ABSTRACT

LYNCH, MOLLY ELIZABETH. Topological and Algebraic Combinatorics of Crystal Posets. (Under the direction of Patricia Hersh).

Crystal bases were introduced by Kashiwara when studying modules of quantum groups. These crystals are combinatorial structures that mirror representations of Lie algebras. Each crystal base has an associated crystal graph. These are edge-colored, directed graphs that give information about representations such as weight space multiplicities and branching rules. Each directed edge in the graph encodes the action of a crystal operator. We study the combinatorics of crystal graphs given by highest weight representations. Much of the structure in these crystal graphs has been revealed by local relations given by Stembridge and Sternberg. However, it has been previously shown that not all relations among crystal operators are generated by these so-called Stembridge and Sternberg relations. In this thesis, we aim to understand and in certain cases, characterize relations that exist among crystal operators.

Crystals coming from highest weight representations have a natural partial order associated to them. Therefore, in this case we often refer to crystal graphs as crystal posets. We use a tool from poset topology known as lexicographic discrete Morse functions to establish a connection between the Möbius function μ of an interval in a crystal poset to the types of relations that can exist among crystal operators within this interval. More specifically, we show that if there is an interval with $\mu(u, v) \neq \{-1, 0, 1\}$, then there is a relation among crystal operators within the interval $[u, v]$ that is not implied by Stembridge or Sternberg relations. We present new relations in doubly laced crystals using this result. In the simply laced case, we prove that any new relation must involve at least three distinct crystal operators. We show via example that this result does not carry over to the doubly laced case.

Crystals corresponding to rank two algebras are often of interest. It is known that crystal posets corresponding to highest weight representations of type A_2 are lattices, while in general crystals of type A_n are not lattices. We show that crystal posets coming from highest weight representations of type B_2 and C_2 are not lattices.

We provide an in depth study of crystal graphs of type A_n coming from highest weight representations with highest weight given by $\lambda = (\lambda_1, \lambda_2)$ and $\lambda = (\lambda_1, 1, \cdots 1)$. We refer to these as two rowed shape crystals and hook shape crystals, respectively. In both cases, we give a characterization of when certain Stembridge relations will occur in the crystal. In hook shape crystals of type A_n, we conjecture that all relations among crystal operators are implied by Stembridge relations. For two rowed shape crystals, we prove that in rank three intervals, all relations among crystal operators are implied by Stembridge relations. It is known that this does not hold for arbitrary rank intervals in two rowed shape crystals of type A_n.

Topological and Algebraic Combinatorics of Crystal Posets

by
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DEDICATION

To my family.

BIOGRAPHY

Molly grew up in Wethersfield, Connecticut and studied mathematics as an undergraduate at the College of the Holy Cross in Worcester, Massachusetts. Following graduation, she began her graduate studies in mathematics at North Carolina State University. When not doing math, she is most likely hanging out with her dog, Mookie. Next year, she will be a Visiting Assistant Professor at Hollins University in Roanoke, Virginia.

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Introduction

Crystal bases are combinatorial structures that mirror representations of Lie algebras. They were introduced independently by two sources around 1990. Lusztig introduced canonical bases from a geometric perspective in [18, 19] and Kashiwara introduced crystal bases when studying modules of quantum groups [15–17]. In this thesis, we use the notion of crystals introduced by Kashiwara. Associated to each crystal base is a directed, edge-colored graph called a crystal graph. In many cases, these graphs are acyclic and have a natural partial order associated with them. Crystals are of interest because one can deduce important information regarding the corresponding representation of the Lie algebra. This includes branching rules, tensor products, and the character.

The main focus of this thesis is to understand the combinatorial structure of crystal graphs (or crystal posets). Each edge in a crystal graph encodes the action of a so-called crystal operator. We can study relations among crystal operators by studying the structure of these crystal posets. Stembridge gave a local characterization of crystal graphs of highest weight representations in the simply laced case in [23]. This characterization implies certain relations that must exist among crystal operators in the crystal graph. Sternberg proved in [24] that additional relations among crystal operators exist in crystals of highest weight representations in the doubly laced case. Although these relations dictate much of the structure of the crystal graph, in [11], Hersh and Lenart proved that there exist relations among crystal operators that are not implied by the Stembridge or Sternberg relations.

In this thesis, we use lexicographic discrete Morse functions to prove a connection between the Möbius function μ of an interval in a crystal poset and the relations that exist among crystal operators within that interval. Namely, we prove that for any interval $[u, v]$ such that $\mu(u, v) \notin \{-1, 0, 1\}$, there must exist a relation among crystal operators within $[u, v]$ that is not implied by Stembridge or Sternberg relations. This was first proven in the simply laced case in [11]. However, the proof given does not extend to the doubly laced case. We give a new proof

using discrete Morse theory that holds in the simply and doubly laced case. Using this result, we find new relations among crystal operators in types B_n and C_n.

We then carry out an in depth study of certain families of crystal posets. Specifically, we look at hook shape crystals and two rowed shape crystals in type A_n. We use a well known combinatorial model to realize these crystal graphs of type A_n where vertices of the graph are semistandard Young tableaux. We use the structure of semistandard Young tableaux to characterize when various types of relations exist among crystal operators. Additionally, we prove that any relation not implied by Stembridge relations in the simply laced case must involve at least three distinct crystal operators. However, we show via example that this result regarding relations needing three distinct operators does not carry over to the doubly laced case.

In the remainder of Chapter 1, we give an introduction to the main objects of study in this thesis. We define crystals, partially ordered sets, and give an introduction to lexicographic discrete Morse functions. We then conclude the chapter by giving further motivation for studying relations among crystal operators.

In Chapter 2, we prove a connection between the Möbius function of an interval in a crystal poset and the relations that exist among crystal operators within that interval. We do so by constructing a lexicographic discrete Morse function on the order complex of the interval. We prove that this discrete Morse function can have at most one critical cell if all relations among crystal operators within the interval are implied by Stembridge or Sternberg relations. We also prove that crystal posets of type B_2 and C_2 are not lattices. We end the chapter by giving examples of relations that exist among crystal operators in the doubly laced case that are not implied by Stembridge or Sternberg relations.

In Chapter 3, we study hook shape crystals and two rowed shape crystals of type A_n. We use the signature of a tableau to characterize when we have a degree two Stembridge relation versus a degree four Stembridge relation upward from a vertex. We prove that hook shape crystals are not in general lattices, although we nonetheless conjecture that Stembridge relations imply all relations among crystal operators in this case. For two rowed shape crystals, we prove that all rank three intervals have the property that no relations exist that are not implied by degree two Stembridge relations. Finally, we give further results regarding the intervals of arbitrary rank that were introduced in [11].

1.1   Crystal bases

In this thesis, we undertake a study of crystals corresponding to highest weight representations of Kac-Moody algebras. In these cases, the directed graphs given by the crystals are acyclic and have a natural partial order associated to them. One of the main goals of this thesis is to

understand the structure of these so-called crystal posets.

Let V be a Euclidean space and $\langle \, , \, \rangle$ the corresponding inner product. For a nonzero vector $\alpha \in V$, we define the *reflection* r_α in the hyperplane orthogonal to α as:

$$r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \text{ where } \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

Definition 1.1.1. A *root system* Φ in V is a nonempty finite set of nonzero vectors in V such that $\mathrm{span}(\Phi) = V$ that satisfy the following:

1. $r_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,

2. $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$,

3. If $\beta \in \Phi$ is a multiple of a $\alpha \in \Phi$, then $\beta = \pm\alpha$.

The elements of Φ are called *roots*, while elements of $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ are called *coroots*. A root system is *simply laced* if every root has the same length. If not all roots have the same length, then the root system is either *doubly laced* or *triply laced* depending on the angles between roots. All root systems in this thesis will be either simply laced or doubly laced.

Given a root system Φ in V, a *weight lattice* is a lattice Λ such that Λ spans V and $\Phi \subset \Lambda$. If $\lambda \in \Lambda$ and $\alpha \in \Phi$, then $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. The elements of Λ are called *weights*.

For a root system Φ, choose a hyperplane through the origin that does not intersect Φ. We call the roots on one side of this hyperplane *positive*, denoted Φ^+, and those on the other side are called *negative* denoted Φ^-. A positive root $\alpha \in \Phi^+$ is called *simple* if it cannot be expressed as a sum of the other positive roots. The set of simple roots $\{\alpha_i \mid i \in I\}$ form a basis for V. Let $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in I\}$. An element of Λ^+ is called a *dominant weight*. For more background on root systems, see [3, 13].

Each crystal is associated with a root system Φ that has an index set I and weight lattice Λ. Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots.

Definition 1.1.2. For a fixed root system Φ with index set I and weight lattice Λ, a *Kashiwara crystal* (crystal for short) of type Φ is a nonempty set $\mathcal{B}$ together with maps

$$e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\},$$
$$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\},$$
$$\mathrm{wt} : \mathcal{B} \to \Lambda,$$

where $i \in I$ and $0 \notin \mathcal{B}$ is an auxilliary element satisfying the following:

(A1) If $x, y \in \mathcal{B}$, then $e_i(y) = x$ if and only if $f_i(x) = y$. Here, we require

$$\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha_i, \quad \varepsilon_i(y) = \varepsilon_i(x) + 1, \quad \varphi_i(y) = \varphi_i(x) - 1$$

(A2) We require that
$$\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$$

for all $x \in \mathcal{B}$ and $i \in I$. In particular, if $\varphi_i(x) = -\infty$, then $\varepsilon_i(x) = -\infty$ as well. If $\varphi_i(x) = -\infty$ then we require $e_i(x) = f_i(x) = 0$.

The map wt is the *weight map*, the operators e_i and f_i are called *Kashiwara or crystal operators*, and the maps φ_i and ε_i are called *string lengths*. For all crystals in this thesis, we have the following:
$$\varphi_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} | f_i^k(x) \neq 0\},$$
$$\varepsilon_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} | e_i^k(x) \neq 0\}.$$

In this case, we say the crystal is *seminormal*.

Definition 1.1.3. A *crystal graph* associated to a crystal $\mathcal{B}$ is a directed, edge-colored (from some index set I) graph with vertex set $\mathcal{B}$ such that:

(S1) Every monochromatic path has finite length,

(S2) For a given vertex x, there is at most one incoming edge colored i and at most one outgoing edge colored i.

We draw an edge colored i from x to y whenever $y = f_i(x)$.

Since our focus is on crystals given by highest weight representations, let us briefly recall a few basics of Lie algebras and Kac-Moody algebras.

Definition 1.1.4. A *Lie algebra* $\mathfrak{g}$ over $\mathbb{C}$ is a vector space equipped with a bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $(x, y) \mapsto [x, y]$ called the *bracket*, that satisfies the following:

- $[x, x] = 0$ for all $x \in \mathfrak{g}$,

- $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ for all $x, y, z \in \mathfrak{g}$.

A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras. The irreducible representations of semisimple Lie algebras are classified by their highest weights. Roughly speaking, these highest weight representations are generated by a single highest weight element. For every dominant weight, there exists a finite dimensional irreducible representation whose highest

weight is that dominant weight. A fundamental result in the representation theory of Lie algebras is that every finite dimensional module can be written as a direct sum of irreducible modules. Kac-Moody algebras are generalizations of finite dimensional semisimple Lie algebras. Many properties related to the structure of a Lie algebra such as its root system and irreducible representations have natural analogs in the setting of Kac-Moody algebras. For a more detailed background on Lie algebras and Kac-Moody algebras, see [9, 13]. For more on the connections between representation theory and crystals, see [4, 12].

When referring to the crystal (or crystal graph) corresponding to the irreducible highest weight representation with highest weight λ, we will say the crystal (or crystal graph) of type Φ of *shape* λ.

For crystals of finite, classical Cartan type there is a combinatorial model, developed in [17], where the vertices of the crystal graph are represented by tableaux. We describe this realization for type A_n as the tableaux model is explicitly used for proofs in Chapter 3. For crystals of types B_n, C_n, and D_n, vertices can be represented by Kashiwara-Nakashima tableaux. For a description of these tableaux see e.g [4, 17].

For crystals of type A_n, each dominant weight λ can be viewed as a partition, i.e. $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. To each partition, we can associate a *Young diagram*. A Young diagram of shape λ is a finite collection of boxes, arranged in left-justified rows such that there are λ_i boxes in row i. We can represent the vertices of the crystal graph by semistandard Young tableaux. A *semistandard Young tableau of shape* λ is a filling of a Young diagram of shape λ where entries across rows read from left to right are weakly increasing, and entries read up columns from top to bottom are strictly increasing. Given a tableau T, the weight of T is:

$$\mathrm{wt}(T) = (\tau_1, \tau_2, ..., \tau_{n+1}),$$

where τ_i is the number of occurrences of i in T.

Definition 1.1.5. The *reading word* of T, denoted $r(T)$, is the word obtained by reading each column from bottom to top and reading columns from left to right.

Definition 1.1.6. The *i-signature* of T, denoted $\sigma_i(T)$, is the subword of $r(T)$ consisting of only the letters i and $i + 1$.

We describe the action of the crystal operators f_i and e_i on a given tableau via a combinatorial rule called the *signature rule*. Let $\mathcal{B}$ be a crystal of type A_n and let x be a vertex of $\mathcal{B}$.

We replace each appearance of i in the i-signature with the symbol $-$ and each appearance of $i + 1$ with the symbol $+$. Then we repeatedly remove any adjacent pairs of $(+-)$ as long as this is possible. In the end, we are left with the *reduced i-signature* of x, denoted $\rho_i(x)$, which

is of the form:

$$\rho_i(x) = \underbrace{- - \cdots -}_{a} \underbrace{+ + \cdots +}_{b}.$$

If $a > 0$ we obtain $f_i(x)$ from x by changing the i in x that corresponds to the rightmost $-$ in $\rho_i(x)$ to an $i+1$. If $a = 0$, then $f_i(x) = 0$. Similarly, if $b > 0$, then $e_i(x)$ is obtained by changing the $i + 1$ in x that corresponds to the leftmost $+$ in $\rho_i(x)$ to an i. If $b = 0$, then $e_i(x) = 0$. For other types, the signature rule for the applications of f_i and e_i are similar. For details see e.g. [4, 12].

Example 1.1.7. Let $\lambda = (9, 6)$ and $x = $

1	1	2	2	2	2	3	3	4
2	3	3	4	4	5			

. We will calculate $f_2(x)$

and $e_2(x)$. We have $r(x) = 213132424252334$ and $\sigma_2(x) = 233222233$. This gives

$$- + + - - - - + +.$$

After removing all $(+-)$ pairs

$$- + \underbrace{+ -}_{} - - - + +$$

we are left with

$$\rho_2(x) = - - - + +,$$

where the rightmost minus sign corresponds to the final appearance of 2 in the first row of x and the leftmost plus sign corresponds to the first appearance of 3 in the first row of x. Then

$$f_2(x) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 \\ \cline{1-6} \end{array},$$

and

$$e_2(x) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 \\ \cline{1-6} \end{array}.$$

1.1.1 Stembridge axioms and Sternberg relations

In [23], Stembridge gives a local characterization of crystals coming from integrable highest weight representations in the simply laced case. In doing so, he provides a list of local relations that exist among crystal operators. He shows that these relations also hold in the doubly laced case, but do not give a complete characterization. In [24], Sternberg shows that for crystals of doubly laced type coming from a highest weight representation, there are additional relations among crystal operators other than those given by the Stembridge axioms. For a complete characterization of doubly laced crystals see [7, 25]. Now, we introduce some notation and the

axioms as seen in [23].

Throughout this section we will let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of a Kac-Moody algebra $\mathfrak{g}$, where I is a finite index set. We recall the following from [23].

We define the *i-string through x* to be:

$$f_i^{-d}(x) \to \cdots \to f_i^{-1}(x) \to x \to f_i(x) \to \cdots \to f_i^r(x).$$

We can then define the *i-rise of x* to be $\vartheta_i(x) := r$ and the *i-depth of x* to be $\delta_i(x) := -d$. To measure the effect of the crystal operators e_i and f_i on the j-rise and j-depth of each vertex, we define the *difference operators* Δ_i and ∇_i to be:

$$\Delta_i \delta_j(x) = \delta_j(e_i(x)) - \delta_j(x), \quad \Delta_i \vartheta_j(x) = \vartheta_j(e_i(x)) - \vartheta_j(x),$$

whenever $e_i(x)$ is defined, and

$$\nabla_i \delta_j(x) = \delta_j(x) - \delta_j(f_i(x)), \quad \nabla_i \vartheta_j(x) = \vartheta_j(x) - \vartheta_j(f_i(x)),$$

whenever $f_i(x)$ is defined.

Definition 1.1.8. We say that an edge-colored, directed graph, X, is *A-regular* if the axioms (S1) and (S2) from Definition 1.1.3 hold as well as (S3)-(S6) and (S5′)-(S6′).

(S3) For a fixed $x \in X$ and $i, j \in I$ such that $e_i(x)$ is defined, we require $\Delta_i \delta_j(x) + \Delta_i \vartheta_j(x) = a_{ij}$,

(S4) For a fixed $x \in X$ and $i, j \in I$ such that $e_i(x)$ is defined, we require $\Delta_i \delta_j(x) \le 0$ and $\Delta_i \vartheta_j(x) \le 0$.

(S5) For a fixed $x \in X$ such that $e_i(x)$ and $e_j(x)$ are both defined, we require that $\Delta_i \delta_j(x) = 0$ implies $e_i e_j(x) = e_j e_i(x) \ne 0$ and $\nabla_j \vartheta_i(y) = 0$ where $y = e_i e_j(x) = e_j e_i(x)$.

(S6) For a fixed $x \in X$ such that $e_i(x)$ and $e_j(x)$ are both defined, we require that $\Delta_i \delta_j(x) = \Delta_j \delta_i(x) = -1$ implies $e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x) \ne 0$ and $\nabla_i \vartheta_j(y) = \nabla_j \vartheta_i(y) = -1$ where $y = e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x)$.

Dually, we have the additional two requirements for X to be *A-regular*,

(S5′) For a fixed $x \in X$, $\nabla_i \vartheta_j(x) = 0$ implies $f_i f_j(x) = f_j f_i(x) \ne 0$ and $\Delta_j \delta_i(y) = 0$ where $y = f_i f_j(x) = f_j f_i(x)$.

(S6′) For a fixed $x \in X$, $\nabla_i \vartheta_j(x) = \nabla_j \vartheta_i(x) = -1$ implies $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x) \ne 0$ and $\Delta_i \delta_j(y) = \Delta_j \delta_i(y) = -1$ where $y = f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

In [23], Stembridge proves the following:

Theorem 1.1.9 ([23])**.** *The crystal graph corresponding to any highest weight representation is A-regular. These axioms characterize crystal graphs in the simply laced case.*

All crystals studied in this thesis are such that the Stembridge axioms hold. The axioms only give a complete characterization in the simply laced case.

Definition 1.1.10. If we have $x \in \mathcal{B}$ such that

$$f_i f_j(x) = f_j f_i(x) \neq 0,$$

then we say there is a *degree two Stembridge relation* upward from x. Similarly, if we have $x \in \mathcal{B}$ such that

$$f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x) \neq 0,$$

then we say that there is a *degree four Stembridge relation* upward from x. Dually, when these relations occur involving the e_i crystal operators, we say we have a degree two or degree four Stembridge relation downward from x, the degree coming from the number of operators.

See Figure 1.1 for visualizations of the degree two and degree four Stembridge relations.

(a)   (b)
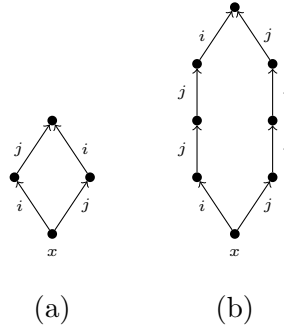
Figure 1.1:   (a) The degree two Stembridge relation, and (b) the degree four Stembridge relation

We now consider the doubly laced case, i.e. crystals corresponding to the root systems of type B_n and C_n. In [24], Sternberg proves a conjecture of Stembridge by providing a description of the local structure of crystals arising from highest weight representations in the doubly laced case.

Theorem 1.1.11 ([24])**.** *Let $\mathcal{B}$ be a crystal coming from a highest weight representation of doubly laced type. Let x be a vertex of $\mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$ where f_i and f_j are two distinct crystal operators. Then exactly one of the following is true:*

1. $f_i f_j(x) = f_j f_i(x)$,

2. $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$,

3. $f_i f_j^3 f_i(x) = f_j f_i f_j f_i f_j(x) = f_j^2 f_i^2 f_j(x)$,

4. $f_i f_j^3 f_i^2 f_j(x) = f_i f_j^2 f_i f_j f_i f_j(x) = f_j f_i^2 f_j^3 f_i(x) = f_j f_i f_j f_i f_j^2 f_i(x)$.

The equivalent statement with the crystal operators e_i and e_j also holds.

Definition 1.1.12. If we have $x \in \mathcal{B}$ such that

$$f_i f_j^3 f_i(x) = f_j f_i f_j f_i f_j(x) = f_j^2 f_i^2 f_j(x),$$

then we say there is a *degree five Sternberg relation* upward from x. Similarly, if we have $x \in \mathcal{B}$ such that

$$f_i f_j^3 f_i^2 f_j(x) = f_i f_j^2 f_i f_j f_i f_j(x) = f_j f_i^2 f_j^3 f_i(x) = f_j f_i f_j f_i f_j^2 f_i(x),$$

then we say that there is a *degree seven Sternberg relation* upward from x. Dually, when these relations occur involving the e_i's, we say we have a degree five or degree seven Sternberg relation downward from x.

See Figure 1.2 for visualizations of the degree five and degree seven Sternberg relations.
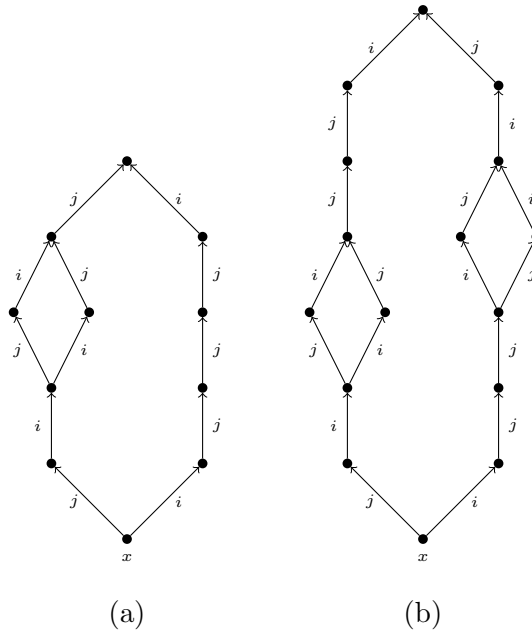

(a)   (b)

Figure 1.2: (a) The degree five Sternberg relation, and (b) the degree seven Sternberg relation

We remark that for both Stembridge and Sternberg relations, there are exactly two distinct crystal operators used. Later we will see there exist relations among three or more distinct crystal operators as well.

1.2 Partially ordered sets

We will now give a brief overview of partially ordered sets, as the main objects of study in this thesis are crystal posets.

Definition 1.2.1. A *partially ordered set P* (or poset) is a set P together with a binary relation $\leq$ such that for all $s, t, u \in P$ we have:

1. reflexivity: $s \leq s$.

2. antisymmetry: if $s \leq t$ and $t \leq s$, then $s = t$.

3. transitivity: if $s \leq t$ and $t \leq u$, then $s \leq u$.

We call "$\leq$" a partial order.

Given a subset $Q \subseteq P$, we say that Q is a *subposet* of P if for $s, t \in Q$, we have $s \leq t$ in Q if and only if $s \leq t$ in P. We say that u is covered by v (or v covers u), denoted by $u \lessdot v$ if $u < v$ and there is no element $w \in P$ such that $u < w < v$. We call these *cover relations*. For finite posets (and more generally for locally finite posets), P is generated by such relations. An *interval* $[u, v]$ is a subposet of P defined by $[u, v] = \{s \in P : u \leq s \leq v\}$ whenever $u < v$, Similarly, an *open interval* (u, v) is defined by $(u, v) = \{s \in P : u < s < v\}$. A poset P is *locally finite* if each interval $[u, v]$ is finite. We say that P has a *minimum element*, denoted $\hat{0}$, if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq u$ for all $u \in P$. Similarly, P has a *maximum element*, denoted $\hat{1}$, if there exists an element $\hat{1} \in P$ such that $u \leq \hat{1}$ for all $u \in P$. A *chain* is a poset in which any two elements x and y are comparable (i.e. $x \leq y$ or $y \leq x$). A subset C of P is a chain if it is a chain when considered as a subposet of P. A *saturated chain* from u to v is a series of cover relations $u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k = v$. We say that a finite poset is *graded* if for all $u \leq v$, every saturated chain from u to v has the same number of cover relations, and we call this number the *rank* of the interval $[u, v]$. The *rank of an element* $x \in P$ is the rank of the interval $[\hat{0}, x]$. The *Hasse diagram* of a finite poset P is the graph whose vertices are elements of P with an edge drawn upward from x to y whenever $x \lessdot y$.

For $s, t \in P$, an *upper bound* of s and t is an element v in P such that $v \geq s$ and $v \geq t$. Similarly, a *lower bound* of s and t is an element u such that $u \leq s$ and $u \leq t$. A *least upper bound* for s and t is an element v such that for any w where $s \leq w \leq v$ and $t \leq w \leq v$, we must have $v = w$. We define a greatest lower bound similarly. If two elements have a unique least

upper bound it is called a *join*. Similarly, if two elements have a unique greatest lower bound, it is called a *meet*. We denote by $s \vee t$ the join of s and t and $s \wedge t$ the meet of s and t. A poset L in which every two elements have a meet and a join is a *lattice*.

Example 1.2.2. Consider the set of subsets of the numbers $\{1, 2, 3\}$ ordered by inclusion. This gives the Boolean lattice seen in Figure 1.3. The minimal element is $\hat{0} = \emptyset$ and the maximal element is $\hat{1} = \{1, 2, 3\}$. We have, for example, that $\{1\} \vee \{3\} = \{1, 3\}$ and $\{1, 3\} \wedge \{2\} = \emptyset$.
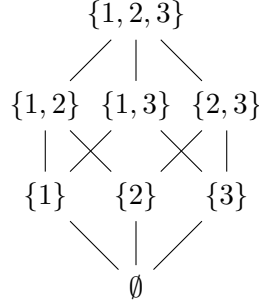
Figure 1.3: Boolean lattice on three elements

The *Möbius function*, μ of a poset P is a function $\mu : P \times P \to \mathbb{Z}$ defined recursively as follows: $\mu(u, u) = 1$, for all $u \in P$, $\mu(u, v) = -\sum_{u \le t < v} \mu(u, t)$, for all $u < v \in P$, and $\mu(u, v) = 0$ otherwise. Given a poset P, the *order complex* $\Delta(P)$ is the abstract simplicial complex whose i-dimensional faces are the chains $x_0 < x_1 < \cdots < x_i$ of P. Let $\Delta(u, v)$ denote the order complex of the subposet consisting of the open interval (u, v).

Example 1.2.3. In Figure 1.4 we have a poset on six elements and its corresponding order complex. Each saturated chain corresponds to a facet of the order complex. For example, the saturated chain $\hat{0} \lessdot a_1 \lessdot a_2 \lessdot \hat{1}$ corresponds to the tetrahedron with vertices $\{\hat{0}, a_1, a_2, \hat{1}\}$.

One reason to be interested in the order complex of a poset is the connection between the Möbius function of a poset P and the Euler characteristic of the order complex $\Delta(P)$, discussed e.g. in [21, 26].

Theorem 1.2.4. *Let P be a poset with $\hat{0}$ and $\hat{1}$. Then $\mu(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$.*

The posets that we study in this thesis come from crystals. More specifically, we study the crystal graphs of crystals of highest weight representations. We view these crystal graphs as posets with exactly the cover relations $u \lessdot v$ for $v = f_i(u)$ for some $i \in I$. This extends transitively to a partial order on the crystal graph, namely $u \le v$ whenever there is a directed
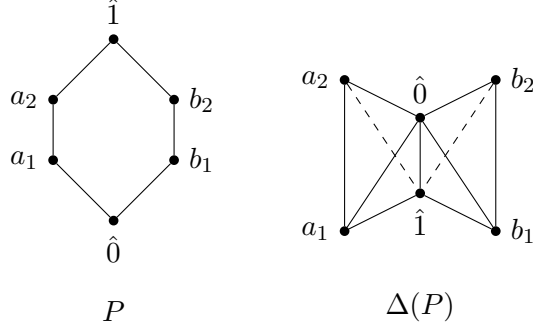
Figure 1.4: A poset P and its order complex $\Delta(P)$

path from u to v. We color the edge of the covering relation given by $f_i(u) = v$ with the color i. This gives the structure of an edge-colored poset. We call these posets *crystal posets*. Note that the crystal graph is the Hasse diagram of the crystal poset. The following definition will be useful later.

Definition 1.2.5. Given $[u, v] \subseteq \mathcal{B}$, for $\mathcal{B}$ a crystal poset, let $C = u \lessdot x_1 \lessdot \cdots \lessdot x_m \lessdot v$ be a saturated chain from u to v. The *edge label sequence* of C is the tuple $(\beta(u \lessdot x_1), ..., \beta(x_m \lessdot v))$ where $\beta(x_k \lessdot x_{k+1}) = i$ if $x_{k+1} = f_i(x_k)$.

Example 1.2.6. Consider the type A_3 crystal graph of shape $\lambda = (2, 1, 1)$. In Figure 1.5 we have the Hasse diagram of the crystal poset. This interval has the following edge label sequences: $(1, 2, 3, 3, 2, 1), (1, 3, 2, 2, 3, 1), (1, 3, 2, 2, 1, 3), (3, 1, 2, 2, 3, 1), (3, 1, 2, 2, 1, 3)$ and $(3, 2, 1, 1, 2, 3)$.

1.3   Discrete Morse theory

Discrete Morse theory was introduced in [8] by Forman as a tool to study the homotopy type and homology groups of (primarily finite) CW complexes. The main idea of discrete Morse theory is that for a given CW complex or simplicial complex Δ, we can construct a more efficient CW complex, while retaining many topological properties of the original complex.

1.3.1   Discrete Morse functions

In this thesis, we will apply discrete Morse theory to simplicial complexes associated to crystal posets. Let Δ be a simplicial complex.

Definition 1.3.1. A *discrete Morse function* on a simplicial complex Δ is a function $f : \Delta \to \mathbb{R}$ such that for each d-dimensional simplex, $\alpha \in \Delta$,
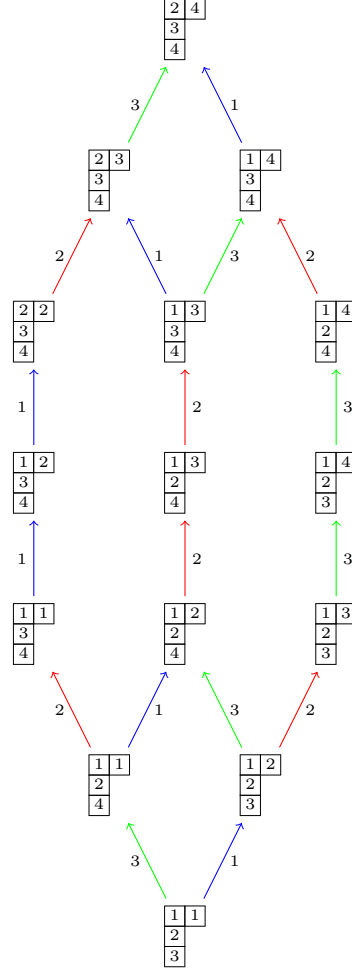
Figure 1.5: Type A_3 crystal poset of shape $\lambda = (2,1,1)$

1. $|\{\beta \supseteq \alpha \mid \dim \beta = d+1 \text{ and } f(\beta) \leq f(\alpha)\}| \leq 1$,

2. $|\{\gamma \subseteq \alpha \mid \dim \gamma = d-1 \text{ and } f(\gamma) \geq f(\alpha)\}| \leq 1$.

In [5], Chari gave a combinatorial reformulation for discrete Morse functions in the case of regular CW complexes. This reformulation is in terms of acyclic matchings on the face posets of the CW complexes, which is what we will use here. A *matching* is a set of edges in a graph with no common vertices. We say a matching on the Hasse diagram of a face poset is *acyclic* if the directed graph obtained by directing matching edges upward and all other edges downward has no directed cycles. It is known, (for example see [10]), that whenever a face poset has an acyclic matching, then there is a nonempty set of associated discrete Morse functions on the corresponding complex. We will be interested in the so-called critical cells of a discrete Morse

function.

Definition 1.3.2. For a discrete Morse function f on a simplicial complex Δ, a simplex $\alpha \in \Delta$ is called a *critical cell* if $|\{\beta \supseteq \alpha \mid \dim \beta = d + 1 \text{ and } f(\beta) \leq f(\alpha)\}| = 0$ and $|\{\gamma \subseteq \alpha \mid \dim \gamma = d - 1 \text{ and } f(\gamma) \geq f(\alpha)\}| = 0$. Equivalently, a simplex α is called a critical cell if it is left unmatched by the matching on the face poset.

Example 1.3.3. Consider the Boolean lattice from Figure 1.3. This is the face poset of the two dimensional simplex Δ_2. In Figure 1.6 we demonstrate a matching on this face poset. We match S with $S \setminus \{1\}$ if $1 \in S$ and $S \cup \{1\}$ othewise. We denote the matching edges in bold. As every vertex is matched, the corresponding discrete Morse function would have no critical cells.

$$\{1, 2, 3\}$$

$$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}$$

$$\{1\} \quad \{2\} \quad \{3\}$$

$$\emptyset$$
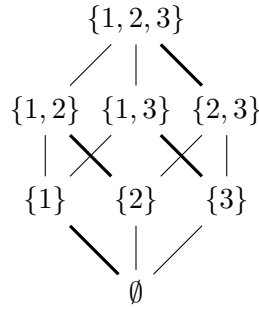
Figure 1.6: Matching on Boolean lattice

The following theorem of Forman illustrates the usefulness of discrete Morse functions.

Theorem 1.3.4 ([8])**.** *Suppose Δ is a simplicial complex with a discrete Morse function f. Then Δ is homotopy equivalent to a CW-complex with exactly one cell of dimension d for each critical cell in Δ of dimension d with respect to f.*

This theorem tells us that discrete Morse functions provide a method of taking a complicated simplicial complex and creating a new simpler one from critical cells that is homotopy equivalent to the original.

We deviate slightly from Forman's conventions in a way that is typical in combinatorics. We allow the empty set to be in the domain of our discrete Morse function f, as well as in the face posets on which we construct acyclic matchings. By doing so, we must express our results in terms of reduced Euler characteristic and reduced homology.

Remark 1.3.5. From Theorem 1.3.4, rephrased to use reduced Betti numbers and Morse numbers, we can immediately deduce that if a discrete Morse function has exactly one critical cell

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of dimension i and no other critical cells, then our original simplicial complex is homotopy equivalent to an i-dimensional sphere.

1.3.2 Lexicographic discrete Morse functions

In [1], Babson and Hersh introduced lexicographic discrete Morse functions as a tool to study the topology of order complexes of partially ordered sets with $\hat{0}$ and $\hat{1}$. This is what we will use to study crystal posets.

Before we describe how to construct lexicographic discrete Morse functions, we explain some of the useful properties they have. Since we attach the facets by lexicographic order on saturated chains, the lexicographic discrete Morse functions will have relatively few critical cells. If the attachment of the facet corresponding to some saturated chain does not change the homotopy of the subcomplex of our order complex built so far, then this step does not introduce any critical cells. Additionally, each facet can contribute at most one critical cell. We describe these critical cells using minimal skipped intervals, which will be discussed shortly.

We will now review lexicographic discrete Morse functions in general. This will rely on a notion of rank within a chain that does not require the poset to be graded. However, the crystal posets we are interested in are graded, as seen in Lemma 2.3.1, simplifying the grading in a chain.

Given a graded poset P of rank n, let β be an integer labeling on the edges of the Hasse diagram of P such that $\beta(u \lessdot v) \neq \beta(u \lessdot w)$ whenever $v \neq w$. Each facet of $\Delta(P)$ corresponds to a saturated chain, $\hat{0} \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot \hat{1}$ in P. For each saturated chain we read off the label sequence $(\beta(\hat{0} \lessdot u_1), \beta(u_1 \lessdot u_2), \cdots, \beta(u_k \lessdot \hat{1}))$ and order these lexicographically. This labeling gives rise to a total order on the facets $F_1, ..., F_k$ of the order complex. By virtue of the fact that we attach facets in a lexicographic order, each maximal face in $\overline{F}_j \cap (\cup_{i<j} \overline{F}_i)$ has rank set of the form $1, ..., i, j, ..., n$ for $j > i+1$ i.e. it omits a single interval of consecutive ranks. We call this rank interval $[i+1, j-1]$ a *minimal skipped interval* of F_j with *support* $i+1, ..., j-1$ and *height* $j - i - 1$. For a given facet F_j, we call the collection of minimal skipped intervals the *interval system of F_j*.

Remark 1.3.6. In order to determine the minimal skipped intervals for a given saturated chain C corresponding to some facet F_j, we consider each cover relation $u \lessdot v$ as we travel up C. At each cover relation $u \lessdot v$, we check if there is a lexicographically earlier cover relation $u \lessdot v'$ upward from u. If so, we obtain a maximal face in $\overline{F}_j \cap (\cup_{i<j} \overline{F}_i)$, and hence a minimal skipped interval, by taking the intersection of $\overline{F}_j$ with the closure of any facet $F_{i'}$ that includes $u \lessdot v'$, that agrees with F_j below u and agrees with F_j above $w \in F_j$ for some $w > v'$ of minimal rank. See Figure 1.7.

When our poset has some natural labeling, like that of our crystal posets, it is often possible
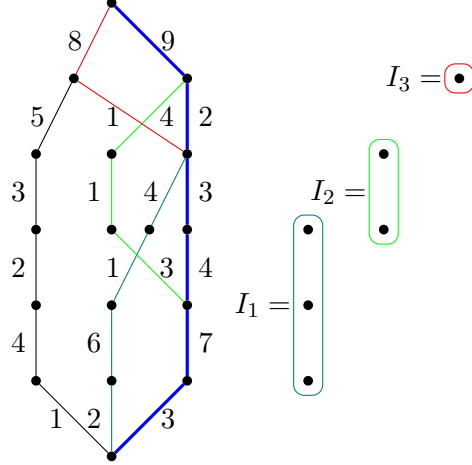
Figure 1.7: Interval system example

to classify its minimal skipped intervals.

Any face in $\overline{F}_j \setminus (\cup_{i<j}\overline{F}_i)$ must include at least one rank from each of the minimal skipped intervals of F_j. For each j, an acyclic matching on the set of faces in $\overline{F}_j \setminus (\cup_{i<j}\overline{F}_i)$ is constructed in [1] in terms of the interval system. The union of these matchings is acyclic on the entire Hasse diagram of the face poset of the order complex of P, and therefore give rise to a family of discrete Morse functions. We describe this acyclic matching after giving a description of the truncation algorithm.

A facet F_j will contribute a critical cell if and only if the interval system of F_j covers all ranks in F_j after the truncation algorithm described below. In this case we say that the corresponding saturated chain is *fully covered*. The dimension of such a critical cell is one less than the number of minimal skipped intervals in the interval system after the truncation algorithm. This truncation algorithm is needed when the interval system of some facet F_j covers all ranks but there are overlapping minimal skipped intervals, i.e. two minimal skipped intervals both cover a common rank. Otherwise the truncated system equals the original system.

Remark 1.3.7. In actuality, we study the order complexes of the proper parts of our posets; if P has a $\hat{0}$ and $\hat{1}$, then $\Delta(P)$ is contractible as it is a cone. We use the $\hat{0}$ and $\hat{1}$ in the lexicographic discrete Morse functions in a bookkeeping role. More specifically, $\hat{0}$ and $\hat{1}$ are needed to record the labels of covering relations upward from $\hat{0}$ and upward towards $\hat{1}$. In particular, when we refer to fully covered saturated chains, the ranks of $\hat{0}$ and $\hat{1}$ are not covered.

For the truncation algorithm, we begin with our interval system, I, and initialize the truncated system, which we call J, to be the empty set. Then, we repeatedly move the minimum interval in I to the truncated system J and truncate all other elements of I to eliminate any

overlap with the minimum interval in I being moved to J at this step (by minimum we mean the minimal skipped interval containing the element of lowest rank). Next, remove any intervals in I that are no longer minimal. Repeat this until there are no longer any minimal skipped intervals in I. We call the truncated, minimal skipped intervals obtained by this algorithm the *J-intervals of F_j*. By construction, these are non-overlapping. If the J-intervals cover all ranks of F_j, then F_j contributes a critical cell. We get this critical cell by taking the lowest rank element of each of the J-intervals. Otherwise F_j does not contribute any critical cells. See Example 1.3.8.

We now give a description of the acyclic matching on $F_j \setminus \cup_{i<j} F_i$. If the interval system I does not cover all ranks of the saturated chain F_j then we match by including and excluding the lowest rank element not covered by I. If the interval system I does cover all ranks, then we consider the interval system J. If J covers all ranks, then we get a critical cell by taking the lowest rank element from each J-interval. We then match any cell that differs from the critical cell on some J-interval by looking at the lowest rank J-interval for which it differs. We match by including and excluding the element of lowest rank in this lowest rank J-interval. If the J-interval system does not cover all ranks, then we match by including and excluding the lowest rank element uncovered by J.

Example 1.3.8. Here we give an example of the interval system of the saturated chain bolded in blue with label sequence $(3, 7, 4, 3, 2, 9)$ seen in Figure 1.8. Following the idea in Remark 1.3.6, we have three minimal skipped intervals in our interval system I: I_1 covers ranks $\{2, 3, 4\}$, I_2 covers ranks $\{4, 5\}$, and I_3 covers rank $\{6\}$. As I_1 and I_2 overlap, we perform the truncation algorithm to get three J-intervals, $J_1 = \{2, 3, 4\}, J_2 = \{5\}$, and $J_3 = \{6\}$. As all proper ranks are covered, the facet F_j would contribute a critical cell coming from the lowest rank elements of each J-interval.

The following example illustrates that it is possible to have a saturated chain that is fully covered by the I-interval system but fails to be fully covered after the truncation algorithm.

Example 1.3.9. We consider the I-interval system $I = \{I_1, I_2, I_3\}$ seen in Figure 1.9. The interval I_1 covers the first and second rank which we denote by $I_1 = \{1, 2\}$. Similarly, we have $I_2 = \{2, 3\}$ and $I_3 = \{3, 4\}$. In order to see if the saturated chain in blue with edge label sequence $(5, 3, 4, 2, 6)$ is fully covered, we must truncate the I-intervals. We set $J_1 = I_1$ and truncate I_2 to remove overlap with I_1 so $I_2' = \{3\}$. However, now $I_2' \subseteq I_3$ implying that I_3 is no longer minimal. As a result, we remove this interval and set $J_2 = I_2'$. Hence, the J-intervals are $J_1 = \{1, 2\}$ and $J_2 = \{3\}$ and the vertex of rank 4 is uncovered.

In Section 2.3, we prove that under certain conditions, if the I-interval system fully covers all ranks, then the J-interval system does as well.
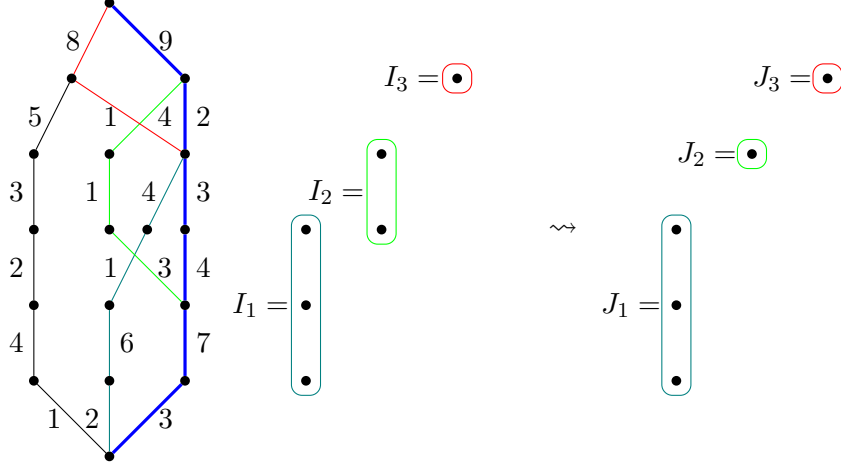
Figure 1.8: Minimal skipped intervals and truncation algorithm
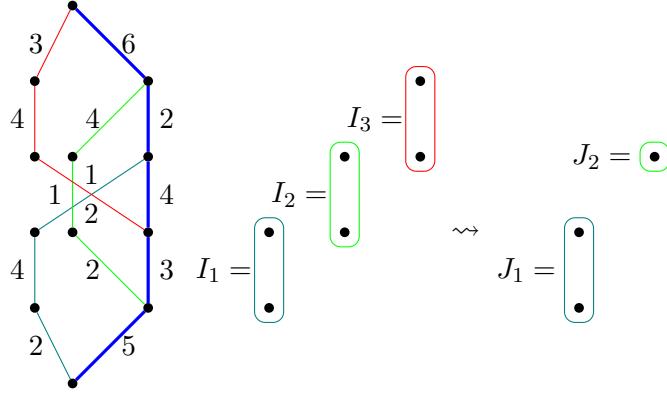
Figure 1.9: Interval system with I fully covering where J-intervals do not fully cover

1.4 Motivation

Much of the inspiration for this work stems from a paper by Hersh and Lenart [11]. Here we recall some of the key definitions and results from their paper as a way to motivate the work done in this thesis.

Definition 1.4.1. A *Stembridge move* on a simply laced crystal is either the replacement of the saturated chain $x \lessdot f_i(x) \lessdot f_j(f_i(x))$ by the saturated chain $x \lessdot f_j(x) \lessdot f_i(f_j(x))$ in the case where $f_i f_j(x) = f_j f_i(x)$ or the replacement of the saturated chain

$$x \lessdot f_i(x) \lessdot f_j(f_i(x)) \lessdot f_j(f_j f_i(x)) \lessdot f_i(f_j f_j f_i(x))$$

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by the saturated chain

$$x \lessdot f_j(x) \lessdot f_i(f_j(x)) \lessdot f_i(f_i f_j(x)) \lessdot f_j(f_i f_i f_j(x))$$

in the case where $f_i f_j f_j f_i(x) = f_i f_j f_j f_i(x)$.

See Figure 1.10 for an illustration of these moves where we replace the saturated chain in blue with the saturated chain in red or vice versa.

(a)   (b)
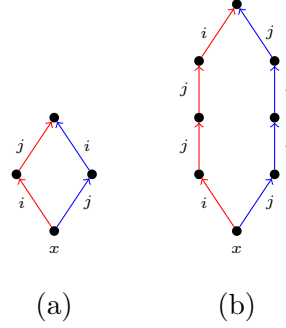
Figure 1.10: Stembridge moves

In [11], they prove the following statement.

Theorem 1.4.2 ([11]). *In the simply laced case, any two saturated chains from $\hat{0}$ to v in a lower interval $[\hat{0}, v]$ are connected by a series of Stembridge moves. In addition, in finite type, the same result holds for upper intervals $[v, \hat{1}]$.*

They remark that the proof will carry over to the doubly laced case. For completeness, we give the proof in the doubly laced case here. To begin, we define a Sternberg move.

Definition 1.4.3. A *degree five Sternberg move* on a doubly laced crystal is the replacement of the saturated chain

$$x \lessdot f_i(x) \lessdot f_j(f_i(x)) \lessdot f_j(f_j f_i(x)) \lessdot f_j(f_j f_j f_i(x)) \lessdot f_i(f_j f_j f_j f_i(x))$$

with either the saturated chain

$$x \lessdot f_j(x) \lessdot f_i(f_j(x)) \lessdot f_j(f_i f_j(x)) \lessdot f_i(f_j f_i f_j(x)) \lessdot f_j(f_i f_j f_i f_j(x))$$

or the saturated chain

$$x \lessdot f_j(x) \lessdot f_i(f_j(x)) \lessdot f_i(f_i f_j(x)) \lessdot f_j(f_i f_i f_j(x)) \lessdot f_j(f_j f_i f_i f_j(x))$$

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in the case where there is a degree five Sternberg relation upward from x.

See Figure 1.11 for an illustration of these moves where we replace the saturated chain in blue with one of the saturated chains in red seen in (a) and (b).

(a)   (b)
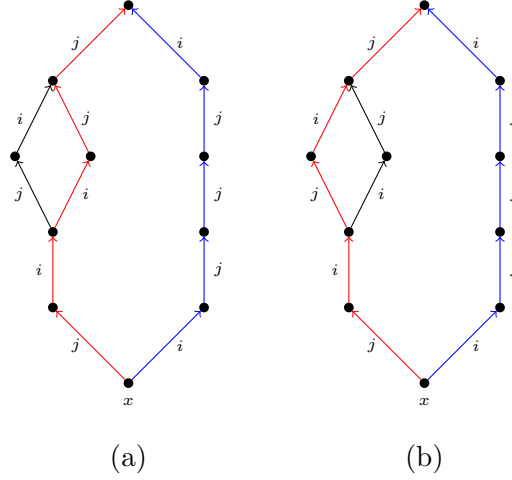
Figure 1.11:   Degree five Sternberg move

Definition 1.4.4. A *degree seven Sternberg move* on a doubly laced crystal is the replacement of the saturated chain

$$x \lessdot f_i(x) \lessdot f_j(f_i(x)) \lessdot f_j(f_j f_i(x)) \lessdot f_j(f_j^2 f_i(x)) \lessdot f_i(f_j^3 f_i(x)) \lessdot f_i(f_i f_j^3 f_i(x)) \lessdot f_j(f_i^2 f_j^3 f_i(x))$$

or

$$x \lessdot f_i(x) \lessdot f_j(f_i(x)) \lessdot f_j(f_j f_i(x)) \lessdot f_i(f_j^2 f_i(x)) \lessdot f_j(f_i f_j^2 f_i(x)) \lessdot f_i(f_j f_i f_j^2 f_i(x)) \lessdot f_j(f_i f_j f_i f_j^2 f_i(x))$$

by either the saturated chain

$$x \lessdot f_j(x) \lessdot f_i(f_j(x)) \lessdot f_j(f_i f_j(x)) \lessdot f_i(f_j f_i f_j(x)) \lessdot f_j(f_i f_j f_i f_j(x)) \lessdot f_j(f_j f_i f_j f_i f_j(x)) \lessdot f_i(f_j^2 f_i f_j f_i f_j(x))$$

or

$$x \lessdot f_j(x) \lessdot f_i(f_j(x)) \lessdot f_i(f_i f_j(x)) \lessdot f_j(f_i^2 f_j(x)) \lessdot f_j(f_j f_i^2 f_j(x)) \lessdot f_j(f_j^2 f_i^2 f_j(x)) \lessdot f_i(f_j^3 f_i^2 f_j(x))$$

in the case where there is a degree seven Sternberg relation upward from x.

See Figure 1.12 for an illustration of these moves where we replace one of the saturated

chains in blue with one of the saturated chains in red. In essence, we can replace any maximal chain with a different maximal chain that only agrees at the end points.
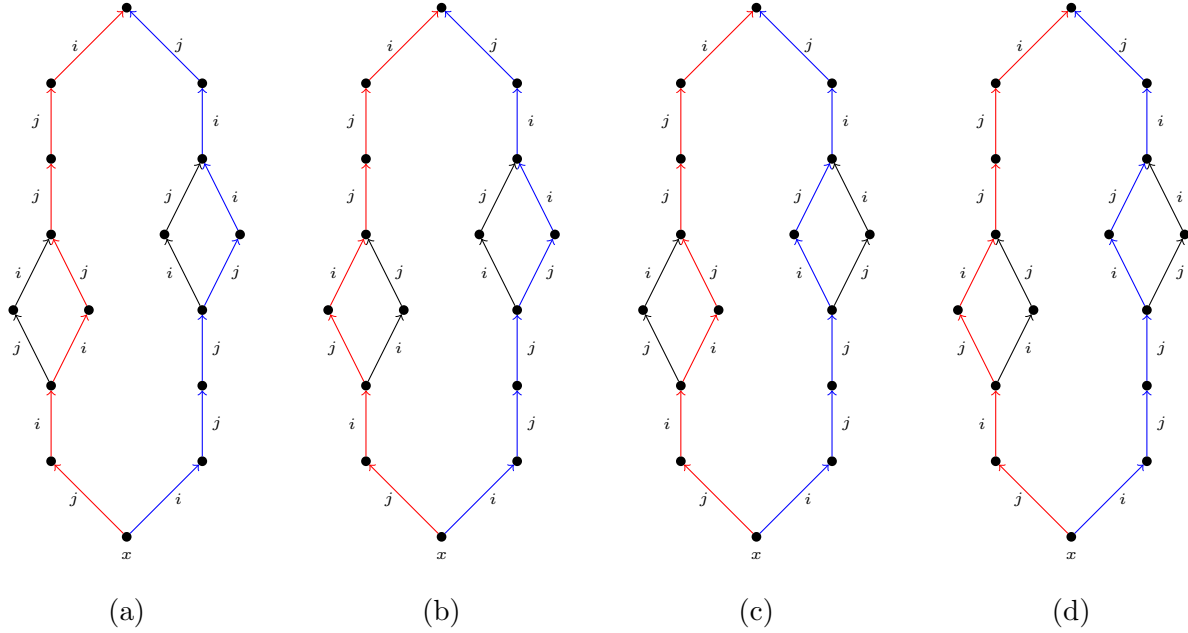
Figure 1.12: Degree seven Sternberg move

With the Sternberg moves defined, we now prove the analogue of Theorem 1.4.2 for doubly laced type. This proof relies on the fact that crystal posets are ranked which we will prove in Lemma 2.3.1.

Theorem 1.4.5. *In the doubly laced case, any two saturated chains from $\hat{0}$ to v in a lower interval $[\hat{0}, v]$ are connected by a sequence of Stembridge and Sternberg moves.*

Proof. We induct on the rank of v. Consider maximal chains C_1 and C_2 with $x \lessdot v$ in C_1 colored i and $y \lessdot v$ in C_2 colored j. Denote by $x \wedge y$ the unique element that either (a) is covered by x and y with $f_i(x \wedge y) = y$ and $f_j(x \wedge y) = x$, (b) is less than both x and y with $f_i f_i f_j(x \wedge y) = y$ and $f_j f_j f_i(x \wedge y) = x$, (c) is less than both x and y with $f_j^3 f_i(x \wedge y) = x$ and $f_j f_i^2 f_j(x \wedge y) = y$, or (d) is less than both x and y with $f_i^2 f_j^3 f_i(x \wedge y) = y$ and $f_j^3 f_i^2 f_j(x \wedge y) = x$. We know these are the only possibilities due to the results of Stembridge and Sternberg.

We have $\hat{0} \le x \wedge y$, so let C be any saturated chain from $\hat{0}$ to $x \wedge y$. Then let C_3 be a saturated chain from $\hat{0}$ to v which includes $x \lessdot v$ and C. Let C_4 be a saturated chain from $\hat{0}$ to v which includes $y \lessdot v$ and C. By induction on the rank of v, we know that C_1 is connected by Stembridge and Sternberg moves to C_3 and that C_2 is connected by Stembridge and Sternberg moves to

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C_4. By construction, we know that C_3 is connected to C_4 by a single Stembridge or Sternberg move. This implies that C_1 is connected to C_2 by Stembridge and Sternberg moves. □

However, Hersh and Lenart go on to prove that this result does not carry over to arbitrary intervals.

Theorem 1.4.6 ([11]). *No finite set of moves suffices to connect the sets of maximal chains in all closed intervals [u,v] of type A_n crystal posets for $n \geq 1$. In particular, there are disconnected open intervals (u,v) of arbitrarily large rank.*

To show this, Hersh and Lenart constructed an infinite family of intervals of arbitrary rank with the property that not all saturated chains were connected by a sequence of Stembridge moves. We construct this infinite family of intervals in Example 1.4.7.

Example 1.4.7. Consider the tableaux u and v shown below as vertices in the type A_n crystal poset of shape $(n+1, n)$. Here we have that $u \leq v$ and in the interval $[u, v]$, there are saturated chains that are not connected by a sequence of Stembridge moves.

$$u := \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & \cdots & n-2 & n-1 \\ \hline 2 & 3 & 4 & 5 & \cdots & n+1 \\ \cline{1-6} \end{array}, \qquad v := \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & \cdots & n-2 & n-1 & n \\ \hline 3 & 4 & 5 & \cdots & n+1 & n+1 \\ \cline{1-6} \end{array},$$

To get from u to v we must increment each of the entries $1, 2, ..., n-1$, in the first row exactly once (here we mean the rightmost 1), and the entries $2, 3, ..., n$ in the second row exactly once. It was shown in [11] that the saturated chain with label sequence

$$(1, 2, 2, 3, 3, ..., n-1, n-1, n)$$

is not connected to the saturated chain with label sequence

$$(n, n-1, n-1, ..., 3, 3, 2, 2, 1).$$

In fact, they prove that every saturated chain in the connected component containing the chain with edge label sequence $(1, 2, 2, 3, 3, ..., n-1, n-1, n)$ begins with the edge label 1 and ends with the edge label n.

In the case when $n = 3$, we get the interval seen in Figure 1.13.

The relation seen in Figure 1.13 appears frequently in crystal graphs among the crystal operators f_{i-1}, f_i and f_{i+1} but was not known to exist until the paper of [11]. This motivated a more in depth study of relations among crystal operators in all types.

Definition 1.4.8. Let $\mathcal{B}$ be the crystal of a highest weight representation. If for $x \in \mathcal{B}$, we have

$$f_{i+1} f_i^2 f_{i-1}(x) = f_i f_{i+1} f_{i-1} f_i(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_{i-1} f_{i^2} f_{i+1}(x),$$
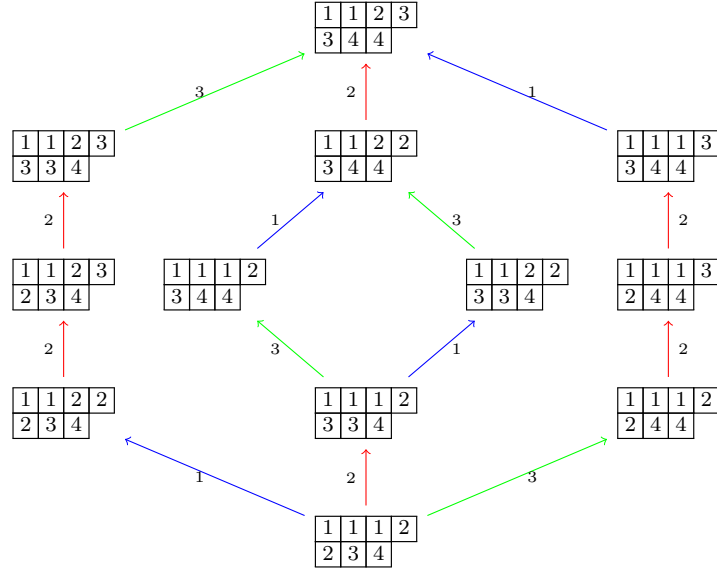
Figure 1.13: Interval when $n = 3$

then we say there is an *HL relation* upward from x. It is possible that only the first three or the final three equalities hold. In this case, we say there is a *partial HL relation* upward from x.

Chapter 2

Understanding relations among crystal operators

In this chapter, we study relations among crystal operators in crystal graphs given by highest weight representations. We begin by giving an overview of the main ideas and results presented in this chapter. We then use the Stembridge axioms introduced in Section 1.1.1 to deduce information about how distinct crystal operators f_i and f_j interact. We use the Sternberg relations to prove that crystals of types B_2 and C_2 are not lattices. In Section 2.3, we prove a connection between the Möbius function of an interval and the relations that exist among crystal operators within that interval. Finally, we use this result as a tool to search for new relations in crystals of doubly laced type that are not implied by Stembridge or Sternberg relations. We end by giving some open questions.

2.1   Introduction

As seen in Section 1.1.1, Stembridge provides a characterization of crystal graphs coming from highest weight representations in the simply laced case. These axioms imply a list of relations that exist among crystal operators (as seen in Definition 1.1.10). These relations also hold in crystal graphs of doubly laced type. However, in this case, in addition to Stembridge relations we have the Sternberg relations seen in Definition 1.1.12. Nevertheless, we saw in Section 1.4 that when viewing these crystal graphs as posets, there exist intervals within the poset where Stembridge relations do not determine the structure of the interval. What this means is that within the interval, there exist saturated chains that are not connected by a sequence of Stembridge moves as defined in Definition 1.4.1. In this chapter, we provide a connection between relations among crystal operators in an interval and the Möbius function of that interval.

The question of what types of relations can exist among crystal operators has been previously

studied by Hersh and Lenart in [11] in the simply laced case. They show that for arbitrary intervals in crystals of simply laced type, there exist relations among crystal operators not implied by Stembridge relations (see Section 1.4). More generally, Hersh and Lenart prove that whenever there is an interval $[u, v]$ in a crystal of finite, simply-laced type with the Möbius function $\mu(u, v) \notin \{-1, 0, 1\}$, then within $[u, v]$ there exists a relation among crystal operators not implied by Stembridge relations. However, the proof technique that is used does not extend to the doubly laced case.

Here, we prove the analogue of this result for crystals of finite, doubly laced type using lexicographic discrete Morse functions (see Section 1.3.2). These functions have been previously used to study certain classes of posets, see e.g. [20, 27]. By using lexicographic discrete Morse functions for crystal posets, we also give a new proof of the result in the simply laced case. More specifically, we show that if we have an interval $[u, v]$ in a crystal poset coming from a highest weight representation of finite Cartan type such that all relations among crystal operators are implied by Stembridge or Sternberg relations, then the Möbius function of this interval must be equal to $-1, 0$, or 1. We do so by constructing a discrete Morse function on the order complex, $\Delta(u, v)$, that has at most one critical cell. We give a procedure for both determining if $[u, v]$ has a critical cell, and for finding this cell, when it exists. If the discrete Morse function has exactly one critical cell, this results in the Möbius function of the interval equalling ± 1, otherwise the Möbius function equals 0.

Danilov, Karzanov, and Koshevoy in [6, 7] studied crystal posets in the case when $n = 2$. They showed that crystals of highest weight representations of type A_2 are lattices. In this chapter, by studying the structure of the Sternberg relations, we prove that crystals of highest weight representations of types B_2 and C_2 are not lattices. Additionally, using SAGE, we are able to search for intervals in crystal posets with Möbius function not equal to $-1, 0$, or 1. As an example, we present new relations among crystal operators in crystals of types B_3 and C_3.

We will now describe and illustrate the main ideas of this chapter, through an example.

The interval $[u, v]$ in Figure 2.1 is a subposet of the crystal of type A_4 of shape $(3, 1)$. We order the saturated chains in our interval according to lexicographic order on their edge label sequences. The critical cells in our lexicographic discrete Morse function come from *fully covered saturated chains* in the interval as defined in Section 1.3.2. Informally, we have a fully covered saturated chain C from u to v when each rank along C, excluding that of u and v, is covered by a "minimal skipped interval". Roughly speaking, we have a skipped interval from some vertex u' to vertex v' consisting of all elements strictly between u' and v' along C if there is a lexicographically earlier chain C' from u' to v'. If there are no strictly smaller (in the sense of number of ranks covered) skipped intervals between u' and v', then we have a minimal skipped interval. The technique we are using is a generalization of a lexicographic shelling. It differs from lexicographic shellings as we allow our minimal skipped intervals to cover more than one
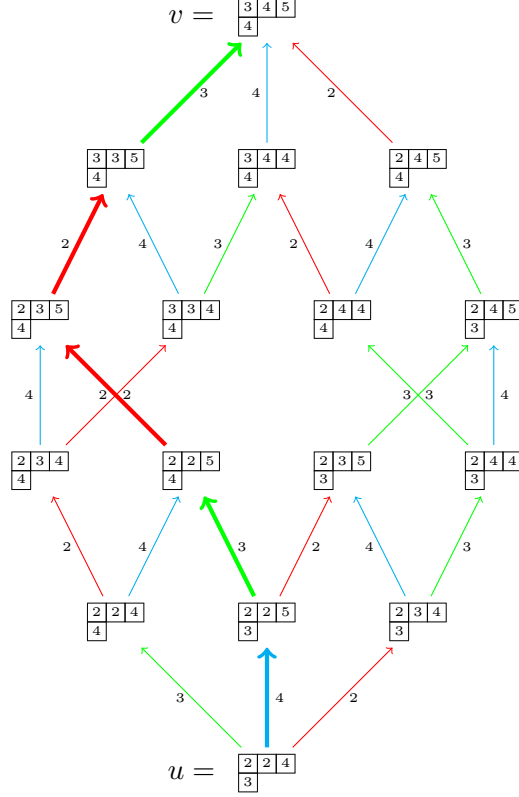
Figure 2.1: Subposet of type A_4 crystal with highest weight $\lambda = (3, 1, 0, 0)$

rank.

Consider the chain in bold in our example. This chain has edge label sequence $(4, 3, 2, 2, 3)$. We can see that this saturated chain is fully covered by looking at it's minimal skipped intervals. For our first minimal skipped interval, instead of traveling up this chain via the edges labeled 4 and 3, we could have traveled up the lexicographically earlier segment via the edges labeled 3 and then 4. Similarly, instead of traveling along the edges labeled by the sequence $(3, 2, 2, 3)$, we could have traveled up the lexicographically earlier segment labeled $(2, 3, 3, 2)$. These two minimal skipped intervals cover all proper ranks of our interval. Therefore, the chain with edge label sequence $(4, 3, 2, 2, 3)$ is fully covered. This is the only fully covered saturated chain within $[u, v]$. As having a fully covered saturated chain gives rise to a critical cell in our discrete Morse function, we are able to deduce that the Möbius function of the interval, $\mu(u, v) = -1$.

We will work in the setting of crystals of highest weight representations where all relations among crystal operators are implied by Stembridge or Sternberg relations. We give an algorithm for finding a fully covered saturated chain from u to v when one exists. In doing so, we prove that there is at most one fully covered saturated chain from u to v. We note that when a fully

covered saturated chain exists, it is not always the lexicographically last chain, though often it is. For such an instance, see Example 2.3.16. We will use this to prove the result regarding the Möbius function. We end by giving new relations in crystals of doubly laced type.

2.2 Consequences of the Stembridge axioms

In this section, we deduce consequences of the Stembridge axioms regarding relations among crystal operators in both the simply laced and doubly laced cases. The axioms give restrictions on which Stembridge/Sternberg relations can occur among two given crystal operators for crystals coming from highest weight representations. In addition, we prove that crystals of types B_2 and C_2 are not lattices due to the asymmetry of the degree five Sternberg relation.

Throughout this section, we will let $A = (a_{ij})_{i,j\in I}$ be the Cartan matrix of a symmetrizable Kac-Moody algebra $\mathfrak{g}$, where I is some finite index set. We first will work with finite simply laced Kac-Moody algebras, namely A_n, D_n, E_6, E_7, and E_8. The Cartan matrices for these types can be seen in Figures 2.2, 2.3, 2.4, 2.5 and 2.6.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix}$$

Figure 2.2: Cartan matrix of type A_n

$$A = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 2 \end{bmatrix}$$

Figure 2.3: Cartan matrix of type D_n

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$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Figure 2.4: Cartan matrix of type E_6

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Figure 2.5: Cartan matrix of type E_7

For the Cartan matrix of type A_n, we note that $a_{i,i+1} = a_{i+1,i} = -1$ and all other off diagonal entries are zero. In particular, $a_{ij} = 0$ for all $i, j \in [n]$ such that $|i - j| > 1$. Therefore, using axioms (S3) and (S4), we have that for any vertex x with $e_i(x) \neq 0$ and $|i - j| > 1$ in a crystal graph of type A_n, $\Delta_i \delta_j(x) = \Delta_i \vartheta_j(x) = 0$. Similarly, we have $\Delta_j \delta_i(x) = \Delta_j \vartheta_i(x) = 0$. As a result, for all x with $f_i(x) \neq 0$ and $f_j(x) \neq 0$, we have the following statement regarding degree two and degree four Stembridge relations.

Proposition 2.2.1. *Let $\mathcal{B}$ by a crystal of type A_n. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If $|i - j| > 1$, then $f_i f_j(x) = f_j f_i(x)$.*

2. *If $|i - j| = 1$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*

The statement remains true if we replace f_i and f_j with the crystal operators e_i and e_j.

Now, consider the Cartan matrix for type D_n from Figure 2.3. We have that $a_{i,i+1} = a_{i+1,i} = -1$ for all $1 \leq i \leq n-2$. In addition, we also have $a_{n-2,n} = a_{n,n-2} = -1$. All other off diagonal entries are equal to zero. In particular, $a_{n-1,n} = a_{n,n-1} = 0$, which differs from the Cartan matrix of type A_n. Hence, using axioms (S3) and (S4), we get the following result.

Proposition 2.2.2. *Let $\mathcal{B}$ be a crystal of type D_n. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If $|i - j| > 1$ and $\{i, j\} \neq \{n - 2, n\}$, then $f_i f_j(x) = f_j f_i(x)$.*

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Figure 2.6: Cartan matrix of type E_8

2. If $\{i,j\} = \{n-1,n\}$, then $f_i f_j(x) = f_j f_i(x)$.

3. If $|i-j| = 1$ and $\{i,j\} \neq \{n-1,n\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

4. If $\{i,j\} = \{n-2,n\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

The statement remains true if we replace f_i and f_j with the crystal operators e_i and e_j.

Finally, we consider the exceptional types E_6, E_7 and E_8. For the Cartan matrix of type E_6, we have $a_{i,i+1} = a_{i+1,i} = -1$ for all i except when $i = 5$. We also have $a_{3,6} = a_{6,3} = -1$. All other off diagonal entries are equal to 0. As a result, we have the following.

Proposition 2.2.3. *Let $\mathcal{B}$ be a crystal coming from a highest weight representation of type E_6. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If $|i-j| > 1$ and $\{i,j\} \neq \{3,6\}$, then $f_i f_j(x) = f_j f_i(x)$.*

2. *If $\{i,j\} = \{5,6\}$, then $f_i f_j(x) = f_j f_i(x)$.*

3. *If $|i-j| = 1$ and $\{i,j\} \neq \{5,6\}$ then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*

4. *If $\{i,j\} = \{3,6\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*

The Cartan matrix of type E_7 is similar to that of E_6 except that here we have $a_{i,i+1} = a_{i+1,i} = -1$ for all i except $i = 6$ and $a_{3,7} = a_{7,3} = -1$. All other off diagonal entries are equal to zero. As a result, we have the following.

Proposition 2.2.4. *Let $\mathcal{B}$ be a crystal coming from a highest weight representation of type E_7. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If $|i-j| > 1$ and $\{i,j\} \neq \{3,7\}$, then $f_i f_j(x) = f_j f_i(x)$.*

2. *If $\{i,j\} = \{6,7\}$, then $f_i f_j(x) = f_j f_i(x)$.*

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3. If $|i-j| = 1$ and $\{i,j\} \neq \{6,7\}$ then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

4. If $\{i,j\} = \{3,7\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

We can similarly study the Cartan matrix of type E_8 to see the following:

Proposition 2.2.5. *Let $\mathcal{B}$ be a crystal coming from a highest weight representation of type E_8. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. If $|i-j| > 1$ and $\{i,j\} \neq \{5,8\}$, then $f_i f_j(x) = f_j f_i(x)$.

2. If $\{i,j\} = \{7,8\}$, then $f_i f_j(x) = f_j f_i(x)$.

3. If $|i-j| = 1$ and $\{i,j\} \neq \{7,8\}$ then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

4. If $\{i,j\} = \{5,8\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

We now move on to consider the doubly laced case. We recall the Cartan matrices of type B_n and type C_n in Figures 2.7 and 2.8.

$$
A = \begin{bmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -2 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2
\end{bmatrix}
$$

Figure 2.7: Cartan matrix of type B_n

$$
A' = \begin{bmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & -2 & 2
\end{bmatrix}
$$

Figure 2.8: Cartan matrix of type C_n

For the Cartan matrix $A = (a_{ij})_{i,j \in I}$ of type B_n, note that $a_{n-1,n} = -2$ and for the Cartan matrix $A = (a_{ij})_{i,j \in I}$ of type C_n, we have $a_{n,n-1} = -2$. All the remaining superdiagonal entries

$a_{i,i+1}$ and remaining subdiagonal entries $a_{i+1,i}$ in A of type B_n and C_n are equal to -1. The remaining off diagonal entries in each Cartan matrix are all zero. Therefore, since crystal graphs of type B_n and C_n are A−regular, for $\{i,j\} \neq \{n-1,n\}$, by axioms (S3) and (S4) we have that for any vertex x, there are only three possibilities for the triples $(a_{ij}, \Delta_i \delta_j(x), \Delta_i \vartheta_j(x))$, namely $(0,0,0), (-1,-1,0)$ or $(-1,0,-1)$. Hence, by axioms (S5)-(S6) and (S5′)-(S6′), we have the following result.

Proposition 2.2.6. *Let $\mathcal{B}$ be a crystal of type B_n or C_n. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If $|i - j| > 1$, then $f_i f_j(x) = f_j f_i(x)$.*

2. *If $|i-j| = 1$ and $\{i,j\} \neq \{n-1,n\}$, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*

3. *If $\{i,j\} = \{n-1,n\}$, then any of the Stembridge or Sternberg relations are possible.*

Therefore, for crystal graphs of doubly laced type, the degree four Stembridge relations can only occur among crystal operators f_i and f_{i+1} as in type A_n, and the degree five and degree seven Sternberg relations can only occur among the crystal operators f_{n-1} and f_n.

Crystals of rank two algebras are often of particular interest. This is due to the result seen in [14] which says that a crystal graph with a unique maximal vertex is the crystal graph of some representation if and only if it decomposes as the disjoint union of crystals of representations relative to the rank two subalgebras corresponding to each pair of edge colors. Therefore, we now consider crystals of type B_2 and C_2. In [6], it is shown that crystals of type A_2 are lattices. We show that this result does not carry over to the doubly laced case.

Theorem 2.2.7. *Crystals of highest weight representations of types B_2 and C_2 are not lattices.*

Proof. This follows from the asymmetry of the degree five Sternberg relations. Let $\mathcal{B}$ be the crystal of a highest weight representation of type B_2 or C_2. Let $x \in \mathcal{B}$ such that there is a degree five Sternberg relation upward from x. Then we have $y \in \mathcal{B}$ such that

$$y = f_1 f_2^3 f_1(x) = f_2 f_1 f_2 f_1 f_2(x) = f_2^2 f_1^2 f_2(x),$$

or

$$y = f_2 f_1^3 f_2(x) = f_1 f_2 f_1 f_2 f_1(x) = f_1^2 f_2^2 f_1(x).$$

In either case, we have that $e_1(y) \neq 0$ and $e_2(y) \neq 0$. As a result, there must be a Stembridge or Sternberg relation downward from y. Hence, $e_1(y)$ and $e_2(y)$ will have two distinct, incomparable greatest lower bounds, one coming from the Stembridge or Sternberg relation downward from y and the other being x.

Similarly, if there exists $y \in \mathcal{B}$ such that there is a degree five Sternberg relation downward from y, then there will exist two vertices that have two distinct, incomparable least upper bounds. Hence, highest weight representations of types B_2 and C_2 are not lattices. □

2.3 Connections between the Möbius function of a poset and relations among crystal operators

In this section we consider crystal posets coming from highest weight representations of simply and doubly laced Cartan type. We prove that whenever there is an interval $[u, v]$ in such a crystal poset whose Möbius function, $\mu(u, v)$, is not equal to $-1, 0$ or 1, then there must be a relation among crystal operators within $[u, v]$ not implied by Stembridge or Sternberg relations. We do so by proving the contrapositive. By "implied" we mean that there exists two saturated chains that are not connected by a sequence of Stembridge or Sternberg moves. Hersh and Lenart showed this result in [11] for crystals of highest weight representations of finite simply laced type. However, the proof used there does not extend to the doubly laced case. In this chapter, we extend the result to crystals of finite doubly laced type, and in doing so, give a new proof for crystals of finite simply laced type. We first develop properties of crystal graphs.

Lemma 2.3.1. *Let $\mathcal{B}$ be the crystal graph of a crystal of type Φ given by a highest weight representation. Let $u, v \in \mathcal{B}$ such that $u < v$. Any saturated chain from u to v uses the same multiset of edge labels. Moreover, we can determine this multiset from $\mathrm{wt}(u)$ and $\mathrm{wt}(v)$.*

Proof. Recall that if $y = f_i(x)$ then $\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha_i$ where α_i is the ith simple root of the root system Φ. Since $u < v$, there exists some sequence of crystal operators $f_{i_1}, f_{i_2}, ..., f_{i_k}$ such that $v = f_{i_k} \cdots f_{i_2} f_{i_1}(u)$. Then we have,

$$\mathrm{wt}(v) = \mathrm{wt}(u) - \sum_{j=1}^{k} \alpha_{i_j}.$$

Suppose by way of contradiction that there exists another distinct sequence of crystal operators $f_{l_1}, f_{l_2}, ..., f_{l_m}$ such that $v = f_{l_m} \cdots f_{l_2} f_{l_1}(u)$. Then we have

$$\mathrm{wt}(u) - \mathrm{wt}(v) = \sum_{j=1}^{k} \alpha_{i_j} = \sum_{n=1}^{m} \alpha_{l_n}.$$

Since the set of simple roots $\{\alpha_i\}_{i \in I}$ is a basis, we must have that $\{\alpha_{i_1}, ..., \alpha_{i_k}\} = \{\alpha_{l_1}, ..., \alpha_{l_m}\}$. Therefore, the same crystal operators are used with the same multiplicities along any saturated chain from u to v. In addition, by writing the vector $\mathrm{wt}(u) - \mathrm{wt}(v)$ as a linear combination of

the simple roots, we can see exactly how many times each crystal operator f_i is applied along any saturated chain from u to v. □

Remark 2.3.2. This implies that crystal posets are graded since every saturated chain in a given interval $[u, v]$ will have the same length.

With Lemma 2.3.1 in mind, we have the following definition.

Definition 2.3.3. Let $\mathcal{B}$ be the crystal graph of a crystal of type Φ given by a highest weight representation and let $[u, v] \subseteq \mathcal{B}$. The *multiset of edge labels of* $[u, v]$ is the multiset of edge labels of any saturated chain C from u to v.

To prove our main result, we will show that for intervals $[u, v] \subseteq \mathcal{B}$ of simply laced (respectively, doubly laced) type with the property that all relations among crystal operators are implied by Stembridge (respectively, Stembridge or Sternberg) relations, we must have that $\mu(u, v) \in \{-1, 0, 1\}$. We do so by constructing a lexicographic discrete Morse function on the order complex $\Delta(u, v)$ that has at most one critical cell. Recall that a saturated chain from u to v contributes a critical cell for $\Delta(u, v)$ if and only if it is fully covered. Therefore, we will give a method to find the unique fully covered saturated chain in the given interval $[u, v]$ when such a chain exists. We lexicographically order the edge label sequences of saturated chains in order to construct the lexicographic discrete Morse function.

Definition 2.3.4. Let $\mathcal{B}$ be the crystal of a highest weight representation and let $[u, v] \subseteq \mathcal{B}$. If $\mathcal{B}$ is of simply laced type and all relations among crystal operators within $[u, v]$ are implied by Stembridge relations, then we say that $[u, v]$ is a *Stembridge only interval*. Similarly, if $\mathcal{B}$ is of doubly laced type and all relations among crystal operators are implied by Stembridge or Sternberg relations, then we say that $[u, v]$ is a *Stembridge and Sternberg only interval*.

Throughout this section, we assume that all intervals are either Stembridge only or Stembridge and Sternberg only intervals. Doing so allows us to control the structure of minimal skipped intervals and construct lexicographic discrete Morse functions. We have that each minimal skipped interval (as described in Remark 1.3.6) in a lexicographic discrete Morse function will arise from a Stembridge or Sternberg relation. Hence, all minimal skipped intervals will be of the forms seen in Figure 2.9 and Figure 2.10.

In the case where $\mathcal{B}$ is the crystal of a highest weight representation of simply laced type, all minimal skipped intervals are of the form seen in Figure 2.9. Assume $i < j$. The saturated chain in red, namely the chain $x \lessdot u_0 \lessdot y$ in the left figure and $x \lessdot u_0 \lessdot u_1 \lessdot u_2 \lessdot y$ in the right figure, represent the pieces of the Stembridge relation that may be on a fully covered saturated chain. This is because it is the lexicographically second chain. The lexicographically earlier chain, (with vertices labeled by the v_i,) will give rise to a minimal skipped interval. In the left

figure, the minimal skipped interval covers the single rank corresponding to the vertex u_0. In the right figure, the minimal skipped interval covers the ranks corresponding to the vertices $u_0, u_1,$ and u_2.
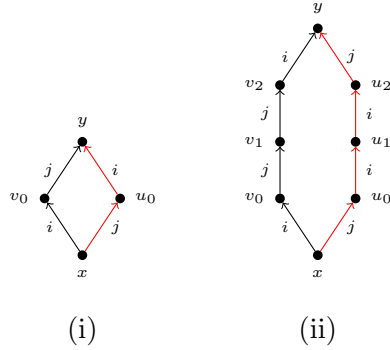
Figure 2.9: Structure of minimal skipped intervals in simply laced case

When a minimal skipped interval arises from a Stembridge relation (as in Figure 2.9), we say the minimal skipped interval *involves* the crystal operators f_i and f_j, (e.g. the minimal skipped intervals in Figure 2.9 involves the crystal operators f_i and f_j). We remark that the possible values for i and j depend on the type of the crystal. For example, if the crystal is of type A_n, then a degree four Stembridge relation can only involve f_i and f_{i+1}. We discussed in Section 2.2 when a degree four Stembridge relation can occur for the different types, i.e. the possible values of i and j for our minimal skipped intervals.

When $\mathcal{B}$ is the crystal of a highest weight representation of doubly laced type, in addition to the Stembridge relations, minimal skipped intervals may also arise from the degree five or degree seven Sternberg relations. By Proposition 2.2.6, we know the degree five and degree seven Sternberg relations can only occur upward from some vertex x if $f_{n-1}(x) \neq 0$ and $f_n(x) \neq 0$. Therefore, we say that minimal skipped intervals arising from Sternberg relations involve the crystal operators f_{n-1} and f_n. The possible Sternberg relations are shown below. The saturated chains with vertices labeled by the u_i (which we marked with red), represent the piece of the Sternberg relation that may be on a fully covered saturated chain, as described above in the simply laced case.

Remark 2.3.5. Note that, unlike in the simply laced case, the chain within the Sternberg relations that is a candidate to be a part of a fully covered saturated chain is not always lexicographically last. This is due to the degree two Stembridge relations sitting inside the degree five and degree seven Sternberg relations.

(i)   or   (ii)
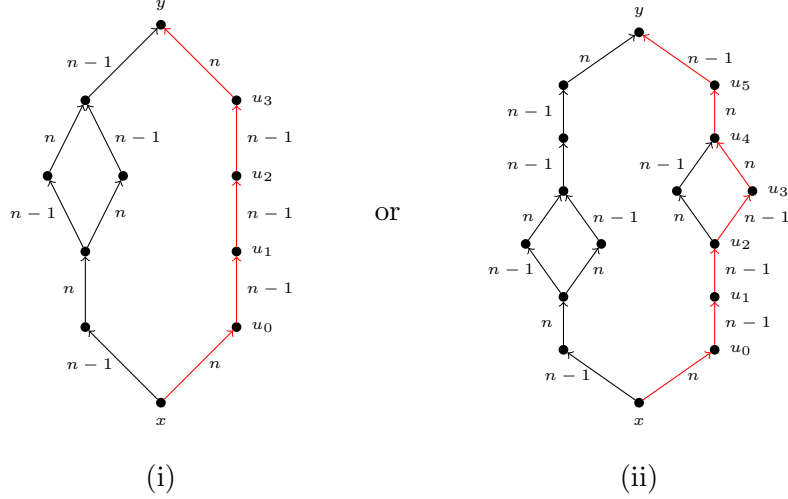
Definition 2.3.6. Let x be a vertex along a saturated chain C in $[u, v]$ such that there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x, where the edge created by applying f_i to x is along C. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. We say that f_j is the *maximal operator for f_i at x* if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

Remark 2.3.7. This is well defined since there is a finite choice of crystal operators and we can always determine which crystal operators will be used along any saturated chain by Lemma 2.3.1. It should be noted that j need not equal l.

Definition 2.3.8. We define a saturated chain C to be *greedily maximal* if given any x along C where the edge created by applying f_i to x along C is the start of a minimal skipped interval involving the crystal operators f_i and f_j, we have that f_j is the maximal operator for f_i at x.

In order to prove our main result connecting the Möbius function of an interval $[u, v]$ with relations among crystal operators within this interval, we first prove a series of lemmas. We begin by proving the following lemma for crystals of highest weight representations of all types. The main idea from this proof is used in several proofs throughout the rest of this chapter.

Lemma 2.3.9. *Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only or a Stembridge and Sternberg only interval, for $\mathcal{B}$ the crystal of a highest weight representation. Let $j = \max\{k \mid k \text{ is in the label sequence of } (u, v)\}$, then f_j must be the first operator applied along a fully covered saturated chain, i.e. j must be the first label in the edge label sequence of any fully covered saturated chain.*

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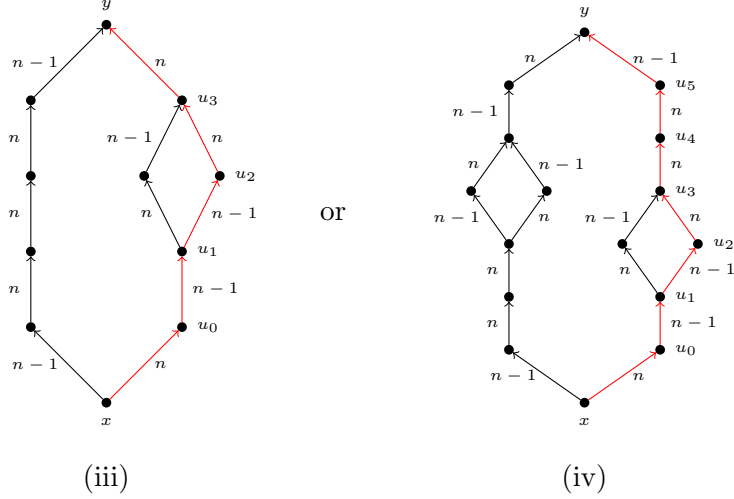
Figure 2.10: Additional minimal skipped intervals in doubly laced case

Proof. Suppose by way of contradiction that there is a fully covered saturated chain, C, from u to v such that f_j is not the first operator applied along C. Consider the first occurrence of the crystal operator f_j as we proceed upward along C from u towards v, namely the first edge colored j. By definition of j, the label k on the edge immediately preceding the edge colored j on C satisfies $k < j$. Since all Stembridge and Sternberg relations involve exactly two crystal operators and all minimal skipped intervals in $[u, v]$ arise from Stembridge or Sternberg relations, the rank corresponding to the vertex labeled x (see Figure 2.11) on the fully covered saturated chain C will not be covered by any minimal skipped intervals, as we justify next. If
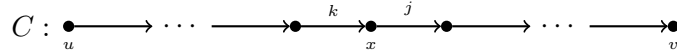
Figure 2.11

the rank corresponding to the vertex labeled x was covered by some minimal skipped interval, the corresponding Stembridge or Sternberg relation must involve the crystal operators f_k and f_j. However, since $k < j$, this piece of the Stembridge or Sternberg relation along C will be lexicographically earlier than the piece with edge label sequence (j, k). Hence, we will not have a minimal skipped interval covering the rank corresponding to the vertex x. This contradicts the saturated chain C being fully covered. □

The interval systems for crystals of simply laced types behave differently than those for

doubly laced types. Namely, no minimal skipped intervals will overlap in the simply laced case, but this does not carry over to the doubly laced case. We first focus on results for simply laced types and then generalize to the doubly laced type.

Lemma 2.3.10. *Let $[u, v] \subseteq \mathcal{B}$ where $\mathcal{B}$ is the crystal of a highest weight representation of simply laced type. Assume $[u, v]$ is a Stembridge only interval. Then no two minimal skipped intervals overlap, i.e. no two minimal skipped intervals cover a common rank.*

Proof. Let C be a saturated chain from u to v and let I be the interval system for C. Any minimal skipped interval in I is of the form seen in Figure 2.9. The first type of minimal skipped interval coming from the degree two Stembridge relation covers exactly one rank. Therefore, any minimal skipped interval arising from this relation cannot overlap with another minimal skipped interval. Hence, we restrict our attention to minimal skipped intervals that arise from the degree four Stembridge relation $f_i f_j f_j f_i(x) = f_j f_i f_i f_j(x)$ where $i < j$.

Suppose we have a vertex $x \in C$ such that there is a minimal skipped interval for the interval system of C beginning at x coming from a degree four Stembridge relation. If there exists another minimal skipped interval that overlaps with the one arising from the degree four Stembridge relation beginning at x, then using the notation from Figure 2.9, it must either begin at the vertex u_0 or the vertex u_1. Since $[u, v]$ is a Stembridge only interval, if we have a minimal skipped interval beginning at u_0 or u_1, it must come from a degree two or degree four Stembridge relation involving f_i and f_j. In fact, it must come from a degree four Stembridge relation. If not, the minimal skipped interval arising from the degree four Stembridge relation beginning at x for the interval system of C would not be minimal.

However, we cannot have a minimal skipped interval beginning at u_0 because the lexicographically last chain in a degree four Stembridge relation does not have an edge label sequence beginning with i, i, j. We also cannot have a minimal skipped interval beginning at u_1 since we have f_i being applied before f_j and therefore we would only see the lexicographically earlier piece of a Stembridge relation on C. As a result, this will not give rise to a minimal skipped interval. Therefore, no two minimal skipped intervals in the interval system of C will overlap. $\square$

Remark 2.3.11. Lemma 2.3.10 tells us that if we have a fully covered saturated chain in a Stembridge only interval in a crystal of a highest weight representation of simply laced type, then the truncation algorithm will not need to be performed.

Now, we prove that any fully covered saturated chain must be greedily maximal, in the sense of Definition 2.3.8. However, for type D_n and the exceptional types E_6, E_7 and E_8, we must amend our definition of "greedily maximal" slightly. We will use this to prove that if there is a fully covered saturated chain in a given interval, then this chain is unique.

Lemma 2.3.12. *Let $\mathcal{B}$ be the crystal of a highest weight representation of type A_n and $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval, then any fully covered saturated chain in $[u, v]$ is greedily maximal.*

Proof. Let C be a fully covered saturated chain from u to v and let $x \in C$ such that the rank of x is the last rank covered by some minimal skipped interval in the interval system for C. Since C is fully covered, by Lemma 2.3.10, x must be the start of a new minimal skipped interval for the interval system of C. Suppose the first edge along C in this minimal skipped interval is labeled i. Let j be the index such that f_j is the maximal operator for f_i at x. Assume by way of contradiction that the minimal skipped interval upward from x involves f_i and f_k where $k \neq j$. Since f_k is not the maximal operator for f_i at x, we know that $k < j$.

We note that since $i > j > k$, we cannot have $k = i - 1$. This implies the minimal skipped interval involving f_i and f_k arises from a degree two Stembridge relation, $f_k f_i(x) = f_i f_k(x)$. Therefore, the next time there is an edge colored j upward from x to v along C, the edge below it on C will have label strictly less than j by definition of maximal operator. This contradicts C being fully covered via the same argument as the proof of Lemma 2.3.9. Namely, there will exist a vertex along the saturated chain C that is not contained in any minimal skipped interval. $\square$

Remark 2.3.13. The idea for crystals of type D_n, E_6, E_7, and E_8 will be similar, but will require some extra care. For example, for crystals of type D_n, the Stembridge relations that can occur among the crystal operators f_{n-2}, f_{n-1} and f_n are different than those that occur in type A_n. Namely, f_{n-1} and f_n can only be involved in a degree two Stembridge relation, while it is possible to have a degree four Stembridge relation involving f_{n-2} and f_n. This is the content of Lemma 2.3.15. See Example 2.3.16 for an illustration of this remark.

Definition 2.3.14. Let $\mathcal{B}$ be the crystal of a highest weight representation of type D_n. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is an *(n,n-2)-special vertex* in C if there is an edge labeled n, upward from x along C which is the start of a minimal skipped interval for the interval system of C and there is another edge labeled n on a saturated chain from $f_n(x)$ to v.

We will now show that any fully covered saturated chain in an interval in a type D_n crystal is greedily maximal.

Lemma 2.3.15. *Suppose that $[u, v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type D_n. Let C be a fully covered saturated chain from u to v. For any $(n, n-2)$-special vertex x along C, let f_{n-2} be the maximal operator for f_n at x. Under this condition, C is greedily maximal.*

Proof. For crystals coming from highest weight representations of type D_n, for all y such that $f_n(y) \neq 0$ and $f_i(y) \neq 0$, we have that $f_n f_i(y) = f_i f_n(y)$ unless $i = n - 2$. In the case $i = n - 2$, it is possible we have $f_n f_{n-2}^2 f_n(y) = f_{n-2} f_n^2 f_{n-2}(y)$.

Suppose by way of contradiction that the fully covered saturated chain C is not greedily maximal. Therefore, there exists a vertex x that is the start of a minimal skipped interval involving f_i and f_j with $i > j$, where f_j is not the maximal operator for f_i at x. As can be seen in Proposition 2.2.2, each crystal operator f_k can be involved in a degree four Stembridge relation with at most one crystal operator f_l, where $l < k$. Additionally, with the exception of f_n and f_{n-2}, all degree four Stembridge relations involve consecutively indexed operators, i.e. f_k and f_{k+1}. Therefore, the case where x is a $(n, n-2)$-special vertex needs to be treated separately.

Assume x is a $(n, n-2)$-special vertex in C and the minimal skipped interval upward from x involves f_n and f_j. Since we are assuming for contradiction that C is not greedily maximal, we must have that f_{n-2} is not the maximal operator for f_n at x (i.e. $j \neq n-2$). Then the minimal skipped interval must arise from a degree two Stembridge relation since f_n commutes with all other operators. Consider the next edge labeled n proceeding upwards along C. Since n is the largest possible edge label occurring on saturated chains from u to v, the edge in C below the edge colored n will have label k for some $k \in [n-1]$. By the nature of Stembridge relations, the rank of the vertex between the k edge and the n edge must be uncovered as seen in the proof of Lemma 2.3.9.

Therefore, the only way to have C be a fully covered saturated chain is if the maximal operator for f_n at x is f_{n-2}. This is because if the minimal skipped interval for C beginning at x comes from a degree four Stembridge relation involving f_n and f_{n-2}, then the next time there is a vertex y on C such that $f_n(y)$ is also along C, it is the start of a new minimal skipped interval and the rank of y is contained in a previous minimal skipped interval. Hence, in order to have a fully covered saturated chain in this case, f_{n-2} must be the maximal operator for f_n. If f_{n-2} is the maximal operator for f_n being applied to an $(n, n-2)$-special vertex, then the proof of a fully covered saturated chain C being greedily maximal is analogous to the type A_n case from Lemma 2.3.12. $\qquad\square$

We now demonstrate via example the ideas of Lemma 2.3.15.

Example 2.3.16. Consider the type D_3 crystal $\mathcal{B}$ of shape $(2, 1, 1)$ and the interval $[u, v]$ shown in Figure 2.12 where

$$u = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bar{3} & \\ \cline{1-1} 3 & \\ \cline{1-1} \end{array}, \quad v = \begin{array}{|c|c|} \hline 2 & \bar{2} \\ \hline \bar{3} & \\ \cline{1-1} \bar{1} & \\ \cline{1-1} \end{array}.$$

One can check that $[u, v]$ is a Stembridge only interval. By Lemma 2.3.9, we know any fully covered saturated chain begins with the application of f_3. By weight considerations, it follows that f_3 needs to be applied again to get from $f_3(u)$ to v. Hence, u is a $(3, 1)$-special vertex,

so f_1 is the maximal operator for f_3 at u. Therefore, the fully covered saturated chain begins $u \lessdot f_3(u) \lessdot f_1 f_3(u)$. The first minimal skipped interval comes from the Stembridge relation $f_1 f_3^2 f_1(u) = f_3 f_1^2 f_3(u)$. The next minimal skipped interval comes from the Stembridge relation $f_2 f_3 (f_1^2 f_3(u)) = f_3 f_2 (f_1^2 f_3(u))$. Therefore, the chain C with label sequence $(3, 1, 1, 3, 2)$ is fully covered. We note that C is not the lexicographically last chain in this interval. The lexicographically last chain has edge label sequence $(3, 2, 1, 1, 3)$.
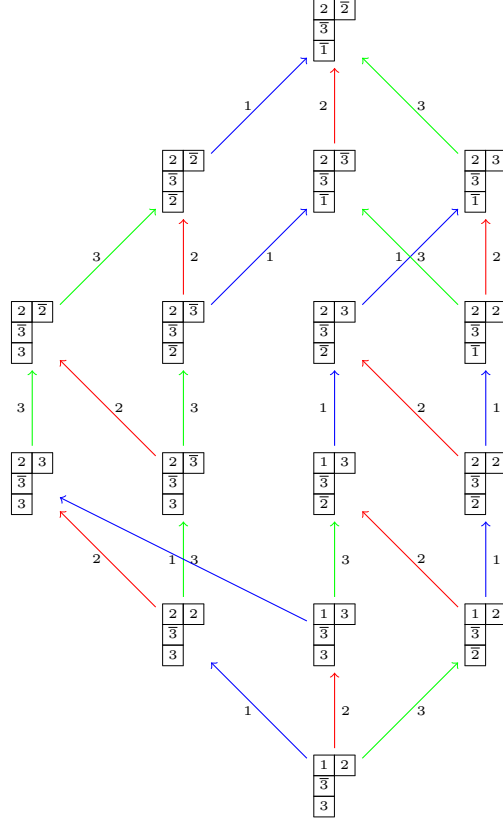
Figure 2.12: Type D_n greedily maximal saturated chain

The same care that was taken with type D_n must also be used for the exceptional types E_6, E_7 and E_8. In each of these cases, we specify special vertices that have certain maximal operators associated with them and prove that any fully covered saturated chain is greedily maximal under this special condition. We begin with type E_6.

Definition 2.3.17. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_6. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to

v. We say that x is a *(6,3)-special vertex* if there is an edge labeled 6 upward from x along C which is the start of a minimal skipped interval for the interval system of C and there exists an edge on a saturated chain from $f_6(x)$ to v labeled 6.

With this definition in mind, we prove that fully covered saturated chains in Stembridge only intervals in crystals of type E_6 are greedily maximal.

Lemma 2.3.18. *Suppose that $[u, v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type E_6. Let C be a fully covered saturated chain from u to v. For any $(6, 3)$-special vertex x along C, f_3 is the maximal operator for f_6 at x. Under this condition, C is greedily maximal.*

Proof. For crystals arising from highest weight representations of type E_6, for all y such that $f_6(y) \neq 0$ and $f_i(y) \neq 0$, we have that $f_i f_6(y) = f_6 f_i(y)$ unless $i = 3$. In the case $i = 3$, it is possible we have $f_6 f_3^2 f_6(y) = f_3 f_6^2 f_3(y)$.

Suppose by way of contradiction that the fully covered saturated chain C is not greedily maximal. Therefore, there exists a vertex x that is the start of a minimal skipped interval involving f_i and f_j with $i > j$, where f_j is not the maximal operator for f_i at x. As can be seen in Proposition 2.2.3, each crystal operator f_k can be in a degree four Stembridge relation with at most one crystal operator, f_l where $l < k$. Additionally, with the exception of f_6 and f_3, all degree four Stembridge relations involve consecutively indexed operators i.e. f_k and f_{k+1}. The case where x is a $(6, 3)$-special vertex needs to be treated separately.

We begin with the case where the minimal skipped interval beginning at x involves the crystal operators f_6 and f_j, where x is a $(6, 3)$-special vertex. Recall we are assuming that f_3 is the maximal operator for f_6 for any $(6, 3)$-special vertex. Since we are assuming for contradiction that C is not greedily maximal, we must have that f_3 is not the maximal operator for f_6 at x (i.e. $j \neq 3$). Then the minimal skipped interval must arise from a degree two Stembridge relation since f_6 commutes with all other operators. However, in this case, if we consider the next time f_6 is applied along our saturated chain, we will get have an uncovered rank as seen in the proof of Lemma 2.3.9. This means C is not fully covered.

If the vertex x is not a $(6, 3)$-special vertex, then the proof is analogous to the type A_n case from Lemma 2.3.12.

As a result, if we have a fully covered saturated chain, it must be greedily maximal. $\square$

We now consider highest weight crystals of type E_7. The arguments will be analogous to those for crystals of type E_6.

Definition 2.3.19. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_7. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to

v. We say that x is a *(7,3)-special vertex* if there is an edge labeled 7 upward from x along C which is the start of a minimal skipped interval for the interval system of C and there exists an edge on a saturated chain from $f_7(x)$ to v labeled 7.

As we have seen in types D_n and E_6, in order to prove a fully covered saturated chain is greedily maximal in type E_7, we must treat $(7,3)$-special vertices as a separate case.

Lemma 2.3.20. *Suppose that $[u,v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type E_7. Let C be a fully covered saturated chain from u to v. For any $(7,3)$-special vertex x along C, f_3 is the maximal operator for f_7 at x. Under this condition, C is greedily maximal.*

Proof. This proof is analogous to the proof of Theorem 2.3.18 with $(7,3)$-special vertices playing the role of $(6,3)$-special vertices. $\square$

Finally, we consider type E_8 crystals.

Definition 2.3.21. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_8. Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is a *(8,5)-special vertex* if there is an edge labeled 8 upward from x along C which is the start of a minimal skipped interval for the interval system of C and there exists an edge on a saturated chain from $f_8(x)$ to v labeled 8.

We give the lemma for type E_8 analogous to Lemma 2.3.18 for type E_6 and Lemma 2.3.20 for type E_7.

Lemma 2.3.22. *Suppose that $[u,v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type E_8. Let C be a fully covered saturated chain from u to v. For any $(8,5)$-special vertex x along C, f_5 is the maximal operator for f_8 at x. Under this condition, C is greedily maximal.*

Proof. This proof is analogous to the proof of Theorem 2.3.18 with $(8,5)$-special vertices playing the role of $(6,3)$-special vertices. $\square$

The five previous lemmas say that for all finite simply laced types, any fully covered saturated chain is greedily maximal. We now give a description of how to find the unique fully covered saturated chain in crystals of finite simply laced type, when it exists.

Theorem 2.3.23. *Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge only interval, for $\mathcal{B}$ the crystal of a highest weight representation of finite simply laced type. Then, there is at most one fully covered saturated chain in $[u,v]$.*

Proof. From Lemma 2.3.9, we know that in order to have a fully covered saturated chain, the chain must start with the application of the crystal operator f_k where

$$k = \max\{i \mid i \text{ is in the multiset of edge labels of } [u, v]\}.$$

Moreover, this says that if $f_k(u) = 0$, then there is no fully covered saturated chain in $[u, v]$. Assuming now that $f_k(u) \neq 0$, we next need to apply f_j where f_j is the maximal operator for f_k at u because any fully covered saturated chain is greedily maximal. If a fully covered saturated chain exists, then it begins with the relations $u \lessdot f_k(u) \lessdot f_j f_k(u)$. In order for the rank of the vertex $f_k(u)$ to be covered by a minimal skipped interval, the chain $u \lessdot f_k(u) \lessdot f_j f_k(u)$ must be contained within a Stembridge relation. In particular, this can only happen if $f_j(u) \neq 0$. If $f_j(u) = 0$, then there is no fully covered saturated chain in this interval because the rank of the vertex $f_k(u)$ will be uncovered. If $f_j(u) \neq 0$, then C must contain the lexicographically later chain in the Stembridge relation upward from u, involving f_k and f_j.

We repeat the process above, beginning at the last uncovered rank. More specifically, this minimal skipped interval described above either ends with the application of f_k (in the case where we have a degree four Stembridge relation between f_k and f_j) or f_j (in the case where we have a degree two Stembridge relation between f_k and f_j). We then see if the maximal operator for the final operator in the previous relation is contained in a Stembridge relation with the final operator beginning at the vertex of the last uncovered rank. If not, there is no fully covered saturated chain in this interval. We continue this process until we reach v. If there is a saturated chain from u to v that is greedily maximal, then we have a fully covered saturated chain. Note that since we chose maximal operators at each step, this chain is uniquely described. □

Using this result, we can say something about the Möbius function of the interval.

Corollary 2.3.24. *For an interval as above, we have $\mu(u, v) \in \{-1, 0, 1\}$.*

Proof. This follows from the correspondence of the reduced Euler characteristic of the order complex of an open interval with the Möbius function of the interval. More specifically, we have the following:

$$\mu(u, v) = \tilde{\chi}(\Delta(u, v)) = \tilde{\chi}(\Delta^M(u, v)),$$

where $\Delta^M(u, v)$ is the CW-complex obtained from the discrete Morse function. Since there is at most one fully covered saturated chain, the discrete Morse function has at most one critical cell. In this case, the cell complex is homotopy equivalent to a sphere with the same dimension as the dimension of the critical cell. Hence, the reduced Euler characteristic will be ± 1 when there is a fully covered saturated chain, and 0 otherwise. □

Remark 2.3.25. The converse of Corollary 2.3.24 is not true. There exist intervals $[u, v]$ in

crystals of highest weight representations of simply laced type such that $\mu(u, v) \in \{-1, 0, 1\}$ where the relations among crystal operators are not implied by Stembridge relations.

In practice, we use the contrapositive of Corollary 2.3.24 to search for new relations among crystal operators as will be seen for the doubly laced case in Section 2.4. We state it here as a corollary.

Corollary 2.3.26. *Let $[u, v] \in \mathcal{B}$, for $\mathcal{B}$ the crystal of a highest weight representation of finite simply laced type. If $\mu(u, v) \notin \{-1, 0, 1\}$, then there exists a relation among crystal operators that is not implied by Stembridge relations.*

We now consider crystals of types B_n and C_n. For these crystals, it is possible to have minimal skipped intervals that overlap.

Lemma 2.3.27. *Suppose $\mathcal{B}$ is the crystal of a highest weight representation of type B_n or C_n. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. If all minimal skipped intervals arise from Stembridge relations or degree five Sternberg relations, then there is no overlap among minimal skipped intervals.*

Proof. The minimal skipped intervals arising from degree two and degree four Stembridge relations remain non-overlapping in the doubly laced case by the same argument used in Lemma 2.3.10. Therefore, we only need to show that if there is a minimal skipped interval for some saturated chain C that comes from a degree five Sternberg relation, then no other minimal skipped intervals overlap with it. The argument is analogous to that of the degree four Stembridge case.

First, suppose that there is a minimal skipped interval coming from (i) in Figure 2.10. In this case, the minimal skipped interval covers the ranks of the vertices $\{u_0, u_1, u_2, u_3\}$. Therefore, we just need to show that no minimal skipped intervals begin at u_0, u_1, or u_2. In each of these cases, the minimal skipped interval would have to involve the crystal operators f_{n-1} and f_n. However, this cannot happen because no saturated chain within a Stembridge or Sternberg relation begins with multiple applications of f_{n-1}, i.e. no edge label sequence begins $(n - 1, n - 1, ...)$. Hence, there is no minimal skipped interval that begins at u_0 or u_1. In addition, there is no minimal skipped interval beginning at u_2 because any Stembridge or Sternberg relation upward from u_2 must involve f_{n-1} and f_n. However, since f_{n-1} is being applied before f_n, due to the lexicographic ordering of chains, we would not have a minimal skipped interval here.

Next, we consider case (iii) from Figure 2.10. As with before, the only possibility for overlap occurs if a minimal skipped interval begins at u_0, u_1, or u_2 and it would need to involve the crystal operators f_{n-1} and f_n. But as before, no saturated chain within a Stembridge or Sternberg relation begins with the repeated application of a single crystal operator so we cannot have a new minimal skipped interval beginning at u_0 or u_2. Also, any chain in a Stembridge or

Sternberg relation involving the operators f_{n-1} and f_n beginning with f_{n-1}, will be the lexicographically earlier chain within that Stembridge or Sternberg relation. As a result, there will be no overlap among minimal skipped intervals coming from a degree five Sternberg relation. □

While there is no overlap among minimal skipped intervals coming from Stembridge relations and degree five Sternberg relations, there can be overlap with a minimal skipped interval coming from a degree seven Sternberg relation. However, we show that if the interval system of a fully covered saturated chain is overlapping, then the truncated interval system still covers all ranks. To do so, we prove a general fact about truncated interval systems for lexicographic discrete Morse functions.

Let P be an edge labeled poset and let $[u, v]$ be an interval in P. We prove that in certain cases if a saturated chain C from u to v is fully covered by the I-intervals but there is overlap among minimal skipped intervals, then the J-intervals will also fully cover C. We will order our I-interval system $I = \{I_1, ..., I_m\}$ so that the lowest rank elements sequentially increase in rank.

Theorem 2.3.28. *Let P be an edge labeled poset and let $[u, v] \subseteq P$. Suppose we have constructed a lexicographic discrete Morse function on $[u, v]$. Let C be a saturated chain from u to v that is fully covered by its I-interval system with the following properties:*

(1) Every minimal skipped interval in I either covers exactly one rank or covers at least three ranks,

(2) For two minimal skipped intervals I_k and I_{k+1}, either $I_k \cap I_{k+1} = \emptyset$ or $I_k \cap I_{k+1}$ contains exactly one element, i.e. any two minimal skipped intervals can overlap on at most one rank.

In this case, after truncation the J-intervals fully cover C.

Proof. We aim to prove that after the truncation algorithm, the J-intervals cover all ranks of C. To do so, we examine what happens at each step of the algorithm. Note that if a minimal skipped interval I_j covers exactly one rank, it cannot overlap with any other intervals and thus will also be a J-interval.

To begin, we set $J_1 = I_1$ since I_1 has the element of minimal rank among all I-intervals. We then need to truncate any I-intervals that overlap with I_1. If $I_1 \cap I_2 = \emptyset$, then set $J_2 = I_2$. Otherwise, if $I_1 \cap I_2 \neq \emptyset$, then there is exactly one rank in this intersection. In this case, we remove the vertex of this rank from I_2 to get an interval I_2' with one fewer element than I_2. If I_2' is still minimal, it becomes a J-interval. We assumed each minimal skipped interval that may have overlap had at least three elements and can overlap with other elements in at most one rank. Therefore, $|I_2'| \geq 2$ and at most one of these elements is contained in another minimal

skipped interval. As a result, all remaining minimal skipped intervals in the I-interval system are still minimal so there are none to throw out. We set $J_2 = I_2'$.

We now repeat the process considering I_3. If $I_2 \cap I_3 = \emptyset$, set $J_3 = I_3$. Otherwise, if $I_2 \cap I_3 \neq \emptyset$, there is at most one element in this intersection. We remove this element from I_3 to get I_3' and by the same argument as before this is not contained in any other minimal skipped interval in I.

Continuing this process gives nonoverlapping J-intervals that fully cover the saturated chain C as desired. $\qquad\square$

We use this result in the next lemma.

Lemma 2.3.29. *Suppose $\mathcal{B}$ is the crystal of a highest weight representation of type B_n or C_n. Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. Let C be a saturated chain from u to v such that its interval system covers all ranks, but with overlap among minimal skipped intervals. Then C remains fully covered after the truncation algorithm.*

Proof. From Lemma 2.3.10 and Lemma 2.3.27, we know that there is no overlap between minimal skipped intervals that arise from Stembridge relations or degree five Sternberg relations. Therefore, we restrict our attention to fully covered saturated chains that have a minimal skipped interval arising from a degree seven Sternberg relation. Let C be one such fully covered saturated chain.

Suppose the minimal skipped interval for C coming from the degree seven Sternberg relation is of the form seen in Figure 2.10 (ii). Say this minimal skipped interval begins at a vertex $x \in C$. If the number of times f_n is applied to get from x to v is greater than three, then in order for C to be fully covered, there is overlap among minimal skipped intervals. Similarly, if the minimal skipped interval comes from the degree seven Sternberg relation seen in Figure 2.10 (iv) and the number of times f_n is applied to get from x to v is greater than four, then there is overlap among minimal skipped intervals. To see why the previous two statements are true, note that in either case, the label sequence of the degree seven Sternberg relation that is contained in C ends with $n-1$. If f_n still needs to be applied along C to reach v, the rank of the first vertex y in C such that $f_n(y)$ is in C is not be contained in a minimal skipped interval unless there is overlap. This is because the edge along C below y is labeled i for some $i \in [n-1]$. The rank of the vertex y is uncovered by the same argument seen in Lemma 2.3.9. To remedy this, there must exist a minimal skipped interval begin with the application of f_n. However, this can only happen if there is overlap.

In either case, the overlap among minimal skipped intervals will include only the rank of the vertex u_5 from Figure 2.10. The proof of why this is the case is analogous to that seen in Lemma 2.3.27. Since all minimal skipped intervals arise from Stembridge or Sternberg relations, a new

minimal skipped interval can only arise off of the degree seven Sternberg relation if it starts at the vertex u_4. Depending on how many times f_n is applied from u to v, the minimal skipped interval may be a degree four Stembridge, degree five or degree seven Sternberg relation. Note that it cannot arise from a degree two Stembridge relation. If this were the case, the original degree seven minimal skipped interval would not in fact be minimal.

Note that each minimal skipped interval either covers exactly one rank or at least three ranks. Additionally, any two minimal skipped intervals are either disjoint, or overlap at exactly one rank. Therefore, by Theorem 2.3.28, if the I-intervals cover all ranks, then the J-intervals do as well. □

The proof that fully covered saturated chains are greedily maximal in the doubly laced case is analogous to the proof for type A_n. The only difference is that there are possibly overlapping intervals. These only occur with degree seven Sternberg relations, which always involve the crystal operators f_{n-1} and f_n. As a result, in this case, the minimal skipped intervals will always involve the maximal operator for f_n. Therefore, we have the following.

Lemma 2.3.30. *Let $\mathcal{B}$ be the crystal of a highest weight representation of type B_n or C_n and $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. Then any fully covered saturated chain in $[u, v]$ is greedily maximal.*

Proof. As stated above, the proof is analogous to that of Lemma 2.3.12. □

We now prove for any interval that is Stembridge and Sternberg only in a highest weight crystal of doubly laced type, there is at most one fully covered saturated chain. The proof is analogous to the simply laced type seen in Theorem 2.3.23.

Theorem 2.3.31. *Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval, for $\mathcal{B}$ the crystal of a highest weight representation of finite doubly laced type. Then there is at most one fully covered saturated chain in $[u, v]$.*

Proof. Let C be a saturated chain from u to v. If there is no overlap among the minimal skipped intervals in the interval system of C, then the argument from Theorem 2.3.23 applies directly. The only difference for doubly laced crystals is that overlap can occur with minimal skipped intervals that arise from degree seven Sternberg relations. Hence, we need only to consider fully covered saturated chains where there is a minimal skipped interval that arises from a degree seven Sternberg relation.

Let C be one such chain. Suppose x is a vertex in C such that there is a minimal skipped interval for the interval system of C beginning at x coming from a degree seven Sternberg relation. In this case, we check if there is overlap among minimal skipped intervals as described in Lemma 2.3.29. Recall that this overlap can occur at exactly one place. In this case, we travel

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up C until the end of the last minimal skipped interval with an overlap. From there, we once again look for the maximal operator as in the proof of Theorem 2.3.23. At each step, there is a unique choice, therefore a fully covered saturated chain from u to v is unique, if it exists. $\qquad\square$

Once again, having at most one fully covered saturated chain in a Stembridge and Sternberg only interval allows us to say something about the Möbius function.

Corollary 2.3.32. *For an interval as above, $\mu(u, v) \in \{-1, 0, 1\}$.*

Proof. The proof is completely analogous to that of Corollary 2.3.24. $\qquad\square$

As in the simply laced case, we use the contrapositive of Corollary 2.3.32 to search for new relations among crystal operators. We state this here as a corollary. For examples illustrating this result, see Section 2.4.

Corollary 2.3.33. *Let $[u, v] \in \mathcal{B}$, for $\mathcal{B}$ the crystal of a highest weight representation of type B_n or C_n. If $\mu(u, v) \notin \{-1, 0, 1\}$, then there exists a relation among crystal operators that is not implied by Stembridge or Sternberg relations.*

2.4   New relations in highest weight crystals of doubly laced type

While trying to find new relations among crystal operators is a difficult task, computing the Möbius function of a given interval is algorithmic and efficient. Specifically, we use SAGE to search for intervals among crystals of finite type with Möbius function not equal to -1, 0, or 1. In general, it is not obvious how to search for new relations among crystal operators. By establishing a relationship between the Möbius function of an interval within our crystal posets and relations among crystal operators within this interval, we have a computational and algorithmic tool to find new relations.

We have found multiple new relations among crystal operators in crystals of type B_n and C_n. We did so by examining intervals where the Möbius function is not equal to $-1, 0$ or 1. See Figure 2.13 for an example of a new relation among crystal operators found in the type C_3 crystal $\mathcal{B}$ of shape $\lambda = (4, 3, 1)$. Namely we have $x \in \mathcal{B}$ such that:

$$f_2 f_3^2 f_2^2 f_1(x) = f_2 f_3 f_2 f_3 f_2 f_1(x) = f_3 f_2^2 f_3 f_1 f_2(x) = f_3 f_2 f_1 f_2 f_3 f_2(x) = f_3 f_2^2 f_1 f_3 f_2(x).$$

See Figure 2.14 for an example of a new relation among crystal operators found in the type B_3 crystal $\mathcal{B}$ of shape $\lambda = (4, 2)$. Note that the open interval has exactly two connected components. It is clear from Figure 2.14 that there is no way to move from the saturated

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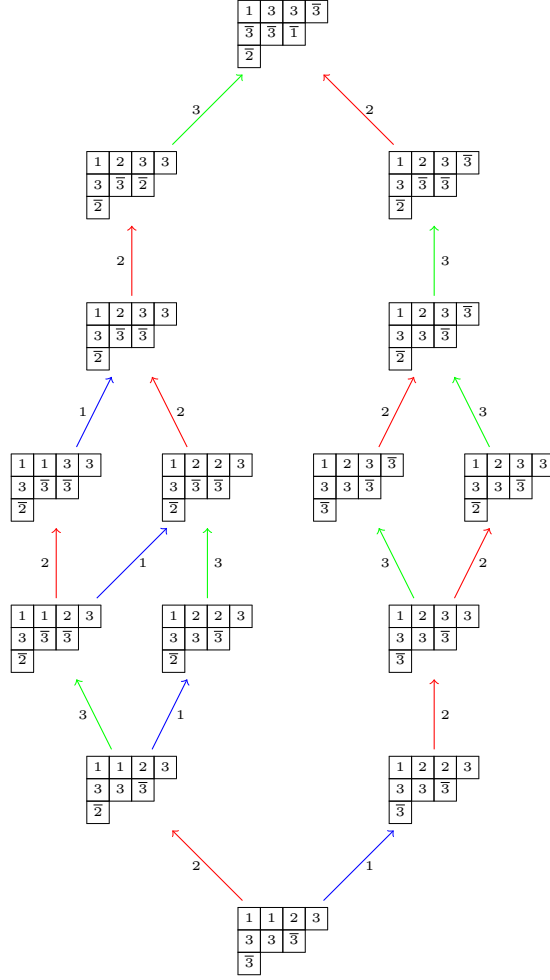
Figure 2.13: New relation in type C_3 crystal $\mathcal{B}$ of shape (4,3,1)

chain with edge label sequence $(2, 3, 3, 1, 1, 2)$ to the saturated chain with edge label sequence $(1, 2, 3, 3, 2, 1)$ using only Stembridge and Sternberg relations. Therefore, this interval gives a new relation among crystal operators. Namely, we have $x \in \mathcal{B}$ such that:

$$f_2 f_1^2 f_3^2 f_2(x) = f_1 f_2 f_3^2 f_2 f_1(x)$$

We note that there are many intervals in crystals of highest weight representations of finite type whose Möbius function is not equal to $-1, 0$ or 1. We have only explored a small number of them. It is likely that there are many unknown relations in the doubly laced case still to be discovered. This chapter gives a tool to find such relations.

Remark 2.4.1. We note that having an interval $[u, v]$ such that $\mu(u, v) \notin \{-1, 0, 1\}$ implies that
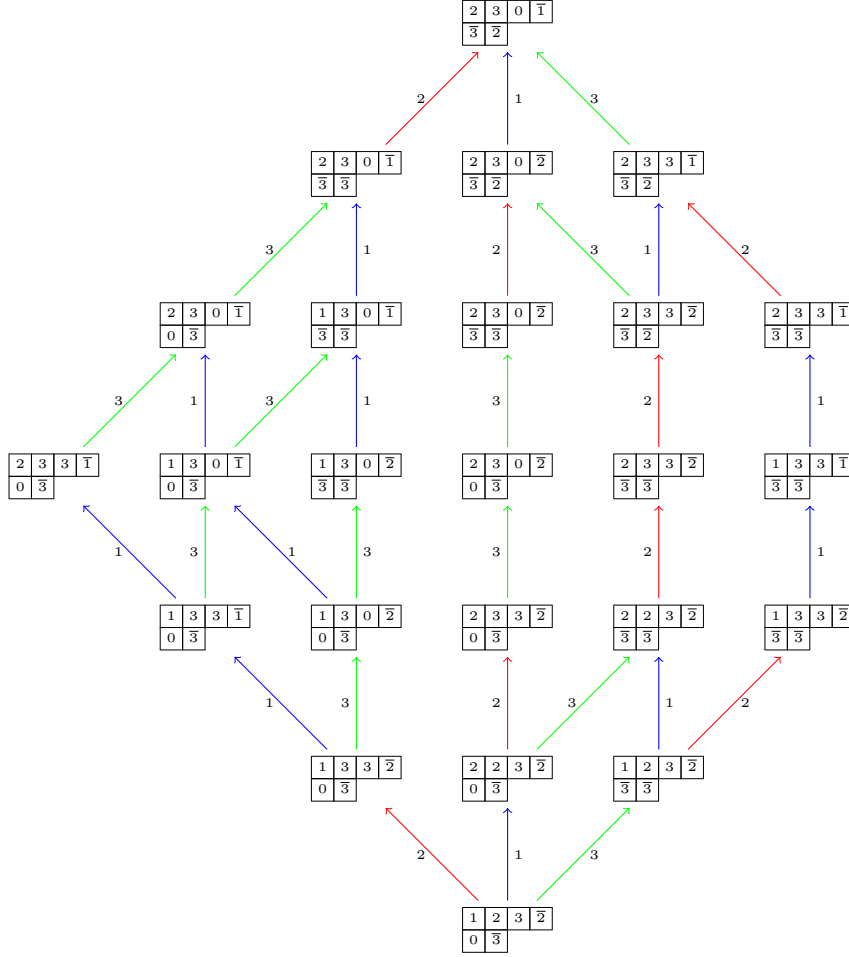
Figure 2.14: New relation in type B_3 crystal $\mathcal{B}$ of shape $(4, 2)$

there exists a new relation among crystal operators, but does not specify what the relation is or how to find it.

2.5   Discussion and open questions

In this chapter, we used lexicographic discrete Morse functions to prove a connection between the Möbius function of an interval and the types of relations that exist among crystal operators. More specifically, for all simply or doubly laced crystals of finite type, we proved that given an interval $[u, v]$ with $\mu(u, v) \notin \{-1, 0, 1\}$, there exists a relation among crystal operators within $[u, v]$ not implied by Stembridge or Sternberg relations. We have begun work extending this result to affine types, which leads to the following question:

Question 2.5.1. *Does $\mu(u,v) \notin \{-1,0,1\}$ for $[u,v] \subseteq \mathcal{B}$ imply the existence of a relation among crystal operators not implied by Stembridge or Sternberg relations for crystals of all affine types?*

While the proof technique used here will likely work for many affine types, it will not work for type $A_n^{(1)}$ as the Dynkin diagram for this type is a cycle. However, it is unknown if a result similar to Corollary 2.3.24 and Corollary 2.3.32 holds in this affine case by using a different proof method.

Another natural question to ask is whether it is possible to classify all possible relations among crystal operators.

Question 2.5.2. *Is there some finite family of relations among crystal operators that implies all relations among crystal operators?*

In general, this question is likely hard to answer. However, in Chapter 3, we use a specific combinatorial model for crystals of type A_n to try and answer this question in certain cases.

In this chapter, we were able to use lexicographic discrete Morse functions because we assumed we had intervals where all relations among crystal operators were implied by Stembridge and Sternberg relations. This controlled the structure of minimal skipped intervals. At the time of this thesis, in type A_n, the only relations apart from Stembridge relations have been those found in [11] (see Section 1.4). It may be interesting to do a similar analysis of intervals where we allow these relations as well.

Question 2.5.3. *Suppose $[u,v] \subseteq \mathcal{B}$ for $\mathcal{B}$ a crystal of type A_n. If all relations among crystal operators are implied by Stembridge relations and HL relations, can we use lexicographic discrete Morse functions to analyze the poset topology of these intervals?*

We have only briefly thought about this question. Unfortunately, we will no longer be guaranteed that there is at most one fully covered saturated chain. However, it may still be possible to classify the structure of fully covered saturated chains and say something about the topology of the order complex of the interval.

Finally, we may also ask if the value of the Möbius function of an interval gives any more information about the interval in the crystal poset. We have found intervals with $\mu(u,v) = 2$ and $\mu(u,v) = 3$. This leads to the following question.

Question 2.5.4. *Is it possible to have an interval $[u,v]$ in a crystal poset where $\mu(u,v) = k$ for all $k \in \mathbb{Z}$, i.e. is every integer the Möbius function of some interval in a crystal poset?*

It is possible that studying Question 2.5.3 may give insight into this question.

Chapter 3

Hook shape and two rowed shape crystals of type A_n

One of the main topics of this thesis is studying the structure of crystal posets. We aim to characterize all relations that occur among crystal operators. However, this is likely very difficult. In this chapter, we focus on two families of crystals of type A_n, namely crystals of shape $(\lambda_1, 1, \cdots, 1)$ and those of shape (λ_1, λ_2). We call a crystal of shape $(\lambda_1, 1, ..., 1)$ a *hook shape crystal*. We call a crystal of shape (λ_1, λ_2) a *two rowed shape crystal*. Throughout this chapter, we will explicitly use the tableaux model for crystals of highest weight representations of type A_n. This model was discussed in Chapter 1.

Our main results in this chapter concern relations among crystal operators in the simply laced case. In Theorem 3.1.1, we prove that any new relation among crystal operators in the simply laced case must involve at least three distinct crystal operators. We note that this result does not carry over to the doubly laced case as can be seen in Example 3.1.3. We conjecture that in hook shape crystals of type A_n, all relations among crystal operators are implied by Stembridge relations. In particular, we prove that HL relations do not occur in hook shape crystals. We know that HL relations do occur in two rowed shape crystals, as can be seen in Figure 1.13. However, in Theorem 3.3.14 we prove that for rank three intervals in two rowed shape crystals, all relations are implied by the degree two Stembridge relation. We end this chapter by studying the arbitrary rank intervals introduced in Section 1.4.

3.1 Introduction

Certain relations among crystal operators are already well known and understood, namely Stembridge relations in the simply laced case and Sternberg relations in the doubly laced case. As seen in Chapter 2, there exist other relations among crystal operators. In this chapter,

we study relations among crystal operators in hook shape crystals of type A_n and two rowed shape crystals of type A_n. Stembridge gave a local characterization of simply laced crystals that come from representations. Every Stembridge relation involves exactly two distinct crystal operators, but as seen in the HL relations of [11], there are relations that involve more than two distinct operators. Using the results of [6, 11, 14], we prove that Stembridge relations are the only relations that use exactly two distinct crystal operators in the simply laced case.

Theorem 3.1.1. *Let $\mathcal{B}$ be a crystal of a highest weight representation of finite simply laced type. Then any relation among crystal operators not implied by Stembridge relations must involve at least three distinct crystal operators.*

Proof. First, recall that by Proposition 2.4.4 of [14], any crystal with a unique maximal vertex is the crystal of a highest weight representation if and only if it decomposes as a disjoint union of such crystals relative to the rank two subalgebras corresponding to each pair of edge colors. In [6], it is shown that crystals of type A_2 are lattices. In addition, in [11] we see that Stembridge upper bounds are least upper bounds. Putting these three results together, it follows that there are no relations among two crystal operators that are not implied by Stembridge relations. $\square$

Remark 3.1.2. We note that Theorem 3.1.1 does not carry over to the doubly laced case. We know that crystals of type B_2 and C_2 are not lattices. The following example gives a relation among crystal operators not implied by Stembridge or Sternberg operators that uses exactly two distinct crystal operators.

Example 3.1.3. Consider $\mathcal{B}$, the type B_2 crystal of shape $(2, 2)$. Let

$$u = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \bar{2} \\ \hline \end{array}, \quad v = \begin{array}{|c|c|} \hline 2 & \bar{2} \\ \hline \bar{2} & \bar{1} \\ \hline \end{array}.$$

Then we have

$$v = f_1 f_2^4 f_1(u) = f_2^2 f_1^2 f_2^2(u) = f_2 f_1 f_2^2 f_1 f_2(u).$$

The interval $[u, v]$ is seen in Figure 3.1. The saturated chain with edge label sequence $(1, 2, 2, 2, 2, 1)$ cannot be connected to the saturated chains with edge label sequences $(2, 1, 2, 2, 1, 2)$ and $(2, 2, 1, 1, 2, 2)$ by a sequence of Stembridge or Sternberg relations.

Let $\mathcal{B}$ be a crystal of a highest weight representation of type A_n realized using the tableaux model. Recall the following definitions introduced in Section 1.1.

Definition 3.1.4. Let T be a tableau. The *reading word* of T, denoted $r(T)$ is the word obtained by reading each column bottom to top and reading columns from left to right.

Definition 3.1.5. Let T be a tableau. The *i-signature* of T, denoted $\sigma_i(T)$, is the subword of $r(T)$ consisting of only the letters i and $i + 1$.
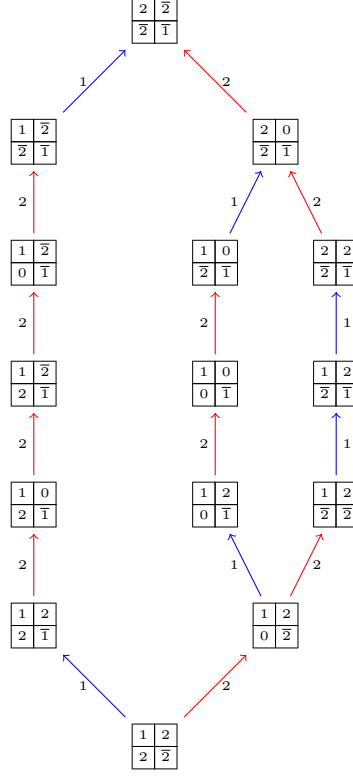
Figure 3.1: New relation in type B_2 crystal $\mathcal{B}$ of shape $(2,2)$

We illustrate these definitions in the following example.

Example 3.1.6. Let $\lambda = (6,3)$ and $x = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} \end{array}$. Then $r(x) = 2131324234$ and $\sigma_2(x) = 233223$.

The application of a crystal operator f_i can only affect certain j-signatures as explained in Remark 3.1.7.

Remark 3.1.7. The application of the crystal operator f_i affects $\sigma_{i-1}(x)$, $\sigma_i(x)$, and $\sigma_{i+1}(x)$. That is to say that $\sigma_j(x)$ equals $\sigma_j(f_i(x))$ unless $j = i-1, i$ or $i+1$. To see this, note that applying f_i changes exactly one i to $i+1$. In terms of $\sigma_{i-1}(x)$, there will be one fewer i in $\sigma_{i-1}(f_i(x))$ than in $\sigma_{i-1}(x)$. For $\sigma_i(x)$, there will be one fewer occurrence of i and one more occurrence of $i+1$ in $\sigma_i(f_i(x))$ compared to that of $\sigma_i(x)$. Similarly, we will have one more occurrence of $i+1$ in $\sigma_{i+1}(f_i(x))$ compared to that of $\sigma_i(x)$. These are the only signatures in which i or $i+1$ occur.

We will use the i-signature to help understand the relations among crystal operators in hook shape and two rowed shape crystals.

3.2 Structure of hook shape crystals of type A_n

In this section, we study relations among crystal operators in hook shape crystals of type A_n. In particular, we work towards showing that in this case, all relations among crystal operators are implied by Stembridge relations.

Remark 3.2.1. When referencing a hook shape tableau T, we refer to the boxes in row one of T as *the row of T*. Similarly, we refer to the boxes in column one of T as *the column of T*.

We propose the following conjecture.

Conjecture 3.2.2 (Lynch). *Suppose $\mathcal{B}$ is a hook shape crystal of type A_n coming from a highest weight representation. Then all relations among crystal operators are implied by Stembridge relations.*

Using the tableaux model for the crystal graph, we make use of the rigid structure of semistandard Young tableaux and the effect this structure has on the i-signature of a vertex. While we do not yet have a proof for Conjecture 3.2.2, we use this section to provide evidence of this conjecture.

3.2.1 Results towards a proof of Conjecture 3.2.2

Let $\mathcal{B}$ be a crystal of a highest weight representation of type A_n. We saw in Chapter 2 that given a vertex $x \in \mathcal{B}$, such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$, we have $f_i f_j(x) = f_j f_i(x)$ whenever $|i-j| > 1$. If $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$, then either $f_i f_{i+1}(x) = f_{i+1} f_i(x)$ or $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$. However, whether we have a degree two or degree four Stembridge relation depends on the location of the vertex x in the crystal. Nevertheless, for hook shape crystals of type A_n, we can characterize when there is a degree two versus a degree four Stembridge relation upward from some vertex x solely from the combinatorics of the semistandard Young tableau for x.

Theorem 3.2.3. *Suppose $\mathcal{B}$ is a hook shape crystal of type A_n. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$. Then there is a degree four Stembridge relation involving f_i and f_{i+1} upward from x if and only if $\sigma_i(x)$ is weakly decreasing.*

Proof. Note that since $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$, there must be at least one i in x and at least one $i+1$ in x. Suppose that $\sigma_i(x)$ is weakly decreasing. This implies that we must have exactly one occurrence of $i+1$ in the column of x and no other occurrences of $i+1$ in x. Since $f_i(x) \neq 0$, there is at least one i in the row of x. This is because if there is an i in the column it is paired in the i-signature of x with the $i+1$ from the column, but $f_i(x) \neq 0$. Therefore, applying f_i to x will change the rightmost i in the row to $i+1$. As there is only one $i+1$ in x, applying f_{i+1} to x will change the $i+1$ in the column of x to $i+2$.

However, when we apply f_{i+1} to $f_i(x)$, the newly created $i + 1$ in the row of $f_i(x)$ will be changed to $i + 2$. Since this changes a different $i + 1$ than when we apply f_{i+1} to x, we have $f_i f_{i+1}(x) \neq f_{i+1} f_i(x)$. Hence, by Stembridge axioms we have that $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$ as desired.

We show the other direction by proving the contrapositive. Here we will show that tthe boxes that are changed by f_i and f_{i+1} are the same regardless of the order the operators are applied. Assume x is such that $\sigma_i(x)$ is not weakly decreasing, and $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$. In this case, there must be an $i + 1$ that occurs after all i in $\sigma_i(x)$. This can only occur if there is at least one $i + 1$ in the row of x. Therefore, applying f_{i+1} to x changes the rightmost $i + 1$ in the row of x. By the signature rule, the i that is changed when f_i is applied to $f_{i+1}(x)$ is the same i changed when f_i is applied to x since the new $i + 1$ occurs after all i's in the i-signature.

Applying f_i to x will change the rightmost i in $\sigma_i(x)$ to $i+1$. However, this $i+1$ will be to the left of the $i+1$ that is changed when applying f_{i+1} to x. Hence, we get that $f_{i+1} f_i(x) = f_i f_{i+1}(x)$. This gives the result. $\qquad\square$

This characterization of when a degree four Stembridge relation can occur allows us to say something about the relations between f_{i-1}, f_i, and f_{i+1} upward from a vertex x. This is especially useful as we know by Theorem 3.1.1 that any new relation among crystal operators must involve at least three distinct crystal operators.

Proposition 3.2.4. *Let $\mathcal{B}$ be a hook shape crystal of type A_n and let $x \in \mathcal{B}$. Suppose that $f_{i-1}(x) \neq 0, f_i(x) \neq 0,$ and $f_{i+1}(x) \neq 0$. Then there can be at most one degree four Stembridge relation upward from x involving the crystal operator f_i, i.e. it is not the case that both $f_{i-1} f_i^2 f_{i-1}(x) = f_i f_{i-1}^2 f_i(x)$ and $f_{i+1} f_i^2 f_{i+1}(x) = f_{i+1} f_i^2 f_{i+1}(x)$.*

Proof. Suppose by way of contradiction that there exists a vertex $x \in \mathcal{B}$ such that $f_{i-1} f_i^2 f_{i-1}(x) = f_i f_{i-1}^2 f_i(x)$ and $f_{i+1} f_i^2 f_{i+1}(x) = f_{i+1} f_i^2 f_{i+1}(x)$. Then by Theorem 3.2.3, we must have that $\sigma_{i-1}(x)$ is weakly decreasing and $\sigma_i(x)$ is weakly decreasing. However, this is not possible as $\sigma_{i-1}(x)$ weakly decreasing implies that there are zero occurrences of i in the row of x, but since $\sigma_i(x)$ is weakly decreasing and $f_i(x) \neq 0$, it follows that the number of occurrences of i in the row of x is strictly greater than zero. The result follows. $\qquad\square$

As of now, the only relations observed in crystal posets of type A_n, besides Stembridge relations are the HL relations described in Section 1.4. We now prove that in hook shape crystals, HL relations cannot occur.

Theorem 3.2.5. *Let $\mathcal{B}$ be a hook shape crystal of type A_n. Then there are no HL relations among crystal operators in $\mathcal{B}$.*

Proof. Recall, given x such that $f_{i-1}(x) \neq 0, f_i(x) \neq 0$, and $f_{i+1}(x) \neq 0$, we say there is a full HL relation upward from x if

$$f_{i+1} f_i^2 f_{i-1}(x) = f_i f_{i+1} f_{i-1} f_i(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_{i-1} f_i^2 f_{i+1}(x)$$

is a least upper bound for $f_{i-1}(x), f_i(x)$ and $f_{i+1}(x)$. In this case, we must have a degree four Stembridge relation upward from x involving the crystal operators f_{i-1} and f_i as well as a degree four Stembridge relation upward from x involving the crystal operators f_i and f_{i+1}. By Proposition 3.2.4, this is not possible. As a result, there are no full HL relations among crystal operators in a hook shape crystal of type A_n.

Similarly, given x such that $f_{i-1}(x) \neq 0, f_i(x) \neq 0$, and $f_{i+1}(x) \neq 0$, we say there is a partial HL relation upward from x if

$$f_{i+1} f_i^2 f_{i-1}(x) \neq f_i f_{i+1} f_{i-1} f_i(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_{i-1} f_i^2 f_{i+1}(x), \tag{3.1}$$

is a least upper bound for $f_i(x)$ and $f_{i+1}(x)$, or

$$f_{i+1} f_i^2 f_{i-1}(x) = f_i f_{i+1} f_{i-1} f_i(x) = f_i f_{i-1} f_{i+1} f_i(x) \neq f_{i-1} f_i^2 f_{i+1}(x). \tag{3.2}$$

is a least upper bound for $f_i(x)$ and $f_{i-1}(x)$.

We will show that neither of these will occur in hook shape crystals. We focus on the equalities in (3.1) first. In order for this relation among crystal operators to exist and not be implied by degree two Stembridge relations, there must be a degree four Stembridge relation upward from x involving the crystal operators f_i and f_{i+1}. This implies that $\sigma_i(x)$ is weakly decreasing. Since $f_{i+1}(x) \neq 0$, there is an $i + 1$ in the column of x. Since $f_i^2 f_{i+1}(x) \neq 0$, there are at least two occurrences of i in x. For any saturated chain upward from x that begins with the label i, the signature rule implies that the $i + 1$ in the column cannot change until after the newly created $i + 1$ in the row has been changed. This says that along any saturated chain beginning $x \lessdot f_i(x)$, f_{i+1} must be applied twice in order to meet up with any saturated chain beginning $x \lessdot f_{i+1}(x)$. As a result, there cannot be any partial HL relations of type (3.1) in a hook shape crystal.

Next, we focus on the equalities in (3.2). In this case, we must have that there is a degree four Stembridge relation upward from x involving the crystal operators f_{i-1} and f_i. This implies that $\sigma_{i-1}(x)$ is weakly decreasing and that there is exactly one occurrence of i in x and this occurrence is in the column of x. Since $f_i(x) \neq 0$, there must be no appearance of $i + 1$ in the column of x. We split the proof into two cases: $f_{i+1}(x) = 0$, and $f_{i+1}(x) \neq 0$.

If $f_{i+1}(x) = 0$, then the application of f_{i+1} to $f_i(x)$ will change the $i + 1$ in the column created by applying f_i to x. However, the application of f_{i+1} to $(f_i^2 f_{i-1})(x)$ will change the

$i + 1$ in the row. Hence, $f_{i+1}f_i^2 f_{i-1}(x) \neq f_i f_{i-1} f_{i+1} f_i(x)$ as in each chain different boxes with $i + 1$ are changed.

Now suppose $f_{i+1}(x) \neq 0$. Then we must have $f_i f_{i+1}(x) = f_{i+1} f_i(x)$ and $f_{i-1} f_{i+1}(x) = f_{i+1} f_{i-1}(x)$. We know that $\sigma_i(x)$ is not weakly decreasing and therefore $\sigma_i(f_{i-1}(x))$ is also not weakly decreasing. Hence, there is a degree two Stembridge relation upward from $f_{i-1}(x)$ involving f_i and f_{i+1}, i.e. $f_{i+1} f_i(f_{i-1}(x)) = f_i f_{i+1}(f_{i-1}(x))$. Similarly, we must have that $\sigma_i(f_i f_{i-1}(x))$ is not weakly decreasing implying that there is a degree two Stembridge relation upward from $f_i f_{i-1}(x)$ involving f_i and f_{i+1}, i.e. $f_{i+1} f_i(f_i f_{i-1}(x)) = f_i f_{i+1}(f_i f_{i-1}(x))$. However, this implies that $f_i f_{i-1}(x) = f_{i-1} f_i(x)$ which contradicts the assumption that there is a degree four Stembridge relation. As a result, we have no partial HL relations in hook shape crystals of type A_n. $\qquad\square$

Recall that we are trying to show that all relations among crystal operators in a hook shape crystal of type A_n are implied by Stembridge relations. So far, we have shown that there are no HL relations in these crystal posets. We know that type A_n crystals are not lattices in general as seen in [11]. This is proven via example by showing that there exist two elements which have two distinct incomparable least upper bounds, namely one arising from a Stembridge relation and one arising from an HL relation. Therefore, since HL relations do not occur in hook shape crystals of type A_n, it is natural to ask whether this subclass of crystal posets are lattices. However, we see in Example 3.2.6 that this is not the case.

Example 3.2.6. Consider the type A_3 crystal $\mathcal{B}$ of shape $\lambda = (3, 1, 1)$. Let

$$x_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \text{ and } x_2 = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}.$$

Then x_1 and x_2 have two distinct, incomparable least upper bounds, namely

$$y_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \text{ and } y_2 = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}.$$

Therefore, x_1 and x_2 do not have a join. See Figure 3.2. We note that for y_1 and y_2 to be comparable, we would need $y_2 = f_1(y_1)$. However, this is not the case. The application of f_1 to u creates a new 2 in the tableau. This results in a different sequence of 2's being changed in the degree four Stembridge relation upward from x than those changed in the degree four Stembridge relation upward from $f_1(x)$.

Note that both least upper bounds from Example 3.2 come from some sequence of Stembridge relations, i.e. all saturated chains in $[u, y_1]$ and $[u, y_2]$ are connected by a sequence of Stembridge moves. Therefore, even though hook shape crystals of type A_n are not lattices, it

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$$y_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array}$$

$$y_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\\hline 3 \\\cline{1-1} 4 \\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array}$$
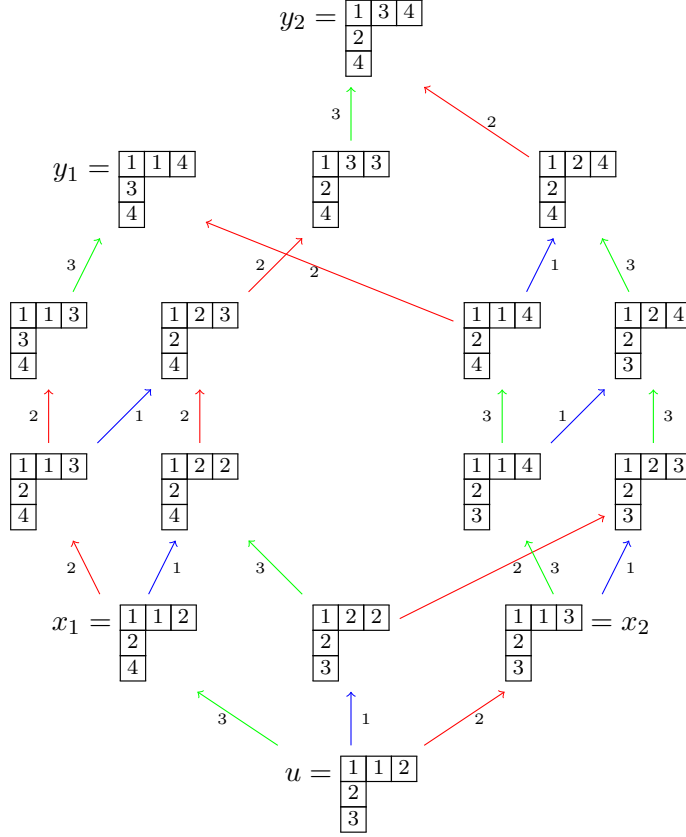
Figure 3.2: Hook shape crystals are not lattices

is still possible that Stembridge relations imply all other relations among crystal operators. We aim to generalize the phenomena seen in Example 3.2.6 and give criteria for when two elements x_1 and x_2 that cover a common element x do not have a unique least upper bound.

Lemma 3.2.7. *Let $\mathcal{B}$ be a hook shape crystal of type A_n with $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$. Assume that $\sigma_i(x)$ is weakly decreasing. If $\sigma_i(x) = i + 1\, i\, i$, with the leftmost i coming from the column of x and $f_{i-1}(x) \neq 0$, then $y = f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$ is not the join of $f_i(x)$ and $f_{i+1}(x)$.*

Proof. Since $\sigma_i(x)$ is weakly decreasing, by Theorem 3.2.3 we know that there is a degree four Stembridge relation upward from x. Therefore,

$$y = f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x) \neq 0$$

is a least upper bound for $f_i(x)$ and $f_{i+1}(x)$. We will show that there must exist a distinct,

incomparable upper bound for $f_i(x)$ and $f_{i+1}(x)$. We claim that

$$y' = f_i f_{i+1}^2 f_i f_{i-1}(x) = f_{i+1} f_i^2 f_{i+1} f_{i-1}(x)$$

is such an upper bound.

To begin, we note that $y' \neq 0$ since applying f_{i-1} to x changes an $i-1$ in the row to an i. As a result, $\sigma_i(f_{i-1})(x)$ is weakly decreasing and there is a degree four Stembridge relation upward from $f_{i-1}(x)$ involving f_i and f_{i+1}. Second, as every saturated chain has the same label sequence, the only way to have $y \leq y'$ is if $y' = f_{i-1}(y)$.

Since $f_{i-1}(x) \neq 0$, by Proposition 3.2.4 we know $\sigma_{i-1}(x)$ is not weakly decreasing and $f_i f_{i-1}(x) = f_{i-1} f_i(x)$. We also must have $f_{i-1} f_{i+1}(x) = f_{i+1} f_{i-1}(x)$. Therefore, we have that $f_i(x) \leq y'$ and $f_{i+1}(x) \leq y'$ implying that y' is an upper bound for $f_i(x)$ and $f_{i+1}(x)$.

Therefore, all that remains to be shown is that y and y' are incomparable. By a rank argument, we cannot have $y' \leq y$, so we only need to prove that $y \nleq y'$. As we are assuming $\sigma_i(x) = i + 1\, i\, i$ where the leftmost i is in the column of x, to get from x to y we must change the i in the row to $i+1$ and then to $i+2$, change the $i+1$ in the column to $i+2$, and the i in the column to $i+1$. However, to get from $f_{i-1}(x)$ to y', both i's that are changed are in the row. Hence, $f_{i-1}(y) \neq y'$ implying $y \nleq y'$ as desired. $\qquad\square$

Corollary 3.2.8. *For $n \geq 3$, hook shape crystals of type A_n are not lattices.*

We would like to prove the converse of Lemma 3.2.7 to characterize when a degree four Stembridge upper bound is not a join.

Conjecture 3.2.9 (Lynch)**.** *Let $\mathcal{B}$ be a type A_n hook shape crystal with $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$ with $\sigma_i(x)$ weakly decreasing. If $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$ is not the join of $f_i(x)$ and $f_{i+1}(x)$, then $f_{i-1}(x) \neq 0$ and $\sigma_i(x) = i + 1\, i\, i$ where the leftmost i comes from the column of x.*

At the time of this thesis, Example 3.2.6 is the only scenario where we have two elements that cover a common element and do not have a unique least upper bound in hook shape crystals of type A_n. It is plausible that this is the only scenario in which two elements who cover a common element do not have a unique least upper bound. Recall Lemma 2.1 from [2].

Lemma 3.2.10 ([2])**.** *Let P be a bounded poset of finite rank such that for any x and y in P, if x and y both cover an element z then the join $x \vee y$ exists. Then P is a lattice.*

Conjecture 3.2.9 together with Lemma 3.2.7 would prove that the only time two elements, x_1 and x_2, cover a common element x such that there is a degree four Stembridge relation above x, where x_1 and x_2 have two distinct incomparable least upper bounds is if:

(1) $\sigma_i(x) = i + 1\ i\ i$, where the leftmost i comes from the column of x, and

(2) $f_{i-1}(x) \neq 0$.

Here we are assuming that $x_1 = f_i(x)$ and $x_2 = f_{i+1}(x)$. In order for there to exist an x that satisfies (1) and (2) above, there needs to be at least two occurrences of $i - 1$ but at most one of these occurrences is in the column. Therefore, the length of the row of λ must be greater than or equal to three. This leads to the following conjecture.

Conjecture 3.2.11 (Lynch). *Let $\mathcal{B}$ be a type A_n crystal with $\lambda = (2, 1, \cdots, 1)$. Then $\mathcal{B}$ is a lattice.*

To prove this, we would also need to show that in this setting, degree two Stembridge upper bounds are joins. This is not true in general as can be seen in the arbitrary rank intervals introduced in [11]. Using SAGE, it can be seen that Conjecture 3.2.11 is true for all hook shape crystals $\mathcal{B}$ of type A_n where the hook is $\lambda = (2, 1, \cdots, 1)$ for $n \leq 10$.

3.3 Structure of two rowed shape crystals of type A_n

We now undertake a similar study of two rowed shape crystals of type A_n. As with hook shape crystals, the i-signature will be useful in determining which Stembridge relation can occur upward from a given vertex. Unlike in hook shape crystals, it is known that HL relations occur in two rowed shape crystals (see Figure 1.13). We aim to characterize when these relations occur. Additionally, we prove that for any rank three interval in a two rowed shape crystal of type A_n, all relations among crystal operators are implied by degree two Stembridge relations. Finally, we end the section by further studying the intervals of arbitrary rank introduced in [11]. These are the intervals used to show that there are relations among crystal operators of arbitrary degree.

3.3.1 Stembridge and HL relations in two rowed shape crystals

Let $\mathcal{B}$ be a two rowed shape crystal of type A_n. Let $x \in B$. We now introduce statistics that will be important for upcoming results.

Definition 3.3.1. Let $\theta_i(x)$ be the largest subword of $\sigma_i(x)$ that begins with the first appearance of $i + 1$ and ends with the last appearance of i. Let $A_i(x)$ be the number of appearances of i in $\theta_i(x)$ and let $B_i(x)$ be the number of appearances of $i + 1$ in $\theta_i(x)$.

We say that $\theta_i(x)$ is empty if there either are no occurrences of i in x, no occurrences of $i + 1$ in x, or the first occurrence of $i + 1$ happens after the last occurrence of i. We illustrate these statistics in the following example.

Remark 3.3.2. We note that if there are no occurrences of i in x, then $f_i(x) = 0$. Similarly, if there are no occurrences of $i + 1$ in x, then $f_{i+1}(x) = 0$. If the first occurrence of $i + 1$ occurs after the last occurrence of i in $\sigma_i(x)$, then we will necessarily have $f_i f_{i+1}(x) = f_{i+1} f_i(x)$ as can be seen by the signature rule. Therefore, the cases where $\theta_i(x)$ are empty are well understood.

Example 3.3.3. Let $x = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} \end{array}$. Then

$$\sigma_2(x) = 2\ 2\ 3\ 2\ 3\ 2\ 2\ 3,$$

and $\theta_2(x) = 32322$. Therefore, $A_2(x) = 3$ and $B_2(x) = 2$.

We use the statistics A_i and B_i on a tableau x to determine whether we have a degree two or a degree four Stembridge relation upward from x, provided that $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$.

Assuming that $A_i(x) \neq 0$ and $B_i(x) \neq 0$, we can translate these statistics from general statements about the reading word to the locations of i and $i + 1$ in the tableau.

Lemma 3.3.4. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n. If $x \in \mathcal{B}$ such that $A_i(x) \neq 0$ and $B_i(x) \neq 0$, then $A_i(x)$ counts the number of times i appears in the first row of x and $B_i(x)$ counts the number of times $i + 1$ appears in the second row of x.*

Proof. Since $A_i(x) \neq 0$, the first appearance of $i + 1$ when read in the order of the i-signature must appear in the second row of x as rows in a semistandard Young tableau are weakly increasing. Similarly, since $B_i(x) \neq 0$, the last appearance of i when read in the order of the i-signature occurs in row one of x. As any appearance of i in row two comes before the first appearance of $i + 1$ in row two, these i's will not be recorded by $\theta_i(x)$. Therefore, $A_i(x)$ will only count occurrences of i in row one. Similarly, since all appearances of $i + 1$ in row one occur after the last i in row one, $B_i(x)$ only counts the number of occurrences of $i + 1$ in the second row. $\qquad\square$

The relative ordering of $A_i(x)$ and $B_i(x)$ tells us how the crystal operator f_i acts on x.

Lemma 3.3.5. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$. If $A_i(x) \leq B_i(x)$, then applying f_i to x changes the rightmost i in the second row of x to $i + 1$. Otherwise, if $A_i(x) > B_i(x)$, then applying f_i to x changes the rightmost occurrence of i in the first row of x to i.*

Proof. This follows directly from the signature rule. Specifically, if $A_i(x) \leq B_i(x)$, then every i in the first row of x is paired with an $i + 1$ from the second row and cannot be changed. However, if $A_i(x) > B_i(x)$, then there is at least one unpaired i in the first row of x and this will appear further to the right in the i-signature than any i coming from the second row of x. $\qquad\square$

With this in mind, we are able to describe whether we have a degree two or degree four Stembridge relation upward from some tableau x. We introduce notation regarding the location of i and $i+1$ in a tableau. For a two rowed shape tableau x, let $\beta_i(x)$ denote the box containing the rightmost i in the first row of x and $\gamma_i(x)$ denote the box containing the rightmost i in the second row of x.

Proposition 3.3.6. *Let $\mathcal{B}$ a two rowed shape crystal of type A_n and let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_{i+1}(x) \neq 0$. Then the following table summarizes when there is a degree two or degree four Stembridge relation above x.*

	$A_i(x) = B_i(x)$	$A_i(x) > B_i(x)$	$A_i(x) < B_i(x)$
$A_{i+1}(x) = B_{i+1}(x)$	*4*	*4*	*2*
$A_{i+1}(x) > B_{i+1}(x)$	*2*	*2*	*2*
$A_{i+1}(x) < B_{i+1}(x)$	*4*	*2*	*2*

Proof. We split the proof into cases. The main idea is to show that the same boxes are affected as we apply crystal operators, regardless of which chain in the Stembridge relation we consider.

<u>Case 1:</u> $A_i(x) = B_i(x), A_{i+1}(x) \leq B_{i+1}(x)$.

By Lemma 3.3.5, applying f_i to x changes the i in $\gamma_i(x)$ to $i+1$. Note that,

$$A_i(f_i(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x),$$
$$B_i(f_i(x)) = B_i(x) + 1, \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x).$$

Next, consider what happens when applying f_{i+1} to x. By Lemma 3.3.5, $f_{i+1}(x)$ changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. Therefore,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_{i+1}(x)) = A_{i+1}(x),$$
$$B_i(f_{i+1}(x)) = B_i(x) - 1, \qquad\qquad B_{i+1}(f_{i+1}(x)) = B_{i+1}(x) + 1.$$

We note that changing an $i+1$ in the second row to $i+2$ results in one fewer pairing of an $i+1$ from row two with an i in row one. Therefore, we can apply f_i to $f_{i+1}(x)$ and get

something nonzero. Since $A_i(f_{i+1}(x)) > B_i(f_{i+1}(x))$, applying f_i will change the i in $\beta_i(x)$ to $i+1$. Similarly, by applying f_{i+1} to $f_i(x)$, we have that f_{i+1} changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$ since $A_{i+1}(f_i(x)) \leq B_{i+1}(f_i(x))$.

To get from x to $f_{i+1}f_i(x)$ we change the i in $\gamma_i(x)$ to $i+1$ and the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. In contrast, to get from x to $f_if_{i+1}(x)$ we change the i in $\beta_i(x)$ to $i+1$ and the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. Since we change a different set of boxes, we must have $f_if_{i+1}(x) \neq f_{i+1}f_i(x)$. Therefore, by the Stembridge axioms we have $f_if_{i+1}^2f_i(x) = f_{i+1}f_i^2f_{i+1}(x)$ as desired.

Case 2: $A_i(x) = B_i(x), A_{i+1}(x) > B_{i+1}(x)$.

As in Case 1, applying f_i to x will change the i in $\gamma_i(x)$ to $i+1$. However, since $A_{i+1}(x) > B_{i+1}(x)$, by Lemma 3.3.5 applying f_{i+1} to x changes the $i+1$ in $\beta_{i+1}(x)$ to $i+2$. Therefore,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x),$$
$$B_i(f_{i+1}(x)) = B_i(x), \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x).$$

As a result, applying f_{i+1} to $f_i(x)$ will also change the $i+1$ in $\beta_{i+1}(x)$ to $i+2$. Similarly, applying f_i to $f_{i+1}(x)$ will change the i in $\gamma_i(x)$ to $i+1$. Hence, $f_if_{i+1}(x) = f_{i+1}f_i(x)$ as desired. This will happen in any case where the application of f_i does not affect the relative order of A_{i+1} with B_{i+1} and the application of f_{i+1} does not affect the relative order of A_i with B_i.

Case 3: $A_i(x) > B_i(x), A_{i+1}(x) = B_{i+1}(x)$.

By Lemma 3.3.5, applying f_i to x changes the i in $\beta_i(x)$ to $i+1$. Similarly, applying f_{i+1} to x changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. Therefore,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x) + 1,$$
$$B_i(f_{i+1}(x)) = B_i(x) - 1, \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x).$$

Next, we consider applying f_{i+1} to $f_i(x)$. Since $A_{i+1}(f_i(x)) > B_{i+1}(f_i(x))$, by Lemma 3.3.5, applying f_{i+1} changes the $i+1$ in $\beta_{i+1}(x)$. Similarly, since $A_i(f_{i+1}(x)) > B_i(f_{i+1}(x))$, applying f_i to $f_{i+1}(x)$ changes the i in $\beta_i(x)$ to $i+1$.

Therefore, to get from x to $f_{i+1}f_i(x)$, we change the i in $\beta_i(x)$ and the $i+1$ in $\beta_{i+1}(x)$. However, to get from x to $f_if_{i+1}(x)$, we change the i in $\beta_i(x)$ and the $i+1$ in $\gamma_{i+1}(x)$. Since we

change a different set of boxes, we must have $f_i f_{i+1}(x) \neq f_{i+1} f_i(x)$. Therefore, by Stembridge axioms $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$ as claimed.

Case 4: $A_i(x) > B_i(x), A_{i+1}(x) < B_{i+1}(x)$.

Applying f_i to x changes the i in $\beta_i(x)$ to $i + 1$. Similarly, applying f_{i+1} to x changes the $i + 1$ in $\gamma_{i+1}(x)$ to $i + 2$. As a result,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x) + 1,$$
$$B_i(f_{i+1}(x)) = B_i(x) - 1, \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x).$$

Therefore, applying f_{i+1} to $f_i(x)$, changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. Applying f_i to $f_{i+1}(x)$ changes the i in $\beta_i(x)$ to $i + 1$. Hence, $f_i f_{i+1}(x) = f_{i+1} f_i(x)$ as claimed.

Case 5: $A_i(x) > B_i(x), A_{i+1}(x) > B_{i+1}(x)$.

By Lemma 3.3.5, applying f_i to x changes the i in $\beta_i(x)$ to $i + 1$, and applying f_{i+1} to x changes the $i + 1$ in $\beta_{i+1}(x)$ to $i + 2$. Hence,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x) + 1,$$
$$B_i(f_{i+1}(x)) = B_i(x), \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x).$$

As a result, applying f_i to $f_{i+1}(x)$ will change the i in $\beta_i(x)$ to $i + 1$. Similarly, applying f_{i+1} to $f_i(x)$ will change the $i + 1$ in $\beta_{i+1}(x)$ to $i + 2$. Since $A_{i+1}(x) > 0$, this is not the $i + 1$ created from the application of f_i. Hence, $f_i f_{i+1}(x) = f_{i+1} f_i(x)$ as desired.

Case 6: $A_i(x) < B_i(x), A_{i+1}(x) \leq B_{i+1}(x)$.

If we apply f_i to x, we change the i in $\gamma_i(x)$ to $i + 1$. Similarly, if we apply f_{i+1} we change the $i + 1$ in $\gamma_{i+1}(x)$ to $i + 2$. Then,

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x),$$
$$B_i(f_{i+1}(x)) = B_i(x) - 1, \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x),$$

Then, applying f_i to $f_{i+1}(x)$ will change the i in $\gamma_i(x)$ to $i+1$ since $A_i(f_{i+1}(x)) \leq B_i(f_{i+1}(x))$. Similarly, applying f_{i+1} to $f_i(x)$ changes the $i+1$ in the $\gamma_{i+1}(x)$ to $i+2$. Since $B_i(x) > 0$, this is not the $i+1$ created from the previous application of f_i. Hence, we have that $f_i f_{i+1}(x) = f_{i+1} f_i(x)$.

<u>Case 7:</u> $A_i(x) < B_i(x), A_{i+1}(x) > B_{i+1}(x)$.

Applying f_i to x will change the i in $\gamma_i(x)$ to $i+1$. When applying f_{i+1} to x we change the $i+1$ in $\beta_{i+1}(x)$ to $i+2$. Then we have

$$A_i(f_{i+1}(x)) = A_i(x), \qquad\qquad A_{i+1}(f_i(x)) = A_{i+1}(x),$$
$$B_i(f_{i+1}(x)) = B_i(x), \qquad\qquad B_{i+1}(f_i(x)) = B_{i+1}(x),$$

The relative order of A_i with B_i and A_{i+1} with B_{i+1} remains the same. Therefore, applying f_i to $f_{i+1}(x)$ will change the i in $\gamma_i(x)$ to $i+1$. Applying f_{i+1} to $f_i(x)$ will change the $i+1$ in $\beta_{i+1}(x)$ to $i+2$. Hence, $f_i f_{i+1}(x) = f_{i+1} f_i(x)$. $\qquad\qquad\square$

Remark 3.3.7. As can be seen in Case 1 and Case 3 of Proposition 3.3.6, any degree four Stembridge relation involving f_i and f_{i+1} is such that the first application of f_{i+1} along the chain with edge label sequence $(i+1, i, i, i+1)$ changes the $i+1$ in γ_{i+1} and the first application of f_i along this chain changes the i in β_i. In fact, in Case 1 the chain with edge label sequence $(i+1, i, i, i+1)$ changes the following boxes in the following order

$$(\gamma_{i+1}, \beta_i, \gamma_i, \gamma_{i+1}).$$

The chain with the edge label sequence $(i, i+1, i+1, i)$ in Case 1 changes the following boxes in the following order

$$(\gamma_i, \gamma_{i+1}, \gamma_{i+1}, \beta_i).$$

In contrast, for Case 3 the chain with edge label sequence $(i+1, i, i, i+1)$ changes the following

boxes in the following order

$$(\gamma_{i+1}, \beta_i, \beta_i, \beta_{i+1}).$$

The chain in Case 3 with edge label sequence $(i, i+1, i+1, i)$ changes the following boxes in the following order

$$(\beta_i, \beta_{i+1}, \gamma_{i+1}, \beta_i).$$

When $\mathcal{B}$ is a two rowed shape crystal of type A_n, we can use the relative ordering of $A_{i-1}(x)$ with $B_{i-1}(x)$, $A_i(x)$ with $B_i(x)$ and $A_{i+1}(x)$ with $B_{i+1}(x)$ to determine when an HL relation will occur upward from a vertex x.

Proposition 3.3.8. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n. Let $x \in \mathcal{B}$ such that $A_i(x) = B_i(x)$ and $f_i(x) \neq 0, f_{i+1}(x) \neq 0, f_{i-1}(x) \neq 0$. Then we have the following:*

(a) If $A_{i+1}(x) = B_{i+1}(x)$ and $A_{i-1}(x) \geq B_{i-1}(x)$, then

$$f_{i-1}f_i^2 f_{i+1}(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_i f_{i+1} f_{i-1} f_i(x) \neq f_{i+1} f_i^2 f_{i-1}(x).$$

(b) If $A_{i+1}(x) < B_{i+1}(x)$ and $A_{i-1}(x) > B_{i-1}(x)$, then

$$f_{i-1}f_i^2 f_{i+1}(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_i f_{i+1} f_{i-1} f_i(x) = f_{i+1} f_i^2 f_{i-1}(x).$$

Proof. We begin by proving the equalities in (a). By Lemma 3.3.5, applying f_i to x changes the i in $\gamma_i(x)$ to $i+1$. As a result, $B_i(f_i(x)) = B_i(x) + 1$ and $B_{i-1}(f_i(x)) = B_{i-1}(x) - 1$. The other quantities remain unchanged. Hence, applying f_{i+1} to $f_i(x)$ changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. As a result, $B_i(f_{i+1}f_i(x)) = B_i(f_i(x)) - 1 = B_i(x)$. Applying f_{i-1} to $f_i(x)$ changes the $i-1$ in $\beta_{i-1}(x)$ to i. This implies $A_i(f_{i-1}f_i(x) = A_i(x) + 1$. By Proposition 2.2.1, $f_{i-1}f_{i+1}(f_i(x)) = f_{i+1}f_{i-1}(f_i(x))$. Finally, applying f_i to $f_{i+1}f_{i-1}f_i(x)$ will change the i in $\beta_i(x)$ as $A_i(f_{i+1}f_{i-1}f_i(x)) = A_i(x) + 1$ and $B_i(f_{i+1}f_{i-1}f_i(x)) = B_i(x)$.

We now consider the chain beginning with the application of f_{i+1}. By Lemma 3.3.5, applying f_{i+1} to x changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. This implies that $B_i(f_{i+1}(x)) = B_i(x) - 1$. Hence, applying f_i to $f_{i+1}(x)$ changes the i in $\beta_i(x)$ to $i+1$. A second application of f_i will then change the i in $\gamma_i(x)$ to $i+1$. It follows that $B_{i-1}(f_i^2 f_{i+1}(x)) = B_{i-1}(f_i f_{i+1}(x)) - 1 = B_{i-1}(x) - 1$. Therefore, applying f_{i-1} changes the $i-1$ in $\beta_{i-1}(x)$. Hence, $f_{i-1}f_i^2 f_{i+1}(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_i f_{i+1} f_{i-1} f_i(x)$ as the order in which the crystal operators is applied does not change which entries are changed.

It remains to be seen that $f_{i+1}f_i^2 f_{i-1}(x) \neq f_{i-1}f_i^2 f_{i+1}(x)$. We split into two cases. First, suppose that $A_{i-1}(x) > B_{i-1}(x)$. Then applying f_{i-1} to x changes the $i-1$ in $\beta_{i-1}(x)$ to i. This implies $A_i(f_{i-1}(x)) = A_i(x) + 1$. Then applying f_i to $f_{i-1}(x)$ changes the i in $\beta_i(x)$. A

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second application of f_i changes the i in $\gamma_i(x)$ to $i+1$. Here we have that $A_{i+1}(f_i^2 f_{i-1}(x)) > B_{i+1}(f_i^2 f_{i-1}(x))$, so applying f_{i+1} changes the $i+1$ in $\beta_{i+1}(x)$. This gives the result, as for in the other three chains, the $i+1$ changed was in $\gamma_{i+1}(x)$.

Now suppose that $A_{i-1}(x) = B_{i-1}(x)$. In this case, applying f_{i-1} to x will change the $i-1$ in $\gamma_{i-1}(x)$ to i. This gives the result, as for in the other three chains, the $i-1$ changed was in $\beta_{i-1}(x)$.

Consider case (b). The proof that $f_{i-1}f_i^2 f_{i+1}(x) = f_i f_{i-1} f_{i+1} f_i(x) = f_i f_{i+1} f_{i-1} f_i(x)$ is the same. Hence, we need to show that if we begin with the crystal operator f_{i-1} that we get equality. Since we are assuming that $A_{i-1}(x) > B_{i-1}(x)$, applying f_{i-1} to x changes the $i-1$ in $\beta_{i-1}(x)$ to i. As a result, $A_i(f_{i-1}(x)) = A_i(x) + 1$ and $B_i(f_{i-1}(x) = B_i(x)$, implying $A_i(f_{i-1}(x)) > B_i(f_{i-1}(x))$. Therefore, applying f_i to $f_{i-1}(x)$ changes the i in $\beta_i(x)$ to $i+1$. Then $A_{i+1}(f_i f_{i-1}(x)) = A_{i+1}(f_{i-1}(x)) + 1 = A_{i+1}(x) + 1$. Applying f_i to $f_i f_{i-1}(x)$ will change the i in $\gamma_i(x)$ to $i+1$. Since $A_{i+1}(f_i^2 f_{i-1}(x)) \le B_{i+1}(f_i^2 f_{i-1}(x))$, applying f_{i+1} to $f_i^2 f_{i-1}(x)$ will change the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. Hence, this sequence of crystal operators matches with the previous and the result follows. $\qquad\square$

Proposition 3.3.8 shows that given certain statistics on a tableau, we are guaranteed to have an HL relation occur. However, it may be possible that there are other circumstances under which an HL relation occurs in a two rowed shape crystal.

Question 3.3.9. *Is it possible to fully characterize when HL relations occur in two rowed crystals of type A_n?*

To do so, we would need to prove the converse of Proposition 3.3.8.

3.3.2   Rank three intervals in two rowed shape crystals of type A_n

In this section, we prove that there are no relations among crystal operators in rank three intervals of two rowed shape crystals of type A_n that are not implied by the degree two Stembridge relation,

$$f_i f_j(x) = f_j f_i(x).$$

Recall the definition of rank given in Section 1.2.

Definition 3.3.10. Let P be a poset and let $u, v \in P$ with $u \le v$. The *rank* of the interval $[u, v]$ is the length of the longest saturated chain in $[u, v]$.

Note that in the definition above, if a poset is graded, the rank of an interval $[u, v]$ is the length of any maximal saturated chain. Crystal posets are graded as seen in Lemma 2.3.1.

Let $[u, v]$ be a rank three interval in a two rowed shape crystal of type A_n. Suppose that there exist two saturated chains $C_1 = u \lessdot x_1 \lessdot x_2 \lessdot v$ and $C_2 = u \lessdot y_1 \lessdot y_2 \lessdot v$, that are not

connected by a sequence of degree two Stembridge moves, see Figure 3.3. Since any saturated chain from u to v uses the same multiset of edge labels, there are at most three distinct crystal operators used along any saturated chain from u to v. The possible edge labelings of C_1 and C_2 are displayed in Figure 3.4. We will prove that the saturated chains C_1 and C_2 must actually be connected by a sequence of degree two Stembridge moves. We split the proof into two lemmas.
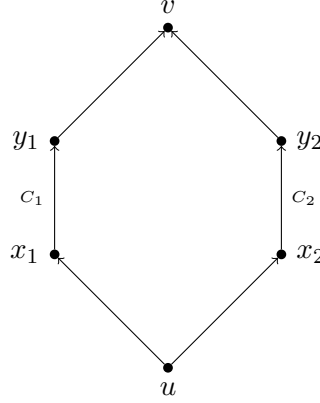

Figure 3.3:   Theorem 3.3.14: Rank three interval

Lemma 3.3.11. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n and let $[u, v] \subseteq \mathcal{B}$ be such that $rk([u, v]) = 3$. Suppose $v = f_k f_j f_i(u) = f_j f_i f_k(u)$ as seen in Figure 3.4 (a). Then all saturated chains from u to v are connected by a sequence of degree two Stembridge moves.*

Proof. We note that case (b) in Figure 3.4 is the same as case (a) under the map sending i to j, j to k and k to i. In order to prove the result, we need to show that $f_i f_k(u) = f_k f_i(u)$, i.e. there is an edge missing in the poset in (a). Recall, given any vertex in a crystal graph, there is at most one outgoing edge labeled i and at most one incoming edge labeled i for any $i \in [n]$. As a result, we must have that $i \neq k$ and $j \neq k$. However, it is possible that $i = j$. Suppose for contradiction that $f_i f_k(u) \neq f_k f_i(u)$. Then, by Stembridge axioms, we must have

$$f_i f_k^2 f_i(u) = f_k f_i^2 f_k(u).$$

From here, we have two possibilities to consider: $i = j$ and $i \neq j$.

First, suppose that $i = j$. This is displayed in Figure 3.5. This contradicts the result seen in [11], namely that Stembridge upper bounds are least upper bounds. Hence, in this case we must have $f_i f_k(u) = f_k f_i(u)$ as desired.
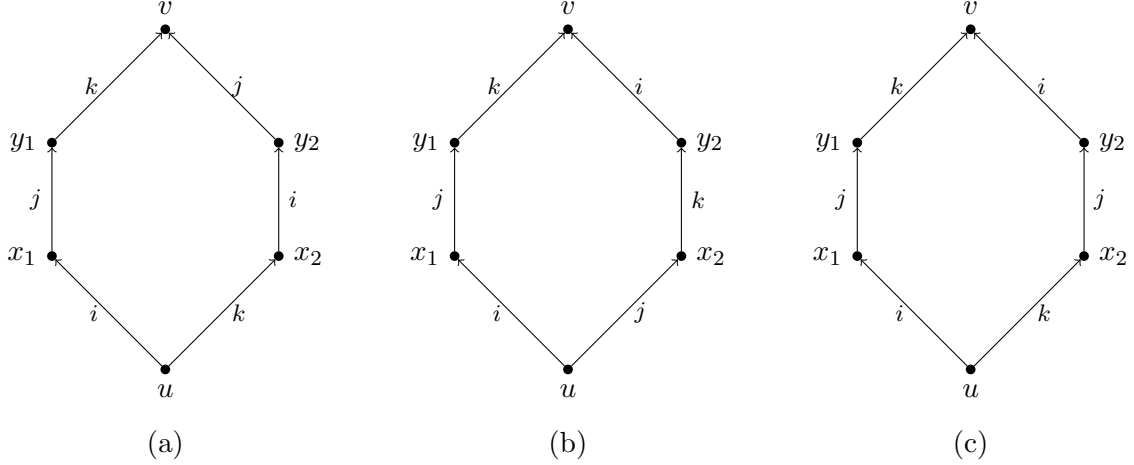
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Figure 3.4: Possible labelings on rank three intervals

Now, consider the case where $i \neq j$. Since there is a degree four Stembridge relation above u, we must have $|i - k| = 1$. First, suppose that $k = i + 1$. By Remark 3.3.7, we know applying f_k to u will change the k in $\gamma_k(u)$ and applying f_i to $f_k(u)$ changes the i in $\beta_i(u)$.

Since $f_j f_i f_k(u) = f_k f_j f_i(u)$ by assumption, we must have that applying f_i to u changes the i in $\beta_i(u)$ to $i + 1$ and applying f_k to $f_j f_i(u)$ changes k in $\gamma_k(u)$. Then applying f_k to $f_i(u)$ will change the k in $\beta_k(u)$, as explained in Remark 3.3.7. As a result, we must have that applying f_j to $f_i(u)$ causes the application of f_k to $f_j f_i(u)$ to change the k in $\gamma_k(u)$. However, since f_k applied to $f_i(u)$ changes k in $\beta_k(u)$, we must have

$$A_k(f_i(u)) > B_k(f_i(u)),$$

and similarly,

$$A_k(f_j f_i(u)) \leq B_k(f_j f_i(u)).$$

As a result, applying f_j to $f_i(u)$ must either

1. decrease the number of appearances of $k = i + 1$ in the row one, or

2. increase the number of appearances of $k + 1 = i + 2$ in row two.

The only way for either of these possibilities to occur is if $j = i + 1$. This contradicts the fact that $k \neq j$. Therefore, we must have $f_i f_k(u) = f_k f_i(u)$ as desired.

Finally, consider the case where $|i - k| = 1$ and $i = k + 1$. Then applying f_i to u changes the i in $\gamma_i(u)$ and applying f_k to $f_i(u)$ changes the k in $\beta_k(u)$. Since $f_k f_j f_i(u) = f_j f_i f_k(u)$, we must have that applying f_i to $f_k(u)$ changes the the i in $\gamma_i(u)$. By Remark 3.3.7, we must
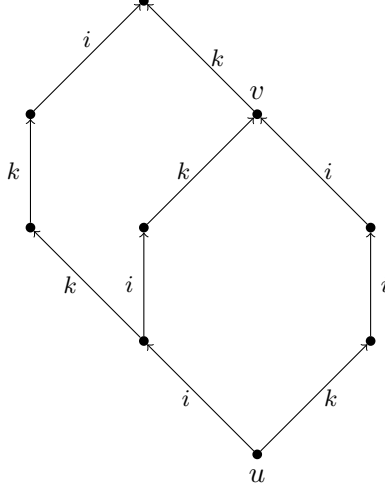
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Figure 3.5:   Lemma 3.3.11: Case where $i = j$

have that applying f_k to u changes the k in $\gamma_k(u)$. Then as in the previous case, we must have that the application of f_j to $f_i(u)$ either decreases the number of k in the first row or increases the number of $k + 1$ in the second row. This can only happen if $k = j$ which is not possible. Therefore, we must have $f_i f_k(u) = f_k f_i(u)$ as desired. □

We now consider the rank three interval seen in Figure 3.4 (c).

Lemma 3.3.12. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n and let $[u, v] \subseteq \mathcal{B}$ be such that $rk([u, v]) = 3$. Suppose $v = f_k f_j f_i(u) = f_i f_j f_k(u)$ as seen in Figure 3.4 (c). Then all saturated chains from u to v are connected by a sequence of degree two Stembridge moves.*

Proof. Recall, every vertex in a crystal graph has at most one incoming edge labeled l and at most one outgoing edge labeled l. Therefore, we must have that $i \neq k$. If either $i = j$ or $k = j$ then the proof is the same as seen in the first case of Lemma 3.3.11. Therefore, assume that i, j, and k are all distinct. Our aim is to show that
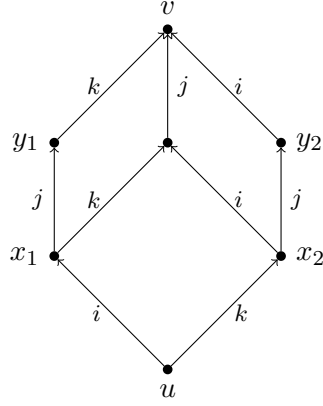
$$f_i f_k(u) = f_k f_i(u), \text{ and } f_j f_k f_i(u) = f_j f_i f_k(u) = f_k f_j f_i(u) = f_i f_j f_k(u), \quad (3.3)$$
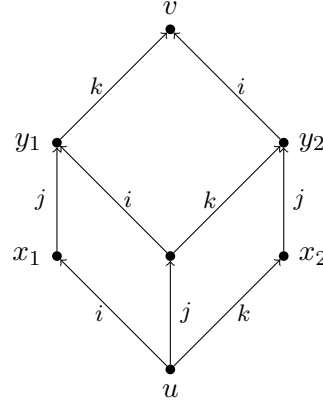
or

$$f_j(u) \neq 0 \text{ and } f_i f_k(f_j(u)) = f_k f_i(f_j(u)). \quad (3.4)$$

These two scenarios are displayed in Figure 3.6. If none of i, j and k are consecutive, then the crystal operators all commute and the result follows.

Suppose that two of the three labels are consecutive and the third is not consecutive with either of the other two. First, assume that i and j are consecutive. Since i and k are not

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Figure 3.6

consecutive, $|i - k| > 1$ and $f_i f_k(u) = f_k f_i(u)$. Similarly, since j and k are not consecutive, we will have $f_j f_k(f_i(u)) = f_k f_j(f_i(u))$ and the desired result occurs, namely Equation 3.3 holds and $[u, v]$ is seen in Figure 3.6 (3.3).

Instead, consider the case where i and k are consecutive. Without loss of generality, assume $k = i + 1$. Suppose that $f_i f_k(u) \neq f_k f_i(u)$. This implies that $f_i f_k^2 f_i(u) = f_k f_i^2 f_k(u)$. In particular, $f_i(f_k(u)) \neq 0, f_j(f_k(u)) \neq 0$ and $f_j(f_i(u)) \neq 0, f_k(f_i(u)) \neq 0$. Since we are assuming i and j are not consecutive and k and j are not consecutive, we have $f_i f_j(f_k(u)) = f_j f_i(f_k(u))$ and $f_j f_k(f_i(u)) = f_k f_j(f_i(u))$. This then gives the result, i.e. Equation (3.3) holds. Therefore, if we assume exactly two of the three crystal operators are consecutive, then the result holds.

Finally, assume that i, j, and k are all consecutive. This splits into two cases: (i) $k < i < j$, i.e. $k = i - 1$ and $j = i + 1$, and (ii) $i < j < k$, i.e. $i = j - 1$ and $k = j + 1$. See Remark 3.3.13 to see why we do not need to consider the case where $j < i < k$.

First, we work through Case (i). If $f_i f_k(u) = f_k f_i(u)$, then we have the desired result since f_k and f_j always commute. Therefore, we need to consider for contradiction the case where $f_i f_k^2 f_i(u) = f_k f_i^2 f_k(u)$. The argument needed is identical to that given in the proof of Lemma 3.3.11 where $i \neq j$. Namely, by Remark 3.3.7, we either must have $j = i$ or $j = k$. As a result, we must have $f_i f_k(u) = f_k f_i(u)$ as desired.

Finally, we work through Case (ii) where the labels i, j, and k are consecutive and $i < j < k$.

We do so by proving that there is one case in which the equalities in Equation (3.3) do not hold, but in this case the equalities in Equation (3.4) do hold. In either case, we have that the two saturated chains are connected by a sequence of degree two Stembridge moves.

Assume that the equations in (3.3) do not hold, namely assume $f_j f_k^2 f_j(f_i(u)) = f_k f_j^2 f_k(f_i(u))$

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and $f_i f_j^2 f_i(f_k(u)) = f_j f_i^2 f_j(f_k(u))$. Then we have the picture seen in Figure 3.7. By Proposition 3.3.6, we must have

$$A_i(f_k(u)) > B_i(f_k(u)) \text{ and } A_j(f_k(u)) = B_j(f_k(u)),$$

or

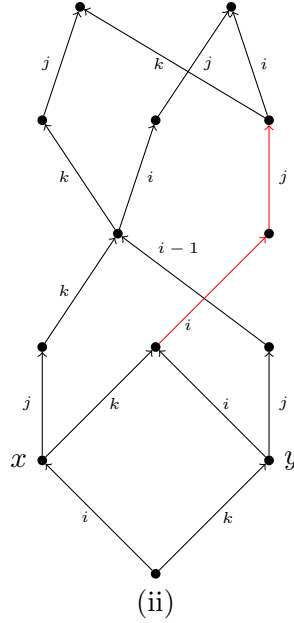$$A_i(f_k(u)) = B_i(f_k(u)) \text{ and } A_j(f_k(u)) \leq B_j(f_k(u)).$$

(ii)

Figure 3.7: Lemma 3.3.12: Case (ii)

Since $f_k f_j f_i(u) = f_i f_j f_k(u)$, the same box labeled j will be incremented with the application of f_j regardless of the chain. By Remark 3.3.7, for any degree four Stembridge relation

$$f_l f_m^2 f_l(x) = f_m f_l^2 f_m(x),$$

where $l < m$ has the property that the first application of f_m along the chain with label sequence (m, l, l, m) changes the m in $\gamma_m(x)$. In our case, this implies that each crystal operator f_j in the chains $f_k f_j f_i(u) = f_i f_j f_k(u)$ changes the j in $\gamma_j(u)$. This is because these edges are also in the chain with the Stembridge relation $f_j f_i^2 f_j(u) = f_i f_j^2 f_i(u)$ where $j > i$.

If we think of the chain in red as part of the chain $f_k f_j^2 f_k(x)$, then the first application

of f_j changes the j in $\beta_j(f_k(x))$ and the second application of f_j changes the j in $\gamma_j(f_k(x))$. This implies that $A_j(x) = B_j(x)$ and $A_k(x) = B_k(x)$. The only way to have these same j's changed when considering the chain in red as part of the chain $f_i f_j^2 f_i(y)$ is if $A_i(y) > B_i(y)$ and $A_j(y) = B_j(y)$ as can be seen in Remark 3.3.7. Otherwise, we would have that when considering the chain in red as part of the chain $f_i f_j^2 f_i(y)$, the first application of f_j changes the j in $\gamma_j(f_k(x))$ where $f_k(x) = f_i(y)$. Therefore, if $A_i(y) \geq B_i(y)$ and $A_j(y) \neq B_j(y)$ then $f_j f_k(x) = f_k f_j(x)$ and $f_j f_i(y) = f_i f_j(y)$ as desired.

Hence, we need to prove that if

$$A_j(x) = B_j(x) \text{ and } A_k(x) = B_k(x)$$

and

$$A_i(y) > B_i(y) \text{ and } A_j(y) = B_j(y),$$

then the equalities in (3.4) hold.

We begin by proving that $f_j(u) \neq 0$. We know applying f_i to u changes the i in $\beta_i(u)$ implying that $A_j(u) < B_j(u)$ since $A_j(u) = A_j(x) - 1$ and $B_j(u) = B_j(x)$. In order to apply f_j and get something nonzero, there must be a j in the row of u. If there is not, then we would have $f_j(f_i(u)) = 0$ and $f_j(f_k(u)) = 0$ since $A_j(x) = B_j(x)$ and $A_j(y) = B_j(y)$. However, we are assuming $f_j(f_i(u)) \neq 0$ and $f_j(f_k(u)) \neq 0$ implying that $f_j(u) \neq 0$ as desired.

Therefore, by Proposition 3.3.6, $f_j f_i(u) = f_i f_j(u)$ since $A_i(u) > B_i(u)$ (because we are assuming $A_i(y) > B_i(y)$ and $y = f_k(u)$) and $A_j(u) < B_j(u)$. Similarly, $f_j f_k(u) = f_k f_j(u)$ since $A_j(u) < B_j(u)$ and $A_k(u) = B_k(u)$ (because we are assuming $A_k(x) = B_k(x)$ and $x = f_i(u)$). This completes the proof. $\square$

Remark 3.3.13. If we assume that $j < i < k$ where $f_i f_j f_k(u) = f_k f_j f_i(u)$ then since $|j - k| > 1$, we have that $f_j(u) \neq 0$ since $f_j(f_k(u)) \neq 0$. If these saturated chains are not connected by a sequence of degree two Stembridge moves, then we must have that f_i and f_j are involved in a degree four Stembridge relation upward from u. This implies that f_i changes i in $\gamma_i(u)$ and f_j applied to $f_i(u)$ changes the j in $\beta_j(u)$. Similarly, f_i and f_k must be in a degree four Stembridge relation upward from u implying that f_k applied to u changes the k in $\gamma_k(u)$. As a result, we must have that $A_i(u) = B_i(u)$ and f_i applied to $f_j f_k(u)$ changes the i in $\beta_i(u)$. As a result, the original degree three relation $f_i f_j f_k(u) = f_k f_j f_i(u)$ would not hold. Therefore, in the case where $j < i < k$, either all the crystal operators commute, or the original degree three relation does not hold.

Combining these two lemmas gives the following theorem.

Theorem 3.3.14. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n and let $[u, v] \subseteq \mathcal{B}$ be such that $rk([u, v]) = 3$. Then all saturated chains from u to v are connected by some sequence of degree*

two Stembridge moves.

In the following theorem, we use the term weak subposet. A *weak subposet* Q of P is a subset of the elements of P with a partial ordering such that if $x \le y$ in Q, then $x \le y$ in P. Recall, a *subposet* Q of P is a subset of the elements of P such that $x \le y$ in Q if and only if $x \le y$ in P, see [22].

Theorem 3.3.15. *Let $\mathcal{B}$ be a two rowed shape crystal of type A_n. Let $u, v \in \mathcal{B}$ such that $[u, v]$ is a weak subposet of $\mathcal{B}$ where $[u, v]$ is one of the rank three intervals seen in Figure 3.8. Then as a subposet $[u, v] \cong B_3$, the Boolean lattice with three atoms.*
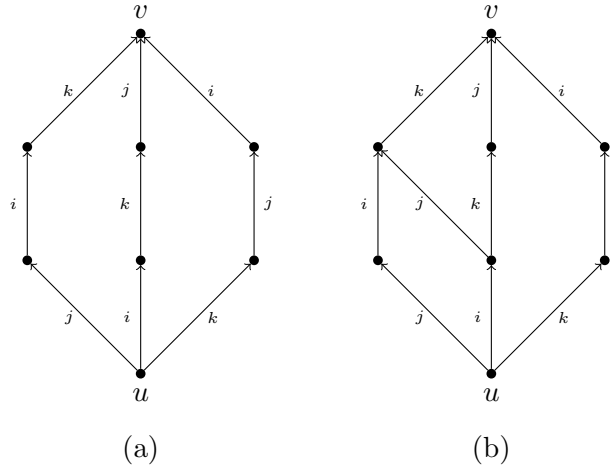

Figure 3.8: Proposition 3.3.15: Rank three intervals isomorphic to B_3

Proof. To begin, we will show that if there is a weak subposet as seen in Figure 3.8 (a), then this implies the additional edge shown in the interval in Figure 3.8 (b). Then it will suffice to show that if Figure 3.8 (b) is a weak subposet, then $[u, v] \cong B_3$ as a subposet.

Let $i, j, k \in [n]$ be arbitrary. Suppose $v = f_k f_j f_i(u) = f_i f_k f_j(u) = f_j f_i f_k(u)$. Since there is at most one outgoing edge labeled i for any $i \in [n]$ and at most one incoming edge labeled i for any $i \in [n]$, we have that i, j, and k are distinct. Therefore, at least one pair of the three labels are not consecutive. Without loss of generality, suppose that $|i - j| > 1$. Then $f_i f_j(u) = f_j f_i(u)$ and the picture becomes that of (b) as desired.

Suppose we are in case (b). If none of i, j and k are consecutive, then the crystal operators f_i, f_j, and f_k all commute with each other and we have $[u, v] \cong B_3$ as desired.

Suppose that two of the i, j and k are consecutive but the third is not consecutive with the

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other two. Without loss of generality, suppose $|i - k| = 1, |i - j| > 1$ and $|j - k| > 1$. Then we have $f_i f_j(u) = f_i f_j(u)$ and $f_j f_k(u) = f_k f_j(u)$. Similarly, we have $e_i e_j(u) = e_j e_i(u)$. This immediately implies that $f_i f_k(u) = f_k f_i(u)$ which gives the result.

Suppose that i, j and k are all consecutive. Then there are two possibilities for what our picture becomes. These possibilities are $j < k < i$ with $k = j + 1, i = j + 2$ and $i < k < j$ with $k = i + 1$ and $j = i + 2$ as seen in Figure 3.9.
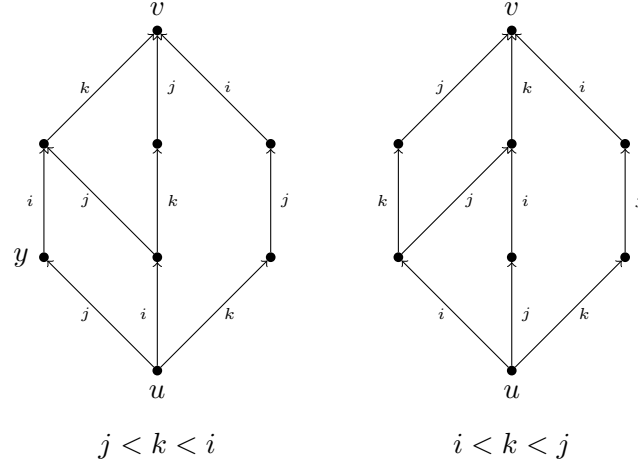

Figure 3.9: Theorem 3.3.15: Case where $i, j,$ and k are all consecutive

While these cases are very similar, there are slightly different proof techniques needed. In either case, we need to prove that f_k commutes with both f_i and f_j at u.

We begin with the case where $j < k < i$ seen on the left in Figure 3.9, with $k = j + 1$ and $i = j + 2$. Note that since $|j - i| = 2$ and $e_i(v) \neq 0$ and $e_j(v) \neq 0$, we have $e_i e_j(v) = e_j e_i(v)$. This directly implies that $f_i f_k(u) = f_k f_i(u)$.

Next, we show that $f_k f_j(u) = f_j f_k(u)$. Suppose by way of contradiction that $f_k f_j(u) \neq f_j f_k(u)$, which implies that $f_k f_j^2 f_k(u) = f_j f_k^2 f_j(u)$. Note that since there is a degree four Stembridge relation above u involving the crystal operators f_j and f_k, with $j < k$ we must have that applying f_k to u changes the k in $\gamma_k(u)$ by Remark 3.3.7. Since we are assuming that $f_k f_i f_j(u) = f_i f_j f_k(u)$, if applying f_j to u changes the rightmost j in row $m \in \{1, 2\}$, then applying f_j to $f_k(u)$ changes the same j in the same row m. This implies that applying f_j to u changes the j in $\beta_j(u)$ by Remark 3.3.7.

Let $y = f_j(u)$. Then applying f_k to y must change the i in $\beta_i(y)$, else we would have that $f_k f_j(u) = f_j f_k(u)$, which we are assuming is not true. Again, as we are assuming that

$f_j f_k(u) \neq f_k f_j(u)$, there must be a degree four Stembridge relation above y involving the crystal operators f_k and f_i. This leads to a contradiction. Consider the action of the crystal operator f_k applied to $f_i(y)$ as part of the chain with edge label sequence (j, i, k). Here, we have f_k changes the k in $\gamma_k(f_i(y))$. However, considering the action of this crystal operator f_k as part of the chain beginning at y with edge label sequence (k, i, i, k), we see that it changes the k in $\beta_k(f_i(y))$ by Remark 3.3.7. As these cannot simultaneously occur, our original assumption must be false and we have $f_k f_j(u) = f_j f_k(u)$ as desired.

Now we turn to the case with $i < k < j$ seen on the right in Figure 3.9, with $k = i + 1$ and $j = i + 2$. We begin by proving that $f_k f_j(u) = f_j f_k(u)$. Suppose for contradiction that $f_k f_j(u) \neq f_j f_k(u)$ implying that $f_k f_j^2 f_k(u) = f_j f_k^2 f_j(u)$. By Remark 3.3.7, this says that the application of f_j to u changes j in $\gamma_j(u)$ and the application of f_k to $f_j(u)$ changes the k in $\beta_k(u)$. Since we are assuming $f_k f_i f_j(u) = f_i f_j f_k(u)$, the application of f_j to $f_k(u)$ also changes the j in $\gamma_j(u)$.

Again, by Remark 3.3.7, this implies that the application of f_k to u changes the k in $\gamma_k(u)$. However, we have that $f_k(f_j(u)) \neq 0$ and $f_i(f_j(u)) \neq 0$ implying there is a Stembridge relation above $f_j(u)$ involving the crystal operators f_k and f_i. As we know that f_k changes the k in $\beta_k(u)$ and $k > i$, this Stembridge relation must be degree two. Namely, we have $f_k f_i(f_j(u)) = f_i f_k(f_j(u))$. However, this can only occur if $f_j f_k(u) = f_k f_j(u)$ as desired.

As in the previous case, we have $f_i f_k(u) = f_k f_i(u)$. This is because $e_i(v) \neq 0$ and $e_j(v) \neq 0$ where $|i - j| > 1$, implying that $e_i e_j(v) = e_j e_i(v)$. This implies that $f_i f_k(u) = f_k f_i(u)$ as desired.

Hence, in either case, if the weak subposet $[u, v]$ is that seen in (a) or (b), we have $[u, v] \cong B_3$ as desired. $\qquad\square$

3.3.3   Analyzing arbitrary rank intervals

Hersh and Lenart gave an example of a family of intervals of arbitrary rank with disjoint saturated chains in [11]. We looked at this example briefly in Section 1.4. These intervals come from a two rowed shape crystal of type A_n. Now that we have a new way to analyze Stembridge relations in two rowed shape crystals, we revisit these intervals.

Recall the two rowed shape crystal $\mathcal{B}$ of type A_n crystal with highest weight $\lambda = (n + 1, n)$ and the interval $[u, v] \in \mathcal{B}$ where

$$
u = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & \cdots & n-2 & n-1 \\
\hline
2 & 3 & 4 & 5 & \cdots & n+1 \\
\cline{1-6}
\end{array}, \qquad
v = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & \cdots & n-2 & n-1 & n \\
\hline
3 & 4 & 5 & \cdots & n+1 & n+1 \\
\cline{1-6}
\end{array},
$$

Note that to get from u to v, we increment $1, 2, 3, ..., n-1$ in the first row once and $2, 3, ..., n-1, n$ in the second row once.

Proposition 3.3.16. *Let $\mathcal{B}$ be the type A_n crystal of shape $(n+1, n)$ and let u, v be as defined above. Then the degree four Stembridge relation does not occur in the interval $[u, v]$.*

Proof. Let $x \in [u, v]$ and suppose that the degree four Stembridge relation holds upward from x, i.e. $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$. We will show that $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x) \notin [u, v]$. Since f_1 and f_n are each applied exactly once along any saturated chain from u to v, we cannot have $i = 1$ or $i + 1 = n$. Therefore we must have that $2 \leq i \leq n - 2$. In order to have a degree four Stembridge relation upward from x, there are three possibilities for the relative values of $A_i(x)$, $B_i(x)$, $A_{i+1}(x)$, and $B_{i+1}(x)$ as seen in Proposition 3.3.6:

1. $A_i(x) = B_i(x)$ and $A_{i+1}(x) = B_{i+1}(x)$,

2. $A_i(x) = B_i(x)$ and $A_{i+1}(x) < B_{i+1}(x)$,

3. $A_i(x) > B_i(x)$ and $A_{i+1}(x) = B_{i+1}(x)$.

For $x \in [u, v]$ and j with $2 \leq j \leq n - 1$ there is at most two appearances of j in the first row of x. Similarly, for any j with $3 \leq j \leq n$ there is at most two appearances of j in the second row of x.

Suppose we are in the case where $A_i(x) = B_i(x)$ and $A_{i+1}(x) = B_{i+1}(x)$. We cannot have $A_{i+1}(x) = 0$ and $B_i(x) = 0$, as this would imply that $f_{i+1}(x) = 0$. So at least one of these must be nonzero. If $A_i(x) = B_i(x) = 0$, then this implies that f_i has been applied along the chain from u to x. But on any saturated chain from u to v, f_i occurs exactly twice. Hence, we cannot have $A_i(x) = B_i(x) = 0$. Similarly, we cannot have $A_{i+1}(x) = B_{i+1}(x) = 0$, as this would imply that f_{i+1} has been applied somewhere along the chain from u to x, but f_{i+1} is applied exactly twice along a saturated chain from u to v. We also cannot have $A_i(x) = B_i(x) = 2$ since having two appearances of $i + 1$ in the second row would imply that f_i has already been applied once along the saturated chain from u to x. Similarly, we cannot have $A_{i+1}(x) = B_{i+1}(x) = 2$ since this implies that f_i has previously been applied.

As a result, we must have $A_i(x) = B_i(x) = 1$ and $A_{i+1}(x) = B_{i+1}(x) = 1$. As seen in Remark 3.3.7, applying f_i to x changes the i in $\gamma_i(x)$ to $i + 1$ and applying f_{i+1} to x changes the $i + 1$ in $\gamma_{i+1}(x)$ to $i + 2$. Applying f_{i+1} to $f_i(x)$ changes the $i + 1$ in $\gamma_{i+1}(x)$ to $i + 2$. When applying f_{i+1} for the second time, there would be two appearances of $i + 2$ in the second row of $f_{i+1} f_i(x)$ compared to one appearance of $i + 1$ in the first row. Therefore by the signature rule, f_{i+1} would change the remaining $i + 1$ in the second row to $i + 2$. This element is no longer in the interval $[u, v]$. Therefore, we cannot have $A_i(x) = B_i(x)$ and $A_{i+1}(x) = B_{i+1}(x)$ and have a degree four Stembridge relation that lies entirely in the interval $[u, v]$.

Now, suppose that $A_i(x) = B_i(x)$ and $A_{i+1}(x) < B_{i+1}(x)$. As before, we must have $A_i(x) = B_i(x) = 1$. There are two possibilities for the relation between A_{i+1} and B_{i+1}. First, we could

have $A_{i+1}(x) = 0$ and $B_{i+1}(x) = 1$ or 2. This would imply that f_{i+1} has been previously applied changing the $i+1$ in the first row to $i+2$. Else, we could have $A_{i+1}(x) = 1$ and $B_{i+1}(x) = 2$. The only way to have two appearances of $i+2$ in the second row of x would be for f_{i+1} to have been previously applied. In either case, we cannot have the entirety of the degree four Stembridge relation in the interval $[u, v]$ since f_{i+1} is applied exactly twice along any saturated chain.

Finally, assume that $A_i(x) > B_i(x)$ and $A_{i+1}(x) = B_{i+1}(x)$. As seen previously, we must have that $A_{i+1}(x) = B_{i+1}(x) = 1$. If $A_i(x) = 1$ and $B_i(x) = 0$, then this implies that f_i has been applied along any saturated chain from u to x. This cannot happen. Instead, assume $A_i(x) = 2$ and $B_i(x) = 1$. Applying f_{i+1} to x changes the $i+1$ in $\gamma_{i+1}(x)$ to $i+2$. There are no occurrences of $i+1$ in row two of $f_{i+1}(x)$ so applying f_i to $f_{i+1}(x)$ changes the rightmost i in $\beta_i(x)$ to $i+1$. A second application of f_i again changes the i in the first row of $f_i f_{i+1}(x)$ to $i+1$. Then we have $f_i^2 f_{i+1}(x) \notin [u, v]$. Therefore, the relation $f_i f_{i+1}^2 f_i(x) = f_{i+1} f_i^2 f_{i+1}(x)$ does not occur within the interval $[u, v]$. $\square$

While we still have more to study with this interval, we do know the number of connected components in the open interval (u, v).

Proposition 3.3.17. *For $n \geq 4$, there are exactly two connected components in the interval $(u, v) \subseteq \mathcal{B}$, the crystal of type A_n of shape $(n+1, n)$ for u and v as defined above. One contains the saturated chain with edge label sequence $(1, 2, 2, ..., n-1, n-1, n)$, the other contains the saturated chain with edge label sequence $(n, n-1, n-1, ..., 2, 2, 1)$.*

Proof. To begin, we recall that in [11], Hersh and Lenart proved that there are at least two connected components since the saturated chain with edge label sequence $(1, 2, 2, .., n-1, n-1, n)$ is in a distinct component from the saturated chain with edge label sequence $(n, n-1, n-1, ...2, 2, 1)$. Additionally, they showed that the connected component containing the saturated chain with edge label sequence $(1, 2, 2, ..., n-1, n-1, n)$ only contains saturated chains that begin with the label 1 and end with the label n.

We will show that for each $i \in \{2, ..., n-1\}$, every saturated chain whose edge label sequence begins with i is in the same connected component as saturated chains whose edge label begins with n. This will show that all saturated chains whose edge label sequence does not begin with 1 are in the same connected component as the saturated chain with edge label sequence $(n, n-1, n-1, ..., 2, 2, 1)$.

For $i \in \{2, , ..., n-2\}$, we have $f_i f_n(u) = f_n f_i(u)$. Therefore, it suffices to prove that $f_i f_n(u) = f_n f_i(u) \leq v$ for all $i \in \{2, ..., n-2\}$. We claim that

$$(i, n, n-1, n-1, ..., i+1, i+1, i-1, i-1, ...2, 2, 1, i)$$

is the edge label sequence of a saturated chain from u to v. Applying f_i to u will change the i in $\gamma_i(u)$ to $i+1$. Similarly, applying f_n to $f_i(u)$ changes the n in $\gamma_n(u)$. For each $j \in \{2, ..., n-2\}$ with $j \neq i$, the first application of f_j changes $\beta_j(u)$ and the second application of f_j changes $\gamma_j(u)$ as can be seen by the signature rule. This includes the application of f_{i-1} since the i in row two was changed to $i+1$ when f_i was applied at the beginning of the chain. Finally, applying f_1 changes the 1 in $\beta_1(u)$ and the final application of f_i changes i in $\beta_i(u)$. The resulting tableau is v. Therefore, we have a saturated chain with edge label sequence $(i, n, n-1, n-1, ..., i+1, i+1, i-1, i-1, ...2, 2, 1, i)$ that is in the same connected component as the saturated chain with edge label sequence $(n, i, n-1, n-1, ..., i+1, i+1, i-1, i-1, ...2, 2, 1, i)$ as desired.

The only remaining task is to show that any saturated chain beginning with the label $n-1$ is in the same connected component as the saturated chain with label sequence $(n, n-1, n-1, ..., 2, 2, 1)$. Using the signature rule, we can see that

$$f_{n-1}f_{n-2}f_n f_{n-1}(u) = f_{n-2}f_{n-1}f_{n-1}f_n(u).$$

Therefore, there exists a saturated chain from u to v with edge label sequence $(n-1, n, n-2, n-1, n-2, n-3, n-3, ..., 2, 2, 1)$. As a result, this interval will have exactly two connected components. $\qquad\square$

Note that this result does not hold when $n = 3$. As can be seen in Figure 1.13, when $n = 3$, the interval (u, v) has exactly three connected components.

3.4   Discussion and open questions

In this chapter, we tried to understand relations among crystal operators in two special cases: hook shape crystals of type A_n and two rowed shape crystals of type A_n. Using the combinatorial tableaux model for crystals, we were able to characterize when different Stembridge relations occur from information about the semistandard Young tableau. From this, we have the following natural question.

Question 3.4.1. *Are there any other shapes or types for which an in depth study of crystal graphs is possible?*

Additionally, we see that the tableaux model gave more combinatorial data to study, namely the structure of the semistandard Young tableau. There are various models that can be used to represent crystals and therefore we ask the following question.

Question 3.4.2. *Would other models for crystals provide insight for further results about relations among crystal operators?*

At this time, we have yet to consider the use of other models.

Finally, we looked into whether hook shape crystal posets were lattices. Although we know that in general, crystals (except type A_2) are not lattices, it may be possible that certain classes of crystal posets are lattices.

Question 3.4.3. *Does there exist a list of suitable conditions under which an interval in a crystal poset is a lattice?*

We have conjectured that hook shape crystals with highest weight $\lambda = (2, 1, ..., 1)$ are lattices. We hope to prove this and find other instances where certain classes of crystals are lattices.

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