Invariant Subspaces, Derivative Arrays, and the Computation of the Drazin Inverse

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Abstract

The Drazin generalized inverse appears in a number of applications including the theory of linear time invariant differential-algebraic equations (DAEs). In this paper we consider its robust computation. Various methods are proposed all of them based on the determination of bases of invariant subspaces connected with the Drazin inverse. We also include comparisons of our methods to some of the other approaches found in the literature.

Keywords Drazin inverse, computation, robustness, invariant subspaces, derivative array

Mathematics Subject Classification (2010) 65F20

1 Introduction

In this paper we will be examining the question of computing the Drazin inverse of a matrix E by exploiting its representation by some associated invariant subspaces. The techniques applied include the use of the derivative array connected with the matrix E as well as the determination of the Jordan structure of the zero eigenvalue of E. From this viewpoint one of the prominent researchers in the area of matrix theory and connected computational methods has been Volker Mehrmann, see, e. g., the book [3] which gives a then up-to-date list of all his publications and covers all his fields of interest.

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Opposed to algebraic approaches such as [4], our interest here is in the development of robust numerical methods. One potential advantage of using the derivative array is that as shown by the work on the continuity properties of the Drazin inverse [5, 6, 8] the derivative array can have a well conditioned rank even if E does not. This robustness is the most important issue due to unavoidable rank decisions in the approaches discussed.

In Section 2 we shall review some of the basic theory of the Drazin inverse. In Section 3 we shall present three methods for the computation of the Drazin inverse based on the determination of certain invariant subspaces. Section 4 lists several different methods from the literature and gives a summary of a number of numerical experiments comparing the various numerical approaches. Finally there are conclusions in Section 5.

2 Preliminaries

Let $\mathbb{K}$ be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. If not stated otherwise, the following known claims hold in both cases. We include proofs since we need parts of the proofs in the algorithms that follow.

Definition 1 *Let $E \in \mathbb{K}^{n,n}$. The quantity*

$$\nu = \min\{k \in \mathbb{N}_0 \mid \operatorname{rank} E^k = \operatorname{rank} E^{k+1}\} \tag{1}$$

is called the index of nilpotency of E. In what follows we write $\nu = \operatorname{ind} E$.

Definition 2 *Let $E \in \mathbb{K}^{n,n}$. A matrix $X \in \mathbb{K}^{n,n}$ satisfying*

$$\begin{array}{ll} \text{(a)} & EX = XE, \\ \text{(b)} & XEX = X, \\ \text{(c)} & XE^{\nu+1} = E^{\nu}, \end{array} \tag{2}$$

is called a Drazin inverse of E.

In the following we use the notation

$$d = \operatorname{rank} E^{\nu} = \dim \operatorname{range} E^{\nu}, \quad a = n - d = \dim \operatorname{kernel} E^{\nu}. \tag{3}$$

The quantity d is sometimes called the core rank of E.

Lemma 3 *Let $E \in \mathbb{K}^{n,n}$. There is a nonsingular $T \in \mathbb{K}^{n,n}$ such that*

$$T^{-1}ET = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}, \tag{4}$$

with $C \in \mathbb{K}^{d,d}$ nonsingular and $N \in \mathbb{K}^{a,a}$ nilpotent.

Proof:
We choose T to be the transformation to (real) Jordan canonical form collecting the nonsingular Jordan blocks in C and the nilpotent Jordan blocks in N. □

Theorem 4 *Every $E \in \mathbb{K}^{n,n}$ possesses a unique Drazin inverse which we denote by E^D.*

Proof:
The matrix E^D defined by

$$T^{-1}E^DT = \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

satisfies the axioms (2). Furthermore, if both X_1 and X_2 satisfy the axioms (2), then we have

$$\begin{aligned} X_1 &= X_1EX_1 = X_1^2E = \cdots = \\ &= X_1^{\nu+1}E^\nu = X_1^{\nu+1}E^{\nu+1}X_2 = \\ &= X_1EX_2 = \\ &= X_1E^{\nu+1}X_2^{\nu+1} = E^\nu X_2^{\nu+1} = \\ &= \cdots = EX_2^2 = X_2EX_2 = X_2. \end{aligned}$$

—

□

Remark 5 *Let $E \in \mathbb{K}^{n,n}$ and*

$$\mathbb{K}^n = \operatorname{range} E^\nu \oplus \operatorname{kernel} E^\nu. \tag{6}$$

Then (4) and (5) say that the linear mapping represented by E is bijective as a mapping from $\operatorname{range} E^\nu$ *onto* $\operatorname{range} E^\nu$ *and that the mapping represented by E^D acts on a vector by taking its part in* $\operatorname{range} E^\nu$ *according to the splitting (6) and then applying the restricted inverse.*

Proof:
From

$$T^{-1}E^\nu T = \begin{bmatrix} C^\nu & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that

$$\operatorname{range} E^\nu = \operatorname{range}\Big(T \begin{bmatrix} I \\ 0 \end{bmatrix}\Big), \quad \operatorname{kernel} E^\nu = \operatorname{range}\Big(T \begin{bmatrix} 0 \\ I \end{bmatrix}\Big),$$

and thus (6) holds. □

Theorem 6 *Let $E \in \mathbb{K}^{n,n}$ with index $\nu = \operatorname{ind} E$. Furthermore, let $W \in \mathbb{K}^{n,d}$ and $K \in \mathbb{K}^{n,a}$ with*

$$\operatorname{range} W = \operatorname{range} E^\nu, \quad \operatorname{range} K = \operatorname{kernel} E^\nu, \tag{7}$$

and set $T = [\, W \;\; K \,]$. Then

$$T^{-1} E^\nu T = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

with some nonsingular $C \in \mathbb{K}^{d,d}$.

Proof:
Splitting $T^{-1}E^\nu T$ according to the splitting of T yields

$$T^{-1} E^\nu T = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

From this we get

$$E^\nu \, [\, W \;\; K \,] = [\, W \;\; K \,] \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

and thus

$$[\, E^\nu W \;\; 0 \,] = [\, W \;\; K \,] \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

The second block column reads

$$[\, W \;\; K \,] \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} = T \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} = 0$$

implying $E_{12} = 0$ and $E_{22} = 0$. For the first block column we obtain

$$E^\nu W = W E_{11} + K E_{21}.$$

Since the columns of $E^\nu W$ lie in $\operatorname{range} E^\nu$, the direct sum (6) yields $K E_{21} = 0$ and thus $E_{21} = 0$ since K is of full column rank. The matrix $E_{11} \in \mathbb{K}^{d,d}$ then has the same rank as E^ν which is d by definition. Hence, $C = E_{11}$ is nonsingular. □

3 Computation of the Drazin inverse

Theorem 6 shows that we can determine a suitable transformation T in the form $T = [W\, K]$ for the representation (4) by constructing bases of range E^ν and kernel E^ν. There are various methods for constructing W and K.

Restricting to the case $\mathbb{K} = \mathbb{R}$, the basic tool for the following is a rank-revealing QR decomposition according to

$$Q^T A \Pi = \begin{bmatrix} R & S \\ 0 & 0 \end{bmatrix}, \tag{9}$$

with Q orthogonal, represented by a sequence of Householder transformations, and R nonsingular upper-triangular, followed by further Householder transformations from the left to get the factorization

$$Q^T A \hat{Q} = \begin{bmatrix} \hat{R} & 0 \\ 0 & 0 \end{bmatrix}, \tag{10}$$

with $\hat{Q}$ orthogonal and $\hat{R}$ nonsingular upper-triangular. If $A \in \mathbb{R}^{m,n}$ and $r = \operatorname{rank} A$, then the last $n - r$ columns of $\hat{Q}$ form an orthonormal basis of kernel A and the first r columns of Q form an orthonormal basis of range A. In case of critical rank decisions, the rank-revealing QR decomposition can be replaced by the singular value decomposition which has the same structure as (10) but with the diagonal matrix of the positive singular values replacing $\hat{R}$. For details see [11].

3.1 Kernel and range from the power of the matrix

The simplest way to obtain matrices W and K is to compute E^ν by successive matrix multiplications and to determine the spaces by rank revealing QR decomposition.

The disadvantage of this approach is that it easily fails for ill-conditioned matrices E.

3.2 Kernel and range from the derivative array

In order to avoid the determination of powers of E we can build the derivative array $M \in \mathbb{R}^{\nu n, \nu n}$ according to

$$M = \begin{bmatrix} E & & & \\ -I & \ddots & & \\ & \ddots & \ddots & \\ & & -I & E \end{bmatrix},$$

see, for example, [7, 16]. Writing M in the form

$$M = \begin{bmatrix} EU^T & 0 \\ -(I-Y) & VE \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & E & & \\ & \ddots & \ddots & \\ & & \ddots & E \\ & & & 0 \end{bmatrix}, \tag{11}$$

where

$$U^T = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \quad V^T = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix},$$

we see that Y is nilpotent so that $I - Y$ is invertible with

$$(I-Y)^{-1} = \begin{bmatrix} I & E & \cdots & E^{\nu-2} \\ & \ddots & \ddots & \vdots \\ & & \ddots & E \\ & & & I \end{bmatrix}.$$

Observe that

$$EU^T(I-Y)^{-1}VE = E^\nu.$$

Column operations on M yield

$$M \to \begin{bmatrix} EU^T(I-Y)^{-1} & 0 \\ -I & VE \end{bmatrix} \to \begin{bmatrix} EU^T(I-Y)^{-1} & E^\nu \\ -I & 0 \end{bmatrix}.$$

Row operations on M yield

$$M \to \begin{bmatrix} EU^T & 0 \\ -I & (I-Y)^{-1}VE \end{bmatrix} \to \begin{bmatrix} 0 & E^\nu \\ -I & (I-Y)^{-1}VE \end{bmatrix}.$$

Hence, the range and kernel of M have the form

$$\hat{W} = \begin{bmatrix} W & * \\ 0 & * \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} * \\ K \end{bmatrix},$$

where W and K are of full column rank. The matrix $\hat{W}$ is row compatible with the original M in (11) and $\hat{K}$ is column compatible with the original M in (11). Thus, we can obtain W from a matrix whose columns span range M by using column operations to produce the zero block according to the form of $\hat{W}$, and we can obtain K simply as the last block of $\hat{K}$.

3.3 Kernel and range from successive determination of the generalized eigenvectors belonging to zero eigenvalues

The kernel of E^ν is spanned by all generalized eigenvectors of E belonging to zero eigenvalues. We therefore can use

$$Ex_{m+1} = x_m$$

to determine a generalized eigenvector x_{m+1} of order $m+1$ from a generalized eigenvector x_m of order m. We can use the same approach to determine the kernel of $(E^T)^\nu$. The range of E^ν is then given as the orthogonal complement of the kernel of $(E^T)^\nu$.

In more detail, we first determine K_1 as a matrix whose columns form a basis of kernel E. Given a matrix K_m of full column rank, we determine K_{m+1} as the upper part of a matrix whose columns form a basis of $[\, E \;\; K_m \,]$ according to

$$[\, E \;\; K_m \,] \begin{bmatrix} K_{m+1} \\ V_{m+1} \end{bmatrix} = EK_{m+1} + K_m V_{m+1} = 0.$$

Thus, if the columns of K_m are generalized eigenvectors of order at most m, the columns of K_{m+1} are generalized eigenvectors of order at most $m+1$. Moreover, the matrix K_{m+1} has full column rank. If there were a nonzero vector y with $K_{m+1}y = 0$ we would have $K_m V_{m+1} y = 0$ implying $V_{m+1}y = 0$ because of the full column rank of K_m. But this is in contradiction to the full column rank of $[\, K_{m+1}^T \;\; V_{m+1}^T \,]^T$. Since all generalized eigenvectors of order at most $m+1$ are related in this way to the generalized eigenvectors of order at most m, we get all possible generalized eigenvectors of order at most $m+1$ and the iterative process becomes stationary when we reach the order of nilpotency of E. The final K_ν then is K in the construction of a suitable T. Accordingly, we can construct W by performing the same procedure for E^T instead of E to get the kernel of $(E^T)^\nu$ followed by the determination of its orthogonal complement.

We should remark here that this procedure is in principle equivalent to the approach in [12] for the determination of the Jordan structure of a given eigenvalue in the Jordan canonical form.

4 Numerical experiments

We compared the above approaches for the determination of the Drazin inverse with various methods suggested in the literature. Since not all of

them are able to determine the index, we provided ν in addition to E. We include a brief summary here of these other methods which is based on the experience from our tests, some of which will be presented below.

4.1 Successive echelon form by elementary row operations

In [2] the following approach was suggested. Starting with the matrix $M_0 = [\,E\,I\,]$, the sequence of matrices $M_k = [\,A_k\,I_k\,]$ is defined by using elementary row operations to bring A_k into echelon form according to

$$[\,A_k\;\;I_k\,] \to \begin{bmatrix} F_k & G_k \\ 0 & H_k \end{bmatrix},$$

with F_k having full row rank, and defining M_{k+1} by

$$M_{k+1} = [\,A_{k+1}\;\;I_{k+1}\,] = \begin{bmatrix} F_k & G_k \\ H_k & 0 \end{bmatrix}.$$

The process stops when $k = \nu$ with $\operatorname{rank} A_\nu = n$. We then have

$$E^D = A_\nu^{-1} I_\nu E^\nu.$$

The main problem here is the use of E^ν in the representation of E^D which can numerically lead to rank drops.

4.2 Successive echelon form by orthogonal row operations

The construction of Section 4.1 can also be performed by using orthogonal transformations, see also [2]. All formulas and comments stay valid.

4.3 Finite sequence 1

In [13, 14] the following approach was suggested. Let $B_0 = I$ and

$$p_k = \operatorname{trace}(EB_k)/(k+1), \quad B_{k+1} = EB_k - p_k I,$$

as long as $p_k \neq 0$. If $p_{k+1} = 0$ for the first time, then

$$E^D = E^\nu B_k^{\nu+1} / p_k^{\nu+1}.$$

With this approach the norm of EB_k can get very large so that the computation of the trace easily suffers from cancellation in which case the decision on $p_{k+1} = 0$ is not reliable.

4.4 Finite sequence 2

In [10] the following approach was suggested. Let $D_0 = I$ and

$$p_k = \operatorname{trace}(E^{\nu+1}D_k)/(k+1), \quad D_{k+1} = E^{\nu+1}D_k - p_k I,$$

as long as $p_k \neq 0$. If $p_{k+1} = 0$ for the first time, then

$$E^D = D_k E^\nu / p_k.$$

Here the norm of $E^{\nu+1}D_k$ can get very large so that the computation of the trace easily suffers from cancellation in which case the decision on $p_{k+1} = 0$ is not reliable. In comparison with the method in Section 4.3 the effects here are even more disastrous because the used trace depends on $E^{\nu+1}$.

4.5 Convergent sequence 1

For sufficiently small $\alpha > 0$ and odd $\kappa \geq \nu$, the sequence $\{X_k\}_{k\in\mathbb{N}_0}$ defined by

$$X_{k+1} = X_k + \alpha(I - EX_k)E^\kappa, \quad X_0 = \alpha E^\kappa,$$

converges to E^D provided the eigenvalues of E are real, see [19], or satisfy some angle condition, see [17].

If E^κ is large in norm, the parameter α must be chosen very small. Together with the linear convergence the method fails easily.

4.6 Convergent sequence 2

For sufficiently small $\alpha > 0$ and odd $\kappa \geq \nu$, the sequence $\{X_k\}_{k\in\mathbb{N}_0}$ defined by

$$X_{k+1} = X_k + X_k(I - EX_k), \quad X_0 = \alpha E^\kappa,$$

converges to E^D provided the eigenvalues of E are as in Section 4.5, see again [19] and [17].

4.7 Convergent sequence 3

For sufficiently small $\alpha > 0$ and odd $\kappa \geq \nu$, the sequence $\{X_k\}_{k\in\mathbb{N}_0}$ defined by

$$X_{k+1} = X_k + E^\kappa(I - EX_k)/(k+2), \quad X_0 = E^\kappa,$$

converges to E^D provided the eigenvalues of E are real, see once more [19].

Even for moderate norm of E^κ the intermediate iterates can easily get very large. Additionally, the convergence rate can be very bad.

4.8 Convergent sequence 4

For sufficiently small $\alpha > 0$ and odd $\kappa \geq \nu$, the sequence $\{X_k\}_{k\in\mathbb{N}_0}$ defined by

$$X_{k+1} = X_k + (2I - E^{\kappa+1}/(k+2))E^{\kappa}(I - EX_k)/(k+2), \quad X_0 = (2I - E^{\kappa})E^{\kappa},$$

converges to E^D provided the eigenvalues of E are real, see [19] for a last time.

Even for moderate norm of E^{κ} the intermediate iterates can easily get very large.

4.9 Convergent sequence 5

The sequence $\{X_k\}_{k\in\mathbb{N}_0}$ defined by

$$\begin{array}{l} P_k = EX_k, \\ Q_k = -29I + P_k(33I + P_k(-15I + 2P_k)), \\ R_k = P_kQ_k, \\ X_{k+1} = X_kQ_k(243I + R_k(27I + R_k)), \end{array} \qquad X_0 = \frac{1}{2\|E^{\nu+1}\|}E^{\nu},$$

converges with ninth order to E^D, see [1].

Under round-off the method appears to be divergent quite often. We therefore take the iterate that leads to the smallest correction as the result provided by this method.

4.10 Successive squaring

For sufficiently small $\beta > 0$ the sequence $\{X_k\}_{k\in\mathbb{N}_0}$ of the upper right blocks of the matrices

$$Z_k = \begin{bmatrix} P & Q \\ 0 & I \end{bmatrix}^{2^k}, \quad P = I - \beta E^{\nu+1},\ Q = \beta E^{\nu},$$

converges to E^D, see [18].

Under round-off the method appears to be divergent quite often. We therefore take the iterate that leads to the smallest correction as the result provided by this method.

4.11 Polynomial representation via eigenvalues

According to [9] we can proceed as follows. Given all nonzero eigenvalues λ_i of E with algebraic multiplicities n_i, $i = 1, \ldots, r$ and let l be the multiplicity

of the zero eigenvalue of E. Furthermore let p be the unique polynomial of the form $p(\lambda) = \lambda^l(\alpha_0 + \alpha_1\lambda + \cdots + \alpha_{m-1}\lambda^{m-1})$, where $m = n_1 + \cdots + n_r$, satisfying

$$p^{(j-1)}(\lambda_i) = (-1)^{j-1}(j-1)!/\lambda_i^j, \quad j = 1, \ldots, n_i, \ i = 1, \ldots, r.$$

Then the Drazin inverse of E is given by $E^D = p(E)$.

The eigenvalue problem may be ill-conditioned, especially when multiple eigenvalues occur, which in the present application is typical for the zero eigenvalue. Furthermore the use of powers in the representation of E^D can numerically lead to rank drops.

4.12 Bordering 1

In [20] the following approach was suggested. Perform the recursive process of Section 4.1 to get the matrix

$$B = \begin{bmatrix} H_1 \\ \vdots \\ H_\nu \end{bmatrix}.$$

Do the same for E^T instead of E to get a corresponding matrix C. Form the matrix

$$M = \begin{bmatrix} E & C^T \\ B & 0 \end{bmatrix}$$

and determine M^{-1} by elementary row operations. The Drazin inverse E^D is then given as the n-by-n upper left submatrix of M^{-1}.

4.13 Bordering 2

In [20] the following modification was also suggested. Determine B and C as in Section 4.12. Apply elementary row operations on $[\, B^T \ I \,]$ and $[\, C^T \ I \,]$ to get F with range F = kernel B and G with kernel G = range C^T. We then have $E^D = F(G^T E F)^{-1} G^T$.

4.14 Results

We constructed the examples by providing C and N in (4) and used a (well-conditioned) random matrix for T. We then computed the reference solution by (5) using quadruple precision.

All the above methods have been carefully implemented and checked. For some methods we experienced bad behavior even for quite simple examples.

Remark 7 *In the case of the approaches that generate infinite convergent sequences in theory, convergence was lost under round-off. Using* $C = \mathrm{diag}(1, \frac{1}{2})$ *and* $N \in \mathbb{R}^{2,2}$ *as the standard Jordan block for the zero eigenvalue, the method of Section 4.6 shows a behavior given in the second column of Table 1. The reason for this behavior is that convergence takes place in a special subspace of* $\mathbb{R}^{n,n}$*. Choosing the initial iterate as stated we start in this subspace and theoretically we stay there. But under round-off this can no longer be guaranteed. In the present example this does not seem crucial since the effect becomes only visible when we already have an approximation of the Drazin inverse of some quality. But for more complicated test examples we may see no convergence behavior at all. To substantiate this interpretation we modified the method of Section 4.6 by projecting every correction from the left and from the right with* $E^D E$ *determined by the method of Section 3.3. The corresponding results are given in the third column of Table 1. We see that there is no crucial difference in the beginning but this time the iteration settles down with an accurate solution. Of course, in practice such a modification does not make sense since we use the quantity which we actually want to compute. Note that the same observation applies to all of the above methods which theoretically generate a convergent sequence.*

Table 1: Norm of the corrections for the method of Section 4.6

iteration	norm for original method	norm for projected method
1	3.322e-03	3.322e-03
2	6.635e-03	6.635e-03
3	1.323e-02	1.323e-02
⋮	⋮	⋮
16	3.202e+00	3.202e+00
17	6.396e-01	6.396e-01
18	1.917e-02	1.917e-02
19	1.626e-05	1.626e-05
20	1.185e-11	1.169e-11
21	2.447e-12	9.159e-15
22	4.894e-12	5.829e-15
23	9.788e-12	2.165e-15
24	1.957e-11	5.940e-15
⋮	⋮	
64	2.983e+00	

Remark 8 *We also experienced problems with the methods using the trace of some matrices. Taking* $C = \mathrm{diag}(100, 10, 1, 0.1, 0.01)$ *and* $N \in \mathbb{R}^{4,4}$ *as the standard Jordan block for the zero eigenvalue without random transformation the method of Section 4.3 yields the sequence* p_k *as given in the second column of Table 2. We observe an increase in the modulus leading to a perturbed value* p_6 *which here was accepted to vanish. Applying a random transformation theoretically leaves the ocurring traces unchanged. But under round-off we here lose three orders of magnitude in the size of* p_6*, see the third column of Table 2. The main problem therefore is that like with a rank decision, we must decide whether a quantity is zero or not. But unlike with a rank decision, orthogonal transformations are not used so that the intermediate quantities are not bounded by the original data.*

Table 2: Values p_k for the method of Section 4.3

k	value of p_k original problem	value of p_k transformed problem
0	1.000e+00	1.000e+00
1	1.111e+02	1.111e+02
2	-1.122e+03	-1.122e+03
3	1.122e+03	1.122e+03
4	-1.111e+02	-1.111e+02
5	1.000e+00	1.000e+00
6	1.023e-07	-3.457e-04
7		-2.500e-02
8		-1.891e+00
9		-1.479e+02

Taking into account that an unsymmetric eigenvalue problem has to be solved in the method of Section 4.11, we expect difficulties for the methods of Section 4.3 through Section 4.11. In the following numerical comparisons we therefore include only those methods which gave the best results. All the following tests are on parametrized families of problems. The corresponding diagrams show the relative errors of the obtained approximation for the Drazin inverse as a function of the parameter.

Example 9 *Taking* $C = \mathrm{diag}(\varepsilon^{-2}, \varepsilon^{-1}, 1, \varepsilon, \varepsilon^2)$*,* $\varepsilon \in (0, 1]$*, and* $N \in \mathbb{R}^{4,4}$ *as single Jordan block for the zero eigenvalue, the conditioning of* C *becomes worse with decreasing* ε*. The diagrams of Figure 1 show that the best methods are those of Section 3.3, Section 4.12, and Section 4.13 performing quite*

similarly. Actually, the methods using bordering are not distinguishable. It follows the method of Section 3.2 based on the derivative array. All other methods are significantly less robust.

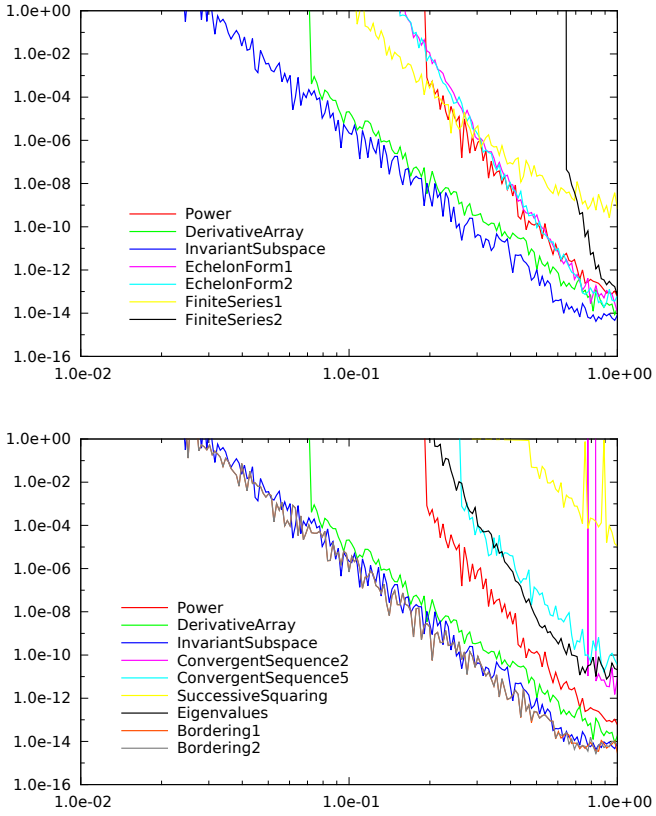

Figure 1: Relative error in E^D for $C = \mathrm{diag}(\varepsilon^{-2}, \varepsilon^{-1}, 1, \varepsilon, \varepsilon^2)$, $N \in \mathbb{R}^{4,4}$ single Jordan block for the zero eigenvalue. Horizontal axis is ϵ.

Example 10 *Let $H \in \mathbb{R}^{4,4}$ be the Hilbert matrix of the stated size, i. e., the matrix with entries $\frac{1}{i+j+1}$, $i, j = 1, \ldots, 4$, and let D be the diagonal matrix with corresponding entries. Taking $C = \alpha D + (1-\alpha)H$ with $\alpha = 1 - 2^{-20t}$, $t \in [0, 1]$, to emphasize the ill-conditioned part, and $N \in \mathbb{R}^{3,3}$ as single Jordan block for the zero eigenvalue, we get the diagrams of Figure 2. Again the best methods are those of Section 3.3, Section 4.12, and Section 4.13 performing quite similarly. The next best methods are those of Section 3.1 and Section 3.2 also performing quite similarly.*

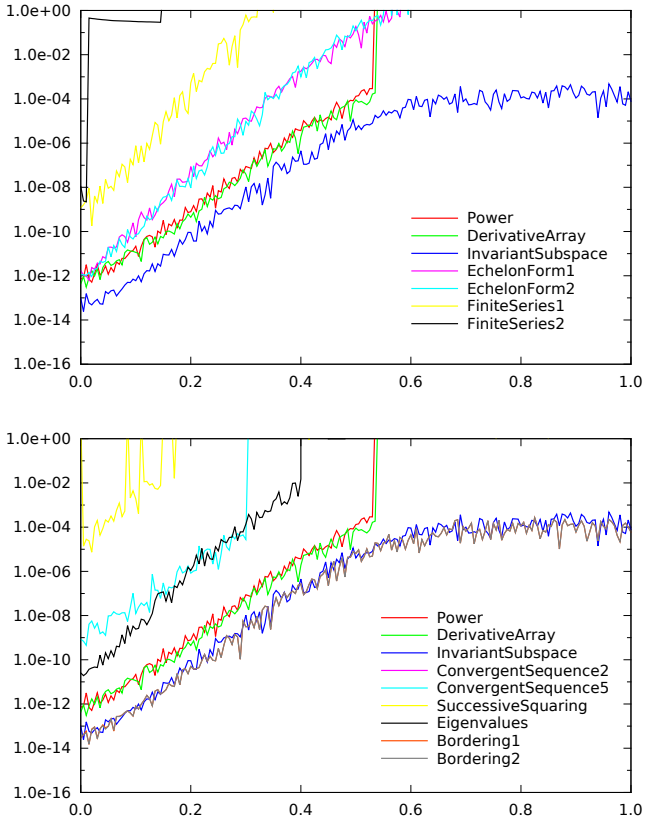

Figure 2: Relative error in E^D, morphing into the Hilbert matrix in $\mathbb{R}^{4,4}$ combined with $N \in \mathbb{R}^{3,3}$ single Jordan block for the zero eigenvalue. Horizontal axis is morphing parameter t.

Example 11 *Taking $C \in \mathbb{R}^{3,3}$ as the Hilbert matrix of the stated size and*

$$N = \begin{bmatrix} 0 & \varphi(t) \\ \varphi(-t) & 0 \end{bmatrix}, \quad \varphi(t) = \begin{cases} e^{-1/t} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases} \qquad t \in [-1, 1],$$

we have a rank change at the origin in E but not in E^ν, and E^D is smooth and of constant rank. The diagrams of Figure 3 show that quite a number of the methods suffer from problems concerning reliable rank decision. Although the method of Section 3.2 is also based on successive rank decisions it seems to be unaffected in this example and performs best. In contrast to the previous examples, the method of Section 4.3 is nearly as good as the best method. The methods of Section 3.1 and Section 3.3 as well as that of Section 4.11

seem to behave independently of t but suffer from the ill-conditioning of the core in combination with the high index.

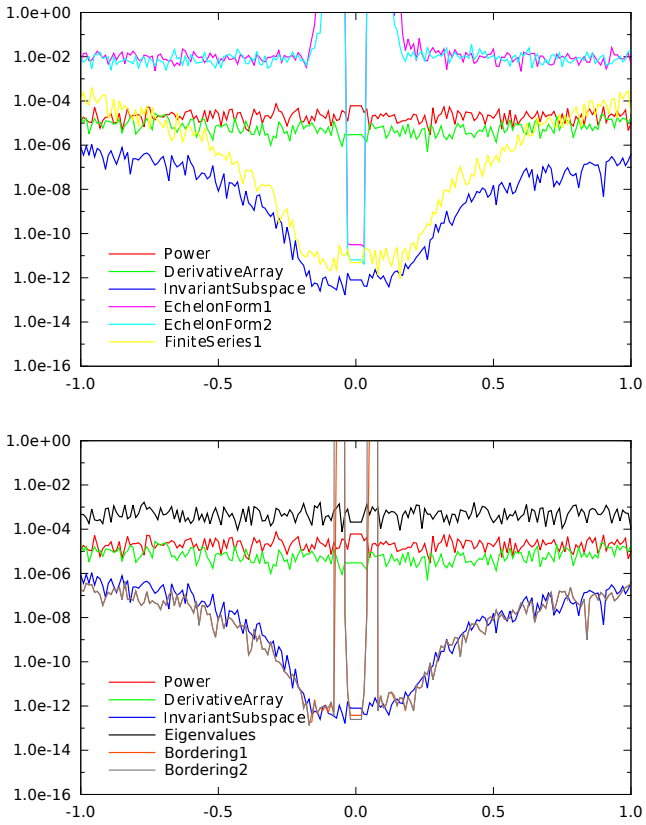

Figure 3: Relative error in E^D, Hilbert matrix in $\mathbb{R}^{3,3}$ combined with $N \in \mathbb{R}^{2,2}$ exhibiting a rank change in parameter t. Horizontal axis is t.

Example 12 *Taking $C \in \mathbb{R}^{3,3}$ again as the Hilbert matrix of the stated size but choosing*

$$N = \begin{bmatrix} 0 & \sin^2(\frac{1}{2}\pi t) & 0 & 0 \\ 0 & 0 & \sin^2(\frac{3}{2}\pi t) & 0 \\ 0 & 0 & 0 & \sin^2(\frac{5}{2}\pi t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

exhibiting several rank changes, we get the diagrams of Figure 4. We see here that all methods using successive rank decisions now have problems near

the rank changes. Again the methods of Section 3.1 and Section 3.2 as well as that of Section 4.11 seem to behave independently of t but suffer from the ill-conditioning of the core in combination with the high index.

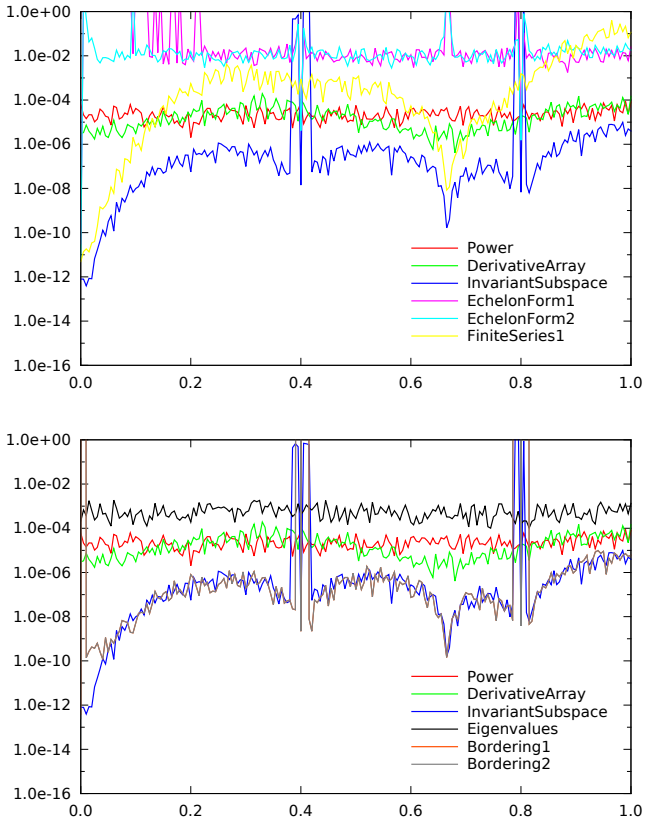

Figure 4: Relative error in E^D, Hilbert matrix in $\mathbb{R}^{3,3}$ combined with $N \in \mathbb{R}^{4,4}$ exhibiting multiple rank changes in terms of parameter t. Horizontal axis is t.

Example 13 *Here we choose the matrix C according to*

$$C = \begin{bmatrix} k & 1 & & & \\ 1 & k-1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -k+1 & 1 \\ & & & 1 & -k \end{bmatrix} \tag{12}$$

combined with $N \in \mathbb{R}^{4,4}$ as a single Jordan block for the zero eigenvalue. It is known from [15] that the matrices of the form of C have a rank drop of one but that the pivots met in Gaussian elimination will not get small for large k due to round-off. This means that rank decisions on the basis of elementary operation are not reliable in this case. The effect is clearly visible in the diagrams of Figure 5. All methods presented in Section 3 behave well. Note especially the difference of the methods of Section 4.1 and Section 4.2 which both transform on echelon form but the first using elementary operations while the second using orthogonal transformations.

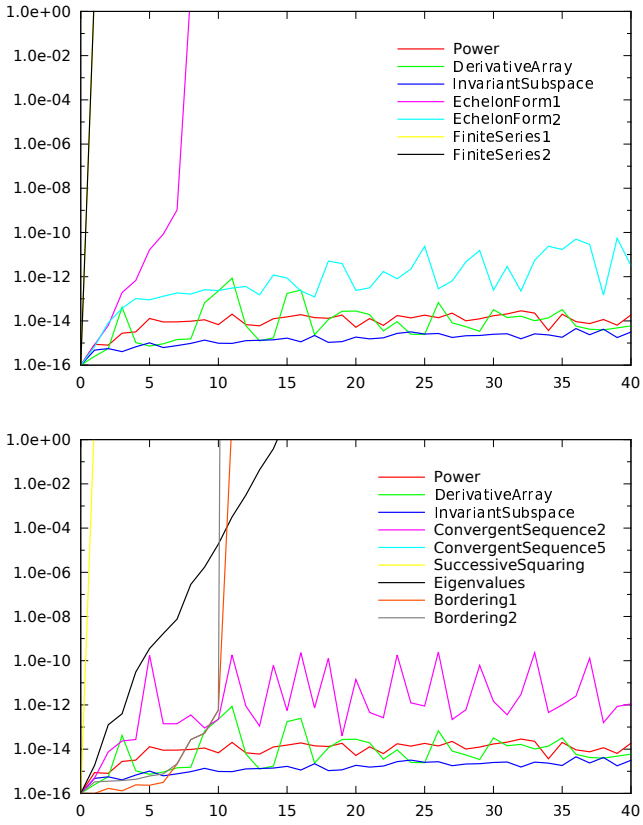

Figure 5: Relative error in E^D, matrix (12) with rank deficiency one combined with $N \in \mathbb{R}^{4,4}$ single Jordan block for the zero eigenvalue. Horizontal axis is size of the matrix (12).

Remark 14 *There is no significant difference in the tested examples when we replace rank-revealing QR by the more robust singular value decomposition.*

5 Conclusions

We suggested three different methods for the computation of the Drazin inverse of a given matrix E. All of them are based on different procedures for the determination of certain invariant spaces connected with E. We compared our approaches with various approaches taken from the literature. Our observations were as follows. The most robust methods are those of Section 3.2 and Section 3.3 as well as those of Section 4.12 and Section 4.13 when rank decisions on the basis of elementary operations are not critical. In the latter case one should base the approach of bordering on orthogonal transformations in the spirit of [12] as has been done for the methods of Section 3.3 and Section 4.2. In the case of rank changes in the nilpotent part the methods of Section 3.1 and Section 3.2 may be preferable.

References

[1] A. S. Al-Fhaid, S. Shateyi, M. Zaka Ullah, and F. Soleymani. A matrix iteration for finding Drazin inverse with ninth-order convergence. *Abstract and Applied Analysis*, vol. 2014, 2014. Article ID 137486, 7 pages.

[2] K. M. Anstreicher and U. G. Rothblum. Using Gauss-Jordan elimination to compute the index, generalized nullspaces, and Drazin inverse. *Linear Algebra Appl.*, 85:221–239, 1987.

[3] P. Benner, M. Bollhöfer, D. Kressner, C. Mehl, and T. Stykel. *Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory.* Springer, Cham, 2015.

[4] F. Bu and Y. Wei. The algorithm for computing the Drazin inverses of two-variable polynomial matrices. *Appl. Math. Comput.*, 147:805–836, 2004.

[5] S. L. Campbell. On the continuity of the Moore-Penrose and Drazin generalized inverses. *Linear Algebra Appl.*, 18:53–57, 1977.

[6] S. L. Campbell. Continuity of the Drazin inverse. *Linear and Multilinear Algebra*, 8:265–268, 1980.

[7] S. L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM J. Math. Anal.*, 18:1101–1115, 1987.

[8] S. L. Campbell and C. D. Meyer. Continuity properties of the Drazin pseudo inverse. *Linear Algebra Appl.*, 10:77–83, 1975.

[9] S. L. Campbell, C. D. Meyer, and N. J. Rose. Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. *SIAM J. Appl. Math.*, 31:411–425, 1976.

[10] Y.-L. Chen. Representation and approximation for the Drazin inverse $A^{(d)}$. *Appl. Math. Comput.*, 119:147–160, 2001.

[11] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, USA, 1983.

[12] G. H. Golub and J. H. Wilkinson. Ill-conditioned eigensystems and the computation of the Jordan canonical form. *SIAM Rev.*, 18:578–619, 1976.

[13] T. N. E. Greville. The Souriau-Frame algorithm and the Drazin pseudoinverse. *Linear Algebra Appl.*, 6:205–208, 1973.

[14] R. E. Hartwig. More on the Souriau-Frame algorithm and the Drazin inverse. *SIAM J. Appl. Math.*, 31:42–46, 1976.

[15] W. Kahan. Numerical linear algebra. *Canad. Math. Bull.*, 9:757–801, 1966.

[16] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

[17] X. Li and Y. Wei. Iterative methods for the Drazin inverse of a matrix with a complex spectrum. *Appl. Math. Comput.*, 147:855–862, 2004.

[18] Y. Wei. Successive matrix squaring algorithm for computing the Drazin inverse. *Appl. Math. Comput.*, 108:67–75, 2000.

[19] Y. Wei and H. Wu. The representation and approximation for Drazin inverse. *J. Comput. Appl. Math.*, 126:417–432, 2000.

[20] X. Zhang and X. Sheng. Two methods for computing the Drazin inverse through elementary row operations. *Filomat*, 30:3759–3770, 2016.