 5.2.

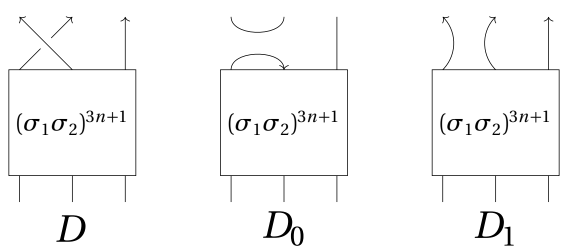
Theorem 5.4.2 (Turner)**.** *For each $q \in \mathbb{Z}$, the Khovanov homology $H(T(3,q))$ of the torus link $T(3,q)$ contains no $\mathbb{Z}_{p^r}$ torsion for $p \neq 2$. That is, there is no $\mathbb{Z}_{p^r}$ torsion for $p \neq 2$ in the Khovanov homology of links of types Ω_0, Ω_1 and Ω_2.*

Now we compute the Khovanov homology of closed 3-braids in Ω_3 over $\mathbb{Q}$ or $\mathbb{Z}_p$ for any prime p. A corollary of this computation is that all torsion in the Khovanov homology of such links is of the form $\mathbb{Z}_{2^r}$.

Theorem 5.4.3. *For $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ for any odd prime p, and any $n \geq 0$,*

$$H(\Delta^{2n+1};\mathbb{F}) \cong H(T(3,3n+1);\mathbb{F})\{-1\} \oplus H(U;\mathbb{F})[-4n-2]\{-12n-5\}.$$

Proof. First observe that $\Delta^{2n+1} = (\sigma_1\sigma_2)^{3n+1}\sigma_1$ where $(\sigma_1\sigma_2)^{3n+1}$ is a braid word for $T(3,3n+1)$. We consider smoothing the top σ_1.



The diagram D_0 is a diagram of the unknot U and D_1 is a diagram of $T(3,3n+1)$. The top σ_1 in Δ^{2n+1} is a negative crossing so we compute $c = n_-(D_0) - n_-(D) = (1+2n) - (6n+3) = -4n-2$. Using (5.2) for each j, and letting $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime, we get a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{i,j}(D;\mathbb{F}) \longrightarrow H^{i+4n+2,j+12n+5}(U;\mathbb{F}) \xrightarrow{\delta_*} H^{i+1,j+1}(T(3,3n+1);\mathbb{F}).$$

For $i \neq -4n-2, -4n-1$, we have $H^{i+4n+2, j+12n+5}(U;\mathbb{F}) = 0 = H^{i+4n+1, j+12n+5}(U;\mathbb{F})$ for every j, so exactness yields $H^{i,j}(D;\mathbb{F}) \cong H^{i,j+1}(T(3,3n+1);\mathbb{F})$ for every j. For $j \neq -12n-5 \pm 1$, the portion of the long exact sequence containing $i = -4n-2$ and $-4n-1$ looks like

$$0 \longrightarrow H^{-4n-2,j+1}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-2,j}(D;\mathbb{F}) \longrightarrow 0$$

$$\xrightarrow{\delta_*} H^{-4n-1,j+1}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-1,j}(D;\mathbb{F}) \longrightarrow 0.$$

Thus $H^{-4n-2,j}(D;\mathbb{F}) \cong H^{-4n-2,j+1}(T(3,3n+1);\mathbb{F}) = 0$ and $H^{-4n-1,j}(D;\mathbb{F}) \cong H^{-4n-1,j+1}(T(3,3n+1);\mathbb{F}) = 0$, where calculations of $H(T(3,3n+1);\mathbb{F})$ are given in Turner's [Tur08].

It remains to check the portion of the long exact sequence containing $i = -4n-2, -4n-1$ in the cases $j = -12n-6, -12n-4$, which are displayed below.

$$0 \longrightarrow H^{-4n-2,-12n-3}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-2,-12n-4}(D;\mathbb{F}) \longrightarrow H^{0,1}(U;\mathbb{F})$$
$$\xrightarrow{\delta_*} H^{-4n-1,-12n-3}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-1,-12n-4}(D;\mathbb{F}) \longrightarrow 0.$$

$$0 \longrightarrow H^{-4n-2,-12n-5}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-2,-12n-6}(D;\mathbb{F}) \longrightarrow H^{0,-1}(U;\mathbb{F})$$
$$\xrightarrow{\delta_*} H^{-4n-1,-12n-5}(T(3,3n+1);\mathbb{F}) \longrightarrow H^{-4n-1,-12n-6}(D;\mathbb{F}) \longrightarrow 0.$$

In [Tur08], Turner also calculates

$$H^{-4n-2,-12n-3}(T(3,3n+1);\mathbb{F}) = 0$$

$$H^{-4n-2,-12n-5}(T(3,3n+1);\mathbb{F}) = 0$$

$$H^{-4n-1,-12n-3}(T(3,3n+1);\mathbb{F}) = \mathbb{F}$$

$$H^{-4n-1,-12n-5}(T(3,3n+1);\mathbb{F}) = \mathbb{F}.$$

And of course $H^{0,\pm1}(U;\mathbb{F}) = \mathbb{F}$ for any field $\mathbb{F}$. Thus we have exact sequences

$$0 \longrightarrow H^{-4n-2,-12n-4}(D;\mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1,-12n-4}(D;\mathbb{F}) \longrightarrow 0 \tag{5.14}$$

and

$$0 \longrightarrow H^{-4n-2,-12n-6}(D;\mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1,-12n-6}(D;\mathbb{F}) \longrightarrow 0. \tag{5.15}$$

From (5.14) and (5.15) it follows that each of the groups $H^{-4n-1,-12n-6}(D;\mathbb{F})$, $H^{-4n-1,-12n-4}(D;\mathbb{F})$, $H^{-4n-2,-12n-6}(D;\mathbb{F})$ and $H^{-4n-2,-12n-4}(D;\mathbb{F})$ is isomorphic to either $\mathbb{F}$ or the trivial group. We argue that all four of them are isomorphic to $\mathbb{F}$.

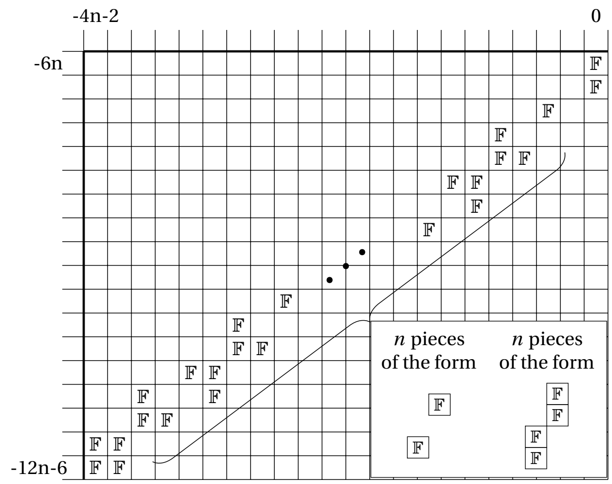


Figure 5.8 The $\mathbb{F}$-Khovanov homology of the braid closure of Δ^{2n+1} over $\mathbb{F}=\mathbb{Q}$ or $\mathbb{Z}_p$ for p an odd prime.

A straight forward calculation of Lee homology using Lemma 5.2.7 yields

$$\dim_{\mathbb{F}}(Lee^{-4n-2}(\Delta^{2n+1};\mathbb{F}))=2.$$

Using Turner's calculation of the Khovanov homology of $T(3,3n+1)$, the long exact sequence (5.4.2) gives $H^{-4n-2,j}(\Delta^{2n+1};\mathbb{F})\cong H^{-4n-2,j+1}(T(3,3n+1);\mathbb{F})=0$ for $j\neq -12n-6,-12n-4$. Therefore, $\dim_{\mathbb{F}} H^{-4n-2,*}(\Delta^{2n+1};\mathbb{F})\leq 2$. Since the Lee spectral sequence has E^1 page the $\mathbb{F}$-Khovanov homology and converges to Lee homology, we must also have $\dim_{\mathbb{F}} H^{-4n-2,*}(\Delta^{2n+1};\mathbb{F})\geq 2$, and so it follows that $H^{-4n-2,-12n-6}(D;\mathbb{F})=\mathbb{F}$ and $H^{-4n-2,-12n-4}(D;\mathbb{F})=\mathbb{F}$. Finally, the non-triviality of these two groups together with (5.14) and (5.15) imply that $H^{-4n-1,-12n-4}(D;\mathbb{F})\cong\mathbb{F}$ and $H^{-4n-2,-12n-6}(D;\mathbb{F})\cong\mathbb{F}$. $\square$

Corollary 5.4.4. *Let L be a link in Ω_3. The Khovanov homology $H(L)$ of L contains no $\mathbb{Z}_{p^r}$ torsion for $p\neq 2$.*

5.4.3 Even Torsion in $\Omega_0,\Omega_1,\Omega_2,\Omega_3$

In this subsection, we use Theorem 5.3.4 and Corollary 5.3.5 to completely compute the torsion in closed braids in $\Omega_0,\Omega_1,\Omega_2$, and Ω_3.

Benheddi [Ben17] computed the reduced $\mathbb{Z}_2$-Khovanov homology of the torus links $T(3,q)$, and from those computations we can recover the unreduced $\mathbb{Z}_2$-Khovanov homology of the torus links $T(3,q)$. These computations encompass the closed 3-braids in Ω_0,Ω_1, and Ω_2. We display the $\mathbb{Q}$-Khovanov homology and $\mathbb{Z}_2$-Khovanov homology of these links in Figures 5.9, 5.10, and 5.11.

The $\mathbb{Z}_2$-Khovanov homology of the closure of braids in Ω_3 is computed from the $\mathbb{Z}_2$-Khovanov homology of $T(3,3n+1)$ in a similar fashion to the analogous computation over $\mathbb{Q}$.

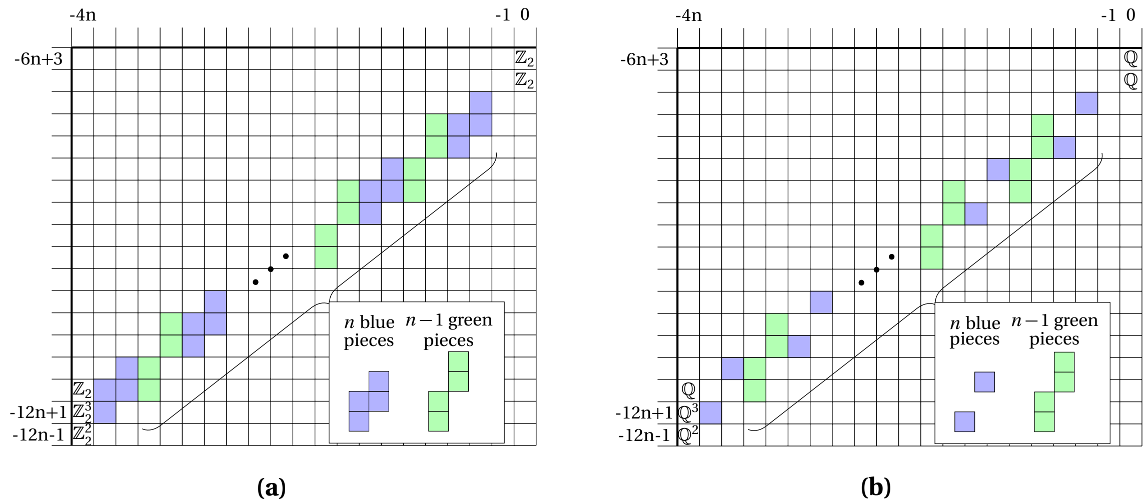


Figure 5.9 In (a) we have the $\mathbb{Z}_2$-Khovanov homology for the three component torus link $T(3,3n)$. Each blue or green box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have the $\mathbb{Q}$-Khovanov homology for the torus link $T(3,3n)$. Each blue or green box represents a copy of $\mathbb{Q}$ which is killed in the Lee spectral sequence.

Theorem 5.4.5. *For any $n \geq 0$,*

$$H(\Delta^{2n+1};\mathbb{Z}_2) \cong H(T(3,3n+1);\mathbb{Z}_2)\{-1\} \oplus H(U;\mathbb{Z}_2)[-4n-2]\{-12n-5\}.$$

Proof. For homological gradings $-4n-1$ through 0, the proof of this theorem is largely the same as the proof of Theorem 5.4.3. We focus on homological grading $-4n-2$. From (5.14) and (5.15) it follows that each of the groups

$$H^{-4n-1,-12n-5\pm1}(D;\mathbb{Z}_2),\ H^{-4n-2,-12n-5\pm1}(D;\mathbb{Z}_2)$$

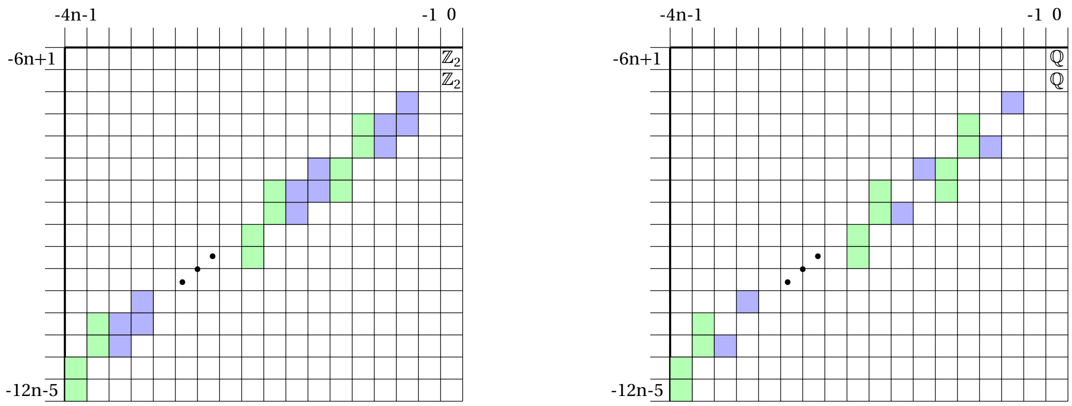
is isomorphic to either $\mathbb{Z}_2$ or the trivial group. We argue that each of these groups is isomorphic to $\mathbb{Z}_2$.

Using Lemma 5.2.3, we find that $\dim_{\mathbb{Z}_2}(BN^{-4n-2}(\Delta^{2n+1})) = 2$. Using Benheddi's calculation [Ben17] of the $\mathbb{Z}_2$-Khovanov homology of $T(3,3n+1)$, shown in Figure 5.10, the long exact sequence (5.4.2) gives

$$H^{-4n-2,j}(\Delta^{2n+1};\mathbb{Z}_2) \cong H^{-4n-2,j+1}(T(3,3n+1);\mathbb{Z}_2) = 0$$

for $j \neq -12n-6, -12n-4$. Therefore, $\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1};\mathbb{Z}_2) \leq 2$. Since the Turner spectral sequence has E^1 page the $\mathbb{Z}_2$-Khovanov homology, and converges to Bar-Natan homology,

$$\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1};\mathbb{Z}_2) \geq 2.$$



(a) **(b)**

Figure 5.10 In (a) we have the $\mathbb{Z}_2$-Khovanov homology for the torus knot $T(3,3n+1)$. Each blue or green box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have the $\mathbb{Q}$-Khovanov homology for the torus knot $T(3,3n+1)$. Each blue or green box represents a copy of $\mathbb{Q}$ which is killed in the Lee spectral sequence.

Therefore it follows that $H^{-4n-2,-12n-6}(D;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H^{-4n-2,-12n-4}(D;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Finally, the non-triviality of these two groups together with (5.14) and (5.15) imply that $H^{-4n-1,-12n-4}(D;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H^{-4n-2,-12n-6}(D;\mathbb{Z}_2) \cong \mathbb{Z}_2$. □

The Khovanov homology with $\mathbb{Z}_2$ and $\mathbb{Q}$ coefficients of the closure of Δ^{2n+1} is depicted in Figure 5.12.

The computations of Khovanov homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for closed braids in $\Omega_0, \Omega_1, \Omega_2$, and Ω_3 leads to the following theorem.

Theorem 5.4.6. *All torsion in the Khovanov homology of a closed 3-braid of type* $\Omega_0, \Omega_1, \Omega_2$ *or* Ω_3 *is* $\mathbb{Z}_2$ *torsion.*

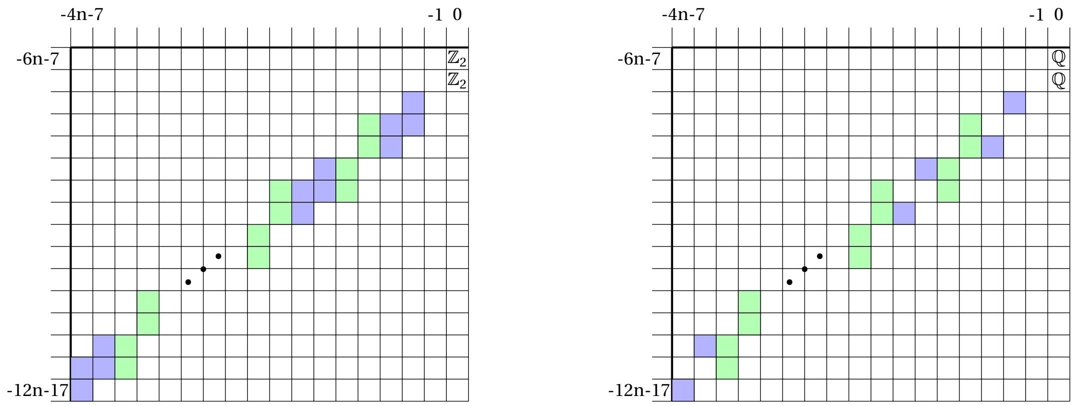
Proof. Let L be a closed braid in $\Omega_0, \Omega_1, \Omega_2$ or Ω_3. Theorem 5.4.2 and Corollary 5.4.4 imply L contains no $\mathbb{Z}_{p^r}$ torsion for any odd prime p. Therefore L satisfies condition 1 of Corollary 5.3.5.

Figures 5.9, 5.10, 5.11, and 5.12 show that all torsion in $H(L)$ occurs in the thin "blue" regions, and moreover, no torsion is supported in the initial homological grading of any thin region. Therefore Corollary 5.3.5 implies that all torsion in $H(L)$ is $\mathbb{Z}_2$ torsion. □

As corollaries, we obtain the integral Khovanov homology of closed 3-braids in $\Omega_0, \Omega_1, \Omega_2$, and Ω_3.

Corollary 5.4.7. *For any* $n \geq 0$,

$$H(\Delta^{2n+1}) \cong H(T(3,3n+1))\{-1\} \oplus H(U)[-4n-2]\{-12n-5\}.$$



(a) **(b)**

Figure 5.11 In (a) we have the $\mathbb{Z}_2$-Khovanov homology for the torus knot $T(3,3n+2)$. Each blue or green box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have the $\mathbb{Q}$-Khovanov homology, and E^1 page of the Lee spectral sequence, for the torus knot $T(3,3n+2)$. Each blue or green box represents a copy of $\mathbb{Q}$ which is killed in the Lee spectral sequence.

Corollary 5.4.8. *The integral Khovanov homology of links in classes* $\Omega_0, \Omega_1, \Omega_2$*, and* Ω_3 *are given in Figures 5.3a, 5.3b, 5.4a, and 5.4b.*

5.4.4 Future Directions

1. One goal of this project is to prove part 1 of Conjecture 5.0.1, that closed 3-braids have only $\mathbb{Z}_2$ torsion in Khovanov homology. Based on Murasugi's classification shown in Theorem 5.0.2, we have confirmed this result for links in the classes Ω_i for $0 \le i \le 3$, leaving only the classes Ω_4, Ω_5, and Ω_6. We now point out examples (see Figures 5.13 and 5.14) from these classes for which the main theorem used in this paper, Theorem 5.3.4, is insufficient. In a future paper, we plan to use these examples as a guide to come up with a stronger version of Theorem 5.3.4 which can be used to deal with these remaining cases, perhaps by showing a relationship between higher order Bockstein and Turner differentials, as suggested by Shumakovitch [Shu18].

2. We would like to find other classes of links for which Theorem 5.3.4 can be applied. For example, based on the results of Manion in [Man14], it appears this may be useful for 3-strand pretzel links.

3. Shumakovitch conjectures that all non-split links besides unknots, Hopf links, and their connect sums, contain $\mathbb{Z}_2$ torsion in Khovanov homology [Shu14, Conjecture 1]. The main result in this chapter gives conditions which restrict the appearance of any torsion besides $\mathbb{Z}_2$-torsion, but does not guarantee the existence of $\mathbb{Z}_2$-torsion. We would like to find a positive result in this direction, as a step towards proving Shumakovitch's conjecture. For example,

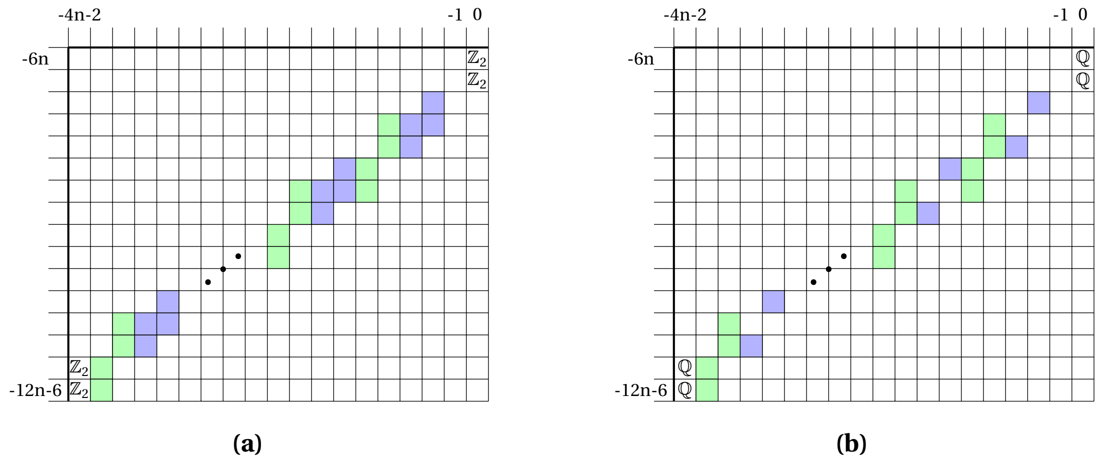


Figure 5.12 In (a) we have the $\mathbb{Z}_2$-Khovanov homology for the two component link Δ^{2n+1}. Each blue or green box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have the $\mathbb{Q}$-Khovanov homology for the link Δ^{2n+1}. Each blue or green box represents a copy of $\mathbb{Q}$ which is killed in the Lee spectral sequence.

one might try to find a condition which forces the appearance of knight moves or $\mathbb{Z}_2$-chains (see Figure 5.6) in the LV decomposition for some region.

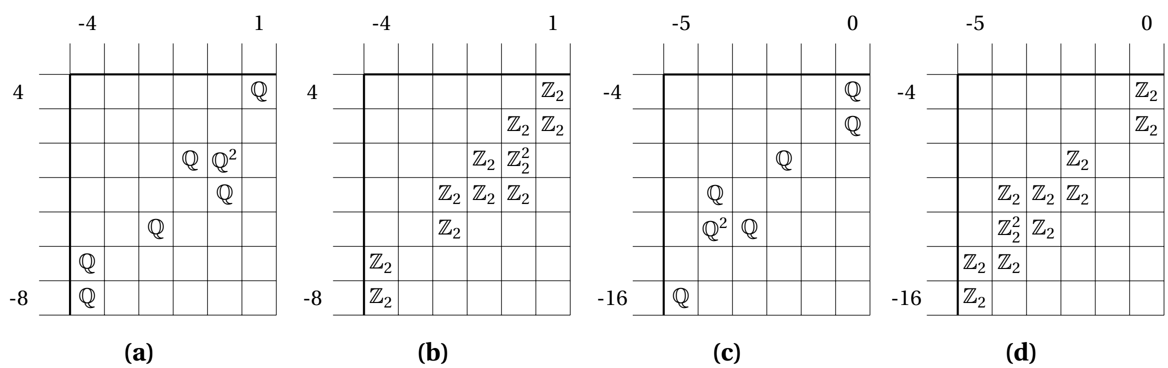
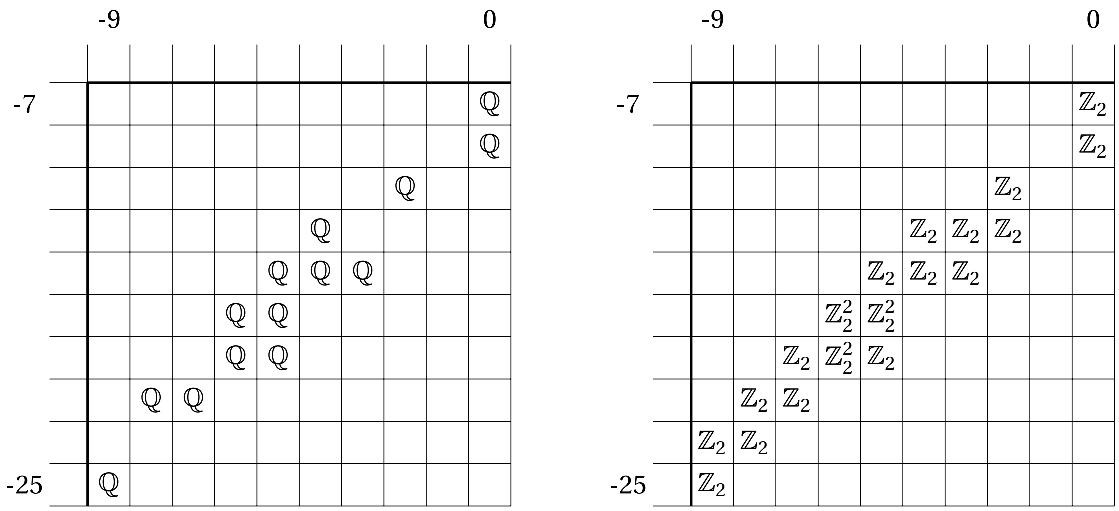


Figure 5.13 In (a) we have the rational Khovanov homology of the closure of the 3-braid $\Delta^2\sigma_1^{-5} \in \Omega_4$. In (b) we have the mod 2 Khovanov homology of the closure of the 3-braid $\Delta^2\sigma_1^{-5} \in \Omega_4$. Theorem 5.3.4 can not be applied here due to the mod 2 homology being supported on 3 diagonals in homological degree 0. In (c) we have the rational Khovanov homology of the closure of the 3-braid $\Delta^2\sigma_2 \in \Omega_5$. In (d) we have mod 2 Khovanov homology of the closure of the 3-braid $\Delta^2\sigma_2 \in \Omega_5$. Again, Theorem 5.3.4 can not be applied due to Khovanov homology being supported on 3 diagonals in homological degree -4.



(a) **(b)**

Figure 5.14 In (a) we have the rational Khovanov homology of the closure of the 3-braid $\Delta^4\sigma_1^{-2}\sigma_2\sigma_1^{-1} \in \Omega_6$. In (b) we have the mod 2 Khovanov homology of the closure of the 3-braid $\Delta^4\sigma_1^{-2}\sigma_2\sigma_1^{-1} \in \Omega_6$. The main theorem 5.3.4 in this paper cannot be applied in this case due to the homology being supported on 3 diagonals in homological degree -5.

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