ABSTRACT

STADNYK, GRACE ELIZABETH. Lexicographic Shellability, the Enriched Tamari Poset, and the Edge-Product Space of Phylogenetic Trees. (Under the direction of Patricia Hersh).

We study the combinatorics and topology of the edge-product space of phylogenetic trees, a topological space arising in evolutionary biology and one that is closely related to a family of topological spaces called toric cubes. Both the edge-product space of phylogenetic trees and toric cubes are known to be regular CW complexes. We explore the question of whether these CW decompositions are shellable by determining whether their face posets are lexicographically shellable.

In doing so, we introduce a new poset that is closely related to the well-known Tamari lattice. We call this poset the enriched Tamari poset and determine some of its properties. The elements of the enriched Tamari poset are the maximal elements of the Tuffley poset, which is isomorphic to the face poset of the edge-product space of phylogenetic trees. Initially, it was our hope that the enriched Tamari poset would give a partial order on the maximal elements of the Tuffley poset such that any linear extension of this partial order would be compatible with a shelling order of the edge-product space itself. However, we show that in fact, there is no shelling order of the edge-product space of phylogenetic trees by showing that the Tuffley poset is not dual CL-shellable.

Throughout this exploration, we have wondered whether we could develop generalizations of some of the well-known and commonly used tools in topological combinatorics to help us answer these questions. To this end, we introduce the notion of a generalized recursive atom ordering. We show that a poset admits a generalized recursive atom ordering if and only if it is CC-shellable, which is analogous to the useful result of Björner and Wachs' stating that a poset admits a (traditional) recursive atom ordering if and only if it is CL-shellable.

Lexicographic Shellability, the Enriched Tamari Poset, and the Edge-Product Space of Phylogenetic Trees

by
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DEDICATION

To Scott.

BIOGRAPHY

Grace was born and raised in Chappaqua, New York. She studied Mathematics and Russian at Hamilton College and graduated in 2010. She worked for several years at a financial data company, both in New York City and in Chicago, before beginning her graduate studies at North Carolina State University. When she is not teaching or learning mathematics, Grace enjoys reading, running, and spending time with her family, friends, and dogs.

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Chapter 1

Introduction

Our research began with a family of topological spaces called toric cubes. Toric cubes are subsets B of standard cubes defined by binomial inequalities such that B is equal to the closure of its strictly positive points (see Definition 1.4.3). Engström, Hersh and Sturmfels showed in [10] that $n-$dimensional toric cubes are the images of d-cubes under maps $f : [0,1]^d \to [0,1]^n$ given by monomials. They also showed that toric cubes have a CW-decomposition. Basu, Gabrielov and Vorobjov showed in [2] that these CW-decompositions are regular. A natural question we ask is whether the face posets of these CW-decompositions are shellable. We thus begin with background on CW complexes, posets, shellability, and poset shellability in Sections 1.1 and 1.2 of Chapter 1.

Our study focuses on a particular class of toric cubes whose monomial maps are determined by trees. The union of all toric cubes derived from trees with a fixed number of leaves labeled by a finite set X defines a topological space that arises in evolutionary biology. We study the combinatorics and topology of this space, which is called the edge-product space of phylogenetic trees on X leaves and which is denoted $\mathcal{E}(X)$ (see Definition 1.4.11). In Section 1.4 of Chapter 1, we present necessary background on toric cubes and the edge-product space of phylogenetic trees. Moulton and Steel showed in [21] that $\mathcal{E}(X)$ has a CW structure with face poset isomorphic to the Tuffley poset, denoted $S(X)$ (see Definition 1.4.14). Taking $X = [n]$ and motivated by the question of finding a dual lexicographic shelling of the Tuffley poset, we present a partial order on the maximal elements of $S([n])$ called the enriched Tamari poset in Chapter 2.

The enriched Tamari poset is a natural extension of the Tamari lattice, a lattice of bracketings of integers from 1 to n. A subset of the cover relations in the enriched Tamari poset are cover relations in the Tamari lattice (see Section 1.5). Instead of requiring the leaves to always be ordered from 1 to n in counterclockwise order, as in the Tamari lattice, this new poset is "enriched" with cover relations that yield different orderings of the leaves. We define the enriched Tamari poset and prove it is, in fact, a poset, in Section 2.2. In Section 2.3, we present some immediate basic properties of this poset.

The initial motivation for introducing the enriched Tamari poset was as a means of giving the maximal elements of the Tuffley poset, $S([n])$, an ordering that is compatible with a recursive coatom ordering. In doing so we aimed to show that $\mathcal{E}([n])$ was shellable in the sense of Definition 1.2.11. However, in Chapter 3, we prove that $S([n]) \cup \{\hat{0}, \hat{1}\}$ is not dual CL-shellable. In particular, we show in Section 3.3 that there does not exist a recursive coatom ordering of $S([n]) \cup \{\hat{0}, \hat{1}\}$. By a result of Björner (see Theorem 1.2.24), this fact shows that the edge-product space $\mathcal{E}([n])$ is not shellable. This is a somewhat surprising result for two main reasons. First, $\mathcal{E}([n])$ is gallery-connected (see Definition 3.2.4), a fact we prove using the enriched Tamari poset in Section 3.2. Second, the existence of a shelling of intervals in $S([n]) \cup \{\hat{0}\}$ was proven by Gill, Linusson, Moulton, and Steel in [12] and an explicit shelling of intervals in $S([n]) \cup \{\hat{0}\}$ is given via an EC-labeling of the uncrossing poset introduced by Hersh and Kenyon in [16]. We describe the relationship between the uncrossing poset and the Tuffley poset in Section 3.1 and use this to show in Theorem 3.1.22 that intervals in $S([n]) \cup \{\hat{0}\}$ are isomorphic to principal order ideals in the uncrossing poset, P_n. This fact also implies that the face poset of any toric cube arising from a tree is shellable.

We look at lexicographic shellability in more detail in Chapter 4. In particular, we modify the notion of a recursive atom ordering to define what we call a generalized recursive atom ordering (see Definition 4.1.4). The main result of this chapter is Theorem 4.1.9, which states that a poset admits a generalized recursive atom ordering if and only if it is CC-shellable. This result is analogous to a result of Björner and Wachs in [5] which states that a poset admits a (traditional) recursive atom ordering if and only if it is CL-shellable.

The remainder of Chapter 4 presents partial order generalizations of shellability results originally given by Björner in [3]. In particular, when a natural total order on facets is not apparent, we wonder whether a partial order on facets with the codimension one property (see Definition 3.2.6) might be enough to give information about the topology of the space. For example, though $\mathcal{E}([n])$ is not shellable, a partial order on its facets satisfying certain conditions might be enough to determine the topology of $\mathcal{E}([n])$. As a first step towards this goal, we look in particular at a generalization of a result of Björner's (see Theorem 4.2.6) in Section 4.2. It turns out that partial orders satisfying certain conditions can give insight into the topology of a space, but these types of partial orders always extend to shelling orders.

1.1 Preliminaries on CW complexes and posets

We begin with background on some of the basic structures that encode mathematical information in combinatorics and topology. We will then describe how these structures relate to each other in the realm of topological combinatorics and what their interaction can tell us about the underlying mathematical objects they encode. We begin by defining partially ordered sets and several concepts that will help us discuss the posets that will appear in later chapters.

Definition 1.1.1. A **poset** or **partially ordered set** is a pair $(P, \leq)$ where P is a set and $\leq$ is an order relation on the elements of P satisfying

(i) *antisymmetry:* If $a \leq b$ and $b \leq a$, then $a = b$.

(ii) *reflexivity:* $a \leq a$ for every $a \in P$.

(iii) *transitivity:* If $a \leq b$ and $b \leq c$, then $a \leq c$.

Definition 1.1.2. If $a \leq b$ and there is no c such that $a \leq c \leq b$, then we write $a \lessdot b$ and we say b **covers** a. If $a \leq b$ or $b \leq a$, then we say a and b are **comparable**. Otherwise, a and b are **incomparable**.

Definition 1.1.3. The **Hasse diagram** of a poset $(P, \leq)$ is a graph such that

(i) elements of P are vertices,

(ii) there is an edge between a and b if $a \lessdot b$ or if $b \lessdot a$,

(iii) if $a < b$, then b is placed higher in the plane than a.

When the order relation $\leq$ is understood, we will sometimes denote the poset $(P, \leq)$ simply by P.

Definition 1.1.4. A **chain** is a totally ordered subset of P. The **length** of a chain is one less than the number of elements in the chain. If all maximal chains in P are of the same length, the poset is said to be **pure**.

Definition 1.1.5. A poset is **bounded** if it has both a unique least element (denoted $\hat{0}$) and a unique greatest element (denoted $\hat{1}$).

Definition 1.1.6. A poset is **graded** if it is bounded and pure.

Definition 1.1.7. A poset P is **connected** if for every pair $a, b \in P$, there is a sequence $a = c_0, c_1, c_2, \ldots c_n = b$ such that c_i and c_{i+1} are comparable for $0 \leq i \leq n-1$ and $c_i \in P$.

One especially interesting and well-behaved type of poset is a lattice, which we describe next.

Definition 1.1.8. Let P be a poset and $u, v \in P$. The **meet** of u and v, if it exists, is the unique greatest common lower bound of u and v. It is denoted $u \wedge v$. The **join** of u and v, if it exists, is the unique least common upper bound of u and v. It is denoted $u \vee v$.

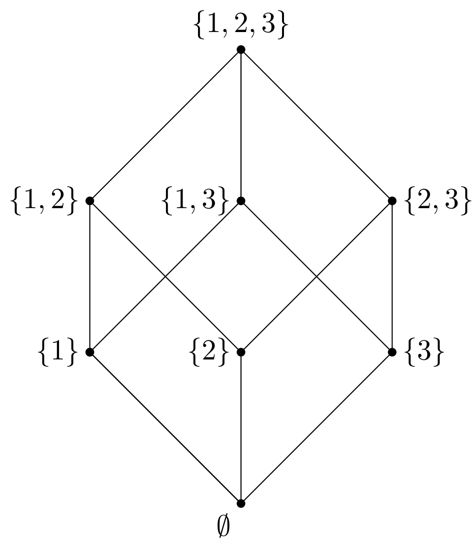


Figure 1.1: The Boolean lattice B_3.

Definition 1.1.9. A **meet-semilattice** is a poset in which every pair of elements of P has a meet. A **join-semilattice** is a poset in which every pair of elements has a join. A **lattice** is a poset in which every pair of elements has both a meet and a join.

Example 1.1.10. The Boolean lattice B_n is a poset whose elements are subsets of the integers from 1 to n. It is bounded. The top element is $\hat{1} = [n] = \{1, 2, \ldots n\}$ and the bottom element is $\hat{0} = \emptyset$. It is also a lattice. If a and b are elements of B_n, their meet is $a \wedge b = a \cap b$, the set of integers in both a or b. Their join is $a \vee b = a \cup b$, the set of integers in either a and b. See Figure 1.1 for the Hasse diagram of B_3.

We can use the following theorem to determine if a poset P is a lattice:

Proposition 1.1.11. *A finite join-semilattice P with a $\hat{0}$ is a lattice. A finite meet-semilattice P with a $\hat{1}$ is a lattice.*

We will need the notion of isomorphic posets in Chapter 3:

Definition 1.1.12. Posets $(P, \leq_P)$ and $(Q, \leq_Q)$ are **isomorphic** if there exists a bijection $f : P \to Q$ such that $x \lessdot_P y$ if and only if $f(x) \lessdot_Q f(y)$.

We now define a basic topological structure called a simplicial complex and the more general notion of a CW complex. In this thesis and in topological combinatorics in general, we often try to elicit information about a topological space that can be decomposed as a simplicial or CW complex using combinatorial data about these decompositions.

Definition 1.1.13. An **abstract simplicial complex** Δ on a finite vertex set V is a nonempty collection of subsets of V such that

(i) $\{v\} \in \Delta$ for all $v \in V$ and

(ii) if $G \in \Delta$ and $F \subseteq G$, then $F \in \Delta$.

Definition 1.1.14. A d-dimensional **geometric simplex** Γ in $\mathbb{R}^n$ is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$ called vertices. For $c < d$, a c-dimensional **face** of Γ is the convex hull of any $c+1$ vertices of Γ.

Definition 1.1.15. A **geometric simplicial complex** K is a nonempty collection of geometric simplices such that

(i) if F is a face of G and $G \in K$, then $F \in K$ and

(ii) if F_1, $F_2 \in K$, then $F_1 \cap F_2$ is a face of both F_1 and F_2.

Theorem 1.1.16. *For every abstract simplicial complex Δ, there is a geometric simplicial complex K such that taking the faces of K as vertex sets yields Δ. In this case, K is called a **geometric realization** of Δ.*

Because of the previous theorem, we will generally not specify whether a simplicial complex is an abstract simplicial complex or a geometric realization.

CW complexes are generalizations of simplicial complexes first formally introduced by J.H.C. Whitehead (see [29]). The basic building block of a CW complex is the open m-cell:

Definition 1.1.17. An **open $m-$cell** is any topological space homeomorphic to the interior of an $m-$ball B^m. An open $0-$cell is a point.

Next, we present the definition of a CW complex. This is an inductive definition given by Hatcher in [13], where the CW complex is constructed by gluing progressively higher dimensional $m-$cells to the collection of previously added, lower dimensional cells. This definition uses a set of continuous functions called **maps**.

Definition 1.1.18. A **CW complex** is defined inductively as follows:

(i) Begin with a discrete set X^0 whose points are regarded as 0-cells.

(ii) Form the $n-$skeleton X^n from X^{n-1} by attaching open $n-$cells, e^n_α, via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. These maps are called **attaching maps**. Each cell e^n_α has a **characteristic map** $\Phi_\alpha : B^n_\alpha \to X$ which extends the attaching map ϕ_α and is a homeomorphism from the interior of B^n_α onto e^n_α.

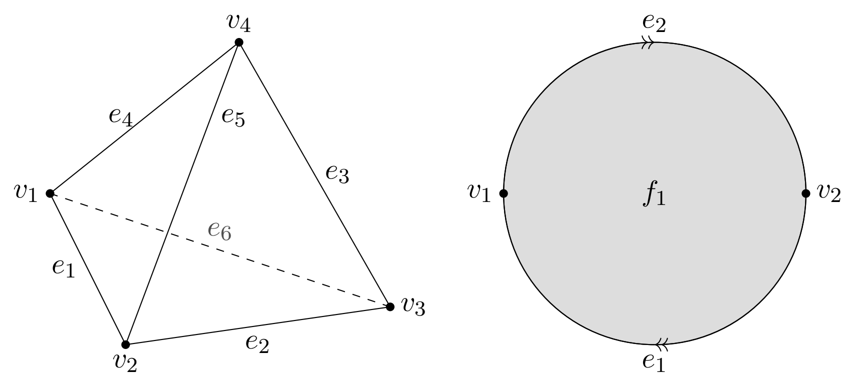


Figure 1.2: The tetrahedron on the left is a regular CW complex. The CW complex on the right is not regular.

Definition 1.1.19. A CW complex K is called **regular** if the closure of each cell of K is homeomorphic to a closed ball. In particular, this implies regular cell complexes have attaching maps that are injective.

Example 1.1.20. The tetrahedron on the left in Figure 1.2 is a regular CW complex. However, $\mathbb{RP}^2$, shown on the right in Figure 1.2, is not a regular CW complex because the attaching map corresponding to the cell denoted e_2 is not injective. Thus the cell e_2 is not homeomorphic to a closed ball.

If a regular CW complex has maximal cells that are all of the same dimension, then this CW complex is more specifically a d-CW complex:

Definition 1.1.21. A finite, regular CW complex K is a **d-CW complex** if every closed cell is a face of a d-dimensional cell. For a cell σ of the d-CW complex K, let $\delta\sigma$ denote the $(d-1)$-CW complex consisting of all proper cells of σ.

We now describe connections between posets and CW complexes arising in the world of topological combinatorics. In general, when a topological property is used to describe a poset, we mean that a simplicial complex derived from the poset, called the order complex of P, has this topological property. We describe the construction of the order complex and introduce some related definitions next.

Definition 1.1.22. Given any poset P, the **order complex** of P is the abstract simplicial complex with $k-$faces corresponding to the chains of length k of P. It is denoted $\Delta(P)$.

Definition 1.1.23. Given any CW complex K, its **face poset** $P(K)$ consists of the cells (sometimes called faces) of K ordered by inclusion with a $\hat{0}$ adjoined. The **augmented face**

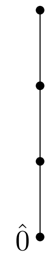


Figure 1.3: The face poset for both a CW decomposition of the real projective plane and a CW decomposition of the 3-ball.

poset, $\hat{P}(K)$, is $P(K)$ with a $\hat{1}$ adjoined, provided that $P(K)$ does not already have a unique maximal element.

If K is a regular CW complex, then $\Delta(P(K) - \hat{0}) \cong K$. Thus the incidence relations of cells as given in $P(K)$ determines the topology of K. This is not true of all CW complexes. See [4] for more details.

Example 1.1.24. Figure 1.3 shows the face poset for a (non-regular) CW decomposition of the real projective plane $\mathbb{R}P^2$ consisting of one 0-cell, one 1-cell, and one 2-cell. It is also the face poset for a non-regular CW decomposition of the ball B^3 consisting of one 0-cell, one 2-cell, and one 3-cell.

Simplicial complexes are regular CW complexes. Given a simplicial complex K, the order complex $\Delta(P(K) - \hat{0})$ is another simplicial complex, specifically the barycentric subdivision of K, $sd(K)$. A simplicial complex is homeomorphic to its barycentric subdivision, thus $K \cong sd(K) \cong \Delta(P(K) - \hat{0})$.

If K is a simplicial complex, one interesting subcomplex of K is the link of a face, F.

Definition 1.1.25. Suppose K is a simplicial complex and F a face of K. The **link** of F is

$$\mathbf{lk}(F) = \{G \in K : F \cup G \in K, F \cap G = \emptyset\}$$

Proposition 1.1.26. *The order complex $\Delta(u, v)$ of (u, v) is the link of some face F in $\Delta([\hat{0}, v])$.*

Proof. For $F \in \Delta(\hat{0}, v)$, $\mathbf{lk}(F) = \{G \in \Delta(\hat{0}, v) : F \cup G \in \Delta(\hat{0}, v), F \cap G = \emptyset\}$. Let $u \leq v$. Consider any saturated chain from $\hat{0}$ to u and add the element v. Let F be the face of $\Delta([\hat{0}, v])$ corresponding to this chain. If G is any chain in the open interval (u, v), $G \cup F$ is a chain in $[\hat{0}, v]$ and thus corresponds to a face in $\Delta([\hat{0}, v])$. Furthermore, $F \cap G = \emptyset$ by construction. Thus $\Delta(u, v)$ is the link of F. $\square$

A question one might ask is when a poset is the face poset of a simplicial complex or a CW complex. This question led Björner to the notion of a CW poset. See [4] for more details on CW posets.

Definition 1.1.27. A poset P is said to be a **CW poset** if the following conditions hold:

(i) P has a least element $\hat{0}$.

(ii) P is nontrivial i.e. P has more than one element.

(iii) for all $x \in P - \{\hat{0}\}$, the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.

Proposition 1.1.28 (Proposition 3.1 [4]). *A poset P is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.*

The following proposition provides criteria that helps us to determine when a poset is a CW poset.

Proposition 1.1.29 (Proposition 2.2 [4]). *Let P be a nontrivial poset such that*

(i) P has a least element $\hat{0}$,

(ii) every interval $[x, y]$ of length two has cardinality four (i.e. has the diamond property), and

(iii) for $x \in P$, every interval $[\hat{0}, x]$ is finite and shellable.

Then P is a CW poset.

The last condition of the previous proposition concerns the shellability of lower intervals in a poset. We give the necessary background on shellability in Section 1.2.

1.2 Shellability

Some of the main results of this thesis surround questions regarding poset shellability. As such, we review the basics of polytopal complex shellability and poset shellability next. Parts of this background on shellability are used in Chapter 3, where we examine whether the face poset of the edge-product space of phylogenetic trees is dual CL-shellable, and in Chapter 4, where we look at extensions of some of the tools introduced in this section.

Definition 1.2.1. A **polytopal complex** is a finite, nonempty collection C of polytopes in $\mathbb{R}^d$ such that

(i) if F is a face of $P \in C$, then $F \in C$ and

(ii) if $P_1, P_2 \in C$, then $P_1 \cap P_2$ is a face of both P_1 and P_2.

A polytopal complex C is of dimension k, denoted $dim(C) = k$, if the largest dimension of a polytope in C is k. A polytopal complex C is **pure** if each of its faces is contained in a face of dimension $dim(C)$. A **facet** of a polytopal complex C is a polytope that is properly contained in no other polytope of C.

Note that by definition, a simplicial complex K is a polytopal complex such that every polytope in K is a simplex.

Definition 1.2.2. Given a polytope P, its **boundary complex** is the complex consisting of all proper faces of P. It is denoted $\delta(P)$. Let $\overline{P}$ be the polytopal complex consisting of P and all its faces.

Definition 1.2.3. Let C be a pure, $k-$dimensional polytopal complex. A **shelling** of C is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of C such that either C is 0-dimensional (so all facets are points), or it satisfies the following conditions:

(i) The boundary complex δF_1 has a shelling.

(ii) For $1 < j \leq s$, the intersection of the facet F_j with the union of the previous facets is nonempty and is the beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of F_j, that is,

$$F_j \bigcap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \cup \ldots \cup G_r$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of δF_j where $1 \leq r \leq t$. In particular, this requires that $F_j \cap \left(\cup_{i=1}^{j-1} F_i \right)$ has a shelling, so it must be pure $(k-1)$-dimensional, and connected for $k > 1$.

A polytopal complex is **shellable** if it is pure and has a shelling.

Every simplex is shellable. Since every face (and thus facet) of a simplicial complex is a simplex, if $F_1, F_2, \ldots F_s$ is a shelling of the simplicial complex K, condition (i) of Definition 1.2.3 is automatically satisfied. Furthermore, every order of the facets of a simplex is a shelling order. Thus $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is necessarily the beginning segment of a shelling of δF_j. Thus condition (ii) of Definition 1.2.3 is satisfied as long as, for $1 < j \leq s$, the intersection of F_j with the union of the previous facets is nonempty and pure $(k-1)-$dimensional. Thus for simplicial complexes, we can specialize Definition 1.2.3 to the following:

Definition 1.2.4. Let K be a simplicial complex. Then K is **shellable** if the facets of K can be ordered $F_1, F_2, \ldots F_s$ such that for $1 < j \leq s$

$$F_j \bigcap \left(\bigcup_{1 \leq i \leq j-1} F_i \right)$$

is nonempty and pure $(dim(F_j) - 1)$-dimensional.

Note that in the above definition, K need not be a pure simplicial complex though simplicial complex shellability was originally defined only for pure complexes. The above generalization to nonpure simplicial complexes was introduced by Björner and Wachs in [6].

We now turn to the shellability of polytopes.

Definition 1.2.5. A polytope is shellable if and only if the boundary complex $\delta(P)$ is shellable.

In [8], Bruggesser and Mani proved that polytopes are shellable using a shelling order construction commonly known as a **line shelling** or a **rocket shelling**, which is used in Theorem 1.2.8.

Definition 1.2.6. Given a polytope P, a facet $F \subseteq P$ is **visible** from x if for every $y \in F$ the closed line segment $[x, y] \cap P = y$.

Definition 1.2.7. A point $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ is in **general position** (or **admissible**) with respect to a polytope P if $\mathbf{x} \notin \text{aff}(F)$ for every face $F \in P$, where $\text{aff}(F)$ is the affine hull of F.

Theorem 1.2.8 (Proposition 2 [8])**.** *Let $P \subseteq \mathbb{R}^d$ be a d-polytope and let $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ be a point outside P. If $\mathbf{x}$ lies in general position (i.e. not in the affine hull of any facet of P), then the boundary complex δP has a shelling in which the facets of P that are visible from $\mathbf{x}$ come first.*

Shellability is not a topological property in general. In other words, if two complexes A and B are homeomorphic, it is possible that A is shellable and B is not shellable. For example, there exist several triangulations of a 3-ball (and more generally, an n-ball) that are not shellable, while there are other triangulations that are shellable (see [30], for example). Likewise, there are simplicial complexes whose underlying spaces are homeomorphic to the 3-sphere that are not shellable (for example as constructed in [27]). The tetrahedron, on the other hand, is a simplicial complex also homeomorphic to the 3-sphere and it is shellable since it is the boundary complex of a polytope.

In particular, though $\Delta(P(K) - \{\hat{0}\}) = sd(K) \cong K$ for a simplicial complex K, $\Delta(P(K))$ being shellable does not imply K is shellable (see [28]). However, we do have the following result:

Theorem 1.2.9. *A pure, shellable, d-dimensional simplicial complex has the homotopy type of a wedge of d-spheres.*

We also have the following generalization when K is a nonpure and shellable simplicial complex:

Theorem 1.2.10 (Theorem 4.1 [6])**.** *A shellable simplicial complex has the homotopy type of a wedge of spheres (in varying dimensions), where for each i, the number of i-spheres is the number of i-facets whose entire boundary is contained in the union of the earlier facets.*

In the previous theorem, any i-facet whose boundary is a subcomplex of the union of earlier facets is called a **homology facet**. These are the facets whose attachments change the homology of the complex. This is most easily seen (and Theorem 1.2.10 is most easily proved) with discrete Morse theory (see Section 1.6).

Homotopy type, however, is a topological property and is thus preserved under homeomorphism. Since a simplicial complex (or regular CW complex) K is homeomorphic to $sd(K)$, if $\Delta(P(K) - \hat{0})$ is shellable, then K has the homotopy type of a wedge of spheres.

In [4], Björner introduces a notion of shellability for d-CW complexes which we will reference in Chapter 3. It is the following:

Definition 1.2.11. An ordering $\sigma_1, \sigma_2, \ldots \sigma_n$ of the d-cells of a d-CW complex K is a **shelling** if $d = 0$ or if $d > 0$ and

(i) $\delta\sigma_1$ has a shelling

(ii) $\delta\sigma_j \cap (\cup_{i<j} \delta\sigma_i)$ is a $(d-1)$-CW complex for $j = 2, 3, \ldots n$

(iii) $\delta\sigma_j$ has a shelling in which the $(d-1)$-cells of $\delta\sigma_j \cap (\cup_{i<j} \delta\sigma_i)$ come first for $j = 2, 3, \ldots n$.

1.2.1 Poset shellability

We now turn to the idea of poset shellability.

Definition 1.2.12. A poset P is said to be **shellable** if its order complex $\Delta(P)$ is shellable.

One tool used to prove that a poset is shellable is lexicographic shellability, of which there are several flavors. Each requires us to come up with an edge labeling (or chain-edge labeling) of the cover relations in the Hasse diagram of the poset and this labeling must satisfy certain conditions.

Let P be a graded poset and let $E(P)$ be the set of edges in the Hasse diagram of P. An **edge labeling** of P is a map $\lambda : E(P) \to Q$ where Q is some poset (usually the set of integers). Each saturated chain of length k of P corresponds to a **label sequence** of length k. In particular, if m is the saturated chain $x_0 \lessdot x_1 \ldots \lessdot x_k$, then we associate to m the label sequence $\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots \lambda(x_{k-1}, x_k)$ where $\lambda(x_{i-1}, x_i)$ represents the label on the edge from x_{i-1} to x_i in the Hasse diagram of P. If $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \ldots \leq \lambda(x_{k-1}, x_k)$ then we say the chain is **increasing**. We can order the maximal chains of P by lexicographically ordering the label sequences associated to the maximal chains.

Definition 1.2.13. An edge labeling is called an **EL-labeling** (or edge lexicographical labeling) if for every interval $[x, y] \in P$

(i) there is a unique increasing maximal chain c in $[x, y]$ and

(ii) the label sequence associated to c lexicographically precedes the label sequences associated to all other maximal chains in $[x, y]$.

If a graded poset P has such a labeling, then P is said to be **EL-shellable**.

The following result of Björner's from [3] is a fundamental result in topological combinatorics and poset topology.

Theorem 1.2.14 (Theorem 2.3 [3])**.** *If P is a graded poset with an EL-labeling, then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$.*

In other words, if P is EL-shellable, then P is shellable.

In [6] and [7], Björner and Wachs introduce the idea of shellability for nonpure posets. In this case, we simply drop the requirement that P is pure in the definitions for edge-labelings.

Example 1.2.15. Let B_n be the Boolean algebra that has as its elements subsets of $[n]$. It is a poset (in particular a lattice) with order given by subset containment. A natural EL-labeling of B_n is to label the cover relation $S \lessdot T$ by the unique element $i \in [n]$ such that $i \in T \cap ([n] \backslash \{S\})$. For any interval $[S, U]$, the unique increasing chain that is lexicographically first is the chain with the label sequence $i_1, i_2, i_3, \ldots, i_k$ where i_j is the jth least element in U that is not in S. The two shaded facets in Figure 1.4 come first in the shelling order of $\Delta(B_3)$.

We now define the notion of CL-shellability, a more general version of EL-shellability and one that figures prominently in this thesis. For a graded poset P of length n, let $E^*(P)$ be the set of edges of maximal chains in the Hasse diagram of P i.e. $E^*(P) = \{(c, x, y) : c \text{ is a maximal chain}, x, y \in c, x \lessdot y\}$.

Definition 1.2.16. Let P and Q be posets. A **chain-edge labeling** of P (or CE-labeling) is a map $\lambda : E^*(P) \to Q$, that satisfies the following condition: If two maximal chains coincide along their first d edges, then their labels also coincide along these edges. In other words, if c is the chain $\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_n = \hat{1}$ and c' is the chain $\hat{0} = x'_0 \lessdot x'_1 \lessdot \ldots \lessdot x'_n = \hat{1}$, then $\lambda(c, x_{i-1}, x_i) = \lambda(c', x'_{i-1}, x'_i)$ whenever $x_i = x'_i$ for $i = 0, 1, \ldots, d$.

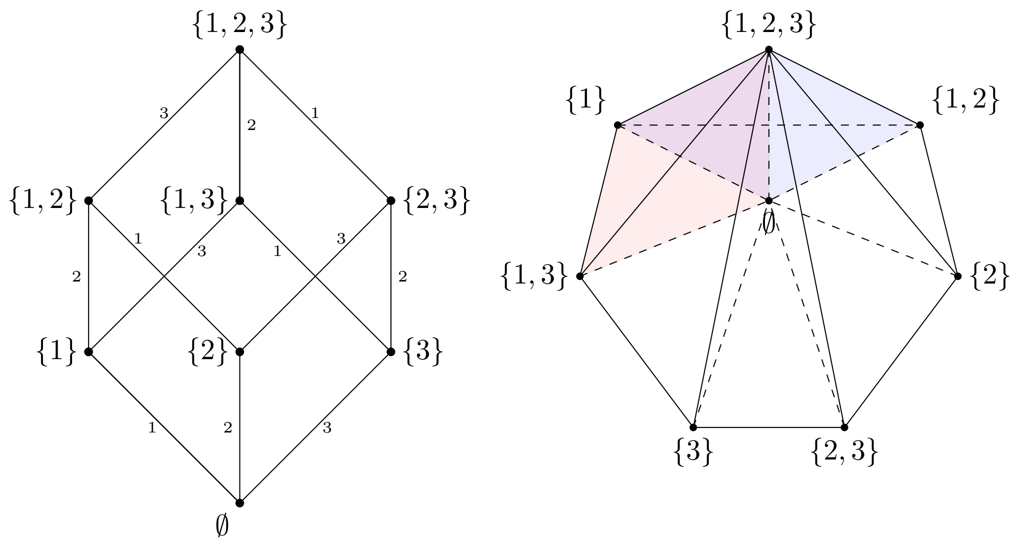


Figure 1.4: On the left, the Boolean lattice B_3 with the EL-labeling described in Example 1.2.15. On the right, the order complex of B_3 with the facets corresponding to the lexicographically first two maximal chains shaded.

Definition 1.2.17. If $[x, y]$ is an interval in P and r is a saturated chain from $\hat{0}$ to x, then the pair $([x, y], r)$ is called a **rooted interval** with root r. It is denoted $[x, y]_r$.

As with edge labelings, given a chain-edge labeling λ of P, each maximal chain of P can be associated with a label sequence. If $m = (\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_n = \hat{1})$ the corresponding label sequence is $\sigma(m) = \lambda(m, x_0, x_1), \lambda(m, x_1, x_2), \ldots \lambda(m, x_{n-1}, x_n)$. By definition of a chain-edge labeling, any maximal chain containing the root r from $\hat{0}$ to x and the saturated chain c in the interval $[x, y]$ will associate the same label sequence to c, which we denote $\sigma_r(c)$. Note that c may be associated to a different label sequence when contained in a maximal chain with a different root r'. This brings us to the following definition:

Definition 1.2.18. A maximal chain c in a rooted interval $[x, y]_r$ is **increasing** if the word $\sigma_r(c)$ is increasing.

Definition 1.2.19. A chain-edge labeling λ is called a **CL-labeling** (chain lexicographical labeling) if for every rooted interval $[x, y]_r$ in P,

(i) there is a unique increasing maximal chain c in $[x, y]_r$ in P and

(ii) the word associated to c lexicographically precedes the words associated to all other maximal chains in $[x, y]_r$.

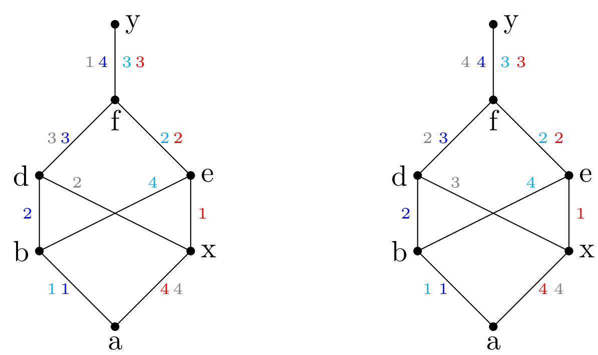


Figure 1.5: A CE-labeling (left) and a CL-labeling (right) of a poset P.

If a graded poset P admits such a labeling, then P is said to be **CL-shellable**.

Example 1.2.20. Figure 1.5 contains two copies of the poset P, which has 4 maximal chains. The labeling of P on the left is a CE-labeling, but it is not a CL-labeling because given root $r = a \lessdot x \lessdot d$, $[d, y]_r$ has no increasing maximal chain. The labeling of P on the right is a CL-labeling.

Note that EL-shellability implies CL-shellability but not conversely. Both EL-shellability and CL-shellability imply shellability.

If P is CL-shellable, then taking maximal chains of P in lexicographic order according to the CL-labeling gives a shelling order of $\Delta(P)$. Since $\Delta(P) = \Delta(P^*)$, it suffices to find a CL-labeling of P^* in order to find a shelling of $\Delta(P)$. A CL-labeling of P^* is called a **dual CL-labeling** and a poset that admits such a labeling is called **dual CL-shellable**.

One helpful way of showing that a poset is CL-shellable is to instead find a recursive atom ordering. The notion of recursive atom (and coatom) orderings was introduced by Björner and Wachs in [5] and reflects the recursive nature of shellability.

Definition 1.2.21. A **recursive atom ordering** of a graded poset P is an ordering $a_1, a_2, \ldots a_t$ of the atoms of P satisfying:

(i) For all $j = 1, \ldots, t$, $[a_j, \hat{1}]$ admits a recursive atom ordering. For $j \neq 1$, the atoms that come first in the ordering are those covering some a_k for $k < j$.

(ii) For all $i < j$ and $y > a_i, a_j$, there exists a $k < j$ and an element z such that $z \gtrdot a_k, a_j$ and $y > z$.

Definition 1.2.22. A poset P admits a **recursive coatom ordering** if its dual poset P^* admits a recursive atom ordering.

The following theorem of Björner and Wachs from [5] makes explicit the connection between recursive atom orderings and CL-shellable posets.

Theorem 1.2.23 (Theorem 3.2 [5])**.** *A graded poset P admits a recursive atom ordering if and only if P is CL-shellable.*

It immediately follows that P admits a recursive coatom ordering if and only if P is dual CL-shellable.

In [4], Björner proves the following result about the shellability of d-CW complexes (see Definition 1.1.21 and Definition 1.2.11):

Proposition 1.2.24 (Proposition 4.2 [4])**.** *A d-CW complex K is shellable if and only if its augmented face poset $\hat{P}(K)$ is dual CL-shellable.*

Another even more general version of poset shellability is called EC-shellability. This method, along with the even more general notion of CC-shellability, was developed by Kozlov in [20]. We use a characterization and language introduced by Hersh in [14] to describe this idea.

Definition 1.2.25. Let λ be an edge labeling on the cover relations of a poset. Let $u \lessdot v \lessdot w$. A **topological ascent** is a pair of labels $(\lambda(u,v),\ \lambda(v,w))$ that is lexicographically earlier than all other pairs of labels on all other saturated chains from u to w. If $(\lambda(u,v), \lambda(v,w))$ is not a topological ascent, then it is a **topological descent**. If c is a saturated chain consisting entirely of topological ascents, then we say that c is **topologically ascending**.

For the following two definitions, we suppose that Q is an arbitrary poset.

Definition 1.2.26. An **EC-labeling** of P is an edge labeling $\lambda : E(P) \to Q$ on the cover relations of P such that every interval $[x, y]$ has a unique saturated chain consisting entirely of topological ascents. If P has an EC-labeling, then P is said to be **EC-shellable**.

Definition 1.2.27. A **CC-labeling** of a poset P is a chain-edge labeling $\lambda : E^*(P) \to Q$ such that every rooted interval $[u, v]_r$ has a unique saturated chain consisting entirely of topological ascents. If P admits a CC-labeling, then P is said to be **CC-shellable**.

As with EL-shellable and CL-shellable posets, an EC-shellable (respectively CC-shellable) poset P is shellable. The shelling order on the facets of $\Delta(P)$ is given by taking the maximal chains in lexicographic order according to their label sequences. The previous definition of CC-shellability by Hersh (from [14]) is a reformulation of the original definition first introduced by Kozlov in [20]. We also include Kozlov's original definition, though we will typically use the language in Definition 1.2.27 when discussing CC-shellability in this thesis.

Definition 1.2.28 (Definition 3.6 [20])**.** A **CC-labeling** of a poset P is a chain-edge labeling $\lambda : E^*(P) \to Q$ such that in any interval

i) all maximal chains have different sequences of labels and

ii) for any maximal chain c in the rooted interval $[u,v]_r$ and any $x, y \in c$ such that $u < x < y < v$, if $c\,|_{[u,y]}$ is lexicographically first in $[u,y]_r$ and $c\,|_{[x,v]}$ is lexicographically first in $[x,v]_{r'}$, then c is lexicographically first in $[u,v]_r$, where $r' = r \cup c\,|_{[u,x]}$.

Definition 1.2.29. Let P be a poset and P^* its dual poset. If P^* admits an EC-labeling (respectively CC-labeling), then P is said to be **dual EC-shellable** (respectively **dual CC-shellable**).

In Chapter 4, we develop a new type of atom ordering of the poset P. As we will see, P admits this new atom ordering if and only if P is CC-shellable, just as P admits a recursive atom ordering if and only if P is CL-shellable.

1.3 The Möbius function of posets

One characteristic of a poset P is its Möbius function μ_P. The Möbius function of P is useful in part because it gives us the reduced Euler characteristic of the order complex of P as we will see shortly in Proposition 1.3.3. For more information on the Möbius function, see [26].

Definition 1.3.1. The Möbius function μ_p of a poset P is defined recursively on intervals of P as follows:

$$\begin{aligned} \mu(x,x) &= 1 \quad \text{for all} \quad x \in P \\ \mu(x,y) &= -\sum_{x \le z < y} \mu(x,z) \quad \text{for all} \quad x < y \in P \end{aligned}$$

Definition 1.3.2. The reduced Euler characteristic of the simplicial complex Δ is

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{dim\Delta} (-1)^i f_i(\Delta)$$

where $f_i(\Delta)$ is the number of $i-$faces of Δ.

Proposition 1.3.3 (Proposition 3.8.5 [26]; Philip Hall's Theorem)**.** *For any poset P, let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$, even if P already has a top and bottom element. Let c_i be the number of chains $\hat{0} = t_0 < t_1 < \ldots < t_i = \hat{1}$ of length i between $\hat{0}$ and $\hat{1}$. Then*

$$\mu(\hat{P}) = c_0 - c_1 + c_2 - c_3 + \ldots = \tilde{\chi}(\Delta(P))$$

The Möbius function of P arises as counting coefficients in inclusion-exclusion formulas. For example, in lexicographically shellable posets, the Möbius function counts maximal chains with certain labelings. See [3] for more on the connection between lexicographic shellability and the Möbius function.

Definition 1.3.4. Given a poset P with EL-labeling λ and a chain $x \lessdot y \lessdot z$, there is a **descent** at y if $\lambda(x, y) > \lambda(y, z)$. A chain $c = x_0 < x_1 < \ldots < x_n$ has **descent set** $D(c) = \{i \in [n-1] : \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\}$.

Definition 1.3.5. Let P be a graded poset of length n. For any subset $S \subseteq \{1, \ldots, n-1\}$, the **rank-selected subposet** P_S is defined as $P_S = \{x \in P : \rho(x) \in S \cup \{0, n\}\}$, where $\rho(x)$ denotes the rank of the element $x \in P$.

Theorem 1.3.6 (Theorem 2.7 [3]). *Let P be a graded poset of length n with an EL-labeling. Then $\mu(P_S) = (-1)^{|S|+1} \cdot \#\{$maximal chains in P with descent set $S\}$. In particular, $\mu(\hat{P}) = (-1)^{rk(P)} \cdot \#\{$maximal decreasing chains in $P\}$.*

Finally, the Möbius function can be used to define Eulerian posets, which are generalizations of face posets of convex polytopes. See [25] and [26] for more on Eulerian posets.

Definition 1.3.7. A finite graded poset P is said to be **Eulerian** if $\mu_P(s, t) = (-1)^{l(s,t)}$ for $s \leq t$, where $l(s, t)$ is the length of the interval $[s, t]$.

Proposition 1.3.8. *If P is a CW poset and has a unique maximal element, then P is Eulerian.*

1.4 Toric cubes

The initial question motivating this research was about the shellability of toric cubes. In this section, we present some preliminaries on toric cubes, which were first introduced by Engström, Hersh, and Sturmfels in [10]. We describe a CW decomposition of this family of topological spaces also from [10]. Toric cubes can be defined as a subset of the standard n-cube satisfying a collection of binomial inequalities and closure requirements. Alternatively, they can be defined as the image of a d-cube under a monomial map i.e. a map whose coordinates are given by monomials in d variables. We present the former definition first, though we will use the latter definition when we consider in more detail those toric cubes defined by trees.

Definition 1.4.1. A **binomial inequality** is an inequality of the form

$$x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \leq x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

where for $1 \leq i \leq n$, x_i are variables and c_i, d_i are nonnegative integers.

Definition 1.4.2. A **toric precube** is a subset of the cube $[0,1]^n$ that satisfies a finite set $\mathcal{B}$ of binomial inequalities.

Definition 1.4.3. A **toric cube** is a toric precube B such that the closure of the strictly positive points of B is equal to B. In other words, $B = \overline{B \cap (0,1]^n}$.

It is proven in [10] that toric cubes can alternatively be defined as the images of monomial maps with variables restricted to $[0,1]$. Let $A = (a_{i,j}) \in \mathbb{Z}^{n \times d}$ and $\mathbf{x} \in [0,1]^d$. Define $A\mathbf{x} = (\mathbf{x}^{a_1}, \mathbf{x}^{a_2}, \ldots, \mathbf{x}^{a_n})$ where $\mathbf{x}^{a_i} = x_1^{a_{i,1}} x_2^{a_{i,2}} \ldots x_d^{a_{i,d}}$ is a monomial in d variables.

Theorem 1.4.4 (Theorem 1 [10]). *Toric cubes are the images of d-cubes under a monomial map*

$$\begin{aligned} f : [0,1]^d &\to [0,1]^n \\ (x_1, x_2, \ldots, x_d) &= (\mathbf{x}^{a_1}, \mathbf{x}^{a_2}, \ldots \mathbf{x}^{a_n}) \\ &= (m_1, m_2, \ldots m_n) \end{aligned}$$

As shown by Engström, Hersh, and Sturmfels in [10], every toric cube can be decomposed as a CW complex. Let $f : [0,1]^d \to [0,1]^n$ be a monomial map. For $(m_1, m_2, \ldots m_n) \in (0,1]^n$, define

$$\begin{aligned} g : (0,1]^n &\to \mathbb{R}^n \\ (m_1, m_2, \ldots m_n) &\mapsto (-\ln(m_1), -\ln(m_2), \ldots, -\ln(m_d)) \end{aligned}$$

The image of $g \circ f \mid_{(0,1]^d}$ is a cone, C. In particular it is the conical hull of the set of rays $\{\lambda r_k : \lambda \geq 0\}_{1 \leq k \leq d}$, where $r_k = (a_{1,k}, a_{2,k}, \ldots, a_{n,k})$. A cross section of C is a polytope T with vertices given as vectors r_k in $\mathbb{R}^d$. If T is simplicial, then $g \circ f$ is a homeomorphism from $(0,1]^n$ onto C. In this case, f suffices as the characteristic map for the $n-$ball in a CW decomposition of the toric cube given by f. More commonly, however, C is not simplicial in which case f is not injective. In this case, subdividing the cone C into smaller, simplicial cones (as first shown in [10] and outlined below) yields a new map $sd(f)$ that serves as the characteristic map for the $n-$ball.

Let $P(T)$ be the face poset of T. If σ_j is a face of T, it is the convex hull of some subset of vectors $\{r_{j_1}, r_{j_2}, \ldots r_{j_k}\}$. Let $\{j_1, j_2, \ldots j_k\}$ denote the element in $P(T)$ corresponding to this face σ_j of T.

Construct a new complex C' of simplicial cones (i.e. cones with simplicial cross section) as follows. For $\sigma \in P(T) \backslash \emptyset$, define $r'_\sigma = \Sigma_{i \in \sigma} r_i$. Note that the barycenter of $\sigma \in T$ is $\lambda r'_\sigma$ for some $\lambda \in \mathbb{R}$. We let $r'_\emptyset = 0$ as a convention. The elements in a maximal chain c of $P(T) \backslash \emptyset$ correspond to a set of vectors $\{r'_\sigma\}_{\sigma\in c}$. The vectors $\{r'_\sigma\}_{\sigma\in c}$ are the vertices of a $(d-1)$-simplex, S_c, since the barycentric subdivision of a polytope T yields a triangulation of T. Hence

$$\bigcup_{c\in P(T)\backslash\emptyset} S_c \cong T$$

For $\sigma \in T$, consider the ray $R'_\sigma = \{\lambda r'_\sigma : \lambda \geq 0\}$. Since for any maximal chain $c \in P(T)\backslash\emptyset$, $\{r'_\sigma\}_{\sigma\in c}$ are the vertices of a $(d-1)$-simplex, $\{R'_\sigma\}_{\sigma\in c}$ is the set of extremal rays of a simplicial d-cone. Call the union of all such simplicial cones C'. Then C' is a collection of simplicial cones such that $C' \cong C$.

Example 1.4.5. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be a monomial map defined by

$$f(x_1, x_2, x_3, x_4, x_5) = (x_1x_2x_3, x_3x_4x_5, x_1^2x_2x_4x_5^2)$$

Note that f is not injective. For example, $f(e^{-1}, 1, 1, e^{-1}, 1) = f(1, e^{-1}, 1, 1, e^{-1})$, though $(e^{-1}, 1, 1, e^{-1}, 1) \neq (1, e^{-1}, 1, 1, e^{-1})$ in $\mathbb{R}^5$. The monomial map $f : [0,1]^5 \to [0,1]^3$ can be given using a matrix A as :

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \end{bmatrix}}_{\text{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$
$$= \begin{pmatrix} x_1x_2x_3 \\ x_3x_4x_5 \\ x_1^2x_2x_4x_5^2 \end{pmatrix}$$

The image of $f\mid_{(0,1]^5}$ is homeomorphic to

$$\begin{aligned} g \circ f\mid_{(0,1]^5} &= -\ln(f\mid_{(0,1]^5}) \\ &= l_1(1,0,2) + l_2(1,0,1) + l_3(1,1,0) + l_4(0,1,1) + l_5(0,1,2) \end{aligned}$$

where $l_i = -\ln(x_i)$. In particular, the image of $C = g \circ f\mid_{(0,1]^5}$ is a (non-simplicial) cone with a pentagonal cross section. The vertices of a cross section of C are $r_1 = (1,0,2)$, $r_2 = (1,0,1)$, $r_3 = (1,1,0)$, $r_4 = (0,1,1)$, and $r_5 = (0,1,2)$. Note that r_i is given by the ith column of A.

The face poset (with the empty set removed) of this pentagonal cross section is given in Figure 1.6. As an example, consider the maximal chain $\{2\} \lessdot \{2,3\} \lessdot \{1,2,3,4,5\}$. This maximal

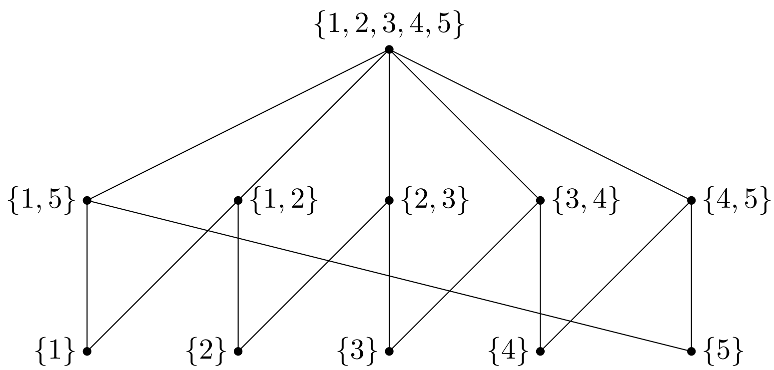


Figure 1.6: The poset $P(T)\backslash\emptyset$ of the cross section T of the non-simplicial cone as in Example 1.4.5.

chain corresponds to a simplicial cone that, for all $\lambda \geq 0$, has as extremal rays:

$$\begin{aligned}
R_{\{2\}} &= \lambda r'_{\{2\}} \\
&= \lambda(1,0,1) \\
&= \lambda(r_2) \\
R_{\{2,3\}} &= \lambda r'_{\{2,3\}} \\
&= \lambda(2,1,1) \\
&= \lambda(r_2 + r_3) \\
R_{\{1,2,3,4,5\}} &= \lambda r'_{\{1,2,3,4,5\}} \\
&= \lambda(3,3,6) \\
&= \lambda(r_1 + r_2 + r_3 + r_4 + r_5)
\end{aligned}$$

In general, $sd(f) : [0,1]^{12} \to [0,1]^3$ is given by the matrix A' shown below, where the column corresponding to $t_{j_1,j_2,\ldots j_k}$ is $[r_{j_1}+r_{j_2}+\ldots+r_{j_k}]^T$.

$$sd(f)\begin{pmatrix} t_{\{1\}} \\ t_{\{2\}} \\ t_{\{3\}} \\ t_{\{4\}} \\ t_{\{5\}} \\ t_{\{1,2\}} \\ t_{\{1,5\}} \\ t_{\{2,3\}} \\ t_{\{3,4\}} \\ t_{\{4,5\}} \\ t_{\{1,2,3,4,5\}} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 1 & 1 & 3 & 6 \end{bmatrix}}_{\text{A}} \begin{bmatrix} t_{\{1\}} \\ t_{\{2\}} \\ t_{\{3\}} \\ t_{\{4\}} \\ t_{\{5\}} \\ t_{\{1,2\}} \\ t_{\{1,5\}} \\ t_{\{2,3\}} \\ t_{\{3,4\}} \\ t_{\{4,5\}} \\ t_{\{1,2,3,4,5\}} \end{bmatrix}$$

$$= \begin{pmatrix} t_{\{1\}}t_{\{2\}}t_{\{3\}}t^2_{\{1,2\}}t^2_{\{1,5\}}t^2_{\{2,3\}}t_{\{3,4\}}t^3_{\{1,2,3,4,5\}} \\ \\ t_{\{3\}}t_{\{4\}}t_{\{5\}}t_{\{1,5\}}t^2_{\{2,3\}}t^2_{\{3,4\}}t^2_{\{4,5\}}t^3_{\{1,2,3,4,5\}} \\ \\ t^2_{\{1\}}t_{\{2\}}t_{\{4\}}t^2_{\{5\}}t^3_{\{1,2\}}t^4_{\{1,5\}}t_{\{2,3\}}t^3_{\{3,4\}}t^3_{\{4,5\}}t^6_{\{1,2,3,4,5\}} \end{pmatrix}$$

$$= \begin{pmatrix} x_1x_2x_3 \\ x_3x_4x_5 \\ x_1^2x_2x_4x_5^2 \end{pmatrix} = f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

1.4.1 The edge-product space of phylogenetic trees and the Tuffley poset

One class of toric cube consists of those whose monomial maps are determined by trees. As these types of toric cubes have CW decompositions whose face posets are already known and are easily described, we restrict our attention to this class for this thesis. Given a tree T, each point of the toric cube derived from T corresponds to a set of weights (which can be viewed as probabilities) between 0 and 1 on the edges of T. Taking the union of all toric cubes determined in this way by trees with leaves in bijection with a finite set X, we construct a topological space that arises in evolutionary biology called the edge-product space of phylogenetic trees (see Definition 1.4.11). A natural question we ask is whether the edge-product space is shellable. In other words, if we adjoin a $\hat{1}$ to the face poset of the edge-product space of phylogenetic trees, is this poset lexicographically shellable? We address this question in Chapter 3. We present the necessary background on the edge-product space of phylogenetic trees in this section. The following definitions and CW complex decomposition were originally presented by Moulton and

Steel in [21].

Let T be a tree with vertex set $V(T)$ and edge set $E(T)$.

Definition 1.4.6. An X**-tree** $\mathcal{T}$ is a pair (T, ϕ) consisting of a tree T and a map $\phi : X \to V(T)$ such that all vertices in $V(T) - \phi(X)$ are of degree greater than 2. An X**-forest** is a set $\mathcal{F} = \{(A, \mathcal{T}_A) : A \in \pi\}$ where π is a set partition of X and $\mathcal{T}_A = (T_A, \phi_A)$ is an A-tree for every block $A \in \pi$.

Definition 1.4.7. Let $\mathcal{T} = (T, \phi)$ be an X-tree and $v \in V(T)$. If $v = \phi(x)$ for some $x \in X$, then we say v is **labeled**. Otherwise v is **unlabeled**.

Given an X-tree $\mathcal{T} = (T, \phi)$, there is a corresponding closed ball $B(\mathcal{T}) = [0,1]^{E(T)}$ and a corresponding open ball $Int(B(\mathcal{T})) = (0,1)^{E(T)}$. We present several maps next which we use to formally define the edge-product space of phylogenetic trees.

Definition 1.4.8. Let $\lambda : E(T) \to [0,1]$ be a map and $\lambda(E(T))$ be the vector in $[0,1]^{E(T)}$ whose e_i entry is $\lambda(e_i)$. For any map $\lambda : E(T) \to [0,1]$, $\lambda(E(T))$ is called an **edge-weight vector**.

For a tree T with leaves labeled by the set X, let p_{xy} denote the set of edges in the unique path between the leaf labeled x and the leaf labeled y in T.

Definition 1.4.9. For $\lambda : E(T) \to [0,1]$ be a map. Define the map $f_{(T,\lambda)} : \binom{X}{2} \to [0,1]$ as

$$\begin{aligned} f_{(T,\lambda)} : \binom{X}{2} &\to [0,1] \\ (x,y) &\mapsto \prod_{e \in p_{xy}} \lambda(e) \end{aligned}$$

Let $f_{(T,\lambda)}\binom{X}{2}$ be the vector in $[0,1]^{\binom{X}{2}}$ whose xy entry is $f_{(T,\lambda)}((x,y))$. For any edge-weight vector λ, call $f_{(T,\lambda)}\binom{X}{2}$ an **edge-product vector**.

See Figure 1.7 for an example of how $f_{(T,\lambda)(x,y)}$ is calculated for a pair of leaves x, y in a tree with leaves labeled by the set $[6]$ and an edge-weighting λ.

Definition 1.4.10. Define Λ_T as the map that sends each edge-weight vector to the corresponding edge-product vector:

$$\begin{aligned} \Lambda_T : [0,1]^{E(T)} &\to [0,1]^{\binom{X}{2}} \\ \lambda(E(T)) &\mapsto f_{(T,\lambda)}\binom{X}{2} \end{aligned}$$

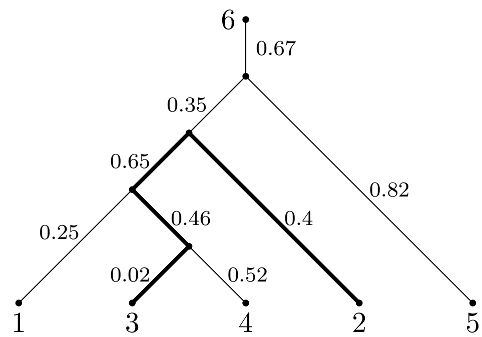


Figure 1.7: A $[6]-$tree $\mathcal{T}$ with an edge-weighting λ. The product of the edge weights on the bold edges gives $f_{(\mathcal{T},\lambda)}(2,3)$.

Definition 1.4.11. Let $\mathcal{E}(X,T)$ be the image of the map Λ_T. Let $\mathbf{T}(X)$ be the set of all trees with leaves labeled bijectively by X. The **edge-product space for trees on** X is

$$\mathcal{E}(X) = \bigcup_{T \in \mathbf{T}(X)} \mathcal{E}(X,T)$$

Several of the above definitions can be naturally extended to forests, as follows: For the $X-$forest $\mathcal{F} = \{(A, \mathcal{T}_A) : A \in \pi\}$ with π a set partition of X,

$$B(\mathcal{F}) = \Pi_{A \in \pi} B(\mathcal{T}_A)$$

and

$$Int(B(\mathcal{F})) = \Pi_{A \in \pi} Int(B(\mathcal{T}_A)).$$

Let $E = \{E(T_A) : (A, \mathcal{T}_A) \in \mathcal{F})\}$ and let $\lambda : E \to [0,1]$ so $\lambda(E) \in [0,1]^E$. Let P be the set of pairs of leaves in $\mathcal{F}$. If we have $x, y \in A$ for some block $A \in \pi$, then we define p_{xy} to be the set of edges in the unique path between the leaf labeled x and the leaf labeled y in $\mathcal{T}_A$. Define the map $\Lambda_{\mathcal{F}}$ as

$$\begin{aligned} \Lambda_{\mathcal{F}} : B(\mathcal{F}) &\to [0,1]^{\binom{X}{2}} \\ \lambda(E) &\mapsto f_{(\mathcal{F},\lambda)}(P) \end{aligned} \tag{1.4.1}$$

where

$$f_{(\mathcal{F},\lambda)} : \binom{X}{2} \to [0,1]$$

$$(x,y) \mapsto \begin{cases} \prod_{e \in p_{xy}} \lambda(e) & \text{if } \exists A \in \pi \text{ with } x,y \in A \\ 0 & \text{otherwise} \end{cases} \tag{1.4.2}$$

The edge-product space of phylogenetic trees has a CW decomposition with face poset that is isomorphic to the Tuffley poset. The Tuffley poset has X-forests as elements and cover relations given by operations on X-forests called edge contraction and safe edge deletion, as we describe next.

Definition 1.4.12. Let u, v be adjacent vertices in the X-forest $\mathcal{F}$. Let $e = [u, v]$ be the edge between them. A **contraction** of the edge e is the elimination of the edge e in $\mathcal{F}$ with the identification of the vertices u and v. If $e \in \mathcal{T}_A = (T_A, \phi_A)$, the resulting vertex is labeled $\phi_A^{-1}(u) \cup \phi_A^{-1}(v)$, i.e. the resulting vertex is labeled by the union of labels on u and v in $\mathcal{F}$.

Definition 1.4.13. Let u, v be adjacent vertices in the X-forest $\mathcal{F}$. Let $e = [u, v]$ be the edge between them. A **deletion** of the edge e is the elimination of edge e in $\mathcal{F}$ with no changes to the vertex set of $\mathcal{T}_A$ for any $\mathcal{T}_A \in \mathcal{F}$. The map of ϕ_A is unchanged for any $(T_A, \phi_A) \in \mathcal{F}$. The deletion of edge $e = [u, v]$ is called a **safe edge deletion** if both u and v are either labeled or of degree greater than 3.

Definition 1.4.14. Let $\mathcal{F} = \{(A, \mathcal{T}_A) : A \in \pi\}$ and $\mathcal{F}' = \{(B, \mathcal{T}_B) : B \in \pi'\}$ be X-forests. Define a partial order $\leq$ on X-forests so that $\mathcal{F}' \leq \mathcal{F}$ if $\mathcal{F}'$ can be obtained from a sequence of contractions and safe deletions of edges of $\mathcal{F}$. The poset $(S(X), \leq)$ is called the **Tuffley poset**.

The following theorem of Moulton and Steel from [21] gives a set of properties of the Tuffley poset $S(X)$, some of which are used extensively throughout the remainder of this thesis.

Theorem 1.4.15 (Theorem 4.2 [21])**.** *Let X be a finite set and $\mathcal{F}, \mathcal{F}' \in S(X)$. Let π be a partition of X and $E(\mathcal{F}), E(\mathcal{F}')$ be the set of edges in the X-forests $\mathcal{F}$ and $\mathcal{F}'$ respectively. The following hold:*

(i) $\mathcal{F}' \leq \mathcal{F}$ if and only if $\mathcal{F}'$ can be obtained from $\mathcal{F}$ by some sequence of contraction and deletion operations. Given this sequence of contraction and deletion operations, $\mathcal{F}'$ can be obtained from $\mathcal{F}$ even if this sequence of operations is reordered so that all contractions occur first and all subsequent deletions are safe.

(ii) $\mathcal{F}'$ is a coatom of $\mathcal{F}$ if and only if $\mathcal{F}'$ can be obtained from $\mathcal{F}$ by a single elementary operation.

(iii) $S(X)$ is pure. The rank of $\mathcal{F} = \{(A, \mathcal{T}_A) : A \in \pi\} \in S(X)$ is $\rho(\mathcal{F}) = |E(\mathcal{F})|$.

(iv) $S(X)$ is thin.

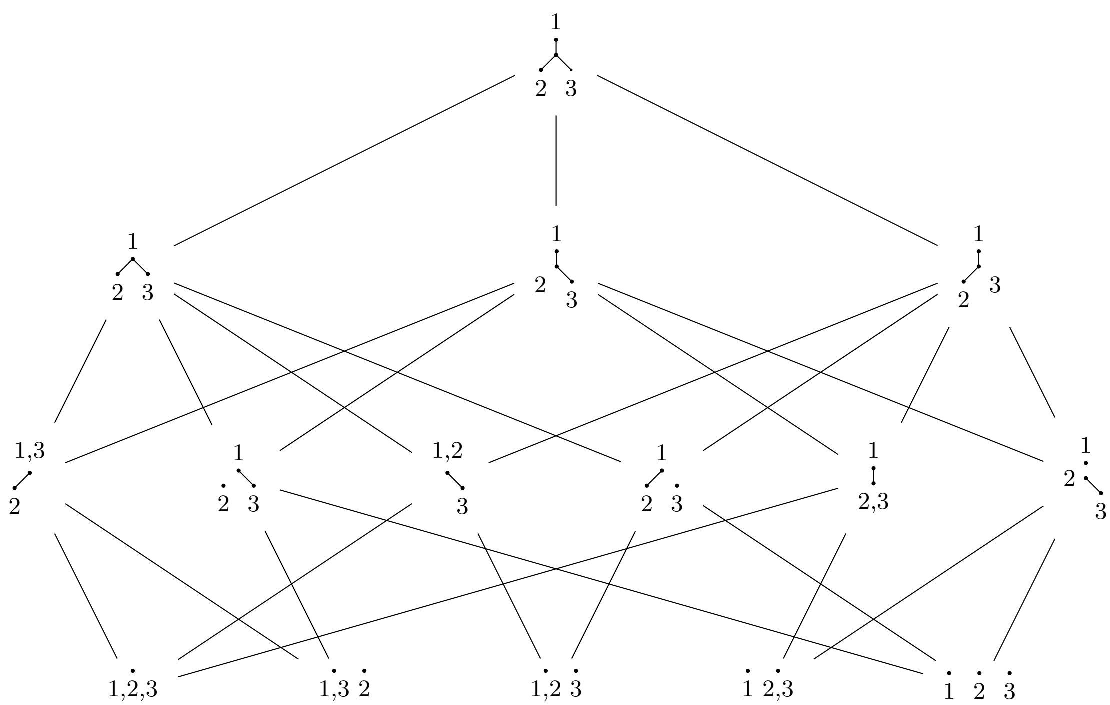


Figure 1.8: The Tuffley poset $S([3])$.

(v) The maximal elements of $S(X)$ are the elements $\mathcal{F}$ for which $\mathcal{F} = \{(X, \mathcal{T})\}$ and $|E(\mathcal{T})| = 2|X| - 3$.

(vi) The minimal elements of $S(X)$ are the elements $\mathcal{F}$ for which $\mathcal{F} = \{(A, \mathcal{T}_A) : A \in \pi\}$ for some partition π of X and $E(\mathcal{T}_A) = \emptyset$ for all $A \in \pi$.

We can use this theorem in particular to determine the intersection of certain maximal elements of $S(X)$, as our next two results show. From Theorem 1.4.15, we know an element x covers an element w (i.e. w is a codimension one face of x) in $S(X)$ if and only if w can be obtained from x by contracting or deleting a single edge. We can say more about the elements covered by a maximal element C_i of $S(X)$, as in the following propositions. Both of these propositions are important to proving results in Chapter 3.

Proposition 1.4.16. *Let C_i be a maximal element of $S(X)$. Any element $w \in S(X)$ covered by C_i is obtained by contracting an edge of C_i.*

Proof. Since C_i is a maximal element of $S(X)$, each internal vertex is of degree exactly 3 and is unlabeled. This means that there are no safe edge deletions in C_i, since deleting any edge results in an unlabeled vertex of degree 2. Thus any element of $S(X)$ covered by C_i is obtained by contracting an edge of C_i. □

The proof of the following lemma describes the full set of maximal elements of $S(X)$ covering the element w, where w is obtained by contracting an edge of the maximal element C_i of $S(X)$.

Lemma 1.4.17. *Let C_i be a maximal element of $S(X)$. If w is obtained from C_i by contracting an internal edge, then two other maximal elements of $S(X)$, C_j and C_k, also cover w. If w is obtained from C_i by contracting a leaf edge, then only C_i covers w.*

Proof. If w is obtained from C_i by contracting an internal edge e, then w has a unique, degree 4, unlabeled vertex v. The maximal element C_j (respectively C_k) is obtained from w by replacing the unique degree 4 vertex, v, with two vertices, v_1 and v_2, and adding a new edge e between them such that v_1 and v_2 both have degree 3 and such that $C_j \neq C_i$ (respectively $C_k \neq C_i$). If v is adjacent to edges e_1, e_2, e_3, and e_4, then e can be added in any of the three following ways:

(i) e_1 and e_2 are adjacent to vertex v_1, and e_3 and e_4 are adjacent to vertex v_2

(ii) e_1 and e_3 are adjacent to vertex v_1, and e_2 and e_4 are adjacent to vertex v_2

(iii) e_1 and e_4 are adjacent to vertex v_1, and e_2 and e_3 are adjacent to vertex v_2

One of the above ways of adding e yields C_i. The other two ways yield C_j and C_k.

If w is obtained from C_i by contracting a leaf edge, then all but one internal vertex of w are of degree 3 and unlabeled. There is precisely one internal vertex v of w that is of degree 2 and labeled. The only maximal element covering w is the element obtained by adding an edge between v and a new leaf, and labeling this new leaf by the label on v in w. This element is C_i. □

The following definition and proposition will be necessary in the proof of Proposition 3.3.13 and ultimately Theorem 3.3.25 in Chapter 3.

Definition 1.4.18. Let w be an X-tree and let x, y be leaves in w. The **distance** between x and y is the number of edges in the unique path between x and y in w. It is denoted $d_w(x, y)$.

Proposition 1.4.19. *Let C_i and C_j be maximal elements of $S(X)$. Let w be an X-tree and $w < C_i, C_j$. For any pair of leaves x, y, we have $d_w(x, y) \leq \min\{d_{C_i}(x, y), d_{C_j}(x, y)\}$.*

Proof. Without loss of generality, assume $\min\{d_{C_i}(x, y), d_{C_j}(x, y)\} = d_{C_i}(x, y)$. If w is an X-tree and $w < C_i$, then w is obtained from C_i by contracting a subset of edges in C_i. Contracting an edge in C_i reduces the distance between some pair of leaves x, y. If $d_w(x, y) > d_{C_i}(x, y)$, then there is no way to contract a subset of edges in C_i to obtain the element w. This contradicts the fact that $w < C_i$. □

We conclude this section by stating the theorem from Moulton and Steel (Theorem 3.3 of [21]) that is the foundation of the questions explored in Chapters 2 and 3.

Theorem 1.4.20 (Theorem 3.3 [21])**.** *The edge-product space $\mathcal{E}(X)$ is a finite CW-complex with cell decomposition $\{\mathbf{B}(\mathcal{F}), \Lambda_{\mathcal{F}}) : \mathcal{F} \in S(X)\}$. The Tuffley poset $(S(X), \leq)$ is isomorphic to the face poset of $\mathcal{E}(X)$ via the map $\mathcal{F} \mapsto \Lambda_{\mathcal{F}}(\mathbf{B}(\mathcal{F}))$.*

1.5 The Tamari lattice

One of the primary results of this thesis is the introduction of the enriched Tamari poset (see Chapter 2, Section 2.2), which can be viewed as an extension of the Tamari lattice. There are many ways to describe the Tamari lattice: we give several here. One natural question is whether the enriched Tamari poset possesses some of the same properties as the Tamari lattice, so we briefly mention some of the interesting properties of the Tamari lattice in this section as well.

Definition 1.5.1. The **Tamari lattice** T_n is defined as the set of all binary bracketings on a fixed sequence of $n+1$ symbols ordered so that $((a, b), c) \lessdot (a, (b, c))$, where a, b, and c are each themselves either symbols or binary bracketings of symbols.

Figure 1.9: A right tree rotation on vertex v in tree T (left) yields tree T' (right).

Remark 1.5.2. Note that the order relation can alternatively be defined by shifting a pair of brackets (or parentheses) to the left instead of to the right, yielding an isomorphic poset.

The Tamari lattice can also be defined in terms of tree rotation operators on full binary trees, as we describe next.

Definition 1.5.3. A **full binary tree** is a tree such that every internal vertex has two children.

In a full binary tree T, any internal vertex v has a **left child** l_v and a **right child** r_v. The left child is the vertex of T adjacent to v that is drawn below and to the left of v. The right child is the vertex of T adjacent to v that is drawn below and to the right of v. The left child (respectively right child) of v is the highest vertex of a maximal subtree of C_i called the **left subtree** (respectively **right subtree**) of C_i, denoted $\downarrow l_v$ (respectively $\downarrow r_v$). It consists of all vertices below l_v (respectively r_v) and all edges between them. The vertex v is called the **parent** of l_v and r_v. Any leaf in either the left subtree or right subtree of the internal vertex v is called a **descendent** of v.

Definition 1.5.4. Let T be a full binary tree. Let v be an internal vertex with left child l_v and right child r_v. A **right tree rotation** replaces vertex v with vertex l_v and replaces vertex r_v with vertex v. The right child of l_v becomes the left child of v, while all other vertex incidences remain the same.

Note that a tree rotation operator maintains the order on the leaves of T. This fact corresponds to the fact that the Tamari lattice can be equivalently defined as an order on bracketings of a fixed sequence of symbols. Figure 1.9 gives an example of right tree rotation.

The characterization of the Tamari lattice in terms of binary trees and tree rotation is closely related to one given by Pallo in [22] and described below.

Definition 1.5.5. The elements of the Tamari lattice T_n are integer $n-$tuples $(w_1, w_2, \ldots, w_n)$ such that

(i) $1 \leq w_i \leq i$ for $1 \leq i \leq n$

(ii) $i - w_i \leq k - w_k$ for $i - w_i + 1 \leq k \leq i$.

If $w_i \geq w'_i$ for all $i = 1, 2, \ldots n$, then $(w_1, w_2, \ldots w_n) < (w'_1, w'_2, \ldots, w'_n)$.

The integer $n-$tuple $(w_1, w_2, \ldots, w_n)$ can be viewed as corresponding to a binary tree T with n internal nodes and with leaves labeled from 1 to $n+1$, where w_i gives the number of leaves in the largest subtree of T for which the leaf labeled i is the rightmost leaf. This sequence is called the **weight sequence** of T. It can also be obtained by labeling each internal vertex v of T by the number of leaves in the left subtree of v and ordering the internal vertices by **inorder**. The inorder on internal vertices of T is $v_1, v_2, \ldots v_n$ and is defined recursively so that if v_i is in the left subtree of v_j then $i < j$ and if v_k is in the right subtree of v_j, then $j < k$. Pallo provides an algorithm in [22] for obtaining all n-tuples $(w_1, w_2, \ldots, w_n)$ corresponding to binary trees. A similar construction is considered in [7], though the sequence of integers is given by the size of the right subtree (instead of the left subtree) of the internal vertices of T ordered by inorder.

Finally, a fourth description of the Tamari lattice is as a poset of triangulations of a convex $(n+2)$-gon, G, ordered by a flip operation that exchanges one diagonal of the triangulation with another. In particular, we label the vertices of G from 0 to $n+1$ clockwise from the top right vertex. We require that vertex labeled 0 and the vertex labeled $n+1$ be connected by a horizontal line (an edge of G) and that all other vertices lie below this horizontal line. Additionally, we require that the vertex labeled i is strictly to the right of the vertex labeled $i+1$, where $0 \leq i \leq n$. To obtain a triangulation of an $(n+2)$-gon G, we draw n distinct chords (also known as diagonals) between vertices across the interior of G so no pair of chords are crossing. Cover relations in this characterization of the Tamari lattice are given by **diagonal flips**. A diagonal flip is an operation on a triangulation T where we remove one chord and reinsert it in the only other possible way that yields a triangulation. See Figure 1.10 for an example. Given two triangulations T and T', $T \lessdot T'$ if T' can be obtained from T by a diagonal flip and if the chord on which the diagonal flip is performed (i.e. the chord that is removed in T and reinserted to obtain T') has a higher slope in T' than in T. See [23] for more on this characterization.

The Tamari lattice has a number of interesting properties. For one, it was shown to be a lattice by Huang and Tamari (in [17]) and by Pallo (in [22]). We give a version of Pallo's proof of this result next.

Theorem 1.5.6 (Corollary to Theorem 2 [22])**.** *The Tamari lattice T_n is a lattice.*

Proof. We will use the weight sequence characterization of the Tamari lattice. It is a bounded poset with $\hat{0} = (1, 2, \ldots, n)$ and $\hat{1} = (1, 1, \ldots 1)$. Given elements $w = (w_1, w_2, \ldots, w_n)$ and

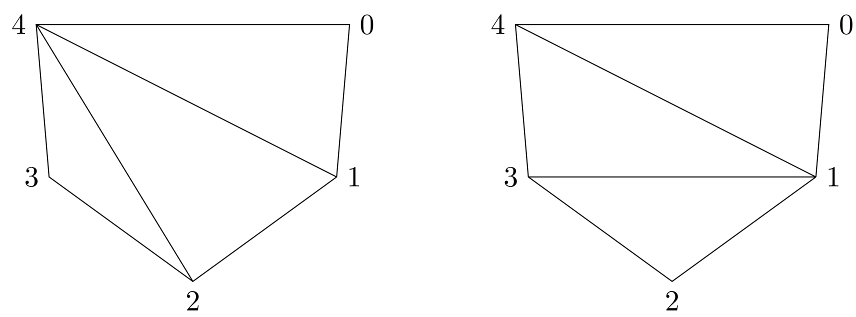


Figure 1.10: A diagonal flip.

$v = (v_1, v_2, \dots v_n)$, we will show $w \vee v = (m_1, m_2, \dots m_n)$ where $m_i = \min\{w_i, v_i\}$. By definition of the cover relations in the Tamari lattice, it is clear that $w < w \vee v$ and $v < w \vee v$. If there were some weight sequence $p = (p_1, p_2, \dots p_n)$ such that $p < w \vee v$ and $p > w, p > v$, then $p_i < m_i$ for some i. However, since $m_i = \min\{w_i, v_i\}$ for all i, this would imply that $p \not> w$ or $p \not> w \vee v$. Thus there can be no $p < w \vee v$ that is a common upper bound of w and v. It remains to show that $w \vee v = (m_1, m_2, \dots m_n)$ is a weight sequence. Since $1 \leq w_i \leq i$ and $1 \leq v_i \leq i$ for $1 \leq i \leq n$, then $1 \leq \min\{w_i, v_i\} = m_i \leq i$. We must now show that if k is such that $i - m_i + 1 \leq k \leq i$, then $i - m_i \leq k - m_k$. Without loss of generality, assume that $m_i = w_i$. Then for k such that $i - m_i + 1 \leq k \leq i$, we have $i - m_i = i - m_k \leq k - w_k$. If $w_k = m_k$, then we have $i - m_i \leq k - m_k$. If $v_k = m_k$, then $k - w_k \leq k - v_k = k - m_k$. Thus we have $i - m_i \leq k - w_k \leq k - m_k$. Thus $m = (m_1, m_2, \dots m_n)$ is a weight sequence. □

Theorem 1.5.7. *The number of elements in the Tamari lattice T_n is the Catalan number $C_n = \frac{1}{2n+1}\binom{2n}{n}$.*

One especially interesting property is that the Tamari lattice is nonpure shellable, as shown by Björner and Wachs. Using Pallo's characterization of the Tamari lattice, we include a version of Björner's and Wachs' labeling and the proof that it is a CL-labeling below, after we introduce two lemmas that will be helpful in the proof.

Lemma 1.5.8. *Let $w = (w_1, w_2, \dots w_n)$ and $v = (v_1, v_2, \dots v_n)$ be elements of T_n such that $w < v$. Let $D = \{j : w_j \neq v_j\} = \{j_1, j_2, \dots j_d\}$, where $j_1 < j_2 < \dots < j_d$. Suppose for $k < d$, u_k is obtained from w by replacing $w_{j_1}, w_{j_2}, \dots w_{j_k}$ with $v_{j_1}, v_{j_2}, \dots v_{j_k}$ respectively. Then u_k is an element of T_n.*

Proof. We will check that $u_k = (u_{k_1}, u_{k_2}, \dots u_{k_n})$ satisfies the two conditions in Definition 1.5.5. Since $u_{k_i} = w_i$ or $u_{k_i} = v_i$ and w and v satisfy condition (i), then $1 \leq u_{k_i} \leq i$ for $1 \leq i \leq n$. Observe that u_k coincides with v up to v_{j_k} and u_k coincides with w after v_{j_k}. Then

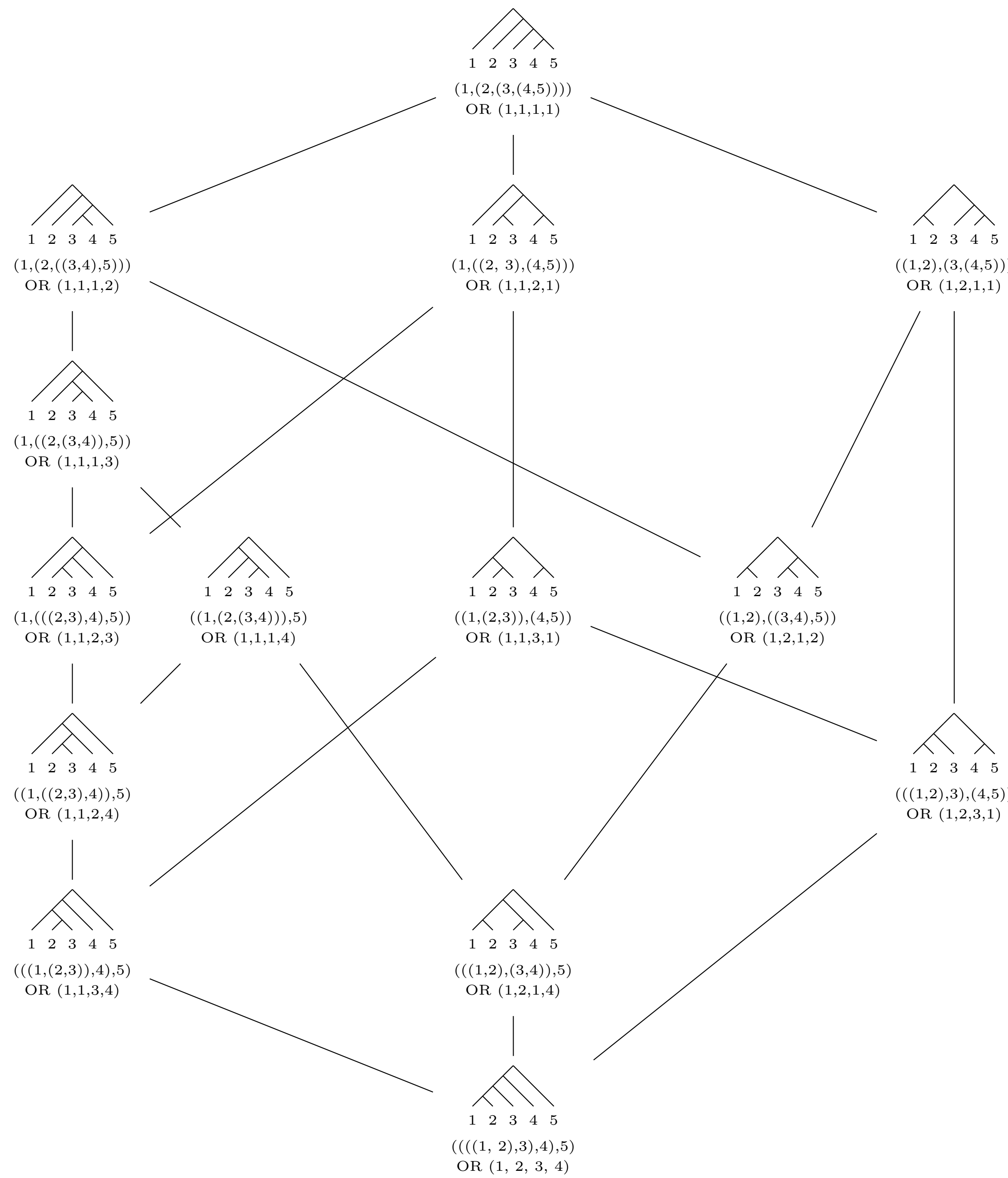


Figure 1.11: The Tamari lattice T_4.

for $1 \leq i \leq j_k$, $i - u_{k_i} = i - v_i \leq l - v_l = l - u_{k_l}$ since v satisfies condition (ii). Now consider i where $j_k + 1 \leq i \leq n$. Observe that if $u_{k_l} = v_l$, then $i - u_{k_i} = i - w_i \leq i - v_i \leq l - v_l$. The first inequality holds since $w_i \geq v_i$ by definition of the order relation and the second inequality holds since v satisfies condition (ii). If, on the other hand, $u_{k_l} = w_l$, then $i - u_{k_i} = i - w_{k_i} \leq l - w_{k_l}$ since w satisfies condition (ii). □

Lemma 1.5.9. *Let* $w = (w_1, w_2, \ldots w_n)$ *and* $v = (v_1, v_2, \ldots v_n)$ *be elements of* T_n *such that* $w \lessdot v$. *Then* $w_i \neq v_i$ *for exactly one* i, *where* $1 \leq i \leq n$.

Proof. Let $i = \min\{j : w_j \neq v_j\}$. Assume k is such that $k > i$ and $w_k \neq v_k$. The element $u = (v_1, v_2, \ldots v_i, w_{i+1}, w_{i+2}, \ldots w_n) \in T_n$ by Lemma 1.5.8. By definition of the order relation in T_n, $w < u < v$. This is a contradiction of $w \lessdot v$. □

Note that the converse is not true. If $w = (w_1, w_2, \ldots w_n)$, $v = (v_1, v_2, \ldots v_n)$ and $w_j \neq v_j$ for exactly one j, we have $w < v$ but it is not necessarily true that $w \lessdot v$.

Theorem 1.5.10 (Theorem 9.2 [7])**.** *The Tamari lattice admits an EL-labeling.*

Proof. Let $w = (w_1, w_2, \ldots w_n)$ and $v = (v_1, v_2, \ldots v_n)$ be elements in T_n and let $w \lessdot v$. By Lemma 1.5.9, there exists a unique i where $1 \leq i \leq n$ such that $w_i \neq v_i$. Define an edge-labeling λ by $\lambda(w \lessdot v) = (i, n - v_i)$, ordered lexicographically.

Let $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$ be elements in T_n such that $r < s$. Let $D = \{i : r_i \neq s_i\} = \{i_1, i_2, \ldots i_d\}$ and let $i_1 < i_2 < \ldots < i_d$. Let r_k be the element in T_n obtained from r by replacing $r_{i_1}, r_{i_2}, \ldots r_{i_k}$ with $s_{i_1}, s_{i_2}, \ldots s_{i_k}$ respectively. By Lemma 1.5.8, r_k is an element of T_n.

Consider the (not necessarily saturated) chain $r = r_0 < r_1 < r_2 < \ldots < r_{d-1} < r_d = s$. Since r_{i-1} and r_i coincide except at one coordinate, the interval $[r_{i-1}, r_i]$ must consist of a single, saturated chain. Since the coordinate at which r_{i-1} and r_i differ must decrease as we move up the chain, the label sequence on the chain is increasing. Concatenating the chains $[r_{i-1}, r_i]$ for $1 \leq i \leq d$ yields a saturated chain m whose label sequence is increasing.

Assume there is another increasing, maximal chain $m' \neq m$ in the interval $[r, s]$. Any increasing chain from r to s must be obtained by first choosing elements that differ from r only in coordinate i_1. Any element that coincides with r except with a smaller entry at coordinate i_1 must be in the interval $[r, r_1]$. Since there is only one saturated chain in the interval $[r, r_1]$, m and m' must coincide on this interval. Any increasing chain from r_1 to s must be obtained by first choosing elements that differ from r_1 only in coordinate i_2. Any element that coincides with r_1 except with a smaller entry at coordinate i_2 must be in the interval $[r_1, r_2]$. Again, since there is only one saturated chain in the interval $[r_1, r_2]$, m and m' must coincide on this interval. Continuing in this way, we see that m and m' coincide in their entirety. This contradicts $m \neq m'$. Thus there is a unique, increasing maximal chain in the interval $[r, s]$. □

This result implies that open intervals in the Tamari lattice are either contractible or homotopy equivalent to a sphere and hence the Möbius function of any interval $[r, s]$ is $\mu(r, s) \in \{-1, 0, 1\}$.

1.6 Discrete Morse theory

Discrete Morse theory was developed by Robin Forman as a way to help determine the topology of CW complexes. In [9], Chari introduced a combinatorial method for utilizing the principal results of discrete Morse theory in the special case of regular CW complexes. It is well known that a lexicographically shellable poset has open intervals that correspond to cells that are either contractible or isomorphic to spheres, but it is is much more straightforward to prove this fact using discrete Morse theory, as we will see in Corollary 1.6.8. We also use discrete Morse theory to attempt to generalize some results on shellable simplicial complexes in Section 4.2. Note that although Forman developed these tools for all CW complexes, we will only consider those CW complexes that are regular.

Definition 1.6.1. Let K be a regular CW complex and σ^d denote a cell of dimension d of K. A function $f : K \to \mathbb{R}$ is a **discrete Morse function** if for every cell σ^d,

(i) $|A| = |\{\tau^{d-1} \subseteq \sigma^d : f(\tau^{d-1}) \geq f(\sigma^d)\}| \leq 1$

(ii) $|B| = |\{\tau^{d+1} \supseteq \sigma^d : f(\tau^{d+1}) \leq f(\sigma^d)\}| \leq 1$

Since these conditions apply to all cells in K, then if either A or B has cardinality 1, the other necessarily has cardinality 0.

Example 1.6.2. The cell complex on the left of Figure 1.12 is labeled with a discrete Morse function. The same cell complex on the right is labeled with a function that fails to be a discrete Morse function because the 2-cell (labeled 4) has two lower-dimensional cells in its boundary that are labeled with labels greater than or equal to 4.

Definition 1.6.3. Given a discrete Morse function f, a cell σ^d is a **critical cell** with respect to f if both

(i) $|A| = |\{\tau^{d-1} \subseteq \sigma^d : f(\tau^{d-1}) \geq f(\sigma^d)\}| = 0$

(ii) $|B| = |\{\tau^{d+1} \supseteq \sigma^d : f(\tau^{d+1}) \leq f(\sigma^d)\}| = 0$

Otherwise σ^d is called **non-critical**.

One of the key results arising from discrete Morse theory is the following by Forman from [11].

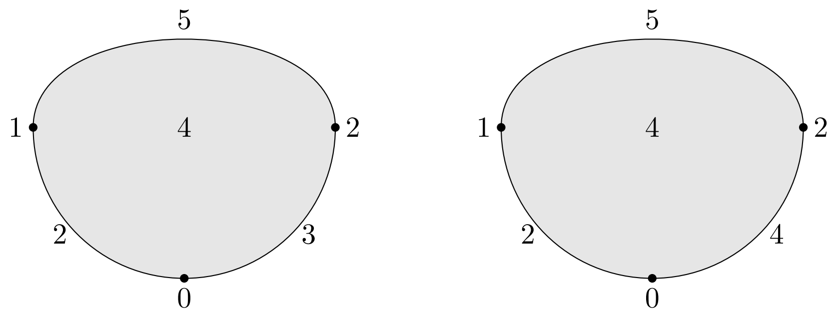


Figure 1.12: A cell complex with a discrete Morse function (left) and the same complex with a function that fails to be a discrete Morse function (right).

Theorem 1.6.4 (Corollary 3.5 [11])**.** *Let K be a regular CW complex with a discrete Morse function. Then K is homotopy equivalent to a CW complex M that has a d-dimensional cell for every $d-$dimensional critical cell of K and no other cells.*

Observe that non-critical cells come in pairs σ^d, τ^{d+1} where σ^d and τ^{d+1} are critical because $f(\sigma^d) > f(\tau^{d+1})$. A discrete Morse function therefore induces a natural matching on the non-critical cells of K. Given any matching on pairs of codimension 1 cells of K, we will represent this matching on the Hasse diagram of the face poset of K by directing those edges corresponding to a matching up and directing all other edges down. The result is a directed graph.

Definition 1.6.5. A matching is **acyclic** if the directed graph obtained from the Hasse diagram has no directed cycles.

Note that if a matching is obtained from a discrete Morse function, then it is always acyclic. This is because as we follow each directed edge, the discrete Morse function decreases. This is impossible if we were to start and end at the same cell. Also note that any acyclic matching corresponds to a matching that arises from some discrete Morse function. This fact is used in the next corollary, which follows from Theorem 1.6.4.

Corollary 1.6.6. *Let K be a simplicial complex and $F(K)$ its face poset. If $F(K)$ has an acyclic matching where the only unmatched cells of K are facets, then K is homotopy equivalent to a wedge of spheres.*

The following lemma appeared independently in both [15] and [18] and is used to prove Corollary 1.6.8. We follow the notation and proof of Hersh in [15]. It is referred to as the Cluster Lemma in [18].

Lemma 1.6.7 (Lemma 4.1 [15])**.** *Let K be a regular CW complex and P a poset with a unique minimal element. Let $D = \{K_\sigma\}_{\sigma \in P}$ be a set of collections of cells of K satisfying:*

(i) Each cell of K belongs to exactly one K_σ.

(ii) For each $\sigma \in P$, $\bigcup_{\tau \leq \sigma} K_\tau$ is a subcomplex of K.

Let M_σ be an acyclic matching on the subposet of $F(K)$ consisting of all cells in K_σ. Then $\bigcup_{\sigma \in P} M_\sigma$ is an acyclic matching on $F(K)$.

Proof. We will call both the elements of D and the corresponding subposets of $F(K)$ components. Let $G(K)$ be the directed graph obtained from the Hasse diagram of the face poset of K by orienting edges between matched cells upward and all other edges downward. Assume that for some pair $\sigma, \tau \in P$ where $\sigma \neq \tau$, there is an edge e oriented downward from a cell in K_σ to a cell in K_τ. Since $\bigcup_{\tau \leq \sigma} K_\tau$ is a subcomplex of K, then $\tau < \sigma$. Since $G(K)$ is obtained by taking a union of acyclic matchings on the components of D and all matching edges are oriented upwards, there is no upward oriented edge between components in $F(K)$. Thus there is no way for a directed path containing e to return to the component K_σ. Thus $\bigcup_{\sigma \in P} M_\sigma$ is an acyclic matching on $F(K)$. $\square$

We will also want the following well-known corollary.

Corollary 1.6.8. *Let $F_1, F_2, \ldots F_n$ be a shelling order on the facets of a simplicial complex K. Then there is an acyclic matching on $F(K)$ whose critical cells are the homology facets of the shelling.*

Proof. Given the shelling order $F_1, F_2, \ldots F_n$, consider the filtration $K_1 \subseteq K_2 \subseteq \ldots K_n$ where $K_j \backslash K_{j-1} = F_j \backslash (\cup_{i<j} F_i)$. Note that $K_j \backslash K_{j-1}$ is a collection of simplices such that $F(K_j \backslash K_{j-1})$ is a Boolean lattice. Recall a Boolean lattice B_n has as elements all the subsets of a set consisting of the integers from 1 to n. For $r \geq 1$, a Boolean lattice B_r admits a natural acyclic matching where any cell corresponding to a set s not containing 1 is matched with the cell corresponding to the set $s \cup \{1\}$. Thus taking $D = \{K_1, K_2, \ldots K_n\}$, we can apply Lemma 1.6.7 to obtain an acyclic matching of $F(K)$. By construction, any unmatched cell is unmatched because K_j consists of a single cell. This occurs only if for some F_j and for all $\sigma \subset \overline{F_j}$, $\sigma \subset \overline{F_i}$ for some $i < j$. Thus any unmatched cell much be a facet. $\square$

It is worth noting that in [1], Eric Babson and Patricia Hersh introduced a method for constructing a nice discrete Morse function for the order complex of any finite bounded poset using any lexicographic order on its saturated chains. They call these types of discrete Morse functions **lexicographic discrete Morse functions**. One question for further research is whether this method will help to determine the topology of the edge-product space of phylogenetic trees, though the face poset of this space is not shellable in its entirety as we show in Chapter 3.

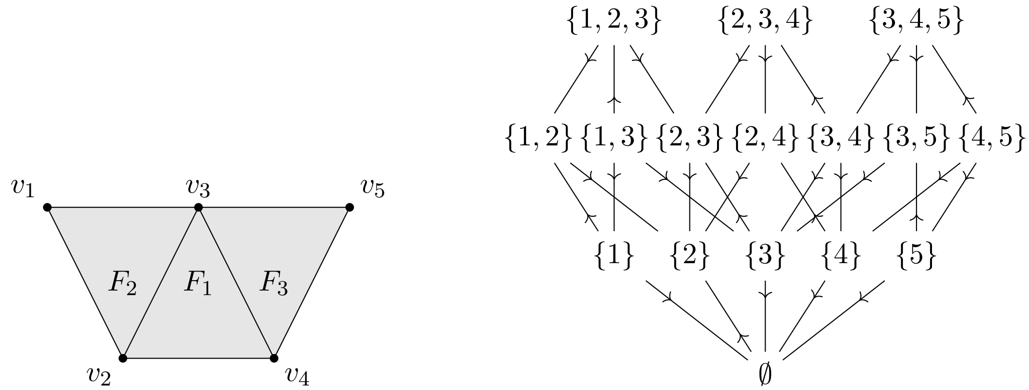


Figure 1.13: A shelling order on a simplicial complex K (shown on the left) yields an acyclic matching on $F(K)$ (shown on the right). There are no critical cells (i.e. unmatched cells).

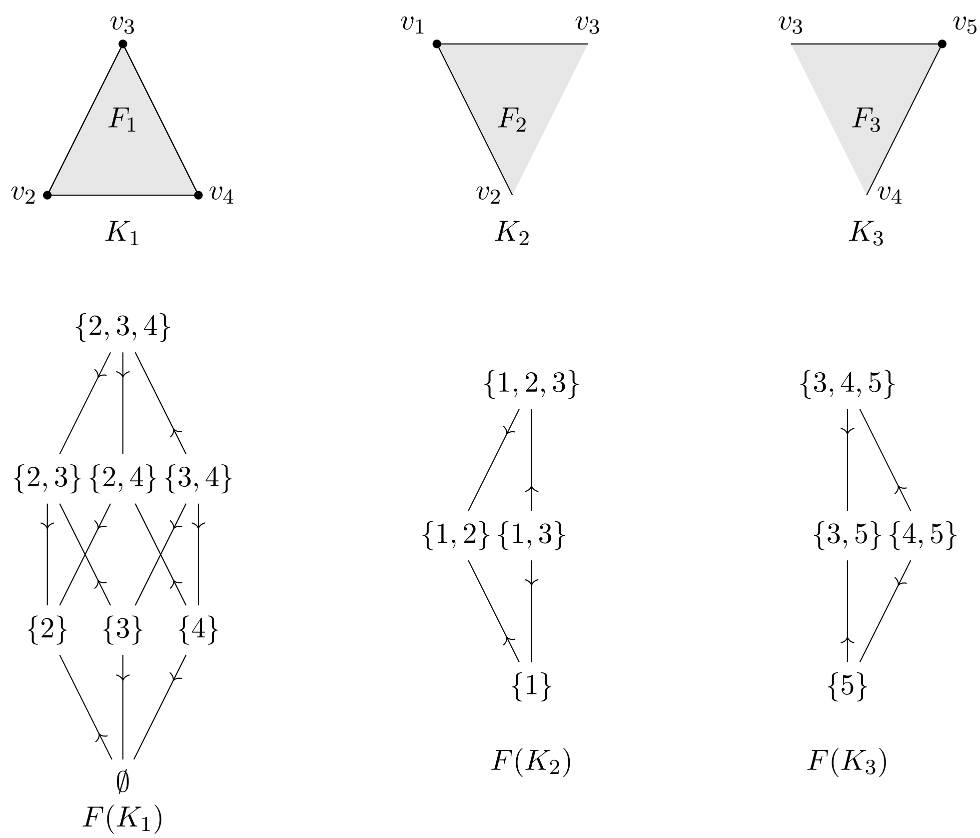


Figure 1.14: A decomposition of the simplicial complex K shown in Figure 1.13. A union of acyclic matchings on the face posets of the components of the decomposition yield an acyclic matching on $F(K)$.

Chapter 2

The enriched Tamari poset

In this chapter, we introduce a new partial order whose elements are the maximal elements of the Tuffley poset $S(X)$. To do so, we first establish a canonical way of embedding the maximal elements of the Tuffley poset in the plane and then describe how we can generate all maximal elements of $S(X)$ from two types of local operations. The initial hope was that we could extend this partial order to a total order that was compatible with a recursive coatom ordering, thereby proving that the augmented Tuffley poset (the Tuffley poset with $\hat{0}$ and $\hat{1}$) was shellable in its entirety. As we will see in Chapter 3, there is no dual CL-shelling of the augmented Tuffley poset. However, given that the enriched Tamari poset is a natural extension of the Tamari lattice, this seems to be an interesting poset to explore in its own right. After defining the enriched Tamari poset (see Definition 2.2.8) and proving it is indeed a poset (see Theorem 2.2.7), we turn to discussing its structure.

2.1 New notions and establishing a framework for them

From Theorem 1.4.15, we know the maximal elements of the Tuffley poset, $S(X)$, are the trees with $|X|$ leaves such that every internal vertex has degree 3. Let the set of maximal elements of $S(X)$ be denoted $\mathcal{C}_X$. If we adjoin a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ to $S(X)$, the elements in $\mathcal{C}_X$ are the coatoms of $S(X) \cup \{\hat{0}, \hat{1}\}$. For simplicity, we will let $X = [n] = \{1, 2, \ldots, n\}$ for the remainder of this thesis.

Definition 2.1.1. Let $x \in S([n])$. A **subforest** of x is any $[n]-$forest that can be obtained from x by deleting a subset of edges and vertices and contracting a subset of edges of x. A **subtree** of x is a subforest that can be obtained from x without deleting any edges. A **maximal subtree** of x is a subtree of x that can be obtained from x without contracting any edges.

Note that a subforest (respectively subtree) is a forest (respectively tree), but need not be an A-forest (respectively A-tree) for some subset $A \subseteq [n]$. In particular, there are no restrictions

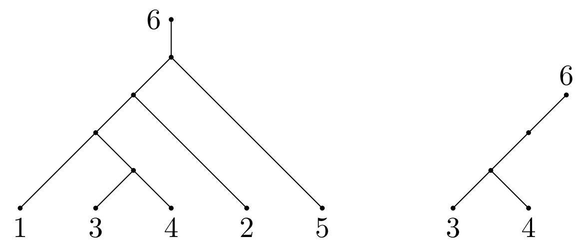


Figure 2.1: A maximal element C_i of $S([6])$ (left) and a subtree of C_i (right).

on the degree of the unlabeled vertices of a subforest (respectively subtree). Also note that a subforest of x is commonly known as a minor of x. See Figure 2.1 for an example of a subtree of an element of $S([6])$.

We now describe a standard embedding of any maximal element of $S([n])$ which will aid in our description of the enriched Tamari poset.

Definition 2.1.2. Let $C_i \in \mathcal{C}_{[n]}$. The leaf labeled n is called the **maximal leaf** of C_i. The unique internal vertex adjacent to the maximal leaf of C_i is called the **maximal internal vertex** and is denoted $\hat{v}$. The edge between the maximal leaf and the maximal internal vertex of C_i we call the **maximal edge**.

Definition 2.1.3. The **standard planar embedding** of $C_i \in \mathcal{C}_{[n]}$, is as follows: The maximal leaf is placed highest in the plane and the maximal internal vertex $\hat{v}$ is drawn directly below it. There are two other edges, e_1 and e_2, adjacent to $\hat{v}$. Let v_1 (respectively v_2) be the endpoint other than $\hat{v}$ of e_1 (respectively e_2). Each of e_1 and e_2 connects $\hat{v}$ to a subtree, say S_1 and S_2 respectively. Choose S_1 to be the subtree among S_1 and S_2 that contains the smallest leaf label among all labels on vertices of C_i. Draw e_1 downward and to the left and e_2 downward and to the right. Repeat this process on the pair of edges adjacent to v_1 and v_2. Continue until all edges in the tree have been drawn either downward and to the left or downward and to the right.

When C_i is depicted with the standard planar embedding, the maximal leaf of C_i is the highest vertex of C_i in the plane and the maximal internal vertex is the highest internal vertex of C_i in the plane, hence the decision to call these vertices maximal. See Figure 2.3 for an example.

Every internal vertex v of $C_i \in \mathcal{C}_{[n]}$ has two **children**. In particular, v has a **left child** l_v and a **right child** r_v. The left child is the vertex of C_i below and to the left of v in the standard planar embedding described above. The right child is the vertex of C_i below and to the right of v. In the standard planar embedding of C_i, the left child (respectively right child) of v is the

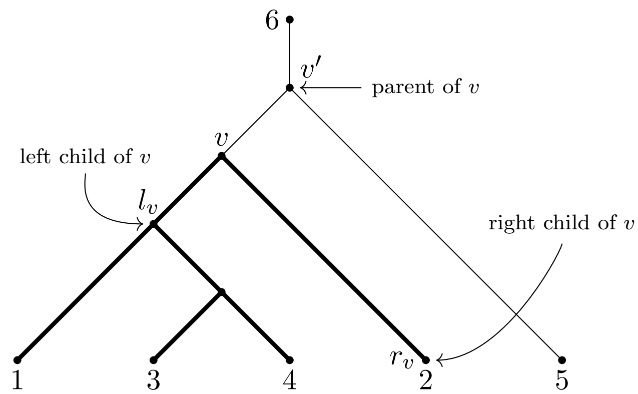


Figure 2.2: A maximal element of $S([6])$. The subtree $\downarrow v$ is indicated with bold edges.

highest vertex of a maximal subtree of C_i called the **left subtree** (respectively **right subtree**) of v. The left subtree of v is denoted $\downarrow l_v$. It consists of all vertices below l_v and all edges between them in the standard planar embedding of C_i. The right subtree of v, denoted $\downarrow r_v$, consists of all vertices below r_v and all edges between them in the standard planar embedding of C_i. The vertex v is called the **parent** of l_v and r_v. For v an internal vertex of C_i, the **downgraph** of v is v with its left subtree and right subtree. It is denoted $\downarrow v$. See Figure 2.2 for an example.

The following proposition says that instead of considering the set of maximal elements of $S([n])$, it suffices to consider instead the set of isomorphism classes of full binary trees (see Definition 1.5.3) with leaves labeled by elements of $[n-1]$.

Proposition 2.1.4. *The maximal elements of $S([n])$ are in bijection with the set of isomorphism classes of full binary trees with leaves labeled by elements of $[n-1]$.*

Proof. Let $C_i \in \mathcal{C}_{[n]}$ and let $\tilde{\mathcal{C}}_{[n]}$ denote the set of isomorphism classes of full binary trees with leaves labeled by elements of $[n-1]$. Define the map $\psi : \mathcal{C}_{[n]} \to \tilde{\mathcal{C}}_{[n]}$ as $\psi(C_i) = \tilde{C}_i$ where $\tilde{C}_i$ is obtained by removing the maximal leaf and edge in C_i. Since the maximal internal vertex is the highest internal vertex in the standard planar embedding of C_i (see Definition 2.1.3) and every internal vertex has degree 3, removing the the maximal leaf of C_i results in a tree such that every internal vertex but $\hat{v}$, has a parent and two children. The maximal internal vertex, $\hat{v}$, has two children and no parent. Thus $\tilde{C}_i \in \tilde{\mathcal{C}}_{[n]}$. The map $\psi : \mathcal{C}_{[n]} \to \tilde{\mathcal{C}}_{[n]}$ is a bijection with inverse $\psi^{-1} : \tilde{\mathcal{C}}_{[n]} \to \mathcal{C}_{[n]}$ defined as follows: $\psi^{-1}(\tilde{C}_i) = \hat{C}_i$ where $\hat{C}_i$ is obtained by adding a vertex labeled n and adding an edge between this new vertex and $\hat{v}$, where $\hat{v}$ is the unique degree 2 internal vertex. It is clear that $\hat{C}_i = C_i$. Thus $\psi : \mathcal{C}_{[n]} \to \tilde{\mathcal{C}}_{[n]}$ is a bijection. $\square$

2.1.1 Encoding maximal elements of the Tuffley poset with words

The following definitions from graph theory are used to define what we call the word of the maximal element C_i of $S([n])$ (see Definition 2.1.9). This word encodes the entire structure of C_i, as we prove in Proposition 2.1.14. It is used to define lexicographic moves and Tamari moves (see Section 2.2) and to prove that the enriched Tamari poset is indeed a poset.

Definition 2.1.5. The **star graph** or **n-star** is a tree with n vertices, one of which has degree $n-1$ and the other $n-1$ of which have degree 1.

Definition 2.1.6. Let C_i be a maximal element of $S([n])$ and let v be an internal vertex of C_i. If v is such that $\downarrow v$ is a star graph, then we call $\downarrow v$ a **star subgraph** of C_i.

Note that since internal vertices of C_i have degree 3, then if $\downarrow v$ is a star subgraph of C_i for some internal vertex v, then $\downarrow v$ is more specifically a 3-star.

Definition 2.1.7. If l is a leaf of the maximal element C_i and l is contained in a star subgraph of C_i, then l is called a **star leaf**.

Any full binary tree with leaves labeled in 1-1 correspondence by elements of $[n-1]$ can be represented by a parenthesization of a permuation of the integers 1 through $n-1$. We present a specific parenthesization of $\tilde{C}_i$ next.

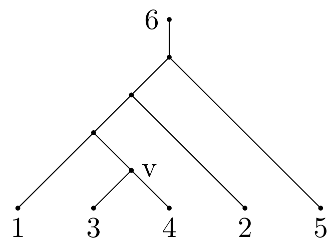
Definition 2.1.8. A **pair** of parentheses is a left parenthesis with its corresponding right parenthesis.

Definition 2.1.9. Let $C_i \in \mathcal{C}_{[n]}$. The **word** of C_i is a word in the alphabet $[n-1] \cup \{(\} \cup \{)\} \cup \{,\}$, obtained recursively as follows. For every internal vertex v, there is a pair of parentheses, denoted p_v, that contains two **entries** separated by a comma. The first entry of p_v is the parenthesization of $\downarrow l_v$ and the second entry is the parenthesization of $\downarrow r_v$. If $\downarrow l_v$ (respectively $\downarrow r_v$) consists of a single leaf then the parenthesization of $\downarrow l_v$ (respectively $\downarrow r_v$) is the leaf label. The word of C_i is obtained recursively by starting at the non-maximal star leaves of C_i. It is denoted $w(C_i)$.

For an internal vertex v, we let $w(\downarrow v)$ denote the word of the downgraph of v, $\downarrow v$. This is well-defined since $\downarrow v$ is a full binary tree. If we are specifically considering the word of the downgraph of v in C_i and C_i is not implied by context, then we write $w(\downarrow v)(C_i)$.

Definition 2.1.10. A **subword** of $w(C_i)$ is any ordered subset of (not necessarily adjacent) letters in the word $w(C_i)$.

Example 2.1.11. The standard planar embedding of a maximal element C_i in $S([6])$ is shown in Figure 2.3. The only star subgraph of C_i is $\downarrow v$. Since v has left child labeled 3 and right child labeled 4, $w(C_i)$ is formed starting with $w(\downarrow v) = (3, 4)$ and continuing as follows:



$(((1,(3,4)),2),5)$

Figure 2.3: A maximal element C_i of $S([6])$ and $w(C_i)$.

(3, 4)
(1,(3, 4))
((1,(3, 4)),2)
(((1,(3, 4)),2),5)

2.1.2 Statistics obtained from $w(C_i)$

Definition 2.1.12. The word $w(C_i)$ has a subword consisting only of the letters in X. This is called the **X-subword of $w(C_i)$** and is denoted $w_X(C_i)$.

Since we consider only when $X = [n]$ in this thesis, we will refer to this statistic as the $[n]$-subword. The $[n]$-subword can be read from the standard planar embedding of C_i by reading the labels on the nonmaximal leaves of C_i from left to right. Also note that by definition of the standard planar embedding, the $[n]$-subword of $w(C_i)$ is lexicographically as early as possible. In other words, there is no way to embed C_i in the plane in such a way that reading off the non-maximal leaf labels in order counterclockwise would give an earlier word in the alphabet $[n]$.

Definition 2.1.13. The ith letter in $w(C_i)$ is said to be in the ith **position**. The word consisting of the positions of open parentheses from left to right in $w(C_i)$ is called the **skew** of C_i and is denoted $Sk(C_i)$.

Proposition 2.1.14. *A maximal element C_i of $S([n])$ is determined by $Sk(C_i)$ and $w_{[n]}(C_i)$.*

Proof. Given $Sk(C_i)$ and $w_{[n]}(C_i)$, the word for the maximal element C_i of $S([n])$ can be recovered by filling the $4n-7$ entries of $w(C_i)$ as follows. If $i \in Sk(C_i)$, place a left parenthesis in the ith entry of $w(C_i)$. If the entry before any left parenthesis is available, place a comma in the entry that immediately precedes the left parenthesis. We then start inserting the letters of $w_{[n]}(C_i) = w_1 w_2 \dots w_n$. We start by placing w_1 in the first available entry from the left. Then:

(i) If the next entry after w_1 is blank and if w_1 is not preceded by a comma, then place a comma in this next entry.

(ii) If the next entry after w_1 is blank and w_1 is preceded by a comma, place a right parenthesis in this next entry.

  (a) If a right parenthesis is placed and corresponds to a left parenthesis that is immediately preceded by a comma, place another right parenthesis.

  (b) If not, place a comma.

(iii) If the next entry after w_1 is not blank, do nothing.

Then continue by placing w_2 according to the same rules. Continue until all letters of $w_{[n]}(C_i)$ have been placed. In the final entry, place a right parenthesis. $\square$

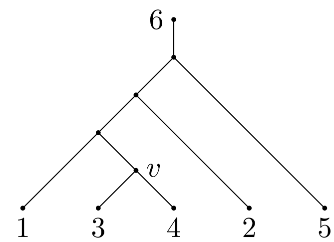
Example 2.1.15. Let $w_{[n]}(C_i) = 1, 3, 4, 2, 5$ and $Sk(C_i) = 1, 2, 5, 12$. The process below shows how we fill the $4(6) - 7 = 17$ entries of $w(C_i)$ and thus determine C_i:

$$
\begin{array}{c}
\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{\quad}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{3}\;\underline{,}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{3}\;\underline{,}\;\underline{4}\;\underline{)}\;\underline{)}\;\underline{,}\;\underline{(}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{3}\;\underline{,}\;\underline{4}\;\underline{)}\;\underline{)}\;\underline{,}\;\underline{(}\;\underline{2}\;\underline{,}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{3}\;\underline{,}\;\underline{4}\;\underline{)}\;\underline{)}\;\underline{,}\;\underline{(}\;\underline{2}\;\underline{,}\;\underline{5}\;\underline{)}\;\underline{\quad} \\
\underline{(}\;\underline{(}\;\underline{1}\;\underline{,}\;\underline{(}\;\underline{3}\;\underline{,}\;\underline{4}\;\underline{)}\;\underline{)}\;\underline{,}\;\underline{(}\;\underline{2}\;\underline{,}\;\underline{5}\;\underline{)}\;\underline{)}
\end{array}
$$

2.2 The enriched Tamari poset

As described in Section 1.5, the Tamari lattice can be viewed as a partial order on the set of full binary trees. Equivalently, it is a partial order on the parenthesizations (or bracketings) of the integers from 1 to n. We can naturally extend this partial order to isomorphism classes of full binary trees with leaves labeled in 1-1 correspondence with the set $[n-1]$ (i.e. to the maximal elements of $S([n])$) as we will describe in this section.

First, we define a total order on any disjoint set of downgraphs in $C_i \in \mathcal{C}_{[n]}$. This is necessary to define the cover relations in what we will call the enriched Tamari poset.



$$w(C_i) = (((1,(3,4)),2),5)$$
$$Sk(C_i) = 1,2,3,6$$

Figure 2.4: A maximal element C_i of the Tuffley poset $S([6])$.

Definition 2.2.1. Let u and v be internal vertices of C_i such that $\downarrow u$ and $\downarrow v$ are disjoint. The **lexicographic order** on disjoint downgraphs is $<_{lex}$ where $u <_{lex} v$ if the smallest integer in $w(\downarrow u)$ is less than the smallest integer in $w(\downarrow v)$.

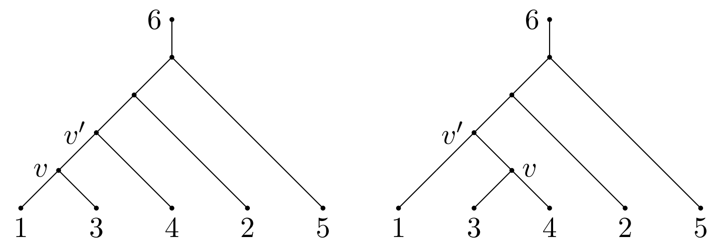
Equivalently, $u <_{lex} v$ if the smallest leaf label in the subtree $\downarrow u$ is less than the smallest leaf label in $\downarrow v$.

The enriched Tamari poset has two types of cover relations, which we define next. The first is called a Tamari move since cover relations in the Tamari lattice are defined similarly. Let C_i be a maximal element of $S([n])$ with the standard planar embedding. Let v' be an internal vertex and v be its left child. Let a and b be the words of the left and right subtrees of v, respectively. Let (a, b) and c be the words of the left and right subtrees of v', respectively.

Definition 2.2.2. Let $b <_{lex} c$. A **Tamari move** on v in C_i changes the subword $((a, b), c)$ of $w(C_i)$ to $(a,(b, c))$.

A Tamari move on v in $w(C_i)$ yields the word $w(C_i')$ where C_i' is a maximal element of $S([n])$. Note that v must be the left child of v' in C_i in order for a Tamari move to be applied to v. We can alternatively characterize a Tamari move by the resulting change in edges adjacent to vertices v and v', as shown in Figure 2.5. If edges x and y are adjacent via vertex v (with x left of y in the standard planar embedding) and edges z and w are adjacent via vertex v' (with z below w in the standard planar embedding), a Tamari move on v results in a tree such that x and w are adjacent and y and z are adjacent. All other edge adjacencies are the same between C_i and C_i'.

Proposition 2.2.3. *If $w(C_i')$ is obtained from $w(C_i)$ by a Tamari move, then $Sk(C_i) < Sk(C_i')$ and the X-subword of $w(C_i')$ equals the X-subword of $w(C_i)$.*



$w(C_i) = ((((1,3),4),2),5)$ $\quad$ $w(C_{i'}) = (((1,(3,4)),2),5)$

Figure 2.5: A Tamari move on vertex v in C_i (left) yields C_i' (right).

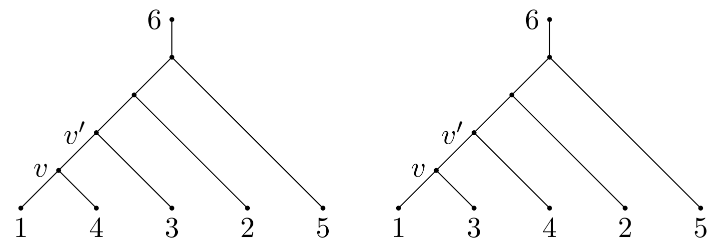
Proof. Assume that $w(C_i')$ is obtained from $w(C_i)$ by applying a Tamari move to the vertex v with parent v'. Assume that $w(\downarrow v')(C_i)$ is $((a,b),c)$ where $a <_{lex} b <_{lex} c$. Applying a Tamari move to vertex v yields $w(C_i')$ where the subword $w(\downarrow v)(C_i)$ is $(a,(b,c))$. Since in both $w(C_i)$ and $w(C_i')$, a comes to the left of b and b comes to the left of c, then the $X-$subword of $w(C_i)$ equals the $X-$subword of $w(C_i')$. However, the position of the open parenthesis for v in $w(C_i')$ is greater than the position of the open parenthesis for v in $w(C_i)$. Since the positions of all other pairs of parentheses remain the same between $w(C_i)$ and $w(C_i')$, $Sk(C_i) < Sk(C_i')$. $\square$

We introduce a new type of move on parenthesizations of a subset of permutations of the integers from 1 to $n-1$. Together with the Tamari move, this type of move generates the set of maximal elements of $S([n])$. As before, let C_i be a maximal element of $S([n])$ with the standard planar embedding. Let v' be an internal vertex and v be its left child. Let a and c be the words of the left and right subtrees of v, respectively. Let (a,c) and b be the words of the left and right subtrees of v', respectively.

Definition 2.2.4. Let $a <_{lex} b <_{lex} c$. A **lexicographic move on** v changes the subword $((a,c),b)$ of $w(C_i)$ to $((a,b),c)$.

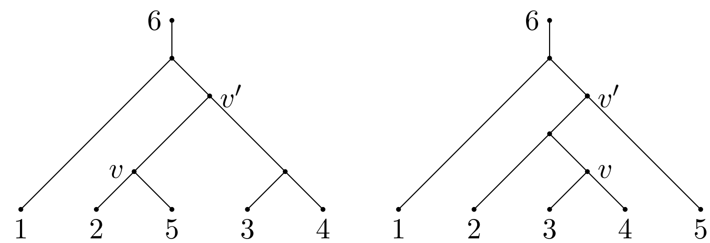
As with a Tamari move, a lexicographic move on v in $w(C_i)$ yields the word $w(C_i')$ where $C_i' \in \mathcal{C}_{[n]}$. The lexicographic move on v swaps the right subtree of v with the right subtree of v' when the right subtree of v is lexicographically later than the right subtree of v'.

As with a Tamari move, a lexicographic move can be characterized by the resulting change in edges that are adjacent to vertices v and v', as in Figure 2.6. If edges x and y are adjacent via vertex v (with x left of y in the standard planar embedding) and edges z and w are adjacent via vertex v' (with z below w in the standard planar embedding), a lexicographic move on v yields a tree such that x and z are adjacent and y and w are adjacent.



$w(C_i) = ((((1,4),3),2),5)$ $\qquad$ $w(C_i') = ((((1,3),4),2),5)$

Figure 2.6: A lexicographic move on vertex v in $w(C_i)$ (left) yields $w(C_i')$ (right).



$w(C_i) = (1, ((2,5),(3,4)))$ $\qquad$ $w(C_{i'}) = (1, ((2,(3,4)),5))$

$w_X(C_i) = 1, 2, 5, 3, 4$ $\qquad$ $w_x(C_i') = 1, 2, 3, 4, 5$

$Sk(C_i) = 1, 3, 4, 7, 8, 11, 12, 13$ $\qquad$ $Sk(C_{i'}) = 1, 3, 4, 6, 9, 10, 12, 13$

Figure 2.7: A lexicographic move on v in C_i (left) yields a coatom $C_{i'}$ (right) such that $w_X(C_{i'}) < w_X(C_i)$ but such that $Sk(C_i) > Sk(C_{i'})$.

Proposition 2.2.5. *A lexicographic move on C_i yields a coatom C_i' such that the X-word of C_i' is lexicographically earlier than the X-word of C_i.*

Proof. Since $((a,b),c) = w(C_i')$ and $((a,c),b) = w(C_i)$, the result follows from the fact that $b <_{lex} c$. $\square$

Note that a lexicographic move on v in $w(C_i)$ yields some $w(C_i')$ where any one of the following holds:

1. $Sk(C_i') < Sk(C_i)$ (as in Figure 2.7)
2. $Sk(C_i') = Sk(C_i)$
3. $Sk(C_i') > Sk(C_i)$.

Definition 2.2.6. Let C_i and C_j be maximal elements of $S([n])$. There is a relation $\leq_{ET_n}$ among the maximal elements of $S([n])$ given by the following: $C_i <_{ET_n} C_j$ if $w(C_j)$ is obtained from $w(C_i)$ by a sequence of Tamari moves and lexicographic moves.

Theorem 2.2.7. *The set of maximal elements of $S([n])$ together with the relation $\leq_{ET_n}$ comprise a partially ordered set.*

Proof. Let $G([n])$ be the directed graph that has elements of $\mathcal{C}_{[n]}$ as vertices and a directed edge from C_i to C_j if and only if C_j is obtained from C_i by a single Tamari or lexicographic move. We will show that $(\mathcal{C}_{[n]}, \leq_{ET_n})$ is a partial order on the maximal elements of $S([n])$ by showing that $G([n])$ is acyclic.

Assume that there is a nontrivial directed path from C_i to C_j in $G([n])$. Then there is a sequence of elements of $\mathcal{C}_{[n]}$, $C_i = C_{i_0} \lessdot_{ET_n} C_{i_1} \lessdot_{ET_n} C_{i_2} \lessdot_{ET_n} \ldots \lessdot_{ET_n} C_{i_t} = C_j$, such that C_{i_k} is obtained from $C_{i_{k-1}}$ by either a Tamari move or a lexicographic move, where $1 \leq k \leq t$.

If this directed path from C_i to C_j is a cycle, then $C_i = C_j$ and thus $w_X(C_i) = w_X(C_j)$. If for any k, C_{i_k} is obtained from $C_{i_{k-1}}$ by a lexicographic move, then $w_X(C_{i_k}) < w_X(C_{i_{k-1}})$ by Proposition 2.2.5. Since neither a lexicographic nor a Tamari move yields an element with a lexicographically later X-word, then for all $r \geq k$, $w_X(C_{i_r}) < w_X(C_{i_{k-1}}) \leq w_X(C_i)$. Thus the directed path from C_i to C_j is not a cycle if for some k, C_{i_k} is obtained from $C_{i_{k-1}}$ by a lexicographic move.

If for all k, C_{i_k} is obtained from $C_{i_{k-1}}$ by a Tamari move, then by Proposition 2.2.3, $Sk(C_{i_k}) > Sk(C_{i_{k-1}})$ and $w_X(C_{i_k}) = w_X(C_{i_{k-1}})$. Then for all k such that $1 \leq k \leq t$, $w_X(C_{i_k}) = w_X(C_i)$ but $Sk(C_{i_k}) > Sk(C_i)$. In particular, $Sk(C_{i_t}) = Sk(C_j) > Sk(C_i)$. Note that if $C_i = C_j$ then $Sk(C_i) = Sk(C_j)$. Since $Sk(C_i) \neq Sk(C_j)$, we have $C_j \neq C_i$. This means there are no cycles in the directed path from C_i to C_j.

Thus $G([n])$ is a directed acyclic graph and so $(\mathcal{C}_{[n]}, \leq_{ET_n})$ is a partial order on the maximal elements of $S([n])$. $\square$

Definition 2.2.8. The poset $(\mathcal{C}_{[n]}, \leq_{ET_n})$ is called the **enriched Tamari poset** and is denoted ET_n.

Remark 2.2.9. The enriched Tamari poset could have been defined in a number of slightly different ways, yielding a family of interrelated partial orders on the maximal elements of $S([n])$. For example, we could have defined a Tamari move so that this type of action on some C_i yields some C_i' such that $w_{[n]}(C_i) < w_{[n]}(C_i')$. We chose the partial order presented in Definitions 2.2.6 and 2.2.8 because it is bounded.

2.3 Properties of the enriched Tamari poset

The Tamari lattice has many nice properties. A natural question to ask is whether the enriched Tamari poset ET_n also has these properties. We are able to immediately determine some of the characteristics of ET_n, which we present here.

Remark 2.3.1. The enriched Tamari poset is bounded. The unique minimal element is the element $\hat{0}$ determined by

$$\begin{aligned} Sk(\hat{0}) &= 1, 2, 3, \dots, n-2 \\ w_{[n]}(\hat{0}) &= 1, n-1, n-2, \dots 3, 2 \end{aligned}$$

The unique maximal element is the element $\hat{1}$ determined by

$$\begin{aligned} Sk(\hat{1}) &= 1, 4, 7, \dots, 1 + 3(n-2) \\ w_{[n]}(\hat{1}) &= 1, 2, 3, \dots, n-1 \end{aligned}$$

Remark 2.3.2. The enriched Tamari poset is not always a lattice.

Example 2.3.3. The elements represented by the words $(((1,2),3),4)$ and $((1,(2,4)),3)$ are both common least upper bounds for the elements x and y represented by the words $(((1,2),4),3)$ and $(((1,3),2),4)$, respectively. Since x and y do not have a unique, common, least upper bound, x and y do not have a join. Thus, the enriched Tamari poset cannot always be a lattice. See Figure 2.8.

Remark 2.3.4. The enriched Tamari poset is not always shellable.

Example 2.3.5. Any interval in a shellable poset is shellable. However, the interval in ET_5 from $\hat{0} = (((1,4),3),2)$ to the element represented by the word $(((1,2),3),4)$ contains two maximal chains of length 3. These two maximal chains correspond to tetrahedra A and B in the order complex of $S([5])$. However, these two chains intersect in a chain of length 1. This chain of length 1 corresponds to a line segment in the order complex of $S([5])$. Thus the intersection of A and B is a face of codimension 2. Thus no ordering on these maximal chains yields a shelling order. See Figure 2.9.

Remark 2.3.6. There are $(2n-5)!!$ elements in the enriched Tamari poset ET_n.

There is one element in ET_3 and this tree has three edges. The number of elements in ET_n can be obtained recursively from the number of elements in ET_{n-1}, as we describe next. For each element of ET_{n-1} and each edge e of this element, we add a leaf labeled n to obtain a unique element of ET_n, in the following way: add an unlabeled vertex at the midpoint of edge e, add an isolated vertex labeled n, and then connect the new unlabeled vertex and the new

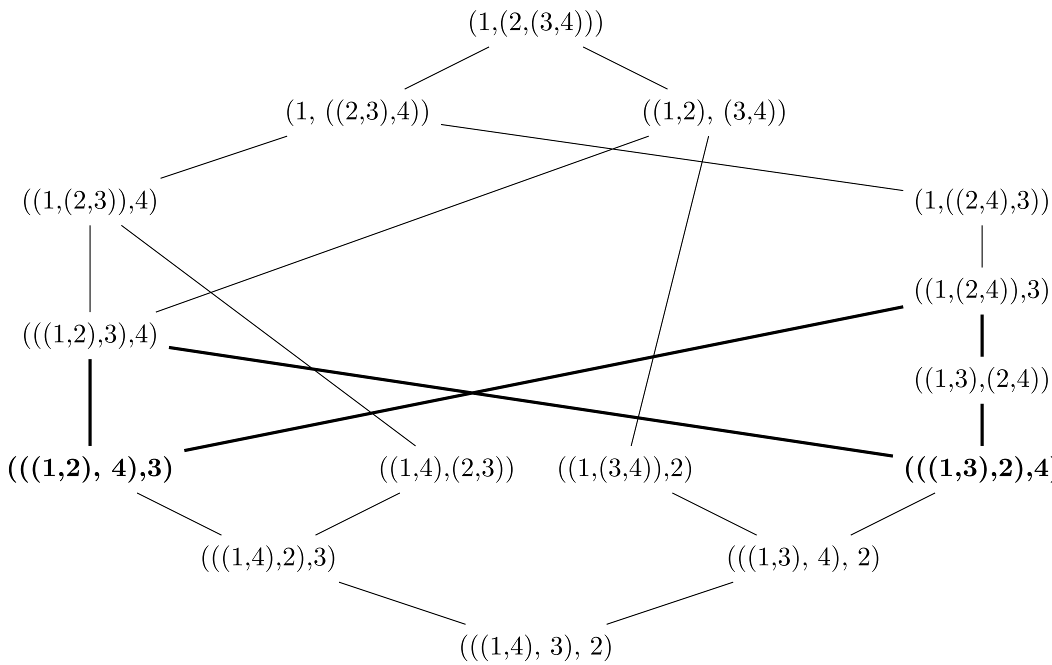


Figure 2.8: The enriched Tamari poset ET_5. The elements represented by the words $(((1,2),4),3)$ and $(((1,3),2),4)$ have no join.

isolated vertex via a new leaf edge. Since there are $2n-5$ edges in any element of ET_{n-1}, the number of elements in ET_n is

$$\begin{aligned}
|ET_n| &= (2n-5)|ET_{n-1}| \\
&= (2n-5)(2(n-1)-5)|ET_{n-2}| \\
&= (2n-5)(2n-7)|ET_{n-2}| \\
&\ \ \vdots \\
&= (2n-5)(2n-7)\dots 3\cdot 1
\end{aligned}$$

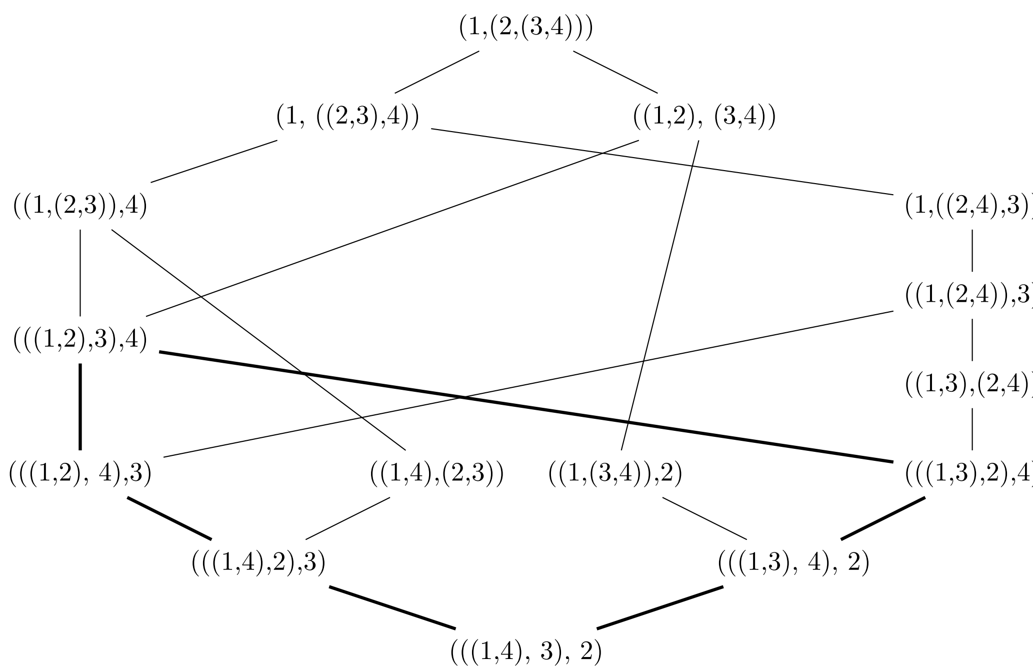


Figure 2.9: The enriched Tamari poset ET_5 is not shellable. The interval marked with bold edges is not shellable.

Chapter 3

The Tuffley poset is not dual CL-shellable

In this chapter, we present one of the primary results of this thesis, which states that the Tuffley poset is not dual CL-shellable in its entirety. We accomplish this by showing that there is no partial or total ordering on the maximal elements of the Tuffley poset that is compatible with a recursive coatom ordering. This is a somewhat surprising result given gallery-connectedness (see Theorem 3.2.5) together with the fact that all intervals in the Tuffley poset with $\hat{0}$ adjoined are shellable, as first proven by Gill, Linusson, Moulton, and Steel in [12]. In this chapter, we let $S_{[n]} = S([n]) \cup \{\hat{0}, \hat{1}\}$ and we let $S^*_{[n]} = (S([n]) \cup \{\hat{0}, \hat{1}\})^*$.

3.1 A shelling for open intervals of the Tuffley poset

This section concerns the uncrossing poset P_n. We show that principal order ideals of the uncrossing poset are isomorphic to intervals $[\hat{0}, \Gamma]$ of $S([n]) \cup \{\hat{0}\}$. Since the uncrossing poset is dual EC-shellable as proven by Hersh and Kenyon via a labeling they introduced in [16], this result leads to an explicit shelling of open intervals of $S([n]) \cup \{\hat{0}\}$. Previously, Gill, Linusson, Moulton and Steel proved in [12] the existence of a shelling for each interval.

3.1.1 The uncrossing poset

We now review the uncrossing poset P_n.

Definition 3.1.1. A **wire diagram** with n wires is a circle with $2n$ nodes placed around the circumference and n wires connecting pairs of nodes such that any node is paired with one and only one other node.

We select one node of every wire diagram to be the **base point**. By convention, we will label the wires of a wire diagram, D, counterclockwise from 1 to n, where 1 coincides with the base point.

We will need the following several definitions both for the definition of the uncrossing poset and for the dual EC-labeling of the uncrossing poset.

Definition 3.1.2. The **uncrossing poset** P_n is a partial order on isotopy classes of wire diagrams, where $C \lessdot D$ if C is obtained from D by uncrossing a pair i, j of wires in D in such a way that each wire crosses every other wire in C at most once.

Definition 3.1.3. The **minimum crossing number** of a wire diagram D is the least number of wire crossings among all representatives of the isotopy class of D.

Definition 3.1.4. The **word** of a wire diagram D is the word $w(D)$ with integer letters from 1 to n obtained by reading the labels on the wires counterclockwise starting from the base point. In doing so, the first instance of an endpoint for wire w is called the **initial node** of wire w and the second instance is called the **final node** of wire w.

Remark 3.1.5. In [16], wires are labeled from 1 to n in clockwise order from the base point and the word of a wire diagram is calculated by reading the labels in clockwise order. We have chosen to label wires in counterclockwise order so that both the minimum labels on the vertices of an $[n]$-tree and the wire labels are increasing in the same direction.

See Figure 3.1 for an example of a cover relation and of calculating the word for a wire diagram.

Remark 3.1.6. The uncrossing poset P_n has a unique maximal element whose word is

$$w(\hat{1}) = 123 \dots n123 \dots n.$$

The uncrossing poset is an example of a poset that admits a dual EC-labeling. This labeling was introduced and proven to be an EC-labeling by Hersh and Kenyon in [16]. We describe this labeling λ next.

Let $C, D \in P_n^*$ and suppose $D \lessdot C$. Suppose further that for $i < j$, $w(D)$ has subsequence i, j, i, j whereas $w(C)$ does not. Hersh and Kenyon label the cover relation $D \lessdot C$ depending on which of the following cases applies to C and $w(C)$:

(i) If $w(C)$ contains the subsequence i, j, j, i, let $\lambda(D \lessdot C) = (i, j)$.

(ii) If $w(C)$ contains the subsequence i, i, j, j, let $\lambda(D \lessdot C) = (j, i)$.

For any coatom $C \in P_n^*$, let $\lambda(C, \hat{1}) = L$.

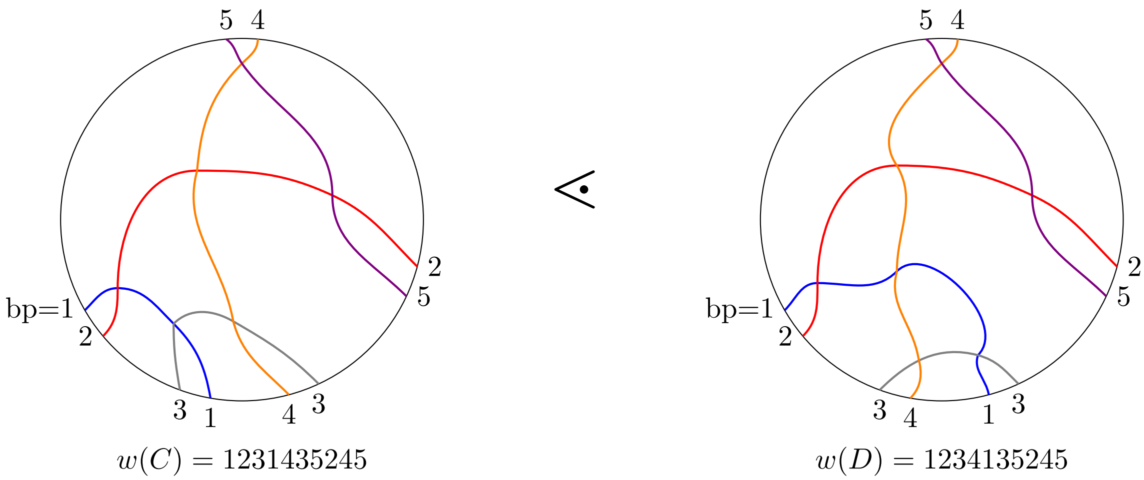


Figure 3.1: A cover relation in the uncrossing poset P_5.

The labels on the cover relations of P_n^* are ordered according to the following total order $<_\lambda$. Suppose $i < j$ as integers. The set of ordered pairs (i, j) are ordered amongst themselves lexicographically:

$$(1,2) <_\lambda (1,3) <_\lambda< (1,4) <_\lambda \ldots (2,1) <_\lambda (2,2) <_\lambda \ldots <_\lambda (n-1,n)$$

The set of ordered pairs (j, i) are ordered amongst themselves reverse lexicographically according to the second coordinate, with ties broken by reverse lexicographic order on the first coordinate:

$$(n,n{-}1) <_\lambda (n,n{-}2) <_\lambda (n{-}1,n{-}2) <_\lambda (n,n{-}3) <_\lambda \ldots <_\lambda (n,1) <_\lambda (n{-}1,1) <_\lambda \ldots <_\lambda (2,1)$$

If $i < j$ and $m > k$, then $(i, j) <_\lambda L <_\lambda (m, k)$.

Theorem 3.1.7 (Theorem 3.18 [16])**.** *The uncrossing poset P_n is dual EC-shellable via the labeling λ.*

3.1.2 The relationship between the uncrossing posets and the Tuffley posets

Now we are ready to describe a relationship between uncrossing posets and Tuffley posets. This relationship is also essentially in results of Kenyon and Wilson in [19]. We first need the following definition:

Definition 3.1.8. An **abstracted element** of $S([n])$, denoted x_a, is obtained by taking an element x of $S([n])$ and replacing all labels but the label 1 with variables.

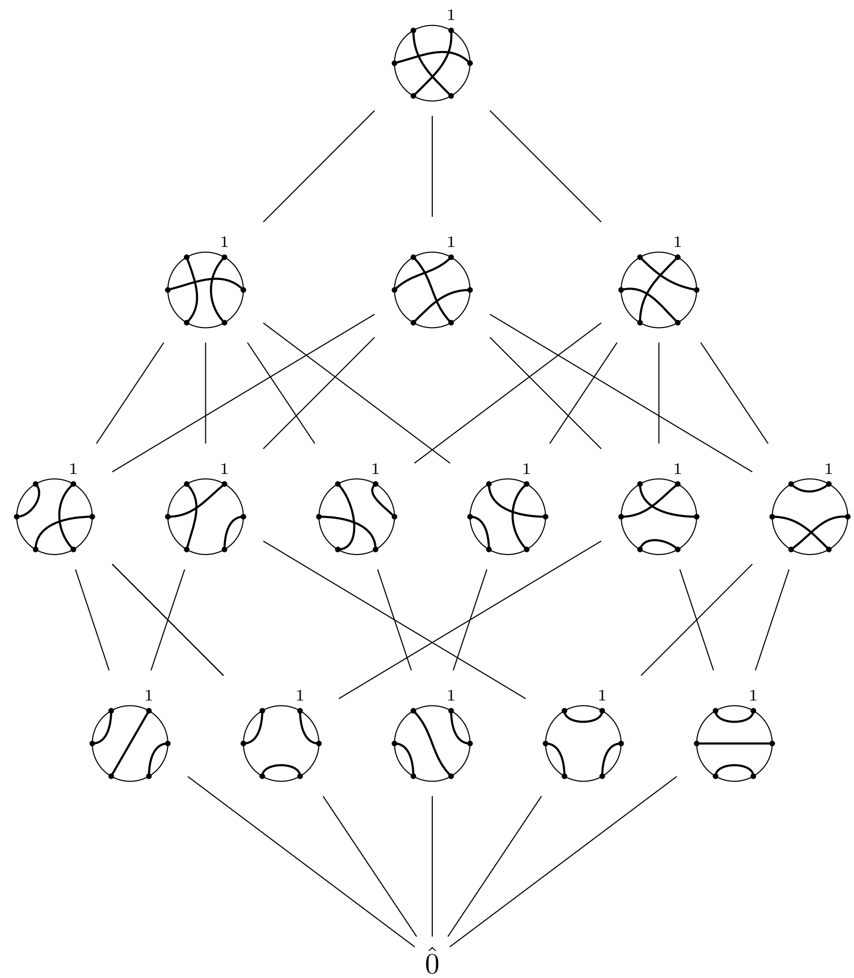


Figure 3.2: The uncrossing poset P_3 with an artificial $\hat{0}$.

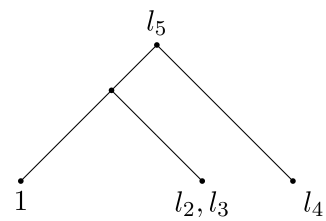


Figure 3.3: An abstracted element x_a of the Tuffley poset $S([5])$. Note that there are several elements $x \in S([5])$ yielding x_a.

See Figure 3.3 for an example of an abstracted element of $S([n])$.

Remark 3.1.9. Cover relations in the Tuffley poset are independent of the specific labels on vertices of the $[n]$-forests. In particular, the safe deletion of edge e in $x \in S([n])$ only requires that the endpoints of e be labeled or of degree greater than 3. The actual labels on the endpoints of e is irrelevant. It follows that an interval $[\hat{0}, \Gamma] \subseteq S([n]) \cup \{\hat{0}\}$ can be completely determined by the abstracted element Γ_a. Thus $[\hat{0}, \Gamma]$ is isomorphic to the corresponding interval of abstracted elements with cover relations unchanged i.e. given by edge contractions and safe edge deletions (see Definition 1.4.13). We call this interval of abstracted elements an **abstracted interval** and will denote it $[\hat{0}, \Gamma]_a$.

We now describe how we can construct a wire diagram $f(x_a)$ from an abstracted element x_a of $S_{[n]}$. We begin by inscribing x_a in a circle K so that any labeled vertex is a point on K and the remainder of x_a appears strictly inside K via an embedding. The edges of x_a separate the interior of K into chambers. A **chamber** is a connected subset of the interior of the circle whose boundary consists of an arc on K with labeled vertices of x_a as endpoints along with a subset of the edges of x_a. See Figure 3.4 for an example of a chamber.

We cross from one chamber to another by crossing some edge e of x_a. The edge e has two endpoints: one to the left as we cross e, say v_{e_l}, and one to the right as we cross e, say v_{e_r}. If v_{e_l} is adjacent to another edge of x_a, say e_l, then crossing e_l is called a **left crossing**. If v_{e_r} is adjacent to another edge of x_a, say e_r, then crossing e_r is called a **right crossing**. See Figure 3.5 for an example of a left crossing and an example of a right crossing.

Let v be a labeled vertex of x_a with m labels. For each label on v, we place a pair of wire endpoints, or **nodes**, on K with one node immediately counterclockwise from v and the other $2m-1$ nodes immediately clockwise from v. As a result, there is a node corresponding to v that is furthest clockwise from v and there is a node corresponding to v that is furthest counterclockwise from v. We call these two nodes the **outermost nodes** corresponding to v. We call the nodes corresponding to v that are not outermost nodes **inner nodes**. For each

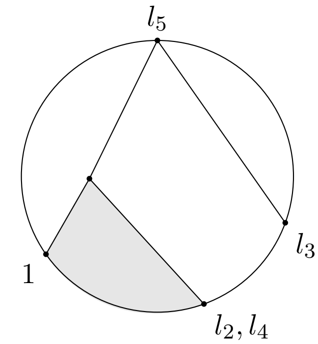


Figure 3.4: An abstracted element of $S([5])$ inscribed in a circle K. One of four chambers is shaded.

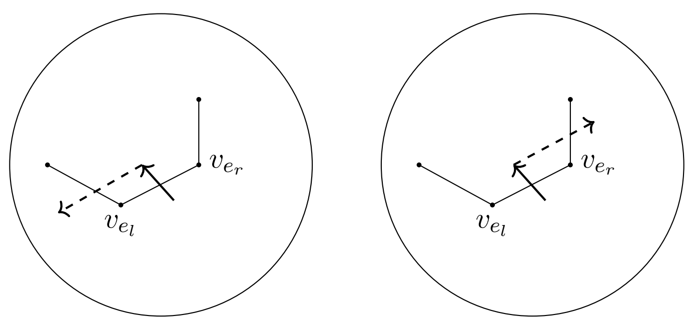


Figure 3.5: A left crossing (left) and a right crossing (right).

labeled vertex of x_a, place nodes on K in this manner.

We now draw wires between pairs of nodes as follows. Suppose w_i is an outermost node for the labeled vertex v. If w_i is clockwise from v, we draw a wire by starting at w_i, crossing the edge adjacent to v and then alternating between left crossings and right crossings, starting with a left crossing. If w_i is counterclockwise from v, we draw a wire by starting at w_i, crossing the edge adjacent to v and then alternating between left crossings and right crossings, starting with a right crossing. We continue crossing edges by alternating between a left crossing and a right crossing until we can no longer perform an alternating crossing. In this case, if our final edge crossing was a right crossing over edge e, this means v_{e_l} is on K and we end the wire at the node furthest clockwise from v_{e_l}. This node is an outermost node corresponding to the labeled vertex v_{e_l}. If our final edge crossing was a left crossing over edge e, then v_{e_r} is on K and we end at the node furthest counterclockwise from v_{e_r}. This node is an outermost node corresponding to the labeled vertex v_{e_r}. See Figure 3.6 for an example. We perform this process on the outermost nodes corresponding to each labeled vertex of x_a so each outermost node is the endpoint of a wire. For a labeled vertex v of x_a with more than one label, we connect remaining pairs of adjacent nodes corresponding to v with wires so that each wire connecting inner nodes for v crosses no other wire. This wire diagram that has x_a superimposed is what we will call the **intermediate diagram** of x_a. We will denote it $I(x_a)$.

Remark 3.1.10. Suppose $I(x_a)$ is the intermediate diagram of x_a. Suppose some wire w crosses edge e of x_a in $I(x_a)$ and then immediately after, crosses edge e'. Then e and e' are adjacent by definition of left crossings and right crossings. Because of this, we can view each wire w as marking the path in x_a between the labeled vertices corresponding to the nodes of w.

Remark 3.1.11. If the vertex v has m labels, then there are at least $m-1$ (and at most m) wires that do not cross any other wire and with the property that both nodes of each wire correspond to the vertex v.

We label the wires counterclockwise in order starting from the node that is clockwise from the vertex labeled 1. This node is the base point. Suppressing x_a, we obtain the wire diagram $f(x_a)$. See Figure 3.11.

Lemma 3.1.12. *Let $D = f(x_a)$ be a wire diagram arising from the abstracted element x_a. Then D has n wires, each of which crosses any other wire at most once.*

Proof. Let w be a wire in $f(x_a)$ with nodes w_i and w_f. The vertices w_i and w_f each correspond to a labeled vertex in a tree t in x_a. Since t is a tree, there is a unique path in t from the vertex corresponding to w_i to the vertex corresponding to w_f. By construction of $I(x_a)$, the wire w crosses each edge of the tree t that is in the path from the vertex corresponding to w_i to the vertex corresponding to w_f in $I(x_a)$. We will show that any wire in $I(x_a)$ crosses any other

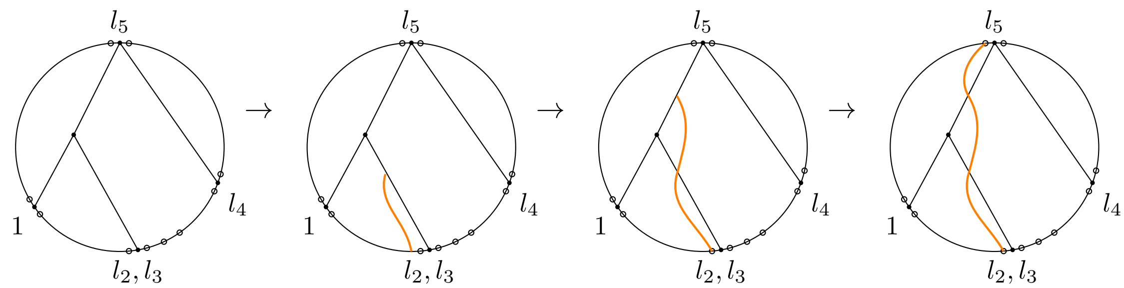


Figure 3.6: We construct a wire diagram $f(x_a)$ from the abstracted element x_a of $S([5])$ by drawing wires. Wire crossings correspond to edges of x_a.

wire at most once, i.e. that no pair of wires has a double crossing. Assume to the contrary that $f(x_a)$ has two wires, w_1 and w_2, that cross at edge e and at edge e' for $e \neq e'$. Since there is exactly one path between any two vertices in x_a, then there is exactly one path from an endpoint v of e to an endpoint v' of e'. Since both wires w_1 and w_2 cross edges e and e', both wires can be viewed as marking a path in x_a that contains both v and v'. There is a unique path from v to v' in x_a, thus every edge between v and v' are also in the paths marked by w_1 and w_2. We will call the set of edges of x_a that are crossed by both w_1 and w_2 in $I(x_a)$ mutual edges. Any vertex v of x_a adjacent to two mutual edges, say e_1 and e_2, must be of degree two and unlabeled, otherwise w_1 and w_2 cannot cross both of these mutual edges. In particular, if w_1 crosses e_1 with a right crossing, it must then cross w_2 with a left crossing. In this case, w_2 must cross e_1 with a left crossing and then cross e_2 with a right crossing. The vertex v being of degree two and unlabeled contradicts x_a being an abstracted element of $S([n])$. Thus w_1 and w_2 cannot cross more than once.

By construction, there are $2n$ nodes, each of which corresponds to a labeled vertex of x_a. Each wire has two nodes, which are attached by a wire that follows the unique path between the labeled vertices corresponding to each node. Thus there are n wires. $\square$

We will need the following definitions before we can define an inverse for the map f:

Definition 3.1.13. Let D be a wire diagram and K the exterior circle. An **arc** of K is any part of K between two adjacent nodes of D.

Definition 3.1.14. Let D be a wire diagram. A **segment** of a wire w is any part of the wire between two wire crossings in D.

Observe that the wires of a wire diagram (respectively an intermediate diagram) subdivide the interior of the circle K.

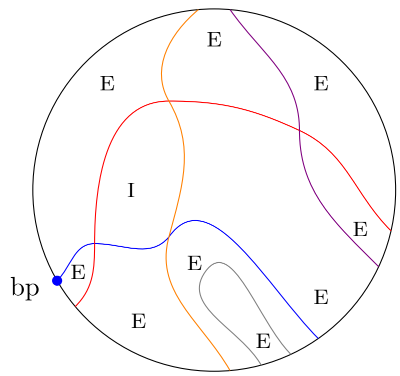


Figure 3.7: A wire diagram $D = f(x_a)$. The interior subdivisions are marked with an I while the exterior subdivisions are marked with an E.

Definition 3.1.15. Let D be a wire diagram. A **subdivision** of D is a connected subset of the interior of the circle K with a subset of segments of wires and a subset of arcs of K as its boundary. We will call any subdivision that does not have an arc of K as part of its boundary an **interior subdivision** and any subdivision that contains an arc of K in its boundary an **exterior subdivision**.

See Figure 3.7 for an example of a wire diagram with the interior and exterior subdivisions marked.

The following propositions will help to justify our construction for the inverse of f.

Proposition 3.1.16. *Let D be a wire diagram and K the exterior circle. There are an even number of arcs of K.*

Proof. Each node is the left endpoint of some arc of K. Since there are an even number of nodes, there are an even number of arcs of K. □

Proposition 3.1.17. *Let x_a be an abstracted element of $S([n])$ and let $I(x_a)$ be its intermediate diagram. If S and S' are adjacent exterior subdivisions, then one and only one of S or S' contains a vertex.*

Proof. Wires are constructed between pairs of nodes, each of which corresponds to a label on a vertex. If S and S' are adjacent exterior subdivisions of $I(x_a)$, then a single wire w separates them. Furthermore, there is a single node w_d (an endpoint of w) between them. This node must correspond to a vertex by construction. If S does not contain a vertex, then w_d must correspond to a vertex in S'. If S' does not contain a vertex, then w_d must correspond to a vertex in S.

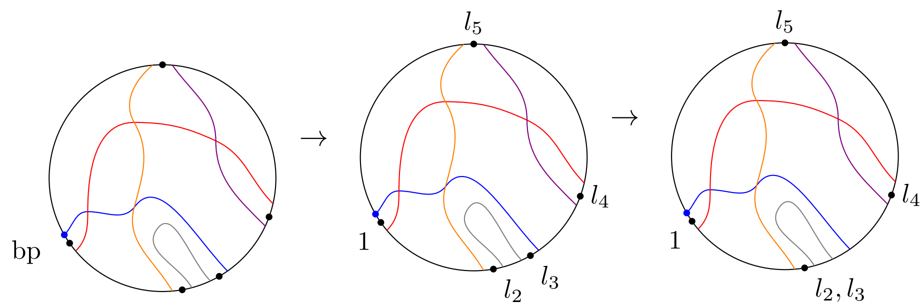


Figure 3.8: Adding labeled vertices to K in the construction of $g(f(a_x))$.

Thus at least one of S and S' contains a vertex. Since a node corresponds to only one vertex, w_d corresponds to either a vertex in S or a vertex in S', but not both. $\square$

Given any wire diagram $D = f(x_a)$ arising from an abstracted Tuffley poset element x_a, there is a map g such that $g(f(x_a)) = x_a$, defined as follows. Place a vertex in the middle of the arc whose left endpoint is the base point and label it with a 1. Then place a vertex in the middle of alternating arcs of K so that for any two adjacent arcs, exactly one contains a vertex. This is possible by Proposition 3.1.16. Label these vertices with variables. If two vertices are on the boundary of the same subdivision, identify them and label this new, identified vertex with the union of the labels on the original vertices. See Figure 3.8 for an example.

Remark 3.1.18. Because there are $2n$ total arcs in $f(x_a)$, we have n total labels.

For any vertex v, v is on the boundary of some exterior subdivision S. Draw an edge e from v through any crossing of wires in the boundary of S. The edge e is contained in the subdivision S and some new subdivision S'. If S' contains a vertex v', let v' be the other endpoint of e. Then repeat the process. If S' does not contain a vertex, then S' is an interior subdivision. Place the other endpoint v' of e in the center of S'. Draw an edge from v' through any intersection of wires contained in the boundary of S' and repeat the process. Continue until an edge is drawn through all wire crossings in D. This diagram (with the wire diagram D superimposed) is called the **intermediate diagram** of $f(x_a)$ and is denoted $I(f(x_a))$. Suppressing D and K we have the abstracted element $g(f(x_a))$ of $S([n])$. See Figure 3.11.

Lemma 3.1.19. *If x_a is an abstracted element of the Tuffley poset, then $g(f(x_a))$ is isomorphic to x_a.*

Proof. The base point of $f(x_a)$ is the node in $I(x_a)$ counterclockwise from the vertex labeled 1. In $I(f(x_a))$, a vertex is placed on the arc of K that has the base point of $f(x_a)$ as its left endpoint. Because of this, Proposition 3.1.17, and the fact that vertices are placed on alternating

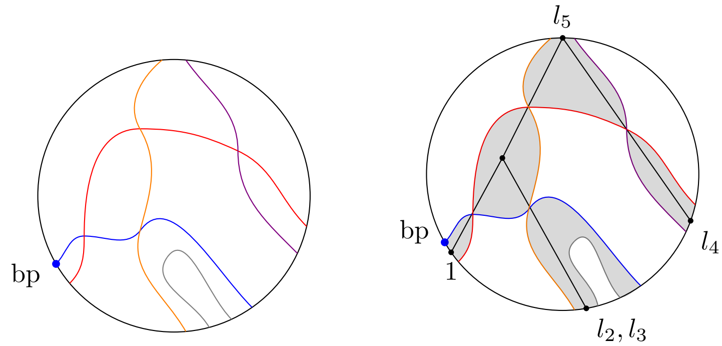


Figure 3.9: A wire diagram $f(x_a)$ and its intermediate diagram $I(f(x_a))$. The vertex subdivisions are shaded.

arcs of K in the construction of $I(f(x_a))$, any exterior subdivision of $I(x_a)$ that contains a vertex corresponds to an exterior subdivision of $I(f(x_a))$ that contains a vertex. Suppose an arc of K in the boundary of the exterior subdivision S contains a labeled vertex v. If there are m arcs of K in the boundary of S, then v is labeled with m labels: in constructing $I(f(x_a))$, we placed a vertex labeled with one label on each arc in the boundary of S and then identified these m vertices.

By construction, every wire crossing in $I(f(x_a))$ coincides with an edge in $I(f(x_a))$ and likewise for $I(x_a)$. If vertex v is adjacent to vertex v' in x_a, then there is a wire crossing in $I(x_a)$. The wire crossings in $I(f(x_a))$ are obtained from the wire crossings in $I(x_a)$. Thus for an edge in x_a from v to v', there is an edge in $I(f(x_a))$, and thus a vertex corresponding to v' in $I(f(x_a))$ by construction of $I(f(x_a))$. Thus every vertex in x_a has a corresponding vertex in $g(f(x_a))$ and if two vertices are adjacent in x_a, there are corresponding vertices in $g(f(x_a))$ that are adjacent. Thus x_a and $g(f(x_a))$ are isomorphic. $\square$

Because x_a and $g(f(x_a))$ are isomorphic as graphs, the intermediate diagram of $f(x_a)$ and the intermediate diagram of x_a are the same. Thus we will use the term intermediate diagram to refer to both $I(f(x_a))$ and $I(x_a)$.

We will need the following definitions for Theorem 3.1.22 and its proof.

Definition 3.1.20. Let D be an intermediate diagram. Any subdivision of D that contains a vertex is called a **vertex subdivision**.

See Figure 3.9 for an example of an intermediate diagram with its vertex subdivisions shaded.

Definition 3.1.21. Let $x \in P$ for some poset P. The **principal order ideal** generated by x is $I(x) = \{y \in P : y \leq x\}$.

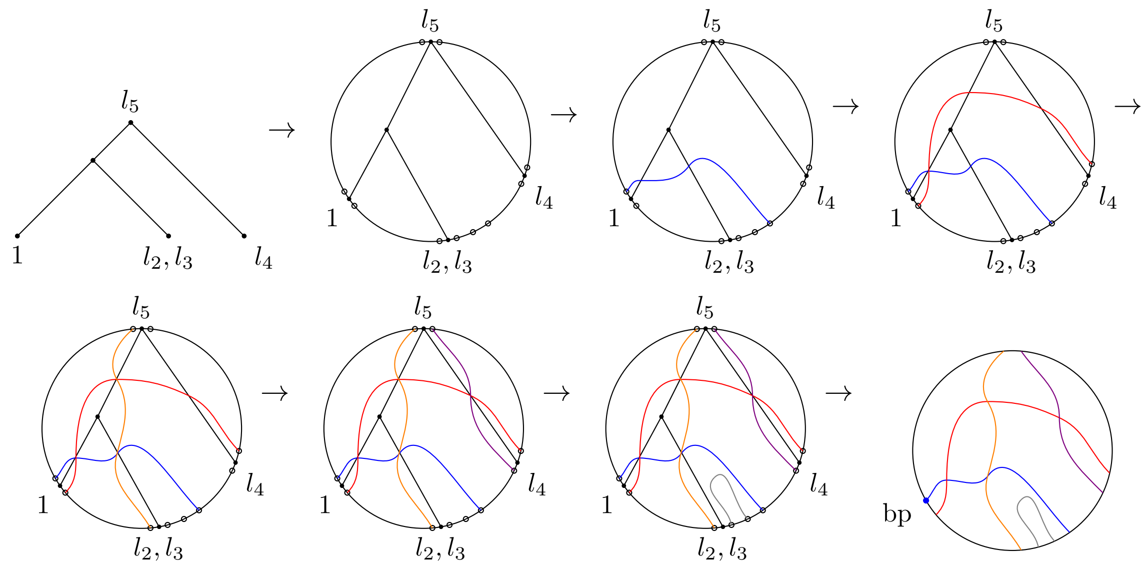


Figure 3.10: We can obtain a wire diagram $D \in P_n$ from any abstracted element of $S([n])$.

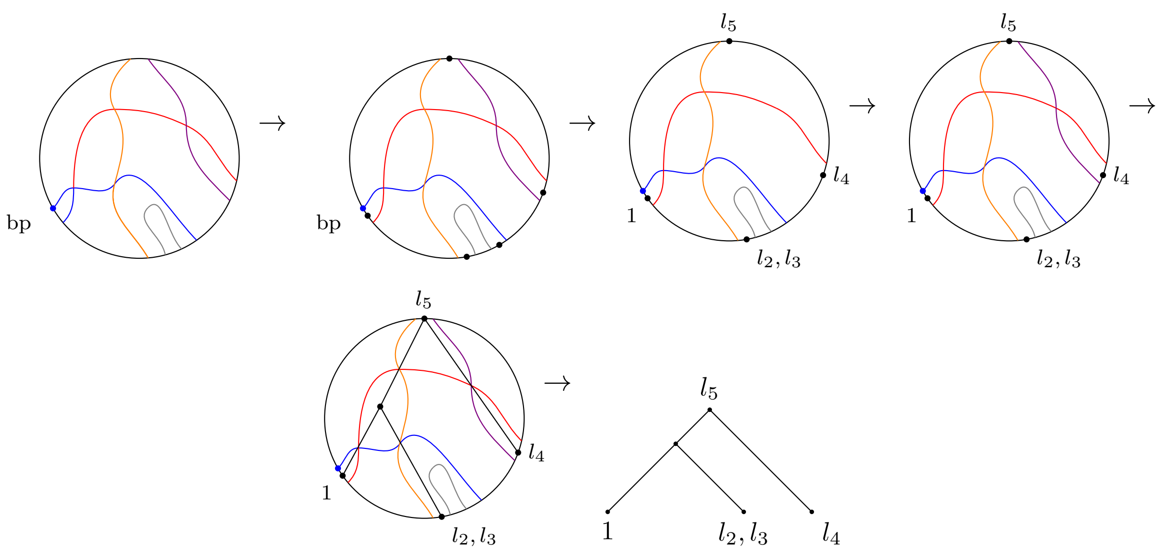
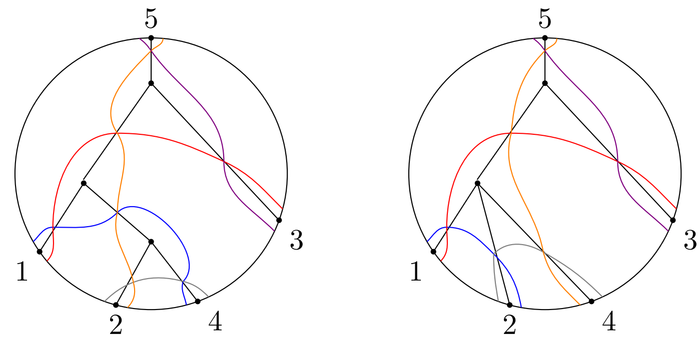
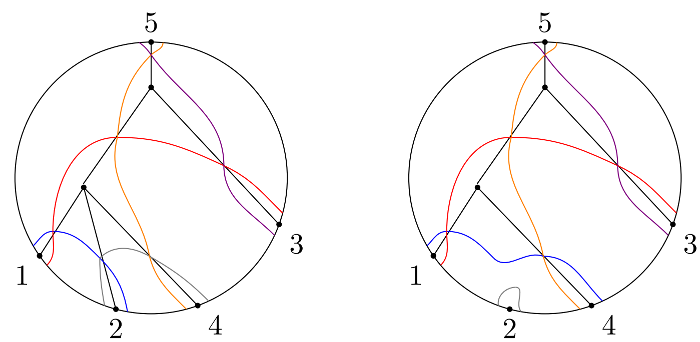


Figure 3.11: From certain wire diagrams $D \in P_n$, we can obtain an abstracted element of $S([n])$.



Edge contraction.



Safe edge deletion.

Figure 3.12: Edge contraction and safe edge deletion yield cover relations in the Tuffley poset. These operations correspond to uncrossing wires in a wire diagram, yielding cover relations in the uncrossing poset.

Theorem 3.1.22. *Intervals* $[\hat{0}, \Gamma] \subseteq S([n]) \cup \{\hat{0}\}$ *are isomorphic to principal order ideals of* P_n.

Proof. By Remark 3.1.9, the interval $[\hat{0}, \Gamma] \subseteq S([n]) \cup \{\hat{0}\}$ is isomorphic to the interval of abstracted elements $[\hat{0}, \Gamma]_a$. The map f is a bijection with inverse g between abstracted elements of $S([n])$ and a subset of wire diagrams of P_n by Lemma 3.1.19. We will show that $x_a \leq y_a$ in $[\hat{0}, \Gamma]_a$ if and only if $f(x_a) \leq f(y_a)$ in P_n. It suffices to show that x_a is obtained from some abstracted element x'_a by an edge contraction or safe edge deletion if and only if $f(x_a)$ is obtained from $f(x'_a)$ by uncrossing a pair of wires so that the minimal crossing number in $f(x_a)$ is one less than the minimal crossing number in $f(x'_a)$.

Let $I(x'_a)$ be the intermediate diagram of x'_a and let $I(x_a)$ be the intermediate diagram of x_a. Let w_1 and w_2 be two wires that cross in $I(x'_a)$ and whose crossing coincides with the edge e of x'_a. The edge e is contained in two subdivisions, say S and S', of $I(x_a)$, where one endpoint of e is in S and the other endpoint is in S'. Suppose x_a is obtained from x'_a by contracting edge e. We obtain $I(x_a)$ from $I(x'_a)$ by uncrossing w_1 and w_2 so that S and S' merge and then contracting edge e. Now suppose x is obtained from x' by a safe deletion of edge e. Then $I(x_a)$ is obtained from $I(x'_a)$ by uncrossing wires w_1 and w_2 so both wires still intersect e and then deleting edge e. In both cases, since $I(x_a)$ has one less edges than $I(x'_a)$, $f(x_a)$ has exactly one less crossing than $f(x'_a)$. Because x_a is an abstracted $X-$tree, $f(x_a)$ has no double crossings by Lemma 3.1.12. Thus the minimal crossing number in $f(x_a)$ is one less than the minimal crossing number in $f(x'_a)$.

On the other hand, consider if $f(x_a) \leq f(x'_a)$ in P_n. We want to show that $x_a \leq x'_a$ as abstracted elements of $S([n])$. It suffices to show that if $f(x_a)$ is obtained from $f(x'_a)$ by uncrossing a pair of wires so that the minimal crossing number of $f(x_a)$ is precisely one less than the minimal crossing number of $f(x'_a)$, then x_a can be obtained from x'_a by an edge contraction or safe edge deletion. Let $I(f(x_a))$ be the intermediate diagram of $f(x_a)$ and let $I(f(x'_a))$ be the intermediate diagram of $f(x'_a)$. Suppose $f(x_a)$ is obtained from $f(x'_a)$ by uncrossing wires w_1 and w_2 and suppose the minimal crossing number of $f(x_a)$ is precisely one less than the minimal crossing number of $f(x'_a)$. This uncrossing in $I(f(x'_a))$ either

(i) decreases the number of vertex subdivisions by exactly one or

(ii) maintains the number of vertex subdivisions.

If (i) holds, then two vertex subdivisions merge in $I(f(x'_a))$ to obtain $I(f(x_a))$. Each of these vertex subdivisions contains a vertex and these vertices are adjacent by the edge e. Since the edge e is entirely contained in a single subdivision in $I(f(x_a))$, we contract edge e to obtain a valid intermediate diagram. Thus $g(f(x_a)) = x_a$ is obtained from $g(f(x'_a)) = x'_a$ by contracting edge e. See Figure 3.13. If (ii) holds, then $g(f(x_a)) = x_a$ and $g(f(x'_a)) = x'_a$ have the same

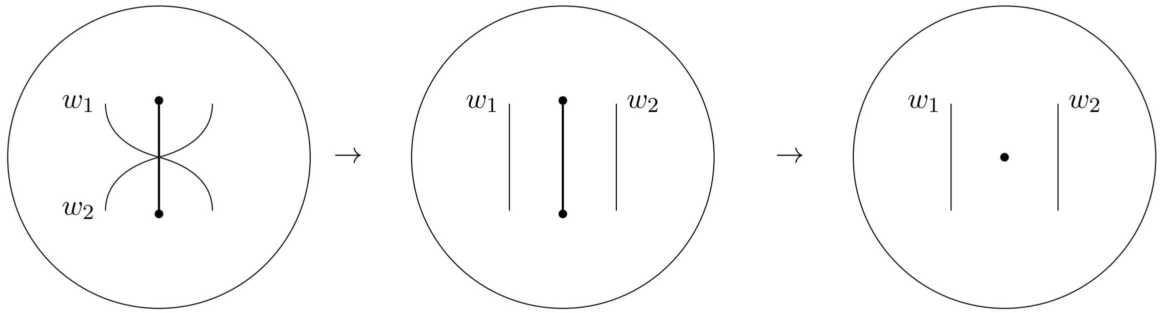


Figure 3.13: If uncrossing wires in $I(f(x'_a))$ decreases the number of vertex subdivisions, this implies an edge is contracted in x'_a.

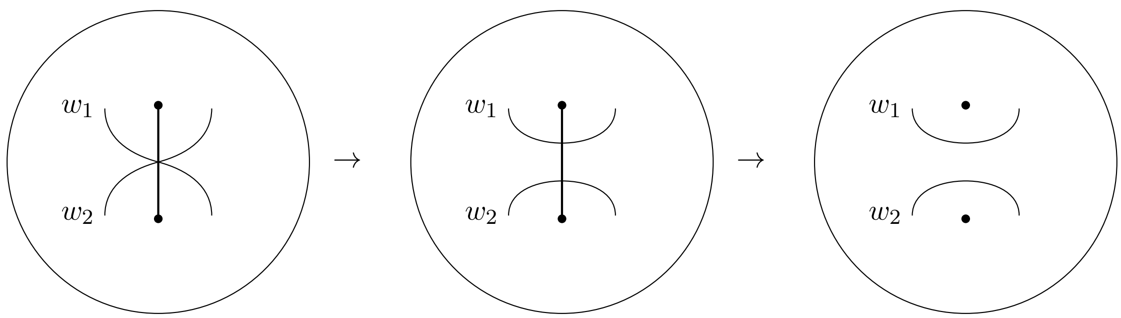


Figure 3.14: If uncrossing wires in $I(f(x'_a))$ maintains the number of vertex subdivisions, this implies an edge is deleted in x'_a.

number of vertices but x_a has one fewer edge than x'_a. Thus x_a is obtained from x'_a by deleting an edge. See Figure 3.14. We must show this is a safe edge deletion. We do so by showing that if the resulting subdivisions after uncrossing are not exterior subdivisions (in which case the corresponding vertex is labeled), then they have at least three wires (and thus three crossings) in their boundaries. Assume that instead there is an interior subdivision S with two wires in its boundary. Then these two wires must cross twice, otherwise S is not bounded by only these wires. Then this wire diagram has an isotopy class representative in which these wires do not cross at all. Since the minimum crossing number of $f(x_a)$ is at least two less than the minimum crossing number of $f(x'_a)$, this implies $f(x_a)$ is not covered by $f(x'_a)$, a contradiction.

□

Figure 3.12 gives an example of how cover relations in P_n correspond to cover relations in $S([n]) \cup \{\hat{0}\}$.

Theorems 3.1.7 and 3.1.22 together imply that the dual EC-labeling of P_n given by Hersh and Kenyon gives an explicit dual EC-labeling of intervals in $S([n]) \cup \{\hat{0}\}$:

Corollary 3.1.23. *Intervals* $[\hat{0}, \Gamma] \subseteq S([n]) \cup \{\hat{0}\}$ *are dual EC-shellable.*

3.2 Partially ordering the facets of $\mathcal{E}([n])$ by the enriched Tamari poset

The initial motivation for developing the enriched Tamari poset was in the hope of finding a shelling order on the facets of the order complex of $S([n])$. Thus this next section investigates whether there is an ordering on the coatoms of $S_{[n]} = S([n]) \cup \{\hat{0}, \hat{1}\}$ that satisfies the conditions of a recursive coatom ordering. As it turns out, no such ordering exists, as we show in Section 3.3. We also discuss why any linear extension of ET_n fails as a shelling order on the edge-product space of phylogenetic trees (in the sense of Definition 1.1.21). We also describe how ET_n can nonetheless give some insight into the structure of $\mathcal{E}([n])$. In particular, it helps us prove that $\mathcal{E}([n])$ is gallery-connected (see Definition 3.2.4).

If the order complex of $S([n])$ is shellable, then the intersection of the closure of any facet with the closures of the previous facets in the shelling order must be pure and codimension one. The next definitions and lemma help us describe the intersection of facets of the order complex of $S([n])$.

Definition 3.2.1. For $x \in S([n])$, $\overline{x}$ is the subcomplex of $\mathcal{E}([n])$ consisting of all faces represented by elements in the principal order ideal generated by x.

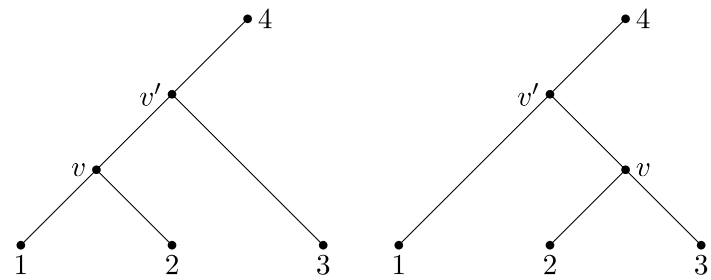
Note that $\overline{x}$ has face poset $I(x)$.

Definition 3.2.2. The **intersection** of elements x and y in the poset $S([n])$ is the subcomplex of $\mathcal{E}([n])$ whose faces correspond to elements in the intersection of the principal order ideal of x and the principal order ideal of y.

Note that the intersection of principal order ideals of a poset is a poset.

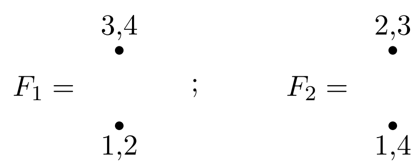
Lemma 3.2.3. *Let C_i and C_j be maximal elements of $S([n])$ such that C_j is obtained from C_i by a Tamari move. Then $I(C_i) \cap I(C_j)$ has a unique maximal element, w. Furthermore, w corresponds to a codimension one face of the faces represented by C_i and C_j in $\mathcal{E}([n])$. The same holds if C_j is obtained from C_i by a lexicographic move or if C_j is obtained by a lexicographic move on a vertex v of C_i, followed by a Tamari move on the same vertex v.*

Proof. If C_j is obtained from C_i by a Tamari move on v with parent v', then C_i and C_j differ only locally. In particular, they differ only by which edges are adjacent to v and which edges are adjacent to v'. As a result, the intersection of C_i and C_j can be obtained by considering the intersection of the elements D_i and D_j, which are shown below and given by the words $w(D_i) = ((1,2),3)$ and $w(D_j) = (1,(2,3))$ respectively, where $1, 2$, 3, and 4 represent disjoint maximal subtrees of C_i. In particular, 1 represents the subtree $\downarrow l_v(C_i)$, 2 represents the subtree $\downarrow r_v(C_i)$, 3 represents $\downarrow r'_v(C_i)$ and 4 represents the subtree obtained from C_i by removing $\downarrow v'$. Note that the intersection $I(D_i) \cap I(D_j)$ can be calculated by hand to verify this lemma, but this is tedious.



D_i $\qquad$ D_j

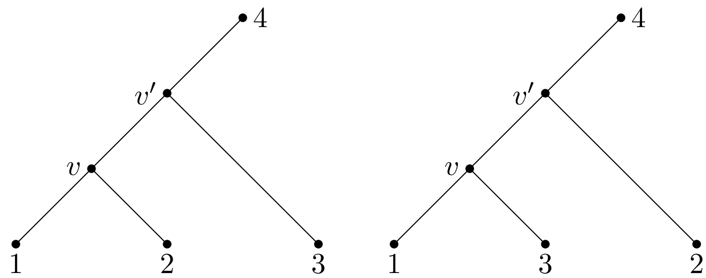
Suppose $y \in I(D_i) \cap I(D_j)$. Then by Theorem 1.4.15, y can be obtained from D_i (respectively D_j) by a series of contractions and safe edge deletions. Furthermore, by Proposition 1.4.16, at least one edge must be contracted in D_i (respectively D_j) to obtain y. Observe that if an element y is disconnected and contains an isolated vertex labeled with only one label, say x, then we can start to obtain y by first contracting edge $e = [v, v']$ and then deleting the edge adjacent to the leaf labeled x. The only disconnected elements of $S([4])$ do not contain at least one isolated vertex with one label are the following two [4]-forests:



It is clear that neither of these [4]-forests is in the intersection $I(D_i) \cap I(D_j)$. In particular, $F_1 \in I(D_i)$ but $F_1 \notin I(D_j)$ while $F_2 \in I(D_j)$ but $F_2 \notin I(D_i)$. Thus any element in $I(D_i) \cap I(D_j)$ is either disconnected and contains at least one isolated vertex labeled by one label, or is connected. We claim that if y is connected, then y can be obtained from D_i (respectively D_j) if we first contract edge $e = [v, v']$. Assume otherwise. then y would be obtainable by contracting only some subset of the edges of D_j that are adjacent to leaves. Let z_j be the element obtained from D_j by contracting all edges adjacent to leaves in D_j. Then $z_j \le y$ and so $z_j \in I(D_i) \cap I(D_j)$. It is not difficult to see, however, that $z_j \not< D_i$ in $S([4])$. This means we must contract edge e to obtain y from D_i (respectively D_j). Thus any element $y \in I(D_i) \cap I(D_j)$ can be obtained by first contracting edge e in D_i (or D_j). Let w' be the element obtained by contracting edge e in D_j. Since w' is obtained by contracting edge e in D_j and D_j is obtained from D_i by a Tamari move on vertex v, Lemma 1.4.17 says that $w' \lessdot D_i$.

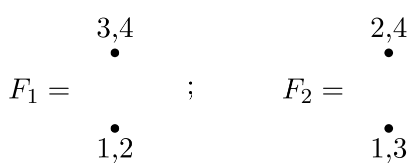
We have shown that if $y \in I(D_i) \cap I(D_j)$, y can be obtained from D_i or D_j by first contracting edge e. Thus if $y \in I(D_i) \cap I(D_j)$, then $y < w'$. This implies w' is the unique maximal element of $I(D_i) \cap I(D_j)$ and thus that $I(C_i) \cap I(C_j)$ has a unique maximal element w obtained by contracting edge $e = [v, v']$ in C_i (or C_j). Since w is obtained by contracting a single edge in either C_i or C_j, w represents a codimension one face within the faces represented by C_i and C_j.

This same argument holds if C_j is obtained from C_i by a lexicographic move on vertex v with parent v'. As above, we only need to consider the intersection of the two elements D_i and D_j in $S([4])$ given by the words $w(D_i) = ((1,2),3)$ and $w(D_j) = (1,(3,2))$, where $1, 2$, 3, and 4 represent disjoint maximal subtrees of C_i. In particular, 1 represents the subtree $\downarrow l_v(C_i)$, 2 represents the subtree $\downarrow r_v(C_i)$, 3 represents $\downarrow r'_v(C_i)$ and 4 represents the subtree obtained from C_i by removing $\downarrow v'$.



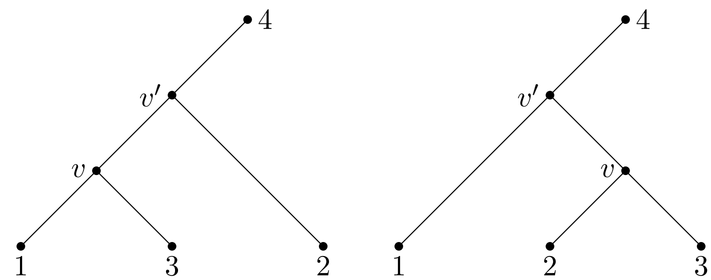
D_i D_j

This time, the only disconnected elements not contained in $I(D_i) \cap I(D_j)$ are the following two $[4]-$forests:



Thus any element in $I(D_i) \cap I(D_j)$ is either disconnected and contains at least one isolated vertex labeled by one label, or is connected. The rest of the argument is identical to the argument above.

Finally, assume C_j is obtained from C_i by a lexicographic move followed by a Tamari move on vertex v with parent v'. As above, we only need consider the intersection of the two elements D_i and D_j in $S([4])$ given by the words $w(D_i) = (1,(2,3))$ and $w(D_j) = ((1,3),2)$, where $1, 2, 3$, and 4 represent disjoint maximal subtrees of C_i. In particular, 1 represents the subtree $\downarrow l_v(C_i)$, 2 represents the subtree $\downarrow r'_v(C_i)$, 3 represents $\downarrow r_v(C_i)$ and 4 represents the subtree obtained from C_i by removing $\downarrow v'$.



D_i D_j

This time, the only disconnected elements not contained in $I(D_i) \cap I(D_j)$ are the following two [4]-forests:

$F_1 =$ 

; $F_2 =$ 

Thus any element in $I(D_i) \cap I(D_j)$ is either disconnected and contains at least one isolated vertex labeled by one label, or is connected. The rest of the argument is identical to the first argument. $\square$

One notion that is helpful in determining the structure of a topological space is that of gallery-connectedness. The following theorem uses Lemma 3.2.3 to show that the edge-product space of phylogenetic trees is gallery-connected.

Definition 3.2.4. A pure CW-complex Δ is **gallery-connected** if for any two d-dimensional facets F_j and F_k, there exists a sequence $F_j = F_0, G_1, F_1, G_2, F_2, \ldots G_{r-1}, F_{r-1}, G_r, F_r = F_k$ where for $1 \leq i \leq r$, F_i is d-dimensional, G_i is (d-1)-dimensional and $F_{i-1} \cap F_i = G_i$.

Theorem 3.2.5. *The edge-product space of phylogenetic trees, $\mathcal{E}([n])$, is gallery-connected.*

Proof. For this proof, the relation $\leq$ will denote the relation given by the enriched Tamari poset. The enriched Tamari poset ET_n is a connected partial order on the maximal elements of the Tuffley poset, $S([n])$. Since each element of the Tuffley poset represents a face of the edge-product space of phylogenetic trees, ET_n gives a partial order on the facets of $\mathcal{E}([n])$. By Lemma 3.2.3, if C_x and C_y are such that $C_x \lessdot C_y \in ET_n$, then $I(C_x) \cap I(C_y)$ has a unique maximal element, say G_{xy} in $S([n])$. This element G_{xy} corresponds to a face of $\mathcal{E}([n])$ that is codimension one within the closures of the faces corresponding to C_x and C_y.

Now let C_i and C_j be elements of ET_n that represent any two facets of $\mathcal{E}([n])$. We consider two cases.

In case 1, let C_i and C_j be comparable in ET_n. Without loss of generality, assume $C_i < C_j$. There is a saturated chain from C_i to C_j: $C_i \lessdot C_{i_1} \lessdot C_{i_2} \lessdot \ldots \lessdot C_{i_{k-1}} \lessdot C_{i_k} = C_j$. The sequence $C_i, G_{i,i_1}, C_{i_1}, G_{i_1,i_2}, C_{i_2} \ldots C_{i_{k-1},j}, C_j$ is the desired sequence.

For case 2, assume C_i and C_j are incomparable in ET_n. Since there exists a unique minimal element in ET_n, $\hat{0}$, there exists a saturated chain from $\hat{0}$ to C_i: $\hat{0} = C_{i_0} \lessdot C_{i_1} \lessdot C_{i_2} \lessdot \ldots \lessdot C_{i_{k_i}} = C_i$. Likewise there exists a saturated chain from $\hat{0}$ to C_j: $\hat{0} = C_{j_0} \lessdot C_{j_1} \lessdot C_{j_2} \lessdot \ldots \lessdot C_{j_{k_j}} = C_j$. Then we can walk along the edges of ET_n from C_i to $\hat{0}$ and then from $\hat{0}$ to C_j and we obtain the desired sequence:

$$C_i, G_{i_{k_i} i_{k_i-1}}, C_{i_{k_i-1}}, \ldots C_{i_1}, G_{i_1,i_0}, \hat{0}, G_{j_0 j_1}, C_{j_1}, \ldots G_{j_{k_j-1} j_k}, C_j.$$

$\square$

Now let us introduce a notion we will use repeatedly.

Definition 3.2.6. Let K be a CW complex and P a partial order on its facets. Let $C_i, C_j \in P$. Then P satisfies the **codimension one property** if the subcomplex $\overline{C_j} \bigcap \left(\cup_{i<j} \overline{C_i}\right)$ of K is pure and codimension one within $\overline{C_j}$.

As we see in the next example, ET_n does not have the codimension one property when viewed as an ordering on the facets of the edge-product space of phylogenetic trees.

Example 3.2.7. Figure 3.15 shows a maximal element in the intersection of the maximal elements C_i and C_j of $S([6])$. However, $F \not< w$ in $S([6])$ for any w with the following properties: $w \in I(C_j) \cap I(C_k)$ for $C_k <_{ET_n} C_j$ and w is obtained from C_j by contracting a single edge. Thus $\overline{C_j} \cap (\cup_{C_i \leq_{ET_n} C_j} \overline{C_i})$ is not pure and codimension one with $\overline{C_j}$.

Remark 3.2.8. Let $\leq_{ET_n}$ denote the order on the facets of the edge-product space of phylogenetic trees given by the enriched Tamari poset. There cannot exist a linear extension $\leq$ of $\leq_{ET_n}$ that satisfies:

$$\overline{C_j} \bigcap \left(\bigcup_{C_i < C_j} \overline{C_i} \right) = \overline{C_j} \bigcap \left(\bigcup_{C_i <_{ET_n} C_j} \overline{C_i} \right) \tag{3.2.1}$$

See Example 3.2.9 for an example of why this equality fails for every linear extension of the enriched Tamari poset.

Example 3.2.9. Let C_i be such that $w(C_i) = (((1,2),4),3)$ and C_j be such that $w(C_j) = (((1,3),2),4)$. There is an element $F \in I(C_i) \cap I(C_j)$ that can be obtained from C_i (respectively C_j) by contracting an internal edge and deleting the edge adjacent to the leaf labeled 3. The elements C_i, C_j, and F are shown in Figure 3.16.

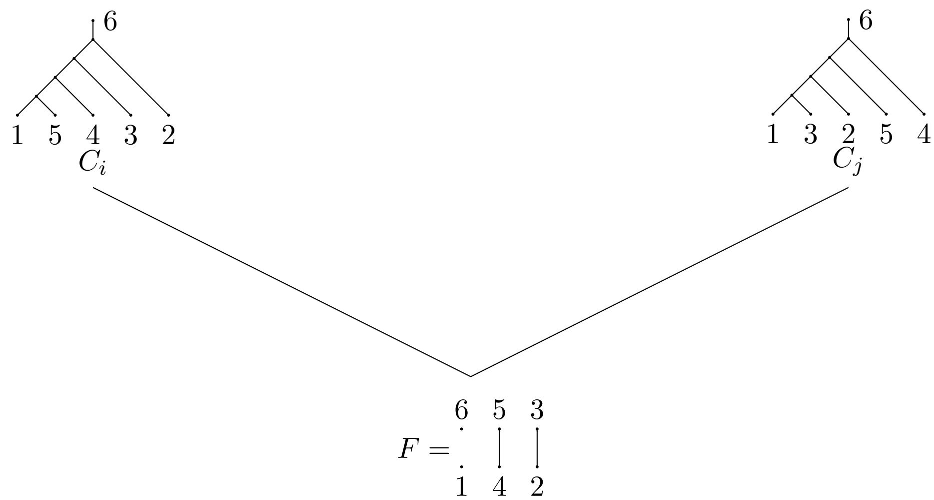


Figure 3.15: The enriched Tamari poset ET_n does not have the codimension one property. The face represented by F is not contained in any face that is represented by an element obtained from C_j by contracting a single edge.

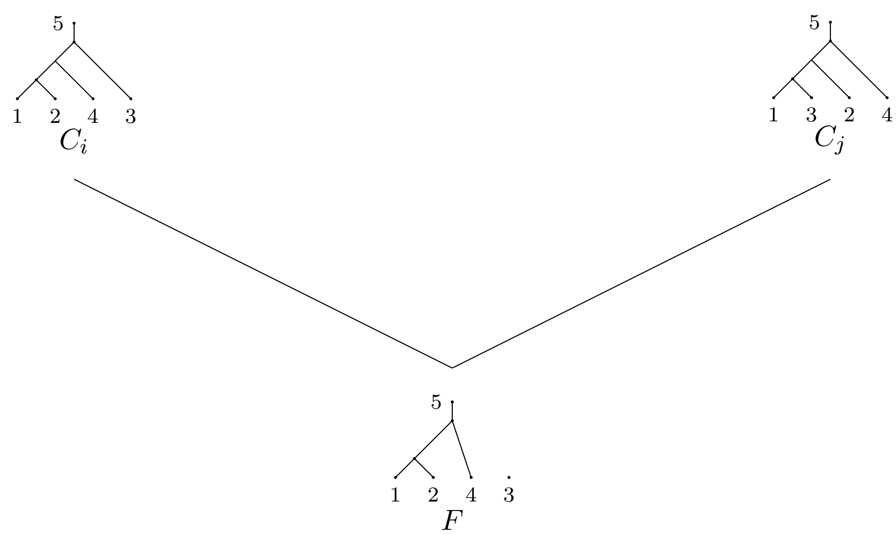


Figure 3.16: The element F is below the elements C_i and C_j in $S([5])$. In ET_5, C_i and C_j are incomparable but for any linear extension $<$ of ET_5, either $C_i < C_j$ or $C_j < C_i$.

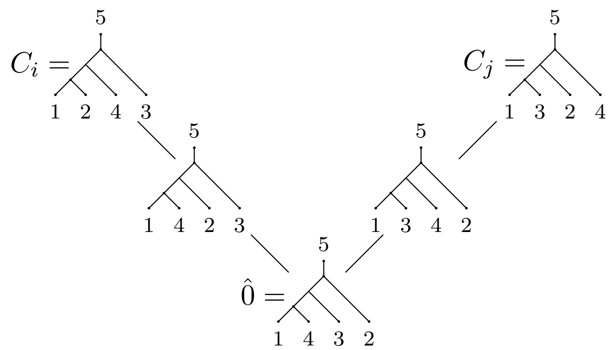


Figure 3.17: The subposet of the enriched Tamari poset consisting of all elements below either C_i or C_j given in Example 3.2.9.

Note that C_i and C_j are not comparable in ET_5. For any linear extension $\leq$ of $\leq_{ET_n}$, either $C_i \leq C_j$ or $C_j \leq C_i$. The set of elements of ET_5 below either C_i or C_j in ET_n is shown in Figure 3.17. It is straightforward to verify that $\overline{F} \nsubseteq I(C_i) \cap I(C_k)$ for any $C_k <_{ET_n} C_i$. Likewise, $\overline{F} \nsubseteq I(C_j) \cap I(C_k)$ for any $C_k <_{ET_n} C_j$. Thus for any linear extension $\leq$ of ET_n, the face represented by F is contained in the subcomplex $\overline{C_j} \cap \left(\cup_{C_i < C_j} \overline{C_i}\right)$. However, F is not contained in $\overline{C_j} \cap \left(\cup_{C_i <_{ET_n} C_j} \overline{C_i}\right)$. Thus 3.2.1 does not hold.

3.3 The poset $S_{[n]}$ does not admit a recursive coatom ordering

This section serves to show that for $n \geq 5$, the Tuffley poset with a $\hat{0}$ and $\hat{1}$ adjoined is not dual CL-shellable. We will show that for $n \geq 5$, there does not exist a total order on the coatoms of $S_{[n]} = S([n]) \cup \{\hat{0}, \hat{1}\}$ that satisfies the conditions of a recursive coatom order. In particular, we show that for any coatom ordering of $S_{[n]}$, (ii) of Definition 1.2.21 fails.

The general idea for this proof is as follows. For $n \geq 5$ and for any ordering Ω on the coatoms of $S_{[n]}$, there must be an element in Figure 3.21 that comes last, say C_i. The two elements adjacent to C_i in Figure 3.21, say C_j and C_k, are both above an element F (specifically defined in Lemma 3.3.14). Supposing C_j comes after C_k in Ω, we prove that there is no coatom that comes before C_j in Ω that covers a common element z with C_j and such that z is above F. This contradicts (ii) of Definition 1.2.21. Two results that are instrumental in proving this fact are Lemma 3.3.22 and Proposition 3.3.23 both of which follow from the idea that we can associate any element that covers F with a vertex of a tree T_F. The tree T_F has the property that two elements covering F are associated with adjacent vertices in T_F if and only if they are covered by a common coatom. Lemma 3.3.22 relies on the fact that since T_F is a tree, it has no cycles. Proposition 3.3.23 relies on the fact that since T_F is a tree, there is a path between any pair of vertices.

We first give a few definitions from elementary graph theory that we will need later in this section.

Definition 3.3.1. A **graph** G is a collection of vertices and edges between pairs of vertices.

Definition 3.3.2. A **path** in the graph G is any sequence of vertices $v_1, v_2, \ldots v_m$ such that adjacent nodes n_i, n_{i+1} in the sequence are connected by an edge G.

Definition 3.3.3. A **cycle** in the graph G is a path $v_1, v_2, \ldots v_m$ where $v_1 = v_m$.

Definition 3.3.4. A **simple cycle** in the graph G is a cycle $n_1, n_2, \ldots n_{m-1}, n_m = n_1$ where no vertex other than the first is repeated in the sequence.

The remainder of the definitions, propositions, and lemmas in this section are used to prove Theorem 3.3.25. As the proofs and notations are somewhat technical, we provide a complete example of the argument for $S_{[5]}$ at the end of this section (Example 3.3.26). We also provide examples throughout the section, especially when new notation is introduced.

We now introduce a graph that will be crucial to proving that $S_{[n]}$ does not admit a recursive coatom ordering.

Definition 3.3.5. Let $H([n])$ be the graph that has coatoms of $S_{[n]}$ (i.e. elements of $\mathcal{C}_{[n]}$) as vertices and an edge between vertex C_i and C_j if and only if C_i and C_j cover a common element in $S([n])$.

Note that $H([n])$ is sometimes called the facet-ridge graph of the CW complex $\mathcal{E}([n])$. See Figure 3.18 for the graph $H([5])$. We use the following definitions to say more about $H([n])$ in Proposition 3.3.7.

Definition 3.3.6. Let $C_i \in \mathcal{C}_{[n]}$ and suppose a vertex v' is the parent of a vertex v in C_i. We say v is **ineligible for a Tamari move** if v is the right child of v'. Otherwise, v is **eligible for a Tamari move**. The vertex v is **ineligible for a lexicographic move** if either v is the right child of v' or $\downarrow r_v <_{lex} \downarrow r_{v'}$. Otherwise, v is **eligible for a lexicographic move**.

Note that if v is ineligible for a Tamari move, then v is also ineligible for a lexicographic move.

Proposition 3.3.7. *If C_i is a vertex of $H([n])$, then C_i is adjacent to $2(n-3)$ other vertices.*

Proof. The maximal element $C_i \in S([n])$ has $(2n-3) - n = n-3$ internal edges. For each internal edge $e = [v, v']$ of C_i, there are two other maximal elements of $S([n])$ that are adjacent to C_i in $H([n])$, which are obtained as follows. Suppose v' is the parent of v. If v is eligible for

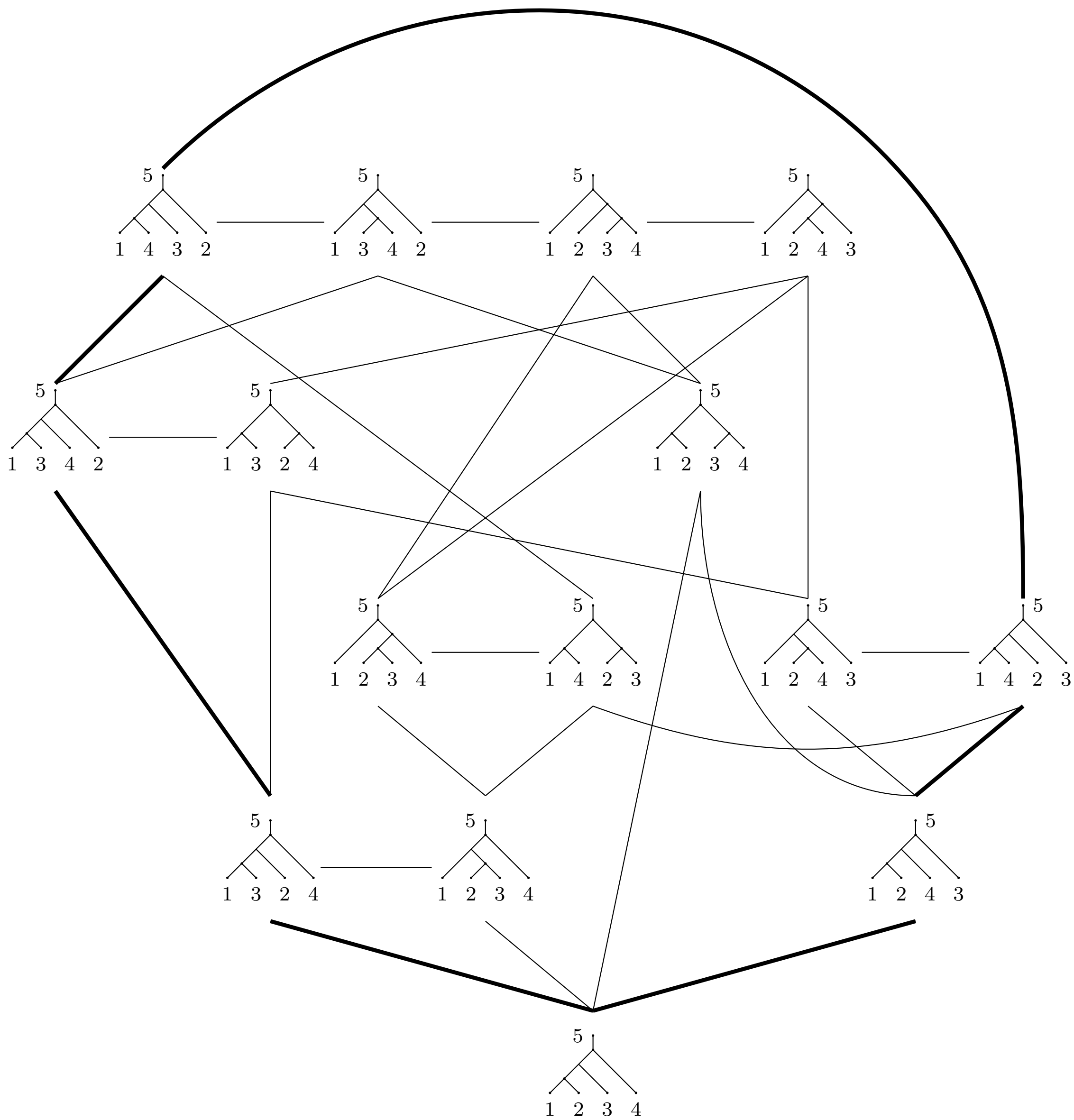


Figure 3.18: The graph $H([5])$. The bold edges mark a nontrivial cycle of $H([5])$.

a lexicographic move, one element is obtained by performing a lexicographic move on vertex v and one element is obtained by performing a lexicographic move followed by a Tamari move on vertex v. If v is ineligible for a lexicographic move but eligible for a Tamari move, one element is obtained by performing a Tamari move on vertex v and the other element is obtained by reversing a lexicographic move on vertex v. If v is ineligible for a Tamari move (and thus also ineligible for a lexicographic move), one element is obtained by reversing a Tamari move on vertex v and the other element is obtained by reversing a Tamari move on vertex v then reversing a lexicographic move on vertex v. $\square$

Definition 3.3.8. A **nontrivial** cycle of $H([n])$ is a simple cycle that contains more than 3 vertices. A **trivial** cycle is a simple cycle that contains 3 vertices.

Remark 3.3.9. By Lemma 1.4.17, any trivial cycle in $H([n])$ has three vertices, each of which is a coatom of $S_{[n]}$ that covers some element w of $S_{[n]}$.

We now relate the graph $H([n])$ to all possible recursive coatom orderings of $S_{[n]}$.

Definition 3.3.10. A coatom order $C_1, C_2, \ldots C_{(2n-5)!!}$ of $S_{[n]}$ is **compatible** with $H([n])$ if for all $j > 1$, a coatom C_j is adjacent in $H([n])$ to some C_i for $i < j$.

Lemma 3.3.11. *If $S_{[n]}$ admits a recursive coatom ordering Ω, then Ω must be compatible with $H([n])$.*

Proof. Suppose $\Omega = C_1, C_2, \ldots, C_{(2n-5)!!}$ is a recursive atom ordering on $S^*_{[n]}$. Assume Ω is not compatible with $H([n])$. Then there is some $j > 1$ such that C_j is not adjacent to any C_i in $H([n])$ for $i < j$. In other words, there is no $z \in S^*_n$ such that $C_j \lessdot z$ and $C_i \lessdot z$ for any $i < j$. In particular, though C_1 and C_j are both less than $\hat{1} \in S^*$, there is no C_k such that $k < j$ and $C_k \lessdot z$, $C_j \lessdot z$ for some $z < \hat{1} \in S^*_n$. This is a contradiction of the assumption that $S_{[n]}$ admits a recursive coatom ordering. $\square$

We will need the following notation for the Definition 3.3.12: Let C_i be a coatom of $S_{[n]}$. If C_i has a leaf labeled x, then the edge adjacent to this leaf is denoted e^x.

Definition 3.3.12. A triple $\{C_i, C_j, C_k\}$ of coatoms of $S_{[n]}$ satisfies the **triplet condition** if the following hold:

Triplet Condition:

1. C_i contains a leaf labeled x such that e^x is adjacent to two internal edges, say e_{x_1} and e_{x_2}
2. C_i and C_k cover F_k obtained by contracting e_{x_1} in C_i
3. C_i and C_j cover F_j obtained by contracting e_{x_2} in C_i.

See Figure 3.19 for an example of a triple (C_i, C_j, C_k) that satisfies the triplet condition.

Proposition 3.3.13. *If $\{C_i, C_j, C_k\}$ satisfies the triplet condition, then C_j and C_k are not adjacent in $H([n])$.*

Proof. For leaves x, y in C_i, let $d_{C_i}(x, y)$ be the number of edges in the unique path between x and y. Since $\{C_i, C_j, C_k\}$ satisfies the triplet condition, there is some leaf labeled x in C_i that is adjacent to internal edges e_{x_1} and e_{x_2}. Furthermore, C_j covers the element obtained by contracting e_{x_1} and C_k covers the element obtained by contracting e_{x_2}. Let y be any leaf for which $d_{C_j}(x, y) < d_{C_i}(x, y)$ (see Definition 1.4.18). Then $d_{C_k}(x, y) = d_{C_j}(x, y) + 2$. If C_j and C_k are adjacent in $H([n])$ then they cover a common element G which can be obtained by contracting a single edge in C_k. However, since $d_{C_k}(x, y) = d_{C_j}(x, y) + 2$, then $d_G(x, y) \leq \min\{d_{C_k}(x, y), d_{C_j}(x, y)\} = d_{C_j}(x, y) = d_{C_k}(x, y) - 2$ by Proposition 1.4.19. Thus at least two edges must be contracted in C_k to obtain G. Hence C_k does not cover G. This is a contradiction, so C_j and C_k are not adjacent in $H([n])$. $\square$

Lemma 3.3.14. *Suppose C_i, C_j, and C_k satisfy the triplet condition. Then there exists a maximal element F of $I(C_j) \cap I(C_k)$ such that $F \lessdot F_j \lessdot C_j$ and $F \lessdot F_k \lessdot C_k$ in $S_{[n]}$.*

Proof. Since C_j and C_k are not adjacent in $H([n])$, they do not cover a common element in $S([n])$. Thus $F_j \neq F_k$. By construction, in each of F_j and F_k there exists a leaf labeled x such that the other vertex of e^x has degree 4. Let F be the $[n]$-forest obtained from either F_j or F_k by deleting e^x. Since F can be obtained by contracting an edge in either C_j or C_k and then deleting e^x, $F \in I(C_j) \cap I(C_k)$. Furthermore, F is maximal since C_j and C_k are not adjacent in $H([n])$ by Proposition 3.3.13 and thus do not cover a common element. $\square$

See Figure 3.19 for an example of an element F obtained as in the proof of Lemma 3.3.14.

Given F constructed from the triple (C_i, C_j, C_k) as in Lemma 3.3.14, we can determine those coatoms of $S_{[n]}$ that are adjacent to C_i and that are above F. We do this in the next lemma, which we will need for the proof of Theorem 3.3.25.

Lemma 3.3.15. *Suppose (C_i, C_j, C_k) are coatoms of $S_{[n]}$ that satisfy the triplet condition and suppose that F is the element of $S_{[n]}$ described as in Lemma 3.3.14. Then there are only four coatoms of $S_{[n]}$, C_j, C_k, D_j, D_k, that are both adjacent to C_i in $H([n])$ and that cover F. Furthermore, C_j and D_j are adjacent and C_k and D_k are adjacent in $H([n])$.*

Proof. By definition, C_i and C_j are adjacent in $H([n])$ because they both cover F_j. Likewise, C_i and C_k are adjacent in $H([n])$ because they both cover F_k. By Lemma 1.4.17, F_j is covered by another coatom D_j and F_k is covered by another coatom D_k. Note that D_j and C_j are adjacent in $H([n])$ and D_k and C_k are adjacent in $H([n])$. Because $F_j \neq F_k$, D_j does not cover

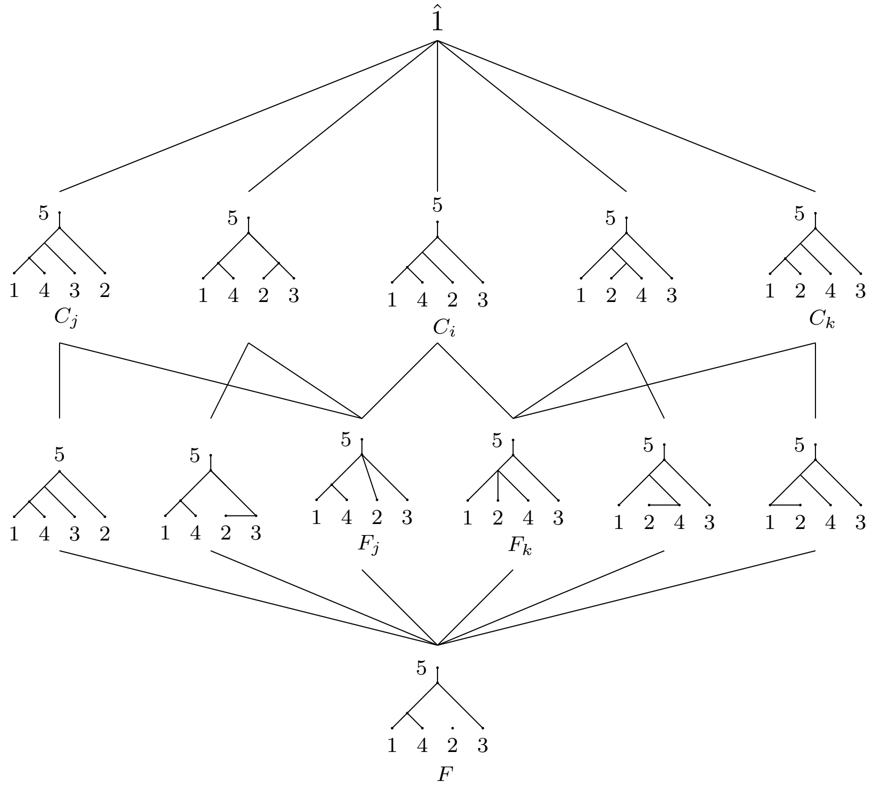


Figure 3.19: An interval $[F, \hat{1}]$ in S_5 obtained as described in Lemma 3.3.14. The element F is a maximal element of $I(C_j) \cap I(C_k)$, where $\{C_i, C_j, C_k\}$ satisfies the triplet condition.

F_k and D_k does not cover F_j. Thus $D_k \neq D_j$. This means there are at least four coatoms of $S_{[n]}$ adjacent to C_i in $H([n])$.

Suppose $F' \in S_{[n]}$ is such that $F \lessdot F' \lessdot C_i$. Then F' must be obtained from C_i by contracting an edge adjacent to e^x, otherwise the leaf labeled x in F' is not adjacent to a degree 4 or labeled vertex, and so deleting e^x in F' would not be a safe deletion. There are only two edges adjacent to e^x: one is e_{x_1} and the other is e_{x_2}. Contracting these edges in C_i yields F_k and F_j respectively. Thus if F' is such that $F \lessdot F' \lessdot C_i$, then $F' = F_j$ or $F' = F_k$. The only coatoms of $S_{[n]}$ covering F_j are C_j and D_j and the only coatoms of $S_{[n]}$ covering F_k are C_k and D_k. Thus there are only four coatoms of $S_{[n]}$ that are both adjacent to C_i in $H([n])$ and above F in $S([n])$. □

Definition 3.3.16. Let $F \in S([n])$. An F**-path** is a path in $H([n])$ all of whose vertices are coatoms of $S_{[n]}$ that are above F. An F**-cycle** is an F-path that is a simple cycle.

For the remainder of this section, suppose (C_i, C_j, C_k) are coatoms of $S_{[n]}$ that satisfy the triplet condition and that F is the element of $S_{[n]}$ obtained from C_i by contracting an edge adjacent to e^x, then deleting e^x.

Definition 3.3.17. The element $F < C_i$ consists of an isolated vertex labeled x and an $([n]) - \{x\}$)-tree. The tree T_F is the $([n] - \{x\})$-tree of F.

The following remark and subsequent definition is crucial for the proofs of Lemma 3.3.22, Proposition 3.3.23 and Proposition 3.3.24.

Remark 3.3.18. For each coatom A of $S_{[n]}$ above F, there is some F' such that $F \lessdot F' \lessdot A$ where F' is obtained from A by contracting an edge of A (see Lemma 1.4.16). If F' is obtained by contracting an internal edge, then there are two other coatoms of $S_{[n]}$ that also cover F' by Lemma 1.4.17. In this case, we associate the element F' to the internal vertex v of T_F having the property that adding an edge between v and the vertex labeled x in F yields F'. If F' is obtained by contracting a leaf edge, we associate the element F' to the leaf v of T_F, where v is such that adding an edge between v and the vertex labeled x in F yields F'. See Figure 3.20 for an example.

Definition 3.3.19. The graph $G(F)$ is the graph that has elements of $S([n])$ that cover F as vertices and an edge between vertices a and b if they are associated to vertices in the tree T_F that are adjacent.

See Figure 3.23 for an example of the graph $G(F)$.

Proposition 3.3.20. *The graph $G(F)$ is isomorphic to T_F.*

Proof. Each vertex of $G(F)$ is an element of $S([n])$. Any element of $S_{[n]}$ covering F can be obtained from F by adding an edge between the isolated vertex labeled x and a vertex of T_F. If

F' covers F in $S_{[n]}$, we associate F' to the vertex $v_{F'}$ of T_F with the following property: adding an edge between the isolated vertex labeled x and $v_{F'}$ in F yields F'. Thus for every vertex of T_F, we have an element of $S_{[n]}$ covering F. By definition of $G(F)$, vertices of $G(F)$ are adjacent precisely when the vertices to which they correspond in T_F are adjacent. Thus $G(F)$ and T_F are isomorphic as graphs. $\square$

Remark 3.3.21. Suppose v_1 and v_2 are adjacent vertices in $G(F)$. There is an element F_1 (respectively F_2) of $S_{[n]}$ obtained from F by adding an edge between the isolated vertex labeled x and the vertex of T_F corresponding to v_1 (respectively v_2). Note $F_1 \gtrdot F$ and $F_2 \gtrdot F$. There is a coatom A of $S_{[n]}$ with the following property: A has two edges adjacent to e^x, say e_1 and e_2, such that contracting e_1 yields F_1 and contracting e_2 yields F_2. Thus if a coatom A covers two distinct elements above F, say F_1 and F_2, then A corresponds to the edge in $G(F)$ between F_1 and F_2. In other words, edges in $G(F)$ correspond to coatoms that cover two elements above F. An example of a coatom with this property is C_i for the triple (C_i, C_j, C_k) satisfying the triplet condition. In this case, C_i covers two distinct elements, F_j and F_k, both of which cover F as in Lemma 3.3.14. Thus C_i corresponds to an edge in $G(F)$.

We will need the following lemma for the proof of Theorem 3.3.25.

Lemma 3.3.22. *There are no nontrivial F-cycles in $H([n])$.*

Proof. If C_r and C_s are both maximal elements of $S([n])$ above F and they are adjacent in $H([n])$, then there is a vertex in $G(F)$ representing the common element they cover in $S([n])$. Assume there is a nontrivial F-cycle $C_1, C_2, C_3, \dots C_q, C_{q+1} = C_1$ in $H([n])$. Then for all t where $1 \leq t < q$, C_t and C_{t+1} cover a common element in $S([n])$ and this element is associated to a vertex $G_{t,t+1}$ in $G(F)$. Then $G_{1,2}, G_{2,3}, \dots G_{q,q+1}, G_{1,2}$ is a cycle in $G(F)$. Since $G(F)$ is isomorphic to T_F and T_F is a tree, this is a contradiction. $\square$

We also have the following proposition which says that any pair of coatoms of $S_{[n]}$ above F are connected by an F-path in $H([n])$. This proposition is important for the proof of Theorem 3.3.25

Proposition 3.3.23. *Suppose the coatoms A and B of $S_{[n]}$ are both above F. Then there is an F-path between A and B in $H([n])$.*

Proof. If A is above F, then there is some F_a in $S_{[n]}$ such that $A \gtrdot F_a \gtrdot F$ and such that F_a is obtained from F by contracting an edge of A. Similarly, if B is above F, then there is some F_b in $S_{[n]}$ such that $B \gtrdot F_b \gtrdot F$. The elements F_a and F_b are vertices in $G(F)$. Since $G(F)$ is a tree by Proposition 3.3.20, there is a path from F_a to F_b in $G(F)$, say $F_a = F_1, F_2, \dots F_q = F_b$. For $1 \leq t < q$, F_t and F_{t+1} are covered by the element $C_{(t,t+1)} \in S_{[n]}$, where $C_{(t,t+1)}$ corresponds to

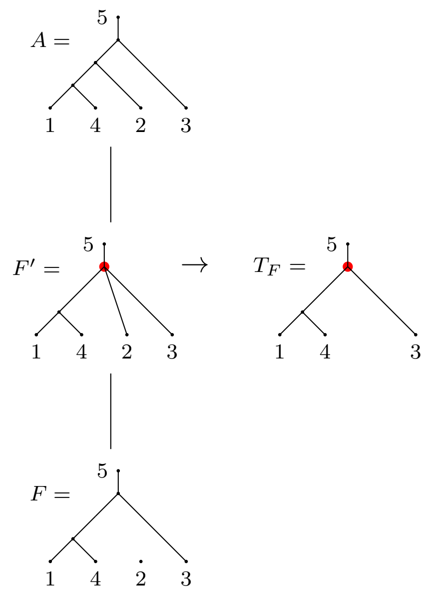


Figure 3.20: In the proof of Lemma 3.3.22, we associate every element covering F with a vertex of T_F.

the edge between F_t and F_{t+1} in $G(F)$, as explained in Remark 3.3.21. Then the following is an F-path from A to B in $H([n])$: $C_{(1,2)}, C_{(2,3)}, \ldots, C_{(q-2,q-1)}, C_{(q-1,q)}$. □

Proposition 3.3.24. *Let A be a coatom of $S_{[n]}$ above F and suppose (C_i, C_j, C_k) are coatoms of $S_{[n]}$ that satisfy the triplet condition. If an F-path from A to C_i contains C_j (respectively C_k), it cannot contain C_k (respectively C_j).*

Proof. Since C_i covers both F_j and F_k, C_i is associated to the edge in $G(F)$ between F_j and F_k by Remark 3.3.21. Since $G(F)$ is a tree by Proposition 3.3.20, there is a unique path between F_j and F_k as vertices in $G(F)$, namely the path consisting of the edge corresponding to C_i. Thus any F-path from C_j to C_k must contain C_i. Thus if there is an F-path that contains both C_j and C_k, it contains C_i. This implies any F-path from A to C_i can contain only one of C_j or C_k. □

We are now ready to prove the main theorem of this chapter:

Theorem 3.3.25. *There does not exist a recursive coatom ordering of $S_{[n]}$.*

Proof. Suppose that Ω is a recursive coatom ordering of $S_{[n]}$. Let K be the cycle of $H([n])$ shown in Figure 3.21. Let C_i be the vertex in K that is latest in Ω. Let C_j be one of the vertices adjacent to C_i in K and let C_k be the other vertex adjacent to C_i in K. Observe that (C_i, C_j, C_k) satisfies the triplet condition. Let F be the maximal element of $I(C_j) \cap I(C_k)$ described in Lemma 3.3.14. Let A be the earliest coatom in Ω that is above F. Since by Lemma 3.3.15, C_i is adjacent in $H([n])$ to only four other coatoms that are above F, namely the atoms we called C_j, C_k, D_j, and D_k, every F-path from A to C_i contains one of these four coatoms. If an F-path from A to C_i contains D_j (respectively D_k), then there exists an F-path from A to C_i that contains C_j (respectively C_k) since D_j and C_j (respectively D_k and C_k) are adjacent in $H([n])$. Thus there exists an F-path in $H([n])$ between A and C_i that contains one of C_k or C_j. By Proposition 3.3.24, no F-path from A to C_i can contain both C_j and C_k. Thus there is an F-path from A to C_i containing exactly one of C_j or C_k. Without loss of generality, let C_k be in the F-path from A to C_i. Let B be the earliest coatom after A in Ω that is above F such that there exists an F-path between B and C_i containing C_j. This B is guaranteed to exist: C_j is in an F-path from C_j to C_i, but if there is a coatom C^* that comes before C_j and after A in Ω such that there exists an F-path from C^* to C_i containing C_j, then we take this earlier coatom C^* to be B.

By definition, A and B are both below F in $S^*_{[n]}$, namely the dual poset to $S_{[n]}$. By definition of a recursive coatom ordering (in particular (ii) of Definition 1.2.21), there must exist some element $C <_\Omega B$ and some element $z \in S^*_{[n]}$ such that $C \lessdot z, B \lessdot z$ and $z < F$. Note that there is no F-path from C to C_i containing C_j, otherwise C (and not B) would have been the first coatom C^* after A with the property that there is an F-path from C^* to C_i containing C_j. Any $C <_\Omega B$ is not adjacent to B in $H([n])$: if C were adjacent to B, this would imply there was a nontrivial F-cycle in $H([n])$ consisting of the union of the F-path from C_i to B (containing C_j), the edge between B and C, and the F-path from C to C_i, contradicting Lemma 3.3.22. Since B and C are not adjacent, they are not covered by a common element in $S^*_{[n]}$. Thus no such z can exist. This is a contradiction to the assumption that Ω is a recursive coatom ordering of $S_{[n]}$. □

Example 3.3.26. Here we provide an example of why a coatom ordering Ω of $S_{[5]}$ fails to be a recursive coatom ordering. Let K be the cycle in $H([5])$ shown in Figure 3.18 with bold edges. Note that this is the same K as in Figure 3.21. Suppose that the element C_i given by the word $w(C_i) = (((1,4),2),3)$ comes last in Ω among all elements of K. One element adjacent to C_i in K is the element C_j given by the word $w(C_j) = (((1,4),3),2)$. The other element adjacent to C_i in K is the element C_k given by the word $w(C_k) = (((1,2),4),3)$. Observe that the triple (C_i, C_j, C_k) satisfies the triplet condition. There is a maximal element F of $I(C_j) \cap I(C_k)$ as described in Lemma 3.3.14. The element F can be seen in Figure 3.19.

The graph shown in Figure 3.22 is the subgraph of $H([5])$ consisting of all the coatoms of $S_{[5]}$ that are above F. Let A be the earliest coatom in Ω that is above F. As an example, suppose $A = C_k$. There is an F-path (i.e. a path in Figure 3.22) from $A = C_k$ to C_i that contains exactly one of C_j or C_k. In particular, it contains C_k. Let B be the first coatom after C_k in Ω that is above F and such that there exists a path between B and C_i containing C_j. As an example, suppose B is the coatom given by the word $w(B) = ((1,4),(2,3))$. Note that B is the coatom at the top left in Figure 3.22.

Both A and B are below F in $S^*_{[5]}$. If Ω is a recursive coatom ordering, there must exist some coatom earlier than B in Ω, say C, and some element $z \in S^*_{[5]}$ such that $C \lessdot z, B \lessdot z$ and $z < F$. Based on how we were required to choose B, C must be in an F-path from A to C_i. In particular, C must be the coatom given by the word $((1,(2,4)),3)$. Note that C is the coatom at the top right in Figure 3.22. However, B and C are not adjacent in $H([5])$, thus they cannot both be covered by an element z in $S^*_{[5]}$. Thus no such z, C, can exist. It follows that Ω is not a recursive coatom ordering.

Gill, Moulton, Linnusson, and Steel showed in [12] that intervals $[\hat{0}, y] \subset S(X) \cup \{\hat{0}\}$ are shellable. They also used this along with results in geometric topology regarding approximation maps by homeomorphisms to prove that the edge-product space is a regular CW complex. Since the edge-product space is pure, the edge-product space is thus a $d-$CW-complex (see Definition 1.1.21).

Remark 3.3.27. Since there does not exist a recursive coatom ordering of $S_{[n]}$, Proposition 1.2.24 implies that the edge-product space $\mathcal{E}([n])$ is not shellable.

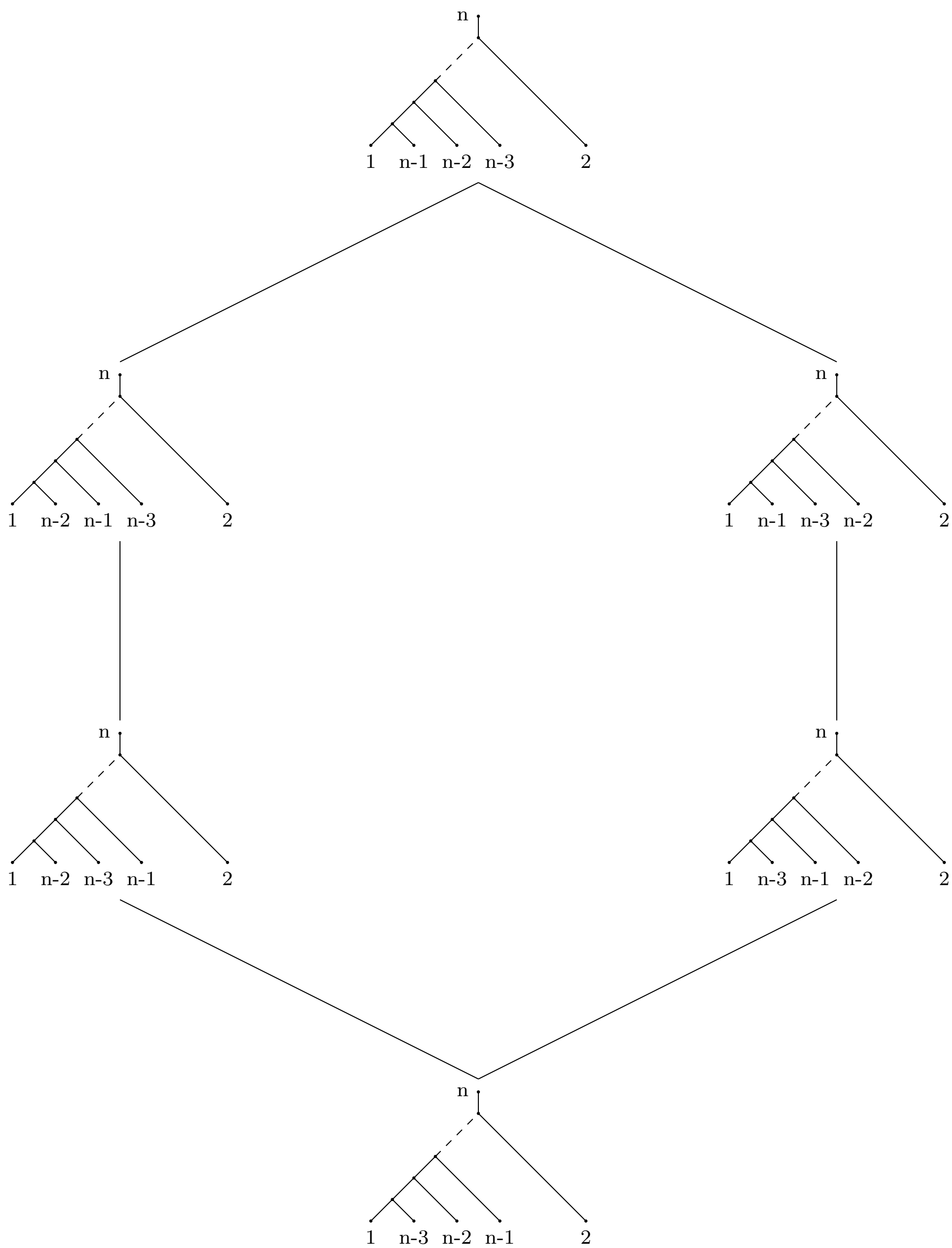


Figure 3.21: The elements in the set K form a nontrivial cycle in $H([n])$.

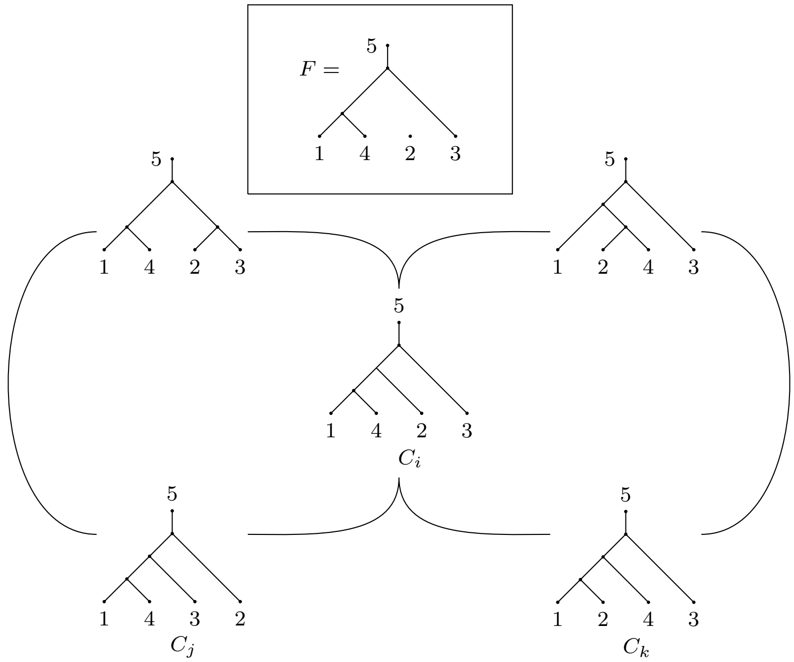


Figure 3.22: The subgraph of $H([5])$ consisting of coatoms in S_5 above F.

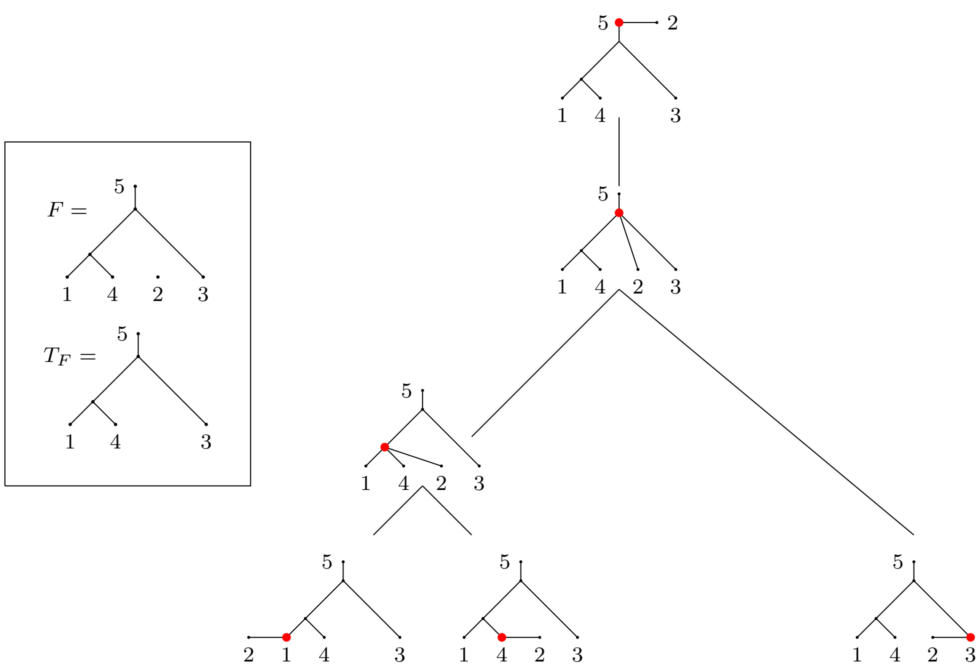


Figure 3.23: The graph $G(F)$ consists of elements of $S([5])$ that cover F. The graph $G(F)$ is isomorphic to T_F.

Chapter 4

Generalizing results on shellability

4.1 Generalized recursive atom orderings

In this chapter we give a generalization of a result of Björner and Wachs from [5] regarding CL-shellable posets by giving a CC-shellable analogue.

Let P be a poset and $E(P)$ the set of cover relations (i.e. the set of edges in the Hasse diagram) of P. Let $E^*(P)$ be the set of pairs $(m, x \lessdot y)$ where m is a maximal chain of P and $x \lessdot y$ is a cover relation in the chain m. Let Q be a poset.

Definition 4.1.1. Let $\lambda : E^*(P) \to Q$ be a CC-labeling of P. Let m be a maximal chain in P containing u and v. Let r be the saturated chain in the interval $[\hat{0}, u]$ of m and $[u, v] = u \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_{k-1} \lessdot v$ be the interval in $[u, v]$ of m. The **label sequence** on m from u to v is

$$\lambda(m, u \lessdot x_1), \lambda(m, x_1 \lessdot x_2) \ldots, \lambda(m, x_{k-1} \lessdot v)$$

If the label sequence on m from u to v is lexicographically earlier than the label sequence on m' from u to v for all other maximal chains m' such that m and m' coincide with r up to u, then we say that m is **lexicographically first** on the rooted interval $[u, v]_r$.

The following lemma and corollary are straightforward but very useful in proving Theorem 4.1.9, the main theorem of the chapter.

Lemma 4.1.2. *Let P be CC-shellable and let c be the unique, topologically ascending chain in $[u, v]_r$. Then c is lexicographically first on the rooted interval $[u, v]_r$.*

Proof. Let c be the chain $u = u_0 \lessdot u_1 \lessdot u_2 \lessdot \ldots \lessdot u_t = v$. Since c is topologically ascending, then $u_{i-2} \lessdot u_{i-1} \lessdot u_i$ is a topological ascent for $i = 2, 3, \ldots t$. Assume the saturated chain c' given by $u = u_0 \lessdot u'_1 \lessdot u'_2 \lessdot \ldots u'_s = v$ is lexicographically first in $[u, v]_r$. Let $i > 0$ be the least value such that $u_i \neq u'_i$. Then $u'_{i-1} \lessdot u'_i \lessdot u'_{i+1}$ is a topological ascent since otherwise, there would exist a lexicographically earlier saturated chain from u'_{i-1} to u'_{i+1}, contradicting the assumption that c' is the lexicographically earliest chain in $[u,v]_r$. Likewise, $u'_i \lessdot u'_{i+1} \lessdot u'_{i+2}$ is a topological ascent. Continuing, we see that $u'_{i+j} \lessdot u'_{i+j+1} \lessdot u'_{i+j+2}$ is a topological ascent for all j such that $0 \leq j \leq s-i-2$. If $i > 1$, then $u'_{i-2} \lessdot u'_{i-1} \lessdot u'_i$ is a topological ascent since otherwise there would be a lexicographically earlier chain in the interval $[u'_{i-2}, u'_i]$. For all $2 < j < i$, $u'_{j-2} \lessdot u'_{j-1} \lessdot u'_j$ is a topological ascent since c and c' coincide on the interval $[u_0, u_j]$. Thus c' consists entirely of topological ascents and is thus a topologically ascending chain in $[u,v]_r$. This is a contradiction of c being the only such chain. □

Corollary 4.1.3. *If P is CC-shellable and c is lexicographically first in $[u,v]_r$, then c is topologically ascending.*

Proof. If c is lexicographically first but not the unique topologically ascending chain in $[u,v]_r$, then the unique topologically ascending chain cannot be lexicographically first. This contradicts Lemma 4.1.2. □

Next, we define a generalized recursive atom ordering. As we show in Theorem 4.1.9, giving a generalized recursive atom ordering for a poset P is equivalent to giving a CC-labeling of P. This is an analogous result to Theorem 3.2 in [5], which concerns CL-shellability and recursive atom orderings.

Definition 4.1.4. The bounded poset P admits a **generalized recursive atom ordering** (GRAO) if the length of P is 1 or if the length of P is greater than 1 and there is an ordering on the atoms $a_1, a_2, \ldots a_t$ of P satisfying:

(i) For all $1 \leq j \leq t$ and w such that $a_j \lessdot x \lessdot w$ for some x, $[a_j, \hat{1}]$ admits a generalized recursive atom ordering with the following property: if any atom of $[a_j, w]$ covers some $i < j$, then the first one does.

(ii) For all $i < j$, if $a_i, a_j < y$, then there exists some $k < j$ and an element z such that $a_k, a_j \lessdot z \leq y$.

Example 4.1.5. The ordering of elements given on the left of Figure 4.1 is a generalized recursive atom ordering but is not a (traditional) recursive atom ordering. In particular, the atom labeled 3 on the left causes (i) of Definition 1.2.21 to fail. The ordering of elements given on the right is a recursive atom ordering.

The main theorem of this chapter is Theorem 4.1.9. Because the proofs are somewhat technical, we will state each direction of this biconditional as a separate result (Theorem 4.1.6 and Theorem 4.1.8). Both theorems follow Björner and Wachs' proof of Theorem 3.2 in [5], with changes being made as necessary to account for topologically ascending chains instead of increasing chains. In both directions, we use induction on the length of P.

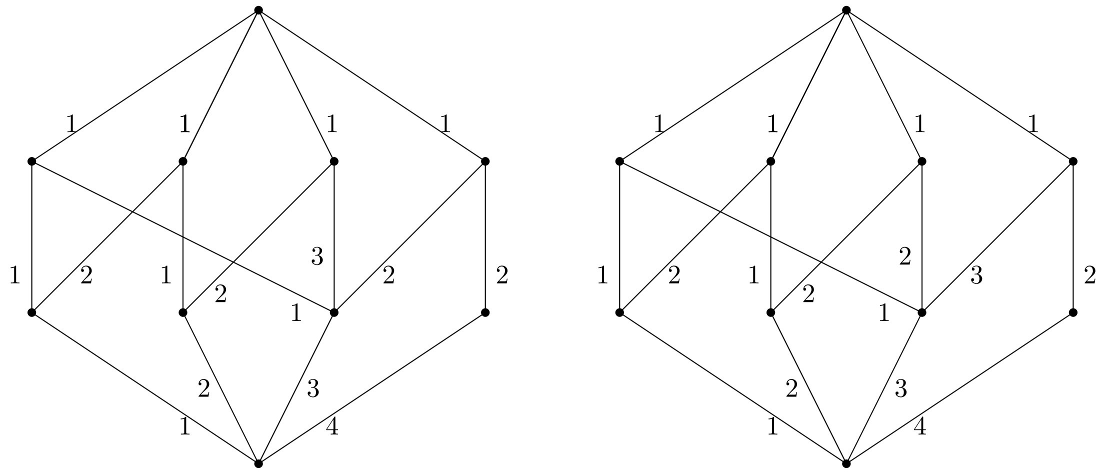


Figure 4.1: An example of a generalized recursive atom ordering that is not a recursive atom ordering (left) and a recursive atom ordering (right).

Theorem 4.1.6. *If a graded poset P admits a generalized recursive atom ordering, then P is CC-shellable.*

Proof. We will prove the following statement by induction on the length of P:

If P admits a generalized recursive atom ordering $a_1, a_2, \ldots a_t$ and if λ is an integer labeling of the bottom edges of P such that $\lambda(\hat{0}, a_i) < \lambda(\hat{0}, a_j)$ for all $i < j$, then λ extends to an integer CC-labeling of P.

Any integer labeling on a bounded poset of length 1 is a CC-labeling. Also, any integer labeling on the t bottom edges of a bounded poset of length 2 by t distinct labels is a CC-labeling. Thus the statement is true for posets of length 1 and length 2.

Let P be a poset of length greater than 2. By (i) of Definition 4.1.4, for each $j = 1, 2, \ldots t$, there is a GRAO of $[a_j, \hat{1}]$. Consider the restriction of this GRAO to the atoms of $[a_j, w]$ for w such that $a_j \lessdot x \lessdot w$ for some x. If any atom of $[a_j, w]$ covers an atom of P that comes earlier than a_j in the GRAO of P, then the first one does. Label the bottom edges of $[a_j, \hat{1}]$ consistently with the GRAO on $[a_j, \hat{1}]$ i.e. so that $\lambda(a_j, x) < \lambda(a_j, y)$ for $x < y$. We will use λ to denote both the integer labeling of the bottom edges of P and the labeling of the bottom edges of $[a_j, \hat{1}]$. Note that because we are showing that this labeling extends to a chain-edge labeling, then the labeling on $[a_j, \hat{1}]$ and the labeling on $[a_k, \hat{1}]$ need not be compatible, even if there are shared cover relations between $[a_j, \hat{1}]$ and $[a_k, \hat{1}]$. By the induction hypothesis, the labeling λ on the bottom edges of $[a_j, \hat{1}]$ extends to a CC-labeling of $[a_j, \hat{1}]$. Extending the labeling at each a_j for $j = 1, 2, \ldots t$, we have a chain-edge labeling of P which restricts to a CC-labeling of $[a_j, \hat{1}]$.

We must check that there is a unique, topologically ascending chain in any interval $[\hat{0}, y]$. Let $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k = y$ be the lexicographically first saturated chain in $[\hat{0}, y]$. Then $x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k = y$ is the lexicographically first saturated chain in $[x_1, y]$ and since λ restricts to a CC-labeling on $[x_1, \hat{1}]$, this chain is topologically ascending by Corollary 4.1.3. The interval $\hat{0} \lessdot x_1 \lessdot x_2$ is lexicographically first in $[\hat{0}, x_2]$ so it is a topological ascent. Thus $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k = y$ is a topologically ascending chain.

Let $\hat{0} \lessdot x'_1 \lessdot x'_2 \lessdot \ldots \lessdot x'_k = y$ be another topologically ascending chain in $[\hat{0}, y]$. Since there is only one topologically ascending chain in $[x_1, y]$, we must have $x_1 \neq x'_1$. Since $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k = y$ is lexicographically first and $\lambda(\hat{0}, a_i) < \lambda(\hat{0}, a_j)$ whenever $i < j$, x_1 comes before x'_1 in the GRAO of P. Since there is a CC-labeling on $[x'_1, \hat{1}]$ by the induction hypothesis, $x'_1 \lessdot x'_2 \lessdot \ldots \lessdot x'_k$ is the lexicographically first maximal chain in $[x'_1, y]$ by Lemma 4.1.2. Since the labeling on the bottom edges of $[x'_1, \hat{1}]$ is consistent with the GRAO on $[x'_1, \hat{1}]$, this means x'_2 is the first atom of $[x'_1, \hat{1}]$ that is less than y in the GRAO of $[x'_1, \hat{1}]$. Since $\hat{0} \lessdot x'_1 \lessdot x'_2$ is topologically ascending, there is no x''_1 such that $\hat{0} \lessdot x''_1 \lessdot x'_2$ is lexicographically earlier. Thus x'_2 covers no atom of P that comes before x'_1 in the GRAO of P. Since x_1 comes before x'_1 in the GRAO of P and both x_1 and x'_1 are below y in P, by (ii) of Definition 4.1.4, there exists some x''_1 that comes before x'_1 and some z such that $x''_1, x'_1 \lessdot z < y$. Likewise, since both x'_2 and z are below y and x'_2 comes before z, there exists some z_1 that comes before z in the GRAO of $[x'_1, \hat{1}]$ and some w_1 such that $z, z_1 \lessdot w_1 < y$. Since z covers an atom that comes earlier than x'_1, by (i) of Definition 4.1.4, the first atom of $[x'_1, w_1]$ must cover some x_{w_1} where x_{w_1} comes before x'_1 in the GRAO of P. Without loss of generality, assume z_1 is this first atom of $[x'_1, w_1]$. The same argument holds to show that there must exist some z_2 that comes before z_1 in the GRAO of $[x'_1, \hat{1}]$ and some w_2 such that $z_1, z_2 \lessdot w_2 < y$. Since z_1 covers x_{w_1}, then the first atom of $[x'_1, w_2]$ must cover some x_{w_2} where x_{w_2} also comes before x'_1 in the GRAO of P. Without loss of generality, assume z_2 is this first atom of $[x'_1, w_2]$. We continue in this way, until we have some z_i such that both z_i and x'_2 are below y, but the only atom of $[x'_1, \hat{1}]$ that comes before z_i and is below y is x'_2. Then z_i and x'_2 must both be covered by some w_{i+1}. Since z_i covers some x_{w_i} where x_{w_i} comes before x'_1 in the GRAO of P, then the first atom of $[x'_1, w_{i+1}]$ must cover some $x_{w_{i+1}}$, where $x_{w_{i+1}}$ comes before x'_1 in the GRAO of P. This contradicts (i) of Definition 4.1.4, since the first atom of $[x'_1, w_i]$ is x'_2, which does not cover an atom of P that comes before x'_1 in the GRAO of P. Thus there is only one topologically ascending chain in any interval $[\hat{0}, y]$.

This shows that if P admits a GRAO, then P is CC-shellable. $\square$

We will need the following definition for the proof of Theorem 4.1.8:

Definition 4.1.7. An atom ordering $a_1, a_2, \ldots a_t$ is **compatible** with a CC-labeling λ if $a_1, a_2, \ldots a_t$ is a linear extension of the partial order on the atoms induced by λ. In other words, an atom ordering is compatible with λ if whenever $(\lambda(\hat{0}, a_i), \lambda(a_i, x))$ is lexicographically

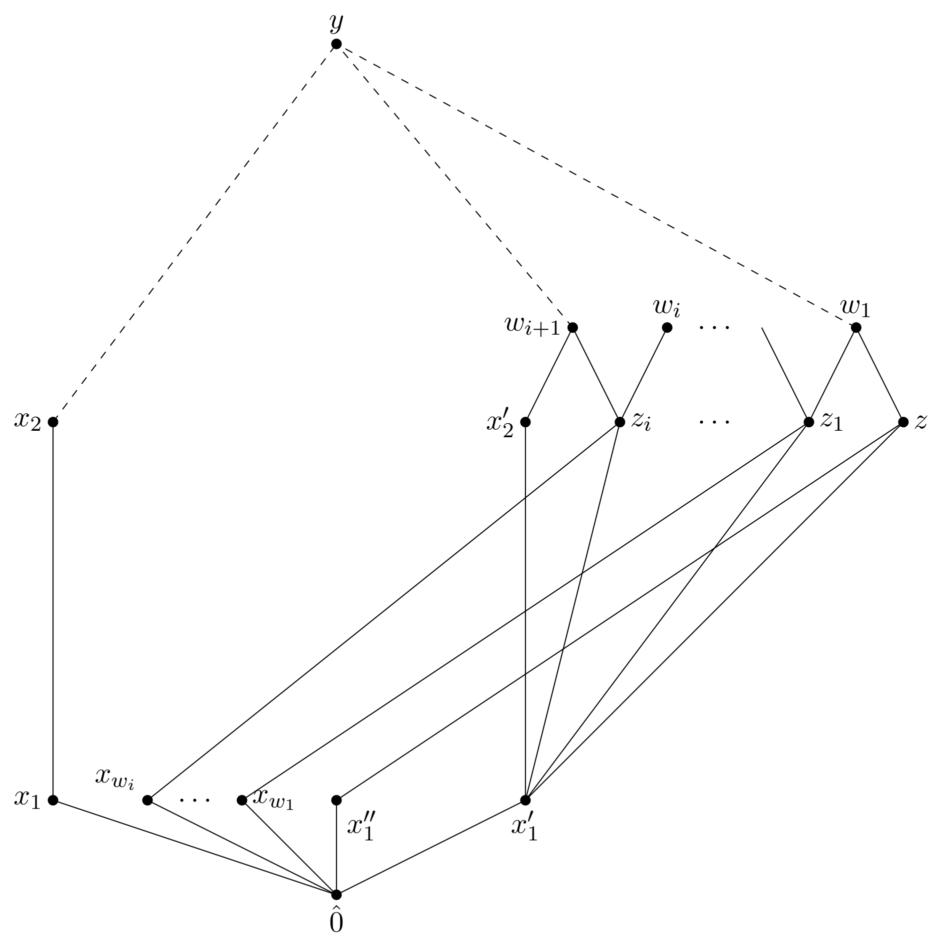


Figure 4.2: If a poset P admits a generalized recursive coatom ordering, then there is a unique topologically ascending chain in any interval $[\hat{0}, y]$.

earlier than $\lambda(\hat{0}, a_j), \lambda(a_j, x)$, then $i < j$.

Theorem 4.1.8. *If a graded poset P is CC-shellable, P admits a generalized recursive atom ordering.*

Proof. Let $\lambda : E^*(P) \to Q$ be a CC-labeling of P where Q is any poset. The CC-labeling λ induces a partial order on the atoms of P, where a_i comes before a_j if $(\lambda(\hat{0}, a_i), \lambda(a_i, x))$ is lexicographically earlier than $(\lambda(\hat{0}, a_j), \lambda(a_j, x))$ for some x covering both a_i and a_j. We will prove the following statement by induction on the length of P:

If an atom ordering $a_1, a_2, \ldots a_t$ is compatible with a CC-labeling λ, then the atom ordering induces a generalized recursive atom ordering.

If P is of length 1, P admits a generalized recursive atom ordering by definition. If the length of P is 2, then any ordering on the atoms is a generalized recursive atom ordering. Let P be a poset of length greater than 2. Suppose $a_1, a_2, \ldots a_t$ is compatible with the CC-labeling λ of P. We must check it is a GRAO.

We first check that the ordering satisfies (ii) of Definition 4.1.4. Consider any $a_i, a_j < y$ for $i < j$. Let $a_j \lessdot z \lessdot \ldots \lessdot y$ be the lexicographically first saturated chain in the rooted interval $[a_j, y]_{\hat{0}}$ (with root $\hat{0}$). Call this rooted chain c. Suppose by way of contradiction that $(\lambda(\hat{0}, a_j), \lambda(a_j, z))$ is a topological ascent, so $\hat{0} \cup c$ is a topologically ascending chain in $[\hat{0}, y]$. Let a_y be the first atom of P in the atom ordering that is below y and suppose $a_y \lessdot z' \lessdot \ldots y$ is the lexicographically first chain in $[a_y, y]_{\hat{0}}$. Our choice of a_y ensures that $(\lambda(\hat{0}, a_y), \lambda(a_y, z))$ is a topological ascent. Since there can be only one topologically ascending chain in $[\hat{0}, y]$, $(\lambda(\hat{0}, a_j), \lambda(a_j, z))$ must be a topological descent. Then there exists some $a_k \lessdot z$ such that $(\lambda(\hat{0}, a_k), \lambda(a_k, z))$ is a topological ascent. Since the atoms of P are ordered compatibly with λ, we have $k < j$. Thus (ii) holds.

We now check (i) of Definition 4.1.4. Since P is CC-shellable via a labeling λ, $[a_j, \hat{1}]$ is CC-shellable for each $j = 1, 2, \ldots t$ also using λ. By the induction hypothesis, any atom ordering of $[a_j, \hat{1}]$ that is compatible with this CC-labeling is a GRAO. We verify that the atoms of $[a_j, \hat{1}]$ ordered compatibly with λ (which itself implies this ordering is a GRAO on $[a_j, \hat{1}]$) gives an atom ordering also satisfying (i) of Definition 4.1.4. We will do this by showing that for w such that $a_j \lessdot x \lessdot w$ for some x, if any atom of $[a_j, w]$ covers some a_i for $i < j$ but y does not, then $(\lambda(a_j, y), \lambda(y, w))$ must be a topological descent. Assume instead under these hypotheses that $(\lambda(a_j, y), \lambda(y, w))$ is a topological ascent. Then for any $x \neq y$ such that $x \gtrdot a_j$ and $x \lessdot w$, $(\lambda(a_j, x), \lambda(x, w))$ is a topological descent. Since $a_1, a_2, \ldots a_t$ is compatible with λ and y does not cover any a_i for $i < j$, $(\lambda(\hat{0}, a_j), \lambda(a_j, y))$ is a topological ascent. This implies that the chain c given by $\hat{0} \lessdot a_j \lessdot y \lessdot w$ is the unique, topologically ascending chain in the interval $[\hat{0}, w]$. Let a_w be the first atom of P below w. Since there exists an atom of $[a_j, w]$ that covers some a_i for

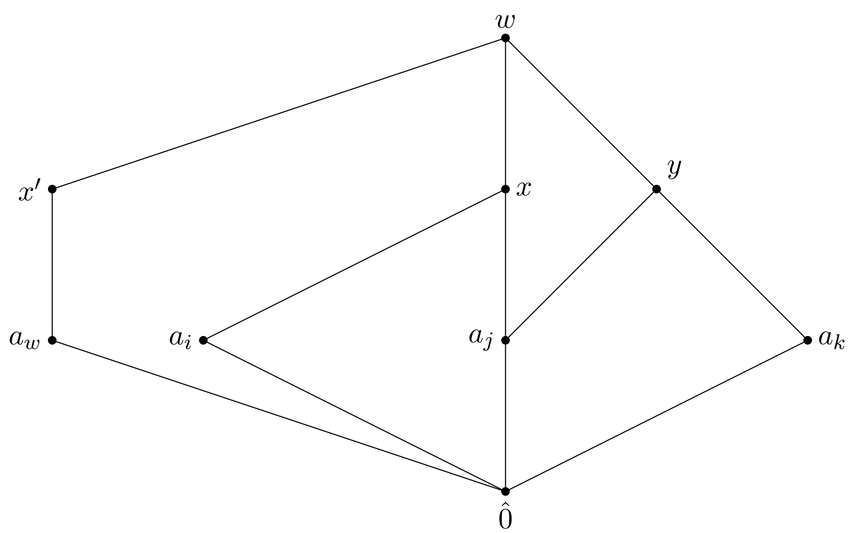


Figure 4.3: If λ is a CC-labeling of P, then for any atom a_j of P, $(\lambda(a_j, y), \lambda(y, w))$ must be a topological descent whenever, for $a_j \lessdot x \lessdot w$, there exists some x that covers some a_i with the property that $(\lambda(\hat{0}, a_i), \lambda(a_i, x))$ is lexicographically earlier than $(\lambda(\hat{0}, a_j), \lambda(a_j, x))$.

$i < j$, we know that $a_w \neq a_j$. Since $a_1, a_2, \dots a_t$ is compatible with λ, $(\lambda(\hat{0}, a_w), \lambda(a_w, y'))$ is a topological ascent for any y' such that $a_w \lessdot y' \lessdot w$. By definition of a CC-labeling, there is a unique, topologically ascending chain in the interval $[a_w, w]$. Let x' be such that $a_w \lessdot x' \lessdot w$ is this topologically ascending chain. Then $\hat{0} \lessdot a_w \lessdot x' \lessdot w$ is a topologically ascending chain in $[\hat{0}, w]$. This is a contradiction to c being the unique topologically ascending chain in $[\hat{0}, w]$. Thus it is not possible for $(\lambda(a_j, y), \lambda(y, w))$ to be a topological ascent if y does not cover some a_i for $i < j$. Therefore $(\lambda(a_j, y), \lambda(y, w))$ must be a descent. See Figure 4.3. This implies that $[a_j, \hat{1}]$ admits an atom ordering that is compatible with λ (hence is a GRAO) and that satisfies (i) of Definition 4.1.4.

Thus if P is CC-shellable, P admits a GRAO. □

Theorem 4.1.9. *A graded poset P admits a generalized recursive atom ordering if and only if P is CC-shellable.*

Proof. This follows from Theorem 4.1.6 and Theorem 4.1.8. □

Reflecting the fact that CC-labelings are more general than CL-labelings, we have the following lemma relating (traditional) recursive atom orderings and generalized recursive atom orderings:

Lemma 4.1.10. *Every recursive atom ordering is a generalized recursive atom ordering.*

Proof. Since part (ii) of Definition 1.2.21 and part (ii) of Definition 4.1.4 are the same, then we only need to check that part (i) of Definition 4.1.4 holds for a recursive atom ordering $a_1, a_2, \ldots a_t$ of P. We do so by induction on the length of P. Since any ordering of the atoms of a poset of length 1 or 2 is a generalized recursive atom ordering, this holds for posets of length 1 or 2. Suppose P is a poset of length greater than 2. By the induction hypothesis, the recursive atom ordering on $[a_j, \hat{1}]$ is a generalized recursive atom ordering for $j = 1, 2, \ldots t$. We must check that for any w such that $a_j \lessdot x \lessdot w$ for some x, if any atom of $[a_j, w]$ covers some a_i for $i < j$, then the first one does. By part (ii) of Definition 1.2.21, the first atoms of $[a_j, \hat{1}]$ are those that cover some a_i for $i < j$. Let $F(a_j)$ be the set of atoms of $[a_j, \hat{1}]$ that cover some a_i for $i < j$ and let A_w be the set of atoms below w for w such that $a_j \lessdot x \lessdot w$ for some x. We see that if $F(a_j) \cap A_w = \emptyset$, then the first atom of $[a_j, w]$ covers some a_i for $i < j$. $\square$

We can also define generalized recursive coatom orderings, as follows.

Definition 4.1.11. Let P be a graded poset and P^* its dual. A **generalized recursive coatom ordering** (GRCO) of P is a generalized recursive atom ordering of P^*.

4.2 A partial order generalization of a result on shelling barycentric subdivisions

In this section, we present a generalization of a result of Björner from [3], namely Theorem 4.2.6 below. While Theorem 4.2.6 concerns shellable simplicial complexes, we wonder whether the same result holds for simplicial complexes with partial orders on their facets satisfying the codimension one property (see Definition 3.2.6). For this we will need the following background and definitions, most of which are from Stanley [24] (see also [3] by Björner).

Definition 4.2.1. Let L be a finite lattice. If $x = a \vee b$ implies $x = a$ or $x = b$, then x is called a **join-irreducible** element of L.

Let $I(L)$ be the set of join-irreducibles of the finite lattice L. Let $\omega : I(L) \to \mathbb{N}$ be any map from the set of join-irreducibles of L to the set of positive integers. Then ω induces an edge labeling λ of L, where

$$\lambda(x \lessdot y) = \min\{\omega(z) : z \in I(L), x < x \vee z = y\}. \tag{4.2.1}$$

Definition 4.2.2. If λ is such that in every interval $[x, y]$ of P, there is a unique, increasing saturated chain, then ω is an **admissible map**.

Definition 4.2.3. A finite lattice L is **admissible** if it is graded and there exists an admissible map $\omega : I(L) \to \mathbb{N}$.

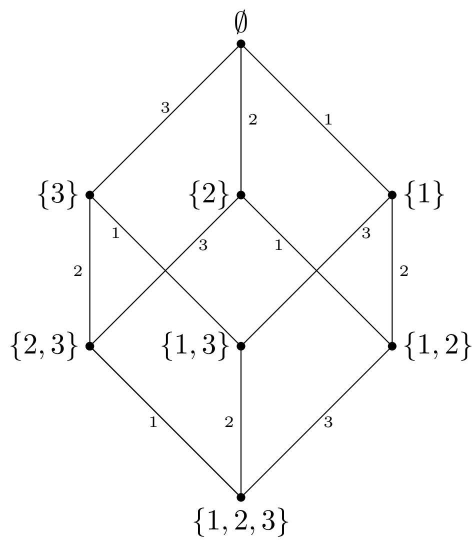


Figure 4.4: The Boolean Lattice B_3 and the EL-labeling induced by the admissible map $\omega : I(B_3) \to \mathbb{N}$ given in Example 4.2.5.

Theorem 4.2.4 (Theorem 3.1 [3])**.** *An admissible lattice is lexicographically shellable.*

Example 4.2.5. The join-irreducible elements of B_n^*, the dual to the Boolean lattice B_n, are precisely the coatoms of the lattice B_n. In particular, they are the subsets of $[n] = \{1, 2, \ldots n\}$ that exclude exactly one integer from 1 to n. We can define an admissible map $\omega : I(B_3^*) \to \mathbb{N}$ by $\omega(x) = x^c$ where x^c is the unique integer in $[n]$ excluded from x. The EL-labeling induced by ω is given for B_3^* in Figure 4.4. A map similar to ω is used in the proof of Theorem 4.2.7.

Theorem 4.2.6 (Theorem 5.1 [3])**.** *Let K be a shellable simplicial complex. Then the barycentric subdivision $sd(K)$ is shellable.*

Our generalization of this result is the following:

Theorem 4.2.7. *Let K be a pure simplicial complex with a partial order on its facets satisfying the codimension one property and with a unique minimal element. Then the barycentric subdivision also has a partial order on facets satisfying the codimension one property (of Definition 3.2.6).*

Proof. Let K be a pure simplicial complex. Let T be the set of facets of K. Suppose the facets of K are arranged in a partial order $P = (T, <_T)$ such that P has the codimension one property and P has a unique minimal element $\hat{0}$. Let $F(K)$ be the face poset of K. We will construct a partial order Ω on the set $\mathcal{M}$ of maximal chains of $F(K)$. Let $f_i \in T$ and let $\mathcal{M}_i$ be the set of all maximal chains of $F(K)$ that contain f_i. Observe that $[\hat{0}, f_i]$ is a finite Boolean lattice on n atoms, where $n = dim(K) + 1$. We label the coatoms of $[\hat{0}, f_i]$ by the labels $c_{i1}, c_{i2}, \ldots, c_{in}$ in such a way that for some k_i, we have the following: whenever $1 \leq j \leq k_i$, c_{ij} is covered by some f_e for some $e < i$ but for $k + 1 \leq j \leq n$, this is not the case. In other words, we label the coatoms of $[\hat{0}, f_i]$ so that all coatoms that come before c_{ik} are covered by a facet that comes before f_i in P and all those that come after c_{ik} are not. Since P has the codimension one property and has a $\hat{0}$, $k_i \geq 1$. Consider the map $\omega : c_{ij} \to j$. Since $[\hat{0}, f_i]$ is a Boolean lattice, the only join-irreducible elements of the dual lattice $[\hat{0}, f_i]^*$ are its atoms. The map ω is an admissible map from the join irreducible elements of the dual lattice $[\hat{0}, f_i]^*$ to the set of positive integers. The edge labeling induced by ω is the natural edge labeling on the dual to a Boolean lattice (see Example 4.2.5). Thus ω induces an EL-labeling of $[\hat{0}, f_i]^*$ by Theorem 4.2.4 and thus a shelling order of $\mathcal{M}_i$ (see [3]).

Let the elements of $\mathcal{M}_i$ be labeled $m_{i1}, m_{i2}, \ldots m_{in!}$ where the second index is compatible with the above shelling order of $\mathcal{M}_i$. Repeat this process of labeling maximal chains containing f_i for all $f_i \in T$. Consider the partial order Ω on $\mathcal{M}$ defined by $m_{ij} \leq m_{kl}$ if and only if one of the following hold:

1. $i = k$ and $j \leq l$
2. $i <_T k$

Observe that this means that m_{ij} and m_{kl} are incomparable if and only if f_i and f_k are incomparable in P.

We claim that Ω has the codimension one property, as we now show. Suppose that we are given $m_{i_0j_0} \in \mathcal{M}$ and that $m_{i_1j_1} <_\Omega m_{i_0j_0}$. We must show that $m_{i_1j_1} \cap m_{i_0j_0} \subseteq m_{i_0j_0} - \{x\}$ for some $x \in m_{i_0j_0}$. We consider two cases:

Case 1: $i_1 = i_0$ and $j_1 \leq j_0$.

In this case, both $m_{i_0j_0}, m_{i_1j_1} \in \mathcal{M}_{i_0}$. Since the second index on elements in $\mathcal{M}_{i_0}$ gives a shelling order of $\mathcal{M}_{i_0}$, there exists some $m_{i_0j_2} \in \mathcal{M}_{i_0}$ such that $m_{i_1j_1} \cap m_{i_0j_0} \subseteq m_{i_0j_2} \cap m_{i_0j_0} = m_{i_0j_0} - \{x\}$ for some $x \in m_{i_0j_0}$.

Case 2: $i_1 <_T i_0$.

Suppose that $m_{i_0j_0}$ is $\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_{n-1} \lessdot x_n = f_{i_0}$. Let $x_{n-1} = c_{i_0z}$. We will subdivide Case 2 into two additional subcases.

Case 2a: $z \leq k_{i_0}$. By definition of k_{i_0}, c_{i_0z} is covered by some f_{i_2} where $f_{i_2} <_T f_{i_0}$. Let $m_{i_2j_2}$ be $\hat{0} = x_0 \lessdot x_1 \ldots \lessdot x_{n-1} \lessdot f_{i_2}$. Observe that $m_{i_2j_2} <_\Omega m_{i_0j_0}$ since $i_2 <_T i_0$. Furthermore, $m_{i_2j_2}$ is such that $m_{i_1j_1} \cap m_{i_0j_0} \subseteq m_{i_0j_0} \cap m_{i_2j_2} = m_{i_0j_0} - \{f_{i_0}\}$. Note that we have $m_{i_1j_1} \cap m_{i_0j_0} \subseteq m_{i_0j_0} \cap m_{i_2j_2}$ because $m_{i_2j_2}$ and $m_{i_0j_0}$ intersect at every $x \leq x_{n-1} = c_{i_0z}$ and $m_{i_1j_1} \cap m_{i_0j_0}$ does not contain f_{i_0} since otherwise we would have $i_1 = i_0$ (Case 1). See Figure 4.5.

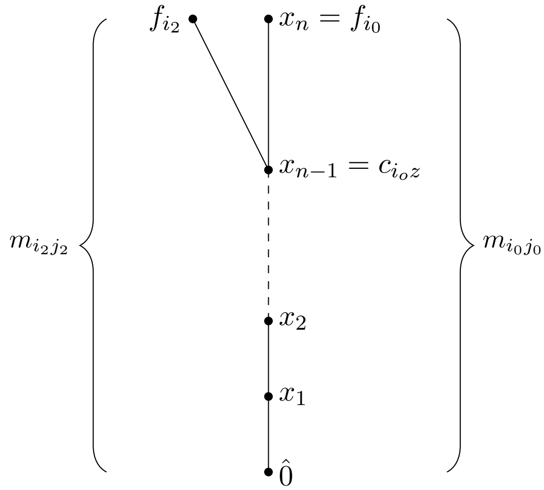


Figure 4.5: Case 2a in the proof of Theorem 4.2.7.

Case 2b: $z > k_{i_0}$. Let $g = \max\{h : x_h \in m_{i_1 j_1} \cap m_{i_0 j_0}\}$. Note that $0 \le g \le n-2$ otherwise we would be in Case 2a. Since P has the codimension one property, there is some $c_{i_0 d}$ such that $x_g < c_{i_0 d} \lessdot f_{i_0}$ where $d \le k_{i_0}$. Let $x_g = y_0 \lessdot y_1 \lessdot \ldots \lessdot y_{n-g-1} = c_{i_0 d}$ be any saturated chain from x_g to $c_{i_0 d}$. Let $m_{i_0 j_2}$ be the chain $x_0 \lessdot x_1 \lessdot \ldots \lessdot x_g = y_0 \lessdot \ldots \lessdot y_{n-g-1} = c_{i_0 d} \lessdot f_{i_0}$. Note that $m_{i_1 j_1} \cap m_{i_0 j_0} \subseteq \{x_0, x_1, \ldots, x_g\} \subseteq m_{i_0 j_2} \cap m_{i_0 j_0}$ because $m_{i_1 j_1}$ agrees with $m_{i_0 j_2}$ up to x_g. Since $d \le k_{i_0}$ and $z > k_{i_0}$, we have $d < z$. Thus $m_{i_0 j_2} <_\Omega m_{i_0 j_0}$. We are now back in Case 1: there exists some $m_{i_0 j_3} \in \mathcal{M}_{i_0}$ such that $j_3 < j_0$ and such that $m_{i_0 j_2} \cap m_{i_0 j_0} \subseteq m_{i_0 j_3} \cap m_{i_0 j_0} = m_{i_0 j_0} - \{x\}$ for some $x \in m_{i_0 j_0}$. Thus $m_{i_0 j_0} \cap m_{i_1 j_1} \subseteq m_{i_0 j_2} \cap m_{i_0 j_0} \subseteq m_{i_0 j_3} \cap m_{i_0 j_0} = m_{i_0 j_0} - \{x\}$. See Figure 4.6. This completes the proof. $\square$

If we have a partial order on the facets of a simplicial complex satisfying the codimension one property and an extra condition on incomparable facets, then any linear extension of this partial order is a shelling order. We show this next.

Lemma 4.2.8. *Let K be a simplicial complex and P be a partial order on the facets of K. Suppose P has the codimension one property and a unique minimal element $\hat{0}$. Suppose that whenever F_i and F_j are incomparable in P and $\emptyset \ne G \in I(F_i) \cap I(F_j)$, then $G \in I(F_k)$ for some $F_k <_P F_i$ or $F_k <_P F_j$. Then any linear extension of P is a shelling order of K.*

Proof. Let T be any linear extension of P. We will show that

$$\overline{F_j} \bigcap \left(\bigcup_{i <_P j} \overline{F_i} \right) = \overline{F_j} \bigcap \left(\bigcup_{i <_T j} \overline{F_i} \right)$$

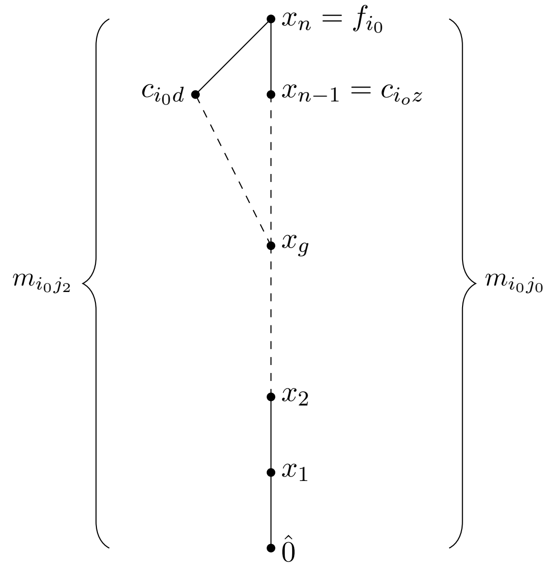


Figure 4.6: Case 2b in the proof of Theorem 4.2.7.

Since P has the codimension one property, proving this equality implies that T is a shelling order of the facets of K.

By definition of a linear extension, if $i \leq_P j$, then $i \leq_T j$. Thus if $G \in \overline{F_j} \bigcap \left(\bigcup_{i<_P j} \overline{F_i} \right)$, then $G \in \overline{F_j} \bigcap \left(\bigcup_{i<_T j} \overline{F_i} \right)$.

Now consider if $G \in \overline{F_j} \bigcap \left(\bigcup_{i<_T j} \overline{F_i} \right)$. This means that $G \in \overline{F_j} \cap \overline{F_i}$ for some $i <_T j$. If $i <_P j$, then $G \in \overline{F_j} \bigcap \left(\bigcup_{i<_P} F_i \right)$. If, however, i and j are incomparable in P, then by hypothesis, $G \in F_k$ where $k <_P i$ or $k <_P j$. If $k <_P j$, then $G \in \overline{F_k} \cap \overline{F_j} \subseteq \overline{F_j} \bigcap \left(\bigcup_{i<_P j} \overline{F_i} \right)$. If $k <_P i$ and k and j are incomparable in P, then again by hypothesis, $G \in \overline{F_{k_1}}$ where $k_1 <_P j$ or $k_1 <_P k$. We repeat this argument until we have some k_n such that $G \in \overline{F_{k_n}}$ and $k_n <_P j$, which must occur since P has a unique minimal element. Then $G \in \overline{F_j} \bigcap \left(\bigcup_{i<_P j} F_i \right)$. □

Lemma 4.2.8 gives the following:

Theorem 4.2.9. *Let K be a simplicial complex and P be a partial order on the facets of K with the codimension one property and a unique minimal element. Suppose that whenever F_i and F_j are incomparable in P and $\emptyset \neq G \in I(F_i) \cap I(F_j)$, then $G \in I(F_k)$ for some $F_k <_P F_i$ or $F_k <_P F_j$. Then K is homotopy equivalent to a wedge of spheres.*

Proof. By Lemma 4.2.8, any linear extension of P is a shelling of K. By Theorem 4.2.10, this implies that K is homotopy equivalent to a wedge of spheres. □

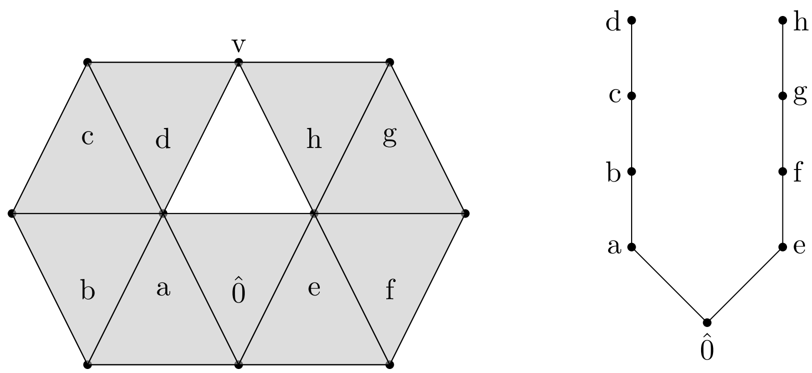


Figure 4.7: A 2-dimensional simplicial complex K (left) and a partial order on its facets (right) satisfying the codimension one property. It is not true that K is homotopy equivalent to a wedge of 2-spheres.

It it natural to ask if we can also generalize the following result to partial orders satisfying the codimension one property:

Theorem 4.2.10. *If the face poset of a simplicial complex K is shellable, then K is homotopy equivalent to a wedge of spheres.*

In particular, it was our hope that we could show that if the face poset of a simplicial complex K has a partial order on its coatoms that satisfies the codimension one property, then K is homotopy equivalent to a wedge of spheres. However, this is not always true, as the example in Figure 4.7 shows.

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