ABSTRACT

SIMON, LILLIAN FAYE PASLEY. Determinantal Representations, Invariance, and the Numerical Range. (Under the direction of Cynthia Vinzant).

We study properties of plane curves that may be certified with a determinantal representation. A well known result due to Helton and Vinnikov relates hyperbolicity of plane curves to definiteness of their representations. We are interested in the properties of hyperbolicity as well as invariance under finite groups. In particular, we guarantee these properties with the structure of its representation.

In Chapter 1 we introduce relevant background information from convex algebraic geometry and group theory. Then we motivate our work with these main questions: 1) what properties of a curve can be guaranteed with a determinantal representation, and 2) how does the geometry of the numerical range relate to properties of its defining plane curve?

In Chapters 2 and 3 we modify a construction of Dixon to produce structured determinantal representations of smooth, invariant hyperbolic plane curves. Moreover, if a plane curve is invariant under the dihedral group, then its representation also has real, linear entries. Plaumann and Vinzant showed to get Hermitian structure, one should use not only the hyperbolic curve, but an interlacing curve as well. Here we describe the explicit polynomial form of a hyperbolic, invariant curve and its interlacing polynomials. The existence of the determinantal representation relies on the geometry of their intersection.

We take a different approach in each of these chapters. In Chapter 2, we examine a special case and choose a specific interlacer. This allows us to prove existence of a structured representation by studying the geometry of their intersection. In Chapter 3, we instead pick any interlacer which satisfies a few additional assumptions to guarantee existence of the representation. We later deal with degenerate cases by discussing topology of curves and their interlacers in general. Finally we discuss hidrances to generalizing the construction to curves of any degree.

By duality, a convex set called the numerical range can be defined in terms of a hyperbolic plane curve. It turns out that the structure of the representation for an invariant, hyperbolic curve is directly related to a matrix whose numerical range is also invariant under rotation. In Chapter 4 we discuss the implications of our result on the geometry of the numerical range. In particular, any numerical range which has rotational symmetry has to coincide with the numerical range of a matrix of certain structure.

Determinantal Representations, Invariance, and the Numerical Range

by
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DEDICATION

"It's amazing the difference a bit of sky can make."

– Shel Silverstein

For Dad.

BIOGRAPHY

Faye was born and raised in Matthews, North Carolina with her sister and a whole bunch of cousins. She attended North Carolina State University for her undergraduate and graduates degrees in Mathematics. During her time at NC State, she played clarinet in the Power Sound of the South marching band as well as the basketball pep band where she met her now husband, Michael. When she isn't doing math, Faye enjoys music, walking, yoga, telling awful jokes, and hanging out with her cats, Zoe and Sam.

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Introduction

The purpose of this dissertation is to study properties of plane curves that may be certified with a *determinantal representation*. More specifically, plane curves satisfying *hyperbolicity* and *invariance* under finite group actions. The interesting part is examining the underlying structure and symmetry of these representations. Moreover, duality then gives us a way to study the geometry of a related convex set called the *numerical range*.

This chapter serves as an overview for the keywords mentioned above. The definitions with respect to plane curves included in this chapter generalize to polynomials in any number of variables. Our main focus in this work will be on projective plane curves, so we restrict the definitions to just three variables for this reason.

1.1 Determinantal Representations and Hyperbolicity

1.1.1 Background

Varieties whose defining equation is the determinant of some matrix, called *determinantal hypersurfaces*, are classically studied [4, 14]. These are defined by polynomials that are the determinant of a matrix with linear entries.

Definition 1.1.1. A **Hermitian determinantal representation** of $f \in \mathbb{R}[t, x, y]_d$ is an expression $f = \det(M)$ where $M = tM_0 + xM_1 + yM_2$ for some Hermitian $M_j \in \mathbb{C}^{d\times d}$. If the matrices M_j are instead (real) symmetric, then $f = \det(M)$ is a (real) symmetric representation. The representation is **definite** if $M(e)$ is positive definite for some $e \in \mathbb{R}^3$.

Example 1.1.2. The curve $f = t^2 - x^2 - y^2$ has a definite Hermitian determinantal representation

$f = \det(M)$ where

$$M = \begin{pmatrix} 5t + 4y & 3it + x + 3iy \\ -3it + x - 3iy & 2t + 2y \end{pmatrix} = \begin{pmatrix} 5 & 3i \\ -3i & 2 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 4 & 3i \\ -3i & 2 \end{pmatrix} y.$$

This is definite because $M(1,0,0) = M_0$ is positive definite.

In convex algebraic geometry, the aim is to study algebraic structure and underlying geometry of convex sets. This area works to synthesize objects and related information studied within algebraic geometry, optimization, and convex geometry. For more see [5]. Definite determinantal representations are of particular interest in the fields of convex algebraic geometry and optimization [3, 24]. In particular, polynomials with definite determinantal representations satisfy a property called hyperbolicity.

Definition 1.1.3. A polynomial $f \in \mathbb{R}[t,x,y]_d$ is **hyperbolic** with respect to $e \in \mathbb{R}^3$ if $f(e) \neq 0$ and the univariate polynomial $f(\lambda e - z) \in \mathbb{R}[\lambda]$ has all real roots for any $z \neq \lambda e$. If all roots of $f(\lambda e - z)$ are also simple for every $z \in \mathbb{R}^3$, then f is **strictly hyperbolic** with respect to $e \in \mathbb{R}^3$. Strict hyperbolicity is equivalent to the condition that $\mathcal{V}_{\mathbb{R}}(f)$ is smooth.

Notice that $f(\lambda e - z) = f(e - (1/\lambda)z)$ for $\lambda \neq 0$ by homogeneity. If the roots of $f(\lambda e - z)$ are all real, then so are the roots of $f(e - \lambda z)$. Any line through $e \in \mathbb{R}^3$ is parametrized by $e - \lambda z$ for $\lambda \in \mathbb{R}$ and hence intersects the hypersurface $\mathcal{V}_{\mathbb{C}}(f)$ in d real points. Topologically, if $\mathcal{V}_{\mathbb{C}}(f)$ is smooth, then f is hyperbolic if and only if $\mathcal{V}_{\mathbb{R}}(f)$ has $\lfloor d/2 \rfloor$ nested ovals as well as a psuedoline when d is odd.

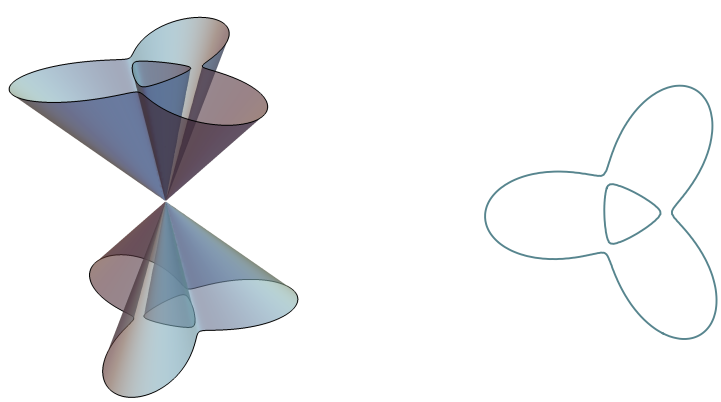

Figure 1.1: A quartic hyperbolic hypersurface in $\mathbb{R}^3$ and $\mathbb{P}^2(\mathbb{R})$.

Denote the set of hyperbolic and strictly hyperbolic forms of degree d by $\mathcal{H}_d$ and $(\mathcal{H}^\circ)_d$ respec-

tively. Nuij [31] showed that $(\mathcal{H}^\circ)_d$ is open and dense in $\mathcal{H}_d$ in the Euclidean topology on $\mathbb{R}[t,x,y]$ meaning that every hyperbolic polynomial is the limit of strictly hyperbolic polynomials.

Hyperbolic polynomials first appeared in the area of partial differential equations due to the work of Gårding [19]. Hyperbolicity has more recently been of interest within convex optimization. Güler [24] and Renegar [33] developed a generalization of semidefinite programming called hyperbolic programming whose feasible sets are hyperbolicity cones. The **hyperbolicity cone** $\mathcal{C}(f,e)$ is the connected component of $\mathbb{R}^3 \setminus \mathcal{V}_{\mathbb{R}}(f)$ containing e. This is a convex cone and f is hyperbolic with respect to any point contained within it [19].

Remark 1.1.4. If $f \in \mathbb{R}[t,x,y]_d$ has a definite determinantal representation $f = \det(M)$ with M_0 positive definite, then $M_0 = UU^*$ for some $U \in \mathbb{C}^{d\times d}$ and $f = \det(M')$ where

$$M' = (U)^{-1}M(U^*)^{-1}$$

and $M_0 = I$. This means f is hyperbolic with respect to $e_0 = (1,0,0)$ since the roots of $f(\lambda e_0 - z) \in \mathbb{R}[\lambda]$ are eigenvalues of the Hermitian matrix $M'(z)$ which are all real for every $z \in \mathbb{R}^3$.

Example 1.1.5. From Example 1.1.5, we have $f = t^2 - x^2 - y^2$ has the determinantal representation $f = \det(M)$ where

$$M = \begin{pmatrix} 5 & 3i \\ -3i & 2 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 4 & 3i \\ -3i & 2 \end{pmatrix} y.$$

The coefficient matrix M_0 is positive definite so we can write $M_0 = UU^*$ where $U = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$. Then $f = \det(M')$ where

$$M' = (U)^{-1}M(U^*)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} -2 & 3i \\ -3i & -4 \end{pmatrix} y.$$

One may ask if the converse statement is true. For which hyperbolic polynomials do there exist definite determinantal representations? The case for plane curves is a well-known result previously conjectured by Lax in 1958. The analogous statement is not true for polynomials in a higher number of variables.

Theorem 1.1.6 (Helton-Vinnikov [28])**.** *Every hyperbolic plane curve has a real, symmetric determinantal representation.*

Dixon gave a construction for symmetric determinantal representations of smooth hyperbolic curves in 1902. Dubrovin and Vinnikov [15, 38] later studied real symmetric and Hermitian determinantal representations of real curves. Definiteness of a representation certifies hyperbolicity, which motivates the following question in a broader context.

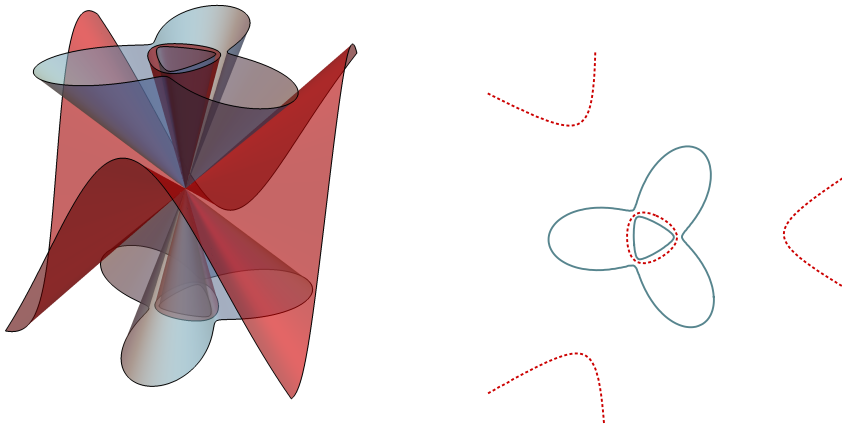

Figure 1.2: A quartic hyperbolic hypersurface and a cubic interlacer in $\mathbb{R}^3$ and $\mathbb{P}^2(\mathbb{R})$.

Question 1.1.7. *What properties of a curve can be certified with a determinantal representation?*

1.1.2 Dixon's Construction for Hermitian Determinantal Representations

In order to produce the desired symmetric representation, Dixon's approach was to first produce a matrix of forms of degree $d-1$ satisfying certain properties. In particular, he observed the structure of a determinantal representation's *adjugate* and utilized this structure within his construction. Plaumann and Vinzant [32] later extended Dixon's construction to produce definite Hermitian determinantal representations of smooth hyperbolic forms. Below we define the adjugate matrix and identify some useful properties for the adjugate of a determinantal representation.

Definition 1.1.8. The **adjugate** of a $d \times d$ matrix M is a matrix of $(d-1) \times (d-1)$ minors defined

$$\operatorname{adj}(M) := \left((-1)^{i+j} M_{ij}^T\right)_{i,j=1}^{d}$$

where M_{ij} is the minor obtained by deleting row i and column j from A.

Definition 1.1.9. Let f and g be real-rooted univariate polynomials of degrees d and $d-1$ where $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_d$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{d-1}$ are the roots of each respectively. The polynomial g **interlaces** f if $\alpha_j \leq \beta_j \leq \alpha_{j+1}$ for every $j \in [n-1]$. If $f \in \mathbb{R}[t,x,y]_d$ is hyperbolic with respect to $e \in \mathbb{R}^3$ we say $g \in \mathbb{R}[t,x,y]_{d-1}$ interlaces f if $g(\lambda e + z)$ interlaces $f(\lambda e + z)$ in $\mathbb{R}[\lambda]$ for every $z \in \mathbb{R}^3$.

Proposition 1.1.10. Let $f \in \mathbb{R}[t,x,y]_d$ and $f = \det(M)$ be a definite Hermitian determinantal representation. Then

(i) $\operatorname{adj}(M)$ is Hermitian.

(ii) each diagonal entry of $\operatorname{adj}(M)$ interlaces f.

(iii) the 2×2 minors of $\operatorname{adj}(M)$ lie in the ideal generated by f [32].

(iv) $\operatorname{adj}(\operatorname{adj}(M)) = f^{d-2}M$.

By constructing a $d \times d$ matrix of forms of degree $d-1$ that satisfy properties (i)–(iv) above, Plaumann and Vinzant were able to prove the following theorem.

Theorem 1.1.11 (Plaumann-Vinzant [32])**.** *Given $f \in \mathcal{H}_d$ with interlacer $g \in \mathbb{R}[t,x,y]_{d-1}$, there exists a definite Hermitian determinantal representation of f such that $f = \det(M)$ and g is the leading $(d-1) \times (d-1)$ diagonal minor of M.*

We later modify this construction in Sections 2.3, 3.1, and 3.2 to address Question 1.1.7 and produce representations with certain structure.

1.2 The Numerical Range

1.2.1 Background

Now we introduce a convex set called the numerical range of a matrix. Also called the field of values, this object originates from functional analysis and gained more popularity in the 1950's due to Kippenhahn [29]. The set is useful in applications related to engineering, numerical analysis, and differential equations [1, 6, 16, 17, 23]. The numerical range can also be defined in terms of a hyperbolic plane curve, so we are also able to examine the convex object by studying this plane curve instead. There exist generalizations of this set, but for simplicity we hold off defining one such generalization until later in Chapter 4.

Definition 1.2.1. The **numerical range** of $A \in \mathbb{C}^{d\times d}$ is

$$\mathcal{W}(A) := \{z^*Az \in \mathbb{C} \mid z \in \mathbb{C}^d, z^*z = 1\}.$$

Proposition 1.2.2. *[29] The set $\mathcal{W}(A)$*

(i) is compact,

(ii) contains the eigenvalues of A,

(iii) is invariant under unitary transformations of A, and

(iv) is convex in $\mathbb{C} \simeq \mathbb{R}^2$ (Toeplitz-Hausdorff [27, 36]).

Proof. (i) The set $\mathcal{W}(A)$ is the image of the unit sphere, which is compact, under the continuous map $z \mapsto z^*Az$.
(ii) If $Az = \lambda z$ with $z^*z = 1$ for some $\lambda \in \mathbb{C}$, then $\lambda = z^*Az \in \mathcal{W}(A)$.
(iii) Suppose $B = U^*AU$ for $U^*U = I$, so A and B are unitarily equivalent. This implies $z^*Bz = (Uz)^*A(Uz)$ where $(Uz)^*Uz = 1$. $\square$

1.2.2 The Boundary Generating Curve

Definition 1.2.3. Let $\mathcal{X} = \mathcal{V}_{\mathbb{C}}(f)$ be an algebraic plane curve. The **dual curve (variety)** is the set of lines tangent to $\mathcal{X}$ defined by

$$\mathcal{X}^* := \overline{\left\{q \in \mathbb{P}^2(\mathbb{C}) \mid q = \nabla f(p) \text{ for some smooth point } p \in \mathcal{X}\right\}}^{\text{Zar}}.$$

Remark 1.2.4. The dual curve is the union of dual curves to each irreducible component of $\mathcal{X}$. However, calling $\mathcal{X}^*$ a "curve" is a bit misleading. If f has a linear factor, then its dual has a point as a corresponding component. Hence if $\mathcal{X}$ is a union of lines, then $\mathcal{X}^*$ is a finite set of points rather than what we traditionally think of as a curve.

For a matrix $A \in \mathbb{C}^{d\times d}$, define

$$f_A(t,x,y) := \det\left(tI + x\left(\frac{A+A^*}{2}\right) + y\left(\frac{A-A^*}{2i}\right)\right). \tag{1.1}$$

Definition 1.2.5. The **boundary generating curve** of $\mathcal{W}(A)$ is the dual variety to $\mathcal{V}_{\mathbb{C}}(f_A)$.

Theorem 1.2.6 (Kippenhahn [29])**.** *Let $\mathcal{X}^*$ be the dual variety to $\mathcal{X} = \mathcal{V}_{\mathbb{C}}(f_A)$. The numerical range of $A \in \mathbb{C}^{d\times d}$ is the convex hull of the real, affine part of $\mathcal{X}^*$. That is,*

$$\mathcal{W}(A) = \operatorname{conv}\left(\left\{(x,y) \in \mathbb{R}^2 \mid [1:x:y] \in \mathcal{X}^*\right\}\right).$$

Notice that given any matrix $A \in \mathbb{C}^{d\times d}$, its numerical range is a convex, semialgebraic set and the plane curve f_A is hyperbolic with respect to $(1,0,0)$. Theorem 1.1.6 gives us a converse statement. Let S be a convex semialgebraic set in $\mathbb{P}^2(\mathbb{C})$ whose boundary is defined by some variety $\mathcal{X}$. If the dual $\mathcal{X}^*$ is hyperbolic with respect to $(1,0,0)$, then Theorem 1.1.6 implies that S is the numerical range of some matrix. Here we aim to find determinantal representations for $f \in \mathbb{R}[t,x,y]_d$ such that $f = f_A$ for $A \in \mathbb{C}^{d\times d}$ with particular structure. Then we can use this to study the geometry of $\mathcal{W}(A)$.

Question 1.2.7. *What does the geometry of $\mathcal{W}(A)$ imply about f_A? Conversely, if f is hyperbolic with respect to $(1,0,0)$ and satisfies some additional property, then can we recover a determinantal representation such that $f = f_A$ where $\mathcal{W}(A)$ has the same underlying geometry?*

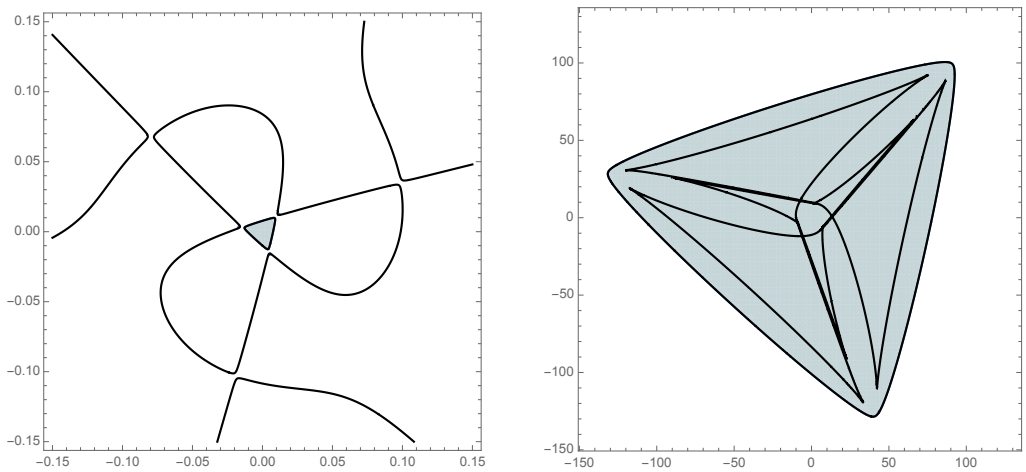

Figure 1.3: The variety $\mathcal{X} = \mathcal{V}_{\mathbb{C}}(f_A)$ which is generated from $A \in \mathbb{C}^{5\times 5}$ in the plane $t = 1$ with a slice of hyperbolicity cone $\mathcal{C}(f_A, (1, 0, 0))$ shaded (left); The dual variety $\mathcal{X}^*$ in the plane $t = 1$ with $\mathcal{W}(A)$ shaded (right).

1.3 Invariant Theory of Finite Groups

1.3.1 Background

Let $\Gamma \subseteq \mathrm{GL}\left(\mathbb{C}^3\right)$ be a finite group. A set of points $\mathcal{X} \in \mathbb{P}^2(\mathbb{C})$ is **invariant** under Γ if $\gamma \cdot p = p\gamma^T \in \mathcal{X}$ for all $p \in \mathcal{X}$. This matrix group acts on elements of $\mathbb{C}[t, x, y]$ where

$$(\gamma \cdot f)(t, x, y) = f((t, x, y)\gamma^T) \text{ for } \gamma \in \Gamma. \tag{1.2}$$

The set

$$\mathbb{C}[t, x, y]^\Gamma := \{f \in \mathbb{C}[t, x, y] \mid \gamma \cdot f = f \text{ for all } \gamma \in \Gamma\} \tag{1.3}$$

is the subset of elements of $\mathbb{C}[t, x, y]$ fixed under the action of Γ, so the ring $\mathbb{C}[t, x, y]^\Gamma$ is **invariant** under the group Γ. The next proposition shows that invariance is preserved by duality.

Proposition 1.3.1. *If $\mathcal{X} = \mathcal{V}_{\mathbb{C}}(f)$ where $f \in \mathbb{R}[t, x, y]^\Gamma$, then $\mathcal{X}^*$ is invariant under Γ.*

Proof. Let $a \in \mathcal{X}^*$. Then $f(p) = 0$ and $a = \nabla f(p)$ for some smooth $p \in \mathcal{X}$. We want to show there exists $q \in \mathcal{X}$ such that $f(q) = 0$ and $a\gamma^T = \nabla f(q)$ for any $\gamma \in \Gamma$. Take $q = p\gamma^{-T}$ for arbitrary $\gamma \in \Gamma$. First $q \in \mathcal{X}$ and $f(q) = \gamma^{-1} \cdot f(p) = 0$ since f is invariant. By Chain Rule,

$$\nabla\left(f\left(p\gamma^T\right)\right) = \nabla f\left(p\gamma^T\right) \cdot \gamma^T.$$

Then $a = \nabla(f(p)) = \nabla\left(f\left(p\gamma^{-T}\right)\right) = \nabla f(p\gamma^{-T}) \cdot \gamma^{-T}$, meaning $a\gamma^T = \nabla f(q)$ and $\gamma \cdot a \in \mathcal{X}^*$. $\square$

In this work we are not only interested in $\mathbb{C}[t,x,y]^\Gamma$, but its set of generators as well. Hilbert proved that the invariant polynomial ring of a finite group in any number of variables always has a finite set of generators.

Corollary 1.3.2 (of Hilbert [35])**.** *The invariant ring $\mathbb{C}[t,x,y]^\Gamma$ is finitely generated.*

One way to check you have a generating set is to use the Hilbert series since its coefficients give the dimension of $\mathbb{C}[t,x,y]^\Gamma_d$ for each d. More specifically, we can write the invariant ring as a graded ring

$$\mathbb{C}[t,x,y]^\Gamma = \bigoplus_{d\geq 0} \mathbb{C}[t,x,y]^\Gamma_d$$

where $\mathbb{C}[t,x,y]^\Gamma_d$ is the space of invariant polynomials that are homogeneous of degree d. The next corollary helps enumerate the dimension of each graded piece. In fact, this result follows from a stronger, analogous statement for an invariant ring in any number of variables.

Corollary 1.3.3 (of Molien [35])**.** *The Hilbert series of $\mathbb{C}[t,x,y]^\Gamma$ is given by*

$$\sum_{d\geq 0} \dim\left(\mathbb{C}[t,x,y]^\Gamma_d\right) z^d = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1}{\det(I-z\gamma)}.$$

Example 1.3.4. Let $\Gamma = \langle R \mid R^2 = I\rangle$ for $R = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. The Hilbert series of $\mathbb{C}[t,x,y]^\Gamma$ is

$$\begin{aligned}
\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1}{\det(I-z\gamma)} &= \frac{1}{2}\left(\frac{1}{\det(I-zR)} + \frac{1}{\det(I-zI)}\right) \\
&= \frac{1}{2(1-z)}\left(\frac{1}{(1+z)^2} + \frac{1}{(1-z)^2}\right) \\
&= 1 + z + 4z^2 + 4z^3 + 9z^4 + 9z^5 \ldots \\
&= \sum_{j\geq 0}(j+1)^2(z^{2j} + z^{2j+1}).
\end{aligned}$$

The monomial t is fixed by R, so $\mathbb{C}[t,x,y]_{2d+1} = \operatorname{span}_{\mathbb{C}}\{t\cdot h \mid h \in \mathbb{C}[t,x,y]^\Gamma_{2d}\}$ since

$$\dim(\mathbb{C}[t,x,y]^\Gamma_{2d}) = \dim(\mathbb{C}[t,x,y]^\Gamma_{2d+1}).$$

The invariant ring $\mathbb{C}[t,x,y]^\Gamma_{2d}$ is spanned by monomials $t^{2d-j-k}x^jy^k$ such that $j+k$ is even and $0 \leq j+k \leq 2d$. For each $d \geq 0$, we can verify this spans $\mathbb{C}[t,x,y]^\Gamma_{2d}$ by counting exactly $(d+1)^2$ monomials which satisfy these constraints. Additionally, $\mathbb{C}[t,x,y]^\Gamma = \mathbb{C}[t,xy,x^2,y^2]$ since $\{t,xy,x^2,y^2\}$ generates each graded part of $\mathbb{C}[t,x,y]^\Gamma$.

1.3.2 The Cyclic and Dihedral Groups

For projective plane curves, we are particularly interested in the case when $\Gamma = C_n$ or D_{2n}. Now we define these groups as subsets of $\mathrm{GL}(\mathbb{C}^3)$, discuss their actions on the polynomial ring $\mathbb{C}[t, x, y]$, and their induced actions on a determinantal representation of $f \in \mathbb{R}[t, x, y]_d$.

Definition 1.3.5. Let

$$\mathrm{rot} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \text{ and } \mathrm{ref} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ where } \theta = 2\pi/n.$$

The **cyclic group of order n** is

$$C_n := \langle \mathrm{rot} \mid \mathrm{rot}^n = \mathrm{id} \rangle$$

generated by a rotation by the angle $2\pi/n$ around the point $[1 : 0 : 0]$. The **dihedral group of order 2n** is

$$D_{2n} := \langle \mathrm{rot}, \mathrm{ref} \mid \mathrm{rot}^n = \mathrm{ref}^2 = (\mathrm{ref} \circ \mathrm{rot})^2 = \mathrm{id} \rangle$$

generated by the same rotation as well as a reflection over the line $y = 0$.

As defined in (1.2), these groups act on $f \in \mathbb{C}[t, x, y]$ in the following way:

$$\begin{aligned} (\mathrm{rot} \cdot f)(t, x, y) &= f(t, \cos(\theta)x + \sin(\theta)y, -\sin(\theta)x + \cos(\theta)y) \\ (\mathrm{ref} \cdot f)(t, x, y) &= f(t, x, -y). \end{aligned}$$

The invariant ring $\mathbb{R}[t, x, y]^{C_n}$ is generated by the polynomials t, $x^2 + y^2$, $\mathrm{Re}[(x+iy)^n]$, and $\mathrm{Im}[(x+iy)^n]$ [21] where

$$\mathrm{Re}[(x+iy)^n] = \frac{(x+iy)^n + (x-iy)^n}{2} \text{ and } \mathrm{Im}[(x+iy)^n] = \frac{(x+iy)^n - (x-iy)^n}{2i}. \tag{1.4}$$

Similarly, $\mathbb{R}[t, x, y]^{D_{2n}} = \mathbb{R}[t, x^2 + y^2 + \mathrm{Re}[(x+iy)^n]]$. We explicitly verify this is a generating set for $\mathbb{R}[t, x, y]^{C_n}_{\leq n}$ later in Section 2.2 using Corollary 1.3.3. For now we can at least verify these polynomials are fixed under rot. Indeed,

$$\begin{aligned} \mathrm{rot} \cdot (x^2 + y^2) &= (\cos(\theta)x + \sin(\theta)y)^2 + (-\sin(\theta)x + \cos(\theta)y)^2 = x^2 + y^2, \\ \mathrm{rot} \cdot \mathrm{Re}[(x+iy)^n] &= \mathrm{Re}[(\cos(\theta)x + \sin(\theta)y + i(-\sin(\theta)x + \cos(\theta)y))^n] \\ &= \mathrm{Re}[(e^{i\theta}x + ie^{i\theta}y)^n] = \mathrm{Re}[(x+iy)^n], \\ \text{and } \mathrm{rot} \cdot \mathrm{Im}[(x-iy)^n] &= \mathrm{Re}[(\cos(\theta)x + \sin(\theta)y - i(-\sin(\theta)x + \cos(\theta)y))^n] \\ &= \mathrm{Re}[(e^{i\theta}x - ie^{i\theta}y)^n] = \mathrm{Re}[(x-iy)^n]. \end{aligned}$$

1.3.3 Invariance and Cyclic Weighted Shift Matrices

Definition 1.3.6. The matrix $A \in \mathbb{C}^{d\times d}$ is a **cyclic weighted shift matrix of order n** if $A_{ij} = 0$ if $i - j \neq n - 1 \mod n$. To abbreviate we say A is a CWS_n matrix.

Example 1.3.7. The matrix $\begin{pmatrix} 0 & 6i & 0 & 0 & 1-7i \\ 0 & 0 & -4 & 0 & 0 \\ 2+8i & 0 & 0 & 5+9i & 0 \\ 0 & -5i & 0 & 0 & 4i \\ 0 & 0 & 3+i & 0 & 0 \end{pmatrix}$ is a CWS_3 matrix.

Complex matrices with this structure in the case $d = n$, also called cyclic weighted shift matrices, have been studied extensively especially with respect to their numerical range [10, 20, 34]. In general, the numerical range of any CWS_n matrix has nice symmetry. Define the group homomorphism $\rho : C_n \to \mathrm{GL}(\mathbb{C}^d)$ such that

$$\rho(\mathrm{rot}) = \Omega^* \text{ where } \Omega := \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^d\right) \text{ and } \omega = e^{2\pi i/n}. \tag{1.5}$$

In general $\mathrm{rot} \cdot A = \Omega^* A \Omega$ and $A_{ij} \mapsto \omega^{j-i} A_{ij}$. If A is a CWS_n matrix, then

$$\Omega^* A \Omega = \mathrm{rot} \cdot A = \omega A$$

and A is unitarily equivalent to ωA. An immediate consequence is that $\mathcal{W}(A)$ is invariant under rotation by the angle $2\pi/n$. Chien and Nakazato [9] were interested in this rotational symmetry of the numerical range with respect to CWS_n matrices, and stated the following result for which we provide an explicit proof below.

Proposition 1.3.8. *[9] If $A \in \mathbb{C}^{d\times d}$ is a CWS_n matrix, then $f_A \in \mathbb{R}[t, x, y]_d^{C_n}$. Moreover, if $A \in \mathbb{R}^{d\times d}$, then $f_A \in \mathbb{R}[t, x, y]_d^{D_{2n}}$.*

Proof. We want to check that $\mathrm{rot} \cdot f_A = f_A$ and if $A \in \mathbb{R}^{d\times d}$, then $\mathrm{ref} \cdot f_A = f_A$. First applying the rotation we have

$$\begin{aligned}
\mathrm{rot} \cdot f_A(t, x, y) &= f_A\left(t, \cos(\theta)x + \sin(\theta)y, -\sin(\theta)x + \cos(\theta)y\right) \\
&= \det\left(tI + (\cos(\theta)x + \sin(\theta)y)\left(\frac{A + A^*}{2}\right) + (-\sin(\theta)x + \cos(\theta)y)\left(\frac{A - A^*}{2i}\right)\right) \\
&= \det\left(tI + \left(\frac{x + iy}{2}\right)(\omega A)^* + \left(\frac{x - iy}{2}\right)(\omega A)\right) \\
&= \det\left(tI + \left(\frac{x + iy}{2}\right)(\Omega A \Omega^*)^* + \left(\frac{x - iy}{2}\right)(\Omega A \Omega^*)\right) \\
&= \det(\Omega) \cdot \det\left(tI + \left(\frac{x + iy}{2}\right)A^* + \left(\frac{x - iy}{2}\right)A\right) \cdot \det(\Omega^*) = f_A(t, x, y).
\end{aligned}$$

If $A \in \mathbb{R}^{d\times d}$, then $A^* = A^T$ and

$$\begin{aligned}\operatorname{ref}\cdot f_A(t,x,y) &= f_A(t,x,-y)\\ &= \det\left(tI_n + \frac{x-iy}{2}A^T + \frac{x+iy}{2}A\right)\\ &= \det\left(\left(tI_n + \frac{x+iy}{2}A^T + \frac{x-iy}{2}A\right)^T\right) = f_A(t,x,y).\end{aligned}$$

□

Chien and Nakazato asked the converse question and provided a position answer in the cases when $d = n = 3, 4$. The authors of [13, 25] studied rotational symmetry of the numerical range of matrices of size $d = 3, 4$. For $\Gamma = C_n, D_{2n}$ denote the set of hyperbolic, invariant forms of degree d by

$$\mathcal{H}_d^\Gamma = \{f \in \mathbb{R}[t,x,y]_d^\Gamma : f(1,0,0) = 1,\ f \text{ is hyperbolic with respect to } (1,0,0)\}. \tag{1.6}$$

We will also be interested in hyperbolic polynomials without any real singularities, which we denote by

$$(\mathcal{H}^\circ)_d^\Gamma = \{f \in \mathcal{H}_d^\Gamma : V_\mathbb{R}(f) \text{ is smooth}\}. \tag{1.7}$$

By [32, Lemma 2.4], this equals the set of invariant polynomials that are strictly hyperbolic with respect to $(1,0,0)$.

Question 1.3.9. *Let $f \in \mathcal{H}_d^\Gamma$. Does there exist a (real)* CWS_n *matrix A such that $f = f_A$?*

The main result of this dissertation, Theorem 3.0.1, gives a positive answer when $d = qn$ for any $q \in \mathbb{Z}_+$. The proof is constructive; we modify Dixon's Hermitian construction for smooth hyperbolic curves and provide an analogous statement of Nuij to deal with singularities.

In Chapter 2 we closely examine Question 1.3.9 in the case for $d = n$. We give an explicit description of these curves and their directional derivatives (in the direction of $(1,0,0)$), compute the dimension of $\mathbb{C}[t,x,y]_d^{C_n}$, and outline the modified construction to produce a CWS_n matrix. For dihedral invariance, we give the explicit unitary transformation necessary to output a real CWS_n matrix. We extend these results to the case when $d = qn$ in Chapter 3. For dihedral invariance here, we describe how to appropriately modify the construction to produce a real CWS_n matrix. Then we discuss generalizing to any degree; the primary obstruction is that curves in $\mathbb{C}[t,x,y]_d^\Gamma$ always have multiple complex singularities when $d \mod n \geq 3$. In Chapter 4 we use these results to address Question 1.2.7. Specifically, we prove Theorem 4.1.2 which states any matrix of size $d = qn$ for some $q \in \mathbb{Z}_+$ with a numerical range invariant under rotation has the same numerical range as some CWS_n matrix of the same size.

Chapter 2

Invariant Curves of Degree $d = n$

The content of this chapter is published in the *Linear Algebra and its Applications* in the paper "Determinantal representations of invariant hyperbolic plane curves" (see [30]). Some notation has been changed to preserve consistency and we have also added Lemma 2.1.1.

Our goal in this chapter is to prove the following theorem, which answers Question 1.3.9 for $d = n$.

Theorem 2.0.1. *Let $f \in \mathbb{R}[t, x, y]_n^\Gamma$ be hyperbolic with repect to $(1, 0, 0)$ and $f(1, 0, 0) = 1$.*

(a) If $\Gamma = C_n$, then there exists a CWS_n matrix $A \in \mathbb{C}^{n\times n}$ such that $f = f_A$.

(b) If $\Gamma = D_{2n}$, then there exists a CWS_n matrix $B \in \mathbb{R}^{n\times n}$ such that $f = f_B$.

In Sections 2.1 and 2.2, we introduce a helpful change of variables that we use heavily in both Chapters 2 and 3. In particular, this $\mathbb{C}$-linear map diagonalizes the action of rotation. Then we discuss generators for the invariant ring $\mathbb{R}[t, x, y]_d^{C_n}$ and prove some necessary facts about curves in this new setting.

In Section 2.3 we provide a proof of Theorem 2.0.1 in the smooth case by modifying Dixon's Hermitian construction introduced in Section 1.1.2. Here we choose a specific interlacer, i.e., the directional derivative of f in the direction $(1, 0, 0)$. For the $d = n$ case, this interlacer is a product of circles and is easier to work with than an arbitrary interlacer.

We extend these results to the singular case in Section 2.5 by reducing the problem to a univariate argument. In Section 2.5 we provide an explicit unitary transformation to produce a real CWS_n matrix in the dihedral case.

2.1 Change of Variables

First let

$$\mathrm{conj} : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C}), \quad [t : x : y] \mapsto [\overline{t} : \overline{x} : \overline{y}] \tag{2.1}$$

denote the action of conjugation with respect to the variables t, x, and y. Consider the change of variables given by the map

$$\begin{aligned} \xi : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C}),\ [t:x:y] \mapsto [t:u:v] = [t:x+iy:x-iy] \\ \xi^{-1} : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C}),\ [t:u:v] \mapsto \Big[t : \frac{u+v}{2} : \frac{u-v}{2i}\Big]. \end{aligned} \tag{2.2}$$

Notice $u \neq \overline{v}$ when $x, y \in \mathbb{C}\backslash\mathbb{R}$, and the action of conjugation in t, u, and v is

$$\xi \circ \mathrm{conj} : [t:x:y] \mapsto \big[\overline{t} : \overline{v} : \overline{u}\big]. \tag{2.3}$$

In terms of group actions of C_n and D_n on $\mathbb{C}[t,u,v]$, our convention is to first apply the actions of rot or ref to points $[t:x:y]$ and then the change of variables ξ. Oftentimes we abuse notation. For example, by $(\mathrm{ref} \circ \mathrm{conj})[t:u:v]$ we really mean $(\xi \circ \mathrm{ref} \circ \mathrm{conj} \circ \xi^{-1})[t:u:v]$. Consequently, the compositions are given by

$$\begin{aligned} \xi \circ \mathrm{rot}^\ell : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C}), [t:x:y] \mapsto \big[t : \omega^\ell u : \omega^{-\ell} v\big] \\ \xi \circ \mathrm{ref} : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C}), [t:x:y] \mapsto [t:v:u] \end{aligned} \tag{2.4}$$

for $\omega = e^{\frac{2\pi i}{n}}$ and some ℓ. These actions give the following equivalent representations:

$$C_n = \langle \xi \circ \mathrm{rot} \mid (\xi \circ \mathrm{rot})^n = \mathrm{id} \rangle \text{ and } D_n = \big\langle \xi \circ \mathrm{rot}, \xi \circ \mathrm{ref} \mid (\xi \circ \mathrm{rot})^n = (\xi \circ \mathrm{rot} \circ \xi \circ \mathrm{ref})^2 = \mathrm{id} \big\rangle \tag{2.5}$$

where $\xi \circ \mathrm{rot} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}$ for $\omega = e^{\frac{2\pi i}{n}}$ and $\xi \circ \mathrm{ref} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ act on points $[t:u:v]$. Under this change of variables, the form f_A in (1.1) becomes

$$f_A\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = \det\left(tI_n + \frac{u}{2}A^* + \frac{v}{2}A\right). \tag{2.6}$$

Define the map

$$\varphi : \mathbb{C}[t,u,v] \to \mathbb{C}[t,u,v] \text{ where } h(t,u,v) \mapsto h(t, \omega u, \omega^{-1} v). \tag{2.7}$$

It has monomial eigenvectors $t^i u^j v^k$ and eigenvalues $1, \omega, \ldots, \omega^{n-1}$. Denote by $\Lambda\big(\omega^\ell\big)$ the eigenspace associated to eigenvalue ω^ℓ for each $\ell = 0, 1, \ldots, n-1$. The restriction $\varphi|_d$ of φ to $\mathbb{C}[t,u,v]_d$ has a finite number of eigenvectors equal to $\dim(\mathbb{C}[t,u,v]_d) = \binom{d+2}{2}$. Denote by

$$\Lambda\big(\omega^\ell\big)_d \tag{2.8}$$

the eigenspace of the restriction $\varphi|_d$ associated to eigenvalue ω^ℓ. Notice $\Lambda(\omega^0)_d = \mathbb{C}[t,u,v]_d^{C_n}$.

Lemma 2.1.1. *For any n, d, and ℓ, the dimension of each eigenspace is*

$$\dim\left(\Lambda(\omega^\ell)_d\right) = \left\lfloor \frac{d-\ell}{2} \right\rfloor + 1 + \sum_{m=1}^{\lfloor \frac{d-\ell}{n} \rfloor} \left(\left\lfloor \frac{d-\ell-mn}{2} \right\rfloor + 1 \right) + \sum_{m=1}^{\lfloor \frac{d+\ell}{n} \rfloor} \left(\left\lfloor \frac{d+\ell-mn}{2} \right\rfloor + 1 \right).$$

Proof. Counting the dimension of each eigenspace is equivalent to counting the number of monomials $t^i u^j v^k$ such that $\varphi(t^i u^j v^k) = \omega^\ell t^i u^j v^k$. In other words, computing the cardinality of the set $\{(j,k) \mid 0 \le j+k \le d, j-k \equiv \ell \mod n\}$. Either $j-k=\ell$, $j-k>\ell$, or $j-k<\ell$. Suppose $j-k=\ell$, so $j=k+\ell$. This implies $0 \le 2k+\ell \le d$ so $0 \le k \le \frac{d-\ell}{2}$, and there are $\lfloor \frac{d-\ell}{2} \rfloor + 1$ possibilities. Suppose $j-k>\ell$, so $j=k+\ell+mn$ where $1 \le m \le \lfloor \frac{d-\ell}{n} \rfloor$ and $0 \le k \le \lfloor \frac{d-\ell-mn}{2} \rfloor$. Then for the second case, there are

$$\sum_{m=1}^{\lfloor \frac{d-\ell}{n} \rfloor} \sum_{k=0}^{\lfloor \frac{d-\ell-mn}{2} \rfloor} 1 = \sum_{m=1}^{\lfloor \frac{d-\ell}{n} \rfloor} \left(\left\lfloor \frac{d-\ell-mn}{2} \right\rfloor + 1 \right)$$

possibilities. Suppose $j-k<\ell$, so $k=j-\ell+mn$ for $1 \le m \le \lfloor \frac{d+\ell}{n} \rfloor$ and $0 \le j \le \lfloor \frac{d+\ell-mn}{2} \rfloor$. Thus for the third case there are

$$\sum_{m=1}^{\lfloor \frac{d+\ell}{n} \rfloor} \sum_{k=0}^{\lfloor \frac{d+\ell-mn}{2} \rfloor} 1 = \sum_{m=1}^{\lfloor \frac{d+\ell}{n} \rfloor} \left(\left\lfloor \frac{d+\ell-mn}{2} \right\rfloor + 1 \right)$$

possibilities. □

2.2 Polynomial Invariants

Under the change of variables

$$\mathbb{R}[t,x,y] = \mathbb{R}\left[t, \frac{u+v}{2}, \frac{u-v}{2i}\right]. \tag{2.9}$$

We often work in $\mathbb{R}[t,(u+v)/2,(u-v)/2i]$ for convenience. Now we only consider polynomials invariant under the action of C_n and later examine the more specific dihedral case in Section 2.5.

Proposition 2.2.1. *The degree n part of the invariant ring $\mathbb{C}[t,u,v]^{C_n}$ has dimension $\lfloor \frac{n}{2} \rfloor + 3$.*

Proof. Let the Hilbert series $H(\mathbb{C}[t,u,v]^{C_n}, z) = \sum_{k=0}^{\infty} \alpha_k z^k$ where $\dim\left(\mathbb{C}[t,u,v]_k^{C_n}\right) = \alpha_k$ for every k.

By Theorem 1.3.3, the Hilbert series is given by

$$\begin{aligned} H(\mathbb{C}[t,u,v]^{C_n}, z) &= \frac{1}{|C_n|} \sum_{\Xi \in C_n} \frac{1}{\det(I - z\,\Xi)} \\ &= \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{\det\left(I - z\,\tilde{\Phi}^{\ell}\right)} = \frac{1}{n}\left(\frac{1}{1-z}\right) \sum_{\ell=0}^{n-1} \frac{1}{(1-\omega^{\ell} z)(1-\omega^{-\ell} z)}. \end{aligned}$$

Expand the inner term, so

$$\frac{1}{(1-\omega^{\ell} z)(1-\omega^{-\ell} z)} = \sum_{i=0}^{\infty} (\omega^{\ell} z)^i \sum_{j=0}^{\infty} (\omega^{-\ell} z)^j = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \omega^{\ell(i-2j)} z^i$$

and the Hilbert series is

$$H(\mathbb{C}[t,u,v]^{C_n}, z) = \frac{1}{n}\left(1 + z + z^2 + \cdots\right) \sum_{\ell=0}^{n-1} \left(\sum_{i=0}^{\infty} \sum_{j=0}^{i} \omega^{\ell(i-2j)} z^i \right).$$

In this expansion, we want to calculate the coefficient α_n. Explicitly,

$$\begin{aligned} \alpha_n &= \frac{1}{n} \sum_{\ell=0}^{n-1} \left(\sum_{i=0}^{n} \sum_{j=0}^{i} \omega^{\ell(i-2j)} \right) \\ &= \frac{1}{n} \sum_{\ell=0}^{n-1} \left((1) + \left(\omega^{\ell} + \omega^{-\ell}\right) + \left(\omega^{2\ell} + 1 + \omega^{-2\ell}\right) + \cdots + \left(\omega^{n\ell} + \omega^{(n-2)\ell} + \cdots + \omega^{-n\ell}\right) \right) \\ &= \frac{1}{n} \sum_{\ell=0}^{n-1} 1 + \sum_{\ell=0}^{n-1} \left(\omega^{\ell} + \omega^{-\ell}\right) + \sum_{\ell=0}^{n-1} \left(\omega^{2\ell} + 1 + \omega^{-2\ell}\right) + \cdots + \sum_{\ell=0}^{n-1} \left(1 + \omega^{(n-2)\ell} + \cdots + 1\right) \\ &= \frac{1}{n}\left(n + 0 + n + \cdots + (n + 0 + \cdots + 0 + n)\right) \\ &= \left\lfloor \frac{n}{2} \right\rfloor + 3. \end{aligned}$$

□

Let

$$\beta_1(t,u,v) = t,\ \beta_2(t,u,v) = uv,\ \beta_3(t,u,v) = \frac{u^n + v^n}{2},\ \beta_4(t,u,v) = \frac{u^n - v^n}{2i} \tag{2.10}$$

where $\beta_i \in \mathbb{C}[t,u,v]^{C_n}$. Then $\dim\left(\mathbb{C}\left[\beta_1, \beta_2, \beta_3, \beta_4\right]_n\right) = \left\lfloor \frac{n}{2} \right\rfloor + 3$ and

$$\mathbb{C}[t,u,v]_n^{C_n} = \mathbb{C}\left[\beta_1, \beta_2, \beta_3, \beta_4\right]_n$$

by Proposition 2.2.1. Therefore, all polynomial invariants of C_n with degree n are generated by

$\beta_1, \beta_2, \beta_3$, and β_4. In general, any $f \in \mathbb{R}[t, (u+v)/2, (u-v)/2i]_n^{C_n}$ can be written

$$f(t,u,v) = t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r}(uv)^r + c_0 \left(\frac{u^n + v^n}{2}\right) + \tilde{c_0} \left(\frac{u^n - v^n}{2i}\right) \tag{2.11}$$

for some coefficients $c_i, \tilde{c_0} \in \mathbb{R}$.

2.3 The Smooth Case

In this section, we describe several properties of f and $\frac{\partial f}{\partial t}$ using the form (2.11). We later use these facts to prove Theorem 2.0.1. The hyperbolicity condition that $f(t, \cos(\theta), \sin(\theta))$ has all real roots for all $\theta \in [0, 2\pi)$ is equivalent to

$$f(\xi(t, \cos(\theta), \sin(\theta))) = \tilde{f}(t, e^{i\theta}, e^{-i\theta}) \tag{2.12}$$

having all real roots for all $\theta \in [0, 2\pi)$ where $f \in \mathbb{R}[t, x, y]$ and $\tilde{f} \in \mathbb{C}[t, u, v]$. The first lemma states that the partial derivative $\frac{\partial f}{\partial t}$ is a product of circles. In the following proof we consider $f \in \mathbb{R}[t, (u+v)/2, (u-v)/2i]_n$ and use the equivalence from (2.12).

Lemma 2.3.1. *The partial derivative is* $\frac{\partial f}{\partial t} = nt^k q_1 q_2 \cdots q_{\lfloor \frac{n-1}{2} \rfloor}$ *where* $k = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ *for* $q_j = t^2 - s_j uv$ *and* $s_j \in \mathbb{R}_{\geq 0}$.

Proof. First write $\frac{\partial f}{\partial t} = nt^{n-1} + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} (n-2r) c_r t^{n-2r-1}(uv)^r$. Assume n is odd. Then $n - 1 = 2m$ for some integer m, so

$$\frac{\partial f}{\partial t} = n\left(t^2\right)^m + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} (n-2r) c_r \left(t^2\right)^{m-r} (uv)^r \in \mathbb{R}[t^2, uv]$$

and factor over $\mathbb{C}$ as $\frac{\partial f}{\partial t} = nq_1 q_2 \cdots q_{\frac{n-1}{2}}$ where $q_j = t^2 - s_j uv$ for some $s_j \in \mathbb{C}$. Since f is hyperbolic with respect to $(1,0,0)$, this means $\frac{\partial f}{\partial t}$ is hyperbolic with respect to $(1,0,0)$ and $\frac{\partial f}{\partial t}(t, e^{i\theta}, e^{-i\theta})$ has all real roots for each $\theta \in [0, 2\pi)$. Thus $q_j(t, e^{i\theta}, e^{-i\theta}) = t^2 - s_j$ has two real roots for every j, so $s_j \in \mathbb{R}_{\geq 0}$ for every j. If n is even, we can factor out t and proceed with the remaining polynomial of odd degree as before. $\square$

The next lemma shows generic $\mathcal{V}_{\mathbb{C}}\left(f, \frac{\partial f}{\partial t}\right)$ cannot contain points with $t = 0$ when n is odd. Specifically, we require at least one of c_0, $\tilde{c_0}$ to be nonzero for genericity. The case where $c_0 = \tilde{c_0} = 0$ occurs when $\mathcal{V}_{\mathbb{C}}(f)$ has singularities and is considered in Section 2.4. With this condition, we establish an explicit description for the points $\mathcal{V}_{\mathbb{C}}\left(f, \frac{\partial f}{\partial t}\right)$.

Lemma 2.3.2. *If n is odd and $f \in \mathbb{R}[t,(u+v)/2,(u-v)/2i]_n^{C_n}$ is hyperbolic with respect to $(1,0,0)$ with at least one of c_0, $\tilde{c_0}$ nonzero, then all points in $\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$ have $t \neq 0$*

Proof. When n is odd, $\deg\left(\frac{\partial f}{\partial t}\right) = n-1$ is even. By Lemma 2.3.1, we can write $\frac{\partial f}{\partial t} = nq_1q_2\cdots q_{\frac{n-1}{2}}$ where $q_j = t^2 - s_juv$ for some $s_j \in \mathbb{R}_{\geq 0}$. If $\frac{\partial f}{\partial t}$ vanishes when $t=0$, then either $u=0$ or $v=0$ as well. Suppose $\frac{\partial f}{\partial t}(0,1,0) = 0$ or $\frac{\partial f}{\partial t}(0,0,1) = 0$. Then, $f(0,1,0) = \frac{c_0 - i\tilde{c_0}}{2}$ or $f(0,0,1) = \frac{c_0+i\tilde{c_0}}{2}$, which are both nonzero under assumption. Therefore, f does not vanish in either case. □

Notice f, $\frac{\partial f}{\partial t} \in \mathbb{R}[t,x,y]^{C_n}$, so if $p \in \mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$, then so is $\mathrm{rot}^\ell(p)$ and $(\mathrm{rot}^\ell \circ \mathrm{conj})(p)$ for each ℓ. Now, if $\mathcal{V}_{\mathbb{R}}\big(f,\frac{\partial f}{\partial t}\big)$ is empty, and in particular, f has no real singularities, then the complex intersection points are distinct.

Proposition 2.3.3. *If $\mathcal{V}_{\mathbb{R}}\big(f,\frac{\partial f}{\partial t}\big)$ is empty and at least one of c_0, $\tilde{c_0}$ is nonzero, then $\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$ consists of $n(n-1)$ distinct points.*

Proof. Suppose $\xi(f)$ and $\xi(\frac{\partial f}{\partial t})$ have a common factor. By Lemma 2.3.1 and since f has no factor of t by assumption, the common factor must be $t^2 - s_juv$ for some $s_j \in \mathbb{R}_{\geq 0}$. Then

$$\xi^{-1}\left(\left[\sqrt{s_j} : 1 : 1\right]\right) = \left[\sqrt{s_j} : 1 : 0\right] \in \mathcal{V}_{\mathbb{R}}\left(f, \frac{\partial f}{\partial t}\right),$$

which is a contradiction. Therefore, f and $\frac{\partial f}{\partial t}$ have no common factors and by Bézout's theorem, $\left|\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)\right| = n(n-1)$. For distinctness, we need to show for any point in $\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$, each orbit under the action of conjugation and rotation is distinct. Suppose for some fixed $\ell \in \{1,\ldots,n-1\}$ that $[1:u:v] = [1:\omega^\ell u : \omega^{-\ell}v] \in \xi\left(\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)\right)$. This implies $\omega^\ell u = u$ and $\omega^{-\ell}v = v$ for $\ell \neq 0$, so $u=v=0$ and $[1:0:0] \in \mathcal{V}_{\mathbb{R}}\big(f,\frac{\partial f}{\partial t}\big)$, which is a contradiction. Now suppose $[1:u:v] = [1:\omega^\ell\overline{v} : \omega^{-\ell}\overline{u}] \in \xi\left(\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)\right)$ for some $\ell \in \{0,\ldots,n-1\}$. These equivalences imply $\omega^{2\ell}u = u$ and $\omega^{-2\ell}v = v$, so either $u=v=0$, $\ell=0$ or $\ell = \frac{n}{2}$. If $u=v=0$, this is a contradiction as in the previous case. If $\ell=0$, then $[1:x:y] = \xi^{-1}([1:u:v]) = \xi^{-1}([1:\overline{v}:\overline{u}]) = [1:\overline{x}:\overline{y}]$, so $x,y \in \mathbb{R}$ and $[1:x:y] \in \mathcal{V}_{\mathbb{R}}\big(f,\frac{\partial f}{\partial t}\big)$, which is a contradiction. The case $\ell = \frac{n}{2}$ can only happen when n is even since $\ell \in \mathbb{Z}$. Assume n be even. If $\ell = \frac{n}{2}$, then $u = -\overline{v}$ and $u\overline{u} = v\overline{v}$. By Proposition 2.3.1, we can write $\frac{\partial f}{\partial t} = tq_1q_2\cdots q_{\frac{n-2}{2}}$ for $q_j = t^2 - s_juv$. For some j this means $0 = q_j(1,-\overline{v},v) = 1 + s_jv\overline{v}$, which is a contradiction. Lastly, suppose $[0:u:v] = [0:\omega^\ell u : \omega^{-\ell}v]$ for some $\ell \in \{1,\ldots,n-1\}$. This gives $\omega^\ell u = u$ and $\omega^{-\ell}v = v$, so $u=v=0$ and this is a contradiction as before.

Now suppose $p \in \mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$ with multiplicity $m_p \geq 2$. Then $\mathrm{rot}^\ell(p)$ and $(\mathrm{rot}^\ell \circ \mathrm{conj})(p)$ are also in $\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$ each with multiplicity m_p. By Lemma 2.3.1, $q_j(p) = 0$ for some factor q_j of $\frac{\partial f}{\partial t}$. Then

$$|\mathcal{V}_{\mathbb{C}}(f,q_j)| = 2n < 2n \cdot m_p,$$

which is a contradiction. Therefore, each point in $\mathcal{V}_{\mathbb{C}}\big(f,\frac{\partial f}{\partial t}\big)$ is distinct. □

Lemma 2.3.2 and Lemma 2.3.3 give an explicit description of the points of intersection which have the form $\mathcal{V}_{\mathbb{C}}\big(f, \frac{\partial f}{\partial t}\big) = S \cup \overline{S}$ where

$$\xi\left(\tilde{S}\right) = \begin{cases} \left\{[1 : u_i : v_i] \mid 1 \leq i \leq \frac{n-1}{2}\right\}, & n \text{ odd} \\ \left\{[1 : u_i : v_i], [0 : u_0 : v_0] \mid 1 \leq i \leq \frac{n-2}{2}\right\}, & n \text{ even} \end{cases} \tag{2.13}$$

is a set of orbit representatives for $\xi(S)$. That is, if $p \in \tilde{S}$, then $\mathrm{rot}^{\ell}(p) \in S$ for any ℓ.

Corollary 2.3.4. *If $\mathcal{V}_{\mathbb{R}}\big(f, \frac{\partial f}{\partial t}\big)$ is empty and at least one of c_0, $\tilde{c}_0$ is nonzero, then $\mathcal{V}_{\mathbb{C}}(f)$ has no singularities.*

Proof. Assume $\mathcal{V}_{\mathbb{R}}\big(f, \frac{\partial f}{\partial t}\big)$ is empty and suppose $\mathcal{V}_{\mathbb{C}}(f)$ has a singularity at the point p. Then the intersection multiplicity of p at f and $\frac{\partial f}{\partial t}$ is at least 2 [18]. By Proposition 2.3.3, $\mathcal{V}_{\mathbb{C}}\big(f, \frac{\partial f}{\partial t}\big)$ consists of $n(n-1)$ distinct points, which gives a contradiction. □

Corollary 2.3.4 implies we need only consider two cases in order to prove Theorem 2.0.1: the case when $\mathcal{V}_{\mathbb{C}}(f)$ is smooth and the case when $\mathcal{V}_{\mathbb{C}}(f)$ has at least one real singularity (Section 2.4). In either case, we consider the map $\varphi|_{n-1}$ as in (2.7). Dixon's idea was to recover M, the determinantal representation of f, by first constructing $\mathrm{adj}(M)$ as discussed in 1.1.2. Our desired representation is Hermitian with added structure, so our goal is to modify the Hermitian construction to reflect this structure. In particular, we require the entries of $\mathrm{adj}(M)$ to lie in the eigenspaces of $\varphi|_{n-1}$. In the next lemma gives the dimension of these eigenspaces.

Lemma 2.3.5. *For any ℓ, the dimension of each eigenspace is*

$$\dim\left(\Lambda\big(\omega^{\ell}\big)_{n-1}\right) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ and } \ell \text{ are even} \\ \frac{n}{2}+1 & \text{if } n \text{ is even and } \ell \text{ is odd.} \end{cases}$$

Proof. Computing the dimension of eigenspace $\Lambda\big(\omega^{\ell}\big)_{n-1}$ is equivalent to counting the number of monomials $t^i u^j v^k \in \mathbb{C}[t,u,v]_{n-1}$ so that $\varphi|_{n-1}(t^i u^j v^k) = \omega^{\ell} t^i u^j v^k$. On the other hand, applying φ here gives $\varphi(t^i u^j v^k) = \omega^{j-k} t^i u^j v^k$. Therefore, we must compute the cardinality of the set $\{(j,k) \mid j-k \equiv \ell \bmod n, 0 \leq j,k,j+k \leq n-1\}$. Let n be odd. Therefore, to prove $\dim(\Lambda\big(\omega^{\ell}\big)_{n-1}) = \frac{n+1}{2}$, we must really show that for some fixed ℓ where $0 \leq \ell \leq n-1$, we have $\#\{(j,k) \mid j-k \equiv \ell \bmod n, 0 \leq j,k,j+k \leq n-1\} = \frac{n+1}{2}$. Let ℓ be even. Then if $j-k \geq 0$, we have $j-k = \ell$. Since $0 \leq j+k \leq n-1$, this implies $0 \leq 2k+\ell \leq n-1$, so $-\frac{\ell}{2} \leq k \leq \frac{n-1-\ell}{2}$. We also have $0 \leq k$ and ℓ is even, so this means $0 \leq k \leq \frac{n-1-\ell}{2}$. Therefore, there are $\frac{n-1-\ell}{2} - 0 + 1 = \frac{n+1-\ell}{2}$ possibilites when $j-k \geq 0$. If $j-k < 0$, then $j-k = \ell - n$ which implies $k-j = n-\ell$ and $k+j = n-\ell+2j$. Since $0 \leq j+k \leq n-1$, we have $0 \leq n-\ell+2j \leq n-1$, so $\frac{\ell-n}{2} \leq j \leq \frac{\ell-1}{2}$. We also have $0 \leq j$ and $\ell-1$

is odd, so these inequalities imply $0 \leq j \leq \frac{\ell-2}{2}$. Therefore, there are $\frac{\ell-2}{2}+1=\frac{\ell}{2}$ possibilities when $j-k<0$. In total, we have counted $\frac{n+1}{2}$ pairs (j,k) when ℓ is even.
Now let ℓ be odd. Then if $j-k \geq 0$, we have $j-k=\ell$. Since $0 \leq j+k \leq n-1$, this implies $0 \leq 2k+\ell \leq n-1$, so $-\frac{\ell}{2} \leq k \leq \frac{n-1-\ell}{2}$. We also have $0 \leq k$ and ℓ is odd, so this means $0 \leq k \leq \frac{n-2-\ell}{2}$. Therefore, there are $\frac{n-2-\ell}{2}-0+1=\frac{n-\ell}{2}$ possibilites when $j-k \geq 0$. If $j-k<0$, then $j-k=\ell-n$ which implies $k-j=n-\ell$ and $k+j=n-\ell+2j$. Since $0 \leq j+k \leq n-1$, we have $0 \leq n-\ell+2j \leq n-1$, so $\frac{\ell-n}{2} \leq j \leq \frac{\ell-1}{2}$. We also have $0 \leq j$ and ℓ is even, so these inequalities imply $0 \leq j \leq \frac{\ell-1}{2}$. Therefore, there are $\frac{\ell-1}{2}+1=\frac{\ell+1}{2}$ possibilities when $j-k<0$. In total, we have counted $\frac{n+1}{2}$ pairs (j,k) when ℓ is odd. Thus, we have shown this set has cardinality $\frac{n+1}{2}$ whether ℓ is even or odd. A similar counting argument follows for the case when n is even.

□

Let

$$\mathcal{I}(S)_d \text{ and } \mathcal{I}\big(\tilde{S}\big)_d \tag{2.14}$$

denote the space of all degree d forms vanishing on the points $\xi(S)$ in (3.2) and $\xi\big(\tilde{S}\big)$ in (2.13) respectively. The next proposition shows there must always exist an eigenvector $h \in \Lambda\big(\omega^\ell\big)_{n-1}$ for every ℓ which vanishes on points of $\xi(S)$.

Proposition 2.3.6. *For all $\ell=0,\ldots,n-1$, $\dim\left(\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1}\right) \geq 1$.*

Proof. First we show $\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1} = \Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}\big(\tilde{S}\big)_{n-1}$. Let $h \in \mathcal{I}(S)_{n-1}$. This means h vanishes on all point of $\xi(S)$, and since $\xi\big(\tilde{S}\big) \subset \xi(S)$, h vanishes on all points of $\xi\big(\tilde{S}\big)$. Therefore, $h \in \mathcal{I}\big(\tilde{S}\big)_{n-1}$ and we have $\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1} \subseteq \Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}\big(\tilde{S}\big)_{n-1}$. For the other direction, suppose $h \in \mathcal{I}\big(\tilde{S}\big)_{n-1} \cap \Lambda\big(\omega^\ell\big)_{n-1}$, we need to show that $h(t,u,v)=0$, for all $[t:u:v] \in \xi(S)$. Every $[t:u:v] \in \xi(S)$ belongs to an orbit under the action of rotation, meaning that there exists $[\tilde{t}:\tilde{u}:\tilde{v}] \in \tilde{S}$ with $[t:u:v]=[\tilde{t}:\omega^\mu\tilde{u}:\omega^{-\mu}\tilde{v}]$ for some integer μ. Then we have $h(t,u,v)=h(\tilde{t},\omega^\mu\tilde{u},\omega^{-\mu}\tilde{v})=\omega^{\mu\ell}h(\tilde{t},\tilde{u},\tilde{v})=0$, so $\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}\big(\tilde{S}\big)_{n-1} \subseteq \Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1}$. Therefore, $\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1} = \Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}\big(\tilde{S}\big)_{n-1}$. More importantly, these sets have the same dimension. When n is odd, $\dim(\Lambda\big(\omega^\ell\big)_{n-1})=\frac{n+1}{2}$ from Lemma 2.3.5 and $|\tilde{S}|=\frac{n-1}{2}$. Then

$$\begin{aligned}\dim(\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1}) &= \dim(\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}\big(\tilde{S}\big)_{n-1}) \\ &\geq \dim(\Lambda\big(\omega^\ell\big)_{n-1}) - |\tilde{S}| \\ &= \frac{n+1}{2} - \frac{n-1}{2} = 1.\end{aligned}$$

When n is even and ℓ is odd, $\dim(\Lambda\big(\omega^\ell\big)_{n-1})=\frac{n}{2}+1$ from Lemma 2.3.5 and $|\tilde{S}|=\frac{n}{2}$. This implies

$$\dim(\Lambda\big(\omega^\ell\big)_{n-1} \cap \mathcal{I}(S)_{n-1}) \geq \dim(\Lambda\big(\omega^\ell\big)_{n-1}) - |\tilde{S}| = \frac{n}{2}+1-\frac{n}{2} = 1.$$

Now consider the case when n and ℓ are both even. We have $\dim(\Lambda(\omega^\ell)_{n-1}) = \frac{n}{2}$ from Lemma 2.3.5, but the monomials $t^i u^j v^k$ so that $j-k=\ell$ and $i+j+k=n-1$ also satisfy $i \geq 1$ since $j-k$ is even and $n-1$ is odd. This means all of the monomials in each eigenspace $\Lambda(\omega^\ell)$ in this case must have a factor of t. So in this case we need not consider the points with $t=0$. Thus $|\xi(\tilde{S})\backslash\{[0:u:v]\}| = \frac{n}{2}-1$ and

$$\dim(\Lambda(\omega^\ell)_{n-1} \cap \mathcal{I}(S)_{n-1}) \geq \dim(\Lambda(\omega^\ell)_{n-1}) - \left(|\tilde{S}|-1\right) = \frac{n}{2} - \left(\frac{n}{2}-1\right) = 1.$$

□

Another requirement for the Hermitian construction is that the 2×2 minors of $\operatorname{adj}(M)$ must lie in the ideal $\langle f\rangle$. To ensure this is possible in addition to the eigenspace requirement, we use a fact due to Max Noether. This fact is developed mainly in the language of divisors. For information on divisors, see [18]. The next lemma not only allows us to write the 2×2 minors as elements of $\langle f\rangle$, but also to choose each entry of $\operatorname{adj}(M)$ in an appropriate eigenspace of $\varphi|_{n-1}$.

Lemma 2.3.7. *Suppose $f \in \Lambda(\omega^0)$, $g \in \Lambda(\omega^0)$ and $h \in \Lambda(\omega^\ell)$ are homogeneous with $\mathcal{V}_{\mathbb{C}}(f)$ smooth, $\deg(h) > \deg(f)$, $\deg(g)$, and f has no irreducible components in common with g. If $\mathcal{V}_{\mathbb{C}}(f,g)$ consists of distinct points and $\mathcal{V}_{\mathbb{C}}(f,g) \subseteq \mathcal{V}_{\mathbb{C}}(f,h)$, then there exists homogeneous polynomials $\hat{a}, \hat{b} \in \Lambda(\omega^\ell)$ so that $h = \hat{a}f + \hat{b}g$ where $\deg(\hat{a}) = \deg(h)-\deg(f)$ and $\deg(\hat{b}) = \deg(h)-\deg(g)$. Additionally, if f, g, and h are real, then $\hat{a}$ and $\hat{b}$ can be chosen real.*

Proof. By Max Noether's fundamental theorem [18], there exists homogeneous $a, b \in \mathbb{C}[t,u,v]$ so that $h = af + bg$ where $\deg(a) = \deg(h)-\deg(f)$ and $\deg(b) = \deg(h)-\deg(g)$ since $\mathcal{V}(f,g) \subseteq \mathcal{V}(f,h)$ and $\mathcal{V}(f,g)$ consists of distinct points. Next, $h \in \Lambda(\omega^\ell)$ implies $h = \omega^{-\ell}\varphi(h)$. Then

$$h = \frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^i(h) = f\cdot\left(\frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^i(a)\right) + g\cdot\left(\frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^i(b)\right).$$

Let $\hat{b} = \frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^i(b)$. Now we show that $\varphi(\hat{b}) = \omega^\ell\hat{b}$. Applying the map, we have

$$\begin{aligned}\varphi(\hat{b}) = \varphi\left(\frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^i(b)\right) &= \frac{1}{n}\sum_{i=0}^{n-1}\omega^{-\ell i}\varphi^{i+1}(b)\\ &= \frac{1}{n}\sum_{i=1}^{n}\omega^{-\ell(i-1)}\varphi^i(b)\\ &= \frac{\omega^\ell}{n}\sum_{i=1}^{n}\omega^{-\ell i}\varphi^i(b)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\omega^\ell}{n} \sum_{i=1}^{n-1} \left(\omega^{-\ell i} \varphi^i(b)\right) + \omega^{-\ell n} \varphi^n(b) \\
&= \frac{\omega^\ell}{n} \sum_{i=1}^{n-1} \left(\omega^{-\ell i} \varphi^i(b)\right) + \omega^0 \varphi^0(b) \\
&= \omega^\ell \hat{b}.
\end{aligned}
$$

This means $\varphi(\hat{b}) = \omega^\ell \hat{b}$ and $\hat{b} \in \Lambda(\omega^\ell)_{\deg(b)}$. The polynomial $\hat{a}$ is chosen in a similar fashion.

If f, g, and $h = \hat{a}f + \hat{g}$ are real, then $h = (1/2)(\hat{a} + \hat{\bar{a}})f + (1/2)(\hat{b} + \hat{\bar{b}})g$ where $\hat{a} + \hat{\bar{a}}$ and $\hat{b} + \hat{\bar{b}}$ are also real. □

Now, with proper choices, we can recover a determinantal representation and its associated cyclic weighted shift matrix for a given curve invariant under C_n. Below we describe the modified construction and prove Theorem 2.0.1 in the case of smooth curves.

Construction 2.3.8. Suppose $f \in \mathbb{R}[t, x, y]_n^{C_n}$ is hyperbolic with respect to $(1, 0, 0)$ where $\mathcal{V}_\mathbb{C}(f)$ is smooth and $f(1, 0, 0) = 1$.

1. Write $f \in \mathbb{R}[t, (u + v)/2, (u - v)/2i]_n^{C_n}$ as in (2.11) and let $g_{11} = \frac{\partial f}{\partial t}$ be of degree $n - 1$.
2. Split the $n(n - 1)$ points of $\mathcal{V}_\mathbb{C}(f, g_{11})$ into two conjugate sets of points $S \cup \overline{S}$ according to C_n-orbits such that $\mathrm{rot}(S) = S$.
3. Extend g_{11} to a linearly independent set $\{g_{11}, g_{12}, \ldots, g_{1n}\}$ of forms in $\mathbb{C}[t, u, v]_{n-1}$ vanishing on all points of S with $\xi(g_{1j}) \in \Lambda(\omega^{1-j})_{n-1}$ for all $j \in [n]$.
4. For $1 < i \leq j$, let g_{ij} be a form for which $g_{11}g_{ij} - \overline{g_{1i}}g_{1j}$ lies in the ideal generated by f, with $\xi(g_{ij}) \in \Lambda(\omega^{i-j})_{n-1}$ and $\xi(g_{ii}) \in \mathbb{R}[t, (u + v)/2, (u - v)/2i]$.
5. For $i < j$, set $g_{ji} = \overline{g_{ij}}$ and define $G = (g_{ij})$ to be the resulting $n \times n$ complex matrix.
6. Define $M = (1/f^{n-2}) \cdot \mathrm{adj}(G)$ such that $\xi(M) \in (\mathbb{C}[t, u, v]_1)^{n \times n}$.
7. Normalize M so all diagonal entries are monic in t.

Proof of Theorem 2.0.1(a) (Smooth case). Assume $\mathcal{V}_\mathbb{C}(f)$ is smooth. The goal is to show there ex-

ists cyclic weighted shift matrix $A \in \mathbb{C}^{n\times n}$ such that

$$f(t,u,v) = f_A(t,u,v) = \det \begin{pmatrix} t & \frac{a_1}{2}v & 0 & \cdots & 0 & \frac{\overline{a_n}}{2}u \\ \frac{\overline{a_1}}{2}u & t & \ddots & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \frac{a_{n-1}}{2}v \\ \frac{a_n}{2}v & 0 & \cdots & 0 & \frac{\overline{a_{n-1}}}{2}u & t \end{pmatrix}. \tag{2.15}$$

We will construct a matrix G of forms of degree $n-1$ and recover the desired representation by taking the adjugate. Let $g_{11} = \frac{\partial f}{\partial t}$. Split $\mathcal{V}_{\mathbb{C}}(f, g_{11})$ into $S \cup \overline{S}$ so S consists of the appropriate number of C_n-orbits of points for odd and even n as in (3.2). Then $\mathrm{rot}(S) = S$ and all of these points are distinct by Proposition 2.3.3. For Step 3, Proposition 2.3.5 allows us to choose g_{1j} such that $\xi(g_{1j}) \in \Lambda\big(\omega^{1-j}\big)$ which vanishes on all points of S for each $j \in [n]$. Now let $g_{j1} = \overline{g_{1j}}$. These entries vanish on all points in $\overline{S}$ and $\mathcal{V}_{\mathbb{C}}(f, g_{11}) \subseteq \mathcal{V}_{\mathbb{C}}(f, g_{1j}g_{j1})$. By Lemma 2.3.7, we can choose g_{ij} for $1 < i < j$ such that $\xi(g_{ij}) \in \Lambda\big(\omega^{i-j}\big)$ and $g_{11}g_{ij} - \overline{g_{1i}}g_{1j} = af$ for some homogeneous $\xi(a) \in \mathbb{C}[t,u,v]$ to complete Step 4. Let $g_{ij} = \overline{g_{ji}}$ for $i > j$ and let $G = (g_{ij})$ be the $n \times n$ matrix of forms of degree $n-1$. Next consider the restriction $\varphi|_1$ of φ to $\mathbb{C}[t,u,v]_1$. The eigenvalues of this restriction are $1, \omega$, and ω^{n-1} with associated eigenspaces $\Lambda\big(\omega^0\big)_{n-1}$, $\Lambda\big(\omega^1\big)_{n-1}$, and $\Lambda\big(\omega^{n-1}\big)_{n-1}$. Since each 2×2 minor of G lies in the ideal $\langle f \rangle$, each entry of $\mathrm{adj}(G)$ will be divisible by f^{n-2} by Theorem 4.6 of [32] and Step 6 is valid. Let $M = (1/f^{n-2}) \cdot \mathrm{adj}(G)$. The entries in G have degree $n-1$, so the entries of $\mathrm{adj}(G)$ have degree $(n-1)^2$. Then f^{n-2} has degree $n(n-2)$ and entries of M satisfy $\xi(M_{ij}) \in \mathbb{C}[t,u,v]_1$. Let $\Omega = \mathrm{diag}(1, \omega, \ldots, \omega^{n-1}$. Applying the map φ to the ij-th entry of M, we have

$$\begin{aligned} \varphi(M)_{ij} &= f^{2-n}\varphi(\mathrm{adj}(G))_{ij} \\ &= (1/f^{n-2})\mathrm{adj}(\varphi(G))_{ij} \\ &= (1/f^{n-2})\mathrm{adj}(\Omega G \Omega^*)_{ij} \\ &= (1/f^{n-2})(\mathrm{adj}(\Omega^*)\mathrm{adj}(G)\mathrm{adj}(\Omega))_{ij} \\ &= (\Omega M \Omega^*)_{ij} \\ &= \overline{\omega^{j-1}}\omega^{i-1}M_{ij} \\ &= \omega^{i-j}M_{ij}. \end{aligned}$$

Therefore, $M_{ij} \in \Lambda\big(\omega^{i-j}\big)_1$ for each i,j. This implies $M_{ij} = 0$ if $|i-j| \neq 0, 1$. Now consider the entries of M so that $i - j = 0$ or 1. These are the entries in the main and upper diagonals of M as well as the M_{1n} entry. For M_{ij} such that $j = i+1$ and $i = 1, j = n$, we have $M_{ij} \in \Lambda\big(\omega^{n-1}\big)_1$

so these are multiples of v. Since M is Hermitian, this implies $M_{ji} \in \Lambda(\omega^1)_1$, so these entries are multiples of u. Also, $M_{ii} \in \Lambda(\omega^0)_1$, so the diagonal elements must be multiples of t. More explicitly, $M_{ii} = c_i t$ for some scalars c_i. To normalize as in Step 7, replace M by DMD where $D = \operatorname{diag}\left(1/\sqrt{c_1}, \ldots, 1/\sqrt{c_n}\right)$ so the coefficients of t are 1. Thus the matrix M can be reduced to the form (2.15). $\square$

Example 2.3.9 ($d = n = 5$)**.** We will work entirely in $\mathbb{C}[t, u, v]$ to compute a determinantal representation of

$$f\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = t^5 - \frac{25}{2}t^3uv + \frac{135}{4}t(uv)^2 - 3\sqrt{3}(1+\sqrt{2})\left(\frac{u^5+v^5}{2}\right) + 3\sqrt{3}(1-\sqrt{2})\left(\frac{u^5-v^5}{2i}\right)$$

and then identify the associated cyclic weighted shift matrix. Let

$$g_{11} = \frac{\partial f}{\partial t} = 5t^4 - \frac{75}{2}t^2uv + \frac{135}{4}(uv)^2$$

and compute the points of $\mathcal{V}_{\mathbb{C}}(f, g_{11})$. Write $\mathcal{V}_{\mathbb{C}}(f, g_{11}) = S \cup \overline{S}$ where

$$\xi(\tilde{S}) = \left\{ \left[1 : -\frac{57012547}{88716765} + \frac{248168479i}{571386778} ; -\frac{62642599}{61332566} - \frac{44957882i}{65129167}\right], \right.$$
$$\left. \left[1 : -\frac{58884041}{140365813} - \frac{5556049i}{389575851} : -\frac{41172728}{111605995} + \frac{6957859i}{554773523}\right] \right\}$$

is a set of orbit representatives of S as in (2.13). Choose

$$g_{12} = -t^3v + 3tuv^2 - \frac{3\sqrt{3}}{2}(1-i)\left(\sqrt{2}+i\right)u^4$$

which lies in $\Lambda(\omega^{-1})_4$ and vanishes on $\xi(S)$. Now for $j \leq 5$, choose $g_{1j} \in \Lambda(\omega^{1-j})_4$ so g_{1j} vanishes on the points of $\xi(S)$ and set $g_{j1} = \overline{g_{1j}}$. Write

$$g_{12}g_{21} = \left(-t^3 + 3tuv\right)f + \left(t^4 - 7t^2uv + 6(uv)^2\right)g_{11} \in \langle f, g_{11} \rangle$$

and let $g_{22} = t^4 - 7t^2uv + 6(uv)^2 \in \Lambda(\omega^0)_4$. For every other $i \leq j$, write $g_{1i}g_{j1} = af + bg_{11}$ for $a, b \in \mathbb{C}[t, u, v]$ where $b \in \Lambda(\omega^{i-j})_4$ and set $g_{ij} = b, g_{ji} = \overline{b}$. Let G be the matrix with g_{ij} entries, take the adjugate of G to get a matrix with entries in $\mathbb{C}[t, u, v]_{16}$, and divide each entry by f^3. One

such representation is $10000 \cdot f(t,u,v) = \det(M)$ where

$$M = \begin{pmatrix} 2t & 2\sqrt{5}v & 0 & 0 & -4i\sqrt{5}u \\ 2\sqrt{5}u & 10t & \frac{15+15i}{\sqrt{2}}v & 0 & 0 \\ 0 & \frac{15-15i}{\sqrt{2}}u & 5t & 5\sqrt{3}v & 0 \\ 0 & 0 & 5\sqrt{3}u & 10t & 5(\sqrt{2}+2i)v \\ 4i\sqrt{5}v & 0 & 0 & 5(\sqrt{2}-2i)u & 10t \end{pmatrix}.$$

Finally, let $D = \operatorname{diag}\left(1/\sqrt{2}, 1/\sqrt{10}, 1/\sqrt{5}, 1/\sqrt{10}, 1/\sqrt{10}\right)$, and normalize M so that

$$f(t, \frac{u+v}{2}, \frac{u-v}{2i}) = \det(DMD) = \det\left(tI_5 + uA^* + vA\right)$$

where A is the cyclic weighted shift matrix with entries $2, 3+3i, \sqrt{6}, \sqrt{2}+2i$, and $-4i$.

2.4 The Singular Case

We have shown this construction holds when $\mathcal{V}_{\mathbb{C}}(f)$ is smooth, but it still remains valid if $\mathcal{V}_{\mathbb{C}}(f)$ is singular. In particular, the case of real singularities may be solved by reducing to a univariate argument. The next lemma is essential in this reduction.

Lemma 2.4.1. *Let $p \in \mathbb{R}[t]$ and $a < b$ for $a, b \in \mathbb{R}$. If $p(t)+a$ and $p(t)+b$ have all real roots, then $p(t)+c$ has all real, distinct roots for $c \in (a,b)$.*

Proof. Suppose $p(t)+a$ and $p(t)+b$ have all real roots. This implies $p'(t)$ must have $n-1$ real roots $r_1, \ldots, r_{n-1}$. Suppose n is even. Then we have $\lim_{t\to-\infty} p(t)+a = \infty$. This means that there must be a local minimum at r_1 due to the shape of the graph of $p(t)+a$. We also know $p(r_1)+a \leq 0$ or else $p(t)+a$ has imaginary roots. Similarly, there must be a local maximum at r_2 and $p(r_2)+a \geq 0$. Continuing in this fashion, $p(r_k)+a \leq 0$ is a local minimum for all odd k and $p(r_k)+a \geq 0$ is a local maximum for all even k where $k = 1, \ldots n-1$. The same argument holds for $p(t)+b$, so we have $p(r_k)+a < p(r_k)+b \leq 0$ for all odd k and $p(r_k)+b > p(r_k)+a \geq 0$ for all even k where $k = 1, \ldots n-1$ because $a < b$. More importantly, we have $c \in (a,b)$, so $p(r_k)+c < 0$ for all odd k and $p(r_k)+c > 0$ for all even k. We have $\lim_{t\to-\infty} p(t)+c = \infty$ and $p(r_1)+c < 0$. By Intermediate Value Theorem, there exists $s_1 \in (-\infty, r_1)$ such that $p(s_1)+c = 0$. Similarly, since $p(r_1)+c < 0$ and $p(r_2)+c > 0$, there exists $s_2 \in (r_1, r_2)$ such that $p(s_2)+c = 0$. Continue in this fashion. We have $p(r_{n-1})+c < 0$ and $\lim_{t\to\infty} p(t)+c = \infty$. Then there exists $s_n \in (r_{n-1}, \infty)$ such that $p(s_n)+c = 0$. We have $p(s_i)+c = 0$ for $i = 1, \ldots, n$, so $p(t)+c$ has n real roots.

Now suppose n is odd. Since $\lim_{t\to-\infty} p(t)+c = -\infty$, we have $p(r_k)+a \geq 0$ is a local maximum for odd k and $p(r_k)+a \leq 0$ is a local minimum for even k where $k = 1, \ldots, n-1$. A similar argument

follows in the same way as the even case. Therefore, $p(t)+c$ has all real roots for $c \in (a,b)$. The roots of $p(t)+c$ are $s_1, s_2, \ldots, s_n$ such that $s_1 < r_1 < s_2 < r_2 < \ldots < r_{n-1} < s_n$ and, more importantly, are distinct. $\square$

The hyperbolicity of $f \in \mathbb{R}[t,x,y]_n^{C_n}$ is equivalent to real rootedness of two univariate polynomials and any real singularities of $\mathcal{V}_{\mathbb{C}}(f)$ are related to repeated roots of these polynomials.

Proposition 2.4.2. *The polynomial f is hyperbolic with respect to $(1,0,0)$ if and only if*

$$t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} \pm \sqrt{c_0^2 + \tilde{c_0}^2}$$

both have all real roots.

Proof. By definition, f is hyperbolic with respect to $(1,0,0)$ if and only if $f(t,\cos(\theta),\sin(\theta))$ has all real roots for all $\theta \in [0,2\pi)$. Then

$$\begin{aligned} f(t,\cos(\theta),\sin(\theta)) &= t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} + c_0 \cos(n\theta) + \tilde{c_0} \sin(n\theta) \\ &= t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} + \sqrt{c_0^2 + \tilde{c_0}^2} \left(\cos(\alpha)\cos(n\theta) + \sin(\alpha)\sin(n\theta)\right) \\ &= t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} + \sqrt{c_0^2 + \tilde{c_0}^2} \cos(\alpha - n\theta) \end{aligned}$$

where $\frac{c_0}{\sqrt{c_0^2+\tilde{c_0}^2}} = \cos(\alpha)$ and $\frac{\tilde{c_0}}{\sqrt{c_0^2+\tilde{c_0}^2}} = \sin(\alpha)$ for some $\alpha \in [0,2\pi)$. By Lemma 2.4.1 and since $-1 \leq \cos(\alpha - n\theta) \leq 1$, it is enough to check if $t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} \pm \sqrt{c_0^2 + \tilde{c_0}^2}$ each have all real roots to show $f(t,\cos(\theta),\sin(\theta))$ has all real roots for every $\theta \in [0,2\pi)$. $\square$

Lemma 2.4.3. *If f is hyperbolic with respect to $(1,0,0)$ where $\mathcal{V}_{\mathbb{C}}(f)$ has a real singularity, then at least one of $t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} \pm \sqrt{c_0^2 + \tilde{c_0}^2}$ has a repeated root.*

Proof. Suppose $t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r} \pm \sqrt{c_0^2 + \tilde{c_0}^2}$ have all distinct real roots. By Lemma 2.4.1, this implies $f\left(t,\cos(\theta),\sin(\theta)\right)$ has all distinct real roots for every $\theta \in [0,2\pi)$. Therefore $f\left(t,\cos(\theta),\sin(\theta)\right)$ and $\frac{\partial f}{\partial t}\left(t,\cos(\theta),\sin(\theta)\right)$ have no common roots in t. This holds for every $\theta \in [0,2\pi)$, so $f\left(t,x,y\right)$ and $\frac{\partial f}{\partial t}\left(t,x,y\right)$ have no real intersection points. In other words, $\mathcal{V}_{\mathbb{R}}\left(f, \frac{\partial f}{\partial t}\right) = \emptyset$, but

$$\mathcal{V}_{\mathbb{R}}\left(f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq \mathcal{V}_{\mathbb{R}}\left(f, \frac{\partial f}{\partial t}\right).$$

Therefore, f has no real singularities. $\square$

Equivalently, if neither of these univariate polynomials have a repeated root, then $\mathcal{V}_{\mathbb{C}}(f)$ contains no real singularities. Now we may prove the remaining singular case of Theorem 2.0.1(a) and again use the equivalent hyperbolicity condition for $f \in \mathbb{R}[t, (u+v)/2, (u-v)/2i]$ as in (2.12) to our advantage.

Proof of Theorem 2.0.1(a) (Singular case). We dealt with the case $\mathcal{V}_{\mathbb{C}}(f)$ smooth in Section 2.3. Suppose $\mathcal{V}_{\mathbb{C}}(f)$ has a singularity and at least one of $c_0, \tilde{c_0}$ are nonzero. Write

$$f = t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r}(uv)^r + c_0 \left(\frac{u^n + v^n}{2} \right) + \tilde{c_0} \left(\frac{u^n - v^n}{2i} \right)$$

as in (2.11). Let $p(t) = t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r}$ and $s = \sqrt{c_0^2 + \tilde{c_0}^2}$. Then by Lemma 2.4.2 and Proposition 2.4.3, each of $p(t) \pm s$ has all real roots and at least one has a repeated root. Either $s \neq 0$ or $s = 0$. If $s \neq 0$ (i.e., at least one of $c_0, \tilde{c_0}$ are nonzero), define

$$f_\varepsilon = t^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} c_r t^{n-2r}(uv)^r + (c_0 - \operatorname{sign}(c_0)\varepsilon) \left(\frac{u^n + v^n}{2} \right) + (\tilde{c_0} - \operatorname{sign}(\tilde{c_0})\varepsilon) \left(\frac{u^n - v^n}{2i} \right),$$

so the limit of f_ε at $s = 0$ is f and by Lemma 2.4.1, $f_\varepsilon(t, e^{i\theta}, e^{-i\theta}) = p(t) + s_\varepsilon$ has all real distinct roots for

$$s_\varepsilon = \sqrt{(c_0 - \operatorname{sign}(c_0)\varepsilon)^2 + (\tilde{c_0} - \operatorname{sign}(\tilde{c_0})\varepsilon)^2}.$$

If $s = 0$, perturb the nonzero coefficients of the univariate polynomial $p(t)$ to get $p_\varepsilon(t)$ with all real distinct roots and $\lim_{\varepsilon \to 0} p_\varepsilon = p$. Let $p_\varepsilon(t)^{\text{hom}}$ be the homogenization of $p_\varepsilon(t)$ with respect to uv and define

$$f_\varepsilon(t, u, v) = p_\varepsilon(t)^{\text{hom}} + \varepsilon \left(\frac{u^n + v^n}{2} \right).$$

Then by Proposition 2.4.1, $f_\varepsilon(t, e^{i\theta}, e^{-i\theta}) = p_\varepsilon(t) + \varepsilon \cos(n\theta)$ has all real distinct roots for every $\theta \in [0, 2\pi)$. In either case, this means $\mathcal{V}_{\mathbb{C}}(f_\varepsilon)$ is smooth and for every $\varepsilon > 0$ there exists cyclic weighted shift matrix $W_\varepsilon \in \mathbb{C}^{n \times n}$ such that $f_\varepsilon = f_{W_\varepsilon}$ by Lemma 2.4.3. Now $f_\varepsilon(t, -1, -1)$ and $f_\varepsilon(t, -i, i)$ are the characteristic polynomials of $\Re(W_\varepsilon)$ and $\Im(W_\varepsilon)$ and converge to the roots of $f(t, -1, -1)$ or $f(t, -i, i)$ respectively. Therefore, the eigenvalues of $\Re(W_\varepsilon)$ and $\Im(W_\varepsilon)$ are bounded, which bounds the sequences $(\Re(W_\varepsilon))_\varepsilon$ and $(\Im(W_\varepsilon))_\varepsilon$. Then

$$(\Re(W_\varepsilon))_\varepsilon + i(\Im(W_\varepsilon))_\varepsilon = (\Re(W_\varepsilon) + i\Im(W_\varepsilon))_\varepsilon = (W_\varepsilon)_\varepsilon$$

which is also bounded. By passing to a convergent subsequence, this means $\lim_{\varepsilon\to 0}(W_\varepsilon)_\varepsilon = W$ and

$$f = \det\left(\lim_{\varepsilon\to 0}\left(tI_n + (u/2)W_\varepsilon^* + (v/2)W_\varepsilon\right)\right) = \det\left(tI_n + (u/2)W^* + (v/2)W\right).$$

□

This proof provides an analogue to the result of Nuij [31] which states every hyperbolic polynomial is the limit of strict hyperbolic polynomials. More specifically, we have the following corollary.

Corollary 2.4.4. *The space of smooth, hyperbolic forms of degree n invariant under C_n is dense in the space of all hyperbolic forms of degree n invariant under C_n. That is, $\mathcal{H}_n^{C_n}$ is the closure of $(\mathcal{H}^\circ)_n^{C_n}$.*

2.5 Dihedral Invariance

Recall that in addition to rotation about the point $[1:0:0]$, the dihedral group of order n is also generated by a reflection such that $(\xi\circ\mathrm{ref})[t:x:y] = [t:v:u]$. Since the polynomial generator $\frac{u^n-v^n}{2i}$ of C_n is not fixed under the action of reflection, polynomials with dihedral invariance have the form

$$f\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = t^n + \sum_{r=1}^{\lfloor \frac{n}{2}\rfloor} c_r t^{n-2r}(uv)^r + c_0\left(\frac{u^n+v^n}{2}\right). \tag{2.16}$$

Now we can prove Theorem 2.0.1(b) which gives a positive answer to Chien and Nakazato's main question in [9].

Proof of Theorem 2.0.1(b). Note $f \in \mathbb{R}[t,x,y]^{D_n} \subseteq \mathbb{R}[t,x,y]^{C_n}$, so by Theorem 2.0.1(a) there exists cyclic weighted shift matrix $A \in \mathbb{C}^{n\times n}$ where $f\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = \det\left(tI_n + (u/2)A^* + (v/2)A\right)$. The polynomial f has the form (2.16), so

$$t^n + 2^{n-1}c_0 = f(t,1,i) = f_A(t,1,i) = \det(tI + A) = t^n - a_1a_2\cdots a_n,$$

which implies $a_1a_2\cdots a_n \in \mathbb{R}$ since $c_0 \in \mathbb{R}$. Write $a_j = r_je^{i\alpha_j}$ with $r_j \in \mathbb{R}$ for each j, so $a_1a_2\cdots a_n \in \mathbb{R}$ and $\alpha_1+\alpha_2+\ldots+\alpha_n = 0$. Let $U = \mathrm{diag}\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\right)$ for some $\theta_i \in [0, 2\pi)$. Since $UU^* = I$, we say A is unitarily equivalent to $B = UAU^*$, so $f = f_A = f_B$. The matrix B is the cylic weighted shift matrix with entries $r_1e^{i(\theta_1-\theta_2+\alpha_1)}, r_2e^{i(\theta_2-\theta_3+\alpha_2)}, \ldots, r_ne^{i(\theta_n-\theta_1+\alpha_n)}$. Choose

$$\begin{aligned}\theta_j &= -\alpha_j - \alpha_{j+1} - \ldots - \alpha_{n-1} \text{ for } 1 \le j < n\\ \theta_n &= -\alpha_1 - \alpha_2 - \ldots - \alpha_n = 0.\end{aligned}$$

This gives $\theta_j - \theta_{j+1} = -\alpha_j$ for $1 \leq j < n$ and $\theta_n - \theta_1 = -\alpha_n$, so the entries of B are $r_1, \ldots, r_n \in \mathbb{R}$. $\square$

Example 2.5.1 ($d = n = 4$)**.** In this example, we work completely in $\mathbb{C}[t, u, v]$. We will compute a determinantal representation of $f(t, u, v) = t^4 - 26t^2uv + 72(uv)^2 - 36(u^4 + v^4)$ and identify an associated cyclic weighted shift matrix. Let $g_{11} = \frac{\partial f}{\partial t} = 4t^3 - 52tuv$ and compute the points of $\mathcal{V}_{\mathbb{C}}(f, g)$. Split the points into $S \cup \overline{S}$ so

$$\tilde{S} = \left\{ \left[1 : \frac{1+i}{\sqrt{39}} : \frac{\sqrt{3}}{2\sqrt{13}}(1-i) \right], [0:1:1] \right\}$$

is a set of orbit representatives for S as in (2.13). Choose $g_{12} = -4t^2v - 36u^3 + 36uv^2$ which lies in $\Lambda\big(\omega^{-1}\big)_3$ and vanishes on S. Now for $j \leq 4$, choose $g_{1j} \in \Lambda\big(\omega^{1-j}\big)_3$ so g_{1j} vanishes on the points of S and set $g_{j1} = \overline{g_{1j}}$. Write

$$g_{12}g_{21} = \left(-4t^2 + 36uv\right) f + \left(t^3 - 18tuv\right) g_{11} \in \langle f, g_{11} \rangle$$

and let $g_{22} = t^3 - 18tuv \in \Lambda\big(\omega^0\big)_3$. For every other $i \leq j$, write $g_{1i}g_{j1} = af + bg_{11}$ for $a, b \in \mathbb{C}[t, u, v]$ where $b \in \Lambda\big(\omega^{i-j}\big)_3$ and set $g_{ij} = b, g_{ji} = \overline{b}$. Denote by G the matrix with g_{ij} entries, take the adjugate of G to get a matrix with entries in $\mathbb{C}[t, u, v]_9$, and divide each entry by f^2. One such representation is $16 \cdot f(t, u, v) = \det(M)$ where

$$M = \begin{pmatrix} 4t & \left(2 + 2\sqrt{3}i\right)v & 0 & 4\sqrt{2}u \\ \left(2 - 2\sqrt{3}i\right)u & t & \left(\frac{3\sqrt{3}+3i}{\sqrt{2}}\right)v & 0 \\ 0 & \left(\frac{3\sqrt{3}-3i}{\sqrt{2}}\right)u & 2t & -6iv \\ 4\sqrt{2}v & 0 & 6iu & 2t \end{pmatrix}.$$

Finally, let $D = \operatorname{diag}\left(1/2, 1, 1/\sqrt{2}, 1/\sqrt{2}\right)$, and normalize M so that

$$f(t, u, v) = \det(DMD) = \det\left(tI_4 + uA^* + vA\right)$$

where

$$A = S\left(2 + 2\sqrt{3}i, 3\sqrt{3} + 3i, -6i, 4\right) = S\left(4e^{\frac{\pi i}{3}}, 6e^{\frac{\pi i}{6}}, 6e^{\frac{\pi i}{2}}, 4e^0\right)$$

is the associated cyclic weighted shift matrix. Let $U = \operatorname{diag}\left(e^0, e^{\frac{\pi i}{3}}, e^{\frac{\pi i}{2}}, e^0\right)$. Then A is unitarily equivalent to $B = UAU^*$ with all real entries $4, 6, 6$, and 4.

Chapter 3

Invariant Curves of Degree $d = qn$

In this chapter our goal is to answer Question 1.3.9 in the case $d = qn$ by proving the following theorem. We choose to focus on this case rather than any d because in the more general case, curves in $(\mathcal{H}^\circ)_d^\Gamma$ and their set of interlacers are often not well-behaved. We discuss the obstructions to the general case in Section 3.4.

Theorem 3.0.1. *Let $d = qn$ for some $q \in \mathbb{Z}_+$ and suppose $f \in \mathcal{H}_d^\Gamma$.*

(a) If $\Gamma = C_n$, then there exists a CWS_n matrix $A \in \mathbb{C}^{d\times d}$ so that $f = f_A$.

(b) If $\Gamma = D_{2n}$, then there exists a CWS_n matrix $A \in \mathbb{R}^{d\times d}$ so that $f = f_A$.

The proof of Theorem 3.0.1 involves extending Construction 2.3.8 by generalizing results from Section 2.3 to higher degree. Now we also do not necessarily choose the directional derivative of f as our interlacer for the construction. In general, f and its directional derivative $\partial f/\partial t$ may have common components. This means our analogue of Max Noether's $af + bg$ Theorem (Lemma 2.3.7) does not hold and we cannot follow through with the construction. In Section 3.2 we also take a different approach to the dihedral case. We alter Construction 3.1.8 to produce a real CWS_n matrix rather than completing the construction to give a complex matrix and applying an explicit unitary transformation.

We later deal with our added assumptions by studying the topology of invariant forms in Section 3.3. We again prove an appropriate analogue of Nuij [31] to handle degenerate curves and their interlacers. Finally we briefly discuss obstructions to completing the construction for curves of any degree in Section 3.4 as previously mentioned.

3.1 The Smooth Case

In this section we aim to prove Theorem 3.0.1, but with some added assumptions about of a curve in $\mathcal{H}_d^\Gamma$ and an interlacer. We prove higher degree analogues of statements from Section 2.3 and modify

Dixon's construction mentioned in Section 1.1.2 to produce a structured definite determinantal representation. Throughout Sections 3.1 and 3.2 we will assume that

A1. $f \in (\mathcal{H}^\circ)_d^\Gamma$ and $\mathcal{V}_\mathbb{C}(f)$ is smooth,

A2. $g \in \mathbb{R}[t,x,y]_{d-1}^\Gamma$ interlaces f,

A3. $\mathcal{V}_\mathbb{C}(f)$ and $\mathcal{V}_\mathbb{C}(g)$ intersect transversely, and

A4. $\left|\mathcal{V}_\mathbb{C}(f,g,t)\right| = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ d & \text{if } n \text{ is even.} \end{cases}$

Specifically, we prove the following theorem.

Theorem 3.1.1. *Let $d = qn$ for some $q \in \mathbb{Z}_+$. Let f and g satisfy (A1)–(A4).*

(a) If $\Gamma = C_n$, then there exists a CWS_n matrix $A \in \mathbb{C}^{d\times d}$ so that $f = f_A$.

(b) If $\Gamma = D_{2n}$, then there exists a CWS_n matrix $A \in \mathbb{R}^{d\times d}$ so that $f = f_A$.

The next lemma is a general statement about complex points. This will allow us to split the intersection points of $f \in \mathcal{H}_d^{C_n}$ and an interlacer into disjoint sets determined by orbits under the action of rotation.

Lemma 3.1.2. *If $|x+iy| = |x-iy|$ for some $(x,y) \in \mathbb{C}^2$, then $x/y \in \mathbb{R}$.*

Proof. By assumption, $x+iy = z(x-iy)$ with $|z| = 1$. Then

$$\frac{x}{y} = \frac{-i(1+z)}{1-z} \cdot \frac{(1-\overline{z})}{(1-\overline{z})} = \frac{-i(1-z+\overline{z}-z\overline{z})}{|1-z|^2} = \frac{-i(-z+\overline{z})}{|1-z|^2} = \frac{2\cdot \mathrm{Im}(z)}{|1-z|^2} \in \mathbb{R}.$$

□

Corollary 3.1.3. *Let f and g satisfy Assumptions (A1)–(A4). Then each C_n-orbit in $\mathcal{V}_\mathbb{C}(f,g)$ is disjoint from its image under conjugation.*

Proof. Let $\mathcal{O}$ be a C_n-orbit of points in $\mathcal{V}_\mathbb{C}(f,g)$ and suppose

$$[t : x+iy : x-iy] \in \xi(\mathcal{O}) \cap (\xi \circ \mathrm{conj})(\mathcal{O})$$

with $t = 1$ or 0. Then

$$[t : x+iy : x-iy] = \left[t : \omega^\ell\left(\overline{x+iy}\right) : \omega^{-\ell}\left(\overline{x-iy}\right)\right] \tag{3.1}$$

for some fixed $\ell \in [n]$. If $t = 1$, (3.1) implies $x+iy = \omega^\ell\left(\overline{x+iy}\right)$ which means $|x+iy| = |x-iy|$, so $x/y \in \mathbb{R}$ by Lemma 3.1.2. If $f(1,x,y) = 0$, then by homogeneity, $f(1/y, x/y, 1) = 0$, meaning that

$1/y$ is a root of the polynomial $f(t, x/y, 1) \in \mathbb{R}[t]$ where $x/y \in \mathbb{R}$ is fixed. The hyperbolicity of f then implies that $1/y \in \mathbb{R}$. Since both x/y and $1/y$ are real, both x and y must be real. Then the point $[1 : x : y] \in \mathcal{V}_{\mathbb{R}}(f, g)$ which is a contradiction to the assumption $\mathcal{V}_{\mathbb{R}}(f, g) = \emptyset$.
If $t = 0$, (3.1) implies $(x + iy)(\overline{x + iy}) = \omega^{2\ell}(x - iy)(\overline{x - iy})$ for some fixed $\ell \in [n]$ which means $|x + iy| = |x - iy|$, so $x/y \in \mathbb{R}$ by Lemma 3.1.2. The point $[0 : x : y] \in \mathcal{V}_{\mathbb{R}}(f, g)$ which is again a contradiction. Since $\xi(\mathcal{O}) \cap (\xi \circ \mathrm{conj})(\mathcal{O})$ is empty, so is $\mathcal{O} \cap \mathrm{conj}(\mathcal{O})$. $\square$

As before, we have the same restriction map

$$\varphi : \mathbb{C}[t, u, v] \to \mathbb{C}[t, u, v] \text{ where } h(t, u, v) \mapsto h(t, \omega u, \overline{\omega} v)$$

with eigenspaces $\Lambda\big(\omega^{\ell}\big)$ as in (2.8) and we can write $\mathbb{C}[t, u, v]_d$ as a decomposition of eigenspaces

$$\mathbb{C}[t, u, v]_d = \bigoplus_{\ell=0}^{n-1} \Lambda\big(\omega^{\ell}\big)_d.$$

Hence we can compute the dimension of each eigenspace using Lemma 2.1.1. In particular, for $d = qn$, we want at least q elements in each eigenspace $\Lambda\big(\omega^{\ell}\big)_{d-1}$ in order to choose linearly independent set of elements in $\mathbb{C}[t, u, v]_{d-1}$ for the first row of the adjugate matrix we wish to construct.

Lemma 3.1.4. *Let $d = qn$ for some $q \in \mathbb{Z}_+$. The dimension of each eigenspace is*

$$\dim\left(\Lambda\big(\omega^{\ell}\big)_{d-1}\right) = \begin{cases} \frac{dq}{2} + \frac{q}{2} & \text{if } n \text{ is odd} \\ \frac{dq}{2} + q & \text{if } n \text{ is even and } \ell \text{ is odd} \\ \frac{dq}{2} & \text{if } n \text{ is even and } \ell \text{ is even.} \end{cases}$$

Proof. Consider the case when $\ell = 0$. Then by Lemma 2.1.1 the dimension is

$$\dim\left(\Lambda\big(\omega^{0}\big)_{d-1}\right) = \left\lfloor \frac{d-1}{2} \right\rfloor + 2q - 1 + 2\sum_{m=1}^{q-1} \left(\left\lfloor \frac{d-1-mn}{2} \right\rfloor \right).$$

Let n be odd, so the parity of $d - 1 - mn$ depends on the parity of m. Suppose d is odd. Then q is odd and the dimension is

$$\dim(\Lambda\big(\omega^{0}\big)_{d-1}) = \frac{d-1}{2} + 2q - 1 + (d-1)(q-1) - n\binom{q}{2} - 2\left(\frac{1}{2}\right)\left(\frac{q-1}{2}\right).$$

Suppose d is even. Then q is even and the dimension is

$$\dim(\Lambda(\omega^0)_{d-1}) = \frac{d-2}{2} + 2q - 1 + (d-1)(q-1) - n\binom{q}{2} - 2\left(\frac{1}{2}\right)\left(\frac{q-2}{2}\right).$$

Let n be even, so d is even and the parity of $d-1-mn$ is always odd. Here d must be even. Then the dimension is

$$\dim(\Lambda(\omega^0)_{d-1}) = \frac{d-2}{2} + 2q - 1 + (d-1)(q-1) - n\binom{q}{2} - 2\left(\frac{1}{2}\right)(q-1).$$

Now consider the case $\ell > 0$. By Lemma 2.1.1 the dimension is

$$\dim\left(\Lambda(\omega^\ell)_{d-1}\right) = \left\lfloor \frac{d-1-\ell}{2} \right\rfloor + \left\lfloor \frac{\ell-1}{2} \right\rfloor + 2q + \sum_{m=1}^{q-1}\left(\left\lfloor \frac{d-1-\ell-mn}{2} \right\rfloor + \left\lfloor \frac{d-1+\ell-mn}{2} \right\rfloor\right).$$

Let n be odd and suppose d is odd, so q is odd. If ℓ is odd, then the dimension is

$$\dim(\Lambda(\omega^\ell)_{d-1}) = \frac{d-\ell-2}{2} + \frac{\ell-1}{2} + 2q + (d-1)(q-1) - n\binom{q-1}{2} - 2\left(\frac{1}{2}\right)(q-1).$$

If ℓ is even, then the dimension is

$$\dim(\Lambda(\omega^\ell)_{d-1}) = \frac{d-\ell-1}{2} + \frac{\ell-2}{2} + 2q + (d-1)(q-1) - n\binom{q-1}{2} - 2\left(\frac{1}{2}\right)(q-1).$$

Let n be even, so d is even. If ℓ is odd, then the dimension is

$$\dim(\Lambda(\omega^\ell)_{d-1}) = \frac{d-\ell-1}{2} + \frac{\ell-1}{2} + 2q + (d-1)(q-1) - n\binom{q-1}{2}.$$

If ℓ is even, then the dimension is

$$\dim(\Lambda(\omega^\ell)_{d-1}) = \frac{d-\ell-2}{2} + \frac{\ell-2}{2} + 2q + (d-1)(q-1) - n\binom{q-1}{2} - 2\left(\frac{1}{2}\right)(q-1).$$

Each of these counts gives the desired result. □

For f and g that satisfy (A1)–(A4), we will again split the points of $\mathcal{V}_{\mathbb{C}}(f,g)$ into $S \cup \overline{S}$ based on orbits under rotation. The next lemma helps enumerate conditions imposed by the set of orbit representatives, and accurately count dimensions later in Lemma 3.1.7.

Lemma 3.1.5. *Let $d = qn$ for some $q \in \mathbb{Z}_+$. If n, d, and ℓ are even, then each monomial in $\Lambda(\omega^\ell)_{d-1}$ has a factor of t.*

Proof. Let $t^i u^j v^k$ be an arbitrary monomial in $\Lambda(\omega^\ell)_{d-1}$. Then $i,j,k \geq 0$, $i+j+k = d-1$, and

$j - k \equiv \ell \mod n$. Since n and ℓ are even, $j - k$ is even. Thus $j - k + 2k = j + k$ is also even, so $i = d - 1 - (j + k)$ is odd and $i \geq 1$. $\square$

By Corollary 3.1.3, $\mathcal{V}_{\mathbb{C}}(f, g)$ may be split into two disjoint sets according to orbits invariant under the action of C_n. More explicitly, write $\mathcal{V}_{\mathbb{C}}(f, g) = S \cup \overline{S}$ as the union of two disjoint conjugate sets. Define $\tilde{S}$ to be a minimal set of orbit representatives from S so that

$$S = \left\{ \mathrm{rot}^{\ell} \cdot p \mid p \in \tilde{S}, \ell \in [n] \right\}. \tag{3.2}$$

The next proposition gives the maximum number of possible conditions imposed by $\tilde{S}$ on an element of $\Lambda\left(\omega^{\ell}\right)_{d-1}$.

Proposition 3.1.6. *Let $d = qn$ for some $q \in \mathbb{Z}_+$ and suppose f and g satisfy (A1)–(A4). Then the number of distinct orbits in S is*

$$\left|\tilde{S}\right| = \begin{cases} q(d-1)/2 & \text{if } n \text{ is odd} \\ qd/2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. The number of orbits in S may be counted as follows. Each point of $\mathcal{V}_{\mathbb{C}}(f, g, t-1)$ generates a C_n-orbit of size n since rot^{ℓ} fixes a point when $\ell \equiv 0 \mod n$. Otherwise, the image of rot^{ℓ} is a point distinct from p. When n is odd, dividing the $d(d-1)/2$ total points of S the size of a C_n orbit gives $q(d-1)/2$ orbits in S. When n is even, the $d(d-2)$ total points in $\mathcal{V}_{\mathbb{C}}(f, g, t-1)$ gives $q(d-2)/2$ orbits in S. Each point in $\mathcal{V}_{\mathbb{C}}(f, g, t)$ generates a C_n-orbit of size $n/2$ since the action of $\mathrm{rot}^{n/2}$ fixes a point. Thus the d total points of $\mathcal{V}_{\mathbb{C}}(f, g, t)$ contribute $(d/2)/(n/2) = q$ orbits to S which means S has a total of $qd/2$ C_n-orbits. $\square$

Denote the space of forms in $\mathbb{C}[t, u, v]_{d-1}$ vanishing on points $\xi(S)$ and $\xi\left(\tilde{S}\right)$ from (3.2) by

$$\mathcal{I}\left(S\right)_{d-1} \text{ and } \mathcal{I}(\tilde{S})_{d-1} \tag{3.3}$$

respectively. Now we can show there are enough elements in each eigenspace to choose a linearly independent set of forms in $\mathbb{C}[t, u, v]_{d-1}$ for the first row in our desired adjugate matrix.

Lemma 3.1.7. *Let $d = qn$ for some $q \in \mathbb{Z}_+$. There exist at least q polynomials which vanish on the points of S in each eigenspace. That is,* $\dim\left(\Lambda\left(\omega^{\ell}\right)_{d-1} \cap \mathcal{I}(S)_{d-1}\right) \geq q$.

Proof. For any ℓ,

$$\dim\left(\Lambda\left(\omega^{\ell}\right)_{d-1} \cap \mathcal{I}(S)_{d-1}\right) = \dim\left(\Lambda\left(\omega^{\ell}\right)_{d-1} \cap \mathcal{I}\left(\tilde{S}\right)_{d-1}\right) \geq \dim\left(\Lambda\left(\omega^{\ell}\right)_{d-1}\right) - \left|\tilde{S}\right|.$$

In the cases when n is odd or n is even with ℓ odd, this count is straightforward due to Lemmas 3.1.4 and 3.1.6. By Lemma 3.1.5, when n and ℓ are even, every monomial in $\Lambda\big(\omega^\ell\big)_{d-1}$ has a factor of t. Thus every element of $\Lambda\big(\omega^\ell\big)_{d-1}$ will already vanish at points with $t = 0$ without adding additional constraints from those in $\mathcal{V}_{\mathbb{C}}(f, g, t)$. In this case we do not take into account the q orbits at infinity and using Lemmas 3.1.4 and 3.1.6 we have

$$\dim\left(\Lambda\big(\omega^\ell\big)_{d-1} \cap \mathcal{I}(S)_{d-1}\right) \geq \dim\left(\Lambda\big(\omega^\ell\big)_{d-1}\right) - \left|\tilde{S}\right| + q \geq q.$$

□

The construction below is similar to Construction 2.3.8 for the $d = n$ case. For normalization of the coefficient matrix of t, however, we must be more careful. Now the variable t appears in off-diagonal entries of the determinantal representation, so we must first block diagonalize the coefficient matrix of t, then normalize with respect to each block separately in order to preserve CWS_n matrix structure.

Construction 3.1.8. Let $d = qn$ for some $q \in \mathbb{Z}_+$ and $\Gamma = C_n$.
Input: Two plane curves f and g satisfying (A1)–(A4).
Output: CWS_n matrix $A \in \mathbb{C}^{d\times d}$ such that $f = f_A$.

1. Set $g_{11} = g$.
2. Split up the distinct $d(d-1)$ points of $\mathcal{V}_{\mathbb{C}}(f, g_{11})$ into two disjoint, conjugate sets $S \cup \overline{S}$ of C_n-orbits such that $\mathrm{rot}(S) = S$.
3. Extend g_{11} to a linearly independent set $\{g_{11}, g_{12}, \ldots, g_{1d}\} \subset \mathbb{C}[t, x, y]_{d-1}$ vanishing on all points of S with $\xi(g_{1j}) \in \Lambda\big(\omega^{1-j}\big)_{d-1}$ for all $j \in [n]$ and set $g_{j1} = \overline{g_{1j}}$ for each j.
4. For $1 < i \leq j$, choose $\xi(g_{ij}) \in \Lambda\big(\omega^{i-j}\big)_{d-1}$ so that $g_{11}g_{ij} - \overline{g_{1i}}g_{1j} \in \langle f\rangle$ and $g_{ii} \in \mathbb{R}[t, x, y]$.
5. For $i < j$, set $g_{ji} = \overline{g_{ij}}$ and define $G = (g_{ij})_{i,j} \in \mathbb{C}^{d\times d}$.
6. Define $M = (1/f^{d-2}) \cdot \mathrm{adj}(G)$.
7. For $\ell \in [d]$, write $\ell - 1 = an + b$ for some integers a and b with $0 \leq b \leq n-1$. Let P be the permutation matrix that takes $\ell = an + b + 1$ to $bq + a + 1$. Define $M' = PMP^T$ as a matrix with $q \times q$ blocks $M' = (M'_{kl})_{k,l=1}^n$
$$\xi\left(\left(M'_{kl}\right)_{ij}\right) \in \Lambda\big(\omega^{k-l}\big)_1 \text{ for } k, l \in [n] \text{ and } i, j \in [q].$$
8. For each k compute the Cholesky decomposition of each diagonal block M'_{kk} and write $(M')^{-1}_{kk} = U_k U_k^*$ for some $U_k \in \mathbb{C}^{q\times q}$.

9. Define $U = \operatorname{diag}(U_1, U_2, \ldots, U_k)$ and output $A = \left(P^T U^* M' U P\right)(0, 1, i)$.

Proof of Theorem 3.0.1(a) with Assumptions (A1)–(A4). Our goal is to show each step of Construction 3.1.8 can be completed and produces a CWS_n matrix $A \in \mathbb{C}^{d\times d}$ of order n such that $f = f_A$ as in (2.11). Let $g_{11} = g$. By Corollary 3.1.3, write $\mathcal{V}_{\mathbb{C}}(f, g_{11}) = S \cup \overline{S}$ where $\operatorname{rot}(S) = S$. All of these points are distinct by assumption. For Step 3, Lemma 3.1.4 allows us to choose linearly independent g_{1j} so $\xi(g_{1j}) \in \Lambda\left(\omega^{1-j}\right)_{d-1}$ each vanish on S for all $j \in [n]$. Now let $g_{j1} = \overline{g_{1j}}$. By Lemma 2.3.7, we can choose g_{ij} such that $\xi(g_{ij} \in \Lambda\left(\omega^{i-j}\right)_{d-1}$ for $1 < i < j$ and $g_{11}g_{ij} - \overline{g_{1i}}g_{1j} = af$ for some homogeneous $\xi(a) \in \mathbb{C}[t, x, y]$ to complete Step 4. Since f, g_{11}, $\overline{g_{1i}}g_{1i} \in \mathbb{R}[t, x, y]$, we can choose $g_{ii} \in \mathbb{R}[t, x, y]$ as well. Let $g_{ji} = \overline{g_{ij}}$ for $i < j$ and define $G = (g_{ij})_{i,j}$ be the $d \times d$ complex matrix of forms of degree $d-1$. Since each 2×2 minor of G lies in the ideal $\langle f\rangle$, each entry of $\operatorname{adj}(G)$ will be divisible by f^{d-2} by Theorem 4.6 of [32] and Step 6 is valid. Let $M = (1/f^{d-2}) \cdot \operatorname{adj}(G)$. The entries in G have degree $d-1$, so entries of its adjugate have degree $(d-1)^2$. Then f^{d-2} has degree $d(d-2)$, so entries of $\xi(M)$ are linear in t, u, and v. Let $\Omega = \operatorname{diag}(1, \omega, \ldots, \omega^{d-1})$. Applying the map φ to the ij-th entry of M, we have

$$\begin{aligned}
(\varphi \circ \xi)(M)_{ij} &= (1/f^{d-2})(\varphi \circ \xi)(\operatorname{adj}(G))_{ij} \\
&= (1/f^{d-2})\operatorname{adj}((\varphi \circ \xi)(G))_{ij} \\
&= (1/f^{d-2})\operatorname{adj}(\Omega G \Omega^*)_{ij} \\
&= (1/f^{d-2})(\operatorname{adj}(\Omega^*)\operatorname{adj}(G)\operatorname{adj}(\Omega))_{ij} \\
&= (\Omega M \Omega^*)_{ij} \\
&= \overline{\omega^{j-1}}\omega^{i-1} M_{ij} \\
&= \omega^{i-j} M_{ij}.
\end{aligned}$$

Therefore, $\xi(M_{ij}) \in \Lambda\left(\omega^{i-j}\right)_1$ for each i, j. The restriction $\varphi\,|_1$ has eigenvalues $1, \omega$, and ω^{n-1} with associated eigenspaces $\Lambda\left(\omega^0\right)_1$, $\Lambda\left(\omega^1\right)_1$, and $\Lambda\left(\omega^{n-1}\right)_1$. This implies $M_{ij} = 0$ if $|i-j| \neq 0, 1$. For M_{ij} such that $i - j \equiv n-1 \mod n$ we have $\xi(M_{ij}) \in \Lambda\left(\omega^{n-1}\right)_1$ so these are multiples of v. Since M is Hermitian, this implies $\xi(M_{ji}) \in \Lambda\left(\omega^1\right)_1$ and these entries are multiples of u. If $i - j \equiv 0 \mod n$ then $i = j$ and $\xi(M_{ij}) \in \Lambda\left(\omega^0\right) \cap \mathbb{R}[t, u, v]$ are multiples of t.

Next we will show by permuting rows and columns of M we may get the identity matrix as the coefficient of t in our representation. Consider M as a matrix of $n \times n$ blocks. Each block is a cyclic weighted shift matrix and there are q^2 blocks in total. For $\ell \in [d]$, write $\ell - 1 = an + b$ for some integers a and b with $0 \le b \le n-1$. Let P be the permutation matrix that takes $\ell = an + b + 1$ to $bq + a + 1$. Define $M' = PMP^T$ as a matrix with $q \times q$ blocks $M' = (M'_{kl})_{k,l=1}^n$ with

$$\xi\left(\left(M'_{kl}\right)_{ij}\right) \in \Lambda\left(\omega^{k-l}\right)_1 \text{ for } k, l \in [n] \text{ and } i, j \in [q]$$

and $M'(1,0,0)$ is a block diagonal matrix.

By Theorem 3.3 of [32], we know $M(1,0,0)$ is definite, thus $M'(1,0,0)$ is definite. For each $k \in [n]$ we can find the decompose $M'_{kk}(1,0,0)$ so $M'_{kk}(1,0,0)^{-1} = U_k^* U_k$ for some $U_k \in \mathbb{C}^{q\times q}$. Define $U = \operatorname{diag}(U_1, U_2, \ldots, U_n)$. Then $M'' = UM'U^*$ is a desired representation of f since $M''(1,0,0) = I_d$ and $f = (1/\lambda)\cdot \det(UMU^*)$ for $\lambda = \det(U)\cdot\det(U^*)$. Lastly, apply the inverse permutation so $f = (1/\lambda)\cdot\det(P^T M'' P)$ and evaluating $(P^T M'' P)(0,1,i)$ gives a cyclic weighted shift matrix of order n. $\square$

Example 3.1.9 ($d = 6$, $n = 3$)**.** We will work entirely in $\mathbb{C}[t,u,v]$ to compute a determinantal representation of

$$\begin{aligned}68096\cdot f\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = {} & 68096t^6 - 908944t^4uv + 385056t^3\left(u^3+v^3\right) - 189476it^3\left(u^3-v^3\right)\\ & + 1227176t^2(uv)^2 - 234844t(uv)\left(u^3+v^3\right) + 241874it(uv)\left(u^3-v^3\right)\\ & - 338630(uv)^3 - 39386\left(u^6+v^6\right) + 83423i\left(u^6-v^6\right)\end{aligned}$$

and then identify an associated 6×6 complex CWS_3 matrix. Choose the interlacer

$$\begin{aligned}g_{11} = {} & 51072t^5 - 454912t^3uv + 111920t^2(u^3+v^3) - 63176it^2(u^3-v^3)\\ & + 264552t(uv)^2 - 37276(uv)(u^3+v^3) - 16758i(uv)(u^3-v^3)\end{aligned}$$

which is invariant under C_3 and compute the points of $\mathcal{V}_{\mathbb{C}}(f, g_{11})$. Write $\mathcal{V}_{\mathbb{C}}(f,g_{11}) = S \cup \overline{S}$ where

$$\begin{aligned}\xi\left(\tilde{S}\right) = \Bigg\{ & \left[1 : \frac{200583326}{64044257} - \frac{285258889i}{233392773} : \frac{244352459}{74607586} + \frac{62709791i}{56915525}\right],\\ & \left[1 : \frac{142597425}{158926708} - \frac{52077399i}{48778709} : \frac{48386204}{51396105} + \frac{42948826i}{30888221}\right],\\ & \left[1 : \frac{90839198}{98990109} + \frac{59647715i}{136443017} : \frac{3720198458}{4953135125} - \frac{32658473i}{71082138}\right],\\ & \left[1 : \frac{65813250}{128179523} - \frac{50995379i}{161970078} : \frac{108140402}{101308711} - \frac{36569392i}{74258181}\right],\\ & \left[1 : \frac{25750408}{63724525} + \frac{23411858i}{489095981} : \frac{93204719}{234123334} - \frac{29029408i}{605365571}\right]\Bigg\}\end{aligned}$$

is a set of orbit representatives of S as in (3.2). Choose

$$\begin{aligned}g_{12} = {} & -(14112+28896i)t^4v + (48784+96848i)t^3u^2 + (61248-103264i)t^2uv^2\\ & + (28612-72316i)tu^3v - (61060-19540i)tv^4 + (3184+42688i)u^5\\ & - (13520-61548i)u^2v^3\end{aligned}$$

$$
\begin{aligned}
g_{13} = {} & (37088 + 52896i)t^4u - (71632 + 116880i)t^3v^2 + (32432 - 209040i)t^2u^2v \\
& + (19992 + 68584i)tu^4 - (16248 - 98624i)tuv^3 + (968 + 73456i)u^3v^2 \\
& - (55928 - 29156i)v^5 \\
g_{14} = {} & (-8512 - 8512i)t^5 + (156528 + 23312i)t^3uv - (140520 - 116208i)t^2u^3 \\
& - (83144 + 38032i)t^2v^3 - (35688 - 42800i)tu^2v^2 + (68482 - 35708i)u^4v \\
& + (45962 + 24276i)uv^4 \\
g_{15} = {} & (26208 - 35168i)t^4v + (-2832 + 87216i)t^3u^2 - (117504 - 71216i)t^2uv^2 \\
& - (11280 + 93436i)tu^3v + (43784 - 1348i)tv^4 - (12836 + 56992i)u^5 \\
& + (65468 - 43444i)u^2v^3 \\
g_{16} = {} & (27360 - 9120i)t^4u + (18128 - 2032i)t^3v^2 - (236800 - 130976i)t^2u^2v \\
& - (31252 + 125404i)tu^4 + (7848 + 32168i)tuv^3 + (76113 - 49517i)u^3v^2 \\
& + (40539 - 8823i)v^5
\end{aligned}
$$

which each lie in $\Lambda\big(\omega^{1-j}\big)_5$ and vanishes on $\xi(S)$ for all $j > 1$. Now set $g_{j1} = \overline{g_{1j}}$. Write $g_{1i}g_{j1} = af + bg_{11}$ for $a, b \in \mathbb{C}[t, u, v]$ where $b \in \Lambda\big(\omega^{i-j}\big)_5$ and $g_{ii} \in \mathbb{R}[t, (u+v)/2, (u-v)/2i]$. Set $g_{ij} = b$ and $g_{ji} = \overline{b}$. Explicitly, the entries can be chosen as

$$
\begin{aligned}
g_{22} = {} & 35840t^5 - 278384t^3uv + 94672t^2(u^3 + v^3) - 62656it^2(u^3 - v^3) \\
& + 167032t(uv)^2 - 49556uv(u^3 + v^3) - 23808iuv(u^3 - v^3) \\
g_{23} = {} & (78848 - 23040i)t^4v + (-50544 + 31712i)t^3u^2 - (250384 - 62928i)t^2uv^2 \\
& - (824 + 34576i)tu^3v + (57608 + 67624i)tv^4 + (14996 - 44928i)u^5 \\
& + (109780 + 23160i)u^2v^3 \\
g_{24} = {} & (23296 + 29792i)t^4u - (15056 + 24544i)t^3v^2 - (99216 + 125888i)t^2u^2v \\
& + (73044 + 2868i)tu^4 - (17124 - 2732i)tuv^3 + (76164 + 39092i)u^3v^2 \\
& - (13856 - 41892i)v^5 \\
g_{25} = {} & 7168it^5 + (111472 + 5984i)t^3uv - (88280 - 17832i)t^2u^3 \\
& - (76712 + 71096i)t^2v^3 - (88432 + 29552i)tu^2v^2 + (39316 - 35208i)u^4v \\
& + (25500 + 48200i)uv^4 \\
g_{26} = {} & -(14336 - 4096i)t^4v + (-23136 + 28368i)t^3u^2 + (182712 - 58280i)t^2uv^2 \\
& + (44280 + 7948i)tu^3v - (67028 + 51904i)tv^4 - (29408 - 42944i)u^5 \\
& - (83634 - 9138i)u^2v^3 \\
g_{33} = {} & 58368t^5 - 414576t^3uv + 92256t^2(u^3 + v^3) - 95264it^2(u^3 - v^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ 245096t(uv)^2 - 3780uv(u^3+v^3) + 43040iuv(u^3-v^3)\\
g_{34} =\ & (31248+105568i)t^3u^2 + (50300-23020i)u^5 - (3648+10336i)t^4v\\
&+ (45256-115856i)tu^3v - (5152+142768i)t^2uv^2 - (17024-28468i)u^2v^3\\
&- (19056-51992i)tv^4\\
g_{35} =\ & (24576+52736i)t^4u - (18160+69024i)t^3v^2 + (51856-128096i)t^2u^2v\\
&- (81384-25488i)tu^4 - (8512-88616i)tuv^3 - (72300-81600i)u^3v^2\\
&- (11028+16104i)v^5\\
g_{36} =\ & -19456t^5 + (254432-12304i)t^3uv - (200232-129432i)t^2u^3\\
&- (93696+71248i)t^2v^3 - (104452-51268i)tu^2v^2 + (74470-45390i)u^4v\\
&+ (29924+46932i)uv^4\\
g_{44} =\ & 25536t^5 - 264752t^3uv + 96640t^2(u^3+v^3) + 66840it^2(u^3-v^3)\\
&+ 166512t(uv)^2 - 16672uv(u^3+v^3) - 38526iuv(u^3-v^3)\\
g_{45} =\ & (17024-4704i)t^4v + (-29168+19424i)t^3u^2 + (49216+18496i)t^2uv^2\\
&- (5916+62672i)tu^3v - (72956+7328i)tv^4 + (1592+21344i)u^5\\
&- (75124+38408i)u^2v^3\\
g_{46} =\ & (-3040+20672i)t^4u - (1232-11296i)t^3v^2 + (58304-194960i)t^2u^2v\\
&+ (10052+84364i)tu^4 - (58336+21616i)tuv^3 - (11613-66879i)u^3v^2\\
&- (56993+1749i)v^5\\
g_{55} =\ & 28672t^5 - 241168t^3uv + 67632t^2(u^3+v^3) - 23352it^2(u^3-v^3)\\
&+ 217656t(uv)^2 + 7940uv(u^3+v^3) + 36200iuv(u^3-v^3)\\
g_{56} =\ & (11520+3840i)t^4v + (10720+24144i)t^3u^2 - (133280+58208i)t^2uv^2\\
&+ (11446-27918i)tu^3v + (61054+15442i)tv^4 + (14150-49450i)u^5\\
&+ (80744+30052i)u^2v^3\\
g_{66} =\ & 29184t^5 - 353440t^3uv + 180200t^2(u^3+v^3) - 82420it^2(u^3-v^3)\\
&+ 161460t(uv)^2 - 64462uv(u^3+v^3) + 25141iuv(u^3-v^3).
\end{aligned}
$$

Let G be the matrix with g_{ij} entries, take the adjugate of G to get a matrix with entries in $\mathbb{C}[t,u,v]_{25}$, and divide each entry by f^4. The definite determinantal representation we obtain is

$f = \det(M)$ where

$$M = 2^{24}\begin{pmatrix} 6t & (-2+4i)v & (-6-7i)u & (2+2i)t & (-6+7i)v & (-8-4i)u \\ (-2-4i)u & 8t & (-10+5i)v & (-7-10i)u & -2it & (-3+3i)v \\ (-6+7i)v & (-10-5i)u & 6t & (-3+6i)v & (-8-8i)u & 4t \\ (2-2i)t & (-7+10i)v & (-3-6i)u & 12t & (-4+8i)v & (-2-10i)u \\ (-6-7i)u & 2it & (-8+8i)v & (-4-8i)u & 10t & (-9+5i)v \\ (-8+4i)v & (-3-3i)u & 4t & (-2+10i)v & (-9-5i)u & 12t \end{pmatrix}.$$

Block diagonalize the coefficient matrix $M(1,0,0)$ of t so $M' = PMP^T$ with

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } M'(1,0,0) = 2^{24}\begin{pmatrix} 8 & -2i & 0 & 0 & 0 & 0 \\ 2i & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & 0 & 0 \\ 0 & 0 & 4 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2+2i \\ 0 & 0 & 0 & 0 & 2-2i & 12 \end{pmatrix}.$$

Decompose each block so we can write $M'(1,0,0) = (U)^{-1}(U^*)^{-1}$ where

$$U^{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix} 4\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{3} & \sqrt{57} & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2\sqrt{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2-2i & 8 \end{pmatrix}.$$

Finally, let $M'' = UM'U^*$ and $P^TM''P = tI + (u/2)A^* + (v/2)A$ is the desired determinantal representation with 6×6 complex CWS_n matrix

$$A = \begin{pmatrix} 0 & -\frac{1-2i}{2\sqrt{3}} & 0 & 0 & -\frac{14-13i}{2\sqrt{57}} & 0 \\ 0 & 0 & -\frac{10-5i}{4\sqrt{3}} & 0 & 0 & \frac{11-i}{4\sqrt{42}} \\ \frac{-6+7i}{6} & 0 & 0 & \frac{4+17i}{24} & 0 & 0 \\ 0 & -\frac{23-24i}{16\sqrt{3}} & 0 & 0 & -\frac{304-21i}{16\sqrt{57}} & 0 \\ 0 & 0 & \frac{(-9+14i)\sqrt{3}}{2\sqrt{19}} & 0 & 0 & -\frac{(15+5i)\sqrt{3}}{4\sqrt{266}} \\ -\frac{6+i}{3\sqrt{14}} & 0 & 0 & \frac{5+34i}{12\sqrt{14}} & 0 & 0 \end{pmatrix}.$$

3.2 Dihedral Invariance

Instead of finding an explicit unitary transformation to output a real CWS_n matrix, we modify Construction 3.1.8 to include the invariance under reflection. We divide the points of $\mathcal{V}_{\mathbb{C}}(f,g)$ based on orbits under rotation, then split according to reflection. Specifically, we require not only that $\mathcal{V}_{\mathbb{C}}(f,g) = S \cup \overline{S}$ where $\mathrm{rot}(S) = S$, but also $\mathrm{ref}\left(\overline{S}\right) = S$ meaning that if $p \in S$, then $\mathrm{ref}(p) \in \overline{S}$.

Corollary 3.2.1. *Every C_n-orbit in $\mathcal{V}_{\mathbb{C}}(f,g)$ is disjoint from its image under reflection when f and g satisfy (A1)–(A4).*

Proof. Let $\mathcal{O}$ be a C_n-orbit in $\mathcal{V}_{\mathbb{C}}(f,g)$. Suppose $[t : x+iy : x-iy] \in \xi(\mathcal{O}) \cap (\xi \circ \mathrm{ref})(\mathcal{O})$ with $t = 0$ or 1. Then

$$[t : x+iy : x-iy] = \Big[t : \omega^{\ell}(x-iy) : \omega^{-\ell}(x+iy)\Big] \tag{3.4}$$

for some fixed $\ell \in [n]$. If $t = 1$, (3.4) implies $x+iy = \omega^{\ell}(x-iy)$ and $[1 : x : y] \in \mathcal{V}_{\mathbb{R}}(f)$ by Lemma 3.1.2 which gives a contradiction. If $t = 0$, then (3.4) implies

$$[0 : x+iy : x-iy] = \Big[0 : \omega^{\ell}(x-iy) : \omega^{-\ell}(x+iy)\Big]$$

for some fixed $\ell \in [n]$. This implies $x+iy = \gamma \cdot \omega^{\ell}(x-iy)$ and $x-iy = \gamma\omega^{\ell}(x+iy)$ for some fixed $\gamma \in \mathbb{C}$. Then

$$x+iy = \gamma \cdot \omega^{\ell}(x-iy) = \gamma \cdot \omega^{\ell}(\gamma \cdot \omega^{-\ell}(x+iy)) = \gamma^2(x+iy)$$

implies $|\gamma| = 1 = |\gamma \cdot \omega^{\ell}|$. Lemma 3.1.2 again gives a contradiction. Since $\xi(\mathcal{O}) \cap (\xi \circ \mathrm{conj})(\mathcal{O})$ is empty, so is $\mathcal{O} \cap \mathrm{conj}(\mathcal{O})$. $\square$

Remark 3.2.2. Corollaries 3.1.3 and 3.2.1 imply that a C_n-orbit $\mathcal{O} \in \mathcal{V}_{\mathbb{C}}(f,g)$ is disjoint from both $\mathrm{conj}(\mathcal{O})$ and $\mathrm{ref}(\mathcal{O})$. However, this tells us nothing about the intersection of orbits $\mathrm{conj}(\mathcal{O})$ and $\mathrm{ref}(\mathcal{O})$. The intersection in this case may be nonempty, hence D_n-orbits in $\mathcal{V}_{\mathbb{C}}(f,g)$ do not always have the same cardinality.

If a matrix has entries in $\mathbb{R}[t,u,v]$, then its adjugate will also have entries in $\mathbb{R}[t,u,v]$. To produce a real CWS_n matrix, we also need the adjugate matrix we construct to have entries in $\mathbb{R}[t,u,v]$.

Remark 3.2.3. The invariant ring $\mathbb{C}[t,u,v]_d^{\langle \mathrm{ref} \circ \mathrm{conj} \rangle} = \mathbb{R}[t,u,v]_d$. Indeed,

$$\begin{aligned}
\sum_{l+j+k=d} c_{ljk} t^l (x+iy)^j (x-iy)^k &= h(t, x+iy, x-iy) \\
&= (\mathrm{ref} \circ \mathrm{conj}) \cdot h(t, x+iy, x-iy) \\
&= \left(\mathrm{ref} \cdot \overline{h}\right)(t, x+iy, x-iy)
\end{aligned}$$

$$= \mathrm{ref} \cdot \sum_{l+j+k=d} \overline{c_{ljk}} t^l (x-iy)^j (x+iy)^k$$
$$= \sum_{l+j+k=d} \overline{c_{ljk}} t^l (x+iy)^j (x-iy)^k.$$

Lemma 3.2.4. *If $S \subset \mathbb{P}^2(\mathbb{C})$ such that $(\mathrm{ref} \circ \mathrm{conj})(S) = S$, then $\Lambda(\omega^\ell)_{d-1} \cap \mathcal{I}(S)_{d-1}$ has a basis in $\mathbb{R}[t,u,v]_{d-1}$.*

Proof. We will argue that each linear subspace is invariant under $\mathrm{ref} \circ \mathrm{conj}$ separately, hence so is their intersection. The subspace $\Lambda(\omega^\ell)_{d-1}$ is invariant under $\mathrm{ref} \circ \mathrm{conj}$ since the action conjugates coefficients of elements in $\mathbb{C}[t,u,v]$, but preserves their support. The subspace $\mathcal{I}(S)_{d-1}$ is invariant under $\mathrm{ref} \circ \mathrm{conj}$ because

$$\mathcal{I}(S)_{d-1} = \mathcal{I}((\mathrm{ref} \circ \mathrm{conj})(S))_{d-1} = (\mathrm{ref} \circ \mathrm{conj})(\mathcal{I}(S)_{d-1}).$$

Each linear subspace is contained in $\mathbb{C}[t,u,v]_{d-1}^{\langle \mathrm{ref} \circ \mathrm{conj} \rangle}$, thus the intersection $\Lambda(\omega^\ell)_{d-1} \cap \mathcal{I}(S)_{d-1}$ has a basis in $\mathbb{R}[t,u,v]_{d-1}$. □

Construction 3.2.5. Let $d = qn$ for some $q \in \mathbb{Z}_+$ and $\Gamma = D_{2n}$.
Input: Two plane curves f and g satisfying (A1)–(A4).
Output: CWS_n matrix $A \in \mathbb{R}^{d \times d}$ such that $f = f_A$.

1. Set $g_{11} = g$.
2. Split up the distinct $d(d-1)$ points of $\mathcal{V}_{\mathbb{C}}(f, g_{11})$ into two disjoint, conjugate sets $S \cup \overline{S}$ of C_n-orbits such that $\mathrm{rot}(S) = S$ and $(\mathrm{ref} \circ \mathrm{conj})(S) = S$.
3. Extend g_{11} to a linearly independent set $\{g_{11}, g_{12}, \ldots, g_{1d}\} \subset \mathbb{C}[t,x,y]_{d-1}$ vanishing on all points of S with $\xi(g_{1j}) \in \Lambda(\omega^{1-j})_{d-1} \cap \mathbb{R}[t,u,v]_{d-1}$ and set $g_{j1} = \overline{g_{1j}}$ for all $j \in [d]$.
4. For $1 < i \leq j$, choose $\xi(g_{ij}) \in \Lambda(\omega^{i-j})_{d-1} \cap \mathbb{R}[t,u,v]_{d-1}$ so that $g_{11} g_{ij} - \overline{g_{1i}} g_{1j} \in \langle f \rangle$ and $g_{ii} \in \mathbb{R}[t,x,y]$.
5. For $i < j$, set $g_{ji} = \overline{g_{ij}}$ and define $G = (g_{ij})_{i,j} \in \mathbb{C}^{d \times d}$.
6. Define $M = (1/f^{d-2}) \cdot \mathrm{adj}(G)$.
7. For $\ell \in [d]$, write $\ell - 1 = an + b$ for some integers a and b with $0 \leq b \leq n-1$. Let P be the permutation matrix that takes $\ell = an + b + 1$ to $bq + a + 1$. Define $M' = PMP^T$ as a matrix with $q \times q$ blocks $M' = (M'_{kl})_{k,l=1}^n$

$$\xi\left(\left(M'_{kl}\right)_{ij}\right) \in \Lambda(\omega^{k-l})_1 \text{ for } k,l \in [n] \text{ and } i,j \in [q].$$

8. For each k compute the Cholesky decomposition of each diagonal block M'_{kk} and write $(M')^{-1}_{kk} = U_k U_k^T$ for some $U_k \in \mathbb{R}^{q\times q}$.

9. Define $U = \operatorname{diag}(U_1, U_2, \ldots, U_k)$ and output $A = \left(P^T U^T M' U P\right)(0,1,i)$.

Proof of Theorem 3.1.1(b). Let $g_{11} = g$. Here we follow Construction 3.1.8, but split the intersection points $\mathcal{V}_{\mathbb{C}}(f,g)$ into $S \cup \overline{S}$ so that $rot(S) = S$ and $(\mathrm{ref} \circ \mathrm{conj})(S) = S$. This is valid due to Corollaries 3.1.3 and 3.2.1. All of these points are distinct by assumption. By Lemma 3.2.4, we can choose linearly independent g_{1j} so $\xi(g_{1j}) \in \Lambda\left(\omega^{1-j}\right)_{d-1} \cap \mathcal{I}(S)_{d-1} \cap \mathbb{R}[t,u,v]$. Now let $g_{j1} = \overline{g_{1j}}$. The polynomials $f, \xi(g_{11}), \xi(\overline{g_{1i}} g_{1j}) \in \mathbb{R}[t,u,v]$, so by Lemma 2.3.7, we are also able to find g_{ij} such that $\xi(g_{ij}) \in \Lambda\left(\omega^{i-j}\right)_{d-1} \cap \mathbb{R}[t,u,v]_d$ for $1 < i < j$. Moreover, $g_{ii} \in \mathbb{R}[t,x,y]$ since $f, g_{11}, g_{1i}\overline{g_{1i}} \in \mathbb{R}[t,x,y]$. Let $g_{ji} = \overline{g_{ij}}$ for $i < j$ and define $G = (g_{ij})_{i,j}$. Notice that $\xi(G) \in \mathbb{R}[t,u,v]_1^{d\times d}$. Complete the construction as before. The matrix $M = (1/f^{d-2}) \cdot \operatorname{adj}(G)$ satisfies $\xi(M) \in \mathbb{R}[t,u,v]_1^{d\times d}$ so $M(1,0,0) \in \mathbb{R}^{d\times d}$.

Next we will show by permuting rows and columns of M we may get the identity matrix as the coefficient of t in our representation. Consider M as a matrix of $n \times n$ blocks. Each block is a cyclic weighted shift matrix and there are q^2 blocks in total. For $\ell \in [d]$, write $\ell - 1 = an + b$ for some integers a and b with $0 \le b \le n-1$. Let P be the permutation matrix that takes $\ell = an + b + 1$ to $bq + a + 1$. Define $M' = PMP^T$ as a matrix with $q \times q$ blocks $M' = (M'_{kl})_{k,l=1}^n$ with

$$\xi\left(\left(M'_{kl}\right)_{ij}\right) \in \Lambda\left(\omega^{k-l}\right)_1 \text{ for } k,l \in [n] \text{ and } i,j \in [q]$$

and $M'(1,0,0)$ is a real block diagonal matrix. By Theorem 3.3 of [32], we know $M(1,0,0)$ is definite, thus $M'(1,0,0)$ is definite. For each $k \in [n]$ write $M'_{kk}(1,0,0)^{-1} = U_k^T U_k$ for some $U_k \in \mathbb{R}^{d\times d}$. Define $U = \operatorname{diag}(U_1, U_2, \ldots, U_n)$. Then $M'' = UM'U^T$ is a representation of f since $M''(1,0,0) = I_d$ and $f = (1/\lambda) \cdot \det(M'')$ for $\lambda = \det(U) \cdot \det(U^T)$. Lastly, apply the inverse permutation so $f = (1/\lambda) \cdot \det(P^T M'' P)$. Evaluating $(P^T M'' P)(0,1,i)$ gives a cyclic weighted shift matrix of order n and it is real because $\xi(P^T M'' P) \in \mathbb{R}[t,u,v]^{d\times d}$. □

Example 3.2.6 ($d = 6$, $n = 3$)**.** We will work entirely in $\mathbb{C}[t,u,v]$ to compute a determinantal representation of

$$f\left(t, \frac{u+v}{2}, \frac{u-v}{2i}\right) = (1/2)\big(16456t^6 - 119539t^4uv + 16168t^3(u^3+v^3) + 105456t^2(uv)^2 + 1398tuv(u^3+v^3) + 162(u^3+v^3)^2 - 7200u^3v^3\big)$$

and then identify an associated 6×6 complex CWS_3 matrix. Choose the interlacer

$$g_{11} = (1/2)\big(2904t^5 - 13597t^3uv + 1914t^2(u^3+v^3) + 9252t(uv)^2 - 324uv(u^3+v^3)\big)$$

which is invariant under C_3 and compute the points of $\mathcal{V}_{\mathbb{C}}(f, g_{11})$. Write $\mathcal{V}_{\mathbb{C}}(f, g_{11}) = S \cup \overline{S}$ where

$$\xi\left(\tilde{S}\right) = \left\{ \left[1 : -\frac{161588550}{14584739} : -\frac{213306845}{52721374}\right], \left[1 : \frac{149867951}{33531022} : \frac{61111733}{77661621}\right], \right.$$
$$\left[1 : -\frac{164124776}{127166883} : -\frac{110044775}{85329829}\right], \left[1 : \frac{51828617}{65864511} : \frac{215475740}{48209919}\right],$$
$$\left. \left[1 : -\frac{103050589}{25470203} : -\frac{322108292}{29073009} - \frac{3373i}{18607772971631}\right] \right\}$$

is a set of orbit representatives of S as in (3.2). Choose

$$\begin{aligned}
g_{12} &= 1452t^5 - \frac{13597}{2}uvt^3 + 957t^2(u^3+v^3) + 4626t(uv)^2 - 162uv(u^3+v^3) \\
g_{12} &= 2024t^4v - 270t^3u^2 - \frac{5553}{2}t^2uv^2 - \frac{375}{2}tu^3v - \frac{837}{2}tv^4 - 9u^5 + 351u^2v^3 \\
g_{13} &= 81v^5 + 294tuv^3 - 844t^3v^2 - 1899u^3v^2 + \frac{4917}{2}t^2u^2v - 165tu^4 + 132t^4u \\
g_{14} &= -968t^5 - 799uvt^3 - 176u^3t^2 + \frac{2229}{2}t^2v^3 + 1581t(uv)^2 - 81uv^4 + 99u^4v \\
g_{15} &= 36u^5 + \frac{1725}{2}tvu^3 + 387t^3u^2 + 216v^3u^2 + 843t^2v^2u - 594t^4v - \frac{783tv^4}{2} \\
g_{16} &= 81v^5 - 2838tuv^3 + 2661t^3v^2 + 261u^3v^2 - \frac{3791}{2}t^2u^2v - 33tu^4 - 528t^4u
\end{aligned}$$

which each lie in $\Lambda\left(\omega^{1-j}\right)_5 \cap \mathbb{R}[t,u,v]$ and vanishes on $\xi(S)$ for all $j > 1$. Now set $g_{j1} = \overline{g_{1j}}$. Write $g_{1i}g_{j1} = af + bg_{11}$ for $a, b \in \mathbb{C}[t,u,v]$ where $b \in \Lambda\left(\omega^{i-j}\right)_5$ and $g_{ii} \in \mathbb{R}[t, (u+v)/2, (u-v)/2i]$. Set $g_{ij} = b$ and $g_{ji} = \overline{b}$. Explicitly, the entries can be chosen as

$$\begin{aligned}
g_{22} &= 1122t^5 - \frac{7199}{4}uvt^3 - 277u^3t^2 - 277v^3t^2 + \frac{453}{2}u^2v^2t + \frac{3uv^4}{2} + \frac{3u^4v}{2} \\
g_{23} &= \frac{189u^5}{2} - \frac{2103}{2}tvu^3 + 1680t^3u^2 + \frac{9v^3u^2}{2} + \frac{447}{2}t^2v^2u + 54tv^4 - 476t^4v \\
g_{24} &= \frac{9v^5}{2} - \frac{39}{2}tuv^3 + 1183t^3v^2 - \frac{411u^3v^2}{2} + \frac{7983}{2}t^2u^2v + 36tu^4 - 4092t^4u \\
g_{25} &= -374t^5 + \frac{3095}{4}uvt^3 + \frac{2945u^3t^2}{2} - 257v^3t^2 + 138u^2v^2t + 3uv^4 - 87u^4v \\
g_{26} &= -\frac{27u^5}{2} + \frac{795}{2}tvu^3 - 2606t^3u^2 + \frac{33v^3u^2}{2} - \frac{3771}{2}t^2v^2u + 51tv^4 + 1530t^4v \\
g_{33} &= 2244t^5 - \frac{30451}{2}uvt^3 + 1584u^3t^2 + 1584v^3t^2 + \frac{23301}{2}u^2v^2t + 405uv^4 + 405u^4v \\
g_{34} &= -1584vt^4 + 1560u^2t^3 + 3235uv^2t^2 - 675u^3vt - \frac{429v^4t}{2} - 900u^2v^3
\end{aligned}$$

$$
\begin{aligned}
g_{35} &= \frac{135v^5}{2} - 1815tuv^3 - 54t^3v^2 - \frac{765u^3v^2}{2} - \frac{2349}{2}t^2u^2v - 297tu^4 + 408t^4u \\
g_{36} &= -748t^5 + \frac{6821}{2}uvt^3 + 252u^3t^2 + 935v^3t^2 - \frac{1419}{2}u^2v^2t + 999uv^4 - 81u^4v \\
g_{44} &= 3388t^5 - \frac{13679}{2}uvt^3 - 164u^3t^2 - 164v^3t^2 + \frac{2141}{2}u^2v^2t + 45uv^4 + 45u^4v \\
g_{45} &= \frac{45u^5}{2} + 594tvu^3 - \frac{3117t^3u^2}{2} - \frac{255v^3u^2}{2} + \frac{7029}{2}t^2v^2u + 72tv^4 + 396t^4v \\
g_{46} &= -54v^5 - 631tuv^3 - 4392t^3v^2 + 126u^3v^2 - 1694t^2u^2v + 2222t^4u - \frac{21tu^4}{2} \\
g_{55} &= 1496t^5 - \frac{36029}{4}uvt^3 + \frac{2623u^3t^2}{2} + \frac{2623v^3t^2}{2} + 1248u^2v^2t - \frac{105uv^4}{2} - \frac{105u^4v}{2} \\
g_{56} &= \frac{27u^5}{2} + 618tvu^3 + 216t^3u^2 - \frac{1833v^3u^2}{2} + \frac{21531}{2}t^2v^2u - 204tv^4 - 2193t^4v \\
g_{66} &= 2992t^5 - \frac{32231}{2}t^3uv + 187t^2u^3 + 187t^2v^3 + \frac{5669}{2}tu^2v^2 - 135u^4v - 135uv.
\end{aligned}
$$

Let G be the matrix with g_{ij} entries, take the adjugate of G to get a matrix with entries in $\mathbb{C}[t,u,v]_{25}$, and divide each entry by f^4. The definite determinantal representation we obtain is $f = \det(M)$ where

$$
M = \frac{1}{2}\begin{pmatrix}
14t & -10v & 2u & 4t & 2v & 0 \\
-10u & 16t & v & 16u & 4t & -5v \\
2v & u & 8t & 3v & u & 2t \\
4t & 16v & 3u & 6t & 4v & -3u \\
2u & 4t & v & 4u & 12t & 7v \\
0 & -5u & 2t & -3v & 7u & 6t
\end{pmatrix}.
$$

Block diagonalize the coefficient matrix $M(1,0,0)$ of t so $M' = PMP^T$ with

$$
P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \text{ and } M'(1,0,0) = \begin{pmatrix}
8 & 2 & 0 & 0 & 0 & 0 \\
2 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 & 0 \\
0 & 0 & 1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 2 \\
0 & 0 & 0 & 0 & 2 & 3
\end{pmatrix}.
$$

Decompose each block so we can write $M'(1,0,0) = (U)^{-1}(U^*)^{-1}$ where

$$U^{-1} = \frac{1}{2\sqrt{7}} \begin{pmatrix} 4\sqrt{14} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{14} & \sqrt{154} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{7} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{7} & \sqrt{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2\sqrt{17} \end{pmatrix}.$$

Finally, let $M'' = UM'U^*$ and $P^T M'' P = tI + (u/2)A^* + (v/2)A$ is the desired determinantal representation with 6×6 real CWS_n matrix

$$A = \begin{pmatrix} 0 & -\frac{5}{2\sqrt{14}} & 0 & 0 & \frac{9}{2\sqrt{154}} & 0 \\ 0 & 0 & \frac{1}{8\sqrt{2}} & 0 & 0 & -\frac{21}{8\sqrt{22}} \\ \frac{1}{2\sqrt{7}} & 0 & 0 & \frac{\sqrt{17}}{4\sqrt{7}} & 0 & 0 \\ 0 & \frac{33}{\sqrt{238}} & 0 & 0 & -\frac{9}{\sqrt{2618}} & 0 \\ 0 & 0 & \frac{3}{8\sqrt{22}} & 0 & 0 & \frac{129}{88\sqrt{2}} \\ -\frac{1}{2\sqrt{77}} & 0 & 0 & -\frac{101}{4\sqrt{1309}} & 0 & 0 \end{pmatrix}.$$

3.3 The Degenerate Case

Here we deal with assumptions (A1)–(A4) posed in Section 3.1. We discuss the topology of curves in $\mathcal{H}_d^\Gamma$ and interlacers and then prove an analogue of Nuij's result concerning the closure of $(\mathcal{H}^\circ)_d^\Gamma$.

Proposition 3.3.1. *For $d = qn$ and $\Gamma = C_n$ or D_{2n}, a generic invariant form $f \in \mathbb{C}[t,x,y]_d^\Gamma$ defines a smooth plane curve $\mathcal{V}_\mathbb{C}(f) \subset \mathbb{P}^2(\mathbb{C})$.*

Proof. Let $\{f_0, \ldots, f_D\}$ be a basis for the $\mathbb{C}$-vector space $\mathbb{C}[t,x,y]_d^\Gamma$, and consider the set

$$\mathcal{X} \;=\; \left\{ (a,p) \in \mathbb{P}^D(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \;:\; \sum_{i=1}^{D} a_i \nabla f_i(p) = (0,0,0) \right\}$$

This is a subvariety of $\mathbb{P}^D(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$. By the Projective Elimination Theorem [26, Theorem 10.6], its image under the projection $\pi_1(a,p) = a$ is a subvariety of $\mathbb{P}^D(\mathbb{C})$. Therefore the image is either all of $\mathbb{P}^D(\mathbb{C})$, meaning that every polynomial in $\mathbb{C}[t,x,y]_d^\Gamma$ defines a singular curve, or it belongs to

a proper subvariety, meaning that a generic polynomial in $\mathbb{C}[t,x,y]_d^\Gamma$ defines a smooth curve. To finish the proof, we note that the polynomial $t^d + (x+iy)^d + (x-iy)^d$ belongs to $\mathbb{C}[t,x,y]_d^\Gamma$ and defines a smooth plane curve. $\square$

Proposition 3.3.2. *For $d = qn$ and $\Gamma = C_n$ or D_{2n} and any $e \in \mathbb{Z}_+$, the plane curves defined by generic invariant forms $f, g \in \mathbb{C}[t,x,y]^\Gamma$ with $\deg(f) = d$ and $\deg(g) = e$ intersect transversely.*

Proof. First, we argue that it suffices to produce one example of a pair of forms $f, g \in \mathbb{C}[t,x,y]^\Gamma$ with $\deg(f) = d$, $\deg(g) = e$ whose plane curves intersect transversely. This is because the intersecting transversely is a Zariski-open condition on f, g. More precisely, consider the subvariety $\mathcal{Y} \subset \mathbb{P}(\mathbb{C}[t,x,y]_d^\Gamma) \times \mathbb{P}(\mathbb{C}[t,x,y]_e^\Gamma) \times \mathbb{P}^2(\mathbb{C})$ defined by

$$\mathcal{Y} = \left\{ (f,g,p) : f(p) = 0, g(p) = 0, \operatorname{rank} \begin{pmatrix} \nabla f(p) \\ \nabla g(p) \end{pmatrix} \leq 1 \right\}.$$

By the Projective Elimination Theorem [26, Theorem 10.6], the image of $\mathcal{Y}$ under the projection $\pi(f,g,p) = (f,g)$ is a Zariski-closed set. By construction, it is the set of pairs (f,g) for which the intersection $\mathcal{V}_\mathbb{C}(f) \cap \mathcal{V}_\mathbb{C}(g)$ is non-transverse. We need to show that this does not occur for all pairs.

First we consider the special case $d = n$ and $e = 1, 2$. Note that since f is invariant under the action of C_n, it has the form

$$f(t,x,y) = a(x+iy)^n + b(x-iy)^n + \sum_{i=0}^{\lfloor n/2 \rfloor} c_i t^{n-i}(x^2+y^2)^i$$

where $a, b, c_0, \ldots, c_{\lfloor n/2 \rfloor} \in \mathbb{C}$ and $a = b$ if $\Gamma = D_{2n}$. Note that when a, b are non-zero, the intersection of $\mathcal{V}_\mathbb{C}(f)$ with $\mathcal{V}_\mathbb{C}(t)$ is transverse for non-zero a, b. Also if a, b are nonzero, then $\mathcal{V}_\mathbb{C}(f)$ and $\mathcal{V}_\mathbb{C}(x^2+y^2)$ have no common points with $t = 0$. Then by Bertini's theorem, for generic $\lambda, \mu \in \mathbb{C}$, the intersection of $\mathcal{V}_\mathbb{C}(f)$ and $\mathcal{V}_\mathbb{C}(\lambda(x^2+y^2) + \mu t^2)$ is transverse [2].

Now we construct the desired pair $f, g \in \mathbb{C}[t,x,y]^\Gamma$ with $\deg(f) = d$ and $\deg(g) = e$. Let f be the product of q generic forms in $\mathbb{C}[t,x,y]_n^\Gamma$ of degree n, and let g be the product of $\lfloor \frac{e}{2} \rfloor$ generic quadratic forms $\mathbb{C}[t,x,y]_2^\Gamma$ and t^δ where $\delta = 2(\frac{e}{2} - \lfloor \frac{e}{2} \rfloor)$. Then by the argument above, $\mathcal{V}_\mathbb{C}(f)$ and $\mathcal{V}_\mathbb{C}(g)$ intersect transversely. $\square$

Proposition 3.3.3. *Let $d = qn$. For $\Gamma = C_n$ or D_{2n} and generic invariant forms f, g in $\mathbb{C}[t,x,y]^\Gamma$ with $\deg(f) = d$ and $\deg(g) = d-1$, the number of intersection points on the line $t = 0$ is given by*

$$|\mathcal{V}_\mathbb{C}(f,g,t)| = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ d & \text{if } n \text{ is even.} \end{cases}$$

Proof. We first prove something slightly different. Let $e \in \mathbb{Z}_+$ be an integer satisfying $e \in 2\mathbb{N} + n\mathbb{N}$ where $q \cdot e$ is even. Then generic invariant forms $f, g \in \mathbb{C}[t, x, y]^\Gamma$ with $\deg(f) = d$ and $\deg(g) = e$ satisfy $\mathcal{V}_\mathbb{C}(f, g, t) = \emptyset$.

By the Projective Elimination Theorem [26, Theorem 10.6], the condition that $\mathcal{V}_\mathbb{C}(f, g, t)$ is non-empty is closed on $(f, g) \in \mathbb{C}[t, x, y]^\Gamma_d \times \mathbb{C}[t, x, y]^\Gamma_e$. Therefore it suffices to show that it is not the whole space.

Let $(a, b) \in \mathbb{N}^2$ so that $2a + nb = e$. Note that if e is even, then we may take b to be even. To see this, note that $e = 2a + bn$, implying that at least one of b and n is even. If $n = 2k$ is even and b is odd, then $b \geq 1$ and we may replace the pair (a, b) with $(a + k, b - 1)$.

For an integer $m \in \mathbb{Z}_+$, let $\chi(m)$ be 0 if m is even and 1 if m is odd. Then consider polynomials

$$f = (u^n + v^n)^{\chi(q)} \prod_{j=1}^{\lfloor q/2 \rfloor} (u^n + r_j v^n)(r_j u^n + v^n) \text{ and } g = (uv)^a (u^n + v^n)^{\chi(b)} \prod_{k=1}^{\lfloor b/2 \rfloor} (u^n + s_k v^n)(s_k u^n + v^n).$$

where $r_1, \ldots, r_{\lfloor q/2 \rfloor}, s_1, \ldots, s_{\lfloor b/2 \rfloor} \in \mathbb{C} \backslash \{0, 1\}$ are all distinct and $u = x + iy$, $v = x - iy$. We claim that both f, g are invariant under the dihedral group and have no common roots with $t = 0$, so long as $\chi(q) \cdot \chi(b) = 0$. For invariance, note that both f, g are invariant under the map $(t, u, v) \mapsto (t, \omega u, \overline{\omega} v)$, when ω is an n-th root of unity, as well as the map $(t, u, v) \mapsto (t, v, u)$.

The zeros of f with $t = 0$ consist of the points $[t : u : v] = [0 : 1 : \lambda\omega]$ where ω is an nth root of unity and $\lambda = 1, r_k, 1/r_k$ for $k = 1, \ldots, \lfloor q/2 \rfloor$. Moreover there is only such a root with $\lambda = 1$ if q is odd. Similarly, the zeros of g with $t = 0$ consist of the points $[t : u : v] = [0 : 1 : 0], [0 : 0 : 1]$ if $a \geq 1$ and $[t : u : v] = [0 : 1 : \lambda\omega]$ where $\lambda = 1, s_k, 1/s_k$ for $k = 1, \ldots, \lfloor b/2 \rfloor$, where $\lambda = 1$ gives a root only if b is odd. Therefore so long as both q and b are not odd, $\mathcal{V}_\mathbb{C}(f, g, t)$ is empty.

Now suppose that $e = d - 1 = qn - 1$ and n is odd. Then qn has the same parity as q, which is different than the parity of e. Furthermore, $e = (n - 1) + (q - 1)n$. Since $n - 1$ is even, this belongs to $2\mathbb{N} + n\mathbb{N}$. The argument from above then shows that $\mathcal{V}_\mathbb{C}(f, g, t) = \emptyset$.

If n is even, then so is d, meaning that $d - 1$ is odd. By Lemma 3.1.5, every polynomial $g \in \mathbb{C}[t, x, y]^\Gamma$ of degree $d - 1$ has a factor of t, meaning that it can be written as $g = t \cdot h$ where $h \in \mathbb{C}[t, x, y]^\Gamma_{d-2}$. Taking $e = d - 2$ above shows that $\mathcal{V}_\mathbb{C}(f, h, t) = \emptyset$. Therefore $\mathcal{V}_\mathbb{C}(f, g, t) = \mathcal{V}_\mathbb{C}(f, t)$. Since f has degree d, this consists of d points generically. □

Having dealt with the algebraic conditions of non-singularity, we address the semi-algebraic conditions of hyperbolicity and interlacing. To understand how the sets $(\mathcal{H}^\circ)^\Gamma_d$ and $\mathcal{H}^\Gamma_d$ relate, we introduce the following linear operator on invariant polynomials. For $s \in \mathbb{R}$, define the linear map $T_s : \mathbb{R}[t, x, y]_d \to \mathbb{R}[t, x, y]_d$ by

$$T_s(f) = f - s^2(x^2 + y^2)\frac{\partial^2 f}{\partial t^2}.$$

Lemma 3.3.4. *For any $s \in \mathbb{R}_{>0}$, the map preserves invariance under Γ and hyperbolicity. That is, $T_s(\mathcal{H}_d^\Gamma) \subset \mathcal{H}_d^\Gamma$. Moreover for any $f \in \mathcal{H}_d^\Gamma$, the polynomial $T_s^d(f)$, obtained by applying T_s d times to f, is strictly hyperbolic with respect to $(1,0,0)$. That is $T_s^d(\mathcal{H}_d^\Gamma) \subset (\mathcal{H}^\circ)_d^\Gamma$.*

Proof. First, note that if $f \in \mathbb{R}[t,x,y]_d^\Gamma$, then so are x^2+y^2 and $\frac{\partial^2 f}{\partial t^2}$, meaning that T_s preserves invariance under Γ.

For the other claims, consider the operator on univariate polynomials $T : \mathbb{R}[t] \to \mathbb{R}[t]$ where $T(p) = p - s^2 p''$. We claim that for any real-rooted polynomial $p \in \mathbb{R}[t]$, $T(p)$ is also real rooted and the roots of $T^d(p)$ where $d = \deg(p)$ are simple. To see this, consider the maps $T_\pm : \mathbb{C}[t] \to \mathbb{C}[t]$ where $T_\pm(p) = p \pm sp'$ for some $s \in \mathbb{R}$. The roots of $T_\pm(p)$ have multiplicity one less than those of p, any repeated roots of $T_\pm(p)$ are also repeated roots of p and any added roots of $T_\pm(p)$ are simple by Lemma of [31]. Let $T = T_+ \circ T_-$ so $T(p) = p - s^2 p''$ for some $s \in \mathbb{R}$. The roots of $T(p)$ have multiplicity two less than those of p, and any repeated roots are also repeated roots of p. Any other roots of $T(p)$ are simple. If $d = \deg(p)$, this implies every root of $T^d(p)$ is simple.

Since for any $(a,b) \in \mathbb{R}^2$, the restriction $T_s(f)(t,a,b)$ equals the image of $p(t) = f(t,a,b)$ under the univariate operator T, the polynomial $T_s(f)$ is hyperbolic with respect to $(1,0,0)$ and $T_s^d(f)$ is strictly hyperbolic. □

Proposition 3.3.5. *For $\Gamma = C_n$ or D_{2n} and any $d \in \mathbb{Z}_+$, $(\mathcal{H}^\circ)_d^\Gamma$ is a full-dimensional open subset of $\mathbb{R}[t,x,y]_d^\Gamma$, and its closure equals $\mathcal{H}_d^\Gamma$.*

Proof. Since strict hyperbolicity with respect to $[1:0:0]$ is an open condition on $\mathbb{R}[t,x,y]_d$, it suffices to show that $(\mathcal{H}^\circ)_d^\Gamma$ is non-empty. An explicit example is $t^\delta \cdot \prod_{i=1}^D (x^2+y^2-r_i t^2)$ where $D = \lfloor \frac{d}{2} \rfloor$, $\delta = 2(\frac{d}{2} - D)$, and $r_1 < \ldots < r_D \in \mathbb{R}_+$.

The set $\mathcal{H}_d^\Gamma$ is closed in $\mathbb{R}[t,x,y]_d^\Gamma$. To see that it is the closure of $(\mathcal{H}^\circ)_d^\Gamma$, let $f \in \mathcal{H}_d^\Gamma$. By Lemma 3.3.4, for $s > 0$, $T_s^d(f)$ is strictly hyperbolic with respect to $(1,0,0)$, meaning that $T_s^d(f)$ belongs to $(\mathcal{H}^\circ)_d^\Gamma$. The limit at $s = 0$ is exactly f. □

Theorem 3.3.6. *For $d = qn$ and $\Gamma = C_n$ or D_{2n}, every polynomial in $\mathcal{H}_d^\Gamma$ is a limit of polynomials $f \in (\mathcal{H}^\circ)_d^\Gamma$ for which there exists $g \in \mathcal{H}_{d-1}^\Gamma$ such that*

(i) $V_\mathbb{C}(f)$ is smooth,

(ii) g interlaces f,

(iii) $V_\mathbb{C}(f) \cap V_\mathbb{C}(g)$ is transverse, and

(iv) $|\mathcal{V}_\mathbb{C}(f,g,t)| = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ d & \text{if } n \text{ is even.} \end{cases}$

Proof. For any strictly hyperbolic $f \in \mathbb{R}[t,x,y]_d$ the set of polynomials $g \in \mathbb{R}[t,x,y]_{d-1}$ that interlace f with respect to $(1,0,0)$ is a full-dimensional set semialgebraic set. Then by Proposition 3.3.5, the set

$$\mathcal{I} = \left\{(f,g) \in (\mathcal{H}_d^\circ)^\Gamma \times \mathcal{H}_{d-1}^\Gamma : g \text{ strictly interlaces } f \text{ with respect to } (1,0,0)\right\}$$

is an open, full dimensional in $\mathbb{R}[t,x,y]_d^\Gamma \times \mathbb{R}[t,x,y]_{d-1}^\Gamma$ whose image under the projection $\pi(f,g) = f$ is all of $(\mathcal{H}_d^\circ)^\Gamma$. By Propositions 3.3.1, 3.3.2, and 3.3.3,

$$\mathcal{U} = \left\{(f,g) \in \mathbb{R}[t,x,y]_d^\Gamma \times \mathbb{R}[t,x,y]_{d-1}^\Gamma : \text{conditions (i),(iii), (iv) are satisfied}\right\}$$

is open and dense in the Euclidean topology on $\mathbb{R}[t,x,y]_d^\Gamma \times \mathbb{R}[t,x,y]_{d-1}^\Gamma$. It follows that $\mathcal{I} \cap \mathcal{U}$ an open set that is dense in $\mathcal{I}$. Since the projection $\pi(\mathcal{I})$ equals $(\mathcal{H}_d^\circ)^\Gamma$, this gives that the projection of $\mathcal{I} \cap \mathcal{U}$ is dense in $(\mathcal{H}_d^\circ)^\Gamma$. Then, by Proposition 3.3.5, we see that

$$\overline{\pi(\mathcal{I} \cap \mathcal{U})} \;=\; \overline{\pi(\mathcal{I})} \;=\; \overline{(\mathcal{H}_d^\circ)^\Gamma} \;=\; \mathcal{H}_d^\Gamma.$$

Therefore every polynomial in $\mathcal{H}_d^\Gamma$ belongs to the closure of the set of polynomials f for which there exists $g \in \mathbb{R}[t,x,y]_{d-1}^\Gamma$ with $(f,g) \in \mathcal{I} \cap \mathcal{U}$. □

Proof of Theorem 3.0.1. Let $f \in \mathcal{H}_d^\Gamma$. By Theorem 3.3.6 f is the limit of some $f_\varepsilon \in (\mathcal{H}^\circ)_d^\Gamma$ satisfying (A1)–(A4). By Theorem 3.1.1, there exists some $d \times d$ complex (real if $\Gamma = D_{2n}$) CWS_n matrix A_ε such that $f_\varepsilon = f_{A_\varepsilon}$. Now $f_\varepsilon(t,-1,0)$ and $f_\varepsilon(t,0,-i)$ are the characteristic polynomials of $\Re(W_\varepsilon)$ and $\Im(W_\varepsilon)$ and converge to the roots of $f(t,-1,0)$ or $f(t,0,-i)$ respectively. Therefore, the eigenvalues of $\Re(A_\varepsilon)$ and $\Im(A_\varepsilon)$ are bounded, which bounds the sequences $(\Re(A_\varepsilon))_\varepsilon$ and $(\Im(A_\varepsilon))_\varepsilon$. Then

$$(\Re(A_\varepsilon))_\varepsilon + i(\Im(A_\varepsilon))_\varepsilon = (\Re(A_\varepsilon) + i\Im(A_\varepsilon))_\varepsilon = (A_\varepsilon)_\varepsilon$$

which is also bounded. By passing to a convergent subsequence, this means $\lim_{\varepsilon\to 0}(A_\varepsilon)_\varepsilon = A$ and

$$f = \det\left(\lim_{\varepsilon\to 0}\left(tI_d + \frac{x+iy}{2}A_\varepsilon^* + \frac{x-iy}{2}A_\varepsilon\right)\right) = \det\left(tI_d + \frac{x+iy}{2}A^* + \frac{x-iy}{2}A\right) = f_A.$$

□

3.4 Generalizing to Any Degree

One could hope to generalize Construction 3.1.8 for a hyperbolic plane curve of any degree. The main obstruction here is with assumption (A1), specifically the requirement that $\mathcal{V}_{\mathbb{C}}(f)$ is smooth. For curves with $d \mod n \geq 3$, it seems there are always multiple singularities at infinity meaning most of these curves do not satisfy (A1). More specifically, there are two complex singularities at the points $[t : x : y] = [0 : 1 : \pm i]$. We conjecture they each have multiplicity $\binom{d \mod n}{2}$.

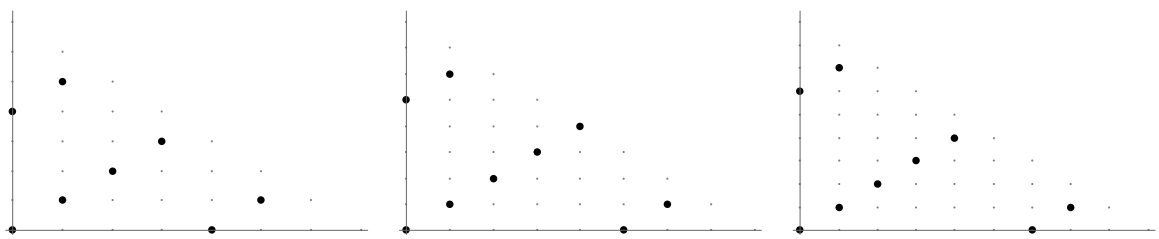

Figure 3.1: The support of elements in $\mathbb{C}[t,u,v]_7^{C_4}$, $\mathbb{C}[t,u,v]_8^{C_5}$, and $\mathbb{C}[t,u,v]_9^{C_6}$ (left to right).

Example 3.4.1 ($d = 7, n = 4$)**.** Consider $f \in \mathbb{C}[t,u,v]_7^{C_4}$. By homogeneity we can view the support of f in the plane $t = 1$ (see Figure 3.1). We see that each monomial of f must be the monomial t^7 or contain two of the variables t, u, and v. This confirms $\mathcal{V}_{\mathbb{C}}(f)$ contains the complex points $[t : u : v] = [0 : 1 : 0]$, $[0 : 0 : 1]$ and that these are singular points of $\mathcal{V}_{\mathbb{C}}(f)$.

Another way to try to construct a determinantal representation is to use Theorem 3.0.1 and hope to further specialize its structure.

Question 3.4.2. *Let $d = qn + m$ for some $q > 1$ and $m \in [n-1]$ and suppose $f \in \mathcal{H}_d^{\Gamma}$. Does there exist a determinantal representation of $t^{n-m}f$ such that*

(i) $t^{n-m}f = f_A$ for some CWS_n *matrix $A \in \mathbb{C}^{(q+1)n \times (q+1)n}$,*

(ii) $A = \mathrm{diag}(A', \boldsymbol{0})$ where $A' \in \mathbb{C}^{d \times d}$ is a CWS_n *matrix and $\boldsymbol{0} \in \mathbb{C}^{(n-m)\times(n-m)}$, and*

(iii) $f = f_{A'}$?

Example 3.4.3. Take $f = (1/8)(8t^4 - 798t^2uv + 1050t(u^3 + v^3) + 425it(u^3 - v^3) + 3860(uv)^2)$. We

can write $t^2 f = f_A$ with the CWS_3 matrix

$$A = \begin{pmatrix} 0 & -2+i & 0 & 0 & 0 & 0 \\ 0 & 0 & -10+5i & 0 & 0 & 0 \\ -6+7i & 0 & 0 & -4+8i & 0 & 0 \\ 0 & -2+10i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then f has a determinantal representation $f = f_{A'}$ given by the leading 4×4 minor of the determinantal representation of $t^2 f$.

Returning to Question 1.1.7, we certified invariance of hyperbolic plane curves of certain degrees under the cyclic and dihedral groups with the existence of a structured determinantal representation in Chapters 2 and 3. A positive answer to Question 3.4.2 would give a positive answer to Question 1.3.9. In general, we conjecture that both are true.

Chapter 4

Invariance of the Numerical Range

4.1 Results on the Classical Numerical Range

There is a lot of research dedicated to studying the geometry of numerical ranges of cyclic weighted shift matrices. The authors of [9, 11, 37] were particularly interested in the relationship between the numerical range and the the curve dual to its boundary generating curve. Specifically, the boundary of the numerical range has a strong correlation with singular points of the dual curve. We will see that this relationship also allows us to use results from Chapters 2 and 3 to characterize matrices whose numerical range is invariant under rotation.

In this chapter, we explore Question 1.2.7 and describe the interaction between invariance of the numerical range, its boundary generating curve, and the dual variety. Then we discuss a generalization of the numerical range and relate back to obstructions discussed in Section 3.4. The content of Theorem 4.1.2 and Corollary 4.2.2 appear in [30] for the case $d = n$.

Remark 4.1.1. Invariance of f_A under rotation implies the same of $\mathcal{W}(A)$ as discussed in Section 1.2. However, the converse does not hold; if $\mathcal{W}(A)$ is invariant under C_n, then this only implies that its boundary is invariant, rather than $\mathcal{V}_{\mathbb{C}}(f_A)$ in its entirety.

However, the invariance of the boundary of $\mathcal{W}(A)$ still gives us information about the dual curve $\mathcal{V}_{\mathbb{C}}(f_A)$. As a result of Theorem 3.0.1 and Corollary 4.2.2, we obtain a result which answers Question 1.2.7 in the case of matrices with CWS_n structure. In particular, the minimal set of irreducible components of the boundary generating curve containing the boundary of $\mathcal{W}(A)$ for CWS_n matrix A has to be invariant.

Theorem 4.1.2. *Suppose $\mathcal{W}(B)$ is invariant under C_n for some arbitrary $B \in \mathbb{C}^{d\times d}$. Then there exists a CWS_n matrix $A \in \mathbb{C}^{Qn\times Qn}$ such that $Q \leq \left\lfloor \frac{d}{n} \right\rfloor + 1$ and $\mathcal{W}(B) = \mathcal{W}(A)$.*

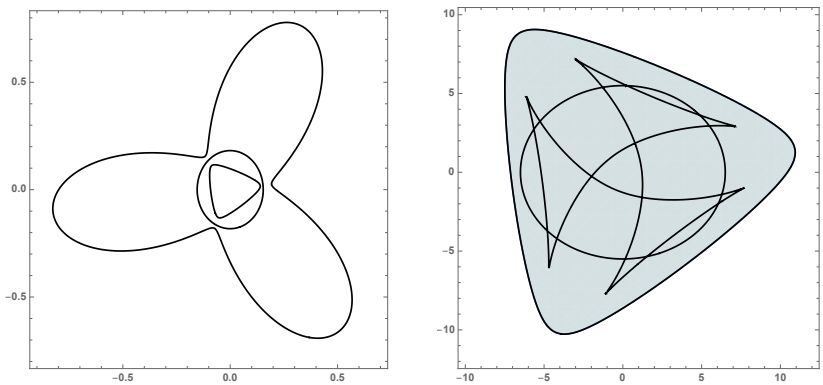

Figure 4.1: The hypersurface $\mathcal{V}_{\mathbb{C}}(f_B)$ in the plane $t = 1$ for $B \in \mathbb{C}^{6\times 6}$ from Example 4.1.3 (right) and $\mathcal{W}(B)$ (left). Although the plane curve and its dual are not invariant under rotation, the numerical range $\mathcal{W}(B)$ is.

Proof. The action of rotation on $\mathcal{W}(B) \in \mathbb{C}^{d\times d}$ is multiplication by $\omega = e^{\frac{2\pi i}{n}}$, so here we assume

$$\mathcal{W}(B) = \omega^{\ell}\mathcal{W}(B) = \mathcal{W}(\omega^{\ell}B)$$

for all $\ell \in [n-1]$. If f_B is irreducible, then $f_B = f_{\omega^{\ell}B}$ for each ℓ by [21, Corollary 2.4], so $f_B \in \mathbb{R}[t,x,y]_d^{C_n}$. By Theorem 3.0.1, there exists a CWS_n matrix A such that $f_B = f_A$, which gives the result.

Now assume f_B is reducible. Let $\mathcal{X} = \mathcal{V}_{\mathbb{C}}(f_B)$ and $S = \partial\mathcal{W}(B)$. Denote by $\mathcal{X}^*_{\min}$ be the minimal union of irreducible components of $\mathcal{X}^*$ which contains S. By assumption, $S \subseteq \mathcal{W}(B)$ is invariant under the action of rotation and is Zariski dense in $\mathcal{X}^*_{\min}$ since S contains an infinite number of points in $\mathcal{X}^*_{\min}$. Together with $S = \omega^{\ell}\, S \subseteq \omega^{\ell}\, \mathcal{X}^*_{\min}$, this implies $\mathcal{X}^*_{\min} \subseteq \omega^{\ell}\, \mathcal{X}^*_{\min}$ for every ℓ. The curve $\omega^{\ell}\mathcal{X}^*_{\min}$ has the same degree as $\mathcal{X}^*_{\min}$, so $\mathcal{X}^*_{\min} = \omega^{k}\mathcal{X}^*_{\min}$ for each ℓ and $\mathcal{X}^*_{\min}$ is invariant under rotation. The set $\mathcal{X}^*_{\min}$ is dual to some union of irreducible components $\mathcal{X}_{\min}$ of $\mathcal{X}$ and since $S \subseteq \mathcal{X}^*_{\min}$, the innermost oval of $\mathcal{X}$ is contained in $\mathcal{X}_{\min}$. Additionally, $\mathcal{X}_{\min}$ is invariant under rotation since by Proposition 1.3.1 this invariance is preserved by duality.

Write $f_A = f_1 f_2$ where $\mathcal{X}_{\min} = \mathcal{V}_{\mathbb{C}}(f_1)$. Then $f_1 \in \mathbb{R}[t,x,y]_D^{C_n}$ since $\mathcal{X}_{\min}$ is invariant under rotation and $0 < D < d$. If $f_2 \in \mathbb{R}[t,x,y]^{C_n}$, then Corollary 4.2.2 gives the result. If not, we want to find $\tilde{f}_2 \in \mathbb{R}[t,x,y]^{C_n}$ such that $f_1\tilde{f}_2$ is hyperbolic with respect to $(1,0,0)$, $\deg(\tilde{f}_2) = d - D$, and $\mathcal{X}_{\min}$ contains the innermost oval of $\mathcal{V}_{\mathbb{R}}(f_1\tilde{f}_2)$ since this will preserve the boundary of $\mathcal{X}^*$. Let $\tilde{f}_2 = t^{d-D}$, so $f_1\tilde{f}_2 \in \mathbb{R}[t,x,y]_d^{C_n}$. By Theorem 3.0.1, there exists a CWS_n matrix $A \in \mathbb{C}^{Qn\times Qn}$ for $Q = \lfloor \frac{D}{n} \rfloor + 1$ such that $f_1\tilde{f}_2 = f_A$, thus $\mathcal{W}(B) = \mathcal{W}(A)$. $\square$

Example 4.1.3. Take $B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -7-6i & 0 & -8-4i & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-2i \\ -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10-2i \\ 0 & -5-10i & 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\mathcal{W}(B)$ is invariant under rotation by the angle $2\pi/3$ (see Figure 4.1). We can write $f_B = f_1 f_2$ where

$$f_1 = (1/8)\left(8t^4 - 798t^2uv + 1050t(u^3+v^3) + 425it(u^3-v^3) + 3860(uv)^2\right),$$
$$f_2 = (1/4)\left(4t^2 + 12u^2 - 145uv + 12v^2\right)$$

and $\mathcal{V}_{\mathbb{C}}(f_1)^*$ contains the boundary of $\mathcal{W}(B)$. Notice that f_B is not invariant under rotation, but the quartic factor f_1 is. By Theorem 3.0.1, we can find a CWS_3 matrix $A \in \mathbb{C}^{6\times 6}$ such that $t^2 f_1 = f_A$ and $\mathcal{W}(A) = \mathcal{W}(B)$. One such matrix is given by

$$A = \begin{pmatrix} 0 & -2+i & 0 & 0 & 0 & 0 \\ 0 & 0 & -10+5i & 0 & 0 & 0 \\ -6+7i & 0 & 0 & -4+8i & 0 & 0 \\ 0 & -2+10i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1.2 gives a characterization for any matrix whose numerical range is invariant. That is, there always exists a larger CWS_n matrix with the same numerical range. If n does not divide d, then the size of the CWS_n matrix is strictly larger. However, a positive answer to Question 1.3.9 would strengthen this result and give a CWS_n matrix whose size depends on the reducibility of the boundary generating curve.

Conjecture 4.1.4. *If $\mathcal{W}(B)$ is invariant under C_n for arbitrary $B \in \mathbb{C}^{d\times d}$, then there exists a CWS_n matrix $A \in \mathbb{C}^{D\times D}$ such that $D \le d$ and $\mathcal{W}(A) = \mathcal{W}(B)$.*

4.2 Results on the k-Higher Rank Numerical Range

In this section we discuss a generalization of the numerical range called the k-higher rank numerical range. This set is also compact and invariant under unitary transformation. Woerdeman [39] showed that the set is convex with the help of [12].

Definition 4.2.1. For $k \in [d]$ the **k-higher rank numerical range** of $A \in \mathbb{C}^{d\times d}$ is

$$\mathcal{W}_k(A) := \left\{ \sum_{j=1}^{k} z_j^* A z_j \in \mathbb{C} \mid \{z_1, z_2, \ldots, z_k\} \text{ is orthonormal in } \mathbb{C}^d \right\}.$$

The classical numerical range is defined by $\mathcal{W}_1(A)$. Much like before, there is a relationship between the geometry of $\mathcal{W}_k(A)$ and the hyperbolic plane curve f_A. Chien and Nakazato described how to compute $\mathcal{W}_k(A)$ using f_A and the boundary generating curve in [8]. They also give conditions for which the k-higher rank numerical range is not given by the numerical range of any matrix when $k > 1$.

Gau and Wu [22] showed that for arbitrary matrices $A, B \in \mathbb{C}^{d\times d}$, $f_A = f_B$ if and only if $\mathcal{W}_k(A) = \mathcal{W}_k(B)$ for $k \in [\lfloor d/2 \rfloor + 1]$. Examples from [7, 13, 25] show there exist matrices A so $f_A \in \mathbb{C}[t, x, y]_d^{C_n}$, but A is not unitarily equivalent to any cyclic weighted shift matrix (with positive weights). Theorem 3.0.1 proves there must exist some CWS_n with the same k-higher rank numerical range of $A \in \mathbb{C}^{d\times d}$, even if the two matrices are not unitarily equivalent.

Corollary 4.2.2. *Suppose $d = qn + m$ for some $q \in \mathbb{Z}_+$ and $m \in [n-1]$. If $f_B \in \mathbb{C}[t, x, y]_d^{C_n}$ for some arbitrary $B \in \mathbb{C}^{d\times d}$, there exists a CWS_n matrix $A \in \mathbb{C}^{(q+1)n\times(q+1)n}$ with $\mathcal{W}_k(A) = \mathcal{W}_k(B)$ for $1 \le k \le \lfloor d/2 \rfloor + 1$.*

Proof. By Theorem 3.0.1, there exists cyclic weighted shift matrix $A \in \mathbb{C}^{(q+1)n\times(q+1)n}$ so that $t^{n-m} f_B = f_A$. Then by [21], this implies $\mathcal{W}_k(A) = \mathcal{W}_k(B)$ for $k \in [\lfloor d/2 \rfloor + 1]$. □

As before, we can strengthen this result with a positive answer to Question 1.3.9. Additionally, Chien and Nakazato proved that for two cyclic weighted shift matrices $A, B \in \mathbb{C}^{n\times n}$ with all nonzero entries that $\mathcal{W}(A) = \mathcal{W}(B)$ implies $\mathcal{W}_k(A) = \mathcal{W}_k(B)$ for all $k \in [d]$ [9]. They also showed if the cyclic weighted shift matrix A has all nonzero entries, then the plane curve f_A has no singularities on the line $y = \pm ix$. We conjecture that the relationship between singularities at the points $[t : x : y] = [0 : 1 : \pm i]$ as discussed in Section 3.4 may disprove the analogous statement that $\mathcal{W}(A) = \mathcal{W}(B)$ implies $\mathcal{W}_k(A) = \mathcal{W}_k(B)$ for $d > n$. Studying CWS_n matrices for $d > n$ in more detail may provide insight into answering Question 1.3.9 and give more information about the geometry of the k-higher rank numerical range in general.

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