

## ABSTRACT

SUMMERS, VICTOR WILLIAM. Torsion in the Khovanov Homology of Links and the Magnitude Homology of Graphs. (Under the direction of Radmila Sazdanović).

In this thesis we investigate the structure of torsion in two bigraded homology theories: the Khovanov homology of knots and links, and the magnitude homology of finite graphs. We analyze interactions of various spectral sequences in order to show that torsion in certain thin regions of the Khovanov homology of a link contains only  $\mathbb{Z}_2$  torsion. This result will be applied to four of seven families of three-stranded braids, resulting in explicit computations of the integral Khovanov homology for those families. We also show that torsion of a given prime order can appear in the magnitude homology of a graph and that there are infinitely many such graphs. Also, we provide a sharp bound on the types of torsion that may arise in the magnitude of a graph and see how this bound connects with geometric information about a graph. Finally, we also show how torsion sizes relate to geodesic properties of graphs, make explicit computations of magnitude homology groups for some families of graphs, with a focus on the ranks of the groups along the main diagonal of magnitude homology.

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Torsion in the Khovanov Homology of Links and the Magnitude Homology of Graphs

by  
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## **DEDICATION**

To my beautiful wife, Kaitlyn.

## **BIOGRAPHY**

Victor Summers was born in Maidenhead, England on the 13th of October, 1989. He loved mathematics from an early age, and first became interested in abstract mathematics as an undergraduate student at the University of Sussex, located in the city of Brighton on the south coast of England. Victor went on to receive a masters degree in mathematics in June of 2013, and is planning on graduating with a Ph.D. in mathematics in May of 2019 at North Carolina State University under the supervision of Dr. Radmila Sazdanović.

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## CHAPTER

# 1

## INTRODUCTION

In 1984, Vaughn Jones introduced a polynomial invariant of knots and links, now referred to as the Jones polynomial [Jon85]. In 1999, Mikhail Khovanov upgraded the Jones polynomial by constructing a chain complex of graded abelian groups whose graded Euler characteristic is the Jones polynomial [Kho00]. Torsion —subgroups of finite order— in Khovanov homology is a phenomenon with the possibility of offering information about knots and links which is not encoded by the Jones polynomial. Throughout this thesis, we say " $\mathbb{Z}_{p^r}$  torsion" or "torsion of order  $p^r$ " to indicate a  $\mathbb{Z}_{p^r}$  summand in the primary decomposition of a finitely generated abelian group. Even torsion is abundant in Khovanov homology, particularly  $\mathbb{Z}_2$  torsion [Shu18], while odd torsion is rare [BN05; Muk17]. To date, we do not have a solid understanding of the structure of torsion, nor its topological implications. Of late there have been various attempts to get a handle on the appearance and structure of torsion in Khovanov homology [Shu18; LS17; PS14]. For example, in 2018 Alexander Shumakovitch showed that all torsion in the Khovanov homology of homologically thin links is  $\mathbb{Z}_2$  torsion [Shu18]. In chapter 3, we extend this result of Shumakovitch by proving Theorem 3.5.6 which states that certain thin regions of Khovanov homology contain only  $\mathbb{Z}_2$  torsion.

**Theorem 3.5.6.** *Suppose that a link  $L$  satisfies:*

1.  $H(L)$  is thin over  $[i_1, i_2]$  for integers  $i_1$  and  $i_2$ ,
2.  $\dim_{\mathbb{Q}} H^{i_1,*}(L; \mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1,*}(L; \mathbb{Z}_p)$  for each odd prime  $p$ , and
3.  $H^{i_1,*}(L)$  is torsion-free.

*Then all torsion in  $H^{i,*}(L)$  is  $\mathbb{Z}_2$  torsion for  $i \in [i_1, i_2]$ .*

An  $n$ -stranded braid is a link obtained by joining up  $n$  interlocking strands into a closed loop (See Figure 3.9). In Theorem 3.6.7 we apply Theorem 3.5.6 to four families of 3-stranded braids, thus giving a partial affirmative answer to Jozef Przytycki and Radmila Sazdanovic's 2012 conjecture (PS braid conjecture) which posits a relationship between torsion in Khovanov homology and the braid index of a link [PS14]. As a consequence of Theorem 3.6.7 and calculations due to Paul Turner [Tur08] and Mounir Benheddi [Ben17], we prove Corollary 3.6.9, giving explicit computations for the integral Khovanov homology groups of the 3-braids of Theorem 3.6.7.

**Theorem 3.6.7.** *The Khovanov homology of a closed braid in  $\Omega_0, \Omega_1, \Omega_2$  or  $\Omega_3$  contains only  $\mathbb{Z}_2$  torsion.*

**Corollary 3.6.9.** *The integral Khovanov homology closed 3-braids in  $\Omega_0, \Omega_1, \Omega_2$ , and  $\Omega_3$  are given in Figures 3.17a, 3.17b, 3.18a, and 3.18b.*

In 2006, Tom Leinster introduced a notion of Euler characteristic for categories [Lei08], yielding a power series invariant of graphs called magnitude [Lei14]. In 2015, Richard Hepworth and Simon Willerton upgraded the magnitude invariant to magnitude homology, a chain complex of graded modules whose graded Euler characteristic is the magnitude power series [HW15]. After calculating the magnitude homology groups for dozens of graphs in their original paper on the subject, Hepworth and Willerton conjectured there to be no torsion in magnitude homology whatsoever. In early 2018, however, Ryuki Kaneta and Masahiko Yoshinaga showed how to construct a graph with  $\mathbb{Z}_2$  torsion in magnitude homology by extracting a graph from a triangulation of the real projective space  $\mathbb{RP}^2$  [KY18]. In chapter 4, we prove Theorem 4.2.6 and Theorem 4.2.8, showing that torsion of a given prime order can be found in certain bigradings of magnitude homology, and prove Theorem 4.2.16 demonstrating that any finitely generated abelian group may be realized as a subgroup of the magnitude homology of some graph  $G$ .

**Theorem 4.2.6.** *For any odd integer  $k \geq 3$ , there is a graph  $G$  such that  $\text{MH}_{k,k+1}(G)$  contains a subgroup isomorphic to  $\mathbb{Z}_2$ .*

**Theorem 4.2.8.** *For each prime  $p$ , there is a graph with torsion of order  $p$  in magnitude homology. More specifically, for each integer  $n \geq 1$  and each prime  $p$ , there is a graph  $G$  such that  $\text{MH}_{3,2n+3}(G)$  contains torsion of order  $p$ .*

**Theorem 4.2.16.** *Let  $M$  be a finitely generated finite abelian group. There exists a graph  $G$  whose magnitude homology contains a subgroup isomorphic to  $M$ .*

We also perform explicit computations of magnitude homology groups for some families of graphs, such as graphs with no 3- or 4-cycle as in Theorem 4.3.4, and certain outerplanar graphs as in Theorem 4.3.11

**Theorem 4.3.4.** *Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . If  $G$  has no 3- or 4-cycles, then the first diagonal in the magnitude homology of  $G$  is torsion-free and satisfies*

$$\mathrm{MH}_{k,k}(G) \cong \begin{cases} \mathbb{Z}^{|V|} & k = 0 \\ \mathbb{Z}^{2|E|} & k > 0 \end{cases}$$

**Theorem 4.3.11.** *Fix a positive integer  $m \geq 3$ . Let  $G$  be an outer planar graph with  $S$  components  $C_4$ , and  $R$  component cycles  $C_{2m}$ , constructed using edge-gluings only. The magnitude homology groups of  $G$  are torsion-free and arrange themselves in diagonals: let  $S_{i,j}^m$  denote the rank of the magnitude homology group of  $G$  in the  $j^{\text{th}}$  entry of the  $i^{\text{th}}$  diagonal, that is,  $S_{i,j}^m = \mathrm{rank}(\mathrm{MH}_{2(i-1)+(j-1), m(i-1)+(j-1)}(G))$ . Then, the magnitude homology groups  $\mathrm{MH}_{k,\ell}$  of  $G$  are all trivial groups except for the groups the aforementioned diagonals, and these satisfy for  $i > 1$*

$$\mathrm{rank}(S_{1,j}^m) = \begin{cases} 2mR + 4S - 2(R + S - 1) & j = 1, \\ 4mR + 4jS - 2(R + S - 1) & j > 1, \end{cases}$$

$$\mathrm{rank}(S_{i,j}^m) = \begin{cases} 2mR + 4S - 2(R + S - 1) & j = 1, \\ 4mR - 2(R + S - 1) & j > 1. \end{cases}$$

As mentioned above, the Jones polynomial is the graded Euler characteristic of Khovanov homology, while the magnitude power series is the graded Euler characteristic of magnitude homology. We say that these homology theories "categorify" their polynomial counterparts. Before diving into our results on the Khovanov and magnitude homologies, in the next chapter, Chapter 2, we give a brief discussion of categorification and provide some illustrative examples.

## CHAPTER

# 2

# CATEGORIFICATION

The Khovanov homology of links and the Magnitude homology of graphs may be viewed as categorifications of their polynomial counterparts, the Jones polynomial and magnitude power series, respectively. In this chapter, we recall the notion of categorification and illustrate the concept with examples of categorifications of the natural numbers, the integers and the single variable Laurent series.

Category theory is a framework for describing abstract structures in mathematics which places an emphasis on the relationships between objects of study, as opposed to the objects themselves. The basic idea of categorification is to replace set-theoretic ideas with category-theoretic ideas. Recovering the original set-theoretic information is called decategorification. Categorification has been widely used in recent times in the context of topological and geometric invariants. The advantage of categorified invariants is that they are much more highly structured, and thus have the capacity to be more powerful invariants.

**Definition 2.0.1** (Category, [Alu09]). *A category  $C$  is a class of objects,  $\text{Ob}(C)$ , and each pair of objects  $x, y \in \text{Ob}(C)$  there is a class of morphisms,  $\text{Hom}(x, y)$ , from  $x$  to  $y$ . For each triple of objects  $x, y$  and  $z$  there is a function  $\text{Hom}(x, y) \times \text{Hom}(y, z) \rightarrow \text{Hom}(x, z)$  assigning to each pair of morphisms  $x \xrightarrow{f} y$  and  $y \xrightarrow{g} z$  their composite morphism  $x \xrightarrow{g \circ f} z$ . Each object  $x \in \text{Ob}(C)$  has an associated identity morphism  $x \xrightarrow{1_x} x$  in  $\text{Hom}(x, x)$ . Further, the composition operation is associative; each triple of morphisms  $x \xrightarrow{f} y, y \xrightarrow{g} z$  and  $z \xrightarrow{h} w$  satisfies  $f \circ (g \circ h) = (f \circ g) \circ h$ . And the identity morphisms act as two-sided identities; for each pair of objects  $x, y \in \text{Ob}(C)$  and morphism  $f \in \text{Hom}(x, y)$ , the*

identity morphisms  $1_x$  and  $1_y$  satisfy  $f \circ 1_x = f$  and  $1_y \circ f = f$ .

Examples of categories include the category of sets and set maps (**Set**); the category of vector spaces over a field  $\mathbb{K}$  and linear maps (**Vect** $_{\mathbb{K}}$ ); the category of  $R$ -modules and  $R$ -linear maps (**R-mod**); the category of topological spaces and continuous maps (**Top**); and the category of chain complexes of vector spaces over a field  $\mathbb{K}$  and chain maps (**Ch(Vect** $_{\mathbb{K}})$ ).

**Definition 2.0.2** ([Alu09]). *Let  $C$  be a category. Objects  $x, y \in \text{Ob}(C)$  are said to be isomorphic in  $C$  if there are morphisms  $f \in \text{Hom}(x, y)$  and  $g \in \text{Hom}(y, x)$  such that  $g \circ f = 1_x$  and  $f \circ g = 1_y$ . Isomorphism is an equivalence relation; we denote the isomorphism class of  $x \in \text{Ob}(C)$  by  $[x]$ , and denote the collection of isomorphism classes of  $C$  by  $C / \sim$ .*

**Definition 2.0.3** ([Bae08]). *A category  $C$  is said to categorify a set  $S$  if there is a surjective map  $p : C / \sim \rightarrow S$ .*

The map  $p$  is commonly referred to as "decategorification." To categorify an algebraic structure  $S$  is to categorify its underlying set in such a way that categorical properties of the category  $C$  are transformed by  $p$  into the algebraic properties of  $S$ .

**Example 2.0.4.** *The category **FinVect** $_{\mathbb{K}}$  of finite dimensional vector spaces over a field  $\mathbb{K}$  categorifies the set of non-negative integers  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  with decategorification map  $p([V]) = \dim(V)$ . Equipped with the usual operations of addition and multiplication,  $\mathbb{N}_0$  becomes a semiring  $(\mathbb{N}_0, +, \cdot)$ . Addition corresponds to the direct sum, while multiplication corresponds to the tensor product:*

$$\dim(V \oplus W) = \dim(V) + \dim(W), \quad \dim(V \otimes W) = \dim(V) \cdot \dim(W)$$

*It is in this sense that the category of finite dimensional vector spaces over a field categorifies the semiring  $\mathbb{N}_0$ .*

Next, we see how to categorify the ring of integers, but first we need to recall the following definitions. Throughout,  $R$  is assumed to be a commutative ring with identity.

**Definition 2.0.5** (Cochain complex, [Wei94]). *A cochain complex  $M^* = (M^*, d^*)$  is a sequence of left  $R$ -modules  $M^n$  ( $n \in \mathbb{Z}$ ) and  $R$ -linear maps (called differentials)  $d^n : M^n \rightarrow M^{n+1}$  satisfying  $d^{n+1} \circ d^n = 0$ . The  $n^{\text{th}}$  cohomology module of  $M^*$  is the quotient module  $H^n(M^*) = \ker(d_n) / \text{Im}(d_{n-1})$ . The cohomology of a cochain complex may be viewed as a graded module  $H(M^*) := \bigoplus_{n \in \mathbb{Z}} H^n(M^*)$ .*

**Remark 2.0.6.** *Using lower indices (i.e.  $M_n, d_n, H_n$ ), and requiring the differentials  $d_n : M_n \rightarrow M_{n-1}$  to be index-decreasing, we recover the definition of a chain complex. As we will see in chapter 3, Khovanov homology is technically the cohomology of a cochain complex and we use upper indices, but it has become commonplace to refer to the Khovanov chain complex and Khovanov homology. We refer to both chain complexes and cochain complexes as chain complexes, and let upper and lower indices to indicate the context.*

**Definition 2.0.7** (Euler characteristic, [BN05]). Let  $M^* = (M^*, d^*)$  be a chain complex of finitely-generated left  $R$ -modules. The Euler characteristic of  $M^*$ , denoted  $\chi(M^*)$ , is the alternating sum of the ranks of its homology modules, and this is equal to the alternating sum of the ranks of its chain modules:

$$\chi(M^*) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rank}(H^i(M^*)) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rank}(M^i)$$

**Definition 2.0.8** (Direct sum of chain complexes, [Wei94]). Let  $M^* = (M^*, d_M^*)$  and  $N^* = (N^*, d_N^*)$  be chain complexes of left  $R$ -modules. The direct sum  $(M \oplus N)^* = ((M \oplus N)^*, d^*)$  is the chain complex with chain modules and differential given by

$$(M \oplus N)^n = M^n \oplus N^n, \quad d^n = d_M^n \oplus d_N^n.$$

**Definition 2.0.9** (Tensor product of chain complexes, [Wei94]). Let  $M^* = (M^*, d_M^*)$  and  $N^* = (N^*, d_N^*)$  be chain complexes of  $RR$ -bimodules. The tensor product  $(M \otimes N)^* = ((M \otimes N)^*, d^*)$  is the chain complex with chain modules and differential given by

$$(M \otimes N)^n = \bigoplus_{i \in \mathbb{Z}} M^i \otimes N^{n-i}, \quad d^n = \bigoplus_{i \in \mathbb{Z}} [d_M^i \otimes 1_{N^{n-1}} + (-1)^i 1_{M_i} \otimes d_N^{n-i}]$$

**Definition 2.0.10** (Mapping cone, [Wei94]). Let  $(M^*, d_M^*)$  and  $(N^*, d_N^*)$  be chain complexes of left  $R$ -modules. The mapping cone of a chain map  $f : (M^*, d_M^*) \rightarrow (N^*, d_N^*)$  is the chain complex  $(C(f)^*, d^*)$  with chain groups and differential given by

$$C(f)^n = M^{n+1} \oplus N^n, \quad d^n(u, v) = (-d_M^{n+1}(u), d_N^n(v) - f^{n+1}(u))$$

**Example 2.0.11.** Consider the category  $\mathbf{Ch}(\mathbf{FinVect}_{\mathbb{K}})$  whose objects are chain complexes  $V^* = (V^*, d^*)$  of finite dimensional vector spaces over a field  $\mathbb{K}$ , and whose morphisms are chain maps. Isomorphic objects in this category have isomorphic homology groups, hence the same Euler characteristic. Consequently, we can define a decategorification map by

$$p[(V^*)] = \chi(V^*)$$

The image of the map  $p$  is precisely the set of integers. Indeed, let  $V$  be a vector space of dimension  $n$  and form the chain complex  $0 \rightarrow V \rightarrow 0$ . Placing  $V$  in homological degree zero gives  $\chi(0 \rightarrow V \rightarrow 0) = n$ , while placing  $V$  in degree one gives  $\chi(0 \rightarrow V \rightarrow 0) = -n$ . As a ring, the integers are categorified by  $\mathbf{Ch}(\mathbf{Vect}_{\mathbb{K}})$  in the following sense. Let  $V^* = (V^*, d_V^*)$  and  $W^* = (W^*, d_W^*)$  be chain complexes with

Euler characteristics  $m$  and  $n$ , respectively, and let  $f : V^* \rightarrow W^*$  be a chain map. Then,

$$\chi((V \oplus W)^*) = m + n,$$

$$\chi(\text{Cone}(f)) = m - n,$$

$$\chi((V \otimes W)^*) = mn.$$

Our final example of categorification will be of the ring of Laurent series  $\mathbb{Z}[[q, q^{-1}]]$ . To this end, let us recall a graded version of the Euler characteristic.

**Definition 2.0.12** (Graded module). *A graded  $R$ -module is a left  $R$ -module  $M$  with a direct sum decomposition  $M = \bigoplus_{j \in \mathbb{Z}} M^j$  into finitely-generated left  $R$ -modules  $M^j$ .*

**Definition 2.0.13** (Degree). *Let  $M$  and  $N$  be graded  $R$ -modules. A linear map  $f : M \rightarrow N$  is said to be of degree  $s \in \mathbb{Z}$  if  $f(M^j) \subseteq N^{j+s}$  for each  $j \in \mathbb{Z}$ .*

**Definition 2.0.14** (Graded rank). *Let  $M$  be a graded  $R$ -module. The graded rank of  $M$  is the Laurent series*

$$\text{qrank}(M) = \sum_{j \in \mathbb{Z}} \text{rank}(M^j) q^j.$$

*If  $R$  is a field, then  $M$  is a graded vector space and  $\text{rank}(M^j) = \dim(M^j)$ . In this case, we refer to the graded dimension of  $M$  and write instead  $\text{qdim}(M)$ .*

**Definition 2.0.15** (Graded Euler characteristic). *Let  $M^* = (M^*, d^*)$  be a chain complex of graded  $R$ -modules  $M^i = \bigoplus_{j \in \mathbb{Z}} M^{i,j}$ . If each differential  $d^i : M^i \rightarrow M^{i+1}$  is map of degree  $s$ , then each homology group  $H^i(M^*)$  is a graded vector space:  $H^i(M^*) = \bigoplus_{j \in \mathbb{Z}} H^i(M^*)^j =: \bigoplus_{j \in \mathbb{Z}} H^{i,j}(M^*)$ . The graded Euler characteristic of such a chain complex is the alternating sum of the graded ranks of its homology groups:*

$$\chi_q(M^*) = \sum_{i \in \mathbb{Z}} (-1)^i \text{qrank}(H^i(M^*)) = \sum_{i,j \in \mathbb{Z}} (-1)^i \text{rank}(H^{i,j}(M^*)) q^j.$$

*If the differential is of degree 0, the graded Euler characteristic is also given by the alternating sum of the graded ranks of the chain groups:*

$$\chi_q(M^*) = \sum_{i \in \mathbb{Z}} (-1)^i \text{qrank}(M^i) = \sum_{i,j \in \mathbb{Z}} (-1)^i \text{rank}(M^{i,j}) q^j.$$

**Example 2.0.16.** *In order to categorify the set of Laurent series, consider the category  $\mathbf{Ch}(\mathbf{grFinVect}_{\mathbb{K}})$  of chain complexes of finite-dimensional graded vector spaces and chain maps. Isomorphic objects in this category have isomorphic homology groups, hence identical graded Euler characteristic. Therefore, we may define a decategorification map  $p : \mathbf{Ch}(\mathbf{grFinVect}_{\mathbb{K}}) / \sim \rightarrow \mathbb{Z}[[q, q^{-1}]]$  by*

$$p([V^*]) = \chi_q(V^*).$$

*There are notions of direct sum and tensor product for chain complexes of graded vector spaces [Wei94], and, similarly to the ring of integers, these categorify the ring operations of the Laurent series.*

This decategorification map plays a key role in viewing Khovanov homology and magnitude homology as categorifications of their polynomial counterparts. Specifically, the graded Euler characteristic of the Khovanov chain complex is the Jones polynomial, and the graded Euler characteristic of the magnitude chain complex is the magnitude power series.

## CHAPTER

# 3

## TORSION IN THIN REGIONS OF THE KHOVANOV LINK HOMOLOGY

In this chapter we prove Theorem 3.5.6, demonstrating that, under certain circumstances, all torsion in locally thin regions of the Khovanov homology of a link is  $\mathbb{Z}_2$  torsion. We begin by recalling our basic objects of study: knots and links, the Jones polynomial and its categorification the Khovanov homology. Then we introduce the analytical tools needed for analyzing torsion in Khovanov homology: long exact sequences; exact couples and the Bockstein spectral sequence; and spectral sequences arising from double complexes emanating from alternate differentials on the Khovanov chain complex. Our approach will be to first use the universal coefficients theorem to provide a condition under which thin regions of Khovanov homology have no torsion of odd order. To show that the only even torsion is  $\mathbb{Z}_2$  torsion, it will remain to show that the  $\mathbb{Z}_2$  Bockstein spectral sequence collapses at the  $E^2$  page in appropriate bigradings. For this, we analyze relationships between various alternate differentials on the Khovanov complex and the spectral sequences they give rise to. We end the chapter with an application of Theorem 3.5.6 to four families of link of braid index 3, thus giving a partial affirmative answer to the PS braid conjecture [PS14]. We end with examples of links of braid index 3 to which our current approach does not work.

### 3.1 Knots and links

A knot may be pictured as a knotted loop of string in three-dimensional space, and a link as several interlocking knots. More formally,

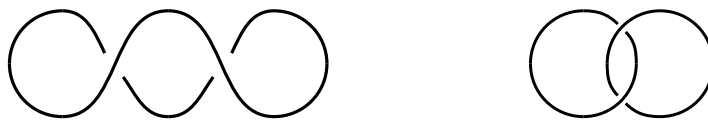
**Definition 3.1.1** ([Lic12]). *A link of  $m$  components is a smooth embedding  $S^1 \sqcup S^1 \sqcup \dots \sqcup S^1 \hookrightarrow S^3$  of  $m$  disjoint circles in  $S^3$ . A knot is a link with a single component.*

We often identify a link with the image of its embedding.

**Definition 3.1.2** ([Lic12]). *Two links  $L_1$  and  $L_2$  are ambient isotopic if there is a one-parameter family of homeomorphisms  $h_t : S^3 \rightarrow S^3$  with  $h_0(L_1) = L_1$  and  $h_1(L_1) = L_2$ , such that  $h_t(L_1)$  is homeomorphic to  $L_1$  for each  $t \in [0, 1]$ . Equivalently,  $L_1$  and  $L_2$  are ambient isotopic if there is an orientation-preserving homeomorphism  $h : S^3 \rightarrow S^3$  with  $h(L_1) = L_2$ .*

The latter definition is the mathematical equivalent of our intuitive notion that two links are "the same" if one can be continuously deformed into the other without tearing or self-intersection; throughout this document we regard links as equivalent if they are ambient isotopic.

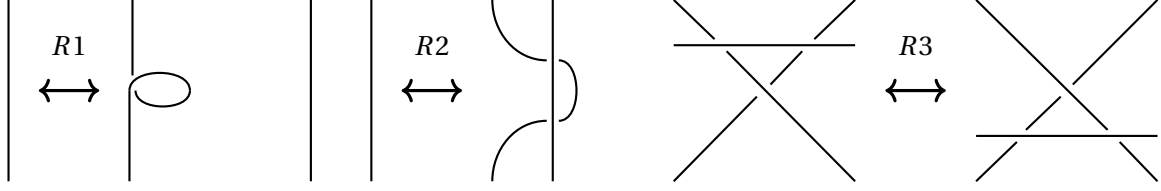
A link invariant is a function from the set of ambient isotopy classes of links into a set of algebraic objects such as numbers, polynomials or groups. In other words, an invariant assigns an object to each link in such a way that equivalent links are mapped to the same object. If the target set is the set of objects in a category, we only require equivalent links to be mapped to isomorphic objects. Rather than dealing with a link  $L$  as a three-dimensional entity, we often project down onto an appropriate plane and consider a two-dimensional "shadow"; the chosen projection must be injective outside of a finite set of points where it is a two-to-one map and the image of a neighborhood of such a double point is a pair of transversely intersection lines. Endowing these transverse double points with over- and under-crossing information, such a projection is called a link diagram  $D$  of  $L$  [Lic12].



**Figure 3.1** On the left, a diagram of the unknot. On the right, a diagram of the Hopf link.

The operations  $R1$ ,  $R2$  and  $R3$  on link diagrams depicted in Figure 3.2 are called Reidemeister moves, named after Kurt Reidemeister.

**Theorem 3.1.3** ([Rei27]). *Link diagrams  $D_1$  and  $D_2$  represent equivalent links if and only if they are related by a finite sequence of Reidemeister moves and plane isotopies.*



**Figure 3.2** Reidemeister moves  $R1$ ,  $R2$  and  $R3$ .

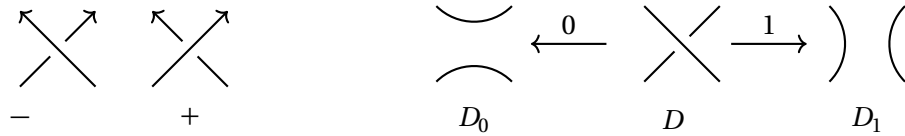
Theorem 3.1.3 enables the definition of link invariants via link diagrams. Indeed, if we assign objects to link diagrams in such a way as to remain unchanged by Reidemeister moves and ambient isotopies, the resulting function is an invariant of links. Many link invariants have been defined in this way, including the Alexander [Ale28], Jones [Jon85] and HOMFLY-PT polynomials [Fre85; PT87]. In Section 3.2 we recall how the Jones polynomial of a link  $L$  is obtained from a link diagram  $D$  by a skein relation that tracks how local changes to crossings affect the polynomial.

## 3.2 The Jones polynomial

The Jones polynomial is an invariant of oriented links with values in the ring  $\mathbb{Z}[q, q^{-1}]$  of Laurent polynomials [Jon85].

**Definition 3.2.1.** *An oriented link  $L$  is a link with a continuous choice of tangent vector for each component.*

In terms of diagrams, an orientation is indicated by a choice of direction for each component. For an example of an oriented link diagram, see Figure 3.4. Each crossing in a link diagram is either positive or negative, and we perform local crossing changes, called 0- and 1-smoothings, according to the conventions displayed in Figure 3.3.



**Figure 3.3** Conventions for crossings and smoothings in link diagrams. The diagram  $D_0$  is obtained by 0-smoothing a single crossing while leaving the rest of the diagram unchanged. Similarly for  $D_1$ .

**Definition 3.2.2** (Kauffman bracket, [Kau87]). *The Kauffman bracket is the function  $\langle \cdot \rangle : \{\text{link diagrams}\} \rightarrow \mathbb{Z}[q, q^{-1}]$  determined by the following properties.*

$$\begin{aligned}
1. \quad & \langle \bigcirc \rangle = q + q^{-1} \\
2. \quad & \langle \bigcirc \sqcup D \rangle = (q + q^{-1}) \langle D \rangle \\
3. \quad & \langle \text{crossing} \rangle = \langle \text{positive crossing} \rangle - q \langle \text{negative crossing} \rangle
\end{aligned}$$

Property 3 is called the skein relation for the Kauffman bracket. The skein relation can be used to calculate the Kauffman bracket of an  $n$  crossing diagram in terms of the Kauffman brackets of two  $n - 1$  crossing diagrams. Repeated application of the skein relation results in a weighted sum of the Kauffman brackets of disjoint collections of circles, which we calculate by appealing to properties 1. and 2.

**Definition 3.2.3** ([Jon85]). *Let  $D$  be a diagram of an oriented link  $L$ , and let  $\langle D \rangle$  be the Kauffman bracket polynomial of  $D$ . If  $D$  has  $n_-$  negative crossings and  $n_+$  positive crossings, then the (unnormalized) Jones polynomial  $\hat{J}(L)$  of  $L$  is defined by*

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

The Jones polynomial is an invariant of oriented links satisfying the skein relation [Jon85]. As we note in Section 3.3, the skein relation lifts to a long exact sequence in Khovanov homology.

### 3.3 Khovanov homology

Khovanov homology is an invariant of oriented links  $L \subset S^3$  with values in the category of bigraded modules over a commutative ring  $R$  with identity [Kho00]. We begin with a construction of Khovanov homology. A bigraded  $R$ -module is an  $R$ -module  $M$  with a direct sum decomposition of the form  $M = \bigoplus_{i,j \in \mathbb{Z}} M^{i,j}$ . The submodule  $M^{i,j}$  is said to have bigrading  $(i, j)$ . For our purposes, we refer to  $i$  as the *homological grading* and to  $j$  as the *polynomial grading*. Given two bigraded  $R$ -modules  $M = \bigoplus_{i,j \in \mathbb{Z}} M^{i,j}$  and  $N = \bigoplus_{i,j \in \mathbb{Z}} N^{i,j}$ , we define the direct sum  $M \oplus_R N$  and tensor product  $M \otimes_R N$  to be the bigraded  $R$ -modules with components  $(M \oplus N)^{i,j} = M^{i,j} \oplus N^{i,j}$  and  $(M \otimes N)^{i,j} = \bigoplus_{k+m=i, l+n=j} M^{k,l} \otimes N^{m,n}$ . We also define homological and polynomial shift operators, denoted  $[\cdot]$  and  $\{\cdot\}$ , respectively, by  $(M[r])^{i,j} = M^{i-r,j}$  and  $(M\{s\})^{i,j} = M^{i,j-s}$ .

Consider the directed graph whose vertex set  $\mathcal{V}(n) = \{0, 1\}^n$  comprises  $n$ -tuples of 0's and 1's and whose edge set  $\mathcal{E}(n)$  contains a directed edge from  $I \in \mathcal{V}(n)$  to  $J \in \mathcal{V}(n)$  if and only if all entries of  $I$  and  $J$  are equal except for one, where  $I$  is 0 and  $J$  is 1. One can think of the underlying graph as the 1-skeleton of an  $n$ -dimensional cube. For example, if  $n = 4$  there are two directed edges

emanating from  $(0, 1, 0, 1)$ : one ending at  $(0, 1, 1, 1)$  and the other ending at  $(1, 1, 0, 1)$ . The height of a vertex  $I = (k_1, k_2, \dots, k_n)$  is  $h(I) = k_1 + k_2 + \dots + k_n$ . In other words,  $h(I)$  is the number of 1's in  $I$ . If  $\varepsilon$  is an edge from  $I$  to  $J$  and they differ in the  $r$ -th entry, the height of  $\varepsilon$  is  $|\varepsilon| := h(I)$ , and the sign of  $\varepsilon$  is  $(-1)^\varepsilon := (-1)^{\sum_{i=1}^{r-1} k_i}$ . In other words, the sign of  $\varepsilon$  is  $-1$  if the number of 1's before the  $r$ -th entry is odd, and is  $+1$  if even.

Let  $D$  be a diagram of a link  $L$  with  $n$  crossings  $c_1, c_2, \dots, c_n$ . To each vertex  $I \in \mathcal{V}(n)$  we associate a collection of circles  $D(I)$ , called a Kauffman state, obtained by 0-resolving those crossings  $c_i$  for which  $k_i = 0$  and 1-resolving those crossings  $c_j$  for which  $k_j = 1$ . The 0- and 1-smoothing conventions are given in Figure 3.3. To each Kauffman state we associate a bigraded  $R$ -module  $C(D(I))$  as follows. Let  $\mathcal{A}_2 = R[X]/X^2 = R1 \oplus RX$  be the bigraded module with generator 1 in bigrading  $(0, 1)$  and generator  $X$  in bigrading  $(0, -1)$ . Denoting the number of circles in  $D(I)$  by  $|D(I)|$ , we define

$$C(D(I)) := \mathcal{A}_2^{\otimes |D(I)|} [h(I)] \{h(I)\}.$$

Here, each tensor factor of  $\mathcal{A}_2$  is understood to be associated to a particular circle of  $D(I)$ . Having associated modules to vertices, we now associate maps to directed edges. Define  $R$ -module homomorphisms  $m : \mathcal{A}_2 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}_2$  and  $\Delta : \mathcal{A}_2 \rightarrow \mathcal{A}_2 \otimes \mathcal{A}_2$  (called multiplication and comultiplication, respectively) by

$$m(1 \otimes 1) = 1, \quad m(1 \otimes X) = X, \quad m(X \otimes 1) = X, \quad m(X \otimes X) = 0,$$

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

To an edge  $\varepsilon$  from  $I$  to  $J$  we associate the map  $d_\varepsilon : C(D(I)) \rightarrow C(D(J))$  defined as follows.

1. If  $|D(J)| = |D(I)| - 1$ , then  $D(J)$  is obtained by merging two circles of  $D(I)$  into one:  $d_\varepsilon$  acts as multiplication  $m$  on the tensor factors associated to the circles being merged, and acts as the identity map on the remaining tensor factors.
2. If  $|D(J)| = |D(I)| + 1$  then  $D(J)$  is obtained by splitting one circle of  $D(I)$  into two:  $d_\varepsilon$  acts as comultiplication  $\Delta$  on the tensor factor associated to the circle being split, and acts as the identity map on the remaining tensor factors.

Suppose  $D$  has  $n_-$  negative crossings and  $n_+$  positive crossings. Define the bigraded module

$$C^{*,*}(D) = \bigoplus_{I \in \mathcal{V}(n)} C(D(I))[-n_-] \{n_+ - 2n_-\}. \quad (3.1)$$

Define maps  $d^i : C^{i,*}(D) \rightarrow C^{i+1,*}(D)$  by  $d^i = \sum_{|\varepsilon|=i} (-1)^\varepsilon d_\varepsilon$ . In [Kho00], Khovanov shows that  $(C(D), d)$  is a (co)chain complex, and since the maps  $d_i$  are polynomial degree-preserving, we get a

bigraded homology  $R$ -module

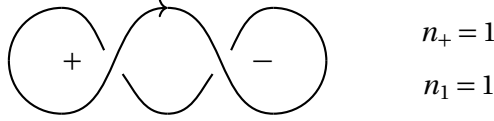
$$H(L; R) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L; R)$$

called the Khovanov homology of the link  $L$  with coefficients in  $R$ , or, more compactly, the  $R$  Khovanov homology of  $L$ . That  $d$  is a differential and that these homology groups are independent of the choice of diagram  $D$  and the ordering of the crossings are shown in [Kho00; Vir04; BN05]. If  $R = \mathbb{Z}$  we simply write  $H(L) = H(L; \mathbb{Z})$ . In this article, we focus on the rings  $R = \mathbb{Z}, \mathbb{Q}$  and  $\mathbb{Z}_p$ , where  $p$  is a prime.

Khovanov homology categorifies the Jones polynomial in the sense that the Jones polynomial of a link  $L$  may be recovered as the graded Euler characteristic its Khovanov homology:

$$\widehat{J}(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i \text{rank}(H^{i,j}(L)) q^j.$$

**Example 3.3.1.** *Let us compute the Khovanov homology of the unknot using the knot diagram of Figure 3.4.*



**Figure 3.4** An oriented two-crossing diagram of the unknot with one positive crossing and one negative.

In Figure 3.6 we list the generators for each bigrading in which the kernel of the Khovanov differential is non-trivial, along with generators of images appropriate for computing the Khovanov homology groups. For example,

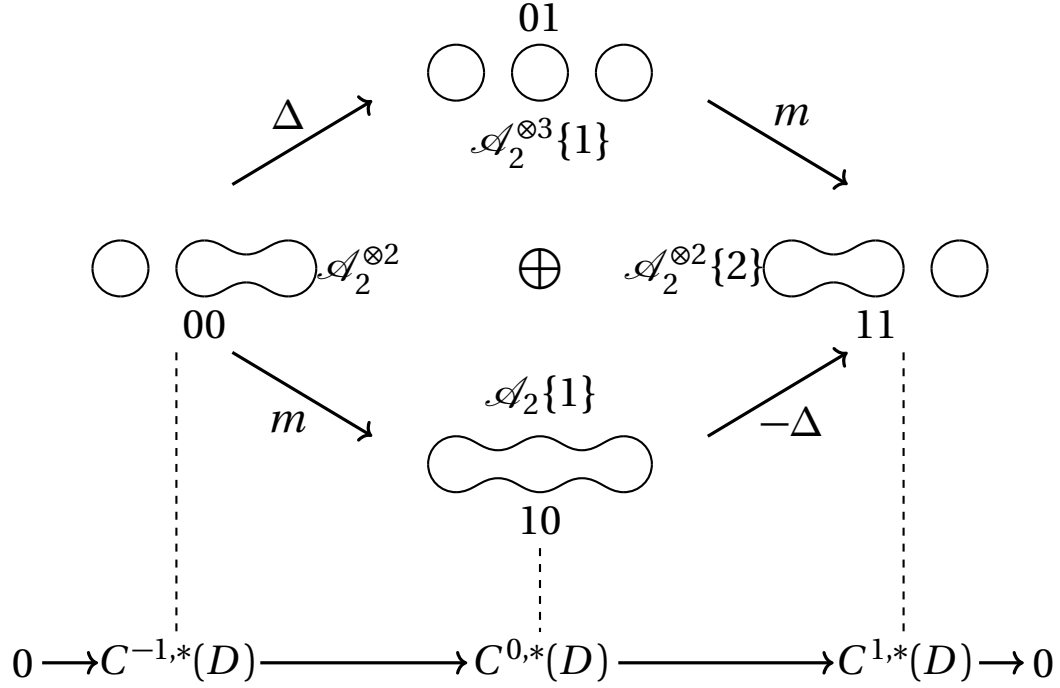
$$H^{1,3}(D) = \frac{\ker(d^{1,3})}{\text{Im}(d^{0,3})} = \frac{\langle 1 \otimes 1 \rangle}{\langle 1 \otimes 1 \rangle} \cong 0$$

and

$$H^{0,1}(D) = \frac{\langle 1 + 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1, 1 + 1 \otimes 1 \otimes X + X \otimes 1 \otimes 1 \rangle}{\langle 1 + 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 \rangle} \cong \langle 1 + 1 \otimes 1 \otimes X + X \otimes 1 \otimes 1 \rangle \cong \mathbb{Z}.$$

From Figure 3.6 we see that the integral Khovanov homology of the unknot is the bigraded module  $H(\text{unknot}) \cong \mathbb{Z}_{(0,1)} \oplus \mathbb{Z}_{(0,-1)}$  with one copy of the integers in bigrading  $(0, 1)$  and one in  $(0, -1)$ . Note that, as expected, the graded Euler characteristic yields the unnormalized Jones polynomial of the unknot:

$$\chi_q(H(\text{unknot})) = (-1)^0 \text{qdim}(\mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(-1)}) = q + q^{-1} = \widehat{J}(\text{unknot}).$$



**Figure 3.5** The Kauffman states and Khovanov chain complex for the knot diagram  $D$  of the unknot given in Figure 3.4.

In Sections 3.3-3.4 we introduce the computational tools needed for proving a series of lemmas leading to our main result (Theorem 3.5.6) on torsion in thin regions of Khovanov homology. In Section 3.3 we recall a long exact sequence relating the Khovanov homology of a diagram  $D$  to that of the diagrams  $D_0$  and  $D_1$  obtained by 0-smoothing and 1-smoothing a chosen crossing. In Section 3.4 we derive the Bockstein spectral sequence from an exact couple, and list properties of the Bockstein spectral sequence which are vital to the study of torsion in homology theories. In Section 3.4 we recall double complexes, their total homology groups, and spectral sequences derived from double complexes that converge to the total homology. We then recall a version of Khovanov homology over  $\mathbb{Z}_2$  called Bar-Natan homology, and see how an alternative differential on the  $\mathbb{Z}_2$  Khovanov complex, called the Turner differential, leads to a spectral sequence converging to Bar-Natan homology [Tur06]. We also recall how Shumakovitch relates a vertical differential on the Khovanov complex to both the Turner and Bockstein differentials [Kho00; Shu18]. In Section 3.4 we recall a version of rational Khovanov homology called Lee homology, and see how the Lee differential on the rational Khovanov complex leads to a spectral sequence converging to Lee homology [Lee05].

Given an oriented link diagram  $D$ , there is a long exact sequence relating  $D$  to the zero and one smoothings  $D_0$  and  $D_1$  at a given crossing of  $D$  [Tur17]. If the crossing is negative, we set

$(i, j)$	$\ker(d^{i,j})$	$\text{Im}(d^{i-1,j})$	$H^{i,j}(D)$
$(1, 3)$	$1 \otimes 1$	$1 \otimes 1$	
$(0, 1)$	$1 + 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1$ $1 + 1 \otimes 1 \otimes X + X \otimes 1 \otimes 1$	$1 + 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1$	$1 + 1 \otimes 1 \otimes X + X \otimes 1 \otimes 1$
$(1, 1)$	$X \otimes 1 \quad 1 \otimes X$	$X \otimes 1 \quad 1 \otimes X$	
$(0, -1)$	$X + X \otimes 1 \otimes X + X \otimes X \otimes 1$ $X + 1 \otimes X \otimes X$ $X \otimes X \otimes 1$	$X + X \otimes 1 \otimes X + X \otimes X \otimes 1$ $X + 1 \otimes X \otimes X$	$X \otimes X \otimes 1$
$(0, -3)$	$X \otimes X \otimes X$	$X \otimes X \otimes X$	

**Figure 3.6** Generators for the kernels, images and homology groups for the knot diagram  $D$  of the unknot given in Figure 3.4.

$c = n_-(D_0) - n_-(D)$  to be the number of negative crossings in  $D_0$  minus the number in  $D$ . For each  $j$  there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(D_1) \longrightarrow H^{i,j}(D) \longrightarrow H^{i-c,j-3c-1}(D_0) \xrightarrow{\delta_*} H^{i+1,j+1}(D_1) \longrightarrow \quad (3.2)$$

Note that in this case  $D_1$  inherits an orientation, and  $D_0$  must be given one. Similarly, if the crossing is positive, we set  $c = n_-(D_1) - n_-(D)$  to be the number of negative crossings in  $D_1$  minus the number in  $D$ . For each  $j$  there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i-c-1,j-3c-2}(D_1) \longrightarrow H^{i,j}(D) \longrightarrow H^{i,j-1}(D_0) \xrightarrow{\delta_*} H^{i-c,j-3c-2}(D_1) \longrightarrow \quad (3.3)$$

and note that in this case  $D_0$  inherits an orientation, and  $D_1$  must be given one.

In Theorem 3.6.4, the long exact sequence 3.2 will play an important role in our calculation of the  $\mathbb{Z}_p$  Khovanov homology of a class of links known as 3-braids of type  $\Omega_3$ .

### 3.4 Spectral sequences

Spectral sequences arise in many different contexts. The spectral sequences relevant to our purposes are spectral sequences of bigraded vector spaces arising from exact couples and double complexes.

**Definition 3.4.1** ([McC01]). A differential bigraded vector space is a pair  $(E, d)$  where  $E$  is a vector space with a direct sum decomposition  $E = \bigoplus_{p,q \in \mathbb{Z}} E^{p,q}$ , and  $d : E \rightarrow E$  is a linear map of bidegree  $(s, 1-s)$  for some integer  $s$ , and satisfying  $d \circ d = 0$ .

**Definition 3.4.2** ([McC01]). A spectral sequence is a sequence  $(E_r, d_r)$  of differential bigraded vector spaces such that each  $E_{r+1}$  is obtained as the homology of  $(E_r, d_r)$ . That is,  $E_{r+1}^{p,q} = H^{p,q}(E_r, d_r)$  for each  $p, q, r \in \mathbb{Z}$ .

If  $(E_r, d_r)$  is a spectral sequence of bigraded vector spaces, and for integers  $p$  and  $q$  there is an integer  $k = k(p, q)$  such that  $E_r^{p,q} \cong E_k^{p,q}$  for each  $r \geq k$ , then we let  $E_\infty^{p,q} = E_k^{p,q}$ . If  $E_\infty^{p,q}$  is well-defined for each  $p, q \in \mathbb{Z}$ , we define  $E_\infty = \bigoplus_{p,q \in \mathbb{Z}} E_\infty^{p,q}$ . The vector space  $E_\infty$  is referred to as the limiting page of the spectral sequence. If  $E_\infty = E_N$  for some integer  $N$ , we say the spectral sequence collapses at the  $E_N$  page.

**Definition 3.4.3** ([McC01]). A (decreasing) filtration  $F$  of a vector space  $H$  is a family of subspaces  $\{F^p H\}$  of  $H$  satisfying,

1.  $F^p H \supseteq F^{p+1} H$  for each  $p \in \mathbb{Z}$ , and
2.  $\bigcup_{p \in \mathbb{Z}} F^p H = H$ .

A filtration is bounded if for each integer  $n$  there exist integers  $s = s(n)$  and  $t = t(n)$  such that

$$H^n = F^s H^n \supseteq F^{s+1} H^n \supseteq \dots \supseteq F^{t-1} H^n \supseteq F^t H^n = \{0\}.$$

A filtered vector space is a pair  $(H, F)$  where  $H$  is a vector space and  $F$  is a filtration on  $H$ . To a filtered vector space  $(H, F)$ , we associate the graded vector space  $E_0 = E_0(H, F)$ , whose homogeneous components are the quotient spaces

$$E_0^p(H, F) = \frac{F^p H}{F^{p+1} H}.$$

If  $H = \bigoplus_{n \in \mathbb{Z}} H^n$  is a graded vector space, then defining  $F^p H^q = F^p H \cap H^q$  yields a bigrading on  $E_0$ :

$$E_0^{p,q}(H, F) = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

The bigraded vector space  $E_0$  is referred to as the associated bigraded vector space for the filtered graded vector space  $(H, F)$ .

**Definition 3.4.4.** A spectral sequence  $(E_r, d_r)$  of bigraded vector spaces is said to converge to the graded vector space  $H = \bigoplus_{n \in \mathbb{Z}} H^n$  if there is a filtration  $F$  of  $H$  such that

$$E_\infty^{p,q} \cong E_0^{p,q}(H, F)$$

for all integers  $p, q \in \mathbb{Z}$ . In such case we write  $E_r \Rightarrow H$ .

The important point here is that a (usually unknown) graded vector space  $H$  can be recovered up to isomorphism from a spectral sequence  $E_r$  converging to  $H$  via

$$H^n \cong \bigoplus_{p \in \mathbb{Z}} E_{\infty}^{p, n-p}.$$

Now that know what spectral sequences are and what it means for a spectral sequence to converge to a graded vector space, let us introduce two situations from which they arise—exact couples and double complexes—and discuss the convergence properties of these particular constructions.

**Definition 3.4.5** ([McC01]). *Let  $D_1$  and  $E_1$  be bigraded vector spaces, and let  $i_1 : D_1 \rightarrow D_1$ ,  $j_1 : D_1 \rightarrow E_1$  and  $k_1 : E_1 \rightarrow D_1$  be linear maps satisfying of bidegrees  $(-1, 1)$ ,  $(0, 0)$  and  $(1, 0)$ , respectively. Suppose further that  $\ker(i_1) = \text{im}(k_1)$ ,  $\ker(j_1) = \text{im}(i_1)$  and  $\ker(k_1) = \text{im}(j_1)$ . The pentuple  $(D_1, E_1, i_1, j_1, k_1)$  is called an exact couple. Succinctly, an exact couple is an exact diagram of vector spaces and linear maps of the following form.*

$$\begin{array}{ccc} D_1 & \xrightarrow{i_1} & D_1 \\ & \swarrow k_1 \quad \searrow j_1 & \\ & E_1 & \end{array}$$

Given an exact couple, we obtain a spectral sequence by the following iterative process. The map  $d_1 : E_1 \rightarrow E_1$  defined by  $d_1 := j_1 \circ k_1$  satisfies  $d_1 \circ d_1 = (j_1 \circ k_1) \circ (j_1 \circ k_1) = j_1 \circ (k_1 \circ j_1) \circ k_1 = 0$ , thus defining a differential on  $E_1$ . Define  $E_2 = H(E_1, d_1)$  to be the homology of the pair  $(E_1, d_1)$ , and define  $D_2 = \text{im}(i_1 : D_1 \rightarrow D_1)$ . Also, define maps  $i_2 : D_2 \rightarrow D_2$ ,  $j_2 : D_2 \rightarrow E_2$  and  $k_2 : E_2 \rightarrow D_2$  by  $i_2 = i_1|_{D_2}$ ,  $j_2(i_1(a)) = j_1(a) + d_1 E_1$  and  $k_2(e + d_1 E_1) = k_1(e)$ . The resulting pentuple  $(D_2, E_2, i_2, j_2, k_2)$  is another exact couple, called the derived exact couple. Setting  $d_2 = j_2 \circ k_2$  yields a differential bigraded vector space  $(E_2, d_2)$ . Repeatedly taking the derived couple, setting  $E_r = H(E_{r-1}, d_{r-1})$  and  $d_r = j_r \circ k_r$  yields a spectral sequences  $(E_r, d_r)$  of bigraded vector spaces.

Our key tool for studying torsion in Khovanov homology is the Bockstein spectral sequence, which arises from an exact couple. For each prime  $p$ , there is a spectral sequence known as the  $\mathbb{Z}_p$  Bockstein spectral sequence. A key attribute of the  $\mathbb{Z}_p$  Bockstein spectral sequence is its ability to proffer information regarding torsion of the form  $\mathbb{Z}_{p^r}$  in homology theories. Given a (co)chain complex  $(C, d)$ , the  $\mathbb{Z}_p$  Bockstein spectral sequence  $(E_B^r, d_B^r)_{r \geq 1}$  (to be defined shortly) enjoys the following properties [McC01].

(B1) The first page is  $(E_B^1, d_B^1) = (H(C; \mathbb{Z}_p), \beta)$ , where  $\beta = (\text{red } p) \circ \partial$ .

- (B2) The limiting page is  $E_B^\infty = (H(C)/\text{Torsion}) \otimes \mathbb{Z}_p$ .
- (B3) If the Bockstein spectral sequence collapses on the  $E^k$  page for a particular bigrading  $(i, j)$ , that is, if  $(E_B^k)^{i,j} = (E_B^\infty)^{i,j}$ , then  $H^{i,j}(C)$  contains no  $\mathbb{Z}_{p^r}$  torsion for  $r \geq k$ .
- (B4) If  $C = C^{*,*}$  is a bigraded complex with differential of degree 0, then each differential  $d_B^r$  has bidegree  $(1, 0)$ .

The construction of the Bockstein spectral proceeds as follows [McC01]. Given a (co)chain complex  $(C, d)$ , let us denote its integral homology by  $H(C)$  and its homology over coefficients in  $\mathbb{Z}_p$  by  $H(C; \mathbb{Z}_p)$ . Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\text{red } p} \mathbb{Z}_p \longrightarrow 0 \quad (3.4)$$

where  $\times p$  is multiplication by  $p$ , and  $\text{red } p$  is reduction modulo  $p$ . Tensoring (3.4) with the cochain complex  $(C, d)$  yields a short exact sequence of cochain complexes

$$0 \longrightarrow C \xrightarrow{\times p} C \xrightarrow{\text{red } p} C \otimes \mathbb{Z}_p \longrightarrow 0.$$

The associated long exact sequence in homology [Hat02] can be viewed as an exact couple

$$\begin{array}{ccc} H(C) & \xrightarrow{\quad \times p \quad} & H(C) \\ & \searrow \partial & \swarrow \text{red } p \\ & H(C; \mathbb{Z}_p) & \end{array} \quad (3.5)$$

where  $\partial$  is the connecting homomorphism.

Let us now describe double complexes and their associated spectral sequences.

**Definition 3.4.6.** Let  $M = \bigoplus_{p,q} M^{p,q}$  be a bigraded vector space. Let  $d' : M \rightarrow M$  and  $d'' : M \rightarrow M$  be linear maps of bidegrees  $(1, 0)$  and  $(0, 1)$ , respectively. Suppose further that  $d' \circ d' = 0$ ,  $d'' \circ d'' = 0$  and  $d' \circ d'' + d'' \circ d' = 0$ . Then, the triple  $(M, d', d'')$  is called a double complex.

A double complex gives rise to a differential graded vector space, called the total complex, whose homology is called the total homology of the double complex. Next, we describe this total complex and see how its total homology can be computed using a spectral sequence associated to the double complex.

**Definition 3.4.7.** The total complex of a double complex  $(M, d', d'')$  is the differential graded vector space  $(\text{Tot}(M), d)$  with homogeneous components

$$\text{Tot}^n(M) = \bigoplus_{p \in \mathbb{Z}} M^{p, n-p}$$

and differential  $d : \text{Tot}^n(M) \rightarrow \text{Tot}^{n+1}(M)$  defined by  $d = d' + d''$ .

In other words, the total complex is obtained by "flattening" (taking direct sums) along diagonals  $q = n - p$ . That  $d$  is indeed a differential follows from the calculation

$$d^2 = (d' + d'')^2 = d' \circ d' + (d' \circ d'' + d'' \circ d') + d'' \circ d'' = 0 + 0 + 0 = 0.$$

A double complex  $(M, d', d'')$  gives rise to a spectral sequence of bigraded vector spaces as follows. Let  $(E_1, d_1) = (H(M, d'), d''_*)$  be the bigraded vector space obtained as the homology of  $(E_1, d_1)$  equipped with differential obtained as the map on homology induced by  $d''$ . Next, let  $E_2 = H(E_1, d_1)$  be the homology of  $(E_1, d_1)$ , equipped with a differential  $d_2 : E_2 \rightarrow E_2$  defined via the diagram chase as given in [McC01]. Iterating this process yields a spectral sequence  $(E_r, d_r)$  of bigraded vector spaces. The decreasing filtration on  $\text{Tot}(M)$  given by  $F^p \text{Tot}^n(M) = \bigoplus_{r \geq p} M^{r, n-r}$  descends to a filtration on  $E_1 = H(M, d')$ . If  $M^{p,q} = 0$  for  $p, q < 0$ , then this filtration is bounded. Consequently, the spectral sequence arising in this manner converges to the total homology  $H(\text{Tot}(M), d)$  [McC01]. By definition 3.4.4, this means

$$E_{\infty}^{p,q} \cong \frac{F^p H^{p+q}(\text{Tot}(M), d)}{F^{p+1} H^{p+q}(\text{Tot}(M), d)}.$$

We then recover the total homology  $H(\text{Tot}(M), d)$  up to isomorphism via

$$H^n(\text{Tot}(M)) \cong \bigoplus_{p+q=n} E_{\infty}^{p,q} = \bigoplus_{p \in \mathbb{Z}} E_{\infty}^{p, n-p}.$$

**Remark 3.4.8.** *In our definition of a double complex, our differentials were assumed to be of bidegrees  $(1, 0)$  and  $(0, 1)$ . However, this construction goes through equally well for differentials of any bidegree. For example, for a double complex  $(M, d', d'')$  with differentials of bidegrees  $(1, 0)$  and  $(1, 2)$  (or, more generally, bidegrees  $(1, 0)$  and  $(1, 2k)$  for some integer  $k$ ), we recover the total homology by taking direct sums along columns of the limiting page:*

$$H^n(\text{Tot}(M)) \cong \bigoplus_{s \in \mathbb{Z}} E_{\infty}^{n,s}$$

*We highlight these particular examples because we next deal with a double complex  $(C(D; \mathbb{Z}_2), d, d_T)$  whose differentials are of bidegrees  $(1, 0)$  and  $(1, 2)$ , and a double complex  $(C(D; \mathbb{F}), d, d_L)$  whose differentials are of bidegrees  $(1, 0)$  and  $(1, 4)$ .*

In [BN05] Bar-Natan defines a variant of  $\mathbb{Z}_2$  Khovanov homology, commonly referred to as Bar-Natan homology. The construction of Bar-Natan mimics the construction of Khovanov homology given in Section 3.3, but with differential defined instead via the multiplication and comultiplication

$$m_B(1 \otimes 1) = 1, \quad m_B(1 \otimes X) = X, \quad m_B(X \otimes 1) = X, \quad m_B(X \otimes X) = X,$$

$$\Delta_B(1) = 1 \otimes X + X \otimes 1 + 1 \otimes 1, \quad \Delta_B(X) = X \otimes X.$$

The resulting singly graded homology theory is a link invariant [BN05] denoted  $BN^*(L)$ . Turner shows the following in [Tur06].

**Theorem 3.4.9** ([Tur06]). *Let  $L$  be an oriented link with  $k$  components. Then,  $\dim_{\mathbb{Z}_2}(BN^*(L)) = 2^k$ . Specifically, if the components of  $L$  are  $L_1, L_2, \dots, L_k$ , then*

$$\dim_{\mathbb{Z}_2}(BN^i(L)) = \# \left\{ E \subset \{1, 2, \dots, k\} \mid \sum_{\ell \in E, m \in E^c} 2 \cdot lk(L_\ell, L_m) = i \right\}$$

where  $lk(L_\ell, L_m)$  is the linking number between  $L_\ell$  and  $L_m$ .

**Example 3.4.10.** *Let  $T(3, q)$  be the  $(3, q)$ -torus link. That is,  $T(3, 3n)$  is the closure of the braid  $\Delta^{2n}$ ,  $T(3, 3n+1)$  is the closure of the braid  $\Delta^{2n}\sigma_1\sigma_2$  and  $T(3, 3n+2)$  is the closure of  $\Delta^{2n}(\sigma_1\sigma_2)^2$ , each oriented so that crossings in  $\Delta$  are negative crossings.*

1. *The torus link  $T(3, 3n)$  has three components and  $BN^0(T(3, 3n)) = \mathbb{Z}_2^2$ ,  $BN^{-4n}(T(3, 3n)) = \mathbb{Z}_2^6$  and  $BN^i(T(3, 3n)) = 0$  for  $i \neq 0, -4n$ .*
2. *The torus knot  $T(3, 3n+1)$  satisfies  $BN^0(T(3, 3n+1)) = \mathbb{Z}_2^2$  and  $BN^i(T(3, 3n+1)) = 0$  for  $i \neq 0$ .*
3. *The torus knot  $T(3, 3n-1)$  satisfies  $BN^0(T(3, 3n-1)) = \mathbb{Z}_2^2$  and  $BN^i(T(3, 3n-1)) = 0$  for  $i \neq 0$ .*

In [Tur06] Turner defines a map  $d_T$  on the  $\mathbb{Z}_2$  Khovanov complex in the same manner as the Khovanov differential  $d$ , but with multiplication and comultiplication given by

$$m_T(1 \otimes 1) = 0, \quad m_T(1 \otimes X) = 0, \quad m_T(X \otimes 1) = 0, \quad m_T(X \otimes X) = X,$$

$$\Delta_T(1) = 1 \otimes 1, \quad \Delta_T(X) = 0.$$

and the map  $d_T$  satisfies  $d_T^2 = 0$  and  $d \circ d_T + d_T \circ d = 0$ .

**Definition 3.4.11** ([Tur06]). *Let  $D$  be a diagram of a link  $L$  and  $(C(D; \mathbb{Z}_2), d)$  be the  $\mathbb{Z}_2$  Khovanov complex. The spectral sequence  $(E_T^r, d_T^r)_{r \geq 1}$  associated to the double complex  $(C(D; \mathbb{Z}_2), d, d_T)$  is called the Turner spectral sequence.*

The Turner spectral sequence satisfies the following properties [Tur06].

- (T1) The first page is  $(E_T^1, d_T^1) = (H(L; \mathbb{Z}_2), d_T^*)$  where  $d_T^* : H(L; \mathbb{Z}_2) \rightarrow H(L; \mathbb{Z}_2)$  is the induced map on homology.
- (T2) Each map  $d_T^r$  is a differential of bidegree  $(1, 2r)$ .

(T3) The Turner spectral sequence converges to Bar-Natan homology:  $E_T^r \implies BN^*(L)$ . In light of Remark 3.4.8, this simply means that taking the direct sum along the  $i$ -th column of the limiting page yields  $BN^i(L)$ .

Our next tool is a ‘vertical’ differential on the  $\mathbb{Z}_2$  Khovanov complex due to Khovanov [Kho00]. Let  $D$  be a diagram of a link  $L$ . A differential  $\nu : C(D; \mathbb{Z}_2) \rightarrow C(D; \mathbb{Z}_2)$  is defined as follows. Recall that for a Kauffman state  $D(I)$ , the algebra  $C(D(I))$  has  $2^{|D(I)|}$  generators of the form  $a_1 \otimes a_2 \otimes \cdots \otimes a_{|D(I)|}$  where  $a_i \in \{1, X\}$  and  $|D(I)|$  is the number of circles in  $D(I)$ . For each Kauffman state we define a map  $\nu_{D(I)} : C(D(I)) \rightarrow C(D(I))$  by sending a generator to the sum of all possible generators obtained by replacing a single  $X$  with a  $1$ . For example,  $X \otimes X \otimes X \mapsto 1 \otimes X \otimes X + X \otimes 1 \otimes X + X \otimes X \otimes 1$ . We then extend  $\nu_{D(I)}$  linearly to all of  $C(D(I))$ , and then to a map  $\nu : C(D; \mathbb{Z}_2) \rightarrow C(D; \mathbb{Z}_2)$ . The properties of this map relevant to our purposes are given below, the proofs of which can be found in Shumakovitch’s [Shu18].

(V1) The map  $\nu$  is a differential of bidegree  $(0, 2)$ .

(V2) The map  $\nu$  commutes with the Khovanov differential  $d$ , and so induces a map (differential)  $\nu^* : H(L; \mathbb{Z}_2) \rightarrow H(L; \mathbb{Z}_2)$  on homology.

(V3) The complex  $(H(L; \mathbb{Z}_2), \nu^*)$  is acyclic, that is, it has trivial homology.

The following lemma of Shumakovitch is the key to proving Theorem 3.5.6. It relates the first  $\mathbb{Z}_2$  Bockstein map with the Turner and vertical differentials. We use the behavior of the Turner spectral sequence and the acyclic homology induced by  $\nu$  to determine when the  $\mathbb{Z}_2$  Bockstein spectral sequence collapses.

**Lemma 3.4.12** ([Shu18]). *Let  $L$  be a link. The Bockstein, Turner and vertical differentials on the  $\mathbb{Z}_2$  Khovanov homology  $H(L; \mathbb{Z}_2)$  of  $L$  are related by  $d_T^* = d_B^1 \circ \nu^* + \nu^* \circ d_B^1$ .*

In [Lee05], Lee defines a variant of  $\mathbb{F}$ -Khovanov homology where  $\mathbb{F}$  is either  $\mathbb{Q}$  or  $\mathbb{Z}_p$  where  $p$  is an odd prime. The Lee differential  $d_L : C(D; \mathbb{F}) \rightarrow C(D; \mathbb{F})$  is defined in the same way as the Khovanov differential  $d$ , but with multiplication and comultiplication given by

$$\begin{aligned} m_L(1 \otimes 1) &= 0, & m_L(1 \otimes X) &= 0, & m_L(X \otimes 1) &= 0, & m_L(X \otimes X) &= 1 \\ \Delta_L(1) &= 0, & \Delta_L(X) &= 1 \otimes 1. \end{aligned}$$

The map  $d_L$  satisfies  $d_L^2 = 0$  and  $d \circ d_L + d_L \circ d = 0$ . The total homology of the double complex  $(C(D; \mathbb{F}), d, d_L)$  is a singly graded  $\mathbb{F}$ -vector space known as Lee homology, denoted  $Lee^*(L)$ . Lee shows the following in [Lee05].

**Theorem 3.4.13** ([Lee05]). *Let  $L$  be an oriented link with  $k$  components. Then,  $\dim_{\mathbb{F}}(\text{Lee}^*(L)) = 2^k$ . Specifically, if the components of  $L$  are  $L_1, L_2, \dots, L_k$ , then*

$$\dim_{\mathbb{F}}(\text{Lee}^i(L)) = \# \left\{ E \subset \{1, 2, \dots, k\} \mid \sum_{\ell \in E, m \in E^c} 2 \cdot lk(L_\ell, L_m) = i \right\}$$

where  $lk(L_\ell, L_m)$  is the linking number between  $L_\ell$  and  $L_m$ .

**Definition 3.4.14.** *Let  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}_p$ ,  $p$  and odd prime. Let  $D$  be a diagram of a link  $L$  and  $(C(D; \mathbb{F}), d)$  be the  $\mathbb{F}$ -Khovanov complex. The spectral sequence  $(E_L^r, d_L^r)_{r \geq 1}$  associated to the double complex  $(C(D; \mathbb{F}), d, d_L)$  is called the  $\mathbb{F}$  Lee spectral sequence.*

The  $\mathbb{F}$  Lee spectral sequence satisfies the following properties [Lee05].

- (L1) The first page is  $(E_L^1, d_L^1) = (H(L; \mathbb{F}), d_L^*)$  where  $d_L^* : H(L; \mathbb{F}) \rightarrow H(L; \mathbb{F})$  is the induced map on homology.
- (L2) Each map  $d_L^r$  is a differential of bidegree  $(1, 4r)$ .
- (L3) The Lee spectral sequence converges to Lee homology:  $E_L^r \implies \text{Lee}^*(L)$ . In light of (L2) and Remark 3.4.8, this simply means that taking the direct sum along the  $i$ -th column of the limiting page yields  $\text{Lee}^i(L)$ .

### 3.5 The main result: Torsion in thin regions of Khovanov homology

In this section we prove our main result, Theorem 3.5.6, providing conditions under which all torsion in certain thin regions of Khovanov homology is  $\mathbb{Z}_2$  torsion. This result extends Theorem 3.5.1 due to Shumakovitch.

**Theorem 3.5.1** ([Shu18]). *If  $L$  is a link whose integral Khovanov homology  $H(L)$  is supported on two adjacent diagonals  $j - 2i = s \pm 1$ , then all torsion in  $H(L)$  is  $\mathbb{Z}_2$  torsion.*

A link  $L$  with integral Khovanov homology supported on two adjacent diagonals is said to be homologically thin. A local version of homological thinness goes as follows.

**Definition 3.5.2.** *Let  $i_1$  and  $i_2$  be integers. For a link  $L$ , let  $H(L; R)^{[i_1, i_2]}$  denote the direct sum*

$$H^{[i_1, i_2]}(L; R) = \bigoplus_{i=i_1}^{i_2} H^{i,*}(L; R).$$

*We say that  $L$  is thin over the interval  $[i_1, i_2]$  if there is an integer  $s$  such that  $H^{[i_1, i_2]}(L; \mathbb{Z}_p)$  is supported in bigradings  $(i, j)$  with  $j - 2i = s \pm 1$  for every prime  $p$ .*

We lead up to the proof of Theorem 3.5.6 by proving several lemmas. Throughout their proofs, we take advantage of the properties of the  $\mathbb{Z}_2$ -Bockstein spectral sequence, Turner spectral sequence, Lee spectral sequence and vertical differential  $\mathcal{V}^*$ , described in Section 3.4.

Our first lemma states that for a link which is thin over  $[i_1, i_2]$ , all torsion in homological gradings  $(i_1, i_2]$  must be supported on the lower diagonal.

**Lemma 3.5.3** ([CLSS19]). *If  $H(L)$  is thin over  $[i_1, i_2]$  where  $H^{[i_1, i_2]}(L; \mathbb{Z})$  is supported in bigradings  $(i, j)$  with  $2i - j = s \pm 1$  for some  $s \in \mathbb{Z}$ , then any torsion summand of  $H^{[i_1, i_2]}(L; \mathbb{Z})$  with homological gradings  $i > i_1$  is supported on the lower diagonal in bigrading  $(i, 2i - s - 1)$ .*

*Proof.* If  $H^{i, 2i-s+1}(L)$  has a nontrivial torsion summand for some  $i \in (i_1, i_2]$ , then the universal coefficient theorem implies that  $H^{i-1, 2i-s+1}(L; \mathbb{Z}_p)$  is nontrivial for some  $p$ , contradicting the fact that  $H(L)$  is locally thin over  $[i_1, i_2]$ . Therefore all torsion in homological gradings  $(i_1, i_2]$  appears in bigradings of the form  $(i, 2i - s - 1)$ .  $\square$

Our next lemma gives a condition to ensure that  $H^{[i_1, i_2]}(L; \mathbb{Z})$  has no odd torsion.

**Lemma 3.5.4** ([CLSS19]). *Suppose that  $H(L)$  is locally thin over  $[i_1, i_2]$  and that*

$$\dim_{\mathbb{Q}} H^{i_1, *}(L; \mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1, *}(L; \mathbb{Z}_p)$$

*for each odd prime  $p$ . Then  $H^{[i_1, i_2]}(L; \mathbb{Z})$  contains no torsion of odd order.*

*Proof.* We show that if  $i \in [i_1, i_2]$ , then there cannot be any torsion in  $H^{i, *}(L)$  of the form  $\mathbb{Z}_{p^r}$  for an odd prime  $p$ . By way of contradiction suppose that for some  $i \in [i_1, i_2]$ , the group  $H^{i, 2i-s-1}(L)$  contains a torsion summand of the form  $\mathbb{Z}_{p^r}$ . The universal coefficient theorem implies that

$$\begin{aligned} \dim_{\mathbb{Q}} H^{i, 2i-s-1}(L; \mathbb{Q}) &< \dim_{\mathbb{Z}_p} H^{i, 2i-s-1}(L; \mathbb{Z}_p) \text{ and} \\ \dim_{\mathbb{Q}} H^{i-1, 2i-s-1}(L; \mathbb{Q}) &< \dim_{\mathbb{Z}_p} H^{i-1, 2i-s-1}(L; \mathbb{Z}_p). \end{aligned}$$

Since  $H(L)$  is thin over  $[i_1, i_2]$ , Theorem 3.4.13 implies the infinity pages of the  $\mathbb{Q}$  and  $\mathbb{Z}_p$  spectral sequences have the same dimension in each bigrading  $(i, 2i - s \pm 1)$  for  $i \in [i_1, i_2]$ . Because the Lee map is of bidegree  $(1, 4)$ , if  $i_1 < i - 1$ ,

$$\dim_{\mathbb{Q}} H^{i-2, 2i-s-5}(L; \mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i-2, 2i-s-5}(L; \mathbb{Z}_p).$$

Since  $H(L)$  is thin over  $[i_1, i_2]$ , it follows that  $H^{i-1, 2i-s-5}(L) = 0$ , and in particular, there is no  $\mathbb{Z}_{p^r}$  torsion summand in bigrading  $(i-1, 2i-s-5)$ . Thus the universal coefficient theorem implies there is a  $\mathbb{Z}_{p^r}$  torsion summand in bigrading  $(i-2, 2i-s-5)$ . A  $\mathbb{Z}_{p^r}$  torsion summand in homological grading  $i$  induces a dimension inequality in homological gradings  $i, i-1$ , and  $i-2$ , and it induces

another  $\mathbb{Z}_{p^r}$  summand in homological grading  $i - 2$ . Since  $i \in [i_1, i_2]$  is arbitrary, repeating this argument for each new  $\mathbb{Z}_{p^r}$  summand implies that

$$\dim_{\mathbb{Q}} H^{i_1,*}(L; \mathbb{Q}) < \dim_{\mathbb{Z}_p} H^{i_1,*}(L; \mathbb{Z}_p).$$

which is a contradiction.  $\square$

Assume that  $H^{[i_1, i_2]}(L; \mathbb{Z})$  contains no torsion of odd order. If  $R = \mathbb{Q}$  or  $\mathbb{Z}_2$ , then  $H^{[i_1, i_2]}(L; R)$  can be decomposed as

$$H^{[i_1, i_2]}(L; R) = H_{<\infty}^{[i_1, i_2]}(L; R) \oplus H_{\infty}^{[i_1, i_2]}(L; R)$$

where  $H_{\infty}^{[i_1, i_2]}(L; R)$  is the submodule that survives to the  $\infty$  page of the Lee or Turner spectral sequence when  $R = \mathbb{Q}$  or  $\mathbb{Z}_2$  respectively. If, in addition,  $H(L)$  is thin on  $[i_1, i_2]$ , then Lemmas 3.4.9 and 3.4.13 imply that for any  $i \in [i_1, i_2]$ ,

$$\begin{aligned} \dim_{\mathbb{Z}_2} H_{\infty}^{i, 2i-s-1}(L; \mathbb{Z}_2) &= \dim_{\mathbb{Q}} H_{\infty}^{i, 2i-s-1}(L; \mathbb{Q}) \text{ and} \\ \dim_{\mathbb{Z}_2} H_{\infty}^{i, 2i-s+1}(L; \mathbb{Z}_2) &= \dim_{\mathbb{Q}} H_{\infty}^{i, 2i-s+1}(L; \mathbb{Q}). \end{aligned} \tag{3.6}$$

Each  $\mathbb{Q}$  summand in  $H(L; \mathbb{Q})$  corresponds to a  $\mathbb{Z}$  summand in  $H(L)$ . Define  $H_{\infty}^{[i_1, i_2]}(L)$  to be the direct sum of the  $\mathbb{Z}$  summands corresponding to the  $\mathbb{Q}$  summands in  $H_{\infty}^{[i_1, i_2]}(L; \mathbb{Q})$ . Also, define  $H_{<\infty}^{[i_1, i_2]}(L)$  to be the direct sum of all torsion in  $H^{[i_1, i_2]}(L)$  with the  $\mathbb{Z}$  summands corresponding to the  $\mathbb{Q}$  summands in  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Q})$ . Then

$$H^{[i_1, i_2]}(L) = H_{<\infty}^{[i_1, i_2]}(L) \oplus H_{\infty}^{[i_1, i_2]}(L).$$

Suppose that  $H^{[i_1, i_2]}(L)$  is supported in bigradings  $(i, j)$  where  $2i - j = s \pm 1$  for some integer  $s$ . The Lee spectral sequence implies that if  $i < i_2$ , then a  $\mathbb{Q}$  summand in bigrading  $(i, 2i - s - 1)$  of  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Q})$  is paired with a  $\mathbb{Q}$  summand in bigrading  $(i + 1, 2i - s + 3)$  of  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Q})$  (and vice versa). We call such pairs *knight move pairs*. The vertical differential implies that for all  $i \in [i_1, i_2]$ , a  $\mathbb{Z}_2$  summand in bigrading  $(i, 2i - s - 1)$  of  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Z}_2)$  is paired with a  $\mathbb{Z}_2$  summand in bigrading  $(i, 2i - s + 1)$  of  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Z}_2)$  (and vice versa). We call such pairs *pawn move pairs*.

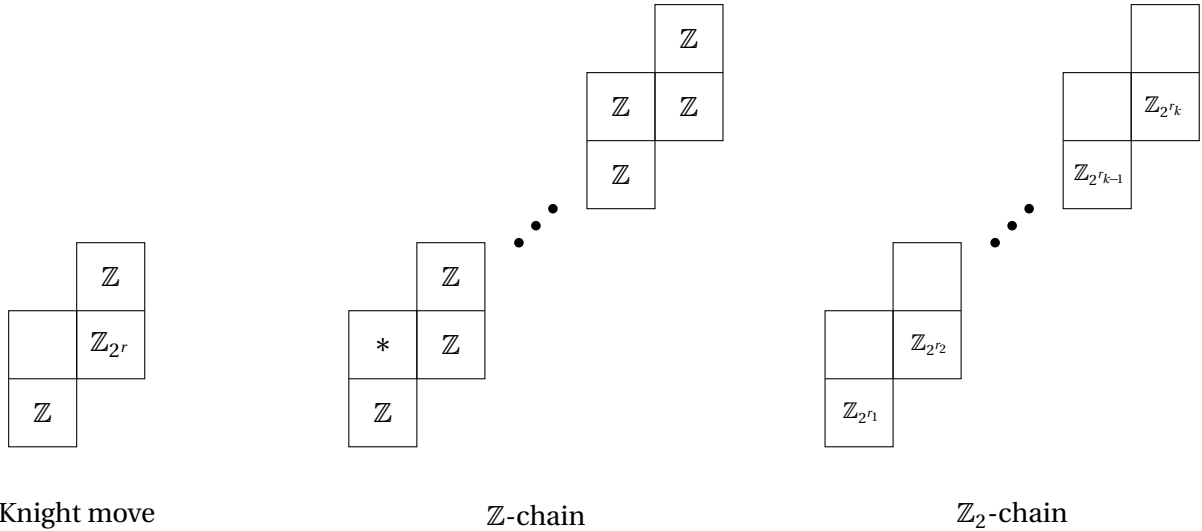
A submodule  $S$  of  $H_{<\infty}^{[i_1, i_2]}(L)$  is *closed under the Lee and vertical differentials* (or LV-closed) if it satisfies the following.

- If  $i \in [i_1, i_2]$ , then  $\text{rank}(S^{i, 2i-s-1}) = \text{rank}(S^{i+1, 2i-s+3})$ .
- If  $i \in [i_1, i_2]$ , then  $\text{rank}(S^{i, 2i-s-1}) + \text{rank}_{\mathbb{Z}_2}(S^{i, 2i-s-1}) = \text{rank}(S^{i, 2i-s+1}) + \text{rank}_{\mathbb{Z}_2}(S^{i+1, 2i-s+1})$ .

Informally, a submodule  $S$  is LV-closed if it induces  $\mathbb{Q}$  summands in  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Q})$  that come in knight move pairs (or partial knight move pairs in the cases that  $i = i_1$  or  $i_2$ ) and if it induces  $\mathbb{Z}_2$  summands

in  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Z}_2)$  that comes in pawn move pairs. A submodule of  $H_{<\infty}^{[i_1, i_2]}(L)$  is *LV-indecomposable* if it is LV-closed and is not the direct sum of two proper LV-closed submodules.

A *knight move* submodule consists of two  $\mathbb{Z}$  summands in bigradings  $(i, 2i - s - 1)$  and  $(i + 1, 2i - s + 3)$  and a  $\mathbb{Z}_{2^r}$  summand in bigrading  $(i + 1, 2i - s + 1)$  where  $i < i_2$ . An  $i_1$ -*knight move* submodule consists of a  $\mathbb{Z}$  summand in bigrading  $(i_1, 2i_1 - s + 1)$  and a  $\mathbb{Z}_{2^r}$  summand in bigrading  $(i_1, 2i_1 - s - 1)$ . An  $i_2$ -*knight move* submodule consists of a  $\mathbb{Z}$  summand in bigrading  $(i_2, 2i_2 - s - 1)$ . Of course, an  $i_1$ -knight move is just the summands of the knight move occurring in the larger homological grading, but supported in homological grading  $i_1$ . Similarly, an  $i_2$ -knight move is just the summand of the knight move occurring in the smaller homological grading, but supported in homological grading  $i_2$ . A  $\mathbb{Z}$ -*chain* submodule consists of  $\mathbb{Z}$  summands in every bigrading  $(i, 2i - s \pm 1)$  except in bigrading  $(i_1, 2i_1 - s + 1)$  where it contains either a  $\mathbb{Z}$  or a  $\mathbb{Z}_{2^r}$  summand. A  $\mathbb{Z}_2$ -*chain* submodule consists of  $\mathbb{Z}_{2^{r_i}}$  summands in bigradings  $(i, 2i - s - 1)$  for  $i \in [i_1, i_2]$ . Knight move,  $i_1$ -knight move,  $i_2$ -knight move,  $\mathbb{Z}$ -chain, and  $\mathbb{Z}_2$ -chain submodules are all LV-indecomposable. See Figure 3.7 for their depictions.



**Figure 3.7** LV-indecomposable submodules of  $H_{<\infty}^{[i,j]}(L)$ .

**Lemma 3.5.5** ([CLSS19]). *If  $H(L)$  is thin over  $[i_1, i_2]$  and  $H^{[i_1, i_2]}(L)$  contains no torsion of odd order, then  $H_{\infty}^{[i_1, i_2]}(L)$  is a direct sum of knight move,  $i_1$ -knight move,  $i_2$ -knight move,  $\mathbb{Z}$ -chain, and  $\mathbb{Z}_2$ -chain submodules.*

*Proof.* By construction,  $H_{\infty}^{[i_1, i_2]}(L)$  is a direct sum of LV-indecomposable submodules. Thus it remains to show that knight move,  $i_1$ -knight move,  $i_2$ -knight move,  $\mathbb{Z}$ -chain, and  $Z_2$ -chain submodules are

the only LV-indecomposable submodules.

Let  $M$  be an LV-indecomposable submodule that contains a  $\mathbb{Z}$  summand on the upper diagonal in bigrading  $(i, 2i - s + 1)$  for some  $i \in [i_1, i_2]$ . If  $i > i_1$ , the Lee spectral sequence then implies that  $M$  contains a  $\mathbb{Z}$  summand in bigrading  $(i - 1, 2i - s - 3)$ . The vertical differential implies that  $M$  must contain either a  $\mathbb{Z}$  summand or  $\mathbb{Z}_{2^r}$  summand in bigrading  $(i, 2i - s - 1)$ . If  $M$  contains a  $\mathbb{Z}_{2^r}$  summand in bigrading  $(i, 2i - s - 1)$ , then these three summands make it LV-closed, and hence it is the knight move submodule. If  $i = i_1$ , then the  $\mathbb{Z}$  and  $\mathbb{Z}_{2^r}$  summands in bigradings  $(i_1, 2i_1 - s \pm 1)$  form an  $i_1$ -knight move. Suppose, on the other hand,  $M$  contains a  $\mathbb{Z}$  summand in bigrading  $(i, 2i - s - 1)$ . Now the vertical differential implies that the  $\mathbb{Z}$  summand in bigrading  $(i - 1, 2i - s - 3)$  needs to pair with something in bigrading  $(i - 1, 2i - s - 1)$ . Since bigrading  $(i - 1, 2i - s - 1)$  is on the upper diagonal, it cannot be a torsion summand unless  $i - 1 = i_1$ . The Lee differential also implies that there is a  $\mathbb{Z}$  summand on the upper diagonal in bigrading  $(i + 1, 2i - s + 3)$ . Each of these  $\mathbb{Z}$  summands on the upper diagonal must induce a summand immediately beneath it on the lower diagonal due to the vertical differential. None of these lower diagonal summands can be  $\mathbb{Z}_{2^r}$  summands because then  $M$  would have a knight move submodule as a proper submodule, making  $M$  LV-decomposable. The resulting submodule structure is thus necessarily a  $\mathbb{Z}$ -chain.

Now let  $M$  be an LV-indecomposable submodule that contains a  $\mathbb{Z}_{2^r}$  summand on the lower diagonal in bigrading  $(i, 2i - s - 1)$  for some  $i \in [i_1, i_2]$ . The vertical differential implies that there is a  $\mathbb{Z}$  summand in bigrading  $(i, 2i - s + 1)$  or a  $\mathbb{Z}_{2^{r'}}$  summand in bigrading  $(i + 1, 2i - s + 1)$ . If there is a  $\mathbb{Z}$  summand in bigrading  $(i, 2i - s + 1)$ , then the Lee spectral sequence implies there is a  $\mathbb{Z}$  summand in bigrading  $(i - 1, 2i - s - 3)$ , resulting in a knight move submodule. If there is a  $\mathbb{Z}_{2^{r'}}$  summand in bigrading  $(i + 1, 2i - s + 1)$ , then we repeat this argument. No  $\mathbb{Z}$  summand can be produced on the upper diagonal in this process because it would induce a knight move submodule of  $M$  contradicting that  $M$  is LV-indecomposable. The resulting submodule structure is thus necessarily a  $\mathbb{Z}_2$ -chain.

Finally, if  $M$  contains a  $\mathbb{Z}$  summand in bigrading  $(i_2, 2i_2 - s - 1)$  and nothing in bigrading  $(i_2, 2i_2 - s + 1)$ , then  $M$  is LV-indecomposable and is an  $i_2$ -knight move.  $\square$

We can now use Lemmas 3.5.3, 3.5.4, and 3.5.5 to prove our main theorem.

**Theorem 3.5.6** ([CLSS19]). *Suppose that a link  $L$  satisfies:*

1.  $L$  is locally thin over  $[i_1, i_2]$  for integers  $i_1$  and  $i_2$ ,
2.  $\dim_{\mathbb{Q}} H^{i,*}(L; \mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i,*}(L; \mathbb{Z}_p)$  for each odd prime  $p$ , and
3.  $H^{i,*}(L)$  is torsion-free.

*Then all torsion in  $H^{i,*}(L)$  is  $\mathbb{Z}_2$  torsion for  $i \in [i_1, i_2]$ .*

*Proof.* Since  $H(L)$  is thin over  $[i_1, i_2]$ , there is an integer  $s$  such that  $H^{[i_1, i_2]}(L)$  is supported in bigradings  $(i, j)$  satisfying  $2i - j = s \pm 1$ . Lemma 3.5.3 implies that all torsion in  $H^{[i_1, i_2]}(L)$  occurs on the

lower diagonal, i.e. in bigradings  $(i, 2i - s - 1)$ . Lemma 3.5.4 implies that  $H^{[i_1, i_2]}(L)$  does not contain any torsion summands of the form  $\mathbb{Z}_{p^r}$  for any odd prime  $p$ . Therefore  $H^{[i_1, i_2]}(L)$  consists of  $\mathbb{Z}$  and  $\mathbb{Z}_{2^r}$  summands, and can be decomposed as

$$H^{[i_1, i_2]}(L) = H_{<\infty}^{[i_1, i_2]}(L) \oplus H_{\infty}^{[i_1, i_2]}(L)$$

where  $H_{\infty}^{[i_1, i_2]}(L)$  is torsion-free.

Lemma 3.5.5 implies that  $H_{<\infty}^{[i_1, i_2]}(L)$  can be decomposed into a direct sum of its LV-indecomposable submodules. Those submodules are knight move submodules,  $i_1$ -knight move submodules,  $i_2$ -knight move submodules,  $\mathbb{Z}$ -chain submodules, or  $\mathbb{Z}_2$ -chain submodules. The assumption that  $H^{i_1, *}(L)$  is torsion-free implies that no  $i_1$ -knight move or  $\mathbb{Z}_2$ -chains occur. Suppose that  $H_{<\infty}^{[i_1, i_2]}(L)$  is the direct sum of  $N_1$  knight move submodules,  $N_2$   $\mathbb{Z}$ -chain submodules, and  $N_3$   $i_2$ -knight move submodules.

Let  $d_T^{[i_1, i_2]}$  denote the sum of the induced Turner maps in homological gradings  $i_1$  to  $i_2 - 1$ , that is

$$d_T^{[i_1, i_2]} = \sum_{i=i_1}^{i_2-1} (d_T^*)^{i, 2i-s-1} + (d_T^*)^{i, 2i-s+1}.$$

Since the  $i_2$ -knight move is entirely supported in homological grading  $i_2$ , it does not have a summand in the domain of  $d_T^{[i_1, i_2]}$  and will not contribute to the rank of  $d_T^{[i_1, i_2]}$ . All of  $H_{<\infty}^{[i_1, i_2]}(L; \mathbb{Z}_2)$  dies at the  $E^2$  page of the Turner spectral sequence. The summands in the  $i_1$  and  $i_2$  homological gradings of a  $\mathbb{Z}$ -chain can either be killed as part of  $d_T^{[i_1, i_2]}$  or from contributions of  $H(L)$  outside of homological gradings  $[i_1, i_2]$ . Therefore, the rank of  $d_T^{[i_1, i_2]}$  when restricted to a  $\mathbb{Z}$ -chain will be at least  $2 \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor$  and at most  $2 \left\lceil \frac{i_2 - i_1}{2} \right\rceil$ . The rank of  $d_T^{[i_1, i_2]}$  when restricted to a knight move submodule is two. Putting all of this together yields the inequality

$$2N_1 + 2N_2 \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor \leq \text{rank}(d_T^{[i_1, i_2]}) \leq 2N_1 + 2N_2 \left\lceil \frac{i_2 - i_1}{2} \right\rceil.$$

Let  $d_B^{[i_1, i_2]}$  denote the sum of the Bockstein maps in homological gradings  $i_1$  to  $i_2 - 1$ , that is

$$d_B^{[i_1, i_2]} = \sum_{i=i_1}^{i_2-1} (d_B^1)^{i, 2i-s+1}.$$

Since the only torsion summands in  $H^{[i_1, i_2]}(L)$  come from knight move submodules, it follows that  $\text{rank}(d_B^{[i_1, i_2]}) \leq N_1$ . Finally, Lemma 3.4.12 and the fact that  $\nu^*$  is an isomorphism imply that  $\text{rank}(d_T^{[i_1, i_2]}) = 2\text{rank}(d_B^{[i_1, i_2]})$ . Therefore

$$N_1 + N_2 \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor \leq \frac{1}{2} \text{rank}(d_T^{[i_1, i_2]}) = \text{rank}(d_B^{[i_1, i_2]}) \leq N_1,$$

which implies that  $\text{rank}(d_B^{[i_1, i_2]}) = N_1$ . Hence the Bockstein spectral sequence in gradings  $i_1 + 1$  to  $i_2$  collapses at the  $E^2$  page, that is  $(E_B^2)^i = (E_B^\infty)^i$  for  $i_1 < i \leq i_2$ , and thus the only torsion in  $H^{i,*}(L)$  is of the form  $\mathbb{Z}_2$  for  $i \in [i_1, i_2]$ .  $\square$

The following corollary will be useful for our 3-braid computations in Section 3.6.

**Corollary 3.5.7** ([CLSS19]). *Let  $L$  be a link satisfying*

1.  $\dim_{\mathbb{Q}} H(L; \mathbb{Q}) = \dim_{\mathbb{Z}_p} H(L; \mathbb{Z}_p)$  for all odd primes  $p$ ,
2. *all torsion summands occur in a homologically thin region, and*
3. *the first homological grading of each homologically thin region is torsion-free.*

*Then all torsion in  $H(L)$  is  $\mathbb{Z}_2$  torsion.*

### 3.6 An application to 3-braids

Various techniques have been used to show that some other families of links only have  $\mathbb{Z}_2$  torsion in their Khovanov homology or only have  $\mathbb{Z}_2$  torsion in certain gradings. In [HGPR06], Helme-Guizon, Przytycki, and Rong established a connection between the Khovanov homology of a link and the chromatic graph homology of graphs associated to diagrams of the link. In [LS17], Lowrance and Sazdanović used this connection to show that in a range of homological gradings, Khovanov homology contains only  $\mathbb{Z}_2$  torsion. This result can now be seen as a corollary to Theorem 3.5.6. In [PS14], Przytycki and Sazdanović obtained explicit formulae for some torsion and proved that the Khovanov homology of semi-adequate links contains  $\mathbb{Z}_2$  torsion if the corresponding Tait-type graph has a cycle of length at least 3.

There are a number of results about the Khovanov homology of closed 3-braids, but a full computation of the Khovanov homology of closed 3-braids remains open. Turner [Tur08] computed the Khovanov homology of the  $(3, q)$  torus links with coefficients in  $\mathbb{Q}$  or  $\mathbb{Z}_p$  for an odd prime  $p$  (see also [Sto09]). Benheddi [Ben17] computed the reduced Khovanov homology of  $T(3, q)$  with coefficients in  $\mathbb{Z}_2$ . Both Turner and Benheddi's computations play a crucial role in our proofs.

The literature on the Khovanov homology of non-torus closed 3-braids is considerably more sparse. Baldwin [Bal08] proved that a closed 3-braid is quasi-alternating if and only if its Khovanov homology is homologically thin. Abe and Kishimoto [AK10] used the Rasmussen  $s$ -invariant to compute the alternation number and dealternation number of many closed 3-braids. Lowrance [Low11] computed the homological width of the Khovanov homology of all closed 3-braids.

Over the next few sections, we prove Theorem 3.6.7, showing that all torsion in the Khovanov homology of a closed braid of type  $\Omega_0, \Omega_1, \Omega_2$ , or  $\Omega_3$  is  $\mathbb{Z}_2$  torsion. This result is a partial address to the conjecture of Przytycki and Sazdanovic known as the PS Braid Conjecture (See Conjecture 3.6.1).

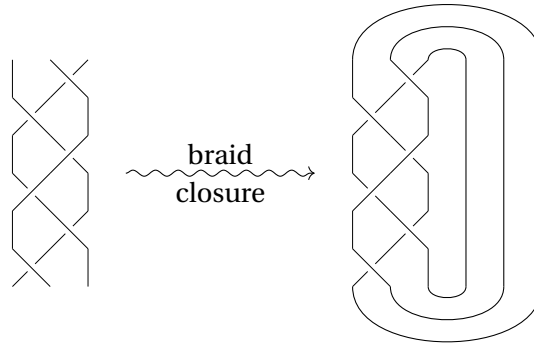
First, we argue that it suffices to prove Theorem 3.6.7 when the exponent  $n$  in  $\Delta^n$  in the braid word is non-negative. Next, we use the long exact sequences (3.2) and (3.3) to compute the Khovanov homology of these closed braids over  $\mathbb{Q}$  and  $\mathbb{Z}_p$  for an odd prime  $p$ . Then, we use Benheddi's results [Ben17] to assist our computations of the Kovanov homology of these closed 3-braids over  $\mathbb{Z}_2$ . Finally, we use Theorem 3.5.6 and Corollary 3.5.7 to complete the proof.

Consider the braid group [Mur74]  $B_s$  on  $s$ -strands whose generators are shown in Figure 3.8. The *half twist*  $\Delta \in B_s$  is defined as  $\Delta = (\sigma_1 \sigma_2 \dots \sigma_{p-1})(\sigma_1 \sigma_2 \dots \sigma_{p-2}) \dots (\sigma_1 \sigma_2)(\sigma_1)$ , and the *full twist* is  $\Delta^2$ .

$$\begin{array}{ccccccc} \sigma_1 = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} \cdots & \sigma_2 = \begin{array}{c} \diagdown \quad \diagup \\ | \quad | \end{array} \cdots & \cdots & \sigma_{s-1} = \begin{array}{c} | \quad | \\ \diagup \quad \diagdown \end{array} \\ \sigma_1^{-1} = \begin{array}{c} \diagdown \quad \diagup \\ | \quad | \end{array} \cdots & \sigma_2^{-1} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} \cdots & \cdots & \sigma_{s-1}^{-1} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} \end{array}$$

**Figure 3.8** Generators and their inverses for the braid group  $B_s$ .

The closure of a braid diagram is a diagram of a link, and a famous result of Alexander states that every link can be represented by the closure of a braid. For convenience, throughout this paper, a braid word will be used to refer to either an element of the braid group or its braid closure depending on the context in which it appears.



**Figure 3.9** A braid diagram and its braid closure.

In [PS14], Przytycki and Sazdanović conjectured the following, connecting torsion in Khovanov homology to braid index.

**Conjecture 3.6.1** ([PS14]).

1. *The Khovanov homology of a closed 3-braid can have only  $\mathbb{Z}_2$  torsion.*
2. *The Khovanov homology of a closed 4-braid cannot have  $\mathbb{Z}_{p^r}$  torsion for  $p \neq 2$ .*
3. *The Khovanov homology of a closed 4-braid can have only  $\mathbb{Z}_2$  and  $\mathbb{Z}_4$  torsion.*
4. *The Khovanov homology of a closed  $n$ -braid cannot have  $\mathbb{Z}_{p^r}$  torsion for  $p > n$  ( $p$  prime).*
5. *The Khovanov homology of a closed  $n$ -braid cannot have  $\mathbb{Z}_{p^r}$  torsion for  $p^r > n$ .*

Counterexamples to parts 2, 3 and 5 are given in [Muk17], and a counterexample to part 4 has recently been constructed by Mukherjee [Muk]. However, part 1 remains open, and computations suggest that part 1 is indeed true. One goal of this (ongoing) project is to prove this.

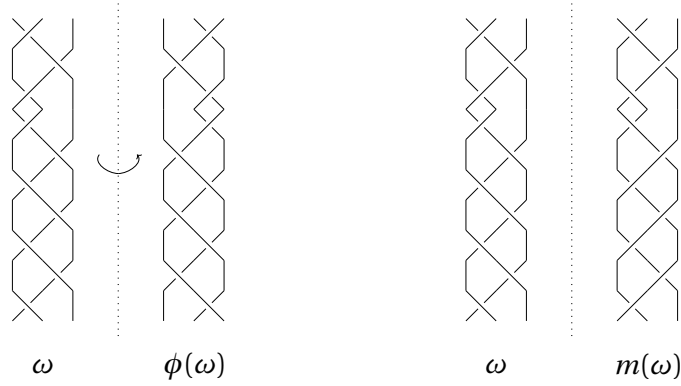
If two elements of a braid group are conjugate, then the corresponding braid closures are isotopic as links. Therefore, it would be convenient to have a classification of elements of the braid group  $B_n$  up to conjugacy. For  $s = 2$ , of course, the classification is trivial. For  $s \geq 4$ , no classification is known. For  $s = 3$ , Kunio Murasugi provides the following [Mur74].

**Theorem 3.6.2** ([Mur74]). *Each element of the braid group  $B_3$  is conjugate to a unique element of one of the following disjoint sets:*

$$\begin{aligned}
\Omega_0 &= \{\Delta^{2n} \mid n \in \mathbb{Z}\}, \\
\Omega_1 &= \{\Delta^{2n} \sigma_1 \sigma_2 \mid n \in \mathbb{Z}\}, \\
\Omega_2 &= \{\Delta^{2n} (\sigma_1 \sigma_2)^2 \mid n \in \mathbb{Z}\}, \\
\Omega_3 &= \{\Delta^{2n+1} \mid n \in \mathbb{Z}\}, \\
\Omega_4 &= \{\Delta^{2n} \sigma_1^{-p} \mid n \in \mathbb{Z}, p \in \mathbb{N}\}, \\
\Omega_5 &= \{\Delta^{2n} \sigma_2^q \mid n \in \mathbb{Z}, q \in \mathbb{N}\}, \\
\Omega_6 &= \{\Delta^{2n} \sigma_1^{-p_1} \sigma_2^{q_1} \cdots \sigma_1^{-p_r} \sigma_2^{q_r} \mid n \in \mathbb{Z}, p_i, q_i \in \mathbb{N}\}.
\end{aligned}$$

Each braid in  $\Omega_0, \Omega_1, \Omega_2$ , or  $\Omega_3$  has a braid word of the form  $\Delta^n \beta$  for some  $\beta \in B_3$ . The following observations imply that we may assume  $n \geq 0$ .

1. The mirror image  $m(D)$  of a link diagram  $D$  is the diagram obtained by changing all crossings. On the level of braid words,  $m : B_3 \rightarrow B_3$  is a group homomorphism satisfying  $m(\sigma_i) = \sigma_i^{-1}$  and  $m(\Delta) = \Delta^{-1}$ . Recall that the torsion in Khovanov homology of a link diagram and the torsion of its mirror image differ only by a homological shift (See Corollary 11 of [Kho00]). So the Khovanov homology of  $L$  has  $\mathbb{Z}_{p^r}$  torsion if and only if the Khovanov homology of its mirror  $m(L)$  has  $\mathbb{Z}_{p^r}$  torsion.
2. Consider the group homomorphism  $\phi : B_3 \rightarrow B_3$  defined on generators by  $\phi(\sigma_1) = \sigma_2$  and  $\phi(\sigma_2) = \sigma_1$ . If the braid word  $\omega$  is a projection of a link  $L$  embedded in  $\{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1\}$  to the plane  $z = 0$ , then the projection of  $L$  to the plane  $z = 1$  is  $\phi(\omega)$ . Thus the map  $\phi$  preserves the isotopy type of the braid word. Therefore the Khovanov homology of the closure of  $\omega$  has  $\mathbb{Z}_{p^r}$  torsion if and only if the Khovanov homology of the closure of  $\phi(\omega)$  has  $\mathbb{Z}_{p^r}$  torsion. Note that the homomorphism  $\phi$  satisfies  $\phi(\Delta) = \Delta$ . See Figure 3.10 for an example of the action of  $\phi$  on a braid diagram.



**Figure 3.10** Left: The braid word  $\omega = \Delta^2 \sigma_1^2 \sigma_2 \sigma_1 \in \Omega_6$  and the corresponding diagram  $\phi(\omega) = \Delta^2 \sigma_2^2 \sigma_1 \sigma_2$ . Think of  $\phi(D)$  as  $D$  rotated about the dotted line. Right: The braid word  $\omega$  and its mirror image  $m(\omega)$ .

The following equalities together with the above two arguments show that in all cases it suffices to determine torsion for  $n \geq 0$ :

$$m(\Delta^{-2n}) = \Delta^{2n} \quad (3.7)$$

$$m(\Delta^{-2n} \sigma_1 \sigma_2) = (\sigma_1 \sigma_2)^{3n-1} \quad (3.8)$$

$$m(\Delta^{-2n} (\sigma_1 \sigma_2)^2) = (\sigma_1 \sigma_2)^{3n-2} \quad (3.9)$$

$$m(\Delta^{-2n-1}) = \Delta^{2n+1} \quad (3.10)$$

$$m\phi(\Delta^{-2n}\sigma_1^{-p}) = \Delta^{2n}\sigma_2^p \quad (3.11)$$

$$m\phi(\Delta^{-2n}\sigma_2^q) = \Delta^{2n}\sigma_1^{-q} \quad (3.12)$$

$$m\phi(\Delta^{-2n}\sigma_1^{-p_1}\sigma_2^{q_1}\dots\sigma_1^{-p_r}\sigma_2^{q_r}) = \Delta^{2n}\sigma_2^{p_1}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r}. \quad (3.13)$$

Note that  $\Delta^{2n}\sigma_2^{p_1}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r} = \sigma_2^{p_1}\Delta^{2n}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r}$  because  $\Delta^{2n}$  is in the center of the braid group. Consequently, the braid closures of  $\Delta^{2n}\sigma_2^{p_1}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r}$  and  $\Delta^{2n}\sigma_1^{-q_1}\dots\sigma_2^{p_r}\sigma_1^{-q_r}\sigma_2^{p_1}$  are equivalent, so the braid in (3.13) is of type  $\Omega_6$ . Although we only address links in  $\Omega_0 - \Omega_3$ , we include these observations for  $\Omega_4 - \Omega_6$  for future reference.

We begin with a theorem, shown by Turner in [Tur08], that will be useful in conjunction with Murasugi's classification of 3-braids and the long exact sequence in Khovanov homology of Section 3.3.

**Theorem 3.6.3** ([Tur08]). *For each  $q \in \mathbb{Z}$ , the Khovanov homology  $H(T(3, q))$  of the torus link  $T(3, q)$  contains no  $\mathbb{Z}_{p^r}$  torsion for  $p \neq 2$ . That is, there is no  $\mathbb{Z}_{p^r}$  torsion for  $p \neq 2$  in the Khovanov homology of links of types  $\Omega_0, \Omega_1$  and  $\Omega_2$ .*

To achieve this result, Turner calculates the  $\mathbb{Q}$  and  $\mathbb{Z}_p$  Khovanov homologies of the torus links  $T(3, q)$ , notes dimensional equality in all bigradings, and appeals to coefficients. We begin Section 3.6 by extending this result to links of type  $\Omega_3$ . Our strategy is also to calculate the Khovanov homology over  $\mathbb{Q}$  and  $\mathbb{Z}_p$ , note dimensional equalities and appeal to universal coefficients. We make use of the long exact sequence and Lee spectral sequence.

**Theorem 3.6.4** ([CLSS19]). *Let  $n \geq 0$  be an integer,  $p$  be an odd prime, and let  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{F} = \mathbb{Z}_p$ . The Khovanov homology  $H(\Delta^{2n+1}, \mathbb{F})$  of the link  $\Delta^{2n+1}$  over coefficients in  $\mathbb{F}$  is as given in Figure 3.11.*

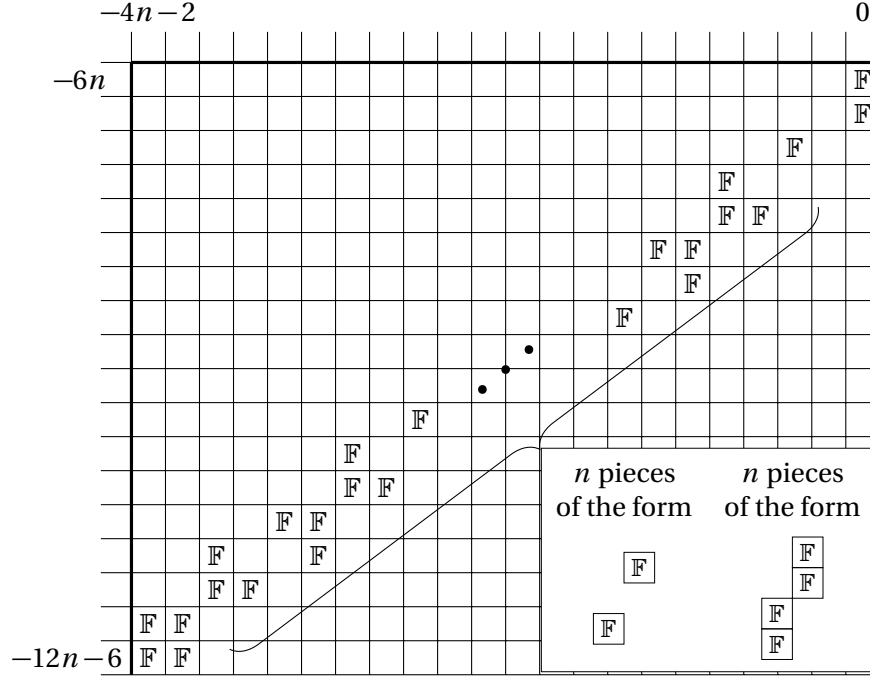
**Corollary 3.6.5.** *There is no  $\mathbb{Z}_{p^r}$  torsion for  $p \neq 2$  in the Khovanov homology of links of types  $\Omega_0, \Omega_1, \Omega_2$  and  $\Omega_3$ .*

*Proof of Theorem 3.6.4.* First observe that  $\Delta^{2n+1} = (\sigma_1\sigma_2)^{3n+1}\sigma_1$  where  $(\sigma_1\sigma_2)^{3n+1}$  is a braid word for  $T(3, 3n+1)$ . We consider smoothing the top  $\sigma_1$  as depicted in Figure 3.12.

The diagram  $D_0$  is a diagram of the unknot  $U$  and  $D_1$  is a diagram of  $T(3, 3n+1)$ . The top  $\sigma_1$  in  $\Delta^{2n+1}$  is a negative crossing so we compute  $c = n_-(D_0) - n_-(D) = (1+2n) - (6n+3) = -4n-2$ . Using (3.2) for each  $j$ , and letting  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}_p$  where  $p$  is an odd prime, we get a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{i,j}(D; \mathbb{F}) \longrightarrow H^{i+4n+2, j+12n+5}(U; \mathbb{F}) \xrightarrow{\delta_*} H^{i+1, j+1}(T(3, 3n+1); \mathbb{F}) \longrightarrow$$

For  $i \neq -4n-2, -4n-1$ , we have  $H^{i+4n+2, j+12n+5}(U; \mathbb{F}) = 0 = H^{i+4n+1, j+12n+5}(U; \mathbb{F})$  for every  $j$ , so exactness yields  $H^{i,j}(D; \mathbb{F}) \cong H^{i,j+1}(T(3, 3n+1); \mathbb{F})$  for every  $j$ . For  $j \neq -12n-5 \pm 1$ , the portion of



**Figure 3.11** The  $\mathbb{F}$ -Khovanov homology of the braid closure of  $\Delta^{2n+1}$  over  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}_p$  for  $p$  an odd prime.

the long exact sequence containing  $i = -4n - 2$  and  $-4n - 1$  looks like

$$0 \longrightarrow H^{-4n-2, j+1}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-2, j}(D; \mathbb{F}) \longrightarrow 0$$

$$\xrightarrow{\delta_*} H^{-4n-1, j+1}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-1, j}(D; \mathbb{F}) \longrightarrow 0.$$

Thus  $H^{-4n-2, j}(D; \mathbb{F}) \cong H^{-4n-2, j+1}(T(3, 3n+1); \mathbb{F}) = 0$  and  $H^{-4n-1, j}(D; \mathbb{F}) \cong H^{-4n-1, j+1}(T(3, 3n+1); \mathbb{F}) = 0$ , where calculations of  $H(T(3, 3n+1); \mathbb{F})$  are given in Turner's [Tur08].

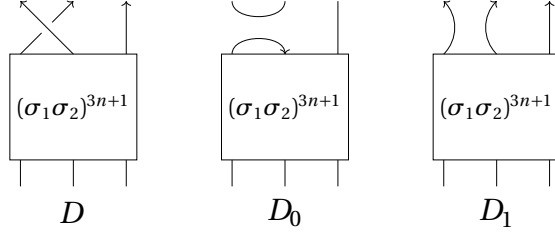
It remains to check the portion of the long exact sequence containing  $i = -4n - 2, -4n - 1$  in the cases  $j = -12n - 6, -12n - 4$ , which are displayed below.

$$0 \longrightarrow H^{-4n-2, -12n-3}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-2, -12n-4}(D; \mathbb{F}) \longrightarrow H^{0,1}(U; \mathbb{F})$$

$$\xrightarrow{\delta_*} H^{-4n-1, -12n-3}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-1, -12n-4}(D; \mathbb{F}) \longrightarrow 0.$$

$$0 \longrightarrow H^{-4n-2, -12n-5}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-2, -12n-6}(D; \mathbb{F}) \longrightarrow H^{0,-1}(U; \mathbb{F})$$

$$\xrightarrow{\delta_*} H^{-4n-1, -12n-5}(T(3, 3n+1); \mathbb{F}) \longrightarrow H^{-4n-1, -12n-6}(D; \mathbb{F}) \longrightarrow 0.$$



**Figure 3.12** A diagram for the braid  $\Delta^{2n+1}$  along with its final  $\sigma_1$  crossing 0- and 1- smoothed.

In [Tur08], Turner also calculates

$$H^{-4n-2, -12n-3}(T(3, 3n+1); \mathbb{F}) = 0$$

$$H^{-4n-2, -12n-5}(T(3, 3n+1); \mathbb{F}) = 0$$

$$H^{-4n-1, -12n-3}(T(3, 3n+1); \mathbb{F}) = \mathbb{F}$$

$$H^{-4n-1, -12n-5}(T(3, 3n+1); \mathbb{F}) = \mathbb{F}.$$

And of course  $H^{0, \pm 1}(U; \mathbb{F}) = \mathbb{F}$  for any field  $\mathbb{F}$ . Thus we have exact sequences

$$0 \longrightarrow H^{-4n-2, -12n-4}(D; \mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1, -12n-4}(D; \mathbb{F}) \longrightarrow 0 \quad (3.14)$$

and

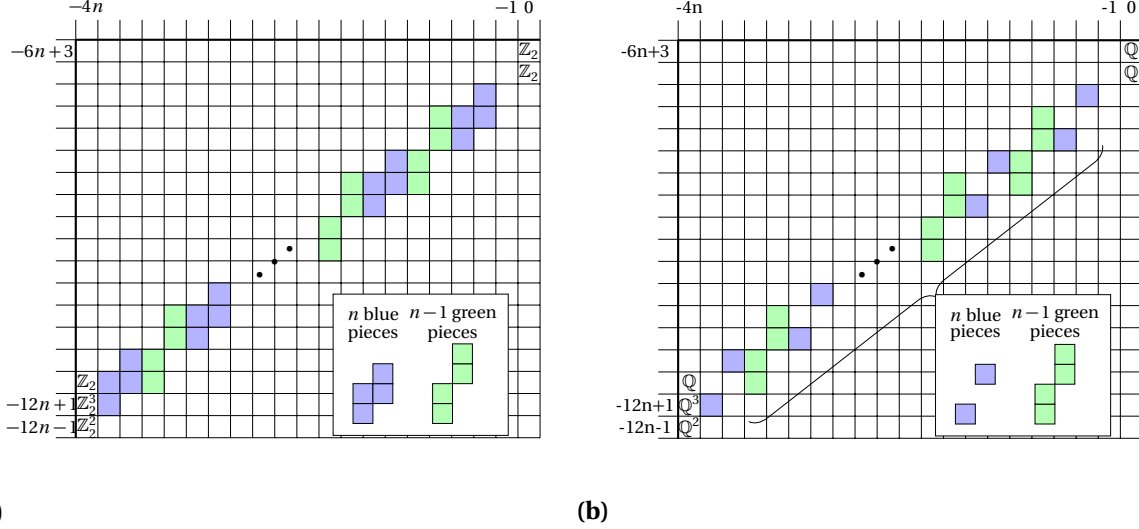
$$0 \longrightarrow H^{-4n-2, -12n-6}(D; \mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1, -12n-6}(D; \mathbb{F}) \longrightarrow 0. \quad (3.15)$$

From (3.14) and (3.15) it follows that  $H^{-4n-1, -12n-5 \pm 1}(D; \mathbb{F})$  and  $H^{-4n-2, -12n-5 \pm 1}(D; \mathbb{F})$  are each isomorphic to  $\mathbb{F}$  or the trivial group. We argue that all four of them are isomorphic to  $\mathbb{F}$  as follows. By Theorem 3.4.13,  $\dim_{\mathbb{F}}(\text{Lee}^{-4n-2}(\Delta^{2n+1}; \mathbb{F})) = 2$ . Using Turner's calculation of the Khovanov homology of  $T(3, 3n+1)$ , the long exact sequence (3.6) gives  $H^{-4n-2, j}(\Delta^{2n+1}; \mathbb{F}) \cong H^{-4n-2, j+1}(T(3, 3n+1); \mathbb{F}) = 0$  for  $j \neq -12n-6, -12n-4$ , so  $\dim_{\mathbb{F}} H^{-4n-2, *}(\Delta^{2n+1}; \mathbb{F}) \leq 2$ . Since the Lee spectral sequence has  $E^1$  page the  $\mathbb{F}$ -Khovanov homology and converges to Lee homology, we must also have  $\dim_{\mathbb{F}} H^{-4n-2, *}(\Delta^{2n+1}; \mathbb{F}) \geq 2$ . Consequently,  $\dim_{\mathbb{F}} H^{-4n-2, *}(\Delta^{2n+1}; \mathbb{F}) = 2$  and it follows that  $H^{-4n-2, -12n-6}(D; \mathbb{F}) = \mathbb{F}$  and  $H^{-4n-2, -12n-4}(D; \mathbb{F}) = \mathbb{F}$ . The non-triviality of these two groups together with sequences (3.14) and (3.15) imply that  $H^{-4n-1, -12n-4}(D; \mathbb{F}) = \mathbb{F}$  and  $H^{-4n-2, -12n-6}(D; \mathbb{F}) = \mathbb{F}$ .  $\square$

Next, we use Theorem 3.5.6 and Corollary 3.5.7 to compute all torsion in closed braids in  $\Omega_0, \Omega_1, \Omega_2$ , and  $\Omega_3$ .

Benheddi [Ben17] computed the reduced  $\mathbb{Z}_2$ -Khovanov homology of the torus links  $T(3, q)$ ,

and from those computations we can recover the unreduced  $\mathbb{Z}_2$ -Khovanov homology of the torus links  $T(3, q)$ . These computations encompass the closed 3-braids in  $\Omega_0, \Omega_1$ , and  $\Omega_2$ . We display the  $\mathbb{Q}$ -Khovanov homology and  $\mathbb{Z}_2$ -Khovanov homology of these links in Figures 3.13, 3.14, and 3.15.



**Figure 3.13** In (a) we have the  $\mathbb{Z}_2$ -Khovanov homology for the three component torus link  $T(3, 3n)$ . Each blue or green box represents a copy of  $\mathbb{Z}_2$  which is killed in the Turner spectral sequence. In (b) we have the  $\mathbb{Q}$ -Khovanov homology for the torus link  $T(3, 3n)$ . Each blue or green box represents a copy of  $\mathbb{Q}$  which is killed in the Lee spectral sequence.

The  $\mathbb{Z}_2$ -Khovanov homology of the closure of braids in  $\Omega_3$  is computed from the  $\mathbb{Z}_2$ -Khovanov homology of  $T(3, 3n + 1)$  in a similar fashion to the analogous computation over  $\mathbb{Q}$ .

**Theorem 3.6.6** ([CLSS19]). *For any  $n \geq 0$ ,*

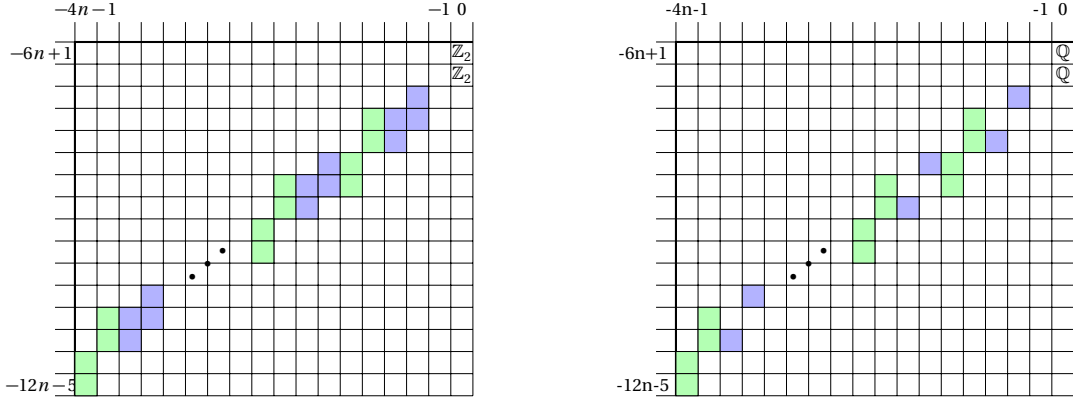
$$H(\Delta^{2n+1}; \mathbb{Z}_2) \cong H(T(3, 3n + 1); \mathbb{Z}_2)\{-1\} \oplus H(U; \mathbb{Z}_2)[-4n - 2]\{-12n - 5\}.$$

*Proof.* For homological gradings  $-4n - 1$  through  $0$ , the proof of this theorem is largely the same as the proof of Theorem 3.6.4. We focus on homological grading  $-4n - 2$ . From (3.14) and (3.15) it follows that each of the groups

$$H^{-4n-1, -12n-5 \pm 1}(D; \mathbb{Z}_2), H^{-4n-2, -12n-5 \pm 1}(D; \mathbb{Z}_2)$$

is isomorphic to either  $\mathbb{Z}_2$  or the trivial group. We argue that each of these groups is isomorphic to  $\mathbb{Z}_2$ .

Using Theorem 3.4.9, we find that  $\dim_{\mathbb{Z}_2}(BN^{-4n-2}(\Delta^{2n+1})') = 2$ . Using Benheddi's calculation



(a)

(b)

**Figure 3.14** In (a) we have the  $\mathbb{Z}_2$ -Khovanov homology for the torus knot  $T(3, 3n+1)$ . Each blue or green box represents a copy of  $\mathbb{Z}_2$  which is killed in the Turner spectral sequence. In (b) we have the  $\mathbb{Q}$ -Khovanov homology for the torus knot  $T(3, 3n+1)$ . Each blue or green box represents a copy of  $\mathbb{Q}$  which is killed in the Lee spectral sequence.

[Ben17] of the  $\mathbb{Z}_2$ -Khovanov homology of  $T(3, 3n+1)$ , shown in Figure 3.14, the long exact sequence (3.6) gives

$$H^{-4n-2,j}(\Delta^{2n+1}; \mathbb{Z}_2) \cong H^{-4n-2,j+1}(T(3, 3n+1); \mathbb{Z}_2) = 0$$

for  $j \neq -12n-6, -12n-4$ . Therefore,  $\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1}; \mathbb{Z}_2) \leq 2$ . Since the Turner spectral sequence has  $E^1$  page the  $\mathbb{Z}_2$ -Khovanov homology, and converges to Bar-Natan homology,

$$\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1}; \mathbb{Z}_2) \geq 2.$$

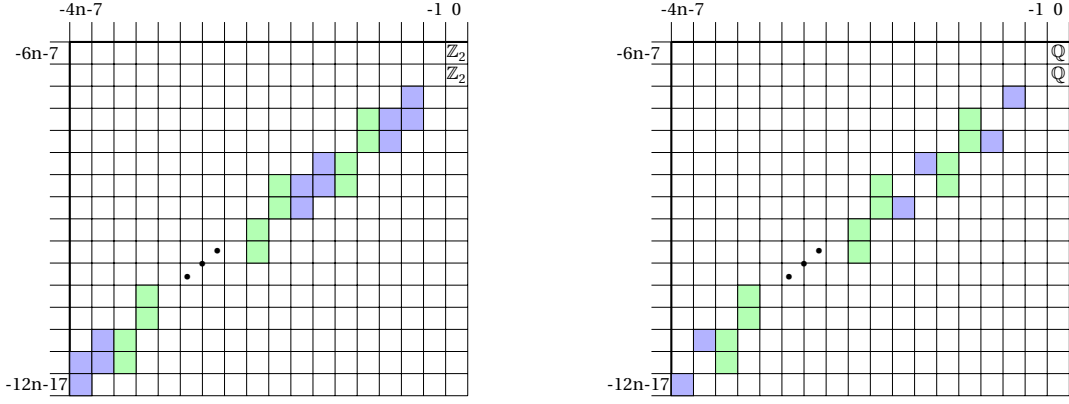
Therefore it follows that  $H^{-4n-2,-12n-6}(D; \mathbb{Z}_2) \cong \mathbb{Z}_2$  and  $H^{-4n-2,-12n-4}(D; \mathbb{Z}_2) \cong \mathbb{Z}_2$ . The non-triviality of these two groups together with sequences (3.14) and (3.15) imply that  $H^{-4n-1,-12n-4}(D; \mathbb{Z}_2) = \mathbb{Z}_2$  and  $H^{-4n-2,-12n-6}(D; \mathbb{Z}_2) = \mathbb{Z}_2$ .  $\square$

The  $\mathbb{Z}_2$  and  $\mathbb{Q}$  Khovanov homology of the closure of  $\Delta^{2n+1}$  is depicted in Figure 3.16.

The computations of  $\mathbb{Q}$  and  $\mathbb{Z}_p$  Khovanov homology for closed braids in  $\Omega_0, \Omega_1, \Omega_2$ , and  $\Omega_3$  leads to the following theorem.

**Theorem 3.6.7** ([CLSS19]). *The Khovanov homology of a closed braid in  $\Omega_0, \Omega_1, \Omega_2$  or  $\Omega_3$  contains only  $\mathbb{Z}_2$  torsion.*

*Proof.* Let  $L$  be a closed braid in  $\Omega_0, \Omega_1, \Omega_2$  or  $\Omega_3$ . Corollary 3.6.5 tells us that  $L$  satisfies condition 1 of Corollary 3.5.7.



(a)

(b)

**Figure 3.15** In (a) we have the  $\mathbb{Z}_2$ -Khovanov homology for the torus knot  $T(3, 3n + 2)$ . Each blue or green box represents a copy of  $\mathbb{Z}_2$  which is killed in the Turner spectral sequence. In (b) we have the  $\mathbb{Q}$ -Khovanov homology, and  $E^1$  page of the Lee spectral sequence, for the torus knot  $T(3, 3n + 2)$ . Each blue or green box represents a copy of  $\mathbb{Q}$  which is killed in the Lee spectral sequence.

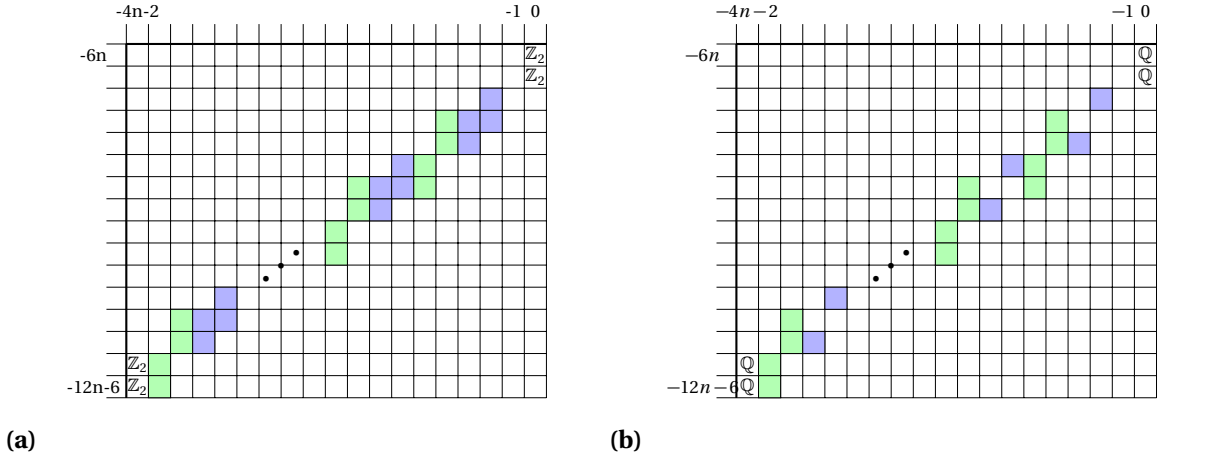
Figures 3.13, 3.14, 3.15, and 3.16 show that all torsion in  $H(L)$  occurs in the locally thin “blue” regions, and moreover, no torsion is supported in the initial homological grading of any locally thin region. Therefore Corollary 3.5.7 implies that all torsion in  $H(L)$  is  $\mathbb{Z}_2$  torsion.  $\square$

As corollaries, we obtain the integral Khovanov homology of closed 3-braids in  $\Omega_0, \Omega_1, \Omega_2$ , and  $\Omega_3$ .

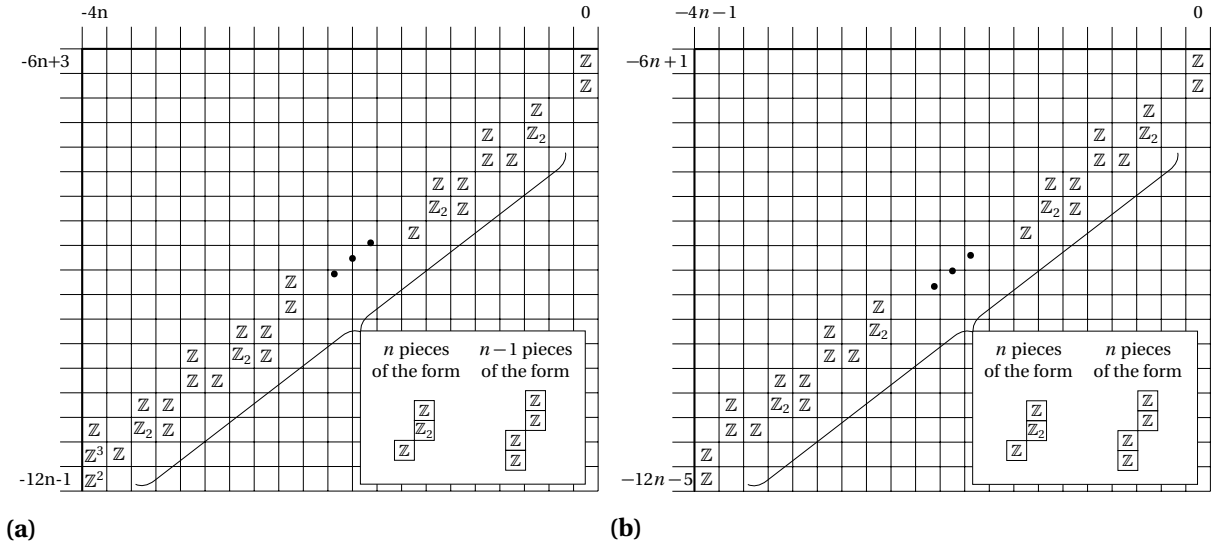
**Corollary 3.6.8** ([CLSS19]). *For any  $n \geq 0$ ,*

$$H(\Delta^{2n+1}) \cong H(T(3, 3n + 1))\{-1\} \oplus H(U)[-4n - 2]\{-12n - 5\}.$$

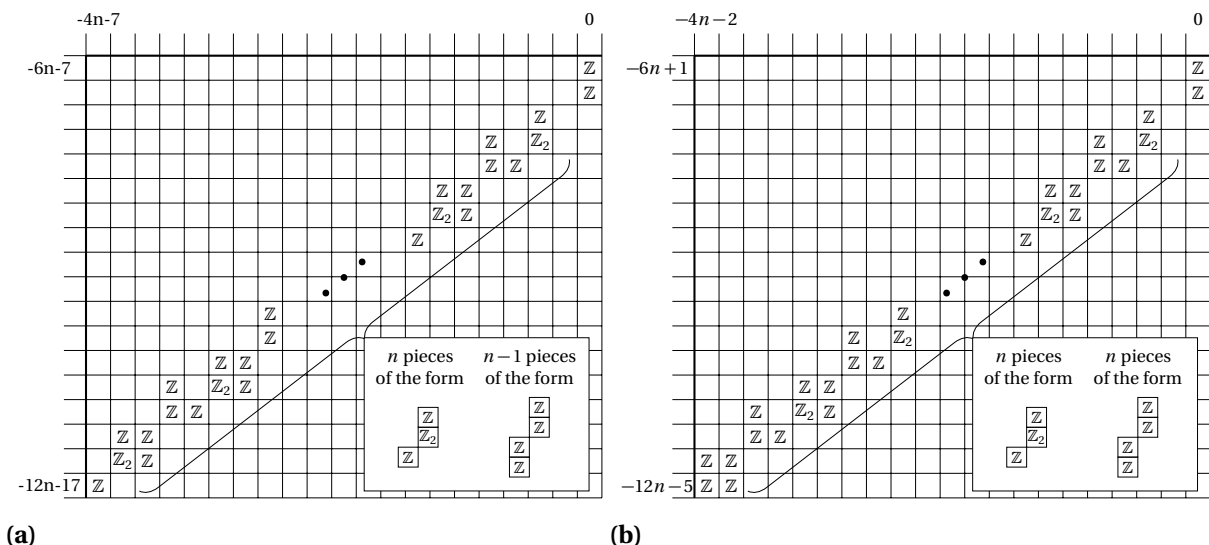
**Corollary 3.6.9** ([CLSS19]). *The integral Khovanov homology closed 3-braids in  $\Omega_0, \Omega_1, \Omega_2$ , and  $\Omega_3$  are given in Figures 3.17a, 3.17b, 3.18a, and 3.18b.*



**Figure 3.16** In (a) we have the  $\mathbb{Z}_2$ -Khovanov homology for the two component link  $\Delta^{2n+1}$ . Each blue or green box represents a copy of  $\mathbb{Z}_2$  which is killed in the Turner spectral sequence. In (b) we have the  $\mathbb{Q}$ -Khovanov homology for the link  $\Delta^{2n+1}$ . Each blue or green box represents a copy of  $\mathbb{Q}$  which is killed in the Lee spectral sequence.



**Figure 3.17** In (a) we have the integral Khovanov homology of  $T(3, 3n)$ . In (b) we have the integral Khovanov homology of  $T(3, 3n + 1)$ .

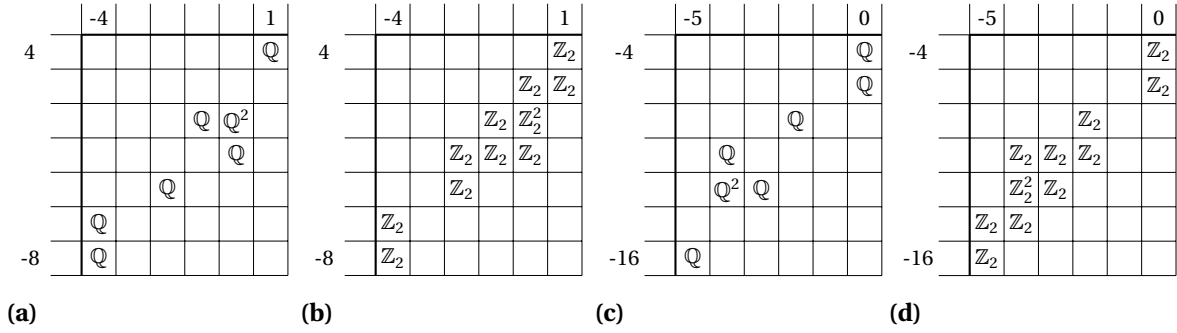


**Figure 3.18** In (a) we have the integral Khovanov homology of  $T(3, 3n + 2)$ . In (b) we have the integral Khovanov homology of the braid closure of  $\Delta^{2n+1}$ .

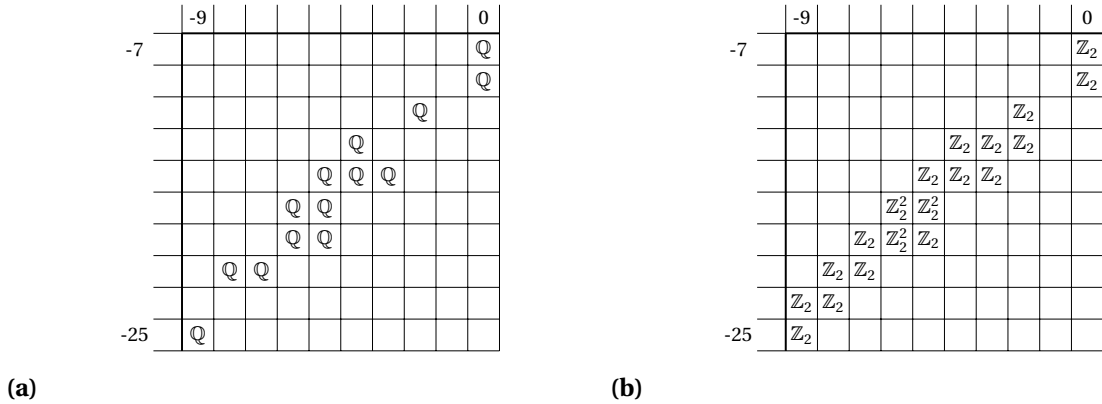
### 3.7 Future work

At the time of writing, we do not know much about relations between topological properties of knots and links on the one hand, and their Khovanov homology groups on the other. This project was an attempt to bridge the gaps between the algebraic and the topological by relating torsion in Khovanov homology to the braid index. We succeeded only partially, in that Theorem 3.5.6 cannot be applied to links of types  $\Omega_4$ ,  $\Omega_5$  or  $\Omega_6$ . Indeed, in Figure 3.19 and Figure 3.20 we give examples of links in these families to which our main theorem is not strong enough to prove their Khovanov homology contains only  $\mathbb{Z}_2$  torsion, for they each have homological intervals over which they are not thin. Computational data on these remaining cases provides evidence of the verity of Przytycki and Szazdanovic's conjecture on  $\mathbb{Z}_2$  torsion in the Khovanov homology of links of braid index 3. One way to get a handle on the remaining cases may involve an as yet unpublished result of Shumakovitch, a relationship between the second Bockstein, Turner and vertical differentials analogous to that of Lemma 3.4.12. This relation dictates that all even torsion in the Khovanov homology of links whose Khovanov homology lies on three consecutive diagonals must be  $\mathbb{Z}_2$  or  $\mathbb{Z}_4$  torsion. Producing a version of Theorem 3.5.6 for regions of Khovanov homology lying on three diagonals would bring us a step closer to proving the PS conjecture. However, Shumakovitch suggests that his proof of the relationship between second-page differentials is extremely difficult and is likely even more so for higher differentials. Consequently, a more fruitful approach may be to employ alternative methods. With its ability to simplify chain complexes, Algebraic Morse Theory is one possible alternative approach. As we note in Chapter 4, Yuzhou Gu defined a family of so-called Morse matching on

a chain complex associated to each cycle graph, and subsequently calculated the (magnitude) homology groups for this entire family [Gu18]. With this in mind, another direction in which to pursue alebro-topological connections in the context of links and Khovanov homology is to search for families of links that have diagrams to whose Khovanov complexes we can define particularly nice Morse matchings, perhaps amenable to the extraction of data on torsion.



**Figure 3.19** In (a) and (b) we have the  $\mathbb{Q}$  and  $\mathbb{Z}_2$  Khovanov homologies of  $\Delta^2 \sigma_1^{-5} \in \Omega_4$ . Theorem 3.5.6 does not imply this link has only  $\mathbb{Z}_2$  in  $\mathbb{Z}$  Khovanov homology, for  $\Delta^2 \sigma_1^{-5}$  is not thin over any interval containing  $i = 0$ . In (c) and (d) we have the  $\mathbb{Q}$  and  $\mathbb{Z}_2$  Khovanov homologies of  $\Delta^2 \sigma_2 \in \Omega_5$ . In this case, Theorem 3.5.6 does not imply all torsion in its  $\mathbb{Z}$  Khovanov homology is  $\mathbb{Z}_2$  torsion, for  $\Delta^2 \sigma_2$  is not thin over any interval containing  $i = -4$ .



**Figure 3.20** In (a) and (b) we have the  $\mathbb{Q}$  and  $\mathbb{Z}_2$  Khovanov homology of  $\Delta^4 \sigma_1^{-2} \sigma_2 \sigma_1^{-1} \in \Omega_6$ . Theorem 3.5.6 does not imply all torsion in its  $\mathbb{Z}$  Khovanov homology is  $\mathbb{Z}_2$  torsion, for this link is not thin over any interval containing  $i = -5$ .

## CHAPTER

# 4

# TORSION IN THE MAGNITUDE HOMOLOGY OF GRAPHS

Tom Leinster introduced a notion of Euler characteristic for finite categories [Lei08]. He then applied his construction to finite graphs by viewing a graph as a metric space, which in turn may be viewed as a certain type of category. This gave rise to the power series invariant of graphs known as magnitude, which Leinster studies extensively in [Lei14]. Magnitude homology is a bigraded homology theory for graphs, introduced by Hepworth and Willerton, whose graded Euler characteristic is the magnitude [HW15]. Hepworth and Willerton initially conjectured that torsion is not to be found in magnitude homology groups, but this was subsequently shown to be false by Yuzhou Gu [Gu18]. Torsion in the magnitude homology groups will be the primary focus of this chapter.

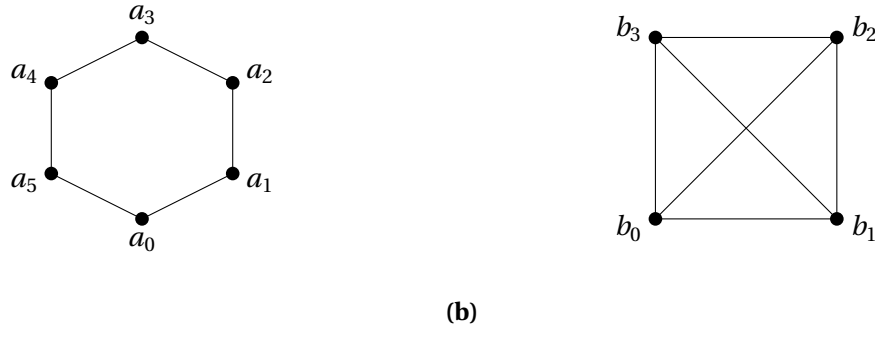
In Section 4.1 we recall the constructions of magnitude and magnitude homology along with their basic properties. In Section 4.2 we review the current state of knowledge of torsion in magnitude homology, show that torsion of arbitrary prime order can be found in magnitude homology (Theorem 4.2.8), and construct infinite families of graphs with a given prime order in magnitude homology (Theorem 4.2.15, Theorem 4.2.14). Finally, in Section 4.3 we extend Gu's computations [Gu18] of the magnitude homology groups of cycle graphs to a family of outerplanar graphs (Theorem 4.3.11), compute the main diagonal of graphs with no induced cycles of length 3 or 4 (Theorem 4.3.4), and put forth several conjectures on the structure of the main diagonal for other families of graphs based on calculations performed in Python.

## 4.1 Magnitude and magnitude homology: construction and properties

A graph is a pair  $G = (V, E)$  where  $V$  is a set of vertices and  $E$  is a set of unordered pairs of vertices, which we think of as a set of undirected edges between vertices. For the purposes of defining magnitude and magnitude homology, we assume all graphs to have no loops and no double edges [Lei14]. Such a graph  $G$  may be viewed as an extended metric space (a metric space with infinity allowed as a distance) whose points are the vertices of  $G$  by declaring each edge to be of length one and defining an extended metric  $d : V \times V \rightarrow [0, \infty]$  by

$$d(x, y) = \min \{ d(x, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, y) : \{x, x_1\}, \{x_{k-1}, y\}, \{x_i, x_{i+1}\} \in E \text{ for } 1 \leq i \leq k-2 \}$$

In other words,  $d(x, y)$  is the length of a shortest path in  $G$  from  $x$  to  $y$ . By convention we let  $d(x, y) = \infty$  if there is no path from  $x$  to  $y$ . For example,  $d(a_1, a_5) = 2$  and  $d(a_0, a_3) = 3$  in the cycle graph  $C_6$  of Figure 4.1, while the distance between distinct vertices of  $K_4$  is 1.



**Figure 4.1** (a) The cycle graph  $C_6$  on six vertices. (b) The complete graph  $K_4$  on four vertices.

**Definition 4.1.1** ([Lei14]). *Let  $G$  be a graph with (ordered) vertex set  $V = \{v_1, v_2, \dots, v_n\}$ . The similarity matrix of  $G$  is the  $n \times n$  square matrix with entries in the polynomial ring  $\mathbb{Z}[q]$  given by*

$$Z_G(q) = \begin{bmatrix} q^{d(v_1, v_1)} & q^{d(v_1, v_2)} & \cdots & q^{d(v_1, v_n)} \\ q^{d(v_2, v_1)} & \ddots & & q^{d(v_2, v_n)} \\ \vdots & & \ddots & \vdots \\ q^{d(v_n, v_1)} & q^{d(v_n, v_2)} & \cdots & q^{d(v_n, v_n)} \end{bmatrix}$$

where it is understood that  $q^\infty = 0$ .

Since  $Z_G(0)$  is the identity matrix, the polynomial  $\det(Z_G(q))$  has constant term 1. Consequently,

$\det(Z_G(q))$  is invertible in the ring  $\mathbb{Z}[[q]]$  of power series in the variable  $q$ . This allows for the following definition due to Leinster.

**Definition 4.1.2** ([Lei14]). *Let  $G$  be a graph and  $Z_G(q)$  its similarity matrix. The magnitude  $\#G = \#G(q)$  of  $G$  is the sum of all entries of the inverse matrix  $Z_G(q)^{-1}$ .*

The magnitude invariant has many cardinality-like properties [Lei08]. For example, the magnitude of a graph  $G$  with no edges is precisely its cardinality as a set:  $\#G = |V(G)|$ . Magnitude is also multiplicative with respect to Cartesian products and, under fairly mild conditions, also satisfies an inclusion-exclusion formula [Lei14]. Another angle from which cardinality-like properties can be seen is the magnitude function. The magnitude function  $f_G(t)$  is the partially defined function of the extended real numbers obtained by setting  $q = e^{-t}$ . Although this function may have singularities, it is known to be increasing for large enough  $t$  and satisfies  $\lim_{t \rightarrow \infty} f_G(t) = |V(G)|$ . These observations and their full details can be found in [LW13; Lei14].

Due to the alternating nature of the formula for Euler characteristic, an alternating sum formula for an invariant is often seen as a potential starting point for a categorification, i.e. can we build a chain complex whose graded Euler characteristic is the invariant in question? In the case of magnitude, it was the alternating sum formula of Proposition 4.1.3 that was the starting point of Hepworth and Willerton's construction of magnitude homology, a categorification of the magnitude power series.

**Proposition 4.1.3** ([Lei14], Proposition 3.9). *For any graph  $G$ ,*

$$\#G(q) = \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \dots \neq x_k} q^{d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k)}$$

*where the  $x_i$  denote vertices of  $G$ . That is, if  $\#G(q) = \sum_{n=0}^{\infty} c_n q^n$ , then the  $c_n$  are given by*

$$c_\ell = \sum_{k=0}^{\ell} (-1)^k |\{(x_0, x_1, \dots, x_k) : x_0 \neq x_1 \neq \dots \neq x_k, d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k) = \ell\}|$$

Let us now recall Hepworth and Willerton's construction [HW15] of the magnitude homology groups of a graph, along with some of its most notable properties.

**Definition 4.1.4.** *A  $k$ -path in  $G$  is a  $(k+1)$ -tuple  $(x_0, x_1, \dots, x_k)$  of vertices in  $G$  with  $x_i \neq x_{i+1}$  and  $d(x_i, x_{i+1}) < \infty$  for each  $0 \leq i \leq k-1$ . The length of a  $k$ -path  $(x_0, x_1, \dots, x_k)$  in  $G$  is*

$$\ell(x_0, x_1, \dots, x_k) = d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k).$$

**Definition 4.1.5** (Magnitude chain complex, [HW15]). *Let  $G$  be a graph. Let  $\text{MC}(G) = \bigoplus_{k,\ell \geq 0} \text{MC}_{k,\ell}(G)$  be the bigraded  $\mathbb{Z}$ -module with components*

$$\text{MC}_{k,\ell}(G) := \mathbb{Z}\{\mathbf{x} = (x_0, x_1, \dots, x_k) : x_0 \neq x_1 \neq \dots \neq x_k, \ell(\mathbf{x}) = \ell\}$$

*That is,  $\text{MC}(G)$  is generated in bigrading  $(k, \ell)$  by all  $k$ -paths in  $G$  of length  $\ell$ . For  $1 \leq i \leq k-1$ , define maps  $\partial_i : \text{MC}_{k,\ell}(G) \rightarrow \text{MC}_{k-1,\ell}(G)$  by*

$$\partial_i(x_0, x_1, \dots, x_k) = \delta^{\ell, \ell(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k)}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$$

*That is,  $\partial_i$  removes vertex  $x_i$  if the length of the path is preserved, and is the zero map otherwise. Defining maps  $\partial : \text{MC}_{k,\ell} \rightarrow \text{MC}_{k-1,\ell}$  by  $\partial = \sum_{i=1}^{k-1} (-1)^i \partial_i$  we find that  $(\text{MC}_{*,\ell}, \partial)$  forms a chain complex for each  $\ell \geq 0$ . The magnitude homology of  $G$  is the bigraded  $\mathbb{Z}$ -module  $\text{MH}(G)$  given in bigrading  $(k, \ell)$  by*

$$\text{MH}_{k,\ell}(G) := H_k(\text{MC}_{*,\ell}(G)).$$

Let us recall some basic properties of magnitude homology. First, magnitude homology categorifies the magnitude invariant in the sense that the magnitude of a graph  $G$  may be recovered as the graded Euler characteristic of its magnitude homology:

$$\begin{aligned} \chi_q(\text{MC}(G)) &= \sum_{\ell \geq 0} \left( \sum_{k \geq 0} (-1)^k \text{rk}(\text{MH}_{k,\ell}(G)) \right) \cdot q^\ell \\ &= \sum_{\ell \geq 0} \left( \sum_{k \geq 0} (-1)^k \text{rk}(\text{MC}_{k,\ell}(G)) \right) \cdot q^\ell \end{aligned} \tag{4.1}$$

$$= \#G(q). \tag{4.2}$$

Line (4.1) holds because the differential is degree 0 (see Definition 2.0.15), while line (4.2) follows by Proposition 4.1.3. Second, magnitude homology is strictly stronger than magnitude. For instance, the Rook(4,4) and Shrikhande graphs have the same magnitude but non-isomorphic magnitude homology groups [Gu18]. Third, the multiplicativity of magnitude with respect to Cartesian products of graphs lifts to a Künneth sequence in magnitude homology [HW15].

**Definition 4.1.6** ([HW15]). *The Cartesian product  $G_1 \square G_2$  of graphs  $G_1$  and  $G_2$  has vertex set  $V(G_1) \times V(G_2)$  and an edge between vertices  $(x_1, x_2)$  to  $(y_1, y_2)$  if either  $x_1 = y_1$  and there is an edge in  $G_2$  between  $x_2$  and  $y_2$ , or  $x_2 = y_2$  and there is an edge between  $x_1$  and  $y_1$  in  $G_1$ .*

**Theorem 4.1.7** ([HW15]). *Let  $G_1$  and  $G_2$  be graphs. Then, magnitude satisfies the formula*

$$\#(G_1 \square G_2) = \#G_1 \cdot \#G_2 \tag{4.3}$$

**Theorem 4.1.8** (Künneth sequence, [HW15]). *Let  $G_1$  and  $G_2$  be graphs. Then, magnitude homology satisfies a short exact sequence of the form*

$$0 \rightarrow \text{MH}_{*,*}(G_1) \otimes \text{MH}_{*,*}(G_2) \rightarrow \text{MH}_{*,*}(G_1 \sqcup G_2) \rightarrow \text{Tor}_1^{\mathbb{Z}}(\text{MH}_{*-1,*}(G_1), \text{MH}_{*,*}(G_2)) \rightarrow 0. \quad (4.4)$$

The Künneth sequence (4.4) lifts equation (4.3) in the sense that taking the graded Euler characteristic of the former yields the latter.

The fourth property noted here is a Mayer-Vietoris-type sequence for magnitude homology. This sequence relates the magnitude homology of a union of two subgraphs to the magnitude of the subgraphs and their intersection. As we will see, this sequence lifts the inclusion-exclusion formula for magnitude and holds for so-called projecting decompositions. The following definitions can be found in [Lei14] and [HW15].

**Definition 4.1.9** (Convex subgraph, [Lei14]). *Let  $H$  be a subgraph of a graph  $G$ , let  $d$  be the shortest path metric on  $G$ , and let  $d_H$  be the shortest path metric  $H$ .  $H$  is said to be convex in  $G$  if  $d_H(x, y) = d(x, y)$  for vertices  $x, y \in H$ .*

**Definition 4.1.10** (Projecting subgraph, [Lei14]). *Let  $H$  be a convex subgraph of a graph  $G$ , and write*

$$V_H(G) = \bigcup_{h \in H} \{x \in G \mid d(x, h) < \infty\}.$$

*In other words,  $V_H(G)$  consists of those vertices of  $G$  in the same connected component of  $H$ .  $G$  is said to project onto  $H$  if for each  $x \in V_H(G)$  there is a vertex  $\pi(x) \in H$  such that for each  $h \in H$ ,*

$$d(x, h) = d(x, \pi(x)) + d(\pi(x), h).$$

If a graph  $G$  projects onto a convex subgraph  $H$ , then we get a projection map  $\pi : V_H(G) \rightarrow V(H)$ ,  $x \mapsto \pi(x)$ , sending each  $x$  to its unique closest neighbor in  $H$ . This map is analogous to the projection map onto a closed, convex subset of Euclidean space.

**Example 4.1.11.** *For a positive integer  $r \geq 2$ , the cycle graph  $C_r$  has vertex set  $V = \mathbb{Z}_r$  and edge set  $E = \{\{i, i+1\} : i \in \mathbb{Z}_r\}$ . Each even cycle graph  $C_{2n}$  projects onto a single edge, but no odd cycle graph  $C_{2n+1}$  projects onto a single edge.*

**Definition 4.1.12** (Projecting decomposition, [HW15]). *Let  $H_1$  and  $H_2$  be subgraphs of a graph  $G$ . The triple  $(G; H_1, H_2)$  is said to be a projecting decomposition of  $G$  if the following hold:*

1.  $G = H_1 \cup H_2$
2.  $H_1 \cap H_2$  is convex in  $G$
3.  $H_1$  projects onto  $H_1 \cap H_2$ .

**Theorem 4.1.13** (Inclusion-exclusion formula, [Lei14]). *Let  $(G; H_1, H_2)$  be a projecting decomposition of a graph  $G$ . Then, magnitude satisfies  $\#G = \#H_1 + \#H_2 - \#(H_1 \cap H_2)$ .*

This inclusion-exclusion formula lifts to a Mayer-Vietoris-type sequence in magnitude homology.

**Theorem 4.1.14** (Mayer-Vietoris for magnitude homology, [HW15]). *Let  $(G; H_1, H_2)$  be a projecting decomposition of a graph  $G$ . Then, magnitude homology satisfies a short exact sequence of the form*

$$0 \rightarrow \mathrm{MH}_{*,*}(H_1 \cap H_2) \rightarrow \mathrm{MH}_{*,*}(H_1) \oplus \mathrm{MH}_{*,*}(H_2) \rightarrow \mathrm{MH}_{*,*}(G) \rightarrow 0 \quad (4.5)$$

*where the middle two maps are induced by inclusions. Moreover, the sequence splits.*

The Mayer-Vietoris sequence lifts the inclusion-exclusion formula in the sense that taking the graded Euler characteristic of the former yields the latter. We make use of the Mayer-Vietoris sequence (4.5) of Section 4.3 when it comes to computing homology groups of a family of outerplanar graphs.

## 4.2 Torsion in Magnitude Homology

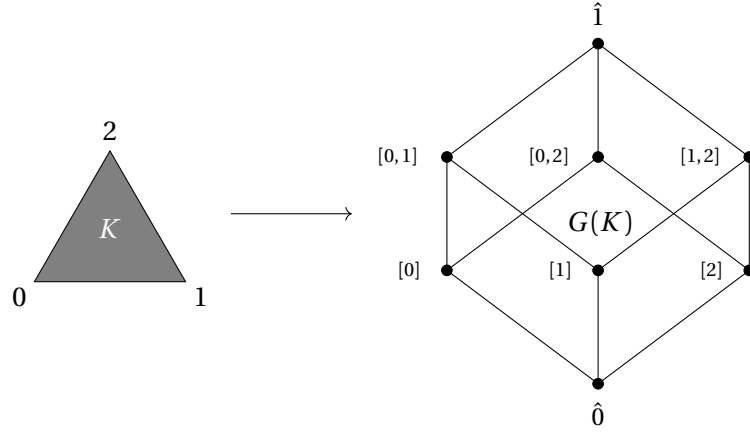
In this section, we review the current state of knowledge regarding torsion in magnitude homology, show that torsion of a prescribed prime order can be found in magnitude homology, and construct infinite families of graphs with a given order of torsion in magnitude homology.

### 4.2.1 Torsion of prime order $p$ in magnitude homology

In 2017, Kaneta and Yoshinaga demonstrated the existence of a graph with torsion of order two in magnitude homology [KY18]. More specifically, they describe a method for constructing a graph from a simplicial complex, such that the homology of the simplicial complex embeds into the magnitude homology of the resulting graph. Next, we describe this construction and extend their result on torsion of order two by constructing graphs whose magnitude homology contains torsion of a given order, and more generally a subgroup isomorphic to a given finitely generated abelian group.

**Definition 4.2.2** (Face poset, [LW06]). *Let  $K$  be a simplicial complex. The face poset of  $K$  is the partially ordered set whose elements are the faces of  $K$ , ordered by inclusion. Denote this poset by  $P(K)$ .*

**Definition 4.2.3** ([KY18]). *Let  $K$  be a simplicial complex of dimension  $m$  such that each face of  $K$  is the face of an  $m$ -simplex. Kaneta and Yoshinaga construct a graph  $G(K)$  as follows. Let  $\widehat{P(K)}$  be the poset obtained from  $P(K)$  by adjoining a minimum element  $\hat{0}$  (if it does not already have one) and a maximum element  $\hat{1}$  (if it does not already have one). Then,  $G(K)$  is the underlying graph of the Hasse diagram of  $\widehat{P(K)}$ .*



**Figure 4.2** A minimal triangulation  $K$  of a disc and the underlying graph  $G(K)$  of the poset  $\widehat{P(K)}$ .

Recall that a triangulation of a topological space  $X$  is a pair  $(K, h)$  where  $K$  is a simplicial complex and  $h : |K| \rightarrow X$  is a homeomorphism of the geometric realization  $|K|$  of  $K$  with  $X$ . By a small abuse of notation, let us write  $K = (K, h)$ . The following lemma is our main tool for producing graphs with torsion in magnitude homology.

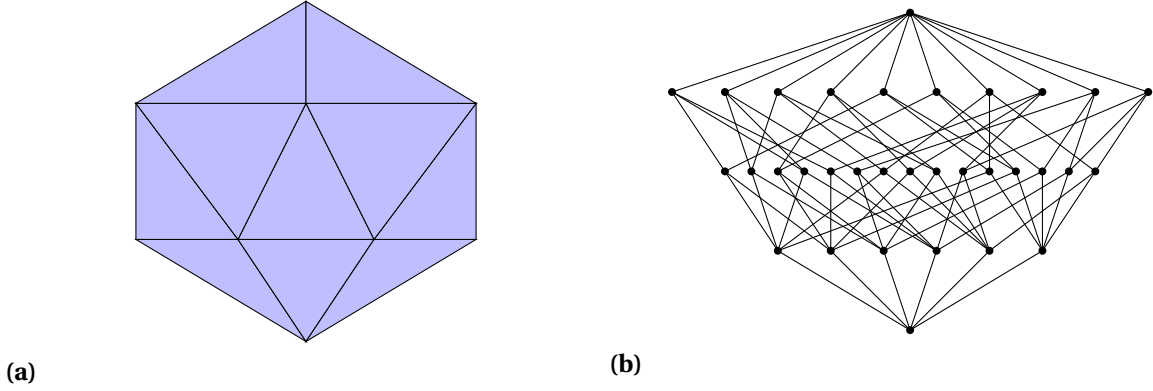
**Theorem 4.2.4** ([KY18], Corollary 5.12 (a)). *Let  $K$  be a triangulation of a manifold  $M$  and let  $\ell = d(\hat{0}, \hat{1})$  be the distance between to the minimum and maximum elements of  $\widehat{P(K)}$ . For each  $k \geq 1$  there is an embedding of the reduced singular homology groups of  $M$  into the magnitude homology of the associated graph  $G(K)$ ,*

$$\tilde{H}_{k-2}(M) \hookrightarrow \text{MH}_{k,\ell}(G(K)). \quad (4.6)$$

**Example 4.2.5** ([KY18]). *Let  $K$  be a minimal triangulation of  $\mathbb{RP}^2$  as shown in Figure 4.3 (a). By Theorem 4.2.4 there is an embedding*

$$\mathbb{Z}_2 \cong H_1(\mathbb{RP}^2) \hookrightarrow \text{MH}_{3,4}(G(K))$$

.



**Figure 4.3** (a) A plane drawing of a minimal triangulation  $K$  of  $\mathbb{RP}^2$ , with outer edges appropriately identified. (b) The graph  $G(K)$  obtained from this triangulation using the approach of Kaneta and Yoshinaga.

We now extend this result of Kaneta and Yoshinaga in two directions. First, we show there exist graphs with torsion of order two in magnitude homology in bigradings other than  $(3, 4)$ . Then, we use lens spaces to show that there are graphs with torsion of a given order in magnitude homology.

**Theorem 4.2.6.** *For any odd integer  $k \geq 3$ , there is a graph  $G$  such that  $\text{MH}_{k,k+1}(G)$  contains a subgroup isomorphic to  $\mathbb{Z}_2$ .*

*Proof.*  $\tilde{H}_{k-2}(\mathbb{RP}^{k-1}) \cong \mathbb{Z}_2$  and for a triangulation  $K$  of  $\mathbb{RP}^{k-1}$ , Theorem 4.2.4 gives an embedding  $\tilde{H}_{k-2}(\mathbb{RP}^{k-1}) \hookrightarrow \text{MH}_{k,k+1}(G(K))$ . □

For coprime integers  $p$  and  $q$ , the lens space  $L(p, q)$  is a three-dimensional triangulable manifold. Since the fundamental group (hence first homology group) of  $L(p, q)$  is isomorphic to  $\mathbb{Z}_p$ , it follows by Kaneta and Yoshinaga's embedding Theorem 4.2.4 that torsion of a given order can show up in the magnitude homology of a graph. However, this approach would only guarantee the existence of graphs with torsion in polynomial degree 5. Using generalized lens spaces, we can prove a more general result.

**Definition 4.2.7** (Generalized lens space, [Li07]). *Let  $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^n |z_i|^2 = 1\}$  be the unit sphere in  $\mathbb{C}^{n+1}$ . Let  $p, q_1, q_2, \dots, q_n$  be integers with  $\gcd(p, q_i) = 1$  for each  $1 \leq i \leq n$ . Consider the action of  $\mathbb{Z}_p$  on  $S^{2n+1}$  defined for each  $g \in \mathbb{Z}_p$  by*

$$g \cdot (z_0, z_1, \dots, z_n) = \left( z_0 e^{\frac{2\pi i g}{p}}, z_1 e^{\frac{2\pi i g q_1}{p}}, z_2 e^{\frac{2\pi i g q_2}{p}}, \dots, z_n e^{\frac{2\pi i g q_n}{p}} \right).$$

*The generalized lens space  $L(p, q_1, q_2, \dots, q_n)$  is the quotient space  $S^{2n+1}/\mathbb{Z}_p$ .*

The lens space  $L(p, q_1, q_2, \dots, q_n)$  is a triangulable  $(2n + 1)$ -dimensional manifold with fundamental group isomorphic to  $\mathbb{Z}_p$  [RT18; Li07].

**Theorem 4.2.8.** *For each prime  $p$  and positive integer  $r$ , there is a graph with  $\mathbb{Z}_{p^r}$  torsion in its magnitude homology. More specifically, for integers  $n, r \geq 1$  and each prime  $p$ , there is a graph  $G$  such that  $\text{MH}_{3,2n+3}(G)$  contains  $\mathbb{Z}_{p^r}$  torsion.*

*Proof.* Let  $G(K)$  the graph obtained from a triangulation  $K$  of the lens space  $L(p^r, q_1, q_2, \dots, q_n)$ . By Theorem 4.2.4, there is an embedding

$$\mathbb{Z}_{p^r} \cong \pi_1(L(p^r, q_1, q_2, \dots, q_n)) \cong \tilde{H}_1(L(p^r, q_1, q_2, \dots, q_n)) \hookrightarrow \text{MH}_{3,2n+3}(G(K)).$$

□

**Definition 4.2.9.** *Let  $r \geq 1$  be a positive integer and  $G$  be a graph. The  $r^{\text{th}}$  diagonal of the magnitude homology of  $G$  is the sequence of groups  $(\text{MH}_{\ell-r+1, \ell}(G))_{\ell \geq r-1}$ . We refer to the  $1^{\text{st}}$  diagonal as the main diagonal.*

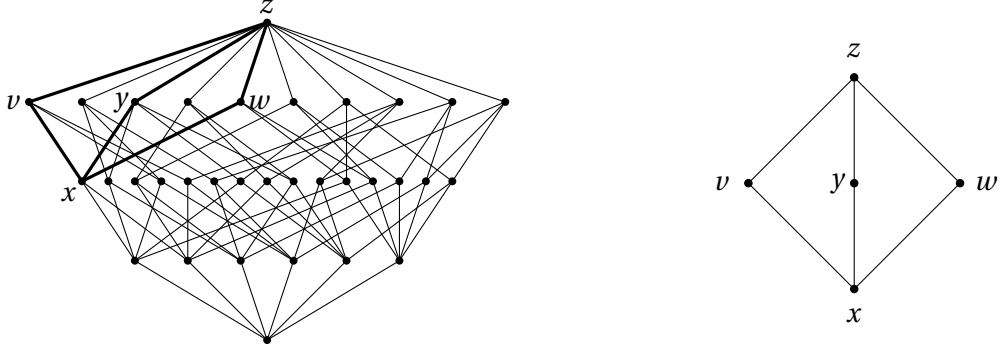
Note that the proof of Theorem 4.2.6 shows there can be torsion of order two in any bigrading along the second diagonal of magnitude homology, while Theorem 4.2.8 shows that, for any odd integer  $r \geq 3$ , torsion of arbitrary prime order can be found in homological degree 3 of the  $r^{\text{th}}$  diagonal.

So far, we have produced a single graph with torsion of a desired order. The following questions arise.

- Is there a graph with torsion in magnitude homology which is not obtained from a triangulation via the method of Kaneta and Yoshinaga?
- Can we produce entire families of graphs with torsion of a given prime order?

**Proposition 4.2.10.** *There is a graph  $G$ , not obtained from a triangulation via the Kaneta-Yoshinaga construction, with torsion of order two in magnitude homology.*

*Proof.* The program `rational_graph_homology_arxiv.py` was uploaded along with Hepworth and Willerton's paper [HW15], and can be used to calculate magnitude homology groups over  $\mathbb{Q}$  and over finite fields  $\mathbb{Z}_p$ . Calculations using this program show that removing a single edge from the graph of Figure 4.3b produces a graph whose magnitude homology has the same rank over  $\mathbb{Q}$  and over  $\mathbb{Z}_2$  coefficients. Consequently, removing a single edge destroys the property leading to  $\mathbb{Z}_2$  torsion in the magnitude homology of this graph. On the other hand, adding a single edge to the same graph sometimes vanishes the  $\mathbb{Z}_2$  torsion and sometimes does not. For instance, the graph in Figure 4.4 contains  $\mathbb{Z}_2$  torsion. □



**Figure 4.4** A graph obtained from that in Figure 4.3b by adding the single edge  $\{x, y\}$ . This graph cannot be obtained via the Kaneta-Yoshinaga construction due to the presence of the forbidden 5-vertex subgraph with vertices  $v, w, x, y$  and  $z$  (bold).

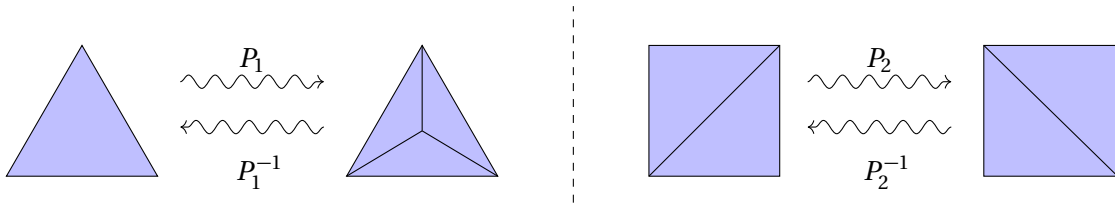
#### 4.2.11 Infinite families of graphs with $\mathbb{Z}_{p^m}$ torsion in magnitude homology

In this section we show that there are infinitely many graphs containing torsion of a prescribed order.

**Definition 4.2.12** ([Pac91]). *Let  $K$  be a triangulation of an  $m$ -manifold  $M$ . Let  $A \subset K$  be a subcomplex of dimension  $m$ , and let  $\varphi : A \rightarrow A' \subset \partial \Delta^{m+1}$  be a simplicial isomorphism. The Pachner move associated to the triple  $(K, A, \varphi)$  is the adjunction space*

$$P_\varphi K := (K - A) \sqcup_\varphi (\partial \Delta^{m+1} - A')$$

In dimension 1, Pachner moves consist of either subdividing an edge into two edges, or the reverse. Pachner moves in dimension 2 are illustrated in Figure 4.5.



**Figure 4.5** Pachner moves on simplicial complexes of dimension 2.

Pachner moves send simplicial complexes to simplicial complexes, and preserve the underlying manifold in the sense that applying a Pachner move leaves the homeomorphism class of the geometric realization unchanged [Pac91]. Applying a Pachner move to a simplicial complex gives rise

to a corresponding change on the level of graphs:  $G(K) \mapsto G(P_\varphi K)$ . As a consequence of Theorem 4.2.4 we have the following.

**Corollary 4.2.13.** *Let  $K$  and  $K'$  be triangulations of a manifold  $M$  related by a finite sequence of Pachner moves. For each  $k \geq 1$ , both  $\text{MH}_{k,*}(G(K))$  and  $\text{MH}_{k,*}(G(K'))$  contain a subgroup isomorphic to  $\tilde{H}_{k-2}(M)$ .*

**Theorem 4.2.14.** *Let  $k \geq 3$  be an integer. There exist infinitely many distinct classes of graphs whose magnitude homology contains  $\mathbb{Z}_2$  torsion in bigrading  $(k, k+1)$ .*

*Proof.* Let  $K$  be a triangulation of  $\mathbb{RP}^{k-1}$  and define a sequence of graphs  $(G_r)_{r \in \mathbb{N}}$  as follows. Let  $K_1 = K$  and for  $r \geq 1$ , define  $K_{r+1} = P_{\varphi_r} K_r$  where  $\varphi_r : A_r \rightarrow \partial \Delta^k$  is a simplicial isomorphism and  $A_r$  is a  $(k-1)$ -simplex of  $K_r$ . Now set  $G_r = G(K_r)$ . These graphs are mutually distinct because, for example, the  $\varphi_r$  have been chosen so that the number of simplices in  $K_{r+1}$  is strictly greater than in  $K_r$ , and correspondingly we have  $|V(G(K_{r+1}))| > |V(G(K_r))|$ . For each  $r \geq 1$ ,  $\text{MH}_{k,k+1}(G_r)$  contains a subgroup isomorphic to  $\mathbb{Z}_2$  by Theorem 4.2.4 and Corollary 4.2.13.  $\square$

**Theorem 4.2.15.** *Let  $p$  be a prime and  $n, m \geq 1$  integers. There exist infinitely many distinct isomorphism classes of graphs whose magnitude homology contains  $\mathbb{Z}_{p^m}$  torsion in bigrading  $(3, 2n+3)$ .*

*Proof.* Let  $p$  be a prime and  $q_1, q_2, \dots, q_n$  be integers coprime to  $p^m$ . Let  $K$  be a triangulation of the lens space  $L(p^m, q_1, q_2, \dots, q_n)$ . Set  $K_1 = K$  and for  $r \geq 1$  define  $K_{r+1} = P_{\varphi_r} K_r$  where  $\varphi_r : A_r \rightarrow \partial \Delta^{2n+2}$  is a simplicial isomorphism and  $A_r$  is a  $(2n+1)$ -simplex of  $K_r$ . By a similar argument as in the proof of Theorem 4.2.14, the graphs  $G_r = G(K_r)$  are mutually distinct, and Theorem 4.2.4 and Corollary 4.2.13 imply that  $\text{MH}_{3,2n+3}(G_r)$  contains a subgroup isomorphic to  $\mathbb{Z}_{p^m}$  for each  $r \geq 1$ .  $\square$

Another way to produce families of graphs with a given prime order torsion was suggested to me by a member of my committee, Dr. Tye Lidman, to whom I am very grateful for the suggestion. It goes as follows. Let  $p$  be a prime and for each  $n \geq 1$ , let  $K_n$  be a triangulation of the lens space  $L(p^n, 1)$ . By Theorem 4.2.4,  $\text{MH}(G(K_n))$  has a subgroup isomorphic to  $\mathbb{Z}_{p^n}$ . Dr. Lidman also pointed out that we can take any finitely generated abelian group and realize it as a subgroup of the singular homology of a topological space. Furthermore, we can always choose such a space that is *triangulable*. As a consequence of the embedding of Theorem 4.2.4, we thus have the following.

**Theorem 4.2.16.** *Let  $M$  be any finitely generated finite abelian group. Then, there exists a graph  $G$  whose magnitude homology  $\text{MH}(G)$  contains a subgroup isomorphic to  $M$ .*

*Proof.* By the fundamental theorem of finitely generated abelian groups [DF04],  $M$  is of the form

$$M \cong \mathbb{Z}^r \oplus \mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{r_m}}$$

for some integers  $r, r_i$  and  $m$ , and primes  $p_i$  (not necessarily distinct). The lens spaces  $L(p^{r_i}, 1)$  have first homology groups  $H_1(L(p^{r_i}, 1)) \cong \mathbb{Z}_{p^{r_i}}$ . For manifolds  $X$  and  $Y$  of the same dimension, the homology groups satisfy  $H_k(X \# Y) \cong H_k(X) \oplus H_k(Y)$  for each  $k \geq 0$ , where  $\#$  denotes the connected sum. This identity extends by induction to  $m$ -fold connected sums. Consequently,

$$H_1(L(p^{r_1}, 1) \# L(p^{r_2}, 1) \# \cdots \# L(p^{r_m}, 1)) \cong \mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{r_m}}.$$

Let  $K$  be a triangulation of the 3-manifold  $L(p^{r_1}, 1) \# L(p^{r_2}, 1) \# \cdots \# L(p^{r_m}, 1)$ . Then, by Theorem 4.2.4, the graph  $G(K)$  obtained via the Kaneta-Yoshinaga construction has magnitude homology with a subgroup isomorphic to  $\mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{r_m}}$ . For any graph  $G$  with at least one edge,  $\text{rank}(\text{MH}_{k,k}(G)) \geq 1$  for every  $k \geq 0$ . Consequently,  $G(K)$  also has a subgroup isomorphic to  $\mathbb{Z}^r$  for every integer  $r$ . Consequently, the magnitude homology of  $G(K)$  has a subgroup isomorphic to  $M$ .  $\square$

### 4.3 Formulas for outerplanar graphs and graphs with no 3- or 4-cycles

In this section we use Gu's computations of the magnitude homology groups of cycle graphs [Gu18] to compute the magnitude homology of a family of outerplanar graphs, compute the main diagonal of all graphs with no cycles of length 3 or 4, and put forth several conjectures regarding the main diagonal for other families of graphs based on calculations performed in Python.

#### 4.3.1 Cycle graphs and graphs without 3- or 4-cycles

**Proposition 4.3.2.** *The magnitude homology of the cycle graph  $C_3$  is torsion-free, supported on the main diagonal, and satisfies  $\text{MH}_{k,k}(C_3) \cong \mathbb{Z}^{3 \cdot 2^k}$ .*

*Proof.* Consider a generator  $(v_0, v_1, \dots, v_k)$  of  $\text{MC}_{k,k}(C_3)$ . Since  $d(v_{j-1}, v_{j+1}) \leq 1$  while  $d(v_{j-1}, v_j) + d(v_j, v_{j+1}) = 2$ , it follows that  $\partial(v_0, v_1, \dots, v_i) = 0$  for every generator. Hence,  $\text{MH}_{k,k}(C_3) \cong \text{MC}_{k,k}(C_3)$ . There are three choices of initial vertex  $v_0$  and two choices for each of the subsequent vertices  $v_1, v_2, \dots, v_k$ , giving a total of  $3 \cdot 2^k$  generators. For the last statement, simply note that for  $l \neq k$  we have  $l(v_0, v_1, \dots, v_k) = k \neq l$ , so that  $\text{MC}_{k,\ell}(C_3) = 0$  for  $l \neq k$ .  $\square$

Using the Künneth sequence, Hepworth and Willerton computed the magnitude homology of the cycle graph  $C_4 = K_2 \square K_2$  as follows.

**Proposition 4.3.3** ([HW15]). *The magnitude homology of the cycle graph  $C_4$  is torsion-free, supported on the main diagonal and satisfies  $\text{MH}_{k,k}(C_4) \cong \mathbb{Z}^{4+4k}$ .*

**Theorem 4.3.4.** *Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . If  $G$  has no 3- or 4-cycles, then the first diagonal in the magnitude homology of  $G$  is torsion-free and satisfies*

$$\text{MH}_{k,k}(G) \cong \begin{cases} \mathbb{Z}^{|V|} & k = 0, \\ \mathbb{Z}^{2|E|} & k > 0. \end{cases}$$

*Proof.* Note that  $\text{MC}_{k+1,k}(G) = 0$ , so  $\text{MH}_{k,k}(G)$  is the kernel of the map  $\partial : \text{MC}_{k,k}(G) \longrightarrow \text{MC}_{k-1,k}(G)$ . For  $k = 0$ , this kernel is generated by the vertices of  $V$ . For  $k = 1$ , there are  $2|E|$  generating tuples  $(x_0, x_1)$  for  $\text{MC}_{k,k}(G)$ , and each has differential zero. For  $k \geq 2$ , assume without loss of generality that  $k$  is odd, and consider the set

$$B = \{(v, w, v, w, \dots, w), (w, v, w, v, \dots, v) \mid \{v, w\} \in E\}.$$

Since  $|B| = 2|E|$ , it suffices to show that  $B$  forms a basis for the kernel of  $\partial : \text{MC}_{k,k}(G) \longrightarrow \text{MC}_{k-1,k}(G)$ . Indeed,  $\text{MC}_{k,k}(G)$  is, by definition, generated by tuples  $(x_0, x_1, \dots, x_k)$  with  $d(x_i, x_{i+1}) = 1$  for each  $0 \leq i \leq k-1$ . For such a tuple to lie in the kernel of  $\partial$ , it must satisfy  $\partial(x_0, x_1, \dots, x_k) = 0$ . This happens if and only iff  $\partial_i(x_0, x_1, \dots, x_k) = 0$  for each  $1 \leq i \leq k-1$ . If there is an index  $i$  with  $x_i \neq x_{i+2}$ , then  $\partial_i(x_0, x_1, \dots, x_k) = 0$  and  $x_i, x_{i+1}$  and  $x_{i+2}$  form a 3-cycle in  $G$ . But  $G$  has no 3-cycles, so no such generators of  $\text{MC}_{k,k}(G)$  are in the kernel of  $\partial$ . It remains to show that no linear combination of generating tuples lies in the kernel of  $\partial$ . Let  $c$  be a linear combination of generating tuples for  $\text{MC}_{k,k}(G)$ , and assume without loss of generality that no tuple in  $c$  belongs to the set  $B$ . Let  $(x_0, x_1, \dots, x_k) \in c$ . For some  $1 \leq i \leq k-1$ , we have  $d(x_{i-1}, x_{i+1}) = 2$ . Since  $\partial(c) = 0$ , the tuple  $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \in \partial(c)$  must cancel with  $\partial_i(x_0, x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_k) \in c - (x_0, x_1, \dots, x_k)$ . Then,  $x_{i-1}, x_i, x_{i+1}$  and  $y$  form a 4-cycle in  $G$ . But  $G$  has no 4-cycles. □

#### 4.3.5 Outerplanar Graphs

This sole dependence of the main diagonal on the number of vertices and edges extends to certain types of outerplanar graphs.

**Definition 4.3.6.** A graph  $G$  is outer planar if it has a plane drawing with no crossings all of whose vertices lies on an outer face of  $G$ . Equivalently, a graph is outer planar if it can be constructed from a finite collection  $\{H_1, H_2, \dots, H_t\}$  of copies of  $K_2$  (the complete graph on two vertices i.e. a single edge) and cycle graphs  $C_n$ , by gluing along single vertices or edges as follows.  $G = H_1 \star H_2 \star \dots \star H_t$  where for each  $1 \leq s \leq t$ ,  $H_1 \star H_2 \star \dots \star H_{s-1} \star H_s$  is formed from  $H_1 \star H_2 \star \dots \star H_{s-1}$  by identifying an edge/vertex of  $H_s$  with an outer-face edge/vertex of  $H_1 \star H_2 \star \dots \star H_{s-1}$ . We refer to the subgraphs  $H_s$  as the components of  $G$ .

**Theorem 4.3.7.** Let  $G$  be an outer planar graph with vertex set  $V$  and edge set  $E$  whose components are either  $K_2$  or even cycles  $C_n$  with  $n \neq 4$ . Then, the main diagonal of the magnitude homology of  $G$

is torsion-free with

$$\text{rank}(\text{MH}_{k,k}(G)) = \begin{cases} |V| & k = 0, \\ 2|E| & k > 0. \end{cases}$$

The proof of this theorem will involve an induction argument, the Mayer-Vietoris sequence, and the following theorem of Hepworth and Willerton.

**Theorem 4.3.8** ([HW15]). *Let  $T$  be a tree with vertex set  $V$  and edge set  $E$ .  $\text{MH}_{*,*}(T)$  is torsion-free group with*

$$\text{rank}(\text{MH}_{k,\ell}(T)) = \begin{cases} |V(T)| & k = \ell = 0, \\ 2|E(T)| & k = \ell \geq 1, \\ 0 & k \neq \ell. \end{cases}$$

*Proof of Theorem 4.3.7.* If  $G$  is a tree, then the result follows from Theorem 4.3.8. Otherwise, suppose  $G = H_1 \star H_2 \star \cdots \star H_t$  and let  $X = H_1 \star H_2 \star \cdots \star H_{t-1}$  and  $Y = H_t$ , where  $H_t = C_n$  for some even integer  $n \geq 6$ . Then,  $(G; X, Y)$  is a projecting decomposition. By the Mayer-Vietoris theorem, there is an isomorphism  $\text{MH}_{*,*}(G) \oplus \text{MH}_{*,*}(X \cap Y) \cong \text{MH}_{*,*}(X) \oplus \text{MH}_{*,*}(Y)$ . In particular,

$$\text{rank}(\text{MH}_{i,i}(G)) = \text{rank}(\text{MH}_{i,i}(X)) + \text{rank}(\text{MH}_{i,i}(Y)) - \text{rank}(\text{MH}_{i,i}(X \cap Y)). \quad (4.7)$$

Since  $X \cap Y$  is a tree, its magnitude homology is torsion-free with

$$\text{rank}(\text{MH}_{k,k}(X \cap Y)) = \begin{cases} |V(X \cap Y)| & k = 0, \\ 2|E(X \cap Y)| & k > 0. \end{cases}$$

By Theorem 4.3.4,  $\text{MH}_{k,k}(Y)$  is torsion-free with

$$\text{rank}(\text{MH}_{k,k}(Y)) = \begin{cases} |V(Y)| & k = 0, \\ 2|E(Y)| & k > 0. \end{cases}$$

And by induction, the magnitude homology of  $X$  is torsion-free with

$$\text{rank}(\text{MH}_{k,k}(X)) = \begin{cases} |V(X)| & k = 0, \\ 2|E(X)| & k > 0. \end{cases}$$

Equation (4.7) then gives

$$\begin{aligned} \text{rank}(\text{MH}_{k,k}(G)) &= \begin{cases} |V(X)| + |V(Y)| - |X \cap Y| & k = 0, \\ 2|E(X)| + 2|E(Y)| - 2|E(X \cap Y)| & k > 0, \end{cases} \\ &= \begin{cases} |V(G)| & k = 0, \\ 2|E(G)| & k > 0. \end{cases} \end{aligned}$$

□

In 2015, Hepworth and Willerton proposed the following recursive formula for the magnitude homology of the even cycle graphs based on experimental data [HW15], which was subsequently proved in 2018 by Yuzhou Gu [Gu18] using the tools of algebraic Morse theory.

**Theorem 4.3.9** ([Gu18]). *Fix an integer  $m \geq 3$ . The magnitude homology of the cycle graph  $C_{2m}$  is described as follows.*

(1) *All groups  $\text{MH}_{k,\ell}(C_{2m})$  are torsion-free.*

(2) *Define a function  $T : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  as*

(a)  *$T(k, \ell) = 0$  if  $k < 0$  or  $\ell < 0$ ;*

(b)  *$T(0, 0) = 2m$ ,  $T(1, 1) = 4m$ ;*

(c)  *$T(k, \ell) = \max\{T(k-1, \ell-1), T(k-2, \ell-m)\}$  for  $(k, \ell) \neq (0, 0), (1, 1)$ .*

*Then,  $\text{rank}(\text{MH}_{k,\ell}(C_{2m})) = T(k, \ell)$ .*

As pointed out by Hepworth and Willerton [HW15], the magnitude homology groups for the even cycle graphs are given by the following equivalent, but more explicit, formula.

**Observation 4.3.10** ([HW15]). *Fix an integer  $m \geq 3$ . The magnitude homology groups of the cycle graph are torsion-free and arrange themselves into diagonals. Let  $T_{i,j}^m$  denote the rank of the magnitude homology group in the  $j^{\text{th}}$  entry of the  $i^{\text{th}}$  diagonal. That is,  $T_{i,j}^m = \text{rank}(\text{MH}_{2(i-1)+(j-1), m(i-1)+(j-1)}(C_{2m}))$ . Then,  $T_{i,1}^m = 2m$  for each  $i \geq 1$ , while  $T_{i,j}^m = 4m$  whenever  $i \geq 1$  and  $j \geq 2$ . See Table 4.1.*

Applying the Mayer-Vietoris sequence (4.5) and an analogous induction argument to that given in the proof of Theorem 4.3.7 yields the following explicit formula for not only the first diagonal of magnitude homology, but all magnitude homology groups for a family of outerplanar graphs constructed from even cycles.

**Theorem 4.3.11.** *Fix a positive integer  $m \geq 3$ . Let  $G$  be an outer planar graph with  $S$  components  $C_4$ , and  $R$  component cycles  $C_{2m}$ , constructed using edge-gluing only. The magnitude homology groups*

	0	1	2	3	4	5	6	7	8	9	10
0	8										
1		16									
2			16								
3				16							
4			8		16						
5				16		16					
6					16		16				
7						16		16			
8					8		16		16		
9						16		16		16	
10							16		16		16

**Table 4.1** The ranks of the torsion-free magnitude homology groups of the cycle graph  $C_8$  [HW15].

of  $G$  are torsion-free and arrange themselves in diagonals: let  $S_{i,j}^m$  denote the rank of the magnitude homology group of  $G$  in the  $j^{\text{th}}$  entry of the  $i^{\text{th}}$  diagonal, that is,  $S_{i,j}^m = \text{rank}(\text{MH}_{2(i-1)+(j-1), m(i-1)+(j-1)}(G))$ . Then, the magnitude homology groups  $\text{MH}_{k,\ell}$  of  $G$  are all trivial groups except for the groups the aforementioned diagonals, and these satisfy for  $i > 1$

$$\text{rank}(S_{1,j}^m) = \begin{cases} 2mR + 4S - 2(R + S - 1) & j = 1, \\ 4mR + 4jS - 2(R + S - 1) & j > 1, \end{cases}$$

$$\text{rank}(S_{i,j}^m) = \begin{cases} 2mR + 4S - 2(R + S - 1) & j = 1, \\ 4mR - 2(R + S - 1) & j > 1. \end{cases}$$

## 4.4 Future work

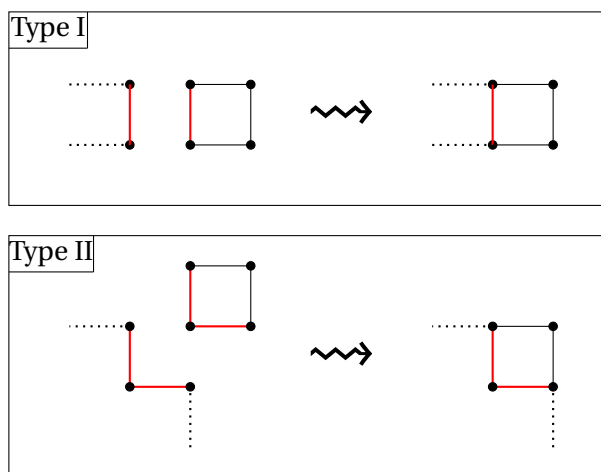
As a very new invariant of graphs, not much is yet known about the relationship between structural properties of graphs and their magnitude homology groups. One such relationship investigated here was a connection between the building blocks (component cycles) of certain outer planar graphs and the ranks of magnitude homology groups. In Theorem 4.3.11, we computed the magnitude homology groups of a family of outer planar graphs built from even cycle graphs. Despite the fact that Gu has also computed the magnitude homology groups of the odd cycle graphs, we cannot extend Theorem 4.3.11 result to the analogous family built instead from odd cycle graphs, or combinations of even and odd cycle graphs. This is due to the fact that the odd cycle graphs do not project onto their edges, and this means we cannot apply the Mayer-Vietoris sequence (Hepworth and Willerton

showed that the "projecting" condition in the statement of the Mayer-Vietoris sequence is strictly necessary [HW15]). Moving forward, we will attempt to extend Gu's algebraic Morse theory approach to compute the magnitude homology of the remaining outer planar graphs. We would then, for example, be in a better position to determine whether magnitude homology detects outer planarity.

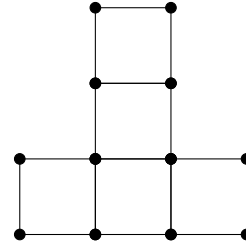
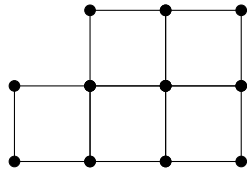
During the course of research, we came to suspect a relationship between the types of torsion found in the magnitude homology of a graph and pairwise counts of geodesics in the graph viewed as a metric space. By a geodesic, we mean a  $k$ -path  $(x_0, x_1, \dots, x_k)$  in a graph  $G$  satisfying  $\ell(x_0, x_1, \dots, x_k) = d(x_0, x_k)$ . In other words, a geodesic is a path of shortest length connecting two vertices. Proving such a relationship is another future avenue.

In pursuit of other connections between graphical structure and magnitude homology, computations performed in Python are an excellent source of data from which we might derive hints. Below, we highlight a few such computations for members of some families of graphs, and offer conjectures for their magnitude homology groups. Many families investigated in this manner over the course of the research for this thesis pertained to graphs obtained by "gluing" cycle graphs of various sizes along edges, collections of edges, or vertices, for example the square polyominoes of Definition 4.4.1. We end by displaying some computations and accompanying conjectures for a small sampling of such families.

**Definition 4.4.1.** *The set  $G_S^4$  of square polyominoes on  $S$  copies of  $C_4$  is defined inductively as follows.  $G_1^4 = C_4$  and each member of  $G_{S+1}^4$  is obtained from a member of  $G_S^4$  by gluing a copy of  $C_4$  via a move of Type I or Type II, shown in Figure 4.6.*



**Figure 4.6** Gluings of Type I and Type II.



**Figure 4.7** Two polyominos of type  $G_4^5$ .

	0	1	2	3	4	5	6
0	11						
1		30					
2			50				
3				70			
4					90		
5						110	
6							130

	0	1	2	3	4	5	6
0	12						
1		32					
2			52				
3				72			
4					92		
5						112	
6							132

**Table 4.2** The ranks of the magnitude homology computations for the two members of the set  $G_5^4$  of polyominos given in Figure 4.7.

Based on the computations given in Table 4.2 along with many others not displayed, we make the following conjecture.

**Conjecture 4.4.2.** *Let  $S$  denote the number of squares  $C_4$  in the square polyomino  $G_S^4$ . The main diagonal of the magnitude homology of a square polyomino is torsion-free and satisfies*

$$\text{rank}(\text{MH}_{k,k}(G_S^4)) \cong \begin{cases} |V(G_S^4)| & k = 0, \\ 2|E(G_S^4)| + 4(i-1)S & k \geq 1. \end{cases}$$

In other words, magnitude homology is counting the number of vertices, edges and squares.

By Theorem 4.3.7, note that Conjecture 4.4.2 holds true for polyominos constructed using exclusively moves of type I. However, we cannot appeal to the Mayer-Vietoris sequence in any simple manner when moves of type II are used for the simple reason that  $C_4$  does not project onto a pair of

its adjacent edges.

Based on the computations in Table 4.3 and others, we posit the following.

**Conjecture 4.4.3.** *The magnitude homology of a graph obtained by gluing two cycle graphs  $C_3$  along single edges to a single cycle graph  $C_4$  has diagonal magnitude homology provided those triangles are not attached to opposite sides of the 4-cycle.*

**Conjecture 4.4.4.** *Gluing any number of cycle graphs  $C_3$  along single edges to a cycle graph  $C_4$  results in a graph with diagonal magnitude homology provided no pair of triangles is glued to opposite faces of the cycle graph  $C_4$ .*



**Figure 4.8** Two graphs obtained by gluing two triangle graphs to a single square graph.

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
0	6								0	6							
1		16							1		16						
2			32						2			32					
3				60					3			2	60				
4					112				4				12	112			
5						212			5					44	212		
6							408		6					2	132	408	
7								796	7						16	356	796

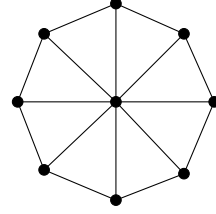
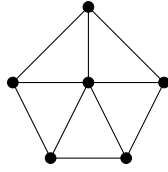
**Table 4.3** The ranks of the magnitude homology of two graphs given in Figure 4.8.

Based on the computations in Table 4.4 we conjecture the following.

**Conjecture 4.4.5.** *Wheel graphs  $W_n$  on  $n$  vertices have diagonal magnitude homology which is*

*torsion-free and satisfies*

$$\text{rank}(\text{MH}_{k,k}(W_n)) \cong \begin{cases} |V(W_n)| & k=0, \\ 2|E(W_n)| \cdot 3^{k-1} & k \geq 1. \end{cases}$$



**Figure 4.9** Wheel graphs  $W_5$  and  $W_8$ .

	0	1	2	3	4	5	6
0	6						
1		20					
2			60				
3				180			
4					540		
5						1620	
6							4860

	0	1	2	3	4	5	6
0	9						
1		32					
2			96				
3				288			
4					864		
5						2592	
6							7776

**Table 4.4** The ranks of the magnitude homology of the wheel graphs  $W_5$  and  $W_8$ .

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