ABSTRACT

CRIFO, SUZANNE ELISE. Some Maximal Dominant Weights and Their Multiplicities for Affine Lie Algebra Representations. (Under the direction of Kailash Misra).

It is known that there are finitely many maximal dominant weights for an integrable highest weight module of an affine Lie algebra $\mathfrak{g}$. However, a description for these maximal dominant weights is known in only select cases. For example, Jayne and Misra studied the maximal dominant weights of the $A_n^{(1)}$-modules $V((k-1)\Lambda_0 + \Lambda_s)$ where $0 \le s \le n$. In this thesis we give an explicit description of all maximal dominant weights for the $\mathfrak{g}$-module $V(k\Lambda_0)$ where $\mathfrak{g}$ is any other affine Lie algebra. After determining the maximal dominant weights, another interesting area to explore is the corresponding weight space. Specifically, we would like to determine the dimension of the corresponding weight space, which is also known as the weight's multiplicity. Using crystal base theory, we determine multiplicities for some of the maximal dominant weights we have found.

Some Maximal Dominant Weights and Their Multiplicities for Affine Lie Algebra
Representations

by
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DEDICATION

In memory of my father, Daniel A. Crifo, who inspired my love for mathematics.

BIOGRAPHY

Suzanne was raised in Glen Rock, New Jersey. She studied mathematics and music at the College of the Holy Cross in Worcester, Massachusetts and wrote a college honors thesis on mathematical concepts present in musical compositions. After graduating, she lived for a year in Mobile, Alabama while volunteering with the Jesuit Volunteer Corps. There, she worked as a case manager for individuals in emergency situations and as a team member at L'Arche. She then moved to Raleigh, North Carolina to begin her graduate studies in mathematics. When she is not doing math, you can find her doing some sort of fitness exercise such as running or yoga. She also founded the LBD 5k: Lace Up To Bring Lewy Down and has co-organized the event for the past three years. She is looking forward to continuing to make North Carolina her home as she starts her new position in Duke University's Academic Resource Center as an Assistant Director - Learning Consultant/Math Specialist.

ACKNOWLEDGEMENTS

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Chapter 1

Introduction

Sophus Lie discovered Lie algebras in the 19th century while studying the symmetries of solutions of differential equations. Since then, Lie algebras have been studied extensively and have been shown to have many connections to other areas in mathematics and mathematical physics. By the end of the 19th century, Wilhelm Killing and Élie Cartan had classified all simple finite-dimensional complex Lie algebras [4]. Independently and simultaneously, Victor Kac [10] and Robert Moody [21] discovered Kac-Moody algebras. Kac-Moody algebras are a generalization of finite dimensional complex semisimple Lie algebras. Kac-Moody algebras are divided into three subcategories: finite dimensional semisimple Lie algebras, infinite dimensional affine Lie algebras of polynomial growth, and infinite dimensional algebras of exponential growth. This thesis is concerned with the second subcategory, which are called affine Lie algebras.

In 1974, Kac [11] introduced the notion of an integrable highest-weight representation, allowing the representation theory of finite dimensional semisimple Lie algebras to expand from the finite dimensional case to affine Lie algebras. To describe the structure of an integrable highest-weight representation, one needs to describe its maximal dominant weights. The maximal weights form a type of "ceiling" to the set of all weights. Any other weight lies on an infinite string off a maximal weight. Any maximal weight is Weyl group conjugate to a maximal dominant weight.

It is known there are finitely many maximal dominant weights for any integrable highest weight representation of an affine Lie algebra. However, determining these maximal dominant weights is a nontrivial task. Tsuchioka found all maximal dominant weights for modules of the form $V(\Lambda_0 + \Lambda_s)$ where $0 \le s < p$ for type $A_{p-1}^{(1)}$ [23]. This work was generalized in 2014 by Jayne and Misra, who found all maximal dominant weights for modules of the form $V((k-1)\Lambda_0 + \Lambda_s)$ where $0 \le s \le n-1$ for type $A_{n-1}^{(1)}$ [7]. In 2017, Kim, Lee, and Oh determined all maximal dominant weights for $V(\Lambda)$ where Λ is of level 2 for types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$, and of level 1 for $C_n^{(1)}$ [18].

Once the maximal dominant weights have been determined, another question arises. Specifically, given a maximal dominant weight, what is the dimension of the corresponding weight space? The dimension of a weight space is known as the *multiplicity* of that weight. These

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multiplicities can be studied using a combinatorial tool that comes out of studying quantum groups. Quantum groups (which are not actually groups), were introduced independently by Drinfel'd [2] and Jimbo [9] in 1985. The quantum group is a q-deformation of the universal enveloping algebra of a symmetrizable Kac-Moody algebra. Lusztig [19] proved that the representation theory of this quantum group is the same as the representation theory of the corresponding Kac-Moody algebra. This means that the weight spaces (and their dimensions) are invariant under this deformation.

The representation theory of quantum groups is more easily studied than that of Kac-Moody algebras due to those combinatorial tools we hinted at earlier. The notion of a *crystal basis* was introduced by Kashiwara [17] in 1990 and is the same as Lusztig's [20] canonical base at $q = 0$. Crystals allow us to study these multiplicities. Specifically, we use *perfect crystals*, whose theory is given in [15] and [14]. These perfect crystals give rise to a path realization [15], [14] of crystals for integrable highest weight modules of affine Lie algebras. This path model obtains a colored, oriented graph, called the *crystal graph*, for irreducible highest weight modules over affine Lie algebras, allowing us to find the multiplicities of maximal dominant weights by counting the number of vertices corresponding to each weight.

This dissertation expands on the results listed above by finding maximal dominant weights for the remaining affine Lie algebras in the case of $V(k\Lambda_0)$ and some of the multiplicities of these weights. Chapter 2 gives the background necessary to find the maximal dominant weights, which are found in Chapter 3. We also include the relevant results from [7] for type $A_n^{(1)}$ and prove our findings are the same as [18] in the case $k = 2$ for the appropriate affine Lie algebras in Chapter 3. Chapter 4 then gives the information needed to study path realizations of integrable highest weight modules. In Chapter 5, we include the results from [15] that are used to study the path realizations of $V(k\Lambda_0)$ in types $B_n^{(1)}, C_n^{(1)}$, and $D_n^{(1)}$, as well as the relevant results from [7] in the case of $A_n^{(1)}$.

Chapter 2

Kac-Moody Algebras and their Representations

In this chapter, we discuss Kac-Moody algebras and their representations. As noted below, Kac-Moody algebras fall into three classes. Of the three classes, we will focus on affine Lie algebras and their representations. Unless otherwise noted, we will assume the field is $\mathbb{C}$.

2.1 Lie Algebras

Definition 2.1.1. A *Lie algebra* over $\mathbb{C}$ is a vector space L equipped with a product, called a Lie bracket, $[\cdot, \cdot] : L \times L \to L$ that satisfies the following properties:

- Bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$,

- Alternativity: $[x, x] = 0$,

- Jacobi Identity: $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$,

for all $x, y, z \in L$ and $a, b \in \mathbb{C}$.

Note that because the Lie bracket is both bilinear and alternative, one can show that it is also skew-symmetric, or $[x, y] = -[y, x]$ for all $x, y \in L$.

While there are many examples of finite dimensional Lie algebras, a familiar one is $\mathbb{R}^3$ with the cross product of vectors serving as the Lie bracket.

Example 2.1.2. For any vector space V, the set of linear operators on V is a Lie algebra, denoted $\mathfrak{gl}(V)$ with the Lie bracket defined by $[x, y] = x \circ y - y \circ x$ where $x \circ y$ is the composition of linear operators. If V has finite dimension n, the elements of this Lie algebra can be thought of as $n \times n$ matrices.

Example 2.1.3. Let $L = \{2 \times 2 \text{ matrices with trace } 0\} \subset \mathbb{C}^{2 \times 2}$

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is a basis for L. L is closed under the commutator bracket: $[x, y] = xy - yx$ for all $x, y \in L$ where xy is matrix multiplication.

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

L is a Lie algebra, called the *special linear Lie algebra*, $\mathfrak{sl}(2, \mathbb{C})$.

Definition 2.1.4. A *subalgebra* of a Lie algebra L is a subspace M of L such that $[x, y] \in M$ for all $x, y \in M$.

As an example, $\mathfrak{sl}(2, \mathbb{C})$ is a subalgebra of the Lie algebra $L = \mathfrak{gl}(n, \mathbb{C}) = \{2 \times 2 \text{ matrices} $ over $\mathbb{C}\}$, equipped with the commutator bracket. Within $\mathfrak{sl}(2, \mathbb{C})$, the span of h, or the set of all diagonal matrices, is itself a subalgebra. We will return to this important type of subalgebra, called a *Cartan subalgebra*, denoted by $\mathfrak{h}$.

Definition 2.1.5. An *ideal* of a Lie algebra is a subspace I of L such that $[x, y] \in I$ for all $x \in L$ and $y \in I$.

As an example, given any Lie algebra L, the *derived algebra* of L, $[L, L]$ is an ideal of L.

Example 2.1.6. Let $L = \mathfrak{gl}(n, \mathbb{C})$, the set of all $n \times n$ matrices with entries in $\mathbb{C}$ equipped with the commutator bracket, also called the *general linear Lie algebra*. Then the derived algebra of L can be found using basis vectors as follows:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

Then if $j \neq k, i \neq l$, $[E_{ij}, E_{kl}] = 0$, if $j = k, i \neq l$, $[E_{ij}, E_{kl}] = E_{il}$, if $j \neq k, i = l$, $[E_{ij}, E_{kl}] = -E_{kj}$, and if $j = k, i = l$ then $[E_{ij}, E_{kl}] = E_{ii} - E_{kk}$. Therefore, the derived algebra of $\mathfrak{gl}(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{C})$, the *special linear Lie algebra*, which is the set of all $n \times n$ matrices with entries in $\mathbb{C}$ and with trace 0.

Definition 2.1.7. A Lie algebra is called *simple* if it is nonabelian (meaning there exist some elements $x, y \in L$ such that $[x, y] \neq 0$) and has no nontrivial proper ideals. A Lie algebra is called *semisimple* if it is the direct sum of simple Lie algebras

$\mathfrak{sl}(2, \mathbb{C})$ is a simple Lie algebra, while $\mathfrak{gl}(n, \mathbb{C})$ is not a simple Lie algebra, since $\mathfrak{sl}(n, \mathbb{C}) = [\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C})]$ is an example of a nontrivial proper ideal.

Definition 2.1.8. A (Lie algebra) *homomorphism* $\phi : L \to M$ is a linear map such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$.

For a Lie algebra L and any element $x \in L$, the linear operator $\mathrm{ad}_x \in \mathfrak{gl}(L)$ where $\mathrm{ad}_x(y) = [x, y]$ gives rise to a linear map $\mathrm{ad}: L \to \mathfrak{gl}(L)$ where $\mathrm{ad}(x) = \mathrm{ad}_x$, which is a homomorphism.

Example 2.1.9. If $L = \mathfrak{sl}(2,\mathbb{C})$ with basis $\{h, e, f\}$, we can represent ad_x as a 3×3 matrix for every $x \in \mathfrak{sl}(2,\mathbb{C})$. We do so for the basis vectors below.

$$ad_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \ ad_e = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ ad_f = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

2.1.1 Representations and Modules

Definition 2.1.10. A *representation* of a Lie algebra L on a vector space V is a homomorphism $\rho : L \to \mathfrak{gl}(V)$. The representation gives rise to an *action* of L on V defined by $x \cdot v = \rho(x)v \in V$ for all $x \in L$ and $v \in V$.

ad:$L \to \mathfrak{gl}(L)$, introduced earlier, is called the adjoint representation. It is a representation of the Lie algebra L on itself. The corresponding adjoint action is the same as the Lie bracket.

Example 2.1.11. Taking $L = \mathfrak{sl}(2,\mathbb{C})$, we see the action of h, for example, under the adjoint representation on each of the basis vectors of $\mathfrak{sl}(2,\mathbb{C})$. In order to do so, we use the ordered basis $\{h, e, f\}$.

$$h \cdot h = \mathrm{ad}(h)h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$h \cdot e = \mathrm{ad}(h)e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2e$$

$$h \cdot f = \mathrm{ad}(h)f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = -2f$$

Definition 2.1.12. An *L-module* V is a vector space equipped with a bilinear map $L \times V \to V$ such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in L$ and $v \in V$.

Note that the module action can be thought of as the action of L on V arising from a representation ρ:

$$\rho([x,y])v = [x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) = \rho(x)\rho(y)v - \rho(y)\rho(x)v = [\rho(x), \rho(y)] \cdot v$$

Then the concepts are equivalent. We often use them interchangeably, though the term "representation" refers to the map while "module" refers to the vector space on which L is acting. In the above example, $\mathfrak{sl}(2,\mathbb{C})$ is itself a $\mathfrak{sl}(2,\mathbb{C})$ module under the adjoint representation.

Definition 2.1.13. A *submodule* of an L-module V is a subspace U of V that is closed under

the action of L, so $xu \in U$ for all $x \in L$ and $u \in U$. V is called *irreducible* if it has no nontrivial proper submodules.

Returning to the adjoint representation, ad: $L \to \mathfrak{gl}(L)$, the corresponding adjoint action is $x \cdot y = \mathrm{ad}(x)y = [x, y]$. Then a submodule of L is a subspace U of L such that $x \cdot u = [x, u] \in U$ for all $x \in L$ and $u \in U$. That is, the submodules of L are the ideals of L. Therefore, the adjoint representation of a Lie algebra L is irreducible if and only if L is simple.

2.2   Kac-Moody Algebras

As detailed in [12], a Lie algebra can be generalized to what is called a *Kac-Moody algebra*. There are three types of such algebras, finite, affine, and indefinite, as classified in Theorem 4.3 in [12].

Theorem 2.2.1.   *[12] Let A be an integral, indecomposable $(n+1) \times (n+1)$ matrix satisfying*

- $a_{ij} \leq 0$ *for* $i \neq j$,

- $a_{ij} = 0$ *implies that* $a_{ji} = 0$,

then one and only one of the following three possibilities holds for A and A^T:

- *(Fin) $\det A \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$,*

- *(Aff) corank $A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$,*

- *(Ind) there exists $u > 0$ such that $Au < 0$; $Av \geq 0, v \geq 0$ imply $v = 0$.*

This thesis will focus on the second type, which gives rise to what are called affine Lie algebras. Note that a square matrix A is called *symmetrizable* if there exists a diagonal matrix D of the same size with all positive entries on the diagonal such that DA is symmetric. We will assume that all matrices are symmetrizable. For our purposes, we have the following definition:

Definition 2.2.2. An integral, symmetrizable $(n+1) \times (n+1)$ matrix A is a *Cartan matrix of affine type* if it satisfies the following conditions:

- $a_{ii} = 2$ for $i = 0, 1, \ldots, n$,

- a_{ij} are nonpositive integers for $i \neq j$,

- $a_{ij} = 0$ implies $a_{ji} = 0$,

- corank(A)=1.

Every Cartan matrix has a corresponding oriented graph called a *Dynkin diagram*, defined as follows.

Definition 2.2.3. The graph $\Gamma(A)$ associated with an affine Cartan matrix A is called a *Dynkin diagram* and is constructed as follows.

- Begin with $n + 1$ nodes

- For $i \neq j$, connect vertices i and j by $\max\{|a_{ij}|, |a_{ji}|\}$ edges.

- For $i \neq j$, if $|a_{ij}| \neq |a_{ji}|$, add an arrow across the edges connecting vertex i and vertex j pointing toward vertex i if $|a_{ij}| > 1$.

Example 2.2.4. Here we show the affine Cartan matrix and corresponding Dynkin diagram associated with the affine Lie algebra $\mathfrak{g} = B_n^{(1)}$. The rows and columns of the Cartan matrix correspond to the $n+1$ node, starting with row (or column) 0 and ending with row (or column) n. Notice that the second node is connected to the 0th, 1st, and 3rd node, as indicated by the corresponding $a_{2i} = a_{i2}$ entries. The $n-1$st and nth node are connected by two edges, since $a_{n,n-1} = -2$ with an arrow across the edges pointing toward the nth node.

$$
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 2 & -1 & 0 & 0 & \cdots & 0 \\
-1 & -1 & 2 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & 2 & -1 & \cdots & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & 0 & 0 & -2 & 2
\end{pmatrix}
$$

A *realization* of an $(n+1) \times (n+1)$ affine Cartan matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$, called the *Cartan subalgebra*, is a complex vector space of dimension $n+2$, $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$, and $\Pi^\vee = \{h_0, h_1, \ldots, h_n\} \subset \mathfrak{h}$ such that Π and $\Pi^\vee$ are linearly independent and $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = a_{ij}$. Let $I = \{0, 1, \ldots, n\}$ be the index set. Following the terminology from the theory of finite-dimensional Lie algebras, Π is the set of *simple roots* and $\Pi^\vee$ the set of *simple coroots*. Then $Q = \sum_{i=0}^{n} \mathbb{Z}\alpha_i$ is called the *root lattice*. We also designate $Q_+ = \sum_{i=0}^{n} \mathbb{Z}_{\geq 0}\alpha_i$. With this notation, we have the partial ordering $\geq$ on $\mathfrak{h}^*$ that $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.

Definition 2.2.5. Let A be an affine Cartan matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Then the associated *affine Lie algebra*, $\mathfrak{g}$ has generators e_i, f_i ($i \in I$) and $\mathfrak{h}$ satisfying the following conditions.

- $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$,

- $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$,

- $[h_i, e_j] = a_{ij} e_j$ for $i, j \in I$,

- $[h_i, f_j] = -a_{ij} f_j$ for $i, j \in I$,

- $(ad(e_i))^{1-a_{ij}}e_j = 0$ for $i \neq j$,

- $(ad(f_i))^{1-a_{ij}}f_j = 0$ for $i \neq j$.

For convenience, we reproduce the tables of Dynkin diagrams associated with all affine Lie algebras from [12]. We give the corresponding Cartan matrix for each affine Lie algebra in the respective section of Chapter 3.

Table Aff 1

$A_1^{(1)}$

$\underset{1}{\circ} \Longleftrightarrow \underset{1}{\circ}$

$A_l^{(1)}(l \geq 2)$

$\overset{\displaystyle \underset{1}{\circ}}{} \quad \underset{1}{\circ} \Longleftarrow \underset{1}{\circ} - \cdots - \underset{1}{\circ} \Longrightarrow \underset{1}{\circ}$

$B_l^{(1)}(l \geq 3)$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{2}{\circ} - \cdots - \underset{2}{\circ} \Longrightarrow \underset{2}{\circ}$ with pendant $\underset{1}{\circ}$

$C_l^{(1)}(l \geq 2)$

$\underset{1}{\circ} \Longrightarrow \underset{2}{\circ} - \cdots - \underset{2}{\circ} \Longleftarrow \underset{1}{\circ}$

$D_l^{(1)}(l \geq 4)$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{2}{\circ} - \cdots - \underset{2}{\circ} - \underset{1}{\circ}$ with pendants $\underset{1}{\circ}$

$G_2^{(1)}$

$\underset{1}{\circ} - \underset{2}{\circ} \Longrightarrow \underset{3}{\circ}$

$F_4^{(1)}$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} \Longrightarrow \underset{4}{\circ} - \underset{2}{\circ}$

$E_6^{(1)}$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} - \underset{2}{\circ} - \underset{1}{\circ}$ with vertical branch $\underset{2}{\circ} - \underset{1}{\circ}$

$E_7^{(1)}$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} - \underset{4}{\circ} - \underset{3}{\circ} - \underset{2}{\circ} - \underset{1}{\circ}$ with branch $\underset{2}{\circ}$

$E_8^{(1)}$

$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} - \underset{4}{\circ} - \underset{5}{\circ} - \underset{6}{\circ} - \underset{4}{\circ} - \underset{2}{\circ}$ with branch $\underset{3}{\circ}$

Table Aff 2

$$A_2^{(2)}$$

$$A_{2l}^{(2)}\,(l \geq 2)$$

$$A_{2l-1}^{(2)}\,(l \geq 3)$$

$$D_{l+1}^{(2)}\,(l \geq 2)$$

$$E_6^{(2)}$$

Table Aff 3

$$D_4^{(3)}$$

Given an affine Lie algebra $\mathfrak{g}$, let a_i be the labeling of the nodes in each Dynkin diagram. $a_i^\vee$ is the labeling in the diagram of the corresponding dual algebra, which is found by reversing all arrows in the Dynkin diagram of $\mathfrak{g}$. For example, if $\mathfrak{g} = B_n^{(1)}$, then its dual algebra is $A_{2n-1}^{(2)}$. Notice if u is the vector with entries a_i then $Au = 0$ where A is the associated Cartan matrix. For example, in the case $\mathfrak{g} = B_n^{(1)}$:

$$Au = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

As in the finite case, an important subalgebra of $\mathfrak{g}$ is the Cartan subalgebra, $\mathfrak{h}$. For every $\alpha \in Q$ we associate a *root space*, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Then $\alpha \neq 0$ is called a *root* if $\mathfrak{g}_\alpha \neq 0$. Associated with $\mathfrak{g}$ is its *Weyl group*, W. W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the following reflections on $\mathfrak{h}^*$.

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \text{ for } \lambda \in \mathfrak{h}^*$$

The r_i are called the *fundamental reflections* [12].

The center of $\mathfrak{g}$ is one dimensional and is spanned by the *canonical central element* $c = \sum_{i=0}^{n} a_i^\vee h_i$. Because $\mathfrak{g}$ is affine, by Theorem 5.6 in [12], which establishes the presence of *imaginary roots* in affine Lie algebras, we have the *null root* $\delta = \sum_{i=0}^{n} a_i \alpha_i \in Q$. A root α is called imaginary if there does not exist a $w \in W$ such that $w(\alpha)$ is a simple root. Every imaginary root in $\mathfrak{g}$ is a nonzero integer multiple of δ. Finally, we have the derivation $d \in \mathfrak{h}$ such that $\langle \alpha_i, d \rangle = 0$ for $i = 1, \ldots, n$ and $\langle \alpha_0, d \rangle = 1$.

We now have all components necessary to define the nondegenerate symmetric bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{h}$:

$$\begin{cases} (h_i | h_j) = \frac{a_j}{a_j^\vee} a_{ij} & i, j \in I, \\ (h_i | d) = 0 & i = 1, \ldots, n, \\ (h_0 | d) = a_0, \\ (d | d) = 0. \end{cases}$$

This can be extended to a bilinear form on $\mathfrak{g}$ by defining $\Lambda_0 \in \mathfrak{h}^*$ by $\langle \Lambda_0, h_i \rangle = \delta_{0i}$ for $i \in I$ and $\langle \Lambda_0, d \rangle = 0$. With Λ_0, $\{\alpha_0, \ldots, \alpha_n, \Lambda_0\}$ forms a basis of $\mathfrak{h}^*$ and so we define the extension of the bilinear form on $\mathfrak{g}$ using this basis:

$$\begin{cases} (\alpha_i | \alpha_j) = \frac{a_i^\vee}{a_i} a_{ij} & i, j \in I \\ (\alpha_i | \Lambda_0) = 0 & i = 1, \ldots, n \\ (\alpha_0 | \Lambda_0) = \frac{1}{a_0} \\ (\Lambda_0 | \Lambda_0) = 0 \end{cases}$$

Finally, we define the *universal enveloping algebra*, $\mathcal{U}(\mathfrak{g})$, to which we will return in Chapter 4. Let $\mathcal{U}(\mathfrak{g})$ be an associative algebra together with a Lie algebra homomorphism $j : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ satisfying the following universal property. If (A, ϕ) is any pair where A is an associative algebra and ϕ is a Lie algebra homomorphism ($\phi : \mathfrak{g} \to A$) then there exists a unique associative algebra homomorphism ψ such that $\psi j = \phi$. This is shown in the following diagram.

2.2.1 Integrable Highest Weight Modules

We now introduce the main object of our study, integrable highest weight modules of affine Lie algebras. To do so, we need to define several notions. We continue with an affine Lie algebra $\mathfrak{g}$,

index set $I = \{0, 1, \ldots, n\}$, and Cartan subalgebra $\mathfrak{h}$. Its *weight lattice* is

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z} \text{ for } i \in I\}.$$

Elements of the weight lattice are called *integral weights*. The *dual weight lattice, $P^\vee$* has basis $\{h_i \in I\} \cup \{d\}$. $P^+ = \{\lambda \in P \mid \langle \lambda, h_i \rangle \geq 0 \text{ for } i \in I\}$ is the set of *dominant integral weights*. Recall that since $\langle \alpha_j, h_i \rangle = a_{ij} \in \mathbb{Z}$, the root lattice, Q is contained within P. We have the *fundamental weights* Λ_i for $i = 0, 1, \ldots, n$ defined by $\Lambda_i(h_j) = \delta_{ij}$. We define the *level, k,* of any $\lambda \in \mathfrak{h}^*$ by $k = \langle \lambda, c \rangle = \sum_{i=0}^n a_i^\vee \lambda(h_i)$ where c is the canonical central element.

Recall that V is a $\mathfrak{g}$-module if the following conditions hold for all $x, y \in \mathfrak{g}, v, w \in V$ and $a, b \in F$:

- $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

- $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

- $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$.

Definition 2.2.6. A $\mathfrak{g}$-module V is a *weight module* if it admits a weight space decomposition $V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu$ where $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \ \forall \ h \in \mathfrak{h}\}$. $\mu \in \mathfrak{h}^*$ is called a *weight* if $V_\mu \neq \{0\}$.

In particular, we are interested in integrable highest weight modules.

Definition 2.2.7. A weight module is a *highest weight module with highest weight λ,* denoted $V(\lambda)$, if there exists $v_\lambda \neq 0$ in $V(\lambda)$ such that

- $\mathcal{U}(\mathfrak{g}) \cdot v_\lambda = V$,

- $e_i \cdot v_\lambda = 0$ for all $i \in I$,

- $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$.

Definition 2.2.8. A weight module V of $\mathfrak{g}$ is called *integrable* if e_i and f_i for $i \in I$ are locally nilpotent. That is, for every $v \in V$ and $i \in I$, there exists an $N \in \mathbb{N}$ dependent on v and i such that $e_i^N v = 0$ and $f_i^N v = 0$.

We use the following important fact:

Lemma 2.2.9 ([12]). *The $\mathfrak{g}$-module $V(\Lambda)$ is integrable if and only if $\Lambda \in P^+$.*

For $\Lambda \in P^+$ there exists a unique (up to isomorphism) irreducible, integrable highest weight module $V(\Lambda)$ generated by a highest weight vector v_Λ.

Fix $\Lambda \in P^+$. Let $P(\Lambda)$ be the set of all weights of $V(\Lambda)$. If $\lambda \in P(\Lambda)$ then $\lambda = \Lambda - \sum_{i=0}^n b_i \alpha_i$ where $b_i \in \mathbb{Z}_{\geq 0}$.

Definition 2.2.10. A weight $\lambda \in P(\Lambda)$ is called *maximal* if $\lambda + \delta \notin P(\Lambda)$. Denote the set of all maximal weights as $\max(\Lambda)$.

Then according to [12],

$$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta | n \in \mathbb{Z}_+\}.$$

That is, the set of maximal weights form a roof to the set of all weights, from which any other weight can be found by subtracting positive integer multiples of δ. Any $\lambda \in \max(\Lambda)$ is W-conjugate to some $\mu \in \max(\Lambda) \cap P^+$, which is known to be a finite set [12]. Then by determining $\max(\Lambda) \cap P^+$, one can describe all of $\max(\Lambda)$ and so all of $P(\Lambda)$. However, only partial results for an explicit description of this set are known. We determine explicit descriptions of $\max(k\Lambda_0) \cap P^+$ for every affine Lie algebra in Chapter 3.

Chapter 3

Maximal Dominant Weights of $V(k\Lambda_0)$

3.1 Introduction

In the following, we will go through each affine Lie algebra $\mathfrak{g}$ and give the corresponding index set and affine Cartan matrix A. Unless otherwise stated, we have the following notation. Let $P, P^\vee$ be the weight lattice and dual weight lattice respectively and $P^+ = \{\lambda \in P | \lambda(h_i) \geq 0$ for all $i \in I\}$ the set of dominant integral weights. For $\Lambda \in P^+$, $V(\Lambda)$ is the integrable highest weight $\mathfrak{g}$-module of level $k = \Lambda(c)$. We know that for any $\mu \in \mathfrak{h}^*$ to be a weight of $V(\Lambda)$, we must have that the corresponding weight space, $V(\Lambda)_\mu = \{v \in V(\Lambda) \mid h \cdot v = \mu(h)v,$ for all $h \in \mathfrak{h}\} \neq \{0\}$. Then μ is of the form $\mu = \Lambda - \sum_{i \in I} b_i \alpha_i$ where $b_i \in \mathbb{Z}_{\geq 0}$ for all $i \in I$ and we say that $\mu \in P(\Lambda)$ where $P(\Lambda)$ is the set of all weights of $V(\Lambda)$.

A weight $\lambda \in V(\Lambda)$ is called a *maximal weight* if $\lambda + \delta$ is not a weight of $V(\Lambda)$. We call the set of all maximal weights of $V(\Lambda)$, $\max(\Lambda)$. This set is important in describing the set of all weights of $V(\Lambda)$ because $P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \mid n \in \mathbb{Z}_{\geq 0}\}$. That is, the set of all maximal weights is like the top of a jellyfish, with all the other weights of $V(\Lambda)$ coming off those maximal weights in strings, or tentacles. If we can describe the set of all maximal weights, then we have a description for the set of all weights. Note that any $\lambda \in \max(\Lambda)$ is W-conjugate to some maximal dominant weight, or $\mu \in \max(\Lambda) \cap P^+$, where W is the Weyl group of the affine Lie algebra $\mathfrak{g}$. Then we need only describe these maximal dominant weights and wish to do so explicitly. To do so, we need to introduce a map . First, $\mathring{\mathfrak{h}}$ is the linear span of $h_1, \ldots, h_n$ over $\mathbb{C}$ (while $\mathring{\mathfrak{h}}_\mathbb{R}$ is the same notion over $\mathbb{R}$.) The duals, $\mathring{\mathfrak{h}}^*$ and $\mathring{\mathfrak{h}}^*_\mathbb{R}$, are defined similarly. Then for any subset S of $\mathfrak{h}^*$, we have its orthogonal projection $\overline{S}$ on $\mathring{\mathfrak{h}}^*$ and the following formula for the map:

$$\overline{\lambda} = \lambda - \langle \lambda, c \rangle \Lambda_0 + (\lambda | \Lambda_0)\delta \text{ where } \lambda \in \mathfrak{h}^*$$

Now we can use Proposition 12.6 in [12]:

Proposition. *[12] The map $\lambda \mapsto \overline{\lambda}$ defines a bijection from $max(\Lambda) \cap P^+$ onto $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$.*

In particular, the set of dominant maximal weights of $V(\Lambda)$ is finite.

Therefore, we can find the set $kC_{af} \cap (\overline{\Lambda + Q})$ and use the inverse of the orthogonal projection map $^-$ to determine the set of maximal dominant weights of $V(\Lambda)$. We focus specifically on the integrable highest weight module $V(k\Lambda_0)$, of level k. From Kac [12], we know that $kC_{af} = \{\overline{\lambda} \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} | \overline{\lambda}(h_i) \geq 0 \text{ for all } i \in I, (\overline{\lambda}|\theta) \leq k\}$. Therefore, to find all maximal dominant weights of $V(\Lambda)$, we need only solve those defining inequalities. Let λ be a weight of $V(\Lambda)$. Then $\lambda = \Lambda - \sum_{i=0}^{n} b_i \alpha_i = k\Lambda_0 - \sum_{i=0}^{n} b_i \alpha_i$ where $b_i \in \mathbb{Z}_{\geq 0}$ for all $i \in I$. Then

$$
\begin{aligned}
\overline{\lambda} &= \overline{k\Lambda_0 - \sum_{i=0}^{n} b_i \alpha_i} \\
&= k\Lambda_0 - \sum_{i=0}^{n} b_i \alpha_i - \langle k\Lambda_0 - \sum_{i=0}^{n} b_i \alpha_i, c \rangle \Lambda_0 - (k\Lambda_0 - \sum_{i=0}^{n} b_i \alpha_i | \Lambda_0)\delta \\
&= -\sum_{i=0}^{n} b_i \alpha_i + b_0 (\sum_{i=0}^{n} a_i \alpha_i) \\
&= \sum_{i=1}^{n} (a_i b_0 - b_i) \alpha_i
\end{aligned}
$$

For convenience, we represent each $\overline{\lambda} \in kC_{af}$ with an n-tuple $\mathbf{x}_{\overline{\lambda}} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^{n}$ where $x_i = a_i b_0 - b_i$. The defining inequalities can then be described succinctly as

$$
\begin{cases}
\mathring{A}\mathbf{x} & \geq 0 \\
(\overline{\lambda}|\theta) & \leq k
\end{cases}
$$

where $\mathring{A}$ is the $n \times n$ matrix obtained by removing row 0 and column 0 from A. Note that $(\overline{\lambda}|\theta) \leq k$ will have different manifestations in the $\mathbf{x}$ vectors for different affine Lie algebras.

We will find all solutions to the set of defining inequalities for each type in terms of $\mathbf{x}$ and then use the definition of each x_i to find all maximal dominant weights of $V(k\Lambda_0)$. Before we find the solutions, we can make a few observations about the system of inequalities. We know that $x_i \in \mathbb{Z}$ for all $i \in I$ since the x_i are linear combinations of the b_i. Because $\mathring{A}$ is of finite type, by Theorem 4.3 in [12], which was presented in Chapter 2, then $\mathring{A}\mathbf{x} \geq 0$ implies that $\mathbf{x} > 0$ or $\mathbf{x} = 0$. Then $\mathbf{x} = 0$ is one solution, though we will concern ourselves with finding the nontrivial solutions.

3.2   Type $A_n^{(1)}$

In this section, we describe the results of Jayne and Misra [7] in the case of $V(k\Lambda_0)$. We begin with the affine Lie algebra $\mathfrak{g} = A_n^{(1)}$ where $n \geq 2$ with index set $I = \{0, 1, \ldots, n\}$ and Cartan

matrix

$$
A = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & -1 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & -1 & 2 & -1 \\
-1 & \cdots & 0 & 0 & -1 & 2
\end{pmatrix}.
$$

Jayne and Misra explicitly determined the maximal dominant weights for the integrable highest weight modules $V((k-1)\Lambda_0 + \Lambda_s), 0 \leq s \leq n, k \geq 2$ and so for our case, we take $s = 0$.

In this case, the defining inequalities are

$$
\begin{cases}
\mathring{A}\mathbf{x} & \geq 0 \\
x_1 + x_n & \leq k
\end{cases}
$$

or

$$
\begin{cases}
2x_1 - x_2 & \geq 0 \\
-x_1 + 2x_2 - x_3 & \geq 0 \\
\quad \vdots & \vdots \\
-x_{n-2} + 2x_{n-1} - x_n & \geq 0 \\
-x_{n-1} + 2x_n & \geq 0 \\
x_1 + x_n & \leq k
\end{cases}
$$

where $x_i = b_0 - b_i$. Jayne and Misra showed that the $\mathbf{x}$ vectors follow a particular pattern in this case. As the entries in $\mathbf{x}$ move from left to right, they increase to a maximum element, l. The value of l may repeat in the entries, and then the entries decrease from l moving to the end of the vector. We list some examples of $\mathbf{x}$ vectors and corresponding maximal dominant weights in tables 3.1, 3.2, and 3.3.

3.3   Type $B_n^{(1)}$

Now we begin discussing our results for explicit descriptions of maximal dominant weights for other affine Lie algebras. We begin with $\mathfrak{g} = B_n^{(1)}$ for $n \geq 3$, index set $I = \{0, 1, \ldots, n\}$, and

Table 3.1: **x** vectors and maximal dominant weights for $\mathfrak{g} = A_2^{(1)}$

k	**x** vector	Element of $\max(k\Lambda_0) \cap P^+$
2	(0,0)	$2\Lambda_0$
	(1,1)	$2\Lambda_0 - \alpha_0$
3	(0,0)	$3\Lambda_0$
	(1,1)	$3\Lambda_0 - \alpha_0$
	(2,1)	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
	(1,2)	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
4	(0,0)	$4\Lambda_0$
	(1,1)	$4\Lambda_0 - \alpha_0$
	(2,1)	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
	(1,2)	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
	(2,2)	$4\Lambda_0 - 2\alpha_0$
5	(0,0)	$5\Lambda_0$
	(1,1)	$5\Lambda_0 - \alpha_0$
	(2,1)	$5\Lambda_0 - 2\alpha_0 - \alpha_2$
	(1,2)	$5\Lambda_0 - 2\alpha_0 - \alpha_1$
	(2,2)	$5\Lambda_0 - 2\alpha_0$
	(3,2)	$5\Lambda_0 - 3\alpha_0 - \alpha_2$
	(2,3)	$5\Lambda_0 - 3\alpha_0 - \alpha_1$

Table 3.2: **x** vectors and maximal dominant weights for $\mathfrak{g} = A_3^{(1)}$

k	**x** vector	Element of $\max(k\Lambda_0) \cap P^+$	k	**x** vector	Element of $\max(k\Lambda_0) \cap P^+$
2	(0,0,0)	$2\Lambda_0$	5	(0,0,0)	$5\Lambda_0$
	(1,1,1)	$2\Lambda_0 - \alpha_0$		(1,1,1)	$5\Lambda_0 - \alpha_0$
	(1,2,1)	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_3$		(1,2,1)	$5\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_3$
3	(0,0,0)	$3\Lambda_0$		(1,2,2)	$5\Lambda_0 - 2\alpha_0 - \alpha_1$
	(1,1,1)	$3\Lambda_0 - \alpha_0$		(2,2,1)	$5\Lambda_0 - 2\alpha_0 - \alpha_3$
	(1,2,1)	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_3$		(2,2,2)	$5\Lambda_0 - 2\alpha_0$
	(1,2,2)	$3\Lambda_0 - 2\alpha_0 - \alpha_1$		(2,3,2)	$5\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_3$
	(2,2,1)	$3\Lambda_0 - 2\alpha_0 - \alpha_3$		(2,4,2)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 2\alpha_3$
4	(0,0,0)	$4\Lambda_0$		(1,2,3)	$5\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
	(1,1,1)	$4\Lambda_0 - \alpha_0$		(3,2,1)	$5\Lambda_0 - 3\alpha_0 - \alpha_2 - 2\alpha_3$
	(1,2,1)	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_3$		(2,3,3)	$5\Lambda_0 - 3\alpha_0 - \alpha_1$
	(1,2,2)	$4\Lambda_0 - 2\alpha_0 - \alpha_1$		(2,4,3)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_3$
	(2,2,1)	$4\Lambda_0 - 2\alpha_0 - \alpha_3$		(3,3,2)	$5\Lambda_0 - 3\alpha_0 - \alpha_3$
	(2,2,2)	$4\Lambda_0 - 2\alpha_0$		(3,4,2)	$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 2\alpha_3$
	(2,3,2)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_3$			
	(2,4,2)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 2\alpha_3$			
	(1,2,3)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$			
	(3,2,1)	$4\Lambda_0 - 3\alpha_0 - \alpha_2 - 2\alpha_3$			

Table 3.3: **x** vectors and maximal dominant weights for $\mathfrak{g} = A_4^{(1)}$

k	**x** vector	Element of $\max(k\Lambda_0) \cap P^+$	k	**x** vector	Element of $\max(k\Lambda_0) \cap P^+$
2	(0,0,0,0)	$2\Lambda_0$	5	(0,0,0,0)	$5\Lambda_0$
	(1,1,1,1)	$2\Lambda_0 - \alpha_0$		(1,1,1,1)	$5\Lambda_0 - \alpha_0$
	(1,2,2,1)	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_4$		(1,2,2,1)	$5\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_4$
3	(0,0,0,0)	$3\Lambda_0$		(1,2,2,2)	$5\Lambda_0 - 2\alpha_0 - \alpha_1$
	(1,1,1,1)	$3\Lambda_0 - \alpha_0$		(1,2,3,2)	$5\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_4$
	(1,2,2,1)	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_4$		(2,2,2,1)	$5\Lambda_0 - 2\alpha_0 - \alpha_4$
	(1,2,2,2)	$3\Lambda_0 - 2\alpha_0 - \alpha_1$		(2,3,2,1)	$5\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_3 - 2\alpha_4$
	(1,2,3,2)	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_4$		(2,2,2,2)	$5\Lambda_0 - 2\alpha_0$
	(2,2,2,1)	$3\Lambda_0 - 2\alpha_0 - \alpha_4$		(2,3,3,2)	$5\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_4$
	(2,3,2,1)	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_3 - 2\alpha_4$		(2,3,4,2)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_4$
4	(0,0,0,0)	$4\Lambda_0$		(2,4,3,2)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_3 - 2\alpha_4$
	(1,1,1,1)	$4\Lambda_0 - \alpha_0$		(2,4,4,2)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 2\alpha_4$
	(1,2,2,1)	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_4$		(1,2,3,3)	$5\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
	(1,2,2,2)	$4\Lambda_0 - 2\alpha_0 - \alpha_1$		(3,3,2,1)	$5\Lambda_0 - 3\alpha_0 - \alpha_3 - 2\alpha_4$
	(1,2,3,2)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_4$		(1,2,3,4)	$5\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
	(2,2,2,1)	$4\Lambda_0 - 2\alpha_0 - \alpha_4$		(2,3,3,3)	$5\Lambda_0 - 3\alpha_0 - \alpha_1$
	(2,3,2,1)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_3 - 2\alpha_4$		(2,3,4,3)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_4$
	(2,2,2,2)	$4\Lambda_0 - 2\alpha_0$		(2,4,4,3)	$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_4$
	(2,3,3,2)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_4$		(2,4,5,3)	$5\Lambda_0 - 5\alpha_0 - 3\alpha_1 - \alpha_2 - 2\alpha_4$
	(2,3,4,2)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_4$		(2,4,6,3)	$5\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2 - 3\alpha_4$
	(2,4,3,2)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_3 - 2\alpha_4$		(3,3,3,2)	$5\Lambda_0 - 3\alpha_0 - \alpha_4$
	(2,4,4,2)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 2\alpha_4$		(3,4,3,2)	$5\Lambda_0 - 4\alpha_0 - \alpha_1 - \alpha_3 - 2\alpha_4$
	(1,2,3,3)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$		(3,4,4,2)	$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 2\alpha_4$
	(3,3,2,1)	$4\Lambda_0 - 3\alpha_0 - \alpha_3 - 2\alpha_4$		(3,5,4,2)	$5\Lambda_0 - 5\alpha_0 - 2\alpha_1 - \alpha_3 - 3\alpha_4$
				(3,6,4,2)	$5\Lambda_0 - 6\alpha_0 - 3\alpha_1 - 2\alpha_3 - 4\alpha_4$
				(4,3,2,1)	$5\Lambda_0 - 4\alpha_0 - \alpha_2 - 2\alpha_3 - 3\alpha_4$

Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & 0 & -2 & 2 \end{pmatrix}.$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

where $x_1 = b_0 - b_1$ and $x_i = 2b_0 - b_i$ for $2 \leq i \leq n$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - x_3 & \geq 0 \\ -x_2 + 2x_3 - x_4 & \geq 0 \\ \quad \vdots & \vdots \\ -x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0 \\ -x_{n-2} + 2x_{n-1} - x_n & \geq 0 \\ -2x_{n-1} + 2x_n & \geq 0 \\ x_2 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 > b_1$ and $b_0 > \frac{b_i}{2}$ for $2 \leq i \leq n$.

Lemma 3.3.1. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

Then $x_i \leq x_{i+1}$ *for* $1 \leq i \leq n-1$. *In addition,* $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. Assume, for the sake of contradiction, that there exists a $2 \leq j \leq n$ such that $x_j < x_{j-1}$. If $j \neq n$, then since $-x_{j-1} + 2x_j - x_{j+1} \geq 0$, we have $x_j - x_{j+1} \geq x_{j-1} - x_j > 0$ by our assumption and so $x_j > x_{j+1}$. Therefore, if the coordinates were to decrease at any point, they would continue to do so up to n, forcing $x_{n-1} > x_n$. However, by $(\mathring{A}\mathbf{x})_n \geq 0$, $2x_n \geq 2x_{n-1}$, and so $x_n \geq x_{n-1}$. Thus, the coordinates never decrease. Additionally, observe that the amount of increase from x_i to x_{i+1} is never more than that from x_{i-1} to x_i for $2 \leq i \leq n-1$, since

$-x_{i-1} + 2x_i - x_{i+1} \geq 0$, implying that $x_i - x_{i-1} \geq x_{i+1} - x_i$. $\qquad \square$

Lemma 3.3.2. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 1 \leq x_2 \leq k,$$
$$x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1, 0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor,$$
$$x_i = x_2 + \sum_{j=3}^{i} l_j,$$
$$0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1, 0 \leq l_n \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3 \text{ for all } 3 \leq i \leq n\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_2 = 0$ gives the solution $\mathbf{x} = 0$. Now, fix x_2 such that $1 \leq x_2 \leq k$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_1 \geq \frac{x_2}{2}$. Since each x_i must be an integer, this implies $x_1 \geq \left\lceil \frac{x_2}{2} \right\rceil$. We have observed that the first n coordinates of $\mathbf{x}$ do not decrease, therefore $x_1 \leq x_2$. Then $\left\lceil \frac{x_2}{2} \right\rceil \leq x_1 \leq x_2$ and so we can write $x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1$ where $0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor$. We will prove the pattern for the remaining coordinates by induction. First, as a base case, we show the expression for x_3. Again, since the coordinates do not decrease, $x_2 \leq x_3$ and from the previous lemma, $x_3 - x_2 \leq x_2 - x_1$. Then $x_3 - x_2 \leq x_2 - (\left\lceil \frac{x_2}{2} \right\rceil + l_1) = \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$. Therefore, $x_2 \leq x_3 \leq x_2 + \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$ and so $x_3 = x_2 + l_3$ where $0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$. Now, assume that x_i is of the form $x_i = x_2 + \sum_{j=3}^{i} l_j$ for all $3 \leq i \leq p < n$ where $l_j = x_j - x_{j-1}$ and so $0 \leq l_p \leq l_{p-1} \leq \cdots \leq l_4 \leq l_3$ by the second result of Lemma 3.3.1. By the same lemma, $x_p \leq x_{p+1} \leq 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_2 + \sum_{j=3}^{p} l_j \leq x_{p+1} \leq 2(x_2 + \sum_{j=3}^{p} l_j) - (x_2 + \sum_{j=3}^{p-1} l_j) = x_2 + (\sum_{j=3}^{p} l_j) + l_p$. Then $x_{p+1} = x_2 + (\sum_{j=3}^{p} l_j) + l_{p+1}$ where $0 \leq l_{p+1} \leq l_p$. Therefore, $x_i = x_2 + \sum_{j=3}^{i} l_j$ where $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$ and so l_i is the amount of increase from x_{i-1} to x_i where $3 \leq i \leq n$. $\qquad \square$

Theorem 3.3.3. *Let $n \geq 3$, $\Lambda = k\Lambda_0, k \geq 2$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n}(2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \right)$ where*

- $1 \leq x_2 \leq k$,

- $l = \max\{x_1, \left\lceil \frac{x_n}{2} \right\rceil\}$,

- $0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor$,

- $0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$,

- $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$.

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Proof. By Proposition 12.6 in [12], the map $\lambda \mapsto \overline{\lambda}$ is a bijection from $\max(\Lambda) \cap P^+$ onto $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$. We first find all elements of $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$ and use this bijection to then find all elements of $\max(\Lambda) \cap P^+$. By definition,

$$kC_{af} \cap (\overline{\Lambda} + \overline{Q}) = \{\overline{\lambda} \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* | \overline{\lambda}(h_i) \geq 0 \text{ for all } i \in \mathring{I}, (\overline{\lambda}|\theta) \leq k\}.$$

We found the set of solutions to these inequalities in Lemma 3.3.2. Recall that by the bijection, $\lambda = \Lambda - \sum_{i=0}^n b_i \alpha_i \in \max\{\Lambda\} \cap P^+$, with $b_i \in \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq n$, maps to $\overline{\lambda} = \sum_{i=1}^n x_i \alpha_i \in kC_{af} \cap (\overline{\Lambda} + \overline{Q})$. The map gives that $x_1 = b_0 - b_1$ and $x_i = 2b_0 - b_i$ for $2 \leq i \leq n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, b_0 - x_1, 2b_0 - x_2, \ldots, 2b_0 - x_n)$. Then if we can determine b_0, we will have the set of all maximal dominant weights since we have already found the pattern for all x_i. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$\vdots$$
$$2b_0 - x_n \geq 0$$

It is clear that $b_0 \geq x_1$ where $x_1 = (\lceil \frac{x_2}{2} \rceil + l_1)$. We have already proven that the first n coordinates of $\mathbf{x}$ never decrease. Therefore, $x_n = \max_{1 \leq i \leq n}\{x_i\}$ and so for the inner inequalities it is sufficient to say that $b_0 \geq \frac{x_n}{2}$. Since x_n may be odd and the b_i are integers, we need $b_0 \geq \lceil \frac{x_n}{2} \rceil$. Then let $l = \max\{x_1, \lceil \frac{x_n}{2} \rceil\}$ and so $b_0 \geq l$. We claim that $b_0 = l$. For the sake of contradiction, suppose $b_0 = l + r$ where $r \in \mathbb{Z}_{>0}$. Then

$$\lambda + \delta = \Lambda - (l+r-1)\alpha_0 - (l+r-x_1-1)\alpha_1 - (2l+2r-x_2-2)\alpha_2 - \cdots - (2l+2r-x_n-2)\alpha_n.$$

Then if each of the subtracted coefficients of the $\alpha_i, 0 \leq i \leq n$ is greater than or equal to 0, we will have $\lambda + \delta \leq \Lambda$ and $\lambda + \delta \in P^+$. First, $l + r - 1 \geq 0$ since $l \geq x_1 \geq 1$ and we assume $r > 0$ then $l + r - 1 > 0$. Second, since $l \geq x_1$ and $r \geq 1$ we have $l + r - x_1 - 1 \geq 0$. Finally, since $l \geq \lceil \frac{x_n}{2} \rceil$, we have $2l \geq 2\lceil \frac{x_n}{2} \rceil$ which means $2l \geq x_n$ if x_n is even or $2l \geq x_n + 1$ if x_n is odd. Either way, $2l \geq x_n \geq x_i$ for all $2 \leq i \leq n$. Since $r \geq 1$ then $2r \geq 2$ and we have $2l + 2r - x_i - 2 \geq 0$ for all $2 \leq i \leq n$. Then $\lambda + \delta$ is a weight of $V(\Lambda)$ which contradicts that $\lambda \in \max\{\Lambda\}$. Therefore, $r = 0$ and $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - \cdots - (2l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.3.2, we have the pattern given above. $\square$

To better understand this theorem, consider some examples for various values of both k and n. We include two examples of $k = 2$ so that the reader may check that our results match those found in [18]. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 1 \leq x_2 \leq k\}$.

Table 3.4: $\mathbf{x}$ vectors and maximal dominant weights for the $\mathfrak{g} = B_5^{(1)}$-module $V(2\Lambda_0)$

x_2	l_1, l_3, l_4, l_5	$\mathbf{x}$ vector	l	Element of $\max(3\Lambda_0) \cap P^+$
0	0,0,0,0	$(0,0,0,0,0)$	0	$2\Lambda_0$
1	0,0,0,0	$(1,1,1,1,1)$	1	$2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,0,0,0	$(1,2,2,2,2)$	1	$2\Lambda_0 - \alpha_0$
2	0,1,0,0	$(1,2,3,3,3)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,1,1,0	$(1,2,3,4,4)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1	$(1,2,3,4,5)$	3	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0	$(2,2,2,2,2)$	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$

Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$.

Example 3.3.4. $B_5^{(1)}, V(\Lambda) = V(2\Lambda_0)$. Using the theorem, we have that

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5\}.$$

Since $1 \leq x_2 \leq k$ and $k = 2$, then we have $x_2 = 1$ or $x_2 = 2$. First, consider $x_2 = 1$. Then $0 \leq l_1 \leq \lfloor \tfrac{x_2}{2} \rfloor = 0$. Then $l_1 = 0$ and so $x_1 = \lceil \tfrac{x_2}{2} \rceil + 0 = 1$ and $0 \leq l_3 \leq 0$. This implies that $l_4 = l_5 = 0$. Therefore, we obtain the $\mathbf{x}$ vector $(1,1,1,1,1)$. Then $l = 1$, giving the maximal dominant weight $2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$.

The only choice we made (and so the only value we can change) was that for x_2. Then consider $x_2 = 2$. This implies that $0 \leq l_1 \leq 1$. Once more, we have a choice. First, let $l_1 = 0$. Then $x_1 = 1$ and $0 \leq l_3 \leq 1$. First, choose $l_3 = 0$, which implies that $l_4 = l_5 = 0$. This gives the $\mathbf{x}$ vector $(1,2,2,2,2)$. By the theorem, this corresponds to the maximal dominant weight $2\Lambda_0 - \alpha_0$ since $l = 1 = x_1 = \lceil \tfrac{x_5}{2} \rceil$. Next, we consider $l_3 = 1$. Then $0 \leq l_5 \leq l_4 \leq 1$. We choose $l_4 = 0$, which implies $l_5 = 0$. This gives the $\mathbf{x}$ vector $(1,2,3,3,3)$, and so $l = \lceil \tfrac{3}{2} \rceil = 2$. The resulting maximal dominant weight is $2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$. If $l_4 = 1$, then $l_5 = 0$ or $l_5 = 1$. $l_5 = 0$ gives the $\mathbf{x}$ vector $(1,2,3,4,4)$ and $l_5 = 1$ gives the $\mathbf{x}$ vector $(1,2,3,4,5)$. The corresponding maximal dominant weights are $2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$ and $2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$ respectively. We have exhausted all possible choices for $l_1 = 0$. Now, consider $l_1 = 1$. Then $0 \leq l_3 \leq \lfloor \tfrac{2}{2} \rfloor - 1 = 0$, which implies that $l_3 = l_4 = l_5 = 0$. This gives the $\mathbf{x}$ vector $(2,2,2,2,2)$ with corresponding maximal dominant weight $2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$. We list these results in Table 3.4. One can check that this set corresponds to that of [18],

$$\{2\Lambda_0, \Lambda_0 + \Lambda_1 - \delta, \Lambda_2 - \delta, \Lambda_3 - 2\delta, \Lambda_4 - 2\delta, 2\Lambda_5 - 3\delta, 2\Lambda_1 - 2\delta\}.$$

Table 3.5: **x** vectors and maximal dominant weights for the $\mathfrak{g} = B_6^{(1)}$-module $V(2\Lambda_0)$

x_2	l_1, l_3, l_4, l_5, l_6	**x** vector	l	Element of $\max(3\Lambda_0) \cap P^+$
0	0,0,0,0,0	$(0,0,0,0,0,0)$	0	$2\Lambda_0$
1	0,0,0,0,0	$(1,1,1,1,1,1)$	1	$2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,0,0,0,0	$(1,2,2,2,2,2)$	1	$2\Lambda_0 - \alpha_0$
2	0,1,0,0,0	$(1,2,3,3,3,3)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,0,0	$(1,2,3,4,4,4)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1,0	$(1,2,3,4,5,5)$	3	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,1,1	$(1,2,3,4,5,6)$	3	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0,0	$(2,2,2,2,2,2)$	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6$

Example 3.3.5. $B_6^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By the theorem, we have

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$

$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5$$

$$- (2l - (x_2 + l_3 + l_4 + l_5 + l_6))\alpha_6\}.$$

$\mathbf{x_2 = 1}$: Then $0 \le l_1 \le 0$ and so $0 \le l_3 \le 0 - 0$, implying $l_1 = l_3 = \cdots = l_6 = 0$ giving the **x** vector $(1,1,1,1,1,1)$ with corresponding maximal dominant weight $2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$.

$\mathbf{x_2 = 2}$: Then $0 \le l_1 \le 1$.

Consider $l_1 = 0$. Then $x_1 = 1$ and $0 \le l_3 \le 1$.

First, consider $l_3 = 0$. Then $x_3 = x_2 = 2$ and $0 \le l_6 \le l_5 \le l_4 \le 0$. This gives the **x** vector $(1,2,2,2,2,2)$ with corresponding maximal dominant weight $2\Lambda_0 - \alpha_0$.

Now consider $l_3 = 1$. Then $x_3 = 3$ and $0 \le l_6 \le l_5 \le l_4 \le 1$.

If $l_4 = 0$, we obtain the **x** vector $(1,2,3,3,3,3)$ with corresponding maximal dominant weight $2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$.

If $l_4 = 1$ and $l_5 = 0$, we obtain the **x** vector $(1,2,3,4,4,4)$ with corresponding maximal dominant weight $2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$.

If $l_4 = l_5 = 1$ and $l_6 = 0$, we have $(1,2,3,4,5,5)$ with maximal dominant weight $2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$.

If $l_4 = l_5 = l_6 = 1$, we have $(1,2,3,4,5,6)$ with maximal dominant weight $2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$.

Finally, consider $l_1 = 1$. Then $0 \le l_3 \le 0$ implies that $0 = l_3 = l_4 = l_5 = l_6$. This gives the **x** vector and maximal dominant weight $(2,2,2,2,2,2)$ and $2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6$. We list these results in Table 3.5.

We show that our formulation for the maximal dominant weights of $V(k\Lambda_0)$ correspond to that of [18] for the case $k = 2$ for arbitrary $n \ge 3$. We can simplify our formulation, splitting it

into sets according to level, in the case of $k = 2$ to the following:

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - \alpha_0 - \sum_{i=2}^{n} \alpha_i\}$$

$$\cup \{2\Lambda_0 - l\alpha_0 - (l - (1 + l_1))\alpha_1 - (2l - 2)\alpha_2 - (\sum_{i=3}^{n}(2l - (2 + \sum_{j=3}^{i} l_j))\alpha_i)\}$$

According to [18], the set of maximal dominant weights for $V(2\Lambda_0)$ is

$$\{(1 + \delta_{2u-1,n})\Lambda_{2u-1} - u\delta \mid 2 \le u \le \left\lfloor \frac{n+1}{2} \right\rfloor\} \cup \{(1 + \delta_{2u,n})\Lambda_{2u} - u\delta \mid 1 \le u \le \left\lfloor \frac{n}{2} \right\rfloor\}$$

$$\cup \{\Lambda_0 + \Lambda_1 - \delta\} \cup \{2\Lambda_0, 2\Lambda_1 - 2\delta\}.$$

To prove that the two formulations are the same, we first write our formulation in terms of the fundamental dominant weights, Λ_i. Note that we must use the entries of the Cartan matrix corresponding to $A_{2n-1}^{(2)}$ as the coefficients of the Λ_i, in order to satisfy that $\sum_{i=0}^{n} a_i \alpha_i = \delta$ and $\alpha_j(h_i) = a_{ij}$. We have the following:

$$\alpha_0 = 2\Lambda_0 - \Lambda_2 + \delta$$
$$\alpha_1 = 2\Lambda_1 - \Lambda_2$$
$$\alpha_2 = -\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3$$
$$\alpha_3 = -\Lambda_2 + 2\Lambda_3 - \Lambda_4$$
$$\vdots$$
$$\alpha_{n-1} = -\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n$$
$$\alpha_n = -\Lambda_{n-1} + 2\Lambda_n$$

Then our formulation becomes:

$$X = \{2\Lambda_0\} \cup \{2\Lambda_0 - (2\Lambda_0 - \Lambda_2 + \delta) - (-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$
$$- \left(\sum_{i=3}^{n-2}(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})\right) - (-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n) - (-\Lambda_{n-1} + 2\Lambda_n)\}$$
$$\cup \{2\Lambda_0 - l(2\Lambda_0 - \Lambda_2 + \delta) - (l - (1 + l_1))(2\Lambda_1 - \Lambda_2) - (2l - 2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$
$$- \left(\sum_{i=3}^{n-2}(2l - (2 + \sum_{j=3}^{i} l_j))(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})\right)$$
$$- (2l - (2 + \sum_{j=3}^{n-1} l_j))(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n)$$
$$- (2l - (2 + \sum_{j=3}^{n} l_j))(-\Lambda_{n-1} + 2\Lambda_n)\}$$

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We now show that

$$X = \{(1 + \delta_{2u-1,n})\Lambda_{2u-1} - u\delta \mid 2 \leq u \leq \left\lfloor \frac{n+1}{2} \right\rfloor\} \cup \{(1 + \delta_{2u,n})\Lambda_{2u} - u\delta \mid 1 \leq u \leq \left\lfloor \frac{n}{2} \right\rfloor\}$$

$$\cup \{\Lambda_0 + \Lambda_1 - \delta\} \cup \{2\Lambda_0, 2\Lambda_1 - 2\delta\}$$

$2\Lambda_0$ is included in both sets so we move on to the remaining weights. Consider the level 1 weight $2\Lambda_0 - (2\Lambda_0 - \Lambda_2 + \delta) - (-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3) - \sum_{i=3}^{n-2}(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}) - (-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n) - (-\Lambda_{n-1} + 2\Lambda_n)$. We immediately see that we have $\Lambda_0 + \Lambda_1 - \delta$ coming from the first three summands. One can check that all other terms cancel, leaving us with $\Lambda_0 + \Lambda_1 - \delta$, matching another of the weights in [18].

We now consider the general term in our expression for W. Since $x_2 = 2$, l_1 can be 0 or 1. If $l_1 = 1$, this implies that $l_n = l_{n-1} = \cdots = l_3 = 0$ which gives the $\mathbf{x}$ vector $(2, 2, \ldots, 2)$, meaning $l = 2$. Simplifying the general expression, we obtain $2\Lambda_0 - 2(2\Lambda_0 - \Lambda_2 + \delta) - 2(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3) - \sum_{i=3}^{n-2} 2(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}) - 2(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n) - 2(-\Lambda_{n-1} + 2\Lambda_n)$. Simplifying this further, we obtain $2\Lambda_1 - 2\delta$, completing the fourth set in the description in [18].

Now consider if $l_1 = l_3 = 0$. This implies $l_n = l_{n-1} = \cdots = l_4 = l_3 = 0$, giving $\mathbf{x}$ vector $(1, 2, \ldots, 2)$. In this case, $l = 1$ and we consider our general term once more: $2\Lambda_0 - (2\Lambda_0 - \Lambda_2 + \delta) = \Lambda_2 - \delta$. This corresponds to the weight in second set for [18] where $u = 1$.

Finally, we consider the remaining weights. Let $l_1 = 0$ and k be the smallest integer between 4 and n such that $l_k = 0$. This implies $l_3 = l_4 = \cdots = l_{k-1} = 1$ and $l_k = l_{k+1} = \cdots = l_n = 0$. Then since $l = \max\{x_1, \left\lceil \frac{x_n}{2} \right\rceil\}$ and the $\mathbf{x}$ vector is nondecreasing by Lemma 3.3.1, $l = \left\lceil \frac{k-1}{2} \right\rceil$. Then our general expression becomes

$$2\Lambda_0 - \left\lceil \frac{k-1}{2} \right\rceil (2\Lambda_0 - \Lambda_2 + \delta) - \left(\left\lceil \frac{k-1}{2} \right\rceil - 1\right)(2\Lambda_1 - \Lambda_2)$$

$$- \left(2\left\lceil \frac{k-1}{2} \right\rceil - 2\right)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$

$$- \sum_{i=3}^{k-1} \left(2\left\lceil \frac{k-1}{2} \right\rceil - (2 + \sum_{j=3}^{i} 1)\right)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- \sum_{i=k}^{n-2} \left(2\left\lceil \frac{k-1}{2} \right\rceil - (2 + \sum_{j=3}^{k-1} 1)\right)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- \left(2\left\lceil \frac{k-1}{2} \right\rceil - (2 + \sum_{j=3}^{k-1} 1)\right)(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n)$$

$$- \left(2\left\lceil \frac{k-1}{2} \right\rceil - (2 + \sum_{j=3}^{k-1} 1)\right)(-\Lambda_{n-1} + 2\Lambda_n)$$

If $k - 1$ is even, we have the following:

$$2\Lambda_0 - \frac{k-1}{2}(2\Lambda_0 - \Lambda_2 + \delta) - \left(\frac{k-1}{2} - 1\right)(2\Lambda_1 - \Lambda_2) - (k - 1 - 2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$

$$-\sum_{i=3}^{k-1}(k-1-(2+\sum_{j=3}^{i}1))(-\Lambda_{i-1}+2\Lambda_i-\Lambda_{i+1})$$

$$-\sum_{i=k}^{n-2}(k-1-(2+\sum_{j=3}^{k-1}1))(-\Lambda_{i-1}+2\Lambda_i-\Lambda_{i+1})$$

$$-(k-1-(2+\sum_{j=3}^{k-1}1))(-\Lambda_{n-2}+2\Lambda_{n-1}-2\Lambda_n)-(k-1-(2+\sum_{j=3}^{k-1}1))(-\Lambda_{n-1}+2\Lambda_n)$$

Collecting like terms, we have

$$2\Lambda_0-(k-1)\Lambda_0+(k-3)\Lambda_0-(k-3)\Lambda_1+(k-3)\Lambda_1$$
$$+\frac{k-1}{2}\Lambda_2+\frac{k-3}{2}\Lambda_2-2(k-3)\Lambda_2+(k-4)\Lambda_2$$
$$+\sum_{i=3}^{k-2}\left((k-i)\Lambda_i-2(k-(i+1))\Lambda_i+(k-(i+2))\Lambda_i\right)+\Lambda_{k-1}-\frac{k-1}{2}\delta$$
$$=\Lambda_{k-1}-\frac{k-1}{2}\delta$$

This corresponds to the second set we have listed except for when $u=\left\lfloor\frac{n}{2}\right\rfloor$ and n is even. Notice we have also left out the case in which n is even and $l_k=1$ for all $3\le k\le n$. These two cases correspond as we show below. In this situation, $l=\max\{x_1,\lceil\frac{x_n}{2}\rceil\}=\max\{1,\frac{n}{2}\}=\frac{n}{2}$.

$$2\Lambda_0-\frac{n}{2}(2\Lambda_0-\Lambda_2+\delta)-\left(\frac{n}{2}-1\right)(2\Lambda_1-\Lambda_2)-(n-2)(-\Lambda_0-\Lambda_1+2\Lambda_2-\Lambda_3)$$
$$-\sum_{i=3}^{n-2}(n-(2+\sum_{j=3}^{i}1))(-\Lambda_{i-1}+2\Lambda_i-\Lambda_{i+1})-(n-(2+\sum_{j=3}^{n-1}1))(-\Lambda_{n-2}+2\Lambda_{n-1}-2\Lambda_n)$$
$$-(n-(2+\sum_{j=3}^{n}1))(-\Lambda_{n-1}+2\Lambda_n)$$

Collecting like terms, this becomes

$$2\Lambda_0-n\Lambda_0+(n-2)\Lambda_0-(n-2)\Lambda_1+(n-2)\Lambda_1$$
$$+\frac{n}{2}\Lambda_2+(\frac{n}{2}-1)\Lambda_2-2(n-2)\Lambda_2+(n-3)\Lambda_2$$
$$+\sum_{i=3}^{n-1}\left((n-(i-1))\Lambda_i-2(n-i)\Lambda_i+(n-(i+1))\Lambda_i\right)+2\Lambda_n-\frac{n}{2}\delta$$
$$=2\Lambda_n-\frac{n}{2}\delta$$

Similarly, we obtain the following if $k-1$ is odd and k is the smallest integer between 4 and n

such that $l_k = 0$:

$$2\Lambda_0 - \frac{k}{2}(2\Lambda_0 - \Lambda_2 + \delta) - (\frac{k}{2} - 1)(2\Lambda_1 - \Lambda_2) - (k-2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$

$$- \sum_{i=3}^{k-1}(k - (2 + \sum_{j=3}^{i} 1))(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- \sum_{i=k}^{n-2}(k - (2 + \sum_{j=3}^{k-1} 1))(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- (k - (2 + \sum_{j=3}^{k-1} 1))(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n) - (k - (2 + \sum_{j=3}^{k-1} 1))(-\Lambda_{n-1} + 2\Lambda_n)$$

$$= 2\Lambda_0 - k\Lambda_0 + (k-2)\Lambda_0 - (k-2)\Lambda_1 + (k-2)\Lambda_1$$

$$+ \frac{k}{2}\Lambda_2 + (\frac{k}{2} - 1)\Lambda_2 - 2(k-2)\Lambda_2 + (k-3)\Lambda_2$$

$$+ \sum_{i=3}^{k-2}\left((k - (i-1))\Lambda_i - 2(k-i)\Lambda_i + (k - (i+1))\Lambda_i\right) + 2\Lambda_{k-1} - 2\Lambda_{k-1} + \Lambda_{k-1} - \frac{k}{2}\delta$$

$$= \Lambda_{k-1} - \frac{k}{2}\delta$$

This corresponds to the first set we have listed, except for the final term when $u = \lfloor \frac{n+1}{2} \rfloor$ and n is odd. This is the case when n is odd and $l_k = 1$ for all $3 \le k \le n$. Then $l = \max\{x_1, \lceil \frac{x_n}{2} \rceil\} = \frac{n+1}{2}$. This gives

$$2\Lambda_0 - \frac{n+1}{2}(2\Lambda_0 - \Lambda_2 + \delta) - (\frac{n+1}{2} - 1)(2\Lambda_1 - \Lambda_2) - (n+1-2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$

$$- \sum_{i=3}^{n-2}(n+1 - (2 + \sum_{j=3}^{i} 1))(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- (n+1 - (2 + \sum_{j=3}^{n-1} 1))(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n)$$

$$- (n+1 - (2 + \sum_{j=3}^{n} 1))(-\Lambda_{n-1} + 2\Lambda_n)$$

$$= 2\Lambda_0 - (n+1)\Lambda_0 + (n-1)\Lambda_0 - (n-1)\Lambda_1 + (n-1)\Lambda_1$$

$$+ \frac{n+1}{2}\Lambda_2 + (\frac{n+1}{2} - 1)\Lambda_2 - 2(n-1)\Lambda_2 + (n-2)\Lambda_2$$

$$+ \sum_{i=3}^{n-1}\left((n+1 - (i-1))\Lambda_i - 2(n+1-i)\Lambda_i + (n+1 - (i+1))\Lambda_i\right)$$

$$+ 4\Lambda_n - 2\Lambda_n - \frac{n+1}{2}\delta$$

$$= 2\Lambda_n - \frac{n+1}{2}\delta$$

Table 3.6: **x** vectors and maximal dominant weights for the $\mathfrak{g} = B_5^{(1)}$-module $V(3\Lambda_0)$

x_2	l_1, l_3, l_4, l_5	**x** vector	l	Element of $\max(3\Lambda_0) \cap P^+$
0	0,0,0,0	$(0,0,0,0,0)$	0	$3\Lambda_0$
1	0,0,0,0	$(1,1,1,1,1)$	1	$3\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,0,0,0	$(1,2,2,2,2)$	1	$3\Lambda_0 - \alpha_0$
2	0,1,0,0	$(1,2,3,3,3)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,1,1,0	$(1,2,3,4,4)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1	$(1,2,3,4,5)$	3	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0	$(2,2,2,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$
3	0,0,0,0	$(2,3,3,3,3)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
3	0,1,0,0	$(2,3,4,4,4)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
3	0,1,1,0	$(2,3,4,5,5)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
3	0,1,1,1	$(2,3,4,5,6)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
3	1,0,0,0	$(3,3,3,3,3)$	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4 - 3\alpha_5$

This finishes the right hand side of the equation. Therefore, the two sets are equal and our formulation matches that of [18] for $k = 2$.

We include several more examples for different values of k below.

Example 3.3.6. $B_5^{(1)}, V(\Lambda) = V(3\Lambda_0)$. According to Theorem 3.3.3,

$$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5$$

where $1 \leq x_2 \leq 3$ and the restrictions on l and the l_i are as listed in the theorem statement. We obtain the results in Table 3.6.

Example 3.3.7. $B_6^{(1)}, V(\Lambda) = V(3\Lambda_0)$. According to Theorem 3.3.3,

$$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5$$
$$- (2l - (x_2 + l_3 + l_4 + l_5 + l_6))\alpha_6$$

where $1 \leq x_2 \leq 3$ and the restrictions on l and the l_i are as listed in the theorem statement. We obtain the results in Table 3.7.

Example 3.3.8. $B_5^{(1)}, V(\Lambda) = V(4\Lambda_0)$. According to Theorem 3.3.3,

$$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5$$

where $1 \leq x_2 \leq 4$ and the restrictions on l and the l_i are as listed in the theorem statement.

Table 3.7: **x** vectors and maximal dominant weights for the $\mathfrak{g} = B_6^{(1)}$-module $V(3\Lambda_0)$

x_2	l_1, l_3, l_4, l_5, l_6	**x** vector	l	Element of $\max(3\Lambda_0) \cap P^+$
0	0,0,0,0,0	$(0,0,0,0,0,0)$	0	$3\Lambda_0$
1	0,0,0,0,0	$(1,1,1,1,1,1)$	1	$3\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,0,0,0,0	$(1,2,2,2,2,2)$	1	$3\Lambda_0 - \alpha_0$
2	0,1,0,0,0	$(1,2,3,3,3,3)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,0,0	$(1,2,3,4,4,4)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1,0	$(1,2,3,4,5,5)$	3	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,1,1	$(1,2,3,4,5,6)$	3	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0,0	$(2,2,2,2,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6$
3	0,0,0,0,0	$(2,3,3,3,3,3)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
3	0,1,0,0,0	$(2,3,4,4,4,4)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
3	0,1,1,0,0	$(2,3,4,5,5,5)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
3	0,1,1,1,0	$(2,3,4,5,6,6)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
3	0,1,1,1,1	$(2,3,4,5,6,7)$	4	$3\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 5\alpha_2 - 4\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
3	1,0,0,0,0	$(3,3,3,3,3,3)$	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4 - 3\alpha_5 - 3\alpha_6$

We obtain the results in Table 3.8.

Example 3.3.9. $B_6^{(1)}, V(\Lambda) = V(4\Lambda_0)$. According to Theorem 3.3.3,

$$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\left\lceil \frac{x_2}{2} \right\rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5$$
$$- (2l - (x_2 + l_3 + l_4 + l_5 + l_6))\alpha_6$$

where $1 \leq x_2 \leq 4$ and the restrictions on l and the l_i are as listed in the theorem statement. We obtain the results in Table 3.9.

3.4 Type $C_n^{(1)}$

We now consider $\mathfrak{g} = C_n^{(1)}$ for $n \geq 2$, index set $I = \{0, 1, \ldots, n\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -2 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 & 0 \\ 0 & \ldots & 0 & 0 & -1 & 2 & -2 \\ 0 & \ldots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Table 3.8: **x** vectors and maximal dominant weights for the $\mathfrak{g} = B_5^{(1)}$-module $V(4\Lambda_0)$

x_2	l_1, l_3, l_4, l_5	**x** vector	l	Element of $\max(4\Lambda_0) \cap P^+$
0	0,0,0,0	$(0,0,0,0,0)$	0	$4\Lambda_0$
1	0,0,0,0	$(1,1,1,1,1)$	1	$4\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,0,0,0	$(1,2,2,2,2)$	1	$4\Lambda_0 - \alpha_0$
2	0,1,0,0	$(1,2,3,3,3)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	0,1,1,0	$(1,2,3,4,4)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1	$(1,2,3,4,5)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0	$(2,2,2,2,2)$	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$
3	0,0,0,0	$(2,3,3,3,3)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
3	0,1,0,0	$(2,3,4,4,4)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
3	0,1,1,0	$(2,3,4,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
3	0,1,1,1	$(2,3,4,5,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
3	1,0,0,0	$(3,3,3,3,3)$	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4 - 3\alpha_5$
4	0,0,0,0	$(2,4,4,4,4)$	2	$4\Lambda_0 - 2\alpha_0$
4	0,1,0,0	$(2,4,5,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
4	0,1,1,0	$(2,4,5,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
4	0,1,1,1	$(2,4,5,6,7)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	0,2,0,0	$(2,4,6,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
4	0,2,1,0	$(2,4,6,7,7)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
4	0,2,1,1	$(2,4,6,7,8)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
4	0,2,2,0	$(2,4,6,8,8)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
4	0,2,2,1	$(2,4,6,8,9)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
4	0,2,2,2	$(2,4,6,8,10)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
4	1,0,0,0	$(3,4,4,4,4)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$
4	1,1,0,0	$(3,4,5,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
4	1,1,1,0	$(3,4,5,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
4	1,1,1,1	$(3,4,5,6,7)$	4	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	2,0,0,0	$(4,4,4,4,4)$	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 4\alpha_5$

Table 3.9: **x** vectors and maximal dominant weights for the $\mathfrak{g} = B_6^{(1)}$-module $V(4\Lambda_0)$

x_2	l_1, l_3, l_4, l_5	**x** vector	l	Element of $\max(4\Lambda_0) \cap P^+$
0	0,0,0,0,0	$(0,0,0,0,0,0)$	0	$4\Lambda_0$
1	0,0,0,0,0	$(1,1,1,1,1,1)$	1	$4\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,0,0,0,0	$(1,2,2,2,2,2)$	1	$4\Lambda_0 - \alpha_0$
2	0,1,0,0,0	$(1,2,3,3,3,3)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,0,0	$(1,2,3,4,4,4)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	0,1,1,1,0	$(1,2,3,4,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
2	0,1,1,1,1	$(1,2,3,4,5,6)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	1,0,0,0,0	$(2,2,2,2,2,2)$	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6$
3	0,0,0,0,0	$(2,3,3,3,3,3)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
3	0,1,0,0,0	$(2,3,4,4,4,4)$	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
3	0,1,1,0,0	$(2,3,4,5,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
3	0,1,1,1,0	$(2,3,4,5,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
3	0,1,1,1,1	$(2,3,4,5,6,7)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 5\alpha_2 - 4\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
3	1,0,0,0,0	$(3,3,3,3,3,3)$	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4 - 3\alpha_5 - 3\alpha_6$
4	0,0,0,0,0	$(2,4,4,4,4,4)$	2	$4\Lambda_0 - 2\alpha_0$
4	0,1,0,0,0	$(2,4,5,5,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
4	0,1,1,0,0	$(2,4,5,6,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
4	0,1,1,1,0	$(2,4,5,6,7,7)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
4	0,1,1,1,1	$(2,4,5,6,7,8)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	0,2,0,0,0	$(2,4,6,6,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
4	0,2,1,0,0	$(2,4,6,7,7,7)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
4	0,2,1,1,0	$(2,4,6,7,8,8)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
4	0,2,1,1,1	$(2,4,6,7,8,9)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
4	0,2,2,0,0	$(2,4,6,8,8,8)$	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
4	0,2,2,1,0	$(2,4,6,8,9,9)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
4	0,2,2,1,1	$(2,4,6,8,9,10)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
4	0,2,2,2,0	$(2,4,6,8,10,10)$	5	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
4	0,2,2,2,1	$(2,4,6,8,10,11)$	6	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5 - \alpha_6$
4	0,2,2,2,2	$(2,4,6,8,10,12)$	6	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$
4	1,0,0,0,0	$(3,4,4,4,4,4)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6$
4	1,1,0,0,0	$(3,4,5,5,5,5)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$
4	1,1,1,0,0	$(3,4,5,6,6,6)$	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
4	1,1,1,1,0	$(3,4,5,6,7,7)$	4	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
4	1,1,1,1,1	$(3,4,5,6,7,8)$	4	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	2,0,0,0,0	$(4,4,4,4,4,4)$	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 4\alpha_5 - 4\alpha_6$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

where $x_i = 2b_0 - b_i$ for $1 \leq i \leq n-1$, and $x_n = b_0 - b_n$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - x_3 & \geq 0 \\ -x_2 + 2x_3 - x_4 & \geq 0 \\ \quad \vdots & \vdots \\ -x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0 \\ -x_{n-2} + 2x_{n-1} - 2x_n & \geq 0 \\ -x_{n-1} + 2x_n & \geq 0 \\ x_1 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \frac{b_i}{2}$ for $1 \leq i \leq n-1$ and $b_0 \geq b_n$.

Lemma 3.4.1. *Let $n \geq 3$ and $\mathbf{x}$ be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

Then $x_i \leq x_{i+1}$ for $1 \leq i \leq n-2$. In addition, $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. This proof is essentially the same as that of Lemma 3.3.1 with the coordinates non-decreasing until $n-2$ rather than $n-1$. In this case we arrive at the contradiction using both $(\mathring{A}\mathbf{x})_{n-1} \geq 0$ and $(\mathring{A}\mathbf{x})_n \geq 0$. □

Lemma 3.4.2. *Let $n \geq 3$ and $\mathbf{x}$ be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

Then $x_i = x_{i-1}$ implies $x_i = 2m$ for some integer m and for $2 \leq i \leq n-1$.

Proof. Assume, for the sake of contradiction, that $x_i = x_{i+1} = 2m+1$ for some $m \in \mathbb{Z}_{\geq 0}$ and for some $1 \leq i \leq n-2$. Then by Lemma 3.4.1, $x_j = 2m+1$ for all $i \leq j \leq n-1$. Therefore, the last two inequalities in $\mathring{A}\mathbf{x} \geq 0$ give $-(2m+1) + 2(2m+1) \geq 2x_n$ and $x_n \geq \lceil \frac{2m+1}{2} \rceil = m+1$ (since the x_i are integers). Therefore, we have $2m+1 \geq 2x_n \geq 2(m+1) = 2m+2$. This is a

contradiction and so odd numbers cannot repeat. Note however, that even numbers do repeat. Let $x_i = x_{i+1} = 2m$ for some m and for some $1 \le i \le n - 2$. By the same reasoning as above, we have $-2m + 4m \ge 2x_n$ and $x_n \ge m$. This gives $2m \ge 2x_n \ge 2m$ which means $x_n = m$ in this case. $\quad\square$

Lemma 3.4.3. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \ge 0 \\ x_1 \le k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\ge 0}^n \mid 1 \le x_1 \le k, x_i = x_1 + \sum_{j=2}^i l_j \text{ for } 2 \le i \le n - 1,$$

$$0 \le l_2 \le x_1, 0 \le l_{n-1} \le l_{n-2} \le \cdots \le l_3 \le l_2,$$

$$\left\lceil \frac{x_{n-1}}{2} \right\rceil \le x_n \le \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor \text{ for } n \ge 3, \text{ and } \left\lceil \frac{x_1}{2} \right\rceil \le x_2 \le x_1 \text{ for } n = 2\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_1 = 0$ gives the solution $\mathbf{x} = 0$. Now, fix x_1 such that $1 \le x_1 \le k$. We prove the pattern for x_2 through x_{n-1} by induction. First, by $(\mathring{A}\mathbf{x})_1 \ge 0$, $x_2 \le 2x_1$. We have observed in Lemma 3.4.1 that the first $n-1$ coordinates of $\mathbf{x}$ do not decrease, therefore $x_2 \ge x_1$. Then $x_1 \le x_2 \le 2x_1$ implies that $x_2 = x_1 + l_2$ where $0 \le l_2 \le x_1$. Now, assume x_i is of the form $x_i = x_1 + \sum_{j=2}^i l_j$ for all $2 \le i \le p < n - 1$ where $l_j = x_j - x_{j-1}$ and so $0 \le l_p \le l_{p-1} \le \cdots \le l_3 \le l_2$ by the second result of Lemma 3.4.1. By the same lemma, $x_p \le x_{p+1} \le 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_1 + \sum_{j=2}^p l_j \le x_{p+1} \le 2(x_1 + \sum_{j=2}^p l_j) - (x_1 + \sum_{j=2}^{p-1} l_j) = x_1 + (\sum_{j=2}^p l_j) + l_p$. Then $x_{p+1} = x_1 + (\sum_{j=2}^p l_j) + l_{p+1}$ where $0 \le l_{p+1} \le l_p$. Therefore, $x_i = x_1 + \sum_{j=2}^i l_j$ where $0 \le l_{n-1} \le l_{n-2} \le \cdots \le l_3 \le l_2 \le x_1$ and so l_i is the amount of increase from x_{i-1} to x_i where $2 \le i \le n-1$. Finally, we need to prove the restrictions on x_n. $(\mathring{A}\mathbf{x})_{n-1} \ge 0$ implies that $2x_n \le -x_{n-2} + 2x_{n-1}$. Using what we just showed, we have $2x_n \le -(x_1 + \sum_{j=2}^{n-2} l_j) + 2(x_1 + \sum_{j=2}^{n-1} l_j) = x_1 + l_{n-1} + \sum_{j=2}^{n-1} l_j = x_{n-1} + l_{n-1}$. $(\mathring{A}\mathbf{x})_n \ge 0$ gives $x_n \ge \frac{x_{n-1}}{2}$. Or, with what we found, we have $x_n \ge \frac{x_1 + \sum_{j=3}^{n-1} l_j}{2}$. Since x_n is an integer, we have $\left\lceil \frac{x_{n-1}}{2} \right\rceil \le x_n \le \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor$. The description for x_2 in the case $n = 2$ follows directly from $\mathring{A}\mathbf{x} \ge 0$. $\quad\square$

Theorem 3.4.4. *Let $n \ge 3$, $\Lambda = k\Lambda_0, k \ge 2$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - \left(\sum_{i=2}^{n-1} (2l - (x_1 + \sum_{j=2}^i l_j))\alpha_i \right)$ where*

- $1 \le x_1 \le k$,

- $0 \le l_2 \le x_1$,

- $0 \le l_{n-1} \le l_{n-2} \le \cdots \le l_3 \le l_2$,

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- $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq l \leq \left\lfloor \frac{x_{n-1}+l_{n-1}}{2} \right\rfloor$.

When $n = 2, \max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1\}$ where $1 \leq x_1 \leq k$ and $\left\lceil \frac{x_1}{2} \right\rceil \leq l \leq 2x_1$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_i = 2b_0 - b_i$ for $1 \leq i \leq n - 1$ and $x_n = b_0 - b_n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, 2b_0 - x_1, 2b_0 - x_2, \ldots, 2b_0 - x_{n-1}, b_0 - x_n)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$\vdots$$
$$2b_0 - x_{n-1} \geq 0$$
$$b_0 - x_n \geq 0$$

We begin with the case $n \geq 3$. We have already proven that the first $n - 1$ coordinates of $\mathbf{x}$ never decrease. Therefore, $x_{n-1} = \max_{1 \leq i \leq n-1}\{x_i\}$ and so for the first $n - 1$ inequalities it is sufficient to say that $b_0 \geq \frac{x_{n-1}}{2}$. Since x_{n-1} may be odd, we need $b_0 \geq \left\lceil \frac{x_{n-1}}{2} \right\rceil$. We showed in Lemma 3.4.3 that $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq x_n \leq \left\lfloor \frac{x_{n-1}+l_{n-1}}{2} \right\rfloor$. Then since $b_0 \geq x_n \geq \left\lceil \frac{x_{n-1}}{2} \right\rceil \geq \frac{x_{n-1}}{2}$, we need only say $b_0 \geq x_n$. Then let $l = x_n$ and so $b_0 \geq l$. Showing $b_0 = l$ is similar to the method used in the proof of Theorem 3.3.3 and we have $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - \cdots - (2l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.4.3, we have the pattern given above.

When $n = 2$, we showed in Lemma 3.4.3 that $x_2 \geq \left\lceil \frac{x_1}{2} \right\rceil$. Then the above holds where $b_0 = l = x_2$. We can again combine this with the solution from Lemma 3.4.3, giving the pattern above. $\qquad \square$

To better understand this theorem, consider some examples for various values of both k and n. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 1 \leq x_1 \leq k\}$. Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$.

Example 3.4.5. $C_5^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.4.4, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - (x_1 + l_2))\alpha_2 - (2l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (2l - (x_1 + l_2 + l_3 + l_4))\alpha_4 + (l - x_5)\alpha_5.$$

We list the $\mathbf{x}$ vectors and corresponding values for l, l_2, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.10.

Table 3.10: **x** vectors and maximal dominant weights for the $\mathfrak{g} = C_5^{(1)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$2\Lambda_0$
(1,2,2,2,1)	1	1,0,0	$2\Lambda_0 - \alpha_0 - \alpha_1$
(1,2,3,4,2)	2	1,1,1	$2\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	1	0,0,0	$2\Lambda_0 - \alpha_0$
(2,3,4,4,2)	2	1,1,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,3)	3	1,1,1	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,2)	2	2,0,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1$
(2,4,5,6,3)	3	2,1,1	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,6,6,3)	3	2,2,0	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,4)	4	2,2,1	$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,4)	4	2,2,2	$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,5)	5	2,2,2	$2\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$

Example 3.4.6. $C_6^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.4.4, an element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - (x_1 + l_2)\alpha_2 - (2l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (2l - (x_1 + l_2 + l_3 + l_4))\alpha_4 - (2l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5 - (l - x_6)\alpha_6.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.11.

Example 3.4.7. $C_5^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.4.4, any element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - (x_1 + l_2))\alpha_2 - (2l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (2l - (x_1 + l_2 + l_3 + l_4))\alpha_4.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.12.

Example 3.4.8. $C_6^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.4.4, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - (x_1 + l_2)\alpha_2 - (2l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (2l - (x_1 + l_2 + l_3 + l_4))\alpha_4 - (2l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.13.

Table 3.11: **x** vectors and maximal dominant weights for the $\mathfrak{g} = C_6^{(1)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$2\Lambda_0$
(1,2,2,2,2,1)	1	1,0,0,0	$2\Lambda_0 - \alpha_0 - \alpha_1$
(1,2,3,4,4,2)	2	1,1,1,0	$2\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	1,1,1,1	$2\Lambda_0 - 3\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	1	0,0,0,0	$2\Lambda_0 - \alpha_0$
(2,3,4,4,4,2)	2	1,1,0,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,6,3)	3	1,1,1,1	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,4,2)	2	2,0,0,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1$
(2,4,5,6,6,3)	3	2,1,1,0	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7,4)	4	2,1,1,1	$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,6,3)	3	2,2,0,0	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,8,4)	4	2,2,1,1	$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,8,4)	4	2,2,2,0	$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9,5)	5	2,2,2,1	$2\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,10,5)	5	2,2,2,2	$2\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,10,6)	6	2,2,2,2	$2\Lambda_0 - 6\alpha_0 - 10\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$

3.5 Type $D_n^{(1)}$

Let $\mathfrak{g} = D_n^{(1)}$ for $n \geq 4$, index set $I = \{0, 1, \ldots, n\}$, and Cartan matrix

$$
A = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\
-1 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & \cdots & 0 & 0 & -1 & 2 & -1 & -1 \\
0 & \cdots & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & \cdots & 0 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}.
$$

The defining inequalities are equivalent to

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
x_2 \leq k
\end{cases}
$$

Table 3.12: **x** vectors and maximal dominant weights for the $\mathfrak{g} = C_5^{(1)}$-module $V(3\Lambda_0)$

x vector	l	l_2, l_3, l_4	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$3\Lambda_0$
(1,2,2,2,1)	1	1,0,0	$3\Lambda_0 - \alpha_0 - \alpha_1$
(1,2,3,4,2)	2	1,1,1	$3\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	1	0,0,0	$3\Lambda_0 - \alpha_0$
(2,3,4,4,2)	2	1,1,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,3)	3	1,1,1	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,2)	2	2,0,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1$
(2,4,5,6,3)	3	2,1,1	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,6,6,3)	3	2,2,0	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,4)	4	2,2,1	$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,4)	4	2,2,2	$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,5)	5	2,2,2	$3\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(3,4,4,4,2)	2	1,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,4,5,6,3)	3	1,1,1	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(3,5,6,6,3)	3	2,1,0	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2$
(3,5,6,7,4)	4	2,1,1	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(3,5,7,8,4)	4	2,2,1	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$
(3,5,7,9,5)	5	2,2,2	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - \alpha_4$
(3,6,6,6,3)	3	3,0,0	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1$
(3,6,7,8,4)	4	3,1,1	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2 - \alpha_3$
(3,6,8,8,4)	4	3,2,0	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$
(3,6,8,9,5)	5	3,2,1	$3\Lambda_0 - 5\alpha_0 - 7\alpha_0 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(3,6,8,10,5)	5	3,2,2	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - 2\alpha_3$
(3,6,8,10,6)	6	3,2,2	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(3,6,9,10,5)	5	3,3,1	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - \alpha_3$
(3,6,9,11,6)	6	3,3,2	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3 - \alpha_4$
(3,6,9,12,6)	6	3,3,3	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3$
(3,6,9,12,7)	7	3,3,3	$3\Lambda_0 - 7\alpha_0 - 11\alpha_1 - 8\alpha_2 - 5\alpha_3 - 2\alpha_4$

Table 3.13: **x** vectors and maximal dominant weights for the $\mathfrak{g} = C_6^{(1)}$-module $V(3\Lambda_0)$

x vector	l	l_2,l_3,l_4,l_5	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$3\Lambda_0$
(1,2,2,2,2,1)	1	1,0,0,0	$3\Lambda_0 - \alpha_0 - \alpha_1$
(1,2,3,4,4,2)	2	1,1,1,0	$3\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	1,1,1,1	$3\Lambda_0 - 3\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	1	0,0,0,0	$3\Lambda_0 - \alpha_0$
(2,3,4,4,4,2)	2	1,1,0,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,6,3)	3	1,1,1,1	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,4,2)	2	2,0,0,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1$
(2,4,5,6,6,3)	3	2,1,1,0	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7,4)	4	2,1,1,1	$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,6,3)	3	2,2,0,0	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,8,4)	4	2,2,1,1	$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,8,4)	4	2,2,2,0	$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9,5)	5	2,2,2,1	$3\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,10,5)	5	2,2,2,2	$3\Lambda_0 - 5\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,10,6)	6	2,2,2,2	$3\Lambda_0 - 6\alpha_0 - 10\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$
(3,4,4,4,4,2)	2	1,0,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,4,5,6,6,3)	3	1,1,1,0	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(3,4,5,6,7,4)	4	1,1,1,1	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(3,5,6,6,6,3)	3	2,1,0,0	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2$
(3,5,6,7,8,4)	4	2,1,1,1	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(3,5,7,8,8,4)	4	2,2,1,0	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$
(3,5,7,8,9,5)	5	2,2,1,2	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(3,5,7,9,10,5)	5	2,2,2,1	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - \alpha_4$
(3,5,7,9,11,6)	6	2,2,2,2	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 7\alpha_2 - 5\alpha_3 - 3\alpha_4 - \alpha_5$
(3,6,6,6,6,3)	3	3,0,0,0	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1$
(3,6,7,8,8,4)	4	3,1,1,0	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2 - \alpha_3$
(3,6,7,8,9,5)	5	3,1,1,1	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(3,6,8,8,8,4)	4	3,2,0,0	$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$
(3,6,8,9,10,5)	5	3,2,1,1	$3\Lambda_0 - 5\alpha_0 - 7\alpha_0 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(3,6,8,10,10,5)	5	3,2,2,0	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - 2\alpha_3$
(3,6,8,10,11,6)	6	3,2,2,1	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(3,6,8,10,12,6)	6	3,2,2,2	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(3,6,8,10,12,7)	7	3,2,2,2	$3\Lambda_0 - 7\alpha_0 - 11\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$
(3,6,9,10,10,5)	5	3,3,1,0	$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - \alpha_3$
(3,6,9,10,11,6)	6	3,3,1,1	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(3,6,9,11,12,6)	6	3,3,2,1	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3 - \alpha_4$
(3,6,9,11,13,7)	7	3,3,2,2	$3\Lambda_0 - 7\alpha_0 - 11\alpha_1 - 8\alpha_2 - 5\alpha_3 - 3\alpha_4 - \alpha_5$
(3,6,9,12,12,6)	6	3,3,3,0	$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3$
(3,6,9,12,13,7)	7	3,3,3,1	$3\Lambda_0 - 7\alpha_0 - 11\alpha_1 - 8\alpha_2 - 5\alpha_3 - 2\alpha_4 - \alpha_5$
(3,6,9,12,14,7)	7	3,3,3,2	$3\Lambda_0 - 7\alpha_0 - 11\alpha_1 - 8\alpha_2 - 5\alpha_3 - 2\alpha_4$
(3,6,9,12,14,8)	8	3,3,3,2	$3\Lambda_0 - 8\alpha_0 - 13\alpha_1 - 10\alpha_2 - 7\alpha_3 - 4\alpha_4 - 2\alpha_5$
(3,6,9,12,15,8)	8	3,3,3,3	$3\Lambda_0 - 8\alpha_0 - 13\alpha_1 - 10\alpha_2 - 7\alpha_3 - 4\alpha_4 - \alpha_5$
(3,6,9,12,15,9)	9	3,3,3,3	$3\Lambda_0 - 9\alpha_0 - 15\alpha_1 - 12\alpha_2 - 9\alpha_3 - 6\alpha_4 - 3\alpha_5$

where $x_1 = b_0 - b_1$, $x_i = 2b_0 - b_i$ for $2 \leq i \leq n-2$, $x_{n-1} = b_0 - b_{n-1}$, and $x_n = b_0 - b_n$ and $x_i \in \mathbb{Z}$ for $i \in I$. That is,

$$
\begin{cases}
2x_1 - x_2 & \geq 0 \\
-x_1 + 2x_2 - x_3 & \geq 0 \\
-x_2 + 2x_3 - x_4 & \geq 0 \\
\quad \vdots & \vdots \\
-x_{n-4} + 2x_{n-3} - x_{n-2} & \geq 0 \\
-x_{n-3} + 2x_{n-2} - x_{n-1} - x_n & \geq 0 \\
-x_{n-2} + 2x_{n-1} & \geq 0 \\
-x_{n-2} + 2x_n & \geq 0 \\
x_2 & \leq k
\end{cases}
$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq b_1, b_{n-1}$ and b_n and $b_0 \geq \frac{b_i}{2}$ for $2 \leq i \leq n-2$.

Lemma 3.5.1. *Let $\mathbf{x}$ be a solution to*

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
x_2 \leq k
\end{cases}
$$

Then $x_i \leq x_{i+1}$ for $1 \leq i \leq n-3$. In addition, $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. This proof is essentially the same as that of Lemma 3.3.1 with the coordinates non-decreasing until $n-3$ rather than $n-1$. We arrive at the contradiction using $(\mathring{A}\mathbf{x})_{n-2} \geq 0$ and $(\mathring{A}\mathbf{x})_{n-1,n} \geq 0$. $\square$

Lemma 3.5.2. *Let $\mathbf{x}$ be a solution to*

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
x_2 \leq k
\end{cases}
$$

Then $x_2 \neq 1$.

Proof. Assume, for the sake of contradiction, that $x_2 = 1$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_1 \geq \frac{1}{2}$. Then $(\mathring{A}\mathbf{x})_3 \geq 0$ implies that $2 - x_1 \geq x_3$ which implies $x_1 = 1$ since as stated earlier, $\mathbf{x} > 0$ and the x_i are integers. We show that this would imply $x_i = 1$ for $1 \leq i \leq n-2$ by induction. We have the base case $x_1 = x_2 = 1$. Assume that $x_{j-1} = x_j = 1$ for all $j \leq p < n-2$. We wish to show $x_{p+1} = 1$. $(A\mathbf{x})_{p+1} \geq 0$ implies that $-1 + 2 \geq x_{p+1}$ which gives $x_{p+1} = 1$. Then we have that $x_i = 1$ for $1 \leq i \leq n-2$. However, by $(\mathring{A}\mathbf{x})_{n-2} \geq 0$, $-1 + 2 \geq x_{n-1} + x_n$. Then either $x_{n-1} = 0$ or $x_n = 0$, which contradicts that $\mathbf{x} = 0$ or $\mathbf{x} > 0$ since $\mathring{A}$ is of finite type. $\square$

Lemma 3.5.3. *Let $\mathbf{x}$ be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

Then $x_i = x_{i-1}$ implies $x_i = 2m$ for some integer m and for $2 \leq i \leq n-2$.

Proof. This proof is essentially the same as Lemma 3.4.2 using the last three inequalities to arrive at a contradiction. □

Lemma 3.5.4. *The set of solutions to*

$$\begin{cases} A\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 2 \leq x_2 \leq k, x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1, 0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor,$$

$$x_i = x_2 + \sum_{j=3}^{i} l_j, 0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1, 0 \leq l_{n-2} \leq l_{n-3} \leq \cdots \leq l_4 \leq l_3$$

$$\text{where } n \geq 5, 3 \leq i \leq n-2, x_{n-1} + x_n \leq x_{n-2} + l_{n-2},$$

$$\min\{x_{n-1}, x_n\} \geq \left\lceil \frac{x_{n-2}}{2} \right\rceil \text{ and } l_2 = x_2 - x_1 \text{ for } n = 4\}.$$

Proof. As stated above since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_2 = 0$ gives the solution $\mathbf{x} = 0$. We have shown that $x_2 \neq 1$.

Now, fix x_2 such that $2 \leq x_2 \leq k$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_1 \geq \frac{x_2}{2}$. Since each x_i must be an integer, this implies $x_1 \geq \left\lceil \frac{x_2}{2} \right\rceil$. We have observed in Lemma 3.5.1 that the first $n-2$ coordinates of $\mathbf{x}$ do not decrease, therefore $x_1 \leq x_2$. Then $\left\lceil \frac{x_2}{2} \right\rceil \leq x_1 \leq x_2$ implies that $x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1$ where $0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor$.

We prove the pattern for x_3 through x_{n-2} by induction. First, by Lemma 3.5.1, $x_2 \leq x_3$ and $x_3 - x_2 \leq x_2 - x_1$. Then $x_3 - x_2 \leq x_2 - (\left\lceil \frac{x_2}{2} \right\rceil + l_1) = \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$. Therefore, $x_2 \leq x_3 \leq x_2 + \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$ and so we have $x_3 = x_2 + l_3$ where $0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1$. Now assume x_i is of the form $x_i = x_2 + \sum_{j=3}^{i} l_j$ for all $3 \leq i \leq p < n-2$ where $l_j = x_j - x_{j-1}$ and so $0 \leq l_p \leq l_{p-1} \leq \cdots \leq l_3$ by the second result of Lemma 3.5.1. By the same lemma, $x_p \leq x_{p+1} \leq 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_2 + \sum_{j=3}^{p} l_j \leq x_{p+1} \leq 2(x_2 + \sum_{j=3}^{p} l_j) - (x_2 + \sum_{j=3}^{p-1} l_j) = x_2 + (\sum_{j=3}^{p} l_j) + l_p$. Then $x_{p+1} = x_2 + (\sum_{j=3}^{p} l_j) + l_{p+1}$ where $0 \leq l_{p+1} \leq l_p$. Therefore, $x_i = x_2 + \sum_{j=3}^{i} l_j$ where $0 \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3$ and so l_i is the amount of increase from x_{i-1} to x_i where $3 \leq i \leq n-2$.

Finally, we need to prove the restrictions on x_{n-1} and x_n. $(\mathring{A}\mathbf{x})_{n-2} \geq 0$ implies that $x_{n-1} + x_n \leq -x_{n-3} + 2x_{n-2}$. Using what we just showed, we have $x_{n-1} + x_n \leq -(x_2 + \sum_{j=3}^{n-3} l_j) + 2(x_2 +$

$\sum_{j=3}^{n-2} l_j) = x_{n-2} + l_{n-2}$. The last two inequalities give $x_{n-1}, x_n \geq \frac{x_{n-2}}{2}$. With what we found, we have $x_{n-1}, x_n \geq \frac{x_2 + \sum_{j=3}^{n-2} l_j}{2}$. Since x_{n-1}, x_n are integers, we have $x_{n-1}, x_n \geq \left\lceil \frac{x_2 + \sum_{j=3}^{n-2} l_j}{2} \right\rceil$. In the case $n = 4$, the descriptions for x_3, x_4, and l_2 follow directly from $\mathring{A}\mathbf{x} \geq 0$. $\qquad\square$

Theorem 3.5.5. *Let $n \geq 5$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n-2} (2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \right) - (l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n\}$ where*

- $2 \leq x_2 \leq k$,

- $l = \max\{x_1, x_{n-1}, x_n\}$,

- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$,

- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$,

- $0 \leq l_{n-2} \leq l_{n-3} \leq \cdots \leq l_4 \leq l_3$,

- $x_{n-1} + x_n \leq x_{n-2} + l_{n-2}$,

- $\min\{x_{n-1}, x_n\} \geq \lceil \frac{x_{n-2}}{2} \rceil$.

When $n = 4, \max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (l - x_3)\alpha_3 - (l - x_4)\alpha_4\}$ where

- $2 \leq x_2 \leq k$,

- $l = \max\{x_1, x_3, x_4\}$,

- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor, l_2 = x_2 - x_1$,

- $x_3 + x_4 \leq x_2 + l_2$,

- $\min\{x_3, x_4\} \geq \lceil \frac{x_2}{2} \rceil$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = b_0 - b_1, x_i = 2b_0 - b_i$ for $2 \leq i \leq n-2, x_{n-1} = b_0 - b_{n-1}$ and $x_n = b_0 - b_n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, b_0 - x_1, 2b_0 - x_2, \ldots, 2b_0 - x_{n-2}, b_0 - x_{n-1}, b_0 - x_n)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$\vdots$$
$$2b_0 - x_{n-2} \geq 0$$

Table 3.14: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_5^{(1)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(\Lambda) \cap P^+$
$(0,0,0,0,0)$	0	$2\Lambda_0$
$(1,2,2,1,1)$	1	$2\Lambda_0 - \alpha_0$
$(1,2,3,2,2)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(2,2,2,1,1)$	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$

$$b_0 - x_{n-1} \geq 0$$
$$b_0 - x_n \geq 0$$

Note that $b_0 \geq \max\{x_1, x_{n-1}, x_n\}$ where $x_1 = (\lceil \frac{x_2}{2} \rceil + l_1)$ by Lemma 3.5.4. We have already proven that the first $n-2$ coordinates of **x** never decrease in Lemma 3.5.1. Therefore, $x_{n-2} = \max_{1 \leq i \leq n-2}\{x_i\}$ and so for the inequalities involving x_i where $2 \leq i \leq n-2$ it is sufficient to say that $b_0 \geq \frac{x_{n-2}}{2}$. Note that due to the $n-1$st and nth rows of $\mathring{A}\mathbf{x} \geq 0$, $x_{n-1}, x_n \geq \lceil \frac{x_{n-2}}{2} \rceil$. We let $l = \max\{x_1, x_{n-1}, x_n\}$ and so $b_0 \geq l$. Showing $b_0 = l$ is similar to the method used in the proof of Theorem 3.3.3 and we have $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - \cdots - (l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.5.4, we have the pattern given above. $\square$

To better understand this theorem, consider some examples for various values of both k and n. We include two examples of $k = 2$ so that the reader may check that our results match those of [18]. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_2 \leq k\}$. Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$.

Example 3.5.6. $D_5^{(1)}, V(\Lambda) = V(2\Lambda_0)$. According to Theorem 3.5.5,

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (l - x_4)\alpha_4 - (l - x_5)\alpha_5\}.$$

We list the **x** vectors, the corresponding l, and the resulting maximal dominant weight in Table 3.14. One can check that our set of maximal dominant weights corresponds to that of [18],

$$\{2\Lambda_0, \Lambda_2 - \delta, \Lambda_4 + \Lambda_5 - 2\delta, 2\Lambda_1 - 2\delta\}$$

by rewriting all α_i in terms of the fundamental weights, Λ_i.

Example 3.5.7. $D_6^{(1)}, V(\Lambda) = V(2\Lambda_0)$. According to Theorem 3.5.5,

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (l - x_5)\alpha_5 - (l - x_6)\alpha_6\}.$$

Table 3.15: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(1)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(\Lambda) \cap P^+$
$(0,0,0,0,0,0)$	0	$2\Lambda_0$
$(1,2,2,2,1,1)$	1	$2\Lambda_0 - \alpha_0$
$(1,2,3,4,2,2)$	2	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(1,2,3,4,2,3)$	3	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
$(1,2,3,4,3,2)$	3	$2\Lambda_0 - 3\alpha_3 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$
$(2,2,2,2,1,1)$	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$

We now list the **x** vectors, their corresponding values of l and maximal dominant weights in Table 3.15. Once more, one can check that our set of maximal dominant weights matches that of [18],

$$\{2\Lambda_0, \Lambda_2 - \delta, \Lambda_4 - 2\delta, 2\Lambda_6 - 3\delta, 2\Lambda_5 - 3\delta, 2\Lambda_1 - 2\delta\}.$$

One can show that our formulation for the maximal dominant weights of $V(k\Lambda_0)$ correspond to that of [18] for the case $k = 2$ for arbitrary $n \geq 4$. According to [18], the set of maximal dominant weights for $V(2\Lambda_0)$ is $\{2\Lambda_0, 2\Lambda_1 - 2\delta\} \cup \{\Lambda_{2u} - u\delta | 1 \leq u \leq \lfloor \frac{n-2}{2} \rfloor\} \cup \{2\Lambda_n - \frac{n}{2}\delta, 2\Lambda_{n-1} - \frac{n}{2}\delta | n \text{ even}\} \cup \{\Lambda_{n-1} + \Lambda_n - \frac{n-1}{2}\delta | n \text{ odd}\}$. To show that the two formulations are the same, we write our formulation in terms of the fundamental dominant weights, Λ_i. Our formulation then becomes

$$\begin{aligned}
X = \{2\Lambda_0 &- l(2\Lambda_0 - \Lambda_2 + \delta) - (l - (1 + l_1))(2\Lambda_1 - \Lambda_2) - (2l - 2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3) \\
&- \left(\sum_{i=3}^{n-3}(2l - (2 + \sum_{j=3}^{i} l_j))\right)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}) \\
&- \left(2l - (2 + \sum_{j=3}^{n-2} l_j)\right)(-\Lambda_{n-3} + 2\Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n) \\
&- (l - x_{n-1})(-\Lambda_{n-2} + 2\Lambda_{n-1}) - (l - x_n)(-\Lambda_{n-2} + 2\Lambda_n)\}.
\end{aligned}$$

Showing this formulation corresponds to the maximal dominant weights found in [18] is similar to how we showed the same for type $B_n^{(1)}$.

Example 3.5.8. $D_5^{(1)}, V(\Lambda) = V(3\Lambda_0)$. According to Theorem 3.5.5,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (l - x_4)\alpha_4 - (l - x_5)\alpha_5\}.$$

We list the **x** vectors, the corresponding l, and the resulting maximal dominant weight in Table 3.16. Recall that we can collect the **x** vectors from $X_{2,5}$ when finding $X_{3,5}$.

Table 3.16: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_5^{(1)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(\Lambda) \cap P^+$
$(0,0,0,0,0)$	0	$3\Lambda_0$
$(1,2,2,1,1)$	1	$3\Lambda_0 - \alpha_0$
$(1,2,3,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(2,2,2,1,1)$	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
$(2,3,4,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
$(2,3,4,2,3)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
$(2,3,4,3,2)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_5$

Table 3.17: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(1)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(\Lambda) \cap P^+$
$(0,0,0,0,0,0)$	0	$3\Lambda_0$
$(1,2,2,2,1,1)$	1	$3\Lambda_0 - \alpha_0$
$(1,2,3,4,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(1,2,3,4,2,3)$	3	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_2 - \alpha_5$
$(1,2,3,4,3,2)$	3	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_2 - \alpha_6$
$(2,2,2,2,1,1)$	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
$(2,3,4,4,2,2)$	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
$(2,3,4,5,3,3)$	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

Example 3.5.9. $D_6^{(1)}, V(\Lambda) = V(3\Lambda_0)$. According to Theorem 3.5.5,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (l - x_5)\alpha_5 - (l - x_6)\alpha_6\}.$$

We list the **x** vectors, the corresponding l, and the resulting maximal dominant weight in Table 3.17. Recall that we can collect the **x** vectors from $X_{2,6}$ when finding $X_{3,6}$.

Example 3.5.10. $D_5^{(1)}, V(\Lambda) = V(4\Lambda_0)$. According to Theorem 3.5.5,

$$\max(4\Lambda_0) \cap P^+ = \{4\Lambda_0\} \cup \{4\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (l - x_4)\alpha_4 - (l - x_5)\alpha_5\}.$$

We list the **x** vectors and the resulting maximal dominant weight in Table 3.18. Recall that we can collect the **x** vectors from $X_{3,5}$ when finding $X_{4,5}$.

Example 3.5.11. $D_6^{(1)}, V(\Lambda) = V(4\Lambda_0)$. According to Theorem 3.5.5,

$$\max(4\Lambda_0) \cap P^+ = \{4\Lambda_0\} \cup \{4\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (l - x_5)\alpha_5 - (l - x_6)\alpha_6\}.$$

Table 3.18: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_5^{(1)}$-module $V(4\Lambda_0)$

x vector	Element of $\max(\Lambda) \cap P^+$
$(0,0,0,0,0)$	$4\Lambda_0$
$(1,2,2,1,1)$	$4\Lambda_0 - \alpha_0$
$(1,2,3,2,2)$	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(2,2,2,1,1)$	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
$(2,3,4,2,2)$	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
$(2,3,4,2,3)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
$(2,3,4,3,2)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_5$
$(2,4,4,2,2)$	$4\Lambda_0 - 2\alpha_0$
$(2,4,5,3,3)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
$(2,4,6,3,3)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
$(2,4,6,3,4)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
$(2,4,6,3,5)$	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
$(2,4,6,5,3)$	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_5$
$(2,4,6,4,3)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_5$
$(2,4,6,4,4)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
$(3,4,4,2,2)$	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
$(3,4,5,3,3)$	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
$(4,4,4,2,2)$	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 2\alpha_4 - 2\alpha_5$

We list the **x** vectors, the corresponding l, and the resulting maximal dominant weight in Table 3.19. Recall that we can collect the **x** vectors from $X_{3,6}$ when finding $X_{4,6}$.

3.6   Type $G_2^{(1)}$

Let $\mathfrak{g} = G_2^{(1)}$, index set $I = \{0, 1, 2\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

where $x_1 = 2b_0 - b_1$ and $x_2 = 3b_0 - b_2$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -3x_1 + 2x_2 & \geq 0 \\ x_1 & \leq k \end{cases}$$

Table 3.19: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(1)}$-module $V(4\Lambda_0)$

x vector	maximal dominant weight
(0,0,0,0,0,0)	$4\Lambda_0$
(1,2,2,2,1,1)	$4\Lambda_0 - \alpha_0$
(1,2,3,4,2,2)	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,3,2)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$
(1,2,3,4,2,3)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,1,1)	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
(2,3,4,5,3,3)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,3,4,4,2,2)	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
(2,4,4,4,2,2)	$4\Lambda_0 - 2\alpha_0$
(2,4,5,6,3,3)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,4,3)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$
(2,4,5,6,3,4)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,3,3)	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
(2,4,6,7,4,4)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,4,4)	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,5,4)	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_6$
(2,4,6,8,4,5)	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,5,5)	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,6,4)	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_6$
(2,4,6,8,4,6)	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$
(3,4,4,4,2,2)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$
(3,4,5,6,3,3)	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
(3,4,5,6,4,3)	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$
(3,4,5,6,3,4)	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(4,4,4,4,2,2)	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5 - 2\alpha_6$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \frac{x_1}{2}$ and $b_0 \geq \frac{x_2}{3}$.

Lemma 3.6.1. *The set of solutions to*

$$
\begin{cases}
2x_1 - x_2 & \geq 0 \\
-3x_1 + 2x_2 & \geq 0 \\
x_1 & \leq k
\end{cases}
$$

is $\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2) \mid 1 \leq x_1 \leq k, \lceil \frac{3x_1}{2} \rceil \leq x_2 \leq 2x_1\}$.

Proof. Note that since $\mathring{A}$ is of finite type, $\mathbf{x} \geq 0$ and so $x_1 \geq 0$ and $x_1 \leq k$ by definition. Therefore, $x_1 = 0$, which implies $\mathbf{x} = 0$, or $x_1 \geq 1$. Now, fix x_1 such that $1 \leq x_1 \leq k$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_2 \leq 2x_1$ and by $(\mathring{A}\mathbf{x})_2 \geq 0, x_2 \geq \frac{3}{2}x_1$. Since x_2 must be an integer, we have $x_2 \geq \lceil \frac{3x_1}{2} \rceil$. This describes all possible solutions. $\qquad\square$

Theorem 3.6.2. *Let* $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$ *where*

- $1 \leq x_1 \leq k$,

- $\lceil \frac{3x_1}{2} \rceil \leq x_2 \leq 2x_1$,

- $l = \lceil \frac{x_2}{3} \rceil$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1$ and $x_2 = 3b_0 - b_2$. Then $(b_0, b_1, b_2) = (b_0, 2b_0 - x_1, 3b_0 - x_2)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 2$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$3b_0 - x_2 \geq 0$$

Then $b_0 \geq \frac{x_2}{2}$ and $b_0 \geq \frac{x_2}{3}$. However, from $(\mathring{A}\mathbf{x})_2 \geq 0$, we have $2x_2 \geq 3x_1$, which implies $\frac{x_2}{3} \geq \frac{x_1}{2}$. Then it suffices to say $b_0 \geq \frac{x_2}{3}$. Then let $l = \lceil \frac{x_2}{3} \rceil$ and so $b_0 \geq l$, since b_0 must be an integer. Showing $b_0 = l$ is similar to the method used in the proof of Theorem 3.3.3 and we have $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$. Combining this with our solutions for the x_i from Lemma 3.6.1, we have the pattern given above. $\qquad\square$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2) \mid \mathring{A}\mathbf{x} \geq 0, 1 \leq x_1 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Table 3.20: **x** vectors and maximal dominant weights for the $\mathfrak{g} = G_2^{(1)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0)	0	$2\Lambda_0$
(1,2)	1	$2\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2$
(2,3)	1	$2\Lambda_0 - \alpha_0$
(2,4)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$

Table 3.21: **x** vectors and maximal dominant weights for the $\mathfrak{g} = G_2^{(1)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0)	0	$3\Lambda_0$
(1,2)	1	$3\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2$
(2,3)	1	$3\Lambda_0 - \alpha_0$
(2,4)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$
(3,5)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_2$
(3,6)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1$

Example 3.6.3. $G_2^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.6.2, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form $2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.20.

Example 3.6.4. $G_2^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.6.2, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form $3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.21.

Example 3.6.5. $G_2^{(1)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.6.2, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form $4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.22. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $W_3 \subset W_4$.

Table 3.22: **x** vectors and maximal dominant weights for the $\mathfrak{g} = G_2^{(1)}$-module $V(4\Lambda_0)$

x vector	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0)	0	$4\Lambda_0$
(1,2)	1	$4\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2$
(2,3)	1	$4\Lambda_0 - \alpha_0$
(2,4)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$
(3,5)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_2$
(3,6)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(4,6)	2	$4\Lambda_0 - 2\alpha_0$
(4,7)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2$
(4,8)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$

Table 3.23: **x** vectors and maximal dominant weights for the $\mathfrak{g} = G_2^{(1)}$-module $V(6\Lambda_0)$

x vector	l	Element of $\max(6\Lambda_0) \cap P^+$
(0,0)	0	$6\Lambda_0$
(1,2)	1	$6\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2$
(2,3)	1	$6\Lambda_0 - \alpha_0$
(2,4)	2	$6\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$
(3,5)	2	$6\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_2$
(3,6)	2	$6\Lambda_0 - 2\alpha_0 - \alpha_1$
(4,6)	2	$6\Lambda_0 - 2\alpha_0$
(4,7)	3	$6\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2$
(4,8)	3	$6\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(5,8)	3	$6\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_2$
(5,9)	3	$6\Lambda_0 - 3\alpha_0 - \alpha_1$
(5,10)	4	$6\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2$
(6,9)	3	$6\Lambda_0 - 3\alpha_0$
(6,10)	4	$6\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 2\alpha_2$
(6,11)	4	$6\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2$
(6,12)	4	$6\Lambda_0 - 4\alpha_0 - 2\alpha_1$

Example 3.6.6. $G_2^{(1)}, V(\Lambda) = V(6\Lambda_0)$. By Theorem 3.6.2, any element of $\max(6\Lambda_0) \cap P^+$ other than $6\Lambda_0$ is of the form $6\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.23. Notice that we can collect the same **x** vectors from the case $V(4\Lambda_0)$ since, as we said earlier, $W_4 \subset W_6$.

3.7 Type $F_4^{(1)}$

Let $\mathfrak{g} = F_4^{(1)}$, index set $I = \{0, 1, 2, 3, 4\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

where $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 4b_0 - b_3$ and $x_4 = 2b_0 - b_4$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - x_3 & \geq 0 \\ -2x_2 + 2x_3 - x_4 & \geq 0 \\ -x_3 + 2x_4 & \geq 0 \\ x_1 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{2}\}$.

Lemma 3.7.1. *Let $\mathbf{x}$ be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

Then $x_i < x_{i+1}$ for $1 \leq i \leq 2$. In addition, $x_i - x_{i-1} \geq x_{i+1} - x_i$ where $x_0 = 0$.

Proof. Assume, for the sake of contradiction, that there exists a $2 \leq j \leq 3$ such that $x_j \leq x_{j-1}$. If $j = 2$, then we have $-x_1 + 2x_2 - x_3 \geq 0$, which implies $x_2 - x_3 \geq x_1 - x_2 \geq 0$, implying that $x_2 \geq x_3$, which gives the same result as if $j = 3$. Therefore, if the coordinates were not to increase at any point, they would continue to do so. Then by $(\mathring{A}\mathbf{x})_3 \geq 0, (x_3 - x_2) \geq x_4$ and our assumption, $x_4 \leq 0$. This is a contradiction since the $x_i = 0$ or $x_i > 0$. Then the $x_i < x_{i+1}$ for $1 \leq i \leq 2$.

Observe that the amount of increase from x_i to x_{i+1} is never more than that from x_{i-1} to x_i for $i = 2$, since $-x_1 + 2x_2 - x_3 \geq 0$, which implies $x_2 - x_1 \geq x_3 - x_2$. Additionally, $x_1 \geq x_2 - x_1$ from the first inequality. $\qquad \square$

Lemma 3.7.2. *The set of solutions to*

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - x_3 & \geq 0 \\ -2x_2 + 2x_3 - x_4 & \geq 0 \\ -x_3 + 2x_4 & \geq 0 \\ x_1 & \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, x_3, x_4) \mid 1 \leq x_1 \leq k, x_i = x_1 + \sum_{k=2}^{i} l_k \text{ for } i = 2, 3, \left\lceil \frac{x_1}{2} \right\rceil \leq l_2 \leq x_1,$$

$$\left\lceil \frac{x_1 + l_2}{3} \right\rceil \leq l_3 \leq l_2 \text{ and } \left\lceil \frac{x_1 + l_2 + l_3}{2} \right\rceil \leq x_4 \leq 2l_3\}.$$

Proof. As stated earlier, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_1 = 0$ gives the solution $\mathbf{x} = 0$.

Now, fix x_1 such that $1 \leq x_1 \leq k$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_2 \leq 2x_1$. We have observed that the first three coordinates of $\mathbf{x}$ increase, therefore $x_1 < x_2$. Then $x_1 < x_2 \leq 2x_1$ and so we can define $x_2 = x_1 + l_2$ where $0 < l_2 \leq x_1$. We've shown that $x_i < x_{i+1}$ and $x_{i+1} - x_i \leq x_i - x_{i-1}$ for $1 \leq i \leq 2$. Therefore, $x_i = x_1 + \sum_{j=2}^{i} l_j$ where $0 < l_3 \leq l_2 \leq x_1$ and so l_i is the amount of increase from x_{i-1} to x_i where $2 \leq i \leq 3$.

We must prove the pattern for x_4. From $\mathring{A}\mathbf{x}_3 \geq 0$, we have $x_4 \leq 2(x_3 - x_2) = 2l_3$. Now we can investigate the lower bounds. From $\mathring{A}\mathbf{x}_4 \geq 0$, we have $x_4 \geq \frac{x_3}{2} = \frac{x_1 + l_2 + l_3}{2}$. Since $x_4 \in \mathbb{Z}$, we must have $x_4 \geq \left\lceil \frac{x_1 + l_2 + l_3}{2} \right\rceil$. Combining the two inequalities for x_4, we obtain $\frac{x_1 + l_2 + l_3}{2} \leq 2l_3$ which implies $\frac{x_1 + l_2}{3} \leq l_3$. Since x_3 must be an integer, we have $\left\lceil \frac{x_1 + l_2}{3} \right\rceil \leq l_3$. Finally, since $l_3 \leq l_2$ and combining with our result for the lower bound of l_3, we obtain $\frac{x_1 + l_2}{3} \leq l_3 \leq l_2$ which implies $\frac{x_1}{2} \leq l_2$. Again, since x_2 must be an integer, we have $\left\lceil \frac{x_1}{2} \right\rceil \leq l_2$. This describes all possible solutions. $\qquad\square$

Theorem 3.7.3. *Let $\Lambda = k\Lambda_0, k \geq 2$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - (x_1 + l_2))\alpha_2 - (4l - (x_1 + l_2 + l_3))\alpha_3 - (2l - x_4)\alpha_4$ where*

- $1 \leq x_1 \leq k$,

- $\left\lceil \frac{x_1}{2} \right\rceil \leq l_2 \leq x_1$,

- $\left\lceil \frac{x_1 + l_2}{3} \right\rceil \leq l_3 \leq l_2$,

- $\left\lceil \frac{x_1 + l_2 + l_3}{2} \right\rceil \leq x_4 \leq 2l_3$,

- $l = \left\lceil \frac{x_4}{2} \right\rceil$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 4b_0 - b_3$, and $x_4 = 2b_0 - b_4$. Then $(b_0, b_1, b_2, b_3, b_4) = (b_0, 2b_0 - x_1, 3b_0 - x_2, 4b_0 - x_3, 2b_0 - x_4)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 4$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$3b_0 - x_2 \geq 0$$
$$4b_0 - x_3 \geq 0$$
$$2b_0 - x_4 \geq 0$$

Then $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{2}\}$. Note that since $x_4 \geq \frac{x_3}{2}$, then $\frac{x_4}{2} \geq \frac{x_3}{4}$. From $(\mathring{A}\mathbf{x})_3 \geq 0$, we have $x_3 \geq \frac{x_4 + 2x_2}{2}$. Combining this with the above result, we have $x_3 \geq \frac{x_3}{4} + x_2$ which implies $\frac{x_3}{4} \geq \frac{x_2}{3}$. Finally, by $(\mathring{A}\mathbf{x})_2 \geq 0$, $x_2 \geq \frac{x_1 + x_3}{2}$, which we combine with the most recent result to

Table 3.24: **x** vectors and maximal dominant weights for the $\mathfrak{g} = F_4^{(1)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0)	0	$2\Lambda_0$
(1,2,3,2)	1	$2\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$
(2,3,4,2)	1	$2\Lambda_0 - \alpha_0$
(2,4,6,3)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,4)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$

Table 3.25: **x** vectors and maximal dominant weights for the $\mathfrak{g} = F_4^{(1)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0)	0	$3\Lambda_0$
(1,2,3,2)	1	$3\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$
(2,3,4,2)	1	$3\Lambda_0 - \alpha_0$
(2,4,6,3)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,4)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$
(3,5,7,4)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$
(3,6,8,4)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,6,9,5)	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 - \alpha_4$
(3,6,9,6)	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3$

obtain $x_2 \geq \frac{x_1}{2} + \frac{2x_2}{3}$, giving $\frac{x_2}{3} \geq \frac{x_1}{2}$. Then it suffices to say $b_0 \geq \frac{x_4}{2}$. Then let $l = \left\lceil \frac{x_4}{2} \right\rceil$ and so $b_0 \geq l$, since b_0 must be an integer. Showing $b_0 = l$ is similar to the method used in the proof of Theorem 3.3.3 and so we have $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (2l - x_4)\alpha_4$. Combining this with our solutions for the x_i from Lemma 3.7.2, we have the pattern given above. $\qquad\square$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2, x_3, x_4) \mid \mathring{A}\mathbf{x} \geq 0, 1 \leq x_1 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Example 3.7.4. $F_4^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.7.3, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form $2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (2l - x_4)\alpha_4$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.24.

Example 3.7.5. $F_4^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.7.3, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form $3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (2l - x_4)\alpha_4$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.25.

Example 3.7.6. $F_4^{(1)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.7.3, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form $4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (2l - x_4)\alpha_4$. We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant

Table 3.26: **x** vectors and maximal dominant weights for the $\mathfrak{g} = F_4^{(1)}$-module $V(4\Lambda_0)$

x vector	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0)	0	$4\Lambda_0$
(1,2,3,2)	1	$4\Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$
(2,3,4,2)	1	$4\Lambda_0 - \alpha_0$
(2,4,6,3)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,4)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$
(3,5,7,4)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$
(3,6,8,4)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,6,9,5)	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 - \alpha_4$
(3,6,9,6)	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 3\alpha_2 - 3\alpha_3$
(4,6,8,4)	2	$4\Lambda_0 - 2\alpha_0$
(4,7,10,5)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$
(4,7,10,6)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$
(4,8,11,6)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_3$
(4,8,12,6)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(4,8,12,7)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 4\alpha_2 - 4\alpha_3 - \alpha_4$
(4,8,12,8)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 4\alpha_2 - 4\alpha_3$

weight in Table 3.26. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

3.8 Type $E_6^{(1)}$

Let $\mathfrak{g} = E_6^{(1)}$, index set $I = \{0, 1, 2, 3, 4, 5, 6\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

where $x_1 = b_0 - b_1, x_2 = 2b_0 - b_2, x_3 = 2b_0 - b_3, x_4 = 3b_0 - b_4, x_5 = 2b_0 - b_5$, and $x_6 = b_0 - b_6$. That is,

$$\begin{cases} 2x_1 - x_3 & \geq 0 \\ 2x_2 - x_4 & \geq 0 \\ -x_1 + 2x_3 - x_4 & \geq 0 \\ -x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\ -x_4 + 2x_5 - x_6 & \geq 0 \\ -x_5 + 2x_6 & \geq 0 \\ x_2 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \max\{x_1, \frac{x_2}{2}, \frac{x_3}{2}, \frac{x_4}{3}, \frac{x_5}{2}, x_6\}$.

Lemma 3.8.1. *Let $\mathbf{x}$ be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

then $x_2 \neq 1$.

Proof. Suppose, for the sake of contradiction, that $x_2 = 1$. Then from $(\mathring{A}\mathbf{x})_2 \geq 0$, we have $x_4 \leq 2$. As observed above, $x_4 > 0$. Then $x_4 = 1$ or $x_4 = 2$. Combining $(\mathring{A}\mathbf{x})_5 \geq 0$ and $(\mathring{A}\mathbf{x})_6 \geq 0$, we have $3x_6 \geq 2x_5 - x_6 \geq x_4$. Then $x_6 \geq \frac{x_4}{3}$. Using this and $(\mathring{A}\mathbf{x})_5 \geq 0$, we obtain $2x_5 \geq x_4 + \frac{x_4}{3} = \frac{4x_4}{3}$. Then $x_5 \geq \frac{2x_4}{3}$. Similarly, by combining $(\mathring{A}\mathbf{x})_1 \geq 0$ and $(\mathring{A}\mathbf{x})_3 \geq 0$ we obtain $x_1 \geq \frac{x_4}{3}$ and $x_3 \geq \frac{2x_4}{3}$. From $(\mathring{A}\mathbf{x})_4 \geq 0$, we have $2x_4 \geq x_2 + x_3 + x_5 \geq 1 + \frac{2x_4}{3} + \frac{2x_4}{3} = 1 + \frac{4x_4}{3}$. Then $x_4 \geq \frac{3}{2}$ and so $x_4 = 2$. However, the x_i must all be integers. Revisiting $(\mathring{A}\mathbf{x})_4 \geq 0$, we find $4 = 2x_4 \geq x_2 + x_3 + x_5 \geq 1 + 2 + 2$ since $x_3 \geq \frac{4}{3}$ and $x_5 \geq \frac{4}{3}$ must both be integers. This is a contradiction and so $x_2 \neq 1$. $\square$

We will need the following:

- For a positive integer n, $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq j < n$ we have

$$\left\lceil \frac{(n+1)(mn + (n-j))}{n} \right\rceil = (n+1)\left\lceil \frac{mn + (n-j)}{n} \right\rceil - j.$$

Begin with the left hand side.

$$\left\lceil \frac{(n+1)(mn + (n-j))}{n} \right\rceil = \left\lceil \frac{mn^2 + n^2 - jn + mn + n - j}{n} \right\rceil$$
$$= \left\lceil mn + n - j + m + 1 - \frac{j}{n} \right\rceil$$

$$= mn + n - j + m + 1.$$

The right hand side is

$$(n+1)\left\lceil \frac{mn+(n-j)}{n} \right\rceil - j = (n+1)\left\lceil m+1-\frac{j}{n} \right\rceil - j$$
$$= (n+1)(m+1) - j$$
$$= mn + m + n + 1 - j.$$

- For a positive integer n, $m \in \mathbb{Z}_{\geq 0}$, and $0 < j \leq n$,

$$\left\lceil \frac{(n-1)(mn+(n-j))}{n} \right\rceil = (n-1)\left\lceil \frac{mn+(n-j)}{n} \right\rceil - j + 1.$$

Begin with the left hand side.

$$\left\lceil \frac{(n-1)(mn+(n-j))}{n} \right\rceil = \left\lceil \frac{mn^2 - mn + n^2 - n - jn + j}{n} \right\rceil$$
$$= \left\lceil mn - m + n - 1 - j + \frac{j}{n} \right\rceil$$
$$= mn - m + n - 1 - j + 1$$
$$= mn - m + n - j.$$

The right hand side is

$$(n-1)\left\lceil \frac{mn+(n-j)}{n} \right\rceil - j = (n-1)\left\lceil m+1-\frac{j}{n} \right\rceil - j + 1$$
$$= (n-1)(m+1) - j + 1$$
$$= mn - m + n - 1 - j + 1$$
$$= mn - m + n - j.$$

Lemma 3.8.2. *The set of solutions to*

$$\begin{cases} 2x_1 - x_3 & \geq 0 \\ 2x_2 - x_4 & \geq 0 \\ -x_1 + 2x_3 - x_4 & \geq 0 \\ -x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\ -x_4 + 2x_5 - x_6 & \geq 0 \\ -x_5 + 2x_6 & \geq 0 \\ x_2 & \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6 \mid 2 \leq x_2 \leq k, 3\left\lceil \frac{x_2}{2} \right\rceil \leq x_4 \leq 2x_2, \left\lceil \frac{2x_4}{3} \right\rceil \leq x_3,$$

$$\left\lceil \frac{2x_4}{3} \right\rceil \leq x_5, x_3 + x_5 \leq 2x_4 - x_2, \left\lceil \frac{x_3}{2} \right\rceil \leq x_1 \leq 2x_3 - x_4,$$

$$and \left\lceil \frac{x_5}{2} \right\rceil \leq x_6 \leq 2x_5 - x_4\}.$$

Proof. As stated earlier, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$. Therefore, $x_2 = 0$, which implies $\mathbf{x} = 0$, or $x_2 \geq 1$. By Lemma 3.8.1, $x_2 \neq 1$. Therefore, $x_2 = 0$ or $2 \leq x_2 \leq k$.

Now, fix x_2 such that $2 \leq x_2 \leq k$. First, notice that by $(\mathring{A}\mathbf{x})_2 \geq 0$, $x_4 \leq 2x_2$. From $(\mathring{A}\mathbf{x})_4 \geq 0$, $(\mathring{A}\mathbf{x})_3 \geq 0$, and $(\mathring{A}\mathbf{x})_5 \geq 0$, we have $2x_4 \geq x_2 + x_3 + x_5 \geq x_2 + \frac{x_1}{2} + \frac{x_4}{2} + \frac{x_4}{2} + \frac{x_6}{2}$. Then $x_4 \geq x_2 + \frac{x_1}{2} + \frac{x_6}{2}$. Notice that $x_1 \geq \frac{x_3}{2} \geq \frac{x_1}{4} + \frac{x_4}{4}$ and so $x_1 \geq \frac{x_4}{3}$ from the first and third inequalities. Similarly, using the fifth and sixth inequalities we obtain $x_6 \geq \frac{x_4}{3}$. Then returning to the lower bound for x_4, we have $x_4 \geq x_2 + 2\frac{x_4}{6}$. Rearranging, we obtain $x_4 \geq \frac{3x_2}{2}$. We will return to this to determine the integer lower bound for x_4.

First, let's consider the lower bounds for x_3 and x_5. The third inequality gives that $x_3 \geq \frac{x_1+x_4}{2}$. Combining this with the first inequality, we have $x_3 \geq \frac{x_4}{2} + \frac{x_3}{4}$. Then $x_3 \geq \frac{2x_4}{3}$. Similarly, using the fifth and sixth inequalities, we have $x_5 \geq \frac{2x_4}{3}$. However, both x_3 and x_5 are integers. As we noted above, $\left\lceil \frac{2x_4}{3} \right\rceil = 2\left\lceil \frac{x_4}{3} \right\rceil$ for all x_4 except $x_4 = 3n + 1$. We now show that the smaller of these two, $\left\lceil \frac{2x_4}{3} \right\rceil$ when $x_4 = 3n + 1$ satisfies the inequalities and is therefore the appropriate lower bound for both x_3 and x_5. Since this is a lower bound, we only need to check the third and fifth inequalities. Checking the third inequality (and the fifth is similar), we have $x_3 = 2n + 1$ since $x_4 = 3n + 1$. Then $2x_3 - x_4 = 2(2n + 1) - 3n + 1 \geq x_1 \geq \frac{x_3}{2} = \frac{2n+1}{2}$. This becomes $n + 1 \geq n + \frac{1}{2}$. Since this is true, this is the appropriate lower bound for x_3 and similarly for x_5.

Recall that x_4 must be an integer and we noted above that the placement of the ceiling matters when x_2 is odd. We now show that when x_2 is odd, $x_4 \neq \left\lceil \frac{3x_2}{2} \right\rceil$, giving the lower bound as indicated in the statement. Assume that $x_2 = 2n + 1$ and, for the sake of contradiction, $x_4 = \left\lceil \frac{3x_2}{2} \right\rceil = \left\lceil \frac{3(2n+1)}{2} \right\rceil = \left\lceil \frac{6n+3}{2} \right\rceil = 3n + 2$. Then from the note above and the lower bounds we just found, $x_3, x_5 \geq 2n + 2$. Then the fourth inequality becomes $-(2n + 1) - (2n + 2) + 2(3n + 2) - (2n + 2) = -1 \geq 0$. Since this is false, $x_4 \neq \left\lceil \frac{3x_2}{2} \right\rceil$. If, instead, $x_4 = 3\left\lceil \frac{x_2}{2} \right\rceil = 3n + 3$, then $x_3, x_5 \geq 2n + 2$ and the fourth inequality is $-(2n + 1) - (2n + 2) + 2(3n + 3) - (2n + 2) = 1 \geq 0$, and so this is the appropriate lower bound for x_4.

The upper bound for the sum of x_3 and x_5 is obtained by rearranging the fourth inequality. The lower bound for both x_1 and x_6 come from the first and sixth inequality and the fact that they must both be integers. The upper bounds for both these values are from rearranging the third and fifth inequality respectively.

This describes all possible solutions. $\qquad \square$

Theorem 3.8.3. *Let $\Lambda = k\Lambda_0, k \geq 2$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (3l - x_4)\alpha_4 - (2l - x_5)\alpha_5 - (l - x_6)\alpha_6$ where*

- $2 \leq x_2 \leq k$,

- $3\left\lceil \frac{x_2}{2} \right\rceil \leq x_4 \leq 2x_2$,

- $\left\lceil \frac{2x_4}{3} \right\rceil \leq x_3$,

- $\left\lceil \frac{2x_4}{3} \right\rceil \leq x_5$,

- $x_3 + x_5 \leq 2x_4 - x_2$,

- $\left\lceil \frac{x_3}{2} \right\rceil \leq x_1 \leq 2x_3 - x_4$,

- $\left\lceil \frac{x_5}{2} \right\rceil \leq x_6 \leq 2x_5 - x_4$,

- $l = \max\{x_1, x_6\}$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = b_0 - b_1, x_2 = 2b_0 - b_2, x_3 = 2b_0 - b_3, x_4 = 3b_0 - b_4, x_5 = 2b_0 - b_5$ and $x_6 = b_0 - b_6$. Then $(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (b_0, b_0 - x_1, 2b_0 - x_2, 2b_0 - x_3, 3b_0 - x_4, 2b_0 - x_5, b_0 - x_6)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 6$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$2b_0 - x_3 \geq 0$$
$$3b_0 - x_4 \geq 0$$
$$2b_0 - x_5 \geq 0$$
$$b_0 - x_6 \geq 0$$

Then $b_0 \geq \max\{x_1, \frac{x_2}{2}, \frac{x_3}{2}, \frac{x_4}{3}, \frac{x_5}{2}, x_6\}$. From the first and sixth inequalities we know that $x_1 \geq \frac{x_3}{2}$ and $x_6 \geq \frac{x_5}{2}$. Combining the first and third inequalities, we have $2x_1 \geq x_3 \geq \frac{x_1+x_4}{2}$ and so $x_1 \geq \frac{x_4}{3}$. Similarly, using the fifth and sixth inequalities, $x_6 \geq \frac{x_4}{3}$. Finally, we can show that $\max\{x_1, x_6\} \geq \frac{x_2}{2}$. Combining the third and fifth inequalities, we have $2x_3 - x_1 + 2x_5 - x_6 \geq 2x_4$. Since $2x_1 \geq x_3$ and $2x_6 \geq x_5$, we have $3x_1 + 3x_6 \geq 2x_4$. Substituting the third and fifth inequalities into the fourth, we have $2x_4 \geq \frac{x_1+x_4}{2} + \frac{x_4+x_6}{2} + x_2$ and so $x_4 \geq \frac{x_1}{2} + \frac{x_6}{2} + x_2$. With these two results we now have $\frac{3x_1+3x_6}{2} \geq \frac{x_1}{2} + \frac{x_6}{2} + x_2$. Then $x_1 + x_6 \geq x_2$. Then we have that $2\max\{x_1, x_6\} \geq x_1 + x_6 \geq x_2$ and so $\max\{x_1, x_6\} \geq \frac{x_2}{2}$.

Then it suffices to say $b_0 \geq \max\{x_1, x_6\}$. Then let $l = \max\{x_1, x_6\}$ and we claim that $b_0 = l$. Showing this is similar to the method used in the proof of Theorem 3.3.3. We obtain $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (3l - x_4)\alpha_4 - (2l - x_5)\alpha_4 - (l - x_6)\alpha_6$. Combining this with our solutions for the x_i from Lemma 3.8.2, we have the pattern given above. $\qquad\square$

Table 3.27: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(1)}$-module $V(2\Lambda_0)$

(x_2, x_4)	(x_1, x_3, x_5, x_6)	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0)	(0,0,0,0)	0	$2\Lambda_0$
(2,3)	(1,2,2,1)	1	$2\Lambda_0 - \alpha_0$
(2,4)	(2,3,3,2)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$

Table 3.28: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(1)}$-module $V(3\Lambda_0)$

(x_2, x_4)	(x_1, x_3, x_5, x_6)	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0)	(0,0,0,0)	0	$3\Lambda_0$
(2,3)	(1,2,2,1)	1	$3\Lambda_0 - \alpha_0$
(2,4)	(2,3,3,2)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$
(3,6)	(2,4,4,2)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
(3,6)	(2,4,5,3)	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - 3\alpha_4 - \alpha_5$
(3,6)	(2,4,5,4)	4	$3\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 5\alpha_2 - 4\alpha_3 - 6\alpha_4 - 3\alpha_5$
(3,6)	(3,5,4,2)	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_2 - \alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
(3,6)	(4,5,4,2)	4	$3\Lambda_0 - 4\alpha_0 - 5\alpha_2 - 3\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2, x_3, x_4, x_5, x_6) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_2 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Example 3.8.4. $E_6^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.8.3, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (3l - x_4)\alpha_4 - (2l - x_5)\alpha_5 - (l - x_6)\alpha_6.$$

We list the **x** vectors in a particular way to show the pattern and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.27.

Example 3.8.5. $E_6^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.8.3, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (3l - x_4)\alpha_4 - (2l - x_5)\alpha_5 - (l - x_6)\alpha_6.$$

We list the **x** vectors in a particular way to show the pattern and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.28. Notice that we can collect the same **x** vectors from the case $V(2\Lambda_0)$ since, as we said earlier, $X_2 \subset X_3$.

Example 3.8.6. $E_6^{(1)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.8.3, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (3l - x_4)\alpha_4 - (2l - x_5)\alpha_5 - (l - x_6)\alpha_6.$$

Table 3.29: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(1)}$-module $V(4\Lambda_0)$

(x_2, x_4)	(x_1, x_3, x_5, x_6)	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0)	(0,0,0,0)	0	$4\Lambda_0$
(2,3)	(1,2,2,1)	1	$4\Lambda_0 - \alpha_0$
(2,4)	(2,3,3,2)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$
(3,6)	(2,4,4,2)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
(3,6)	(2,4,5,3)	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - 3\alpha_4 - \alpha_5$
(3,6)	(2,4,5,4)	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 5\alpha_2 - 4\alpha_3 - 6\alpha_4 - 3\alpha_5$
(3,6)	(3,5,4,2)	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_2 - \alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
(3,6)	(4,5,4,2)	4	$4\Lambda_0 - 4\alpha_0 - 5\alpha_2 - 3\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$
(4,6)	(2,4,4,2)	2	$4\Lambda_0 - 2\alpha_0$
(4,7)	(3,5,5,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$
(4,8)	(3,6,6,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_4$
(4,8)	(3,6,6,4)	4	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 2\alpha_3 - 4\alpha_4 - 2\alpha_5$
(4,8)	(4,6,6,3)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 2\alpha_3 - 4\alpha_4 - 2\alpha_5 - \alpha_6$
(4,8)	(4,6,6,4)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 2\alpha_3 - 4\alpha_4 - 2\alpha_5$

We list the **x** vectors in a particular way to show the pattern and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.29. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

3.9 Type $E_7^{(1)}$

Let $\mathfrak{g} = E_7^{(1)}$, index set $I = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and Cartan matrix

$$
A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
$$

The defining inequalities are equivalent to

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
x_1 \leq k
\end{cases}
$$

where $x_1 = 2b_0 - b_1, x_2 = 2b_0 - b_2, x_3 = 3b_0 - b_3, x_4 = 4b_0 - b_4, x_5 = 3b_0 - b_5, x_6 = 2b_0 - b_6$, and $x_7 = b_0 - b_7$. That is,

$$
\begin{cases}
2x_1 - x_3 & \geq 0 \\
2x_2 - x_4 & \geq 0 \\
-x_1 + 2x_3 - x_4 & \geq 0 \\
-x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\
-x_4 + 2x_5 - x_6 & \geq 0 \\
-x_5 + 2x_6 - x_7 & \geq 0 \\
-x_6 + 2x_7 & \geq 0 \\
x_1 & \leq k
\end{cases}
$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \frac{x_5}{3}, \frac{x_6}{2}, x_7\}$.

In addition to the ceiling facts we included for $E_6^{(1)}$, we need the following equality:

- For a positive integer n, $m \in \mathbb{Z}_{\geq 0}$, and $0 < j \leq n$,

$$
\left\lceil \frac{(n+1)(mn + (n-j))}{n} \right\rceil = (n+1) \left\lceil \frac{mn + (n-j)}{n} \right\rceil + (n-j).
$$

Begin with the left hand side:

$$
\left\lceil \frac{(n+1)(mn + (n-j))}{n} \right\rceil = \left\lceil mn + n - j + m + 1 - \frac{j}{n} \right\rceil
$$
$$
= mn + n - j + m
$$
$$
= (n+1)m + n - j.
$$

The right hand side is

$$
(n+1) \left\lceil \frac{mn + (n-j)}{n} \right\rceil + (n-j) = (n+1) \left\lceil m + 1 - \frac{j}{n} \right\rceil + (n-j)
$$
$$
= (n+1)m + n - j.
$$

Lemma 3.9.1. *Let* $\mathbf{x}$ *be a solution to*

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
x_1 \leq k
\end{cases}
$$

then $x_3 \geq 3 \left\lceil \frac{x_1}{2} \right\rceil$.

Proof. First, we add the third, fourth, and fifth inequalities and obtain $-x_1 - x_2 - x_6 + x_3 + x_5 \geq 0$

and so $x_3 + x_5 \geq x_1 + x_2 + x_6$. Similarly, we add the fifth, sixth, and seventh and obtain $-x_4 + x_5 + x_7 \geq 0$ and so $x_5 + x_7 \geq x_4$. Notice that combining this with the sixth inequality, we obtain $x_6 \geq \frac{x_4}{2}$. Now combining the fourth and second inequalities, we have $2x_4 \geq x_2 + x_3 + x_5 \geq \frac{x_4}{2} + x_3 + x_5$ and so $\frac{3x_4}{2} \geq x_3 + x_5$. Using our first two results and combining them with the second inequality again we have $\frac{3x_4}{2} \geq x_3 + x_5 \geq x_1 + x_2 + x_6 \geq x_1 + x_2 + \frac{x_4}{2} \geq x_1 + \frac{x_4}{2} + \frac{x_4}{2}$ and so $\frac{x_4}{2} \geq x_1$. Then visiting the third inequality, we have $x_3 \geq \frac{x_4}{2} + \frac{x_1}{2} \geq x_1 + \frac{x_1}{2} = \frac{3x_1}{2}$.

Since x_3 must be an integer, we consider the case when x_1 is odd. From our ceiling notes, we have to show the appropriate lower bound for x_3 is $3\left\lceil \frac{x_1}{2} \right\rceil$ in this case. Note that if $x_1 = 2n+1$ then from above $x_3 \geq \frac{3(2n+1)}{2}$. Assume, for the sake of contradiction, that $x_3 \geq \left\lceil \frac{3(2n+1)}{2} \right\rceil = 3n+2$. Then there can be a solution where $x_3 = 3n+2$. From the fourth and second inequality, we have $2x_4 \geq x_2 + x_3 + x_5 \geq \frac{x_4}{2} + x_3 + x_5$. From the fifth inequality this becomes $2x_4 \geq \frac{x_4}{2} + x_3 + \frac{x_4}{2} + \frac{x_6}{2}$. From the previous paragraph we know $x_6 \geq \frac{x_4}{2}$ and so $2x_4 \geq x_4 + x_3 + \frac{x_4}{4}$. Rearranging, we obtain $x_4 \geq \frac{4x_3}{3}$. Now, if $x_3 = 3n+2$ then we have $\frac{4(3n+2)}{3} \leq x_4 \leq 2(3n+2) - (2n+1)$ by combining our most recent result with the third inequality. This simplifies to $4n + \frac{8}{3} \leq x_4 \leq 4n+3$ and so $x_4 = 4n+3$ in this case. Looking at the fifth inequality and combining with $x_6 \geq \frac{x_4}{2}$ we have $x_5 \geq \frac{x_4}{2} + \frac{x_6}{2} \geq \frac{3x_4}{4}$. Then $x_2 \leq 2x_4 - x_3 - x_5 \leq 2x_4 - x_3 - \frac{3x_4}{4} = \frac{5x_4}{4} - x_3$ from the fourth inequality. Returning to the case of $x_4 = 4n+3$, we have $\frac{x_4}{2} \leq x_2 \leq \frac{5x_4}{4} - x_3$ which becomes $\frac{4n+3}{2} \leq x_2 \leq \frac{5(4n+3)}{4} - (3n+2)$. Simplifying, this is $2n + \frac{3}{2} \leq x_2 \leq 2n + \frac{7}{4}$. Since x_2 must be an integer, this is impossible. Then $x_3 \neq \left\lceil \frac{3(2n+1)}{2} \right\rceil = 3n+2$.

Note that it is possible for $x_3 = 3\left\lceil \frac{2n+1}{2} \right\rceil$. This means $x_3 = 3n+3$. Following our above argument, this means $\frac{4(3n+3)}{3} \leq x_4 \leq 2(3n+3) - (2n+1)$, which simplifies to $4n+4 \leq x_4 \leq 4n+5$. In the case of $x_4 = 4n+4$ then $\frac{4n+4}{2} \leq x_2 \leq \frac{5(4n+4)}{4} - (3n+3)$ simplifies to $2n+2 \leq x_2 \leq 2n+2$ and so $x_2 = 2n+2$. In the case of $x_4 = 4n+5$, then $\frac{4n+5}{2} \leq x_2 \leq \frac{5(4n+5)}{4} - (3n+3)$ simplifies to $2n + \frac{5}{2} \leq x_2 \leq 2n + \frac{13}{4}$ and so x_2 can be $2n+3$. $\qquad \square$

Lemma 3.9.2. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

then $x_1 \neq 1$.

Proof. Suppose, for the sake of contradiction, that $x_1 = 1$. Then from $(\mathring{A}\mathbf{x})_1 \geq 0$, we have $x_3 \leq 2$ and from Lemma 3.9.1, we have $x_3 \geq 3\left\lceil \frac{1}{2} \right\rceil = 3$. Since this is impossible, $x_1 \neq 1$. $\qquad \square$

Lemma 3.9.3. *The set of solutions to*

$$\begin{cases} 2x_1 - x_3 & \geq 0 \\ 2x_2 - x_4 & \geq 0 \\ -x_1 + 2x_3 - x_4 & \geq 0 \\ -x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\ -x_4 + 2x_5 - x_6 & \geq 0 \\ -x_5 + 2x_6 - x_7 & \geq 0 \\ -x_6 + 2x_7 & \geq 0 \\ x_1 & \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_6) \in \mathbb{Z}_{>0}^7 \mid 2 \leq x_1 \leq k, 3\left\lceil \frac{x_1}{2} \right\rceil \leq x_3 \leq 2x_1,$$

$$w \leq x_4 \leq 2x_3 - x_1, \left\lceil \frac{x_4}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{5x_4}{4} \right\rfloor - x_3, \left\lceil \frac{3x_4}{4} \right\rceil \leq x_5 \leq 2x_4 - x_2 - x_3,$$

$$\left\lceil \frac{2x_5}{3} \right\rceil \leq x_6 \leq 2x_5 - x_4, \ \ and \ \left\lceil \frac{x_6}{2} \right\rceil \leq x_7 \leq 2x_6 - x_5\}$$

where $w = \left\lceil \frac{4x_3}{3} \right\rceil + 1$ if $x_3 = 2 \mod 3$ and $w = \left\lceil \frac{4x_3}{3} \right\rceil$ otherwise.

Proof. As stated earlier, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = \mathbf{0}$. Therefore, $x_1 = 0$, which implies $\mathbf{x} = 0$, or $x_1 \geq 1$. By Lemma 3.9.2, $x_1 \neq 1$. Therefore, $x_1 = 0$ or $2 \leq x_1 \leq k$.

Now, fix x_1 such that $2 \leq x_1 \leq k$. We found the lower bound for x_3 in Lemma 3.9.1 and the upper bound comes from the first inequality. The remaining bounds fall out from the inequalities and are best discovered from the end.

The bounds for x_7 come from the sixth and seventh inequalities and the fact that x_7 must be an integer.

The upper bound for x_6 comes from the fifth inequality while the lower bound combines the sixth and seventh: $x_6 \geq \frac{x_5}{2} + \frac{x_7}{2} \geq \frac{x_5}{2} + \frac{x_6}{4}$ and so $x_6 \geq \frac{2x_5}{3}$. Recall that x_6 must be an integer. As noted earlier, $\left\lceil \frac{2x_5}{3} \right\rceil = 2\left\lceil \frac{x_5}{3} \right\rceil$ except when $x_5 = 3n + 1$. We claim that in that case, the smaller of the two options is the true lower bound for x_6. Since this is a lower bound, we need only check the sixth inequality. We have that $x_6 \geq \left\lceil \frac{2x_5}{3} \right\rceil = \left\lceil \frac{2(3n+1)}{3} \right\rceil = 2n + 1$. Then $x_7 \geq \left\lceil \frac{2n+1}{2} \right\rceil = n + 1$ and $-x_5 + 2x_6 = -(3n+1) + 2(2n+1) = n + 1 \geq x_7$. So this is a possible value for x_6 with $x_7 = n + 1$.

The upper bound for x_5 comes from the fourth inequality and the lower bound combines the lower bound for x_6 with the fifth inequality: $x_5 \geq \frac{x_4}{2} + \frac{x_6}{2} \geq \frac{x_4}{2} + \frac{x_5}{3}$. This implies $x_5 \geq \frac{3x_4}{4}$. Recall that x_5 must be an integer and from above, $\left\lceil \frac{3x_4}{4} \right\rceil < 3\left\lceil \frac{x_4}{4} \right\rceil$ when $x_4 = 4n + 1$ or $x_4 = 4n + 2$ for some $n \in \mathbb{Z}_{\geq 0}$. Then we need to check that the lower of the two, $\left\lceil \frac{3x_4}{4} \right\rceil$ satisfies the inequalities in both of these cases.

Case 1: $x_4 = 4n + 1$. Then $x_5 \geq \left\lceil \frac{3x_4}{4} \right\rceil = 3n + 1$. Since this is a lower bound, we need only check the fifth inequality: $-(4n + 1) + 2(3n + 1) \geq x_6$ gives $2n + 1 \geq x_6$. From our bounds for x_6 we have $\left\lceil \frac{2(3n+1)}{3} \right\rceil = 2n + 1 \leq x_6$. Then $x_6 = 2n + 1$ and the inequalities are satisfied.

Case 2: $x_4 = 4n + 2$. Then $x_5 \geq \left\lceil \frac{3x_4}{4} \right\rceil = 3n + 2$. Since this is a lower bound, we need only check the fifth inequality: $-(4n + 2) + 2(3n + 2) \geq x_6$ gives $2n + 2 \geq x_6$. From our bounds for x_6 we have $\left\lceil \frac{2(3n+2)}{3} \right\rceil = 2n + 2 \leq x_6$. Then $x_6 = 2n + 2$ and the inequalities are satisfied.

Therefore, the lower bound for x_5 is as given in the statement of the lemma.

The lower bound for x_2 comes from the second inequality and the fact that x_2 must be an integer. We use the fourth inequality and the lower bound for x_5 to obtain the upper bound for x_2: $x_2 \leq 2x_4 - x_3 - x_5 \leq 2x_4 - x_3 - \frac{3x_4}{4}$. Once more, we must determine the correct upper bound since x_2 must be an integer. Since this is an upper bound and we claim that the larger of the two options, $\left\lfloor \frac{5x_4}{4} \right\rfloor$, is the correct choice, we need only show that the bounds for x_5 (and so the fourth inequality) are satisfied in this case. We need to check for each of the cases when $\left\lfloor \frac{5x_4}{4} \right\rfloor \neq 5 \left\lfloor \frac{x_4}{4} \right\rfloor$.

Case 1: $x_4 = 4n + 1$. Then $x_2 \leq \left\lfloor \frac{5x_4}{4} \right\rfloor = 5n + 1 - x_3$. Then when $x_2 = 5n + 1 - x_3$ we have $3n + 1 \leq x_5 \leq 2(4n + 1) - (5n + 1 - x_3) - x_3 = 3n + 1$. Then $x_5 = 3n + 1$ and the bounds are satisfied.

Case 2: $x_4 = 4n + 2$. Then $x_2 \leq \left\lfloor \frac{5x_4}{4} \right\rfloor = 5n + 2 - x_3$. Then when $x_2 = 5n + 2 - x_3$ we have $3n + 2 \leq x_5 \leq 2(4n + 2) - (5n + 2 - x_3) - x_3 = 3n + 2$. Then $x_5 = 3n + 2$ and the bounds are satisfied.

Case 3: $x_4 = 4n + 3$. Then $x_2 \leq \left\lfloor \frac{5x_4}{4} \right\rfloor = 5n + 3 - x_3$. Then when $x_2 = 5n + 3 - x_3$ we have $3n + 3 \leq x_5 \leq 2(4n + 3) - (5n + 3 - x_3) - x_3 = 3n + 3$. Then $x_5 = 3n + 3$ and the bounds are satisfied.

Therefore, the upper bound for x_2 is as given.

The upper bound for x_4 comes directly from the third inequality. The lower bound combines the fourth and second inequality with the lower bound for x_5: $x_4 \geq \frac{x_2}{2} + \frac{x_3}{2} + \frac{x_5}{2} \geq \frac{x_4}{4} + \frac{x_3}{2} + \frac{3x_4}{8}$, giving $x_4 \geq \frac{4x_3}{3}$. Again, since we need x_4 to be an integer and this is a lower bound, we need to check the least value x_4 takes on based on the placement of the ceiling. Unfortunately, as stated in Lemma 3.9.1, this depends on the remainder of x_3 when divided by 3. We check the appropriate lower bound based on the bounds for x_2.

Case 1: If $x_3 = 3n$ for $n \in \mathbb{Z}_{\geq 0}$, $\left\lceil \frac{4x_3}{3} \right\rceil = 4 \left\lceil \frac{x_3}{3} \right\rceil$, so we need not check this. We can then say $x_4 \geq \left\lceil \frac{4x_3}{3} \right\rceil$ in this case.

Case 2: If $x_3 = 3n + 1$, then if $x_4 = \left\lceil \frac{4x_3}{3} \right\rceil = 4n + 2$ we have $\left\lceil \frac{4n+2}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{5(4n+2)}{4} \right\rfloor - (3n + 1)$ which simplifies to $2n + 1 \leq x_2 \leq 2n + 1$. Then $x_2 = 2n + 1$ and the bounds are satisfied. In this case, $x_4 \geq \left\lceil \frac{4x_3}{3} \right\rceil$.

Case 3: If $x_3 = 3n + 2$ and if $x_4 = \left\lceil \frac{4x_3}{3} \right\rceil = 4n + 3$, we have $\left\lceil \frac{4n+3}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{5(4n+3)}{4} \right\rfloor - (3n + 2)$ which simplifies to $2n + 2 \leq x_2 \leq 2n + 1$. Then there is no possible value for x_2. However, if we have $x_4 = 4 \left\lceil \frac{3n+2}{3} \right\rceil = 4n + 4$ then $\left\lceil \frac{4n+4}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{5(4n+4)}{4} \right\rfloor - (3n + 2)$, which simplifies to $2n + 2 \leq x_2 \leq 2n + 3$. Then $x_2 = 2n + 2$ or $x_2 = 2n + 3$. Note that in this case, $x_4 = \left\lceil \frac{4x_3}{3} \right\rceil + 1$

since $x_3 = 2 \mod 3$, as stated in the claim. $\qquad \square$

Theorem 3.9.4. *Let* $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (3l - x_3)\alpha_3 - (4l - x_4)\alpha_4 - (3l - x_5)\alpha_5 - (2l - x_6)\alpha_6 + (l - x_7)\alpha_7$ *where*

- $2 \leq x_1 \leq k$,

- $3\lceil \frac{x_1}{2} \rceil \leq x_3 \leq 2x_1$,

- $w = \lceil \frac{4x_3}{3} \rceil + 1$ *if* $x_3 = 2 \mod 3$ *and* $w = \lceil \frac{4x_3}{3} \rceil$ *otherwise,*

- $w \leq x_4 \leq 2x_3 - x_1$,

- $\lceil \frac{x_4}{2} \rceil \leq x_2 \leq \lfloor \frac{5x_4}{4} \rfloor - x_3$,

- $\lceil \frac{3x_4}{4} \rceil \leq x_5 \leq 2x_4 - x_2 - x_3$,

- $\lceil \frac{2x_5}{3} \rceil \leq x_6 \leq 2x_5 - x_4$,

- $\lceil \frac{x_6}{2} \rceil \leq x_7 \leq 2x_6 - x_5 \}$,

- $l = \max\{\frac{x_2}{2}, x_7\}$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1, x_2 = 2b_0 - b_2, x_3 = 3b_0 - b_3, x_4 = 4b_0 - b_4, x_5 = 3b_0 - b_5, x_6 = 2b_0 - b_6$ and $x_7 = b_0 - b_7$. Then $(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (b_0, 2b_0 - x_1, 2b_0 - x_2, 3b_0 - x_3, 4b_0 - x_4, 3b_0 - x_5, 2b_0 - x_6, b_0 - x_7)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 7$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$3b_0 - x_3 \geq 0$$
$$4b_0 - x_4 \geq 0$$
$$3b_0 - x_5 \geq 0$$
$$2b_0 - x_6 \geq 0$$
$$b_0 - x_7 \geq 0$$

Then $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \frac{x_5}{3}, \frac{x_6}{2}, x_7\}$. We use the inequalities from $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$ to eliminate some of these options. From the seventh inequality, we know that $x_7 \geq \frac{x_6}{2}$. Combining the seventh and sixth inequalities we have $4x_7 \geq 2x_6 \geq x_5 + x_7$ and so $x_7 \geq \frac{x_5}{3}$. We add the fifth, sixth, and seventh inequalities to obtain $x_5 + x_7 - x_4 \geq 0$ and using this with the last sentence we have $4x_7 \geq x_5 + x_7 \geq x_4$, giving $x_7 \geq \frac{x_4}{4}$. Using the lower bound (though perhaps

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Table 3.30: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_7^{(1)}$-module $V(2\Lambda_0)$

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0)	0	$2\Lambda_0$
(2,2,3,4,3,2,1)	1	$2\Lambda_0 - \alpha_0$
(2,3,4,6,5,4,2)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(2,3,4,6,5,4,3)	3	$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 5\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$

not the greatest lower bound) for x_4 we have $x_7 \geq \frac{x_4}{4} \geq \frac{x_3}{3}$. Finally, using the lower bound for x_3 we have $x_7 \geq \frac{x_3}{3} \geq \frac{x_1}{2}$.

Notice that there is an instance when $\left\lceil \frac{x_2}{2} \right\rceil > x_7$. If $x_3 = 2 \mod 3$ then $x_3 = 3n+2$ for some $n \in \mathbb{Z}_{\geq 0}$. Then by the claim, $x_4 = 4n + 4$ and we can have $x_2 = \left\lfloor \frac{5x_4}{4} \right\rfloor - x_3 = 2n + 3$. Allowing $x_5, x_6,$ and x_7 to take their lower bounds, we get $x_5 = \left\lceil \frac{3x_4}{4} \right\rceil = 3n + 3, x_6 = \left\lceil \frac{2x_5}{3} \right\rceil = 2n + 2$, and $x_7 = \left\lceil \frac{x_6}{2} \right\rceil = n + 1$. Then $\left\lceil \frac{x_2}{2} \right\rceil = n + 2 > x_7$.

Then it suffices to say $b_0 \geq \max\{\left\lceil \frac{x_2}{2} \right\rceil, x_7\}$. Then let $l = \max\{\left\lceil \frac{x_2}{2} \right\rceil, x_7\}$ and we claim that $b_0 = l$. Showing this is similar to the method used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (2l-x_1)\alpha_1 - (2l-x_2)\alpha_2 - (3l-x_3)\alpha_3 - (4l-x_4)\alpha_4 - (3l-x_5)\alpha_4 - (2l-x_6)\alpha_6 - (l-x_7)\alpha_7$. Combining this with our solutions for the x_i from Lemma 3.9.3, we have the pattern given above. $\qquad \square$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid \mathring{A}\mathbf{x} > 0, 2 \leq x_1 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Example 3.9.5. $E_7^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.9.4, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (3l - x_3)\alpha_3 - (4l - x_4)\alpha_4$$
$$- (3l - x_5)\alpha_5 - (2l - x_6)\alpha_6 - (l - x_7)\alpha_7.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.30.

Example 3.9.6. $E_7^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.9.4, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (3l - x_3)\alpha_3 - (4l - x_4)\alpha_4$$
$$- (3l - x_5)\alpha_5 - (2l - x_6)\alpha_6 - (l - x_7)\alpha_7.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.31. Notice that we can collect the same **x** vectors from the case $V(2\Lambda_0)$ since, as we said earlier, $X_2 \subset X_3$.

Table 3.31: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_7^{(1)}$-module $V(3\Lambda_0)$

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0)	0	$3\Lambda_0$
(2,2,3,4,3,2,1)	1	$3\Lambda_0 - \alpha_0$
(2,3,4,6,5,4,2)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(2,3,4,6,5,4,3)	3	$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 5\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$
(3,4,6,8,6,4,2)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,5,6,9,7,5,3)	3	$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2 - 3\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$

Table 3.32: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_7^{(1)}$-module $V(4\Lambda_0)$

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0)	0	$4\Lambda_0$
(2,2,3,4,3,2,1)	1	$4\Lambda_0 - \alpha_0$
(2,3,4,6,5,4,2)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(2,3,4,6,5,4,3)	3	$4\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 3\alpha_2 - 5\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$
(3,4,6,8,6,4,2)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(3,5,6,9,7,5,3)	3	$4\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2 - 3\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$
(4,4,6,8,6,4,2)	2	$4\Lambda_0 - 2\alpha_0$
(4,5,7,10,8,6,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(4,5,7,10,8,6,4)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 3\alpha_2 - 5\alpha_3 - 6\alpha_4 - 4\alpha_5 - 2\alpha_6$
(4,6,8,12,9,6,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_3$
(4,6,8,12,10,7,4)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 2\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5 - \alpha_6$
(4,6,8,12,10,8,4)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 2\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5$
(4,6,8,12,10,8,5)	5	$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 4\alpha_2 - 7\alpha_3 - 8\alpha_4 - 5\alpha_5 - 2\alpha_6$
(4,6,8,12,10,8,6)	6	$4\Lambda_0 - 6\alpha_0 - 8\alpha_1 - 6\alpha_2 - 10\alpha_3 - 12\alpha_4 - 8\alpha_5 - 4\alpha_6$
(4,7,8,12,9,6,3)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - \alpha_2 - 4\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7$

Example 3.9.7. $E_7^{(1)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.9.4, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - (3l - x_3)\alpha_3 - (4l - x_4)\alpha_4$$
$$- (3l - x_5)\alpha_5 - (2l - x_6)\alpha_6 - (l - x_7)\alpha_7.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.32. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

3.10 Type $E_8^{(1)}$

Let $\mathfrak{g} = E_8^{(1)}$, index set $I = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_8 \leq k \end{cases}$$

where $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 4b_0 - b_3, x_4 = 6b_0 - b_4, x_5 = 5b_0 - b_5, x_6 = 4b_0 - b_6, x_7 = 3b_0 - b_7$ and $x_8 = 2b_0 - b_8$. That is,

$$\begin{cases} 2x_1 - x_3 & \geq 0 \\ 2x_2 - x_4 & \geq 0 \\ -x_1 + 2x_3 - x_4 & \geq 0 \\ -x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\ -x_4 + 2x_5 - x_6 & \geq 0 \\ -x_5 + 2x_6 - x_7 & \geq 0 \\ -x_6 + 2x_7 - x_8 & \geq 0 \\ -x_7 + 2x_8 & \geq 0 \\ x_8 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{6}, \frac{x_5}{5}, \frac{x_6}{4}, \frac{x_7}{3}, \frac{x_8}{2}\}$.

We will once more require the ceiling and floor facts we obtained in the discussion for types $E_6^{(1)}$ and $E_7^{(1)}$.

Lemma 3.10.1. *The set of solutions to*

$$
\begin{cases}
2x_1 - x_3 & \geq 0 \\
2x_2 - x_4 & \geq 0 \\
-x_1 + 2x_3 - x_4 & \geq 0 \\
-x_2 - x_3 + 2x_4 - x_5 & \geq 0 \\
-x_4 + 2x_5 - x_6 & \geq 0 \\
-x_5 + 2x_6 - x_7 & \geq 0 \\
-x_6 + 2x_7 - x_8 & \geq 0 \\
-x_7 + 2x_8 & \geq 0 \\
x_8 & \leq k
\end{cases}
$$

is

$$
\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8_{>0} \mid 2 \leq x_8 \leq k, 3\left\lceil \frac{x_8}{2} \right\rceil \leq x_7 \leq 2x_8,
$$

$$
a \leq x_6 \leq 2x_7 - x_8, b \leq x_5 \leq 2x_6 - x_7, c \leq x_4 \leq 2x_5 - x_6, \left\lceil \frac{x_4}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{4x_4}{3} \right\rfloor - x_5,
$$

$$
\left\lceil \frac{2x_4}{3} \right\rceil \leq x_3 \leq 2x_4 - x_2 - x_5 \text{ and } \left\lceil \frac{x_3}{2} \right\rceil \leq x_1 \leq 2x_3 - x_4\}
$$

where

$$
a = \begin{cases} 4\left\lceil \frac{x_7}{3} \right\rceil & \text{if } x_7 = 3n + 2 \\ \left\lceil \frac{4x_7}{3} \right\rceil & \text{otherwise} \end{cases}, \quad
b = \begin{cases} 5\left\lceil \frac{x_6}{4} \right\rceil & \text{if } x_6 = 4n + 3 \\ \left\lceil \frac{5x_6}{4} \right\rceil & \text{otherwise} \end{cases},
$$

and

$$
c = \begin{cases} 6\left\lceil \frac{x_5}{5} \right\rceil & \text{if } x_5 = 5n + 4 \\ \left\lceil \frac{6x_5}{5} \right\rceil & \text{otherwise} \end{cases}.
$$

Proof. As stated earlier, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$. Therefore, $x_8 = 0$, which implies $\mathbf{x} = 0$, or $x_8 \geq 1$. We will prove at the end $x_8 \neq 1$.

Now, fix x_8 such that $2 \leq x_8 \leq k$. The bounds for x_1 come from the first and third inequalities and the fact that x_1 must be an integer.

The upper bound for x_3 comes from the fourth inequality while the lower bound combines the third and the lower bound for x_1: $2x_3 \geq x_4 + x_1 \geq x_4 + \frac{x_3}{2}$. Then $x_3 \geq \frac{2x_4}{3}$. Recall that x_3 must be an integer. As noted earlier, $\left\lceil \frac{2x_4}{3} \right\rceil = 2\left\lceil \frac{x_4}{3} \right\rceil$ except when $x_4 = 3n + 1$. We claim that in that case, the smaller of the two options is the true lower bound for x_3. Since this is a lower bound, we need only check the third inequality. We have that $x_3 \geq \left\lceil \frac{2x_4}{3} \right\rceil = \left\lceil \frac{2(3n+1)}{3} \right\rceil = 2n + 1$. Then $x_1 \geq \left\lceil \frac{2n+1}{2} \right\rceil = n + 1$ and $-x_4 + 2x_3 = -(3n + 1) + 2(2n + 1) = n + 1 \geq x_1$. So this is a possible value for x_3 with $x_1 = n + 1$.

The lower bound for x_2 comes from the second inequality and the fact that x_2 must be

an integer. The upper bound combines the lower bound for x_3 with the fourth inequality: $x_2 \leq 2x_4 - x_3 - x_5 \leq 2x_4 - \frac{2x_4}{3} - x_5 = \frac{4x_4}{3} - x_5$. Recall that x_2 must be an integer and from earlier $\lfloor \frac{4x_4}{3} \rfloor = 4 \lfloor \frac{x_4}{3} \rfloor + (x_4 \mod 3)$. Then we need to check that the greater of the two, $\lfloor \frac{4x_4}{3} \rfloor$ satisfies the inequalities in both of the cases for $x_4 \neq 0 \mod 3$. We return to this after establishing the bounds for the other variables.

The upper bound for x_4 comes from the fifth inequality. We use the fourth inequality and the lower bounds for x_2 and x_3 to obtain the lower bound for x_4: $2x_4 \geq x_2 + x_3 + x_5 \geq \frac{x_4}{2} + \frac{2x_4}{3} + x_5$. This gives $x_4 \geq \frac{6x_5}{5}$. Once more, we must determine the correct lower bound since x_4 must be an integer. From earlier, we claim that the smaller value, $\lceil \frac{6x_5}{5} \rceil$ is the greatest lower bound except in the case $x_5 = 4n + 4$. Since this is a lower bound, we need to show that the fourth inequality is satisfied in this case. We need to check for each of the cases when $x_5 \neq 0 \mod 5$.

Case 1: $x_5 = 5n + 1$. Then $x_4 \geq \lceil \frac{6(5n+1)}{5} \rceil = 6n + 2$. So $2x_4 \geq 12n + 4 \geq x_2 + x_3 + (5n + 1) \geq \lceil \frac{6n+2}{2} \rceil + \lceil \frac{2(6n+2)}{3} \rceil + (5n+1) = 12n + 4$. Then when $x_4 = 6n + 2, x_2 = 3n + 1$ and $x_3 = 4n + 2$, satisfying the inequalities.

Case 2: $x_5 = 5n + 2$. Then $x_4 \geq \lceil \frac{6(5n+2)}{5} \rceil = 6n + 3$. So $2x_4 \geq 12n + 6 \geq x_2 + x_3 + (5n + 2) \geq \lceil \frac{6n+3}{2} \rceil + \lceil \frac{2(6n+3)}{3} \rceil + (5n+2) = 12n + 6$. Then when $x_4 = 6n + 3, x_2 = 3n + 2$ and $x_3 = 4n + 2$, satisfying the inequalities.

Case 3: $x_5 = 5n + 3$. Then $x_4 \geq \lceil \frac{6(5n+3)}{5} \rceil = 6n + 4$. So $2x_4 \geq 12n + 8 \geq x_2 + x_3 + (5n + 3) \geq \lceil \frac{6n+4}{2} \rceil + \lceil \frac{2(6n+4)}{3} \rceil + (5n+3) = 12n + 8$. Then when $x_4 = 6n + 4, x_2 = 3n + 2$ and $x_3 = 4n + 3$, satisfying the inequalities.

Case 4: $x_5 = 5n + 4$. Consider the smaller of the two values. Then $x_4 \geq \lceil \frac{6(5n+4)}{5} \rceil = 6n + 5$. So $2x_4 \geq 12n + 10 \geq x_2 + x_3 + (5n + 4) \geq \lceil \frac{6n+5}{2} \rceil + \lceil \frac{2(6n+5)}{3} \rceil + (5n+4) = 12n + 11$. This is impossible. Instead, we consider the lower bound being $6 \lceil \frac{5n+4}{5} \rceil$. Then $x_4 \geq 6(n+1) = 6n + 6$. So $2x_4 \geq 12n + 12 \geq x_2 + x_3 + (5n + 4) \geq \lceil \frac{6n+6}{2} \rceil + \lceil \frac{2(6n+6)}{3} \rceil + (5n+4) = 12n + 11$. Then x_2 and x_3 can take on various values. Considering the bounds for each we have $3n + 3 \leq x_2 \leq \lfloor \frac{4(6n+6)}{3} \rfloor - (5n+4) = 3n + 4$ and $4n + 4 \leq x_3 \leq 2(6n+6) - (5n+4) - x_2 = 7n + 8 - x_2$. Then when $x_4 = 6n + 6$, we can have $x_2 = 3n + 3$ and $x_3 = 4n + 4$ or $x_3 = 4n + 5$, or $x_2 = 3n + 4$ and $x_3 = 4n + 4$. In each case, the inequalities are satisfied.

The upper bound for x_5 comes directly from the sixth inequality. The lower bound combines the fifth inequality with the lower bound for x_4: $2x_5 \geq x_4 + x_6 \geq \frac{6x_5}{5} + x_6$, giving $x_5 \geq \frac{5x_6}{4}$. Again, since we need x_5 to be an integer and this is a lower bound, we need to check the least value x_5 takes on based on the placement of the ceiling. Unfortunately, this depends on the remainder of x_6 when divided by 4. Since this is a lower bound, we need to show that the fifth inequality is satisfied for each case when $x_6 \neq 0 \mod 4$.

Case 1: $x_6 = 4n + 1$. Then $x_5 \geq \lceil \frac{5(4n+1)}{4} \rceil = 5n + 2$. So $2x_5 \geq 10n + 4 \geq x_4 + (4n + 1) \geq \lceil \frac{6(5n+2)}{5} \rceil + (4n+1) = 10n + 4$. Then when $x_5 = 5n + 2, x_4 = 6n + 3$ satisfying the inequalities.

Case 2: $x_6 = 4n + 2$. Then $x_5 \geq \lceil \frac{5(4n+2)}{4} \rceil = 5n + 3$. So $2x_5 \geq 10n + 6 \geq x_4 + (4n + 2) \geq$

$\left\lceil \frac{6(5n+3)}{5} \right\rceil + (4n+2) = 10n+6$. Then when $x_5 = 5n+3, x_4 = 6n+4$ satisfying the inequalities.

Case 3: $x_6 = 4n+3$. Consider the smaller of the two values. Then $x_5 \geq \left\lceil \frac{5(4n+3)}{4} \right\rceil = 5n+4$. So $2x_5 \geq 10n+8 \geq x_4 + (4n+3) \geq 6\left\lceil \frac{5n+4}{5} \right\rceil + (4n+3) = 10n+9$, using the bound we proved for x_4 above. This is impossible. Instead, we consider the lower bound being $5\left\lceil \frac{4n+3}{4} \right\rceil$. Then $x_5 \geq 5(n+1) = 5n+5$. So $2x_5 \geq 10n+10 \geq x_4 + (4n+3) \geq \left\lceil \frac{6(5n+5)}{5} \right\rceil + (4n+3) = 10n+9$. Then when $x_5 = 5n+5$, we can have $x_4 = 6n+6$ or $x_4 = 6n+7$. In each case, the inequalities are satisfied.

The upper bound for x_6 comes directly from the seventh inequality. The lower bound combines the sixth inequality with the lower bound for x_5: $2x_6 \geq x_5 + x_7 \geq \frac{5x_6}{4} + x_7$, giving $x_6 \geq \frac{4x_7}{3}$. Again, since we need x_6 to be an integer and this is a lower bound, we need to check the least value x_6 takes on based on the placement of the ceiling. This depends on the remainder of x_7 when divided by 3. Since this is a lower bound, we need to show that the sixth inequality is satisfied for each case when $x_7 \neq 0 \mod 3$.

Case 1: $x_7 = 3n+1$. Then $x_6 \geq \left\lceil \frac{4(3n+1)}{3} \right\rceil = 4n+2$. So $2x_6 \geq 8n+4 \geq x_5 + (3n+1) \geq \left\lceil \frac{5(4n+2)}{4} \right\rceil + (3n+1) = 8n+4$. Then when $x_6 = 4n+2, x_5 = 5n+3$ satisfying the inequalities.

Case 2: $x_7 = 3n+2$. Consider the smaller of the two values. Then $x_6 \geq \left\lceil \frac{4(3n+2)}{3} \right\rceil = 4n+3$. So $2x_6 \geq 8n+6 \geq x_5 + (3n+2) \geq 5\left\lceil \frac{4n+3}{4} \right\rceil + (3n+2) = 8n+7$, using the bound we proved for x_5 above. This is impossible. Instead, we consider the lower bound being $4\left\lceil \frac{3n+2}{3} \right\rceil$. Then $x_6 \geq 4(n+1) = 4n+4$. So $2x_6 \geq 8n+8 \geq x_5 + (3n+2) \geq \left\lceil \frac{5(4n+4)}{4} \right\rceil + (3n+2) = 8n+7$. Then when $x_6 = 4n+4$, we can have $x_5 = 5n+5$ or $x_5 = 5n+6$. In each case, the inequalities are satisfied.

The upper bound for x_7 comes directly from the eighth inequality. The lower bound combines the seventh inequality with the lower bound for x_6: $2x_7 \geq x_6 + x_8 \geq \frac{4x_7}{3} + x_8$, giving $x_7 \geq \frac{3x_8}{2}$. Since x_7 must be an integer, we need to check the placement of the ceiling when x_8 is odd. Let $x_8 = 2n+1$. Consider if $x_7 = \left\lceil \frac{3(2n+1)}{2} \right\rceil = 3n+2$. Then the seventh inequality becomes $2(3n+2) = 6n+4 \geq 4\left\lceil \frac{3n+2}{3} \right\rceil + (2n+1) = 6n+5$. This is false. Instead, if $x_7 = 3\left\lceil \frac{2n+1}{2} \right\rceil = 3n+3$, the seventh inequality holds: $2(3n+3) = 6n+6 \geq \left\lceil \frac{4(3n+3)}{3} \right\rceil + (2n+1) = 6n+5$, allowing x_6 to be either $4n+4$ or $4n+5$. As a result, $x_8 \neq 1$ since then the bounds for x_7 give $3 \leq x_7 \leq 2$. Therefore, $x_8 \geq 2$.

We revisit the upper bound for x_2 to ensure it is an integer. First, note that since $x_8 \geq 2, x_7 \geq 3, x_6 \geq 4, x_5 \geq 5$, and so $x_4 \geq 6$. Recall that we wish to show that the greater of the two possible values for x_2, $\left\lfloor \frac{4x_4}{3} \right\rfloor - x_5$ satisfies the inequalities and is therefore the upper bound for x_2. Consider the bounds for x_2: $\left\lceil \frac{x_4}{2} \right\rceil \leq x_2 \leq \frac{4x_4}{3} - x_5$. Following all of our bounds we have established, this becomes $\left\lceil \frac{x_4}{2} \right\rceil \leq x_2 \leq \frac{4x_4}{3} - x_5 \leq \frac{4x_4}{3} - \frac{5x_6}{4} \leq \frac{4x_4}{3} - \frac{5x_7}{3} \leq \frac{4x_4}{3} - \frac{5x_8}{2} \leq \frac{4x_4}{3} - 5$. Since we are testing the upper bound for x_2, we need only show that the second inequality, or its lower bound, is satisfied.

Case 1: $x_4 = 3n+1$. Then $\left\lceil \frac{3n+1}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{4(3n+1)}{3} \right\rfloor - 5 = 4n-4$. Then $3n+1 \leq 8n-8$, giving $n \geq \frac{9}{5}$. Since $n \geq 2$ because $x_4 \geq 6$, this is satisfied.

Case 2: $x_4 = 3n + 2$. Then $\left\lceil \frac{3n+2}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{4(3n+2)}{3} \right\rfloor - 5 = 4n - 3$. Then $3n + 2 \leq 8n - 6$, giving $n \geq \frac{8}{5}$. Since $n \geq 2$ because $x_4 \geq 6$, this is satisfied.

Therefore, the greater value is acceptable as an upper bound for x_2. $\qquad\square$

Theorem 3.10.2. *Let* $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (6l - x_4)\alpha_4 - (5l - x_5)\alpha_5 - (4l - x_6)\alpha_6 + (3l - x_7)\alpha_7 - (3l - x_8)\alpha_8$ *where*

- $2 \leq x_8 \leq k$,

- $3\left\lceil \frac{x_8}{2} \right\rceil \leq x_7 \leq 2x_8$,

- $a = \begin{cases} 4\left\lceil \frac{x_7}{3} \right\rceil & \text{if } x_3 = 3n + 2 \\ \left\lceil \frac{4x_7}{3} \right\rceil & \text{otherwise} \end{cases}$,

- $b = \begin{cases} 5\left\lceil \frac{x_6}{4} \right\rceil & \text{if } x_6 = 4n + 3 \\ \left\lceil \frac{5x_6}{4} \right\rceil & \text{otherwise} \end{cases}$,

- $c = \begin{cases} 6\left\lceil \frac{x_5}{5} \right\rceil & \text{if } x_5 = 5n + 4 \\ \left\lceil \frac{6x_5}{5} \right\rceil & \text{otherwise} \end{cases}$,

- $a \leq x_6 \leq 2x_7 - x_8$,

- $b \leq x_5 \leq 2x_6 - x_7$,

- $c \leq x_4 \leq 2x_5 - x_6$,

- $\left\lceil \frac{x_4}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{4x_4}{3} \right\rfloor - x_5$,

- $\left\lceil \frac{2x_4}{3} \right\rceil \leq x_3 \leq 2x_4 - x_2 - x_5$,

- $\left\lceil \frac{x_3}{2} \right\rceil \leq x_1 \leq 2x_3 - x_4$,

- $l = \max\{\left\lceil \frac{x_1}{2} \right\rceil, \left\lceil \frac{x_2}{3} \right\rceil\}$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 4b_0 - b_3, x_4 = 6b_0 - b_4, x_5 = 5b_0 - b_5, x_6 = 4b_0 - b_6, x_7 = 3b_0 - b_7$, and $x_8 = 2b_0 - b_8$. Then $(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8) = (b_0, 2b_0 - x_1, 3b_0 - x_2, 4b_0 - x_3, 6b_0 - x_4, 5b_0 - x_5, 4b_0 - x_6, 3b_0 - b_7, 2b_0 - b_8)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 8$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$3b_0 - x_2 \geq 0$$
$$4b_0 - x_3 \geq 0$$

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Table 3.33: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_8^{(1)}$-module $V(2\Lambda_0)$

$(x_1 \ldots, x_8)$	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0,0)	0	$2\Lambda_0$
(2,3,4,6,5,4,3,2)	1	$2\Lambda_0 - \alpha_0$
(4,5,7,10,8,6,4,2)	2	$2\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8$

$$6b_0 - x_4 \geq 0$$

$$5b_0 - x_5 \geq 0$$

$$4b_0 - x_6 \geq 0$$

$$3b_0 - x_7 \geq 0$$

$$2b_0 - x_8 \geq 0$$

Then $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{6}, \frac{x_5}{5}, \frac{x_6}{4}, \frac{x_7}{3}, \frac{x_8}{2}\}$. We use the lower bounds from the lemma to eliminate some of these options. Combining all of the lower bounds we have $\frac{x_1}{2} \geq \frac{x_3}{4} \geq \frac{x_4}{6} \geq \frac{x_5}{5} \geq \frac{x_6}{4} \geq \frac{x_7}{3} \geq \frac{x_8}{2}$. Similarly, $\frac{x_2}{3} \geq \frac{x_4}{6}$ and the rest of the line follows.

Notice that it is possible for $\frac{x_2}{3} \geq \frac{x_1}{2}$. If $x_1 = \frac{x_3}{2} = \frac{x_4}{3}$ (e.g. $x_1 = 10, x_3 = 20, x_4 = 30$) and $x_2 = \frac{4x_4}{3} - x_5$, then $x_2 = 4x_1 - x_5$. Following the chain of inequalities in the previous paragraph, this implies $\frac{x_2}{3} = \frac{4x_1}{3} - \frac{x_5}{3} \geq \frac{4x_1}{3} - \frac{5x_1}{6} = \frac{x_1}{2}$.

Then it suffices to say $b_0 \geq \max\{\lceil \frac{x_1}{2} \rceil, \lceil \frac{x_2}{3} \rceil\}$. Then let $l = \max\{\lceil \frac{x_1}{2} \rceil, \lceil \frac{x_2}{3} \rceil\}$ and we claim that $b_0 = l$. To show this, the same method is used as in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (6l - x_4)\alpha_4 - (5l - x_5)\alpha_4 - (4l - x_6)\alpha_6 - (3l - x_7)\alpha_7 - (2l - x_8)\alpha_8$. Combining this with our solutions for the x_i from Lemma 3.10.1, we have the pattern given above. $\square$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_8 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Example 3.10.3. $E_8^{(1)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.10.2, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (6l - x_4)\alpha_4$$
$$- (5l - x_5)\alpha_4 - (4l - x_6)\alpha_6 - (3l - x_7)\alpha_7 - (2l - x_8)\alpha_8.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.33.

Example 3.10.4. $E_8^{(1)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.10.2, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (6l - x_4)\alpha_4$$

Table 3.34: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_8^{(1)}$-module $V(3\Lambda_0)$

$(x_1,\ldots,x_8)$	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0,0)	0	$3\Lambda_0$
(2,3,4,6,5,4,3,2)	1	$3\Lambda_0 - \alpha_0$
(4,5,7,10,8,6,4,2)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8$
(4,6,8,12,10,8,6,3)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_8$
(5,8,10,15,12,9,6,3)	3	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 3\alpha_6 - 3\alpha_7 - 3\alpha_8$

Table 3.35: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_8^{(1)}$-module $V(4\Lambda_0)$

$(x_1,\ldots,x_8)$	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0,0,0,0)	0	$4\Lambda_0$
(2,3,4,6,5,4,3,2)	1	$4\Lambda_0 - \alpha_0$
(4,5,7,10,8,6,4,2)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8$
(4,6,8,12,10,8,6,3)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_8$
(4,6,8,12,10,8,6,4)	2	$4\Lambda_0 - 2\alpha_0$
(5,8,10,15,12,9,6,3)	3	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 3\alpha_6 - 3\alpha_7 - 3\alpha_8$
(6,8,11,16,13,10,7,4)	3	$4\Lambda_0 - 3\alpha_0 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - 2\alpha_8$
(6,9,12,18,15,12,8,4)	3	$4\Lambda_0 - 3\alpha_0 - \alpha_7 - 2\alpha_8$
(7,10,14,20,16,12,8,4)	4	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 4\alpha_4 - 4\alpha_5 - 4\alpha_6 - 4\alpha_7 - 4\alpha_8$
(8,10,14,20,16,12,8,4)	4	$4\Lambda_0 - 4\alpha_0 - 2\alpha_2 - 2\alpha_3 - 4\alpha_4 - 4\alpha_5 - 4\alpha_6 - 4\alpha_7 - 4\alpha_8$

$$- (5l - x_5)\alpha_4 - (4l - x_6)\alpha_6 - (3l - x_7)\alpha_7 - (2l - x_8)\alpha_8.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.34. Notice that we can collect the same **x** vectors from the case $V(2\Lambda_0)$ since, as we said earlier, $X_2 \subset X_3$.

Example 3.10.5. $E_8^{(1)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.10.2, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (4l - x_3)\alpha_3 - (6l - x_4)\alpha_4$$
$$- (5l - x_5)\alpha_4 - (4l - x_6)\alpha_6 - (3l - x_7)\alpha_7 - (2l - x_8)\alpha_8.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.35. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

3.11 Type $A_{2n}^{(2)}$

Let $\mathfrak{g} = A_{2n}^{(2)}$ for $n \geq 2$, index set $I = \{0, 1, \ldots, n\}$, and Cartan matrix

$$
A = \begin{pmatrix}
2 & -2 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & \cdots & 0 & 0 & -1 & 2 & -2 \\
0 & \cdots & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
$$

The defining inequalities are equivalent to

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
2x_1 \leq k
\end{cases}
$$

where $x_i = b_0 - b_i$ for $1 \leq i \leq n-1$, and $x_n = \frac{1}{2}b_0 - b_n$. That is,

$$
\begin{cases}
2x_1 - x_2 & \geq 0 \\
-x_1 + 2x_2 - x_3 & \geq 0 \\
-x_2 + 2x_3 - x_4 & \geq 0 \\
\quad \vdots & \vdots \\
-x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0 \\
-x_{n-2} + 2x_{n-1} - 2x_n & \geq 0 \\
-x_{n-1} + 2x_n & \geq 0 \\
2x_1 & \leq k
\end{cases}
$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq b_i$ for $1 \leq i \leq n-1$ and $b_0 \geq 2b_n$.

Lemma 3.11.1. *Let* $\mathbf{x}$ *be a solution to*

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
2x_1 \leq k
\end{cases}
$$

Then $x_i \leq x_{i+1}$ *for* $1 \leq i \leq n-2$. *In addition,* $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. This proof is essentially the same as that of Lemma 3.3.1 with the coordinates non-decreasing until $n-2$ rather than $n-1$. We arrive at the contradiction using $(\mathring{A}\mathbf{x})_{n-1} \geq 0$ and $(\mathring{A}\mathbf{x})_n \geq 0$. $\qquad \square$

Lemma 3.11.2. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k \end{cases}$$

Then $x_i = x_{i-1}$ *implies* $x_i = 2m$ *for some integer* m *and for* $2 \leq i \leq n-1$.

Proof. This proof is essentially the same as Lemma 3.4.2. □

Lemma 3.11.3. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k \end{cases}$$

is the same as those to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k-1 \end{cases}$$

if k *is odd.*

Proof. Assume k is odd. Notice that the only difference between the two sets of inequalities is the last in each: $2x_1 \leq k$ and $2x_1 \leq k-1$. Since k is odd, in the first set we have $2x_1 \leq k$ which implies $x_1 \leq \frac{k}{2}$, and since x_1 must be an integer, we have $x_1 \leq \lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$. Then since the solution for x_1 are the same in both sets and the x_i never decrease by Lemma 3.11.1, the solution sets must be the same. □

Lemma 3.11.4. *Given* $\mathring{A}$ *the Cartan matrix for type* C_n *of finite type, the set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 1 \leq x_1 \leq \left\lfloor \frac{k}{2} \right\rfloor, x_i = x_1 + \sum_{j=2}^{i} l_j \text{ for } 2 \leq i \leq n-1,$$

$$0 \leq l_2 \leq x_1, 0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_3 \leq l_2, \left\lceil \frac{x_{n-1}}{2} \right\rceil \leq x_n \leq \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor$$

$$\text{where } l_{n-1} = x_1 \text{ for } n = 2\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$. Therefore, $x_1 = 0$, which implies $\mathbf{x} = 0$, or $x_1 \geq 1$.

Now, fix x_1 such that $1 \leq x_1 \leq \lfloor \frac{k}{2} \rfloor$. We prove the pattern for x_2 through x_{n-1} by induction. First, by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_2 \leq 2x_1$. We have observed that the first $n-1$ coordinates of $\mathbf{x}$ do not

decrease, therefore $x_2 \geq x_1$. Then $x_1 \leq x_2 \leq 2x_1$ implies that $x_2 = x_1 + l_2$ where $0 \leq l_2 \leq x_1$. Now assume x_i is of the form $x_i = x_1 + \sum_{j=2}^{i} l_j$ for all $2 \leq i \leq p < n-1$ where $l_j = x_j - x_{j-1}$ and so $0 \leq l_p \leq l_{p-1} \leq \cdots \leq l_2$ by the second result of Lemma 3.11.1. By the same lemma, $x_p \leq x_{p+1} \leq 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_1 + \sum_{j=2}^{p} l_j \leq x_{p+1} \leq 2(x_1 + \sum_{j=2}^{p} l_j) - (x_1 + \sum_{j=2}) = x_1 + (\sum_{j=2}^{p} l_j) + l_p$. Then $x_{p+1} = x_1 + (\sum_{j=2}^{p} l_j) + l_{p+1}$ where $0 \leq l_{p+1} \leq l_p$. Therefore, $x_i = x_1 + \sum_{j=2}^{i} l_j$ where $0 \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$ and so l_i is the amount of increase from x_{i-1} to x_i where $2 \leq i \leq n-1$.

Finally, we need to prove the restrictions on x_n. By $(\mathring{A}\mathbf{x})_{n-1} \geq 0$, $2x_n \leq -x_{n-2} + 2x_{n-1}$. Using what we just showed, we have $2x_n \leq -(x_1 + \sum_{j=3}^{n-2} l_j) + 2(x_1 + \sum_{j=3}^{n-1} l_j) = x_1 + l_{n-1} + \sum_{j=3}^{n-1} l_j = x_{n-1} + l_{n-1}$. The last inequality gives $x_n \geq \frac{x_{n-1}}{2}$. Or, with what we found, we have $x_n \geq \frac{x_1 + \sum_{j=3}^{n-1} l_j}{2}$. Since x_n is an integer, we have $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq x_n \leq \left\lfloor \frac{x_{n-1}+l_{n-1}}{2} \right\rfloor$. $\qquad \square$

Theorem 3.11.5. *Let $n \geq 2$, $\Lambda = k\Lambda_0$, $k \geq 2$, $m \in \mathbb{Z}_{\geq 0}$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - x_1)\alpha_1 - \left(\sum_{i=2}^{n-1}(l - (x_1 + \sum_{j=2}^{i} l_j))\alpha_i \right)$ where*

- $1 \leq x_1 \leq \left\lfloor \frac{k}{2} \right\rfloor$,

- $0 \leq l_2 \leq x_1$,

- $0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_3 \leq l_2$ *and* $l_{n-1} = x_1$ *for* $n = 2$,

- $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq l \leq \left\lfloor \frac{x_{n-1}+l_{n-1}}{2} \right\rfloor$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_i = b_0 - b_i$ for $1 \leq i \leq n-1$ and $x_n = \frac{1}{2}b_0 - b_n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, b_0 - x_1, b_0 - x_2, \ldots, b_0 - x_{n-1}, \frac{1}{2}b_0 - x_n)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$b_0 - x_2 \geq 0$$
$$\vdots$$
$$b_0 - x_{n-1} \geq 0$$
$$\frac{1}{2}b_0 - x_n \geq 0$$

We have already proven in Lemma 3.11.1 that the first $n-1$ coordinates of $\mathbf{x}$ never decrease. Therefore, $x_{n-1} = \max_{1 \leq i \leq n-1}\{x_i\}$ and so for the first $n-1$ inequalities it is sufficient to say that $b_0 \geq x_{n-1}$. We showed in Lemma 3.11.4 that $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq x_n \leq \left\lfloor \frac{x_{n-1}+l_{n-1}}{2} \right\rfloor$. Then since $b_0 \geq 2x_n \geq 2 \cdot \left\lceil \frac{x_{n-1}}{2} \right\rceil \geq x_{n-1}$, we need only say $b_0 \geq 2x_n$. Then let $l = 2x_n$ and so $b_0 \geq l$. One can show $b_0 = l$ using the same method as that used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (l - x_2)\alpha_2 - \cdots - (l - x_{n-1})\alpha_{n-1} - (\frac{1}{2}l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.11.4, we have the pattern given above. $\qquad \square$

Table 3.36: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{10}^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$2\Lambda_0$
(1,2,2,2,1)	1	1,0,0	$2\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,2)	2	1,1,1	$2\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$

Table 3.37: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{12}^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$2\Lambda_0$
(1,2,2,2,2,1)	1	1,0,0,0	$2\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,4,2)	2	1,1,1,0	$2\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	1,1,1,1	$2\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$

To better understand this theorem, consider some examples for various values of both k and n. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_2 \leq k\}$. Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$.

Example 3.11.6. $A_{10}^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.11.5, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.36.

Example 3.11.7. $A_{12}^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.11.5, an element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (l - (x_1 + l_2 + l_3 + l_4))\alpha_4 - (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.37.

One can show that our formulation for the maximal dominant weights of $V(k\Lambda_0)$ correspond to that of [18] for the case $k = 2$ for arbitrary $n \geq 3$. According to [18], the set of maximal dominant weights for $V(2\Lambda_0)$ is $\{(1 + \delta_{2u,n-1})\Lambda_{2u} - 2u\delta \mid 0 \leq u \leq \lfloor \frac{n}{2} \rfloor\}$. We run into an issue for both $u = 0$ and $u = \frac{n-1}{2}$ though we propose this is due to a small generalization error in [18]. Instead, we prove that our set matches what we expect should be the set in [18]: $\{2\Lambda_0\} \cup \{\Lambda_{2u} - 2u\delta \mid 1 \leq u \leq \lfloor \frac{n}{2} \rfloor\}$. To show that the two formulations are the same, we write our formulation in terms of the fundamental dominant weights, Λ_i. With this conversion, we

Table 3.38: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{10}^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_2, l_3, l_4	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$3\Lambda_0$
(1,2,2,2,1)	1	1,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,2)	2	1,1,1	$3\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$

Table 3.39: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{12}^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$3\Lambda_0$
(1,2,2,2,2,1)	1	1,0,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,4,2)	2	1,1,1,0	$3\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	1,1,1,1	$3\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$

have

$$X = \{2\Lambda_0\} \cup \{2\Lambda_0 - l(2\Lambda_0 - \Lambda_1 + \delta) - (l - (1))(-2\Lambda_0 + 2\Lambda_1 - \Lambda_2)$$

$$-\Big(\sum_{i=2}^{n-1}(l - (1 + \sum_{j=2}^{i} l_j))\Big)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})\}$$

Showing this set corresponds to the expected set from [18] is similar to how we showed the same for type $B_n^{(1)}$.

Example 3.11.8. $A_{10}^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.11.5, any element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.38.

Example 3.11.9. $A_{12}^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.11.5, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (l - (x_1 + l_2 + l_3 + l_4))\alpha_4 - (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.39.

Example 3.11.10. $A_{10}^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.11.5, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4.$$

Table 3.40: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{10}^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_2, l_3, l_4	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$4\Lambda_0$
(1,2,2,2,1)	1	1,0,0	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,2)	2	1,1,1	$4\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	1	0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,3,4,4,2)	2	1,1,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,3)	3	1,1,1	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,2)	2	2,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1$
(2,4,5,6,3)	3	2,1,1	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,6,6,3)	3	2,2,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,4)	4	2,2,1	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,4)	4	2,2,2	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,5)	5	2,2,2	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$

We list the **x** vectors and corresponding values for l, l_2, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.40. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_{3,5} \subset X_{4,5}$.

Example 3.11.11. $A_{12}^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.11.5, an element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 - (l - (x_1 + l_2 + l_3))\alpha_3$$
$$- (l - (x_1 + l_2 + l_3 + l_4))\alpha_4 - (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

We list the **x** vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.41. Note again that since $X_{3,6} \subset X_{4,6}$, we can collect the **x** vectors from the example of $A_{12}^{(2)}, V(3\Lambda_0)$.

3.12 Type $A_{2n-1}^{(2)}$

Let $\mathfrak{g} = A_{2n-1}^{(2)}$ for $n \geq 3$, index set $I = \{0, 1, \ldots, n\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -2 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Table 3.41: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{12}^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$4\Lambda_0$
(1,2,2,2,2,1)	1	1,0,0,0	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,4,4,2)	2	1,1,1,0	$4\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	1,1,1,1	$4\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	1	0,0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,3,4,4,4,2)	2	1,1,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,6,3)	3	1,1,1,1	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,4,2)	2	2,0,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1$
(2,4,5,6,6,3)	3	2,1,1,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7,4)	4	2,1,1,1	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,6,3)	3	2,2,0,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,8,4)	4	2,2,1,1	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,8,4)	4	2,2,2,0	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9,5)	5	2,2,2,1	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,10,5)	5	2,2,2,2	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,10,6)	6	2,2,2,2	$4\Lambda_0 - 12\alpha_0 - 10\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

where $x_1 = b_0 - b_1, x_i = 2b_0 - b_i$ for $2 \leq i \leq n-1$, and $x_n = b_0 - b_n$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - x_3 & \geq 0 \\ -x_2 + 2x_3 - x_4 & \geq 0 \\ \quad \vdots & \vdots \\ -x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0 \\ -x_{n-2} + 2x_{n-1} - 2x_n & \geq 0 \\ -x_{n-1} + 2x_n & \geq 0 \\ x_2 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq b_1, b_0 \geq b_n$, and $b_0 \geq \frac{b_i}{2}$ for $2 \leq i \leq n-1$.

Lemma 3.12.1. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

Then $x_i \leq x_{i+1}$ *for* $1 \leq i \leq n-2$. *In addition,* $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. This proof is essentially the same as that of Lemma 3.3.1 using $(\mathring{A}\mathbf{x})_{n-1} \geq 0$ and $(\mathring{A}\mathbf{x})_n \geq 0$ to arrive at a contradiction. $\square$

Lemma 3.12.2. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

Then $x_2 \neq 1$.

Proof. This proof is essentially the same as that of Lemma 3.5.2 using $(\mathring{A}\mathbf{x})_{n-1} \geq 0$ to arrive at the contradiction. $\square$

Lemma 3.12.3. *Let* $\mathbf{x}$ *be a solution to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

Then $x_i = x_{i-1}$ *implies that* $x_i = 2m$ *for some integer* m *and for* $2 \leq i \leq n-1$.

Proof. This proof is essentially the same as that of Lemma 3.4.2. $\square$

Lemma 3.12.4. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 2 \leq x_2 \leq k, x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1, 0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor,$$

$$x_i = x_2 + \sum_{j=3}^{i} l_j \text{ for } 3 \leq i \leq n-1,$$

$$0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1, 0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3 \text{ for } n > 3,$$

$$\text{and } \left\lceil \frac{x_{n-1}}{2} \right\rceil \leq x_n \leq \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor \text{ where } l_{n-1} = x_2 - x_1 \text{ for } n = 3\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$. Therefore, $x_2 = 0$, which implies $\mathbf{x} = 0$, or $x_2 \geq 2$ by Lemma 3.12.2.

Now, fix x_2 such that $2 \leq x_2 \leq k$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_1 \geq \frac{x_2}{2}$. Since each x_i must be an integer, this implies $x_1 \geq \lceil \frac{x_2}{2} \rceil$. We have observed that the first $n-1$ coordinates of $\mathbf{x}$ do not decrease, therefore $x_1 \leq x_2$. Then $\lceil \frac{x_2}{2} \rceil \leq x_1 \leq x_2$ and so we can say $x_1 = \lceil \frac{x_2}{2} \rceil + l_1$ where $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$. We prove the pattern for x_3 through x_{n-1} by induction. We've shown that $x_2 \leq x_3$ and $x_3 - x_2 \leq x_2 - x_1$. Then $x_3 - x_2 \leq x_2 - (\lceil \frac{x_2}{2} \rceil + l_1) = \lfloor \frac{x_2}{2} \rfloor - l_1$. Therefore, $x_2 \leq x_3 \leq x_2 + \lfloor \frac{x_2}{2} \rfloor - l_1$ and so $x_3 = x_2 + l_3$ where $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$. Now assume $x_i = x_2 + \sum_{j=3}^{i} l_j$ for all $3 \leq i \leq p < n-1$ where $l_j = x_j - x_{j-1}$ and so $0 \leq l_p \leq l_{p-1} \leq \cdots \leq l_3$ by the second result of Lemma 3.12.1. By the same lemma, $x_p \leq x_{p+1} \leq 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_2 + \sum_{j=3}^{p} l_j \leq x_{p+1} \leq 2(x_2 + \sum_{j=3}^{p} l_j) - (x_2 + \sum_{j=3}^{p-1} l_j) = x_2 + (\sum_{j=3}^{p} l_j) + l_p$. Then $x_{p+1} = x_2 + (\sum_{j=3}^{p} l_j) + l_{p+1}$ where $0 \leq l_{p+1} \leq l_p$. Therefore, $x_i = x_2 + \sum_{j=3}^{i} l_j$ where $0 \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$ and so l_i is the amount of increase from x_{i-1} to x_i where $3 \leq i \leq n-1$.

Finally, we need to prove the restrictions on x_n. $(\mathring{A}\mathbf{x})_{n-1} \geq 0$ gives that $2x_n \leq -x_{n-2} + 2x_{n-1}$. Using what we just showed, we have $2x_n \leq -(x_2 + \sum_{j=3}^{n-2} l_j) + 2(x_2 + \sum_{j=3}^{n-1} l_j) = x_2 + l_{n-1} + \sum_{j=3}^{n-1} l_j = x_{n-1} + l_{n-1}$. The last inequality gives $x_n \geq \frac{x_{n-1}}{2}$. Or, with what we found, we have $x_n \geq \frac{x_2 + \sum_{j=3}^{n-1} l_j}{2}$. Since x_n is an integer, we have $\lceil \frac{x_{n-1}}{2} \rceil \leq x_n \leq \lfloor \frac{x_{n-1} + l_{n-1}}{2} \rfloor$. $\qquad \square$

Theorem 3.12.5. *Let $n \geq 3$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n-1} (2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \right) - (l - x_n)\alpha_n$ where*

- $2 \leq x_2 \leq k$,

- $l = \max\{x_1, x_n\}$,

- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$,

- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$ *for $n > 3$,*

- $0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3$, $l_2 = x_2 - x_1$ *for $n = 3, l_2 = 0$ else,*

- $\lceil \frac{x_{n-1}}{2} \rceil \leq x_n \leq \lfloor \frac{x_{n-1} + l_{n-1}}{2} \rfloor$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = b_0 - b_1$, $x_i = 2b_0 - b_i$ for $2 \leq i \leq n-1$, and $x_n = b_0 - b_n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, b_0 - x_1, 2b_0 - x_2, \ldots, 2b_0 - x_{n-1}, b_0 - x_n)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$2b_0 - x_2 \geq 0$$
$$\vdots$$

Table 3.42: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_9^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_1, l_3, l_4	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$2\Lambda_0$
(1,2,2,2,1)	1	0,0,0	$2\Lambda_0 - \alpha_0$
(1,2,3,4,2)	2	0,1,1	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	2	1,0,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$

$$2b_0 - x_{n-1} \geq 0$$
$$b_0 - x_n \geq 0$$

It is clear that $b_0 \geq x_1$ where $x_1 = (\lceil \frac{x_2}{2} \rceil + l_1)$. We have already proven that the first $n - 1$ coordinates of **x** never decrease. Therefore, $x_{n-1} = \max_{1 \leq i \leq n-1}\{x_i\}$ and so for the inner inequalities it is sufficient to say that $b_0 \geq \frac{x_{n-1}}{2}$. Since x_{n-1} may be odd, we need $b_0 \geq \lceil \frac{x_{n-1}}{2} \rceil$. We showed in Lemma 3.12.4 that $\lceil \frac{x_{n-1}}{2} \rceil \leq x_n \leq \lfloor \frac{x_{n-1}+l_{n-1}}{2} \rfloor$. Then since $b_0 \geq x_n \geq \lceil \frac{x_{n-1}}{2} \rceil$, we need only say $b_0 \geq x_n$ to satisfy the last $n - 1$ inequalities. Then let $l = \max\{x_1, x_n\}$ and so $b_0 \geq l$. We claim that $b_0 = l$. This can be shown using the same method as that used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (2l - x_2)\alpha_2 - \cdots - (2l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.12.4, we have the pattern given above. $\qquad\square$

To better understand this theorem, consider some examples for various values of both k and n. We include two examples of $k = 2$ and prove that our results match those of Lee's. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_2 \leq k\}$. Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$.

Example 3.12.6. $A_9^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.12.5,

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 + (l - x_5)\alpha_5.$$

We list the **x** vectors and corresponding values for $l, l_1, l_3,$ and l_4 in addition to the resulting maximal dominant weight in Table 3.42.

Example 3.12.7. $A_{11}^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By the theorem,

$$\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5 - (l - x_6)\alpha_6.$$

We list the **x** vectors and corresponding values for $l, l_1, l_3, l_4,$ and l_5 in addition to the resulting maximal dominant weight in Table 3.43.

One can show that our formulation for the maximal dominant weights of $V(k\Lambda_0)$ correspond

Table 3.43: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{11}^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_1, l_3, l_4, l_5	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$2\Lambda_0$
(1,2,2,2,2,1)	1	0,0,0,0	$2\Lambda_0 - \alpha_0$
(1,2,3,4,4,2)	2	0,1,1,0	$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	0,1,1,1	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	2	1,0,0,0	$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$

to that of [18] for the case $k = 2$ for arbitrary $n \geq 3$. According to [18], the set of maximal dominant weights for $V(2\Lambda_0)$ is $\{\Lambda_0 + \Lambda_{2u} - u\delta | 0 \leq u \leq \lfloor \frac{n}{2} \rfloor\} \cup \{2\Lambda_1 - 2\delta\}$. We run into an in issue with the first set for $u > 0$, though we propose this is due to a small generalization error in [18]. For example, for $u = 1$, the paper gives the maximal dominant weight $\Lambda_0 + \Lambda_2 - \delta$, which is of level 3. Since it is in $V(2\Lambda_0)$, it must be of level 2. Instead, we expect the intended set in [18] is: $\{2\Lambda_0\} \cup \{\Lambda_{2u} - u\delta | 0 < u \leq \lfloor \frac{n}{2} \rfloor\} \cup \{2\Lambda_1 - 2\delta\}$. To prove that this formulation matches ours, first write our formulation in terms of the fundamental dominant weights, Λ_i. This conversion gives

$$X = \{2\Lambda_0 - l(2\Lambda_0 - \Lambda_2 + \delta) - (l - (1 + l_1))(2\Lambda_1 - \Lambda_2) - (2l - 2)(-\Lambda_0 - \Lambda_1 + 2\Lambda_2 - \Lambda_3)$$
$$- \Big(\sum_{i=3}^{n-3}(2l - (2 + \sum_{j=3}^{i} l_j)))\Big)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$
$$- \Big(2l - (2 + \sum_{j=3}^{n-2} l_j)\Big)(-\Lambda_{n-3} + 2\Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n)$$
$$- (l - x_{n-1})(-\Lambda_{n-2} + 2\Lambda_{n-1}) - (l - x_n)(-\Lambda_{n-2} + 2\Lambda_n)\}.$$

To show the two sets are the same, the process is similar to that used when showing the same for type $B_n^{(1)}$.

Example 3.12.8. $A_9^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.12.5,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 + (l - x_5)\alpha_5.$$

We list the **x** vectors and corresponding values for $l, l_1, l_3,$ and l_4 in addition to the resulting maximal dominant weight in Table 3.44. Notice that we can collect the same **x** vectors from the case $V(2\Lambda_0)$ since, as we said earlier, $X_{2,n} \subset X_{3,n}$.

Example 3.12.9. $A_{11}^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.12.5,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5 - (l - x_6)\alpha_6.$$

Table 3.44: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_9^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_1, l_3, l_4	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$3\Lambda_0$
(1,2,2,2,1)	1	0,0,0	$3\Lambda_0 - \alpha_0$
(1,2,3,4,2)	2	0,1,1	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	2	1,0,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(2,3,4,4,2)	2	0,1,0	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
(2,3,4,5,3)	3	0,1,1	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

Table 3.45: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{11}^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_1, l_3, l_4, l_5	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$3\Lambda_0$
(1,2,2,2,2,1)	1	0,0,0,0	$3\Lambda_0 - \alpha_0$
(1,2,3,4,4,2)	2	0,1,1,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	0,1,1,1	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	2	1,0,0,0	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$
(2,3,4,4,4,2)	2	0,1,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
(2,3,4,5,6,3)	3	0,1,1,1	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

We list the **x** vectors and corresponding values for l, l_1, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.45. Note again that since $X_{2,6} \subset X_{3,6}$, we can collect the **x** vectors from the example of $A_{11}^{(2)}, V(2\Lambda_0)$.

Example 3.12.10. $A_9^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.12.5,

$$\max(4\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 + (l - x_5)\alpha_5.$$

We list the **x** vectors and corresponding values for l, l_1, l_3, and l_4 in addition to the resulting maximal dominant weight in Table 3.46. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_{3,5} \subset X_{4,5}$.

Example 3.12.11. $A_{11}^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.12.5,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (\lceil \tfrac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - (2l - (x_2 + l_3))\alpha_3$$
$$- (2l - (x_2 + l_3 + l_4))\alpha_4 - (2l - (x_2 + l_3 + l_4 + l_5))\alpha_5 - (l - x_6)\alpha_6.$$

We list the **x** vectors and corresponding values for l, l_1, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.46. Note again that since $X_{3,6} \subset X_{4,6}$, we can collect the **x** vectors from the example of $A_{11}^{(2)}, V(3\Lambda_0)$.

Table 3.46: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_9^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_1,l_3,l_4	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0	$4\Lambda_0$
(1,2,2,2,1)	1	0,0,0	$4\Lambda_0 - \alpha_0$
(1,2,3,4,2)	2	0,1,1	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,2,2,2,1)	2	1,0,0	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(2,3,4,4,2)	2	0,1,0	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
(2,3,4,5,3)	3	0,1,1	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,2)	2	0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,4,5,6,3)	3	0,1,1	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,6,6,3)	3	0,2,0	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
(2,4,6,7,4)	4	0,2,1	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,4)	4	0,2,2	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,5)	5	0,2,2	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(3,4,4,4,2)	3	1,0,0	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$
(3,4,5,6,3)	3	1,1,1	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
(4,4,4,4,2)	4	2,0,0	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5$

Table 3.47: **x** vectors and maximal dominant weights for the $\mathfrak{g} = A_{11}^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_1,l_3,l_4,l_5	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0	$4\Lambda_0$
(1,2,2,2,2,1)	1	0,0,0,0	$4\Lambda_0 - \alpha_0$
(1,2,3,4,4,2)	2	0,1,1,0	$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,3)	3	0,1,1,1	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,1)	2	1,0,0,0	$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$
(2,3,4,4,4,2)	2	0,1,0,0	$4\Lambda_0 - 2\alpha_0 - \alpha_2$
(2,3,4,5,6,3)	3	0,1,1,1	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,4,2)	2	0,0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,4,5,6,6,3)	3	0,1,1,0	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7,4)	4	0,1,1,1	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,6,3)	3	0,2,0,0	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
(2,4,6,7,8,4)	4	0,2,1,1	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,8,4)	4	0,2,2,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9,5)	5	0,2,2,1	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,10,5)	5	0,2,2,2	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,10,6)	6	0,2,2,2	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$
(3,4,4,4,4,2)	3	1,0,0,0	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$
(3,4,5,6,6,3)	3	1,1,1,0	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
(3,4,5,6,7,4)	4	1,1,1,1	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(4,4,4,4,4,2)	4	2,0,0,0	$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 4\alpha_5 - 2\alpha_6$

3.13   Type $D_{n+1}^{(2)}$

Let $\mathfrak{g} = D_{n+1}^{(2)}$ for $n \geq 2$, index set $I = \{0, 1, \ldots, n\}$, and Cartan marix

$$
A = \begin{pmatrix}
2 & -2 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & \cdots & 0 & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & 0 & 0 & -2 & 2
\end{pmatrix}
$$

The defining inequalities are equivalent to

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
2x_1 \leq k
\end{cases}
$$

where $x_i = b_0 - b_i$ for $1 \leq i \leq n$. That is,

$$
\begin{cases}
2x_1 - x_2 & \geq 0 \\
-x_1 + 2x_2 - x_3 & \geq 0 \\
-x_2 + 2x_3 - x_4 & \geq 0 \\
\quad\vdots & \vdots \\
-x_{n-2} + 2x_{n-1} - x_n & \geq 0 \\
-2x_{n-1} + 2x_n & \geq 0 \\
2x_1 & \leq k
\end{cases}
$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq b_i$ for $1 \leq i \leq n$.

Lemma 3.13.1. *Let* $\mathbf{x}$ *be a solution to*

$$
\begin{cases}
\mathring{A}\mathbf{x} \geq 0 \\
2x_1 \leq k
\end{cases}
$$

Then $x_i \leq x_{i+1}$ *for* $1 \leq i \leq n-1$. *In addition,* $x_i - x_{i-1} \geq x_{i+1} - x_i$.

Proof. This proof is essentially the same as that of Lemma 3.3.1 using $(\mathring{A}\mathbf{x})_n \geq 0$ to arrive at the contradiction. $\qquad\square$

Lemma 3.13.2. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k \end{cases}$$

is the same as those to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k - 1 \end{cases}$$

if k is odd.

Proof. This proof is essentially the same as that of Lemma 3.11.3. $\square$

Lemma 3.13.3. *The set of solutions to*

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ 2x_1 \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 1 \leq x_1 \leq \left\lfloor \frac{k}{2} \right\rfloor, x_i = x_1 + \sum_{j=2}^{i} l_j, 0 \leq l_2 \leq x_1,$$

$$0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_3 \leq l_2 \text{ where } 2 \leq i \leq n\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so if $x_1 = 0$ then $\mathbf{x} = 0$.

Now, fix x_1 such that $1 \leq x_1 \leq \lfloor \frac{k}{2} \rfloor$ since x_1 must be an integer. We prove the pattern for the remaining coordinates by induction. First, by $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_2 \leq 2x_1$. We have observed that the coordinates of $\mathbf{x}$ do not decrease, therefore $x_1 \leq x_2$. Then $x_1 \leq x_2 \leq 2x_1$ and so we can say $x_2 = x_1 + l_2$ where $0 \leq l_2 \leq x_1$. Now assume $x_i = x_1 + \sum_{j=2}^{i} l_j$ for all $2 \leq i \leq p < n$ where $l_j = x_j - x_{j-1}$ and so $0 \leq l_p \leq l_{p-1} \leq \cdots \leq l_2$ by the second result of Lemma 3.13.1. By the same lemma, $x_p \leq x_{p+1} \leq 2x_p - x_{p-1}$. Applying the induction hypothesis, we have $x_1 + \sum_{j=2}^{p} l_j \leq x_{p+1} \leq 2(x_1 + \sum_{j=2}^{p} l_j) - (x_1 + \sum_{j=2}^{p-1} l_j) = x_1 + (\sum_{j=2}^{p} l_j) + l_p$. Then $x_{p+1} = x_1 + (\sum_{j=2}^{p} l_j) + l_{p+1}$ where $0 \leq l_{p+1} \leq l_p$. Therefore, $x_i = x_1 + \sum_{j=2}^{i} l_j$ where $0 \leq l_n \leq \cdots \leq l_3 \leq l_2$ and so l_i is the amount of increase from x_{i-1} to x_i where $2 \leq i \leq n$. $\square$

Theorem 3.13.4. *Let $n \geq 2$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - x_1)\alpha_1 - \left(\sum_{i=2}^{n}(l - (x_1 + \sum_{j=2}^{i} l_j))\alpha_i \right)$ where*

- $1 \leq x_1 \leq \lfloor \frac{k}{2} \rfloor$,

- $l = x_n$,

- $0 \leq l_2 \leq x_1$,

- $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_i = b_0 - b_i$ for $1 \leq i \leq n$. Then $(b_0, b_1, \ldots, b_n) = (b_0, b_0 - x_1, b_0 - x_2, \ldots, b_0 - x_{n-1}, b_0 - x_n)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq n$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$b_0 - x_1 \geq 0$$
$$b_0 - x_2 \geq 0$$
$$\vdots$$
$$b_0 - x_{n-1} \geq 0$$
$$b_0 - x_n \geq 0$$

We have already proven that the coordinates of $\mathbf{x}$ never decrease. Therefore, $x_n = \max_{1 \leq i \leq n}\{x_i\}$ and so it is sufficient to say that $b_0 \geq x_n$. We claim that $b_0 = l$. One can show this using the same method that is used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (l - x_1)\alpha_1 - (l - x_2)\alpha_2 - \cdots - (l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n$. Combining this with our solutions for the x_i from Lemma 3.13.3, we have the pattern given above. $\qquad\square$

To better understand this theorem, consider some examples for various values of both k and n. We include two examples of $k = 2$ and prove that our results match those of [18]. For ease in our future examples, define $X_{k,n} = \{(x_1, \ldots, x_n) \mid \mathring{A}\mathbf{x} \geq 0, 1 \leq x_1 \leq \lfloor \frac{k}{2} \rfloor\}$. Note that $X_{l,n} \subseteq X_{k,n}$ whenever $l \leq k$ and $X_{k,n} = X_{k+1,n}$ whenever k is even.

Example 3.13.5. $D_6^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.13.4,

$$
\begin{aligned}
\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 & - l\alpha_0 - (l - (x_1))\alpha_1 - (l - (x_1 + l_2))\alpha_2 \\
& - (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4 \\
& - (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.
\end{aligned}
$$

We list the $\mathbf{x}$ vectors and corresponding values for l, l_2, l_3, l_4, and l_5 in addition to the resulting maximal dominant weight in Table 3.48.

Example 3.13.6. $D_7^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.13.4,

$$
\begin{aligned}
\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 & - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2 \\
& - (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4 \\
& - (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5 - (l - (x_1 + l_2 + l_3 + l_4 + l_5 + l_6))\alpha_6.
\end{aligned}
$$

We list the $\mathbf{x}$ vectors and corresponding values for l, l_2, l_3, l_4, l_5, and l_6 in addition to the resulting maximal dominant weight in Table 3.49.

Table 3.48: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0,0	$2\Lambda_0$
(1,1,1,1,1)	1	0,0,0,0	$2\Lambda_0 - \alpha_0$
(1,2,2,2,2)	2	1,0,0,0	$2\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3)	3	1,1,0,0	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4)	4	1,1,1,0	$2\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5)	5	1,1,1,1	$2\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

Table 3.49: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_7^{(2)}$-module $V(2\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5, l_6	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0,0	$2\Lambda_0$
(1,1,1,1,1,1)	1	0,0,0,0,0	$2\Lambda_0 - \alpha_0$
(1,2,2,2,2,2)	2	1,0,0,0,0	$2\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3,3)	3	1,1,0,0,0	$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4,4)	4	1,1,1,0,0	$2\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,5)	5	1,1,1,1,0	$2\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(1,2,3,4,5,6)	6	1,1,1,1,1	$2\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$

One can show that our formulation for the maximal dominant weights of $V(k\Lambda_0)$ correspond to that of [18] for the case $k = 2$ for arbitrary $n \geq 2$. According to [18], the set of maximal dominant weights for $V(2\Lambda_0)$ is $\{1 + \delta_{u,n}\Lambda_u - u\delta | 0 \leq u \leq n\}$. We run into an issue since for $u = 0$ the maximal dominant weight is Λ_0, not $2\Lambda_0$, though we propose this is due to a small generalization error in [18]. To show that this formulation matches ours, first write our formulation in terms of the fundamental dominant weights, Λ_i. With this conversion, our set becomes

$$X = \{2\Lambda_0\} \cup \{2\Lambda_0 - l(2\Lambda_0 - \Lambda_1 + \delta) - (l-1)(-2\Lambda_0 + 2\Lambda_1 - \Lambda_2)$$

$$- \left(\sum_{i=2}^{n-2}(l - (1 + \sum_{j=2}^{i} l_j)) \right)(-\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1})$$

$$- \left(l - (1 + \sum_{j=2}^{n-1} l_j) \right)(-\Lambda_{n-2} + 2\Lambda_{n-1} - 2\Lambda_n)$$

$$- (l - \left(1 + \sum_{j=2}^{n} l_j \right))(-\Lambda_{n-1} + 2\Lambda_n).$$

Showing these two sets are the same is very similar to how we showed the same for type $B_n^{(1)}$.

Example 3.13.7. $D_6^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.13.4,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - (x_1))\alpha_1 - (l - (x_1 + l_2))\alpha_2$$

Table 3.50: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0,0	$3\Lambda_0$
(1,1,1,1,1)	1	0,0,0,0	$3\Lambda_0 - \alpha_0$
(1,2,2,2,2)	2	1,0,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3)	3	1,1,0,0	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4)	4	1,1,1,0	$3\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5)	5	1,1,1,1	$3\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

Table 3.51: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_7^{(2)}$-module $V(3\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5, l_6	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0,0	$3\Lambda_0$
(1,1,1,1,1,1)	1	0,0,0,0,0	$3\Lambda_0 - \alpha_0$
(1,2,2,2,2,2)	2	1,0,0,0,0	$3\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3,3)	3	1,1,0,0,0	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4,4)	4	1,1,1,0,0	$3\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,5)	5	1,1,1,1,0	$3\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(1,2,3,4,5,6)	6	1,1,1,1,1	$3\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$

$$- (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4$$
$$- (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

Recall that the set of **x** vectors for $V(3\Lambda_0)$ is the same as that for $V(2\Lambda_0)$. Then we have the results listed in Table 3.50.

Example 3.13.8. $D_7^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.13.4,

$$\max(3\Lambda_0) \cap P^+ = \{3\Lambda_0\} \cup \{3\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2$$
$$- (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4$$
$$- (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5 - (l - (x_1 + l_2 + l_3 + l_4 + l_5 + l_6))\alpha_6.$$

Recall that the set of **x** vectors for $V(3\Lambda_0)$ is the same as that for $V(2\Lambda_0)$. Then we have the results listed in Table 3.51.

Example 3.13.9. $D_6^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.13.4, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$\max(4\Lambda_0) \cap P^+ = \{4\Lambda_0\} \cup \{4\Lambda_0 - l\alpha_0 - (l - (x_1))\alpha_1 - (l - (x_1 + l_2))\alpha_2$$
$$- (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4$$
$$- (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5.$$

We list the **x** vectors and corresponding values for $l, l_2, l_3, l_4,$ and l_5 in addition to the resulting

Table 3.52: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_6^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0)	0	0,0,0,0	$4\Lambda_0$
(1,1,1,1,1)	1	0,0,0,0	$4\Lambda_0 - \alpha_0$
(1,2,2,2,2)	2	1,0,0,0	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3)	3	1,1,0,0	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4)	4	1,1,1,0	$4\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5)	5	1,1,1,1	$4\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,2,2,2,2)	2	0,0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,3,3,3,3)	3	1,0,0,0	$4\Lambda_0 - 3\alpha_0 - \alpha_1$
(2,3,4,4,4)	4	1,1,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,5)	5	1,1,1,0	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(2,3,4,5,6)	6	1,1,1,1	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,4,4,4)	4	2,0,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1$
(2,4,5,5,5)	5	2,1,0,0	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - \alpha_2$
(2,4,5,6,6)	6	2,1,1,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7)	7	2,1,1,1	$4\Lambda_0 - 7\alpha_0 - 5\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,6,6)	6	2,2,0,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,7)	7	2,2,1,0	$4\Lambda_0 - 7\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$
(2,4,6,7,8)	8	2,2,1,1	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,8,8)	8	2,2,2,0	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9)	9	2,2,2,1	$4\Lambda_0 - 9\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - \alpha_4$
(2,4,6,8,10)	10	2,2,2,2	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$

maximal dominant weight in Table 3.52. Note that since $X_{3,5} \subset X_{4,5}$ and so we include all **x** vectors from the previous $n = 5$ example.

Example 3.13.10. $D_7^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.13.4, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$\max(4\Lambda_0) \cap P^+ = \{4\Lambda_0\} \cup \{4\Lambda_0 - l\alpha_0 - (l - x_1)\alpha_1 - (l - (x_1 + l_2))\alpha_2$$
$$- (l - (x_1 + l_2 + l_3))\alpha_3 - (l - (x_1 + l_2 + l_3 + l_4))\alpha_4$$
$$- (l - (x_1 + l_2 + l_3 + l_4 + l_5))\alpha_5 - (l - (x_1 + l_2 + l_3 + l_4 + l_5 + l_6))\alpha_6.$$

We list the **x** vectors and corresponding values for $l, l_2, l_3, l_4, l_5,$ and l_6 in addition to the resulting maximal dominant weight in Table 3.53. Note that since $X_{3,6} \subset X_{4,6}$ and so we include all **x** vectors from the previous $n = 6$ example.

Table 3.53: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_7^{(2)}$-module $V(4\Lambda_0)$

x vector	l	l_2, l_3, l_4, l_5, l_6	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0,0,0)	0	0,0,0,0,0	$4\Lambda_0$
(1,1,1,1,1,1)	1	0,0,0,0,0	$4\Lambda_0 - \alpha_0$
(1,2,2,2,2,2)	2	1,0,0,0,0	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(1,2,3,3,3,3)	3	1,1,0,0,0	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(1,2,3,4,4,4)	4	1,1,1,0,0	$4\Lambda_0 - 4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(1,2,3,4,5,5)	5	1,1,1,1,0	$4\Lambda_0 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(1,2,3,4,5,6)	6	1,1,1,1,1	$4\Lambda_0 - 6\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,2,2,2,2,2)	2	0,0,0,0,0	$4\Lambda_0 - 2\alpha_0$
(2,3,3,3,3,3)	3	1,0,0,0,0	$4\Lambda_0 - 3\alpha_0 - \alpha_1$
(2,3,4,4,4,4)	4	1,1,0,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - \alpha_2$
(2,3,4,5,5,5)	5	1,1,1,0,0	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$
(2,3,4,5,6,6)	6	1,1,1,1.0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,3,4,5,6,7)	7	1,1,1,1,1	$4\Lambda_0 - 7\alpha_0 - 5\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,4,4,4,4)	4	2,0,0,0,0	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1$
(2,4,5,5,5,5)	5	2,1,0,0,0	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - \alpha_2$
(2,4,5,6,6,6)	6	2,1,1,0,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$
(2,4,5,6,7,7)	7	2,1,1,1,0	$4\Lambda_0 - 7\alpha_0 - 5\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,5,6,7,8)	8	2,1,1,1,1	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,6,6,6)	6	2,2,0,0,0	$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$
(2,4,6,7,7,7)	7	2,2,1,0,0	$4\Lambda_0 - 7\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$
(2,4,6,7,8,8)	8	2,2,1,1,0	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
(2,4,6,7,8,9)	9	2,2,1,1,1	$4\Lambda_0 - 9\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,8,8)	8	2,2,2,0,0	$4\Lambda_0 - 8\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$
(2,4,6,8,9,9)	9	2,2,2,1,0	$4\Lambda_0 - 9\alpha_0 - 7\alpha_1 - 5\alpha_2 - 3\alpha_3 - \alpha_4$
(2,4,6,8,9,10)	10	2,2,2,1,1	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
(2,4,6,8,10,10)	10	2,2,2,2,0	$4\Lambda_0 - 10\alpha_0 - 8\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
(2,4,6,8,10,11)	11	2,2,2,2,1	$4\Lambda_0 - 11\alpha_0 - 9\alpha_1 - 7\alpha_2 - 5\alpha_3 - 3\alpha_4 - \alpha_5$
(2,4,6,8,10,12)	12	2,2,2,2,2	$4\Lambda_0 - 12\alpha_0 - 10\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$

3.14 Type $E_6^{(2)}$

Let $\mathfrak{g} = E_6^{(2)}$, index set $I = \{0, 1, 2, 3, 4\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

where $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 2b_0 - b_3$ and $x_4 = b_0 - b_4$. That is,

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - 2x_3 & \geq 0 \\ -x_2 + 2x_3 - x_4 & \geq 0 \\ -x_3 + 2x_4 & \geq 0 \\ x_1 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{2}, x_4\}$.

Lemma 3.14.1. *The set of solutions to*

$$\begin{cases} 2x_1 - x_2 & \geq 0 \\ -x_1 + 2x_2 - 2x_3 & \geq 0 \\ -x_2 + 2x_3 - x_4 & \geq 0 \\ -x_3 + 2x_4 & \geq 0 \\ x_1 & \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, x_3, x_4) \mid 2 \leq x_1 \leq k, 3\left\lceil\frac{x_1}{2}\right\rceil \leq x_2 \leq 2x_1, \left\lceil\frac{2}{3}x_2\right\rceil \leq x_3 \leq x_2 - \left\lceil\frac{x_1}{2}\right\rceil,$$

$$and \left\lceil\frac{x_3}{2}\right\rceil \leq x_4 \leq 2x_3 - x_2\}.$$

Proof. As stated above, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_1 = 0$, which implies

$\mathbf{x} = 0$, or $x_1 \geq 1$. Consider the possibility of $x_1 = 1$. Then by $(\mathring{A}\mathbf{x})_1 \geq 0$, we have $x_2 \leq 2$. x_2 cannot be 0 as we have observed. If $x_2 = 1$, this would force $x_3 = 0$, which leads to a contradiction as well. Finally, if $x_2 = 2$, then $x_3 = 1$, forcing x_4 to be 0, another contradiction. Therefore, $x_1 \neq 1$.

Now, fix x_1 such that $2 \leq x_1 \leq k$. First, notice that by $(\mathring{A}\mathbf{x})_3 \geq 0$, $x_3 \geq \frac{x_2}{2} + \frac{x_4}{2}$. Combining this with $(\mathring{A}\mathbf{x})_4 \geq 0$, we obtain $x_3 \geq \frac{x_2}{2} + \frac{x_3}{4}$. Then $3x_3 \geq 2x_2$ or $\frac{x_3}{2} \geq \frac{x_2}{3}$. We now can establish the bounds for x_2. By $(\mathring{A}\mathbf{x})_1 \geq 0$, $x_2 \leq 2x_1$. From $(\mathring{A}\mathbf{x})_2 \geq 0$, we have $x_2 \geq \frac{x_1 + 2x_3}{2}$. Since x_2 must be an integer, this is equivalent to $x_2 \geq \left\lceil \frac{x_1 + 2x_3}{2} \right\rceil$. It is a quick calculation to check that $\left\lceil \frac{x_1 + 2x_3}{2} \right\rceil = \left\lceil \frac{x_1}{2} \right\rceil + x_3$. Combining this with our observation above, we obtain $x_2 \geq \left\lceil \frac{x_1}{2} \right\rceil + \frac{2}{3}x_2$. This implies $\frac{x_2}{3} \geq \left\lceil \frac{x_1}{2} \right\rceil$ and so $x_2 \geq 3\left\lceil \frac{x_1}{2} \right\rceil$ as desired. Moving on to the bounds for x_3, we already have the lower bound from our observation; since $3x_3 \geq 2x_2$ and x_3 must be an integer, we have $x_3 \geq \left\lceil \frac{2}{3}x_2 \right\rceil$. From $(\mathring{A}\mathbf{x})_2 \geq 0$, we obtain $2x_3 \leq 2x_2 - x_1$ and so $x_3 \leq \left\lfloor \frac{2x_2 - x_1}{2} \right\rfloor$ since x_3 must be an integer. It is a quick calculation to check that $\left\lfloor \frac{2x_2 - x_1}{2} \right\rfloor = x_2 - \left\lceil \frac{x_1}{2} \right\rceil$. Finally, we establish the bounds for x_4. From $(\mathring{A}\mathbf{x})_4 \geq 0$ and the fact that x_4 must be an integer, we have $x_4 \geq \left\lceil \frac{x_3}{2} \right\rceil$. Finally, $(\mathring{A}\mathbf{x})_3 \geq 0$ gives us that $x_4 \leq 2x_3 - x_2$. This describes all possible solutions. $\qquad \square$

Theorem 3.14.2. *Let* $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3$ *where*

- $2 \leq x_1 \leq k$,

- $3\left\lceil \frac{x_1}{2} \right\rceil \leq x_2 \leq 2x_1$,

- $\left\lceil \frac{2}{3}x_2 \right\rceil \leq x_3 \leq x_2 - \left\lceil \frac{x_1}{2} \right\rceil$,

- $\left\lceil \frac{x_3}{2} \right\rceil \leq l \leq 2x_3 - x_2$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1, x_2 = 3b_0 - b_2, x_3 = 2b_0 - b_3$, and $x_4 = b_0 - b_4$. Then $(b_0, b_1, b_2, b_3, b_4) = (b_0, 2b_0 - x_1, 3b_0 - x_2, 2b_0 - x_3, b_0 - x_4)$. By definition, the $b_i \geq 0$ for $0 \leq i \leq 4$. Therefore, the following must be true:

$$b_0 \geq 0$$
$$2b_0 - x_1 \geq 0$$
$$3b_0 - x_2 \geq 0$$
$$2b_0 - x_3 \geq 0$$
$$b_0 - x_4 \geq 0$$

Then $b_0 \geq \max\{\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{2}, x_4\}$. Recall that $x_4 \geq \frac{x_3}{2}$. We also observed in our proof of Lemma 3.14.1 that $\frac{x_3}{2} \geq \frac{x_2}{3}$. Finally, we found in the proof of the same lemma that $\frac{x_2}{3} \geq \left\lceil \frac{x_1}{2} \right\rceil$ and so $\frac{x_2}{3} \geq \frac{x_1}{2}$. Then it suffices to say $b_0 \geq x_4$. Then let $l = x_4$ and we claim that $b_0 = l$.

Table 3.54: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(2)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0,0,0)	0	$2\Lambda_0$
(2,3,2,1)	1	$2\Lambda_0 - \alpha_0$
(2,4,3,2)	2	$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - \alpha_3$

Table 3.55: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(2)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0,0,0)	0	$3\Lambda_0$
(2,3,2,1)	1	$3\Lambda_0 - \alpha_0$
(2,4,3,2)	2	$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - \alpha_3$
(3,6,4,2)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1$

This can be shown using the same method that is used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3 - (l - x_4)\alpha_4$. Combining this with our solutions for the x_i from Lemma 3.14.1, we have the pattern given above. □

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2, x_3, x_4) \mid \mathring{A}\mathbf{x} \geq 0, 2 \leq x_1 \leq k\}$. Note that $X_l \subseteq X_k$ whenever $l \leq k$.

Example 3.14.3. $E_6^{(2)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.14.2, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.54.

Example 3.14.4. $E_6^{(2)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.14.2, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.55.

Example 3.14.5. $E_6^{(2)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.14.2, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant

Table 3.56: **x** vectors and maximal dominant weights for the $\mathfrak{g} = E_6^{(2)}$-module $V(4\Lambda_0)$

x vector	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0,0,0)	0	$4\Lambda_0$
(2,3,2,1)	1	$4\Lambda_0 - \alpha_0$
(2,4,3,2)	2	$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - \alpha_3$
(3,6,4,2)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(4,6,4,2)	2	$4\Lambda_0 - 2\alpha_0$
(4,7,5,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2 - \alpha_3$
(4,8,6,3)	3	$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$
(4,8,6,4)	4	$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 4\alpha_2 - 2\alpha_3$

weight in Table 3.56. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

3.15 Type $D_4^{(3)}$

Let $\mathfrak{g} = D_4^{(3)}$, index set $I = \{0, 1, 2\}$, and Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

The defining inequalities are equivalent to

$$\begin{cases} A\mathbf{x} \geq 0 \\ x_1 \leq k \end{cases}$$

where $x_1 = 2b_0 - b_1$ and $x_2 = b_0 - b_2$. That is,

$$\begin{cases} 2x_1 - 3x_2 & \geq 0 \\ -x_1 + 2x_2 & \geq 0 \\ x_1 & \leq k \end{cases}$$

Recall that we are looking for the nontrivial solutions, $\mathbf{x} > 0$. Note as well that this means $b_0 \geq \frac{x_1}{2}$ and $b_0 \geq x_2$.

Lemma 3.15.1. *The set of solutions to*

$$\begin{cases} 2x_1 - 3x_2 & \geq 0 \\ -x_1 + 2x_2 & \geq 0 \\ x_1 & \leq k \end{cases}$$

is

$$\{\mathbf{x} = \mathbf{0}\} \cup \left\{\mathbf{x} = (x_1, x_2) \mid 2 \le x_1 \le k, \left\lceil \frac{x_1}{2} \right\rceil \le x_2 \le \left\lfloor \frac{2x_1}{3} \right\rfloor\right\}.$$

Proof. As stated earlier, since $\mathring{A}$ is of finite type, $\mathbf{x} > 0$ or $\mathbf{x} = 0$ and so $x_1 = 0$, which implies $\mathbf{x} = 0$, or $x_1 \ge 1$. However, by $(\mathring{A}\mathbf{x})_1 \ge 0$ and $(\mathring{A}\mathbf{x})_2 \ge 0$, $\frac{x_1}{2} \le x_2 \le \frac{2}{3}x_1$ and since x_2 must be an integer, $x_1 \ne 1$.

Now, fix x_1 such that $2 \le x_1 \le k$. Then by $(\mathring{A}\mathbf{x})_1 \ge 0$, $x_2 \le \frac{2}{3}x_1$ and by $(\mathring{A}\mathbf{x})_2 \ge 0, x_2 \ge \frac{1}{2}x_1$. Since x_2 must be an integer, we have $\left\lceil \frac{x_1}{2} \right\rceil \le x_2 \le \left\lfloor \frac{2x_1}{3} \right\rfloor$. This describes all possible solutions. $\qquad\square$

Theorem 3.15.2. *Let* $\Lambda = k\Lambda_0, k \ge 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1$ *where*

- $2 \le x_1 \le k$,
- $\left\lceil \frac{x_1}{2} \right\rceil \le l \le \left\lfloor \frac{2x_1}{3} \right\rfloor$.

Proof. Most of this proof is identical to that of Theorem 3.3.3. In this case, the map gives that $x_1 = 2b_0 - b_1$ and $x_2 = b_0 - b_2$. Then $(b_0, b_1, b_2) = (b_0, 2b_0 - x_1, b_0 - x_2)$. Then if we can determine b_0, we will have the set of all maximal dominant weights since we have already found the pattern for x_1 and x_2. By definition, the $b_i \ge 0$ for $0 \le i \le 2$. Therefore, the following must be true:

$$b_0 \ge 0$$
$$2b_0 - x_1 \ge 0$$
$$b_0 - x_2 \ge 0$$

Then $b_0 \ge \frac{x_2}{2}$ and $b_0 \ge x_2$. However, from $(\mathring{A}\mathbf{x})_2 \ge 0$, we have $x_2 \ge \frac{x_1}{2}$. Then it suffices to say $b_0 \ge x_2$. Then let $l = x_2$ and so $b_0 \ge l$. We claim that $b_0 = l$. This can be shown using the same method that is used in the proof of Theorem 3.3.3. Then $\lambda = \Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (l - x_2)\alpha_2$. Combining this with our solutions for the x_i from Lemma 3.15.1, we have the pattern given above. $\qquad\square$

To better understand this theorem, consider some examples for various values of k. For ease in our future examples, define $X_k = \{(x_1, x_2) \mid \mathring{A}\mathring{x} \ge 0, 2 \le x_1 \le k\}$. Note that $X_l \subseteq X_k$ whenever $l \le k$.

Example 3.15.3. $D_4^{(3)}, V(\Lambda) = V(2\Lambda_0)$. By Theorem 3.15.2, any element of $\max(2\Lambda_0) \cap P^+$ other than $2\Lambda_0$ is of the form

$$2\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1.$$

We list the $\mathbf{x}$ vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.57.

Table 3.57: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_4^{(3)}$-module $V(2\Lambda_0)$

x vector	l	Element of $\max(2\Lambda_0) \cap P^+$
(0,0)	0	$2\Lambda_0$
(2,1)	1	$2\Lambda_0 - \alpha_0$

Table 3.58: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_4^{(3)}$-module $V(3\Lambda_0)$

x vector	l	Element of $\max(3\Lambda_0) \cap P^+$
(0,0)	0	$3\Lambda_0$
(2,1)	1	$3\Lambda_0 - \alpha_0$
(3,2)	2	$3\Lambda_0 - 2\alpha_0 - \alpha_1$

Example 3.15.4. $D_4^{(3)}, V(\Lambda) = V(3\Lambda_0)$. By Theorem 3.15.2, an element of $\max(3\Lambda_0) \cap P^+$ other than $3\Lambda_0$ is of the form

$$3\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.58.

Example 3.15.5. $D_4^{(3)}, V(\Lambda) = V(4\Lambda_0)$. By Theorem 3.15.2, any element of $\max(4\Lambda_0) \cap P^+$ other than $4\Lambda_0$ is of the form

$$4\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.59. Notice that we can collect the same **x** vectors from the case $V(3\Lambda_0)$ since, as we said earlier, $X_3 \subset X_4$.

Example 3.15.6. $D_4^{(3)}, V(\Lambda) = V(6\Lambda_0)$. By Theorem 3.15.2, any element of $\max(6\Lambda_0) \cap P^+$ other than $6\Lambda_0$ is of the form

$$6\Lambda_0 - l\alpha_0 - (2l - x_1)\alpha_1.$$

We list the **x** vectors and corresponding value for l in addition to the resulting maximal dominant weight in Table 3.60.

Table 3.59: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_4^{(3)}$-module $V(4\Lambda_0)$

x vector	l	Element of $\max(4\Lambda_0) \cap P^+$
(0,0)	0	$4\Lambda_0$
(2,1)	1	$4\Lambda_0 - \alpha_0$
(3,2)	2	$4\Lambda_0 - 2\alpha_0 - \alpha_1$
(4,2)	2	$4\Lambda_0 - 2\alpha_0$

Table 3.60: **x** vectors and maximal dominant weights for the $\mathfrak{g} = D_4^{(3)}$-module $V(6\Lambda_0)$

x vector	l	Element of $\max(6\Lambda_0) \cap P^+$
(0,0)	0	$6\Lambda_0$
(2,1)	1	$6\Lambda_0 - \alpha_0$
(3,2)	2	$6\Lambda_0 - 2\alpha_0 - \alpha_1$
(4,2)	2	$6\Lambda_0 - 2\alpha_0$
(5,3)	3	$6\Lambda_0 - 3\alpha_0 - \alpha_1$
(6,3)	3	$6\Lambda_0 - 3\alpha_0$
(6,4)	4	$6\Lambda_0 - 4\alpha_0 - 2\alpha_1$

Chapter 4

Crystal Bases

4.1 Introduction

In Chapter 5, we investigate the dimension of the weight spaces corresponding to some of the maximal dominant weights we found in Chapter 3. To do so, we introduce a combinatorial object called a *crystal base*. We first need the notion of a quantum deformation of the universal enveloping algebra, $\mathcal{U}_q(\mathfrak{g})$, which is known as a *quantum group*.

In the following, we let q be an indeterminate such that $q^m \neq 1$ for any nonzero m. A q-analog is a way to generalize aspects of mathematics such that as the classical limit of q approaches 1, we obtain the original aspect. For example, a q-integer, denoted $[n]_q$ is defined to be $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. We can check that as $q \to 1$, this expression becomes n.

$$\lim_{q \to 1} \frac{q^n - q^{-n}}{q - q^{-1}} = \lim_{q \to 1} \frac{nq^{n-1} + nq^{-n-1}}{1 + q^{-2}}$$
$$= \frac{2n}{2}$$
$$= n$$

The first step above is obtained by applying L'Hôpital's rule. Note that we obtain the following:

$$[1]_q = 1$$
$$[2]_q = \frac{q^2 - q^{-2}}{q - q^{-1}} = \frac{(q - q^{-1})(q + q^{-1})}{q - q^{-1}} = q + q^{-1}$$
$$[3]_q = \frac{q^3 - q^{-3}}{q - q^{-1}} = \frac{(q - q^{-1})(q^2 + 1 + q^{-2})}{q - q^{-1}} = q^2 + 1 + q^{-2}$$
$$\vdots$$
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}$$

A q-factorial is defined as follows:

$$[n]_q! = [n]_q[n-1]_q[n-2]_q \cdots [2]_q[1]_q \text{ for all } n \in \mathbb{Z}_{>0}$$

The q-binomial coefficient is then $[\binom{m}{n}]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$.

For convenience, we remind the reader of a universal enveloping algebra, which was mentioned in Chapter 2, before introducing the quantum group.

Definition 4.1.1. A *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ is a pair $(\mathcal{U}(\mathfrak{g}), j)$ where $\mathcal{U}(\mathfrak{g})$ is an associative algebra with unity over $\mathbb{C}$ and $j : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is a linear map satisfying $j([x,y]) = j(x)j(y) - j(y)j(x)$ for all $x, y \in \mathfrak{g}$. If A is any other associative algebra with a linear map ϕ satisfying $\phi([x,y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for all $x, y \in \mathfrak{g}$ then there exists a unique homomorphism of algebras $\psi : \mathcal{U}(\mathfrak{g}) \to A$ such that $\psi \circ j = \phi$.

Again, we work with a realization, $(\mathfrak{h}, \Pi, \Pi^\vee)$ of an affine Cartan matrix A, corresponding weight lattice P, and dual weight lattice $P^\vee$. Let $D = \text{diag}(s_i \in \mathbb{Z}_+ | i \in I)$ be the diagonal matrix making DA symmetric. Then the quantum deformation of $\mathcal{U}(\mathfrak{g})$ is defined as follows.

Definition 4.1.2. The *quantum group*, $\mathcal{U}_q(\mathfrak{g})$, is the associative algebra over $\mathbb{C}(q)$ with unity generated by the elements $e_i, f_i,$ and q^h where $i \in I$ and $h \in P^\vee$ and satisfying the following relations:

- $q^0 = 1, q^h q^{h'} = q^{h+h'}$,

- $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$,

- $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,

- $e_i f_j - f_j e_i = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$,

- $\sum_{k=0}^{1-a_{ij}} (-1)^k [\binom{1-a_{ij}}{k}]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

- $\sum_{k=0}^{1-a_{ij}} (-1)^k [\binom{1-a_{ij}}{k}]_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$,

where $h, h' \in P^\vee, i, j \in I$, and $q_i = q^{s_i}$.

As in the case of an affine Lie algebra, we can discuss weight modules of $\mathcal{U}_q(\mathfrak{g})$. A $\mathcal{U}_q(\mathfrak{g})$-module V^q is called a *weight module* if it has a weight space decomposition $V^q = \oplus_{\mu \in P} V_\mu^q$ where $V_\mu^q = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$. If $V_\mu^q \neq 0$, then μ is called a *weight* of V^q, V_μ^q the corresponding *weight space*, $\dim V_\mu^q$ the *multiplicity* of μ, and $0 \neq u \in V_\mu^q$ is a *μ-weight vector*.

A weight module $V^q(\lambda)$ is called a *highest weight module* with *highest weight* $\lambda \in P$ if there exists a nonzero $v_\lambda \in V^q(\lambda)$ such that

- $V^q(\lambda) = \mathcal{U}_q(\mathfrak{g}) \cdot v_\lambda$,

- $e_i \cdot v_\lambda = 0$ for all $i \in I$,

- $h \cdot v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$.

In this case, v_λ is called the *highest weight vector*.

We are interested in $\mathcal{U}_q(\mathfrak{g})$ highest weight modules because of the following theorem from [19]:

Theorem 4.1.3. *For $\lambda \in P^+$, let $V^q(\lambda)$ $(V(\lambda))$ be the irreducible highest weight $\mathcal{U}_q(\mathfrak{g})$ (respectively $\mathcal{U}(\mathfrak{g})$) module with highest weight λ and highest weight vector v_λ. Then*

$$ch(V^q(\lambda)) = ch(V(\lambda))$$

This implies that for every $\mu \in P$, $\dim V^q(\lambda)_\mu = \dim V(\lambda)_\mu$. Since the representation theory of $\mathcal{U}(\mathfrak{g})$ is the same as that of $\mathfrak{g}$, we have $\dim V^q(\lambda)_\mu = V(\lambda)_\mu$ where $V(\lambda)$ is considered as a $\mathfrak{g}$-module. Crystal bases, which we will introduce shortly, are combinatorial objects that can be used to determine the dimensions of weight spaces in the case of $V^q(\lambda)$. As a result, we will know the corresponding dimensions in the case of the $\mathfrak{g}$-module $V(\lambda)$.

4.2 Crystal Bases

The notion of a crystal basis was introduced by Kashiwara [17]. First, let $V^q = \oplus_{\lambda \in P} V^q_\lambda$ be an integrable $\mathcal{U}_q(\mathfrak{g})$-module such that the dimensions of all the weight spaces are finite and there exist $\lambda_1, \ldots, \lambda_s \in P$ such that the set of all weights of V^q is contained within $D(\lambda_1) \cup \cdots \cup D(\lambda_s)$ where $D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}$. Then for each $i \in I$, every vector $v \in V^q_\lambda$ can be written in the form

$$v = v_0 + f_i^{(1)} v_1 + \cdots + f_i^{(N)} v_N$$

where N is a nonnegative integer, $v_k \in V^q_{\lambda + k\alpha_i} \cap \ker e_i$ for all $k = 0, 1, \ldots, N$, and $f_i^{(k)} = \frac{f_i^k}{[k]_{q_i}!}$.

The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$ on V^q are defined by

$$\tilde{e}_i v = \sum_{k=1}^{N} f_i^{(k-1)} v_k$$

$$\tilde{f}_i v = \sum_{k=0}^{N} f_i^{(k+1)} v_k$$

They act on the weight spaces of V^q in the following way:

$$\tilde{e}_i V^q_\lambda \subset V^q_{\lambda + \alpha_i}$$
$$\tilde{f}_i V^q_\lambda \subset V^q_{\lambda - \alpha_i}$$

Consider the principal ideal domain $\mathbb{A}_0$ with $\mathbb{C}(q)$ its fraction field:

$$\mathbb{A}_0 = \left\{ \frac{g(q)}{h(q)} \mid g(q), h(q) \in \mathbb{C}[q], h(0) \neq 0 \right\}.$$

Then we have

Definition 4.2.1. [5] Let V^q be an integrable $\mathcal{U}_q(\mathfrak{g})$-module. A free $\mathbb{A}_0$-submodule $\mathcal{L}$ of V^q is called a *crystal lattice* if

- $\mathcal{L}$ generates V^q as a vector space over $\mathbb{C}(q)$

- $\mathcal{L} = \oplus_{\lambda \in P} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap V_\lambda^q$ for all $\lambda \in P$

- $\tilde{e}_i \mathcal{L} \subset \mathcal{L}, \tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$

With the notion of a crystal lattice, we can now present the definition of a crystal basis.

Definition 4.2.2. [5] A *crystal basis* or *crystal base* of an integrable $\mathcal{U}_q(\mathfrak{g})$-module V^q is a pair $(\mathcal{L}, \mathcal{B})$ such that

- $\mathcal{L}$ is a crystal lattice of V^q

- $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}_0} \mathcal{L}$

- $\mathcal{B} = \sqcup_{\lambda \in P} \mathcal{B}_\lambda$ where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$

- $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}, \tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ for all $i \in I$

- for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$

To each crystal base we can associate a directed graph. We take $\mathcal{B}$ as the set of vertices and we define I-colored arrows on $\mathcal{B}$ by

$$b \xrightarrow{i} b' \text{ if and only if } \tilde{f}_i b = b'$$

The resulting oriented graph with coloring I is called the *crystal graph* of V^q. This graph allows us to determine the multiplicities of the maximal dominant weights we found in Chapter 3, because of the following theorem.

Theorem 4.2.3. *[5] Let V^q be an integrable $\mathcal{U}_q(\mathfrak{g})$-module and let $(\mathcal{L}, \mathcal{B})$ be a crystal basis of V^q. Then for all $\lambda \in P$,*

$$mult\lambda = \#\mathcal{B}_\lambda$$

To determine the multiplicity of each $\lambda \in \max(k\Lambda_0) \cap P^+$, we need only know the number of elements in $\mathcal{B}_\lambda$. In order to determine the crystal graph $\mathcal{B}(k\Lambda_0)$, we need to define perfect crystals, which are used in the path realization of $\mathcal{B}(k\Lambda_0)$.

To do so, we introduce an alternative way to view the crystal graph, $\mathcal{B}$. We need several more maps.

$\mathrm{wt} : \mathcal{B} \to P$ so that $b \in \mathcal{B}_\lambda$ maps to its corresponding weight, that is, $\mathrm{wt}(b) = \lambda$

$\varepsilon_i : \mathcal{B} \to \mathbb{Z}$ defined by $\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\}$ for all $i \in I$

$\phi_i : \mathcal{B} \to \mathbb{Z}$ defined by $\phi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}$ for all $i \in I$

Definition 4.2.4. Given an (affine) Cartan matrix A with corresponding $\Pi, \Pi^\vee, P, P^\vee$, a *crystal* is a set $\mathcal{B}$ together with the maps

$$\mathrm{wt} : \mathcal{B} \to P$$
$$\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$$
$$\varepsilon_i, \phi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$$

for all $i \in I$ satisfying the following properties:

- $\phi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$ for all $i \in I$,

- $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt} b + \alpha_i$ if $\tilde{e}_i b \in \mathcal{B}$,

- $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt} b - \alpha_i$ if $\tilde{f}_i b \in \mathcal{B}$,

- $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \phi_i(\tilde{e}_i b) = \phi_i(b) + 1$ if $\tilde{e}_i b \in \mathcal{B}$,

- $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \phi_i(\tilde{f}_i b) = \phi_i(b) - 1$ if $\tilde{f}_i b \in \mathcal{B}$,

- $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}, i \in I$,

- if $\phi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Then we can refer to $\mathcal{B}$ as simply a $\mathcal{U}_q(\mathfrak{g})$-crystal. In this case, $\mathcal{B}_\lambda = \{b \in \mathcal{B} \mid \mathrm{wt}(b) = \lambda\}$ so that $\mathcal{B} = \sqcup_{\lambda \in P} \mathcal{B}_\lambda$.

Later, we will need to consider the *tensor product of crystals* and the action of the Kashiwara operators on such an object.

Definition 4.2.5. The *tensor product*, $\mathcal{B}_1 \otimes \mathcal{B}_2$, of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is the set $\mathcal{B}_1 \times \mathcal{B}_2$ with crystal structure defined by

- $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

- $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1) \rangle)$,

- $\phi_i(b_1 \otimes b_2) = \max(\phi_i(b_2), \phi_i(b_1) + \langle h_i, \mathrm{wt}(b_2) \rangle)$,

- $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \phi_i(b_1) < \varepsilon_i(b_2) \end{cases}$,

- $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$.

We also consider maps between crystals.

Definition 4.2.6. [5] Let $\mathcal{B}_1, \mathcal{B}_2$ be crystals. A *crystal morphism* $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that:

- $\Psi(0) = 0$,

- if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b), \varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\phi_i(\Psi(b)) = \phi_i(b)$ for all $i \in I$,

- if $b, b' \in \mathcal{B}_1, \Psi(b), \Psi(b') \in \mathcal{B}_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.

A crystal morphism is called an *isomorphism* if it is a bijection from $\mathcal{B}_1 \cup \{0\}$ to $\mathcal{B}_2 \cup \{0\}$.

4.3   Perfect Crystals

Consider the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $e_i, f_i, q_i^{h_i}$, and $q_i^{-h_i}$ for $i \in I$, denoted $\mathcal{U}'_q(\mathfrak{g})$. Let $\overline{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$ and $\overline{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \overline{P}^\vee$. We consider the α_i and Λ_i as linear functionals on $\overline{\mathfrak{h}}$ and define $\overline{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$. The elements of $\overline{P}$ are called the *classical weights* and $\mathcal{U}'_q(\mathfrak{g})$ is the quantum group associated with $A, \Pi, \Pi^\vee, \overline{P}$ and $\overline{P}^\vee$. We also consider the *classical dominant weights*, $\overline{P}^+ = \{\lambda \in \overline{P} \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$. A classical dominant weight has *level* $l \geq 0$ if $\lambda(c) = l$.

We can define a finite dimensional $\mathcal{U}'_q(\mathfrak{g})$-module, V^q, as we did for $\mathcal{U}_q(\mathfrak{g})$-modules. The corresponding crystal $\mathcal{B}$ is called a *classical crystal*. Given an element b of a classical crystal $\mathcal{B}$, we define

$$\varepsilon(b) = \sum_i \varepsilon_i(b) \Lambda_i$$

$$\phi(b) = \sum_i \phi_i(b) \Lambda_i$$

Notice that $\overline{\mathrm{wt}}(b) = \phi(b) - \varepsilon(b)$. Finally, for $l \in \mathbb{Z}_{>0}$, let $\overline{P}_l^+ = \{\lambda \in \overline{P}^+ \mid \langle c, \lambda \rangle = l\}$.

We now define perfect crystals, which are used to study the path realizations.

Definition 4.3.1. [14], [15] $\mathcal{B}$ is a *perfect crystal of level l* if it satisfies the following:

- there exists a finite dimensional $U'_q(\mathfrak{g})$-module with crystal basis $\mathcal{B}$,

- $\mathcal{B} \otimes \mathcal{B}$ is connected,

- there exists a classical weight λ_0 such that $\overline{\mathrm{wt}}(\mathcal{B}) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0}\alpha_i$ with $\#(\mathcal{B}_{\lambda_0}) = 1$,

- for any $b \in \mathcal{B}$, we have $\langle c, \varepsilon(b) \rangle \geq l$,

- for each $\lambda \in \overline{P}_l^+$ there exist unique vectors $b^\lambda \in \mathcal{B}$ and $b_\lambda \in \mathcal{B}$ such that $\varepsilon(b^\lambda) = \lambda = \phi(b_\lambda)$.

Given a perfect crystal $\mathcal{B}$, define $\mathcal{B}^{\min} = \{b \in \mathcal{B} \mid \langle c, \varepsilon(b) \rangle = l\}$, making the maps $\varepsilon, \phi :$ $\mathcal{B}^{\min} \to \overline{P}_l^+$ bijections.

We include an example of a perfect crystal, obtained via SageMath [22], in Figure 4.1. Recall the edge color i corresponds to the action of $\tilde{f}_i$ on an element $b \in \mathcal{B}$.

4.4   Path Realizations

We now have all the notions required to explore the realization of the $\mathcal{U}_q'(\mathfrak{g})$-crystal $\mathcal{B}(\lambda)$. First, we fix our level $l > 0$ and let $\mathcal{B}$ be a perfect crystal of level l. It is known [14], [15] that for any $\lambda \in \overline{P}_l^+$, there exists a unique crystal isomorphism

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B} \text{ given by } v_\lambda \mapsto v_{\varepsilon(b_\lambda)} \otimes b_\lambda$$

where b_λ is the unique vector in $\mathcal{B}$ such that $\phi(b_\lambda) = \lambda$. We set

$$\lambda_0 = \lambda, \lambda_{k+1} = \varepsilon(b_{\lambda_k}), b_0 = b_\lambda, b_{k+1} = b_{\lambda_{k+1}}.$$

Then we obtain a sequence

$$\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_2 \otimes b_1 \otimes b_0 \in \cdots \otimes B \otimes B \otimes B,$$

called the λ-*ground state path.*

Example 4.4.1. In the perfect crystal in Figure 4.1, $b_0 = \boxed{\overline{1}\ \overline{1}}$ and $\varepsilon(b_0) = 2\Lambda_1$. Then $b_1 = \boxed{1\ 1}$. and $\varepsilon(b_1) = 2\Lambda_0$. Then the $2\Lambda_0$-ground state path is

$$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\overline{1}\ \overline{1}} \otimes \boxed{1\ 1} \otimes \boxed{\overline{1}\ \overline{1}}$$

A λ-*path* in $\mathcal{B}$ is a sequence

$$\mathbf{p} = (\mathbf{p}_k)_{k=0}^\infty = \cdots \otimes \mathbf{p}_2 \otimes \mathbf{p}_1 \otimes \mathbf{p}_0$$

such that $\mathbf{p}_k \in \mathcal{B}$ for all k and $\mathbf{p}_j = b_j$ for all $j \gg 0$. Let $\mathcal{P}(\lambda)$ be the set of all λ-paths. Let $\mathcal{B}(\lambda)$ be the crystal associated with $V(\lambda)$ with highest weight vector v_λ. We use the following
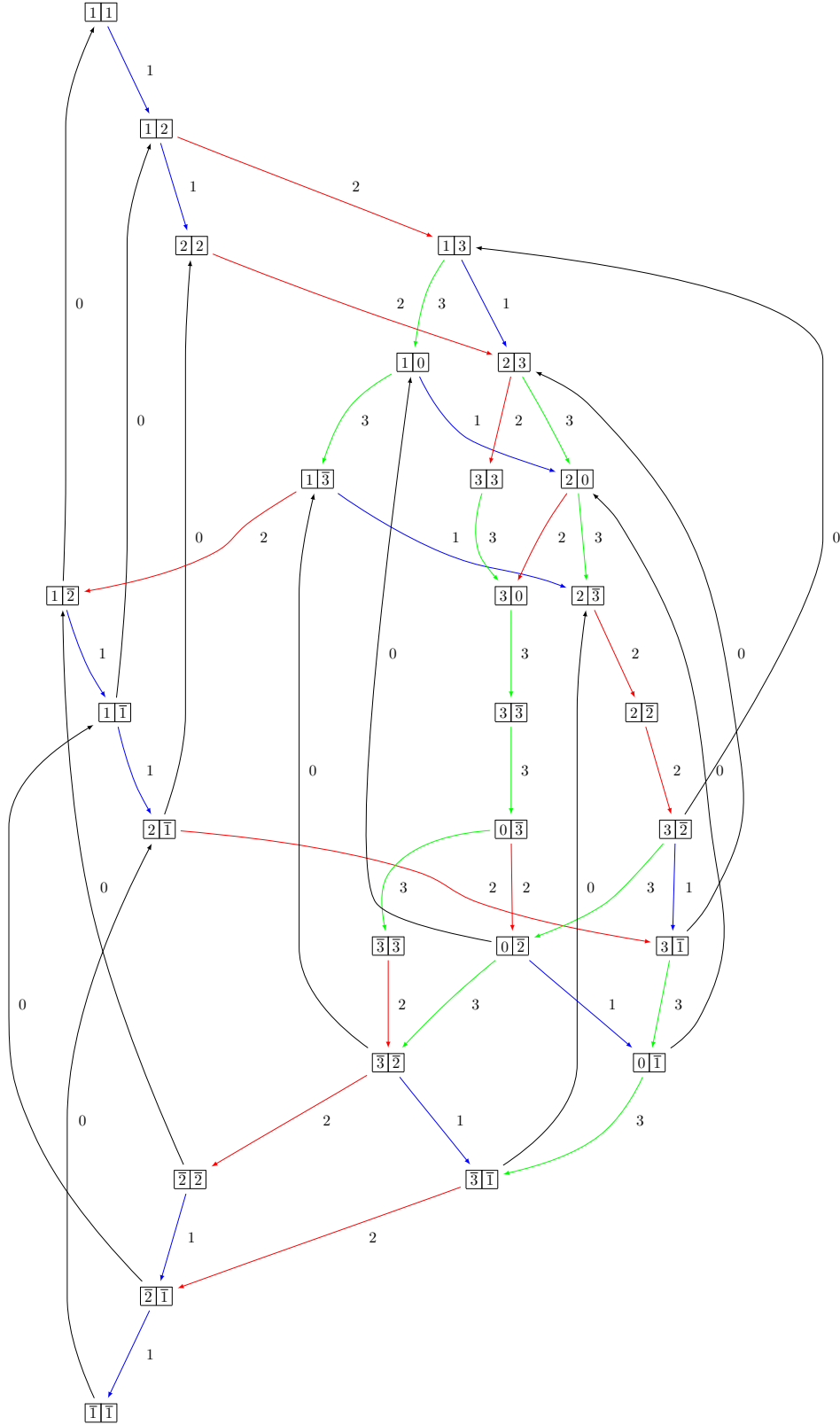
Figure 4.1: The $B_3^{(1)}$ perfect crystal of level 2.

important fact [14], [15], [5] to realize $\mathcal{B}(\lambda)$:

$$\mathcal{B}(\lambda) \cong \mathcal{P}(\lambda)$$
$$v_\lambda \mapsto \mathbf{p}_\lambda = \cdots \otimes b_2 \otimes b_1 \otimes b_0$$

We use the tensor product rule to determine the edges in the corresponding crystal graph.

We give an example in Figure 4.2 of the part of the beginning of the path realization $\mathcal{P}(2\Lambda_0)$ in the case $\mathcal{B}(2\Lambda_0)$ using the perfect crystal $\mathcal{B}$ from Figure 4.1. Because of space constraints, we only include edges and vertices that lead to maximal dominant weights. Notice that the weight of a particular path in this path realization is $2\Lambda_0 - \sum_{i=0}^{3} b_i \alpha_i$ where b_i is the number of i-arrows required to arrive at the path from the ground state path. This is due to the fact that the arrows are the action of $\tilde{f}_i$ and $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$. We would like to determine the weight of any path p for any $\mathfrak{g}$ without looking at a crystal graph in this way.
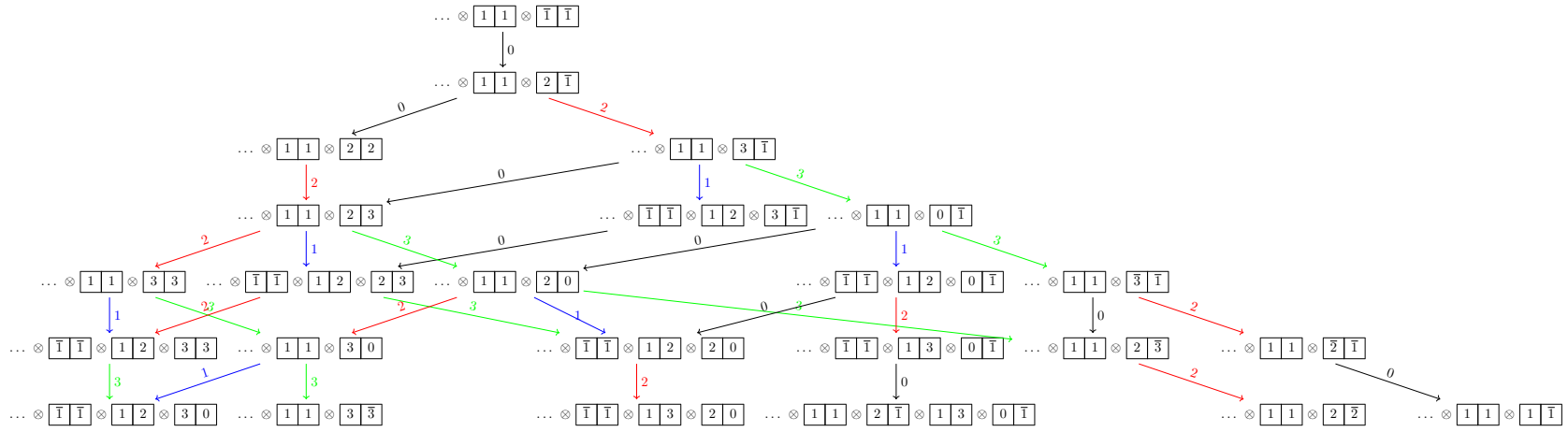
Figure 4.2: Part of the path realization of the $\mathcal{U}_q'(B_3^{(1)})$-crystal, $\mathcal{B}(2\Lambda_0)$.

To determine the affine weight of a path $\mathbf{p}$ within $\mathcal{P}(\lambda)$, we need to introduce another concept. Given a finite dimension $\mathcal{U}'_q(\mathfrak{g})$-module V^q with crystal basis $(\mathcal{L}, \mathcal{B})$, we have the following definition.

Definition 4.4.2. [14], [15], [5] An *energy function* on $\mathcal{B}$ is a $\mathbb{Z}$-valued function $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying the following conditions:

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \phi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \phi_0(b_1) < \varepsilon_0(b_2) \end{cases}$$

for all $i \in I$, $b_1 \otimes b_2 \in \mathcal{B} \otimes \mathcal{B}$ with $\tilde{e}_i(b_1 \otimes b_2) \in \mathcal{B} \otimes \mathcal{B}$.

Then the affine weight of $\mathbf{p}$ is given by the formula [14], [15]

$$\text{wt}(\mathbf{p}) = \lambda + \sum_{k=0}^{\infty} (\overline{\text{wt}}\mathbf{p}_k - \overline{\text{wt}}b_k)$$
$$- \left(\sum_{k=0}^{\infty} (k+1)(H(\mathbf{p}_{k+1} \otimes \mathbf{p}_k) - H(b_{k+1} \otimes b_k)) \right) \delta,$$

where $\overline{\text{wt}}$ denotes the classical weight. This formula can be used to verify the weight of a given path. Our goal is to use it to determine the paths corresponding to a specific maximal dominant weight. Some of the paths listed in Chapter 5 are found using this formula, while others were found using SageMath [22].

Chapter 5

Weight Multiplicities

5.1 Introduction

In this chapter, we collect results for some multiplicities of the maximal dominant weights we found in Chapter 3 of the $\mathfrak{g}$-module $V(k\Lambda_0)$ for specific affine Lie algebras $\mathfrak{g}$. We begin by including the results in [7], which are found using extended Young diagrams. We then include the perfect crystal, from [13] required to produce the path realization for $V(k\Lambda_0)$ in types $B_n^{(1)}, C_n^{(1)}$, and $D_n^{(1)}$. The tables of multiplicities for maximal dominant weights were produced using SageMath [22].

Note that $k\Lambda_0$ is a dominant integral weight of level k. As discussed in Chapter 4, the corresponding crystal, $\mathcal{B}(k\Lambda_0)$ is isomorphic to a semi-infinite tensor product of perfect crystals of level k, $\cdots \otimes \mathcal{B}_k \otimes \mathcal{B}_k \otimes \mathcal{B}_k$. In this isomorphism, the highest weight vector $v_{k\Lambda_0}$ is mapped to the path $\cdots \otimes b_2 \otimes b_1 \otimes b_0$ where each $b_j \in \mathcal{B}_k^{\min} = \{b \in \mathcal{B}_k \mid \varepsilon(b) = k\}$ is given by the following:

$$
\begin{aligned}
\lambda_0 &= k\Lambda_0 & b_0 &= b_{k\Lambda_0} \\
\lambda_{j+1} &= \varepsilon(b_{\lambda_j}) & b_{j+1} &= b_{\lambda_{j+1}}
\end{aligned}
$$

where b_λ is the unique vector in $\mathcal{B}_k^{\min}$ such that $\phi(b_\lambda) = \lambda$ and $\phi, \varepsilon : \mathcal{B}_k^{\min} \to \overline{P}_l^+$ are the bijections defined in Chapter 4. We will also need the maps $\tilde{e}_i, \tilde{f}_i : \mathcal{B}_k \to \mathcal{B} \cup \{0\}, \varepsilon_i, \phi_i : \mathcal{B}_k \to \mathbb{Z}$, and $\overline{wt} : \mathcal{B}_k \to \overline{P}$. Recall as well that the affine weight of a path $\mathbf{p} \in \mathcal{P}(k\Lambda_0)$ is given by the formula

$$
\begin{aligned}
\mathrm{wt}(\mathbf{p}) = k\Lambda_0 &+ \sum_{j=0}^{\infty} (\overline{\mathrm{wt}}(p_j) - \overline{\mathrm{wt}}(b_j)) \\
&- \Big(\sum_{j=0}^{\infty} (j+1)\big(H(p_{j+1} \otimes p_j) - H(b_{j+1} \otimes b_j)\big) a_0^{-1} \Big) \delta
\end{aligned}
$$

Notice that $a_0 = 1$ in all types except $A_{2n}^{(2)}$, which is not discussed here.

5.2 Type $A_n^{(1)}$

The multiplicities of maximal dominant weights of the form $\{k\Lambda_0 - \gamma_l \mid 1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor\}$ where $\gamma_l = l\alpha_0 + (l-1)\alpha_1 + (l-2)\alpha_2 + \cdots + \alpha_{l-1} + \alpha_{(n+1)-l+1} + \cdots + (l-2)\alpha_{n-1} + (l-1)\alpha_n$ were studied in [7]. They used extended Young diagrams $Y = (-l, -l, -l, \ldots, -l, 0, 0, \ldots)$, containing l instances of $-l$, and drew a sequence of $k-1$ lattice paths, $p_1, p_2, \ldots, p_{k-1}$. The sequence was drawn "from the lower left to the upper right corner of the square, moving only up and to the right in such a way that for each color, the number of colored boxes of that same color below p_i is greater than or equal to the number of colored boxes of that same color below p_{i-1}" citeJayneMisra. They define t_i^j for $i \geq 2$ to be the number of j-colored boxes between p_{i-1} and p_{i-2}. They have the following rules to deem a sequence of lattice paths *admissible*.

- The first path, p_1, cannot not cross the diagonal $y = x - l$.

- For every i such that $3 \leq i \leq k-1$,

  - $t_i^j \leq \min\left\{t_{i-1}^j, l - |j| - t_2^j - \sum_{a=2}^{i-1} t_a^j\right\}$
  - for $j > 0, t_i^j \leq t_i^{j-1} \leq t_i^{j-2} \leq \cdots \leq t_i^1 \leq t_i^0$ and for $j < 0, t_i^j \leq t_i^{j+1} \leq t_i^{j+2} \leq \cdots \leq t_i^{-1} \leq t_i^0$.

Then they define $\mathcal{T}_l^k$ to be the set of admissible sequences of $(k-1)$ paths in an $l \times l$ square.

Theorem 5.2.1. *[7] Consider the maximal dominant weights $k\Lambda_0 - \gamma_l \in \max(k\Lambda_0) \cap P^+$, where $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$. The multiplicity of $k\Lambda_0 - \gamma_l$ in $V(k\Lambda_0)$ is equal to $|\mathcal{T}_l^k|$.*

Further study has been done for the multiplicities of other maximal dominant weights in this type, see [8].

5.3 Type $B_n^{(1)}$

In this case [13],

$$\mathcal{B}_k = \{(x_1, \ldots, x_n, x_0, x_{\bar{n}}, x_{\overline{n-1}}, \ldots, x_{\bar{1}}) \in \mathbb{Z}^{2n} \times \{0, 1\} \mid x_0 = 0 \text{ or } 1,$$

$$x_i, x_{\bar{i}} \geq 0, x_0 + \sum_{i=1}^n x_i + \sum_{i=1}^n x_{\bar{i}} = k\}.$$

Given an element $b = (x_1, \ldots, x_n, x_0, x_{\bar{n}}, \ldots, x_{\bar{1}}) \in \mathcal{B}_k$, the actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ for $i = 0, 1, \ldots, n$ are defined as follows:

$$\tilde{e}_0(b) = \begin{cases} (x_1, x_2 - 1, x_3, \ldots, x_{\bar{2}}, x_{\bar{1}} + 1) & \text{if } x_2 > x_{\bar{2}} \\ (x_1 - 1, x_2, \ldots, x_{\bar{3}}, x_{\bar{2}} + 1, x_{\bar{1}}) & \text{if } x_2 \leq x_{\bar{2}} \end{cases}$$

$$\tilde{e}_i(b) = \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} > x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} + 1, x_{\bar{i}} - 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} \leq x_{\overline{i+1}} \end{cases} \text{ for } 1 \leq i \leq n-1$$

$$\tilde{e}_n(b) = \begin{cases} (x_1, \ldots, x_n, x_0 + 1, x_{\bar{n}} - 1, \ldots, x_{\bar{1}}) & \text{if } x_0 = 0 \\ (x_1, \ldots, x_n + 1, x_0 - 1, x_{\bar{n}}, \ldots, x_{\bar{1}}) & \text{if } x_0 = 1 \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} (x_1, x_2 + 1, x_3, \ldots, x_{\bar{2}}, x_{\bar{1}} - 1) & \text{if } x_2 \geq x_{\bar{2}} \\ (x_1 + 1, x_2, \ldots, x_{\bar{3}}, x_{\bar{2}} - 1, x_{\bar{1}}) & \text{if } x_2 < x_{\bar{2}} \end{cases}$$

$$\tilde{f}_i(b) = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} \geq x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} - 1, x_{\bar{i}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} < x_{\overline{i+1}} \end{cases} \text{ for } 1 \leq i \leq n-1$$

$$\tilde{f}_n(b) = \begin{cases} (x_1, \ldots, x_n - 1, x_0 + 1, x_{\bar{n}}, \ldots, x_{\bar{1}}) & \text{if } x_0 = 0 \\ (x_1, \ldots, x_n, x_0 - 1, x_{\bar{n}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_0 = 1 \end{cases}$$

We also define the maps $\varepsilon_i, \phi_i : \mathcal{B}_k \to \mathbb{Z}$. We use the notation that $(y)_+ = \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$.

$$\varepsilon_0(b) = x_1 + (x_2 - x_{\bar{2}})_+$$
$$\phi_0(b) = x_{\bar{1}} + (x_{\bar{2}} - x_2)_+$$
$$\varepsilon_i(b) = x_{\bar{i}} + (x_{i+1} - x_{\overline{i+1}})_+$$
$$\phi_i(b) = x_i + (x_{\overline{i+1}} - x_{i+1})_+$$
$$\varepsilon_n(b) = 2x_{\bar{n}} + x_0$$
$$\phi_n(b) = 2x_n + x_0$$

where $1 \leq i \leq n - 1$. Then we have

$$\mathcal{B}_k^{\min} = \{(x_1, \ldots, x_n, 0, x_n, \ldots, x_2, x_{\bar{1}}) \in \mathbb{Z}^{2n} \times \{0\} \mid x_{\bar{1}}, x_i \geq 0, x_1 + x_{\bar{1}} + 2\sum_{i=2}^n x_i = k\}$$

$$\cup \{(x_1, \ldots, x_n, 1, x_n, \ldots, x_2, x_{\bar{1}}) \in \mathbb{Z}^{2n} \times \{1\} \mid x_{\bar{1}}, x_i \geq 0, x_1 + x_{\bar{1}} + 2\sum_{i=2}^n x_i = k\}$$

So the $k\Lambda_0$-ground state path is

$$\cdots \otimes (k, 0, \ldots, 0) \otimes (0, \ldots, 0, k) \otimes (k, 0, \ldots, 0) \otimes (0, \ldots, 0, k)$$

which we can represent with tableaux, as in Chapter 4. In this case, if $x_i = j$, then we have j boxes colored i and similar for $x_{\bar{i}}$.

Finally, we need the formula for the energy function $H : \mathcal{B}_k \otimes \mathcal{B}_k \to \mathbb{Z}$ in this case. Given

Table 5.1: Elements of $\max(2\Lambda_0)\cap P^+$ and their corresponding multiplicities and paths for the $B_3^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$2\Lambda_0$	1	$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{2\ \bar{1}}$
$2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3$	1	$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{0\ \bar{1}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 3} \otimes \boxed{2\ 0}$ $\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 2} \otimes \boxed{3\ 0}$ $\cdots \otimes \boxed{1\ 1} \otimes \boxed{2\ \bar{1}} \otimes \boxed{1\ 3} \otimes \boxed{0\ \bar{1}}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3$	3	$\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{1\ \bar{1}}$ $\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{2\ \bar{2}}$ $\cdots \otimes \boxed{1\ 1} \otimes \boxed{\bar{1}\ \bar{1}} \otimes \boxed{1\ 1} \otimes \boxed{3\ \bar{3}}$

$b, b' \in \mathcal{B}_k$,

$$H(b \otimes b') = \max(\{\theta_j(b \otimes b'), \theta'_j(b \otimes b') \mid 1 \leq j \leq n-1\}$$
$$\{\eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leq j \leq n-1\})$$

where

$$\theta_j(b \otimes b') = \sum_{l=1}^{j}(x_{\bar{l}} - x'_{\bar{l}})$$

$$\theta'_j(b \otimes b') = \sum_{l=1}^{j}(x'_l - x_l)$$

$$\eta_j(b \otimes b') = \sum_{l=1}^{j}(x_{\bar{l}} - x'_{\bar{l}}) + (x'_{\bar{j}} - x_j)$$

$$\eta'_j(b \otimes b') = \sum_{l=1}^{j}(x'_l - x_l) + (x_j - x'_{\bar{j}})$$

The reader can verify the results in Table 5.1 by finding paths corresponding to each maximal dominant weight in Figure 4.2.

Table 5.2: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $B_4^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths							
$2\Lambda_0$	1	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{1}}$
$2\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4$	1	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{0}\,\boxed{\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{2}\,\boxed{4}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{2}$	$\otimes$	$\boxed{3}\,\boxed{4}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{4}\,\boxed{\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$	3	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{2}\,\boxed{0}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{2}$	$\otimes$	$\boxed{3}\,\boxed{0}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{0}\,\boxed{\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$	4	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{1}\,\boxed{\bar{1}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{2}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{3}\,\boxed{\bar{3}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{4}\,\boxed{\bar{4}}$

Table 5.3: Elements of $\max(3\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $B_3^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths				
$3\Lambda_0$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	
$3\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{1}\,\bar{1}}$	
$3\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{0\,\bar{1}\,\bar{1}}$	
$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{2\,0\,\bar{1}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,2}$	$\otimes \boxed{3\,0\,\bar{1}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{0\,\bar{1}\,\bar{1}}$	
$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{1\,\bar{1}\,\bar{1}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{2}\,\bar{1}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{3\,\bar{3}\,\bar{1}}$	
$3\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,0\,\bar{1}}$	
$3\Lambda_0 - 2\alpha_0 - \alpha_2$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,3\,\bar{1}}$	
$3\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{1\,0\,\bar{1}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,0\,\bar{2}}$	
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{3\,0\,\bar{3}}$	

5.4 Type $C_n^{(1)}$

In this case [13],

$$\mathcal{B}_k = \{(x_1,\ldots,x_n, x_{\overline{n}}, x_{\overline{n-1}},\ldots, x_{\overline{1}}) \in \mathbb{Z}^{2n} \mid x_i, x_{\overline{i}} \geq 0, \sum_{i=1}^n x_i + \sum_{i=1}^n x_{\overline{i}} \leq 2k,$$

$$\sum_{i=1}^n x_i + \sum_{i=1}^n x_{\overline{i}} \in 2\mathbb{Z}\}.$$

Given an element $b = (x_1,\ldots,x_n,x_{\overline{n}},\ldots,x_{\overline{1}}) \in \mathcal{B}_k$, define $s(b) = \sum_{i=1}^n x_i + \sum_{i=1}^n x_{\overline{i}}$. The actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ for $i = 0, 1, \ldots, n$ are defined as follows:

$$\tilde{e}_0(b) = \begin{cases} (x_1 - 2, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}}) & \text{if } x_1 \geq x_{\overline{1}} + 2 \\ (x_1 - 1, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}} + 1) & \text{if } x_1 = x_{\overline{1}} + 1 \\ (x_1, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}} + 2) & \text{if } x_1 \leq x_{\overline{1}} \end{cases}$$

$$\tilde{e}_i(b) = \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_{\overline{1}}) & \text{if } x_{i+1} > x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} + 1, x_{\overline{i}} - 1, \ldots, x_{\overline{1}}) & \text{if } x_{i+1} \leq x_{\overline{i+1}} \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$\tilde{e}_n(b) = (x_1, \ldots, x_n + 1, x_{\overline{n}} - 1, \ldots, x_{\overline{1}})$$

$$\tilde{f}_0(b) = \begin{cases} (x_1 + 2, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}}) & \text{if } x_1 \geq x_{\overline{1}} \\ (x_1 + 1, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}} - 1) & \text{if } x_1 = x_{\overline{1}} - 1 \\ (x_1, x_2, \ldots, x_{\overline{2}}, x_{\overline{1}} - 2) & \text{if } x_1 \leq x_{\overline{1}} - 2 \end{cases}$$

$$\tilde{f}_i(b) = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_{\overline{1}}) & \text{if } x_{i+1} \geq x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} - 1, x_{\overline{i}} + 1, \ldots, x_{\overline{1}}) & \text{if } x_{i+1} < x_{\overline{i+1}} \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$\tilde{f}_n(b) = (x_1, \ldots, x_n - 1, x_{\overline{n}} + 1, \ldots, x_{\overline{1}})$$

We also define the maps $\varepsilon_i, \phi_i : \mathcal{B}_k \to \mathbb{Z}$. We use the notation that $(y)_+ = \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$.

$$\varepsilon_0(b) = k - \frac{1}{2}s(b) + 2(x_1 - x_{\overline{1}})_+$$

$$\phi_0(b) = k - \frac{1}{2}s(b) + 2(x_{\overline{1}} - x_1)_+$$

$$\varepsilon_i(b) = x_{\overline{i}} + (x_{i+1} - x_{\overline{i+1}})_+$$

$$\phi_i(b) = x_i + (x_{\overline{i+1}} - x_{i+1})_+$$

$$\varepsilon_n(b) = x_{\overline{n}}$$

$$\phi_n(b) = x_n$$

where $1 \leq i \leq n-1$. Then we have

$$\mathcal{B}_k^{\min} = \{(x_1, \ldots, x_n, x_n, \ldots, x_2, x_1) \in \mathbb{Z}^{2n} \mid x_i \geq 0, \sum_{i=1}^{n} x_i \leq k\}$$

Then the $k\Lambda_0$-ground state path is

$$\cdots \otimes (0, \ldots, 0) \otimes (0, \ldots, 0) \otimes (0, \ldots, 0) \otimes (0, \ldots, 0)$$

Again, we can represent paths using tableaux similar to those described in type $B_n^{(1)}$. Since our ground state path does not have any entries, we do not color the corresponding tableaux.

Finally, we need the formula for the energy function $H : \mathcal{B}_k \otimes \mathcal{B}_k \to \mathbb{Z}$ in this case. Given $b, b' \in \mathcal{B}_k$,

$$H(b \otimes b') = \max(\{\theta_j(b \otimes b'), \theta'_j(b \otimes b') \mid 1 \leq j \leq n\}$$
$$\{\eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leq j \leq n\})$$

where

$$\theta_j(b \otimes b') = \sum_{l=1}^{j-1} (x_{\bar{l}} - x'_{\bar{l}}) + \frac{1}{2}(s(b') - s(b))$$

$$\theta'_j(b \otimes b') = \sum_{l=1}^{j} (x'_l - x_l) + \frac{1}{2}(s(b) - s(b'))$$

$$\eta_j(b \otimes b') = \sum_{l=1}^{j} (x_{\bar{l}} - x'_{\bar{l}}) + (x'_{\bar{j}} - x_j) + \frac{1}{2}(s(b') - s(b))$$

$$\eta'_j(b \otimes b') = \sum_{l=1}^{j} (x'_l - x_l) + (x'_j - x'_{\bar{j}}) + \frac{1}{2}(s(b) - s(b'))$$

Table 5.4: Elements of $\max(2\Lambda_0)\cap P^+$ and their corresponding multiplicities and paths for the $C_2^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$2\Lambda_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \square$
$2\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;2}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;2\;2}$

Table 5.5: Elements of $\max(3\Lambda_0)\cap P^+$ and their corresponding multiplicities and paths for the $C_2^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$3\Lambda_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \square$
$3\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;2}$
$3\Lambda_0 - \alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1}$
$3\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;2\;2}$
$3\Lambda_0 - 2\alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;2}$
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;2\;2\;2}$

Table 5.6: Elements of $\max(4\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $C_2^{(1)}$-module $V(4\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$4\Lambda_0$	1	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ }$
$4\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,2}$
$4\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1}$
$4\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1} \otimes \boxed{2\,2}$ $\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,2\,2}$
$4\Lambda_0 - 2\alpha_0 - \alpha_1$	1	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,2}$
$4\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1} \otimes \boxed{1\,2\,2\,2}$ $\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,2\,2\,2}$
$4\Lambda_0 - 2\alpha_0$	1	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,1}$
$4\Lambda_0 - 3\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1} \otimes \boxed{1\,1\,2\,2}$ $\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,1\,2\,2}$
$4\Lambda_0 - 4\alpha_0 - 4\alpha_1$	3	$\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,1\,2\,2\,2\,2}$ $\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1} \otimes \boxed{1\,1\,2\,2\,2\,2}$ $\cdots \otimes \boxed{\ } \otimes \boxed{\ } \otimes \boxed{1\,1\,1\,1} \otimes \boxed{2\,2\,2\,2}$

Table 5.7: Elements of $\max(5\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $C_2^{(1)}$-module $V(5\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$5\Lambda_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \square$
$5\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;2}$
$5\Lambda_0 - \alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1}$
$5\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;2\;2}$
$5\Lambda_0 - 2\alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;2}$
$5\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;2\;2\;2}$
$5\Lambda_0 - 2\alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1}$
$5\Lambda_0 - 3\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;1\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1\;2\;2}$
$5\Lambda_0 - 4\alpha_0 - 4\alpha_1$	3	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1\;2\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;1\;2\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1} \otimes \boxed{2\;2\;2\;2}$
$5\Lambda_0 - 5\alpha_0 - 5\alpha_1$	3	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1\;1\;2\;2\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;1\;1\;2\;2\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1} \otimes \boxed{1\;2\;2\;2\;2\;2}$
$5\Lambda_0 - 3\alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1\;1\;2}$
$5\Lambda_0 - 4\alpha_0 - 3\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\;1} \otimes \boxed{1\;1\;1\;2\;2\;2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\;1\;1\;1\;1\;2\;2\;2}$

Table 5.8: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $C_3^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$2\Lambda_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \square$
$2\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{2}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1} \otimes \boxed{2}\,\boxed{3}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{3}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1} \otimes \boxed{2}\,\boxed{2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{2}$
$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	4	$\cdots \otimes \square \otimes \square \otimes \boxed{2}\,\boxed{2} \otimes \boxed{1}\,\boxed{1}\,\boxed{3}\,\boxed{3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{2} \otimes \boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1}\,\boxed{1} \otimes \boxed{2}\,\boxed{2}\,\boxed{3}\,\boxed{3}$ $\cdots \otimes \square \otimes \boxed{1}\,\boxed{1} \otimes \boxed{2}\,\boxed{2} \otimes \boxed{3}\,\boxed{3}$

5.5 Type $D_n^{(1)}$

In this case [13],

$$\mathcal{B}_k = \{(x_1, \ldots, x_n, x_{\bar{n}}, x_{\overline{n-1}}, \ldots, x_{\bar{1}}) \in \mathbb{Z}^{2n} \mid x_i, x_{\bar{i}} \geq 0, \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_{\bar{i}} = k\}.$$

Given an element $b = (x_1, \ldots, x_n, x_{\bar{n}}, \ldots, x_{\bar{1}}) \in \mathcal{B}_k$, the actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ for $i = 0, 1, \ldots, n$ are defined as follows:

$$\tilde{e}_0(b) = \begin{cases} (x_1, x_2 - 1, x_3, \ldots, x_{\bar{2}}, x_{\bar{1}} + 1) & \text{if } x_2 > x_{\bar{2}} \\ (x_1 - 1, x_2, \ldots, x_{\bar{3}}, x_{\bar{2}} + 1, x_{\bar{1}}) & \text{if } x_2 \leq x_{\bar{2}} \end{cases}$$

$$\tilde{e}_i(b) = \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} > x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} + 1, x_{\bar{i}} - 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} \leq x_{\overline{i+1}} \end{cases} \quad \text{for } 1 \leq i \leq n-2$$

$$\tilde{e}_{n-1}(b) = \begin{cases} (x_1, \ldots, x_{n-1} + 1, x_n - 1, x_{\bar{n}}, \ldots, x_{\bar{1}}) & \text{if } x_n > 0, x_{\bar{n}} = 0 \\ (x_1, \ldots, x_n, x_{\bar{n}} + 1, x_{\overline{n-1}} - 1,, \ldots, x_{\bar{1}}) & \text{if } x_n = 0, x_{\bar{n}} \geq 0 \end{cases}$$

Table 5.9: Elements of $\max(3\Lambda_0)\cap P^+$ and their corresponding multiplicities and paths for the $C_3^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$3\Lambda_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \square$
$3\Lambda_0 - \alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,2}$
$3\Lambda_0 - \alpha_0$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1}$
$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{2\,3}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,2\,3}$
$3\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{2\,2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,2\,2}$
$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	5	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,2\,2\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{2\,2} \otimes \boxed{1\,1\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,2} \otimes \boxed{1\,2\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{2\,2\,3\,3}$ $\cdots \otimes \square \otimes \boxed{1\,1} \otimes \boxed{2\,2} \otimes \boxed{3\,3}$
$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$	5	$\cdots \otimes \square \otimes \square \otimes \boxed{2\,2} \otimes \boxed{1\,1\,1\,2\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,2} \otimes \boxed{1\,1\,2\,2\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{1\,2\,2\,2\,3\,3}$ $\cdots \otimes \square \otimes \boxed{1\,1} \otimes \boxed{2\,2} \otimes \boxed{1\,2\,3\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,1\,1\,2} \otimes \boxed{2\,2\,3\,3}$
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2$	3	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,1\,2\,2\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,2} \otimes \boxed{1\,1\,2\,3}$ $\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{1\,2\,2\,3}$
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2	$\cdots \otimes \square \otimes \square \otimes \boxed{1\,1} \otimes \boxed{1\,2\,2\,2}$ $\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,1\,2\,2\,2}$
$3\Lambda_0 - 2\alpha_0 - \alpha_1$	1	$\cdots \otimes \square \otimes \square \otimes \square \otimes \boxed{1\,1\,1\,2}$

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Table 5.10: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $C_4^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$2\Lambda_0$	1
$2\Lambda_0 - \alpha_0 - \alpha_1$	1
$2\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	3
$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$	5
$2\Lambda_0 - \alpha_0$	1
$2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2
$2\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2
$2\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$	10
$2\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	4

Table 5.11: Elements of $\max(3\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $C_4^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$3\Lambda_0$	1
$3\Lambda_0 - \alpha_0 - \alpha_1$	1
$3\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	3
$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$	6
$3\Lambda_0 - \alpha_0$	1
$3\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2
$3\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2
$3\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$	16
$3\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	5
$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$	5
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2$	3
$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - \alpha_3$	12
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2
$3\Lambda_0 - 2\alpha_0 - \alpha_1$	1
$3\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	4
$3\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3$	23
$3\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - 2\alpha_3$	18
$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2 - \alpha_3$	7
$3\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$	10

Table 5.12: Elements of $\max(4\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $C_4^{(1)}$-module $V(4\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$4\Lambda_0$	1
$4\Lambda_0 - \alpha_0 - \alpha_1$	1
$4\Lambda_0 - 2\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	3
$4\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$	6
$4\Lambda_0 - \alpha_0$	1
$4\Lambda_0 - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2
$4\Lambda_0 - 2\alpha_0 - 2\alpha_1$	2
$4\Lambda_0 - 4\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$	17
$4\Lambda_0 - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	5
$4\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$	6
$4\Lambda_0 - 3\alpha_0 - 3\alpha_1 - \alpha_2$	3
$4\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - \alpha_3$	17
$4\Lambda_0 - 3\alpha_0 - 3\alpha_1$	2
$4\Lambda_0 - 2\alpha_0 - \alpha_1$	1
$4\Lambda_0 - 3\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	4
$4\Lambda_0 - 6\alpha_0 - 9\alpha_1 - 6\alpha_2 - 3\alpha_3$	40
$4\Lambda_0 - 5\alpha_0 - 7\alpha_1 - 4\alpha_2 - 2\alpha_3$	24
$4\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 2\alpha_2 - \alpha_3$	8
$4\Lambda_0 - 4\alpha_0 - 5\alpha_1 - 3\alpha_2 - \alpha_3$	11
$4\Lambda_0 - 7\alpha_0 - 10\alpha_1 - 6\alpha_2 - 3\alpha_3$	50
$4\Lambda_0 - 6\alpha_0 - 8\alpha_1 - 4\alpha_2$	11
$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 2\alpha_2$	8
$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 2\alpha_2$	6
$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 2\alpha_2 - \alpha_3$	10
$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 3\alpha_2 - \alpha_3$	15
$4\Lambda_0 - 6\alpha_0 - 8\alpha_1 - 5\alpha_2 - 2\alpha_3$	34
$4\Lambda_0 - 4\alpha_0 - 4\alpha_1$	3
$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - 2\alpha_2 - \alpha_3$	7
$4\Lambda_0 - 7\alpha_0 - 10\alpha_1 - 6\alpha_2 - 2\alpha_3$	44
$4\Lambda_0 - 6\alpha_0 - 8\alpha_1 - 4\alpha_2 - 2\alpha_3$	31
$4\Lambda_0 - 4\alpha_0 - 4\alpha_1 - \alpha_2$	4
$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 3\alpha_2$	8
$4\Lambda_0 - 3\alpha_0 - 2\alpha_1$	2
$4\Lambda_0 - 6\alpha_0 - 8\alpha_1 - 4\alpha_2 - \alpha_3$	21
$4\Lambda_0 - 2\alpha_0$	1
$4\Lambda_0 - 5\alpha_0 - 6\alpha_1 - 4\alpha_2 - 2\alpha_3$	22
$4\Lambda_0 - 8\alpha_0 - 12\alpha_1 - 8\alpha_2 - 4\alpha_3$	66
$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - \alpha_2$	2

$$\tilde{e}_n(b) = \begin{cases} (x_1, \ldots, x_n + 1, x_{\bar{n}}, x_{\overline{n-1}} - 1, \ldots, x_{\bar{1}}) & \text{if } x_n \geq 0, x_{\bar{n}} = 0 \\ (x_1, \ldots, x_{n-1} + 1, x_n, x_{\bar{n}} - 1, \ldots, x_{\bar{1}}) & \text{if } x_n = 0, x_{\bar{n}} > 0 \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} (x_1, x_2 + 1, x_3, \ldots, x_{\bar{2}}, x_{\bar{1}} - 1) & \text{if } x_2 \geq x_{\bar{2}} \\ (x_1 + 1, x_2, \ldots, x_{\bar{3}}, x_{\bar{2}} - 1, x_{\bar{1}}) & \text{if } x_2 < x_{\bar{2}} \end{cases}$$

$$\tilde{f}_i(b) = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} \geq x_{\overline{i+1}} \\ (x_1, \ldots, x_{\overline{i+1}} - 1, x_{\bar{i}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_{i+1} < x_{\overline{i+1}} \end{cases} \quad \text{for } 1 \leq i \leq n-2$$

$$\tilde{f}_{n-1}(b) = \begin{cases} (x_1, \ldots, x_{n-1} - 1, x_n + 1, x_{\bar{n}}, \ldots, x_{\bar{1}}) & \text{if } x_n \geq 0, x_{\bar{n}} = 0 \\ (x_1, \ldots, x_n, x_{\bar{n}} - 1, x_{\overline{n-1}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_n = 0, x_{\bar{n}} > 0 \end{cases}$$

$$\tilde{f}_n(b) = \begin{cases} (x_1, \ldots, x_n - 1, x_{\bar{n}}, x_{\overline{n-1}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_n > 0, x_{\bar{n}} = 0 \\ (x_1, \ldots, x_{n-1} - 1, x_n, x_{\bar{n}} + 1, \ldots, x_{\bar{1}}) & \text{if } x_n = 0, x_{\bar{n}} \geq 0 \end{cases}$$

We also define the maps $\varepsilon_i, \phi_i : \mathcal{B}_k \to \mathbb{Z}$. We use the notation that $(y)_+ = \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$.

$$\varepsilon_0(b) = x_1 + (x_2 - x_{\bar{2}})_+$$
$$\phi_0(b) = x_{\bar{1}} + (x_{\bar{2}} - x_2)_+$$
$$\varepsilon_i(b) = x_{\bar{i}} + (x_{i+1} - x_{\overline{i+1}})_+$$
$$\phi_i(b) = x_i + (x_{\overline{i+1}} - x_{i+1})_+$$
$$\varepsilon_{n-1}(b) = x_{\overline{n-1}} + x_n$$
$$\phi_{n-1}(b) = x_{n-1} + x_{\bar{n}}$$
$$\varepsilon_n(b) = x_{\overline{n-1}} + x_{\bar{n}}$$
$$\phi_n(b) = x_{n-1} + x_n$$

where $1 \leq i \leq n-2$. Then we have

$$\mathcal{B}_k^{\min} = \{(x_1, \ldots, x_{n-1}, x_n, 0, x_{n-1}, \ldots, x_2, x_{\bar{1}}) \in \mathbb{Z}^{2n} \mid x_{\bar{1}}, x_i \geq 0, x_1 + x_{\bar{1}} + 2\sum_{i=2}^{n-1} x_i + x_n = k\}$$

$$\cup \{(x_1, \ldots, x_{n-1}, 0, x_{\bar{n}}, x_{n-1}, \ldots, x_2, x_{\bar{1}}) \in \mathbb{Z}^{2n} \mid x_{\bar{1}}, x_{\bar{n}}, x_i \geq 0, x_1 + x_{\bar{1}} + 2\sum_{i=2}^{n-1} x_i + x_{\bar{n}} = k\}$$

Then the $k\Lambda_0$-ground state path is

$$\cdots \otimes (k, 0, \ldots, 0) \otimes (0, \ldots, 0, k) \otimes (k, 0, \ldots, 0) \otimes (0, \ldots, 0, k)$$

which we can represent with tableaux, as with type $B_n^{(1)}$.

Finally, we need the formula for the energy function $H : \mathcal{B}_k \otimes \mathcal{B}_k \to \mathbb{Z}$ in this case. Given $b, b' \in \mathcal{B}_k$,

$$H(b \otimes b') = \max(\{\theta_j(b \otimes b'), \theta'_j(b \otimes b') \mid 1 \leq j \leq n-2\}$$
$$\{\eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leq j \leq n\})$$

where

$$\theta_j(b \otimes b') = \sum_{l=1}^{j} (x_{\bar{l}} - x'_{\bar{l}}) \text{ for } j = 1, \cdots, n-2$$

$$\theta'_j(b \otimes b') = \sum_{l=1}^{j} (x'_l - x_l) \text{ for } j = 1, \cdots, n-2$$

$$\eta_j(b \otimes b') = \sum_{l=1}^{j} (x_{\bar{l}} - x'_{\bar{l}}) + (x'_{\bar{j}} - x_j) \text{ for } j = 1, \cdots, n-1$$

$$\eta'_j(b \otimes b') = \sum_{l=1}^{j} (x'_l - x_l) + (x_j - x'_{\bar{j}}) \text{ for } j = 1, \cdots, n-1$$

$$\eta_n(b \otimes b') = \sum_{l=1}^{n-1} (x_{\bar{l}} - x'_{\bar{l}}) + x_n$$

$$\eta'_n(b \otimes b') = \sum_{l=1}^{n-1} (x'_l - x_l) - x_n$$

Table 5.13: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $D_4^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$2\Lambda_0$	1	$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;1} \otimes \boxed{2\;\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - \alpha_3 - \alpha_4$	3	$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;1} \otimes \boxed{1\;\bar{1}}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;1} \otimes \boxed{2\;\bar{2}}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;1} \otimes \boxed{3\;\bar{3}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;3} \otimes \boxed{2\;4}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;2} \otimes \boxed{3\;4}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{2\;\bar{1}} \otimes \boxed{1\;3} \otimes \boxed{4\;\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_4$	3	$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;3} \otimes \boxed{2\;\bar{4}}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{\bar{1}\;\bar{1}} \otimes \boxed{1\;2} \otimes \boxed{3\;\bar{4}}$
		$\cdots \otimes \boxed{1\;1} \otimes \boxed{2\;\bar{1}} \otimes \boxed{1\;3} \otimes \boxed{\bar{4}\;\bar{1}}$

Table 5.14: Elements of $\max(3\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $D_4^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths			
$3\Lambda_0$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - \alpha_3 - \alpha_4$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{1\,\bar{1}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{2}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{3\,\bar{3}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{2\,4\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,2}$	$\otimes \boxed{3\,4\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{4\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_4$	3	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{2\,\bar{4}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,2}$	$\otimes \boxed{3\,\bar{4}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,3}$	$\otimes \boxed{\bar{4}\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - \alpha_2$	1	$\cdots \otimes \boxed{1\,1\,1}$	$\otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$	$\otimes \boxed{1\,1\,1}$	$\otimes \boxed{2\,3\,\bar{1}}$

Table 5.15: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $D_5^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths							
$2\Lambda_0$	1	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$
$2\Lambda_0 - \alpha_0$	1	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{1}}$
$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$	4	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{1}\,\boxed{\bar{1}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{2}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{3}\,\boxed{\bar{3}}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{4}\,\boxed{\bar{4}}$
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{2}\,\boxed{4}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{\bar{1}}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{2}$	$\otimes$	$\boxed{3}\,\boxed{4}$
		$\cdots \otimes$	$\boxed{1}\,\boxed{1}$	$\otimes$	$\boxed{2}\,\boxed{\bar{1}}$	$\otimes$	$\boxed{1}\,\boxed{3}$	$\otimes$	$\boxed{4}\,\boxed{\bar{1}}$

Table 5.16: Elements of $\max(3\Lambda_0) \cap P^+$ and their corresponding multiplicities and paths for the $D_5^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Mult.	Paths
$3\Lambda_0$	1	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - \alpha_0$	1	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$	4	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{1\,\bar{1}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{2}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{3\,\bar{3}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{4\,\bar{4}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{2\,4\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,2} \otimes \boxed{3\,4\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{4\,\bar{1}\,\bar{1}}$
$3\Lambda_0 - 2\alpha_0 - \alpha_2$	1	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,1} \otimes \boxed{2\,3\,\bar{1}}$
$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_5$	6	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{2\,3\,\bar{5}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{2\,4\,\bar{5}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{3\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{2\,\bar{5}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,2} \otimes \boxed{3\,4\,\bar{5}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{3\,\bar{5}\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{4\,\bar{5}\,\bar{1}}$
$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	6	$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{2\,3\,5}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{2\,4\,5}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{3\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{2\,5\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{\bar{1}\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,2} \otimes \boxed{3\,4\,5}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,4} \otimes \boxed{3\,5\,\bar{1}}$
		$\cdots \otimes \boxed{1\,1\,1} \otimes \boxed{2\,\bar{1}\,\bar{1}} \otimes \boxed{1\,1\,3} \otimes \boxed{4\,5\,\bar{1}}$

Table 5.17: Elements of $\max(2\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $D_6^{(1)}$-module $V(2\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$2\Lambda_0$	1
$2\Lambda_0 - \alpha_0$	1
$2\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3
$2\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	10
$2\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	10

Table 5.18: Elements of $\max(3\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $D_6^{(1)}$-module $V(3\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$3\Lambda_0$	1
$3\Lambda_0 - \alpha_0$	1
$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3
$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$3\Lambda_0 - 2\alpha_0 - \alpha_2$	1
$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	15
$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	15
$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	6

Table 5.19: Elements of $\max(4\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $D_6^{(1)}$-module $V(4\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$4\Lambda_0$	1
$4\Lambda_0 - \alpha_0$	1
$4\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3
$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3
$4\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$4\Lambda_0 - 2\alpha_0 - \alpha_2$	1
$4\Lambda_0 - 2\alpha_0$	1
$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	15
$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$	6
$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$	1
$4\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	15
$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$	10
$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_6$	37
$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$	1
$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	21
$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	6
$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$	37
$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	21
$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$	70
$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	10
$4\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5 - 2\alpha_6$	15
$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	10
$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$	16
$4\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_6$	70

Table 5.20: Elements of $\max(5\Lambda_0) \cap P^+$ and their corresponding multiplicities for the $D_6^{(1)}$-module $V(5\Lambda_0)$.

Maximal Dominant Weight	Multiplicity
$5\Lambda_0$	1
$5\Lambda_0 - \alpha_0$	1
$5\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$	3
$5\Lambda_0 - 3\alpha_0 - 1\alpha_1 - 2\alpha_2 - \alpha_3$	3
$5\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$5\Lambda_0 - 2\alpha_0 - \alpha_2$	1
$5\Lambda_0 - 2\alpha_0$	1
$5\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	15
$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$	6
$5\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$	1
$5\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	15
$5\Lambda_0 - 3\alpha_0 - \alpha_2$	1
$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$	10
$5\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_6$	46
$5\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$	1
$5\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	21
$5\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	6
$5\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$	46
$5\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	21
$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 3\alpha_2 - \alpha_3$	3
$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	10
$5\Lambda_0 - 4\alpha_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5 - 2\alpha_6$	15
$5\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$	22
$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	10
$5\Lambda_0 - 4\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	6
$5\Lambda_0 - 5\alpha_0 - 2\alpha_1 - 5\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_6$	28
$5\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5$	115
$5\Lambda_0 - 5\alpha_0 - 2\alpha_1 - 5\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	28
$5\Lambda_0 - 5\alpha_0 - 2\alpha_1 - 5\alpha_2 - 3\alpha_3 - \alpha_4$	15
$5\Lambda_0 - 4\alpha_0 - 3\alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$	5
$5\Lambda_0 - 6\alpha_0 - 4\alpha_1 - 8\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_6$	115
$5\Lambda_0 - 4\alpha_0 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	1
$5\Lambda_0 - 6\alpha_0 - 3\alpha_1 - 7\alpha_2 - 5\alpha_3 - 3\alpha_4 - \alpha_5$	70
$5\Lambda_0 - 6\alpha_0 - 3\alpha_1 - 7\alpha_2 - 5\alpha_3 - 3\alpha_4 - \alpha_6$	70

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