ABSTRACT

HOSSAIN, CHETAK. Quotients Derived from Posets in Algebraic and Topological Combinatorics. (Under the direction of Patricia Hersh and Nathan Reading.)

The sequence OEIS A071356 is the coefficient sequence of an ordinary generating function similar to the Catalan numbers. We show using generating function arguments that the sequence counts several families of combinatorial objects. Specifically, a family of underdiagonal lattice paths, a family of pattern avoiding permutations, and a family of pattern avoiding inversion sequences. Furthermore, we construct bijections between the families. More specifically, we construct two surjective maps from all permutations to the lattice paths, whose fibers are intervals in the weak order and the equivalence classes of an equivalence relation on posets defined using pattern avoidance. In one case, the bottom elements of the intervals are the pattern avoiding permutations in question. In the other case, the reverse Lehmer codes of the top elements of the intervals are the pattern avoiding inversion sequences.

The family of pattern avoiding permutations has the structure of a quotient lattice of the weak order and is the basis for a Hopf algebra. We use a bijection between the permutations and lattice paths to impose a lattice and Hopf algebra structure on the lattice paths. Specifically, we find a description of the cover relations in the lattice of the lattice paths. We give a complete description of the product of the Hopf algebra of lattice paths.

We also consider the type B partition lattice Π_n^B. The type B Coxeter group S_n^B acts in a natural way on the order complex $\Delta(\Pi_n^B)$. We consider the resulting quotient complex $\Delta(\Pi_n^B)/S_n^B$. We show that the resulting quotient complex is not Cohen-Macaulay nor shellable for large enough n. We do this by demonstrating via geometric constructions that the link of a face of the quotient complex has nontrivial homology at a particular rank. We also describe a combinatorial model for the cells of the quotient complex that may be used to eventually give a CC-labeling for $\Delta(\Pi_n^B)/S_n^B$.

Quotients Derived from Posets in Algebraic and Topological Combinatorics

by
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DEDICATION

For Warsong Commander and Ice Block.

BIOGRAPHY

Chetak grew up in Wilmington, Delaware. He earned a B.A. in mathematics and a B.S. in electrical engineering and computer science from the University of California, Berkeley. After spending a semester in Budapest, he started graduate studies in mathematics at North Carolina State University. In his spare time, he enjoys playing Hearthstone, cooking, and sleeping (usually not at the same time).

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

In Chapter 2 of this thesis, we will give bijections between three families of combinatorial objects. Let us now define these objects before further elaborating our results.

Definition 1.0.1. A permutation, or *pattern*, π is said to be contained in another permutation σ, if σ has a subsequence whose terms are order isomorphic to (i.e. have the same relative ordering as) π. If σ does not contain π, we say that σ *avoids* π.

Definition 1.0.2. A vincular pattern is a pattern with certain entries underlined as a way of specifying adjacency conditions. In this thesis, we will consider the special vincular patterns that have exactly one pair of adjacent entries underlined. Given a pattern $\pi = \pi_1 \cdots \pi_n$, suppose that $\pi_i \pi_{i+1}$ is underlined. Suppose that σ contains a the pattern π with instance $\sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_n}$. We say that σ contains the vincular pattern π if $j_i + 1 = j_{i+1}$.

Definition 1.0.3. A *Dyck path* of semilength n is a lattice path from $(0,0)$ to (n,n) that stays on or below the line segment from $(0,0)$ to (n,n) and consists of north and east steps of length 1 . Let $\mathcal{D}_n$ be defined as the set of all Dyck paths of semilength n.

Definition 1.0.4. A *Schröder path* of semilength n is a lattice path from $(0,0)$ to (n,n) that stays on or below the line segment from $(0,0)$ to (n,n) and consists of north and east steps of length 1 and diagonal northeast (length $\sqrt{2}$) steps . Let $\mathcal{SP}_n$ be defined as the set of Schröder paths of semilength n.

Definition 1.0.5. Let $\mathcal{RSP}_n$ be the set of Schröder paths of semilength n with no diagonal steps on the line segment from $(0,0)$ to (n,n), and such that every diagonal step is followed by an east step. We call these *restricted Schröder paths*.

Definition 1.0.6. The set of *inversion sequences* of length $n \geq 0$ is

$$I_n = \{(e_1, e_2, \cdots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i \leq i - 1\}.$$

Definition 1.0.7. A *pattern* in an inversion sequence is a subsequence of length m that satisfies $m - 1$ binary relations. The binary relations can come from the set $\{<, >, \leq, \geq, =, \neq\}$. We will consider inversion sequences that avoid certain patterns and we will use the following notation. Given one binary relation ρ, one can define $I_n(e_i \rho e_j)$ as the set of inversion sequences of length n that have no subsequence of the form $e_i \rho e_j$ for all $1 \leq i < j \leq n$. Given two binary relations, ρ_1, ρ_2, one can define $I_n(e_i \rho_1 e_j \rho_2 e_k)$ as the set of inversion sequences of length n that have no subsequence of the form (e_i, e_j, e_k) satisfying $e_i \rho_1 e_j$ and $e_j \rho_2 e_k$ for all $1 \leq i < j < k \leq n$.

In Chapter 2, we consider the following three families of combinatorial objects. The first family is the restricted Schröder paths as defined above. The second family is the set $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ of permutations avoiding the patterns $2\underline{41}3, 3\underline{41}2, \underline{41}23$, and $\underline{41}32$. The third family is the set of pattern avoiding inversion sequences $I_n(e_i > e_j \leq e_k)$. We show that these families are all counted by the sequence OEIS A071356 in Section 2.3. Furthermore, we develop an explicit bijection between the first and second families at the end of Section 2.5. We develop an explicit bijection between the first and third families at the end of Section 2.6.

In Chapter 3, we use the bijection between the first and second families to impose algebraic structures on the set of restricted Schröder paths. The symmetric group is known to have a lattice structure called the weak order. In [26], Reading shows that the elements of $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ are special representatives of the congruence classes of a quotient lattice of the weak order. The bijection allows us to transfer a lattice structure onto the set of restricted Schröder paths. In Section 3.2 we explicitly describe the cover relations in this lattice.

For a fixed field $\mathbb{K}$, let the permutations of S_n serve as the basis for an $n!$-dimensional vector space, which we denote as $\mathbb{K}[S_n]$. The graded vector space $\bigoplus_{n\geq 0} \mathbb{K}[S_n]$ has a combinatorial Hopf algebra structure known as the Malvenuto-Reutenauer Hopf algebra of permutations (MR). Let $\mathrm{Av}_n = S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$, and let $\bigoplus_{n\geq 0} \mathbb{K}[\mathrm{Av}_n]$ be the vector space with basis $\bigcup_{n=0}^{\infty} \mathrm{Av}_n$. In [26, Section 9], Reading shows that $\bigoplus_{n\geq 0} \mathbb{K}[\mathrm{Av}_n]$ embeds as a sub-Hopf algebra of MR. We use the bijection between the first and second families to impose a Hopf algebra structure on the set of restricted Schröder paths.

Specifically, in Section 3.3, we give an intrinsic description of the product.

In Chapter 4, we undertake a study of a family of quotient cell complexes of poset order complexes. Specifically, we consider the quotient complex obtained from the order complex of the type B_n partition lattice by letting the cells be the orbits of a group action of the hyperoctahedral group on poset chains, namely on faces of the order complex. We prove that the resulting quotient complex for $n \geq 8$ is not shellable nor homotopy Cohen-Macaulay in Section 4.4. To do this we use a geometric construction involving a certain model of the chain orbits that make up the quotient complex. In [17], Hultman develops a model using labeled trees. In Section 4.3, we develop a different model using balls and labeled splitters and show how it corresponds to Hultman's model. This new model is maybe convenient for potential future work on giving a partitioning of the quotient complex.

Chapter 2

Combinatorics of the Boolean-Catalan Numbers

2.1 Introduction

The sequence OEIS A071356 has been associated with many families of combinatorial objects. These include trees [1], underdiagonal lattice paths [1], pattern avoiding permutations [26], inversion sequences [22], outerplanar maps [13], and preimages of pattern avoiding permutations under the classic stack sorting algorithm [12]. Here we focus on Schröder paths with the restriction that every diagonal step is followed by an east step ($\mathcal{RSP}_n$), pattern avoiding permutations avoiding four vincular patterns (Av_n), and pattern avoiding inversion sequences that are almost unimodular ($I_n(e_i > e_j \leq e_k)$).

The sequence OEIS A071356 is the coefficient sequence of an ordinary generating function similar to the Catalan numbers. Specifically, the generating function is

$$A(x) = \frac{1 + 2x - \sqrt{1 - 4x - 4x^2}}{4x}. \tag{2.1}$$

We propose the name Boolean-Catalan numbers for the coefficients of the ordinary generating function $A(x)$ for two related reasons. One reason is that $2A(x) + 2x + 1$ is equal to $\sum_{s=0}^{\infty} c_s x^s (x+1)^{s+1}$ for the Catalan numbers c_s. The second reason is that each restricted Schröder path counted by the sequence has a fingerprint Dyck path, and the number of restricted Schröder paths that have the same fixed fingerprint Dyck path is 2^k for some integer k. In Section 2.3, we give proofs that three families of combinatorial objects are counted by the Boolean-Catalan numbers. In Section 2.3.1, we give a direct proof of a fact noted by Aguiar and Moreira that Equation (2.1) is the generating function for certain underdiagonal lattice paths. In Section 2.3.2, we prove that Equation (2.1) is the generating function for the family of pattern avoiding permutations described above. In Section 2.3.3, we prove that Equation (2.1) is the generating function for the family of pattern avoiding inversion sequences described above, answering a question of Martinez and Savage [22].

In Section 2.4, we construct a surjective map τ from S_n to all Dyck paths whose fibers agree with a map from [8]. Moreover, the fibers of τ are intervals in the weak order and coincide with a lattice congruence on S_n. In Section 2.5, we construct a surjective map ρ from S_n to restricted Schröder paths whose fibers are intervals in the weak order and coincide with a different lattice congruence on S_n. When ρ is restricted to the bottom or top elements of the interval, we get a bijection between the pattern avoiding permutations and restricted Schröder paths.

In Section 2.6, we construct a different surjective map ω from S_n to the restricted Schröder paths. We prove that the fibers of ω are intervals in the weak order. While the fibers of ω do not coincide with a lattice congruence on the weak order, they do refine the fibers of τ. We show that the reverse Lehmer codes of the top elements of the intervals are elements of $I_n(e_i > e_j \leq e_k)$. So ω restricted to the top elements is the desired bijection.

In Section 2.7, we rephrase the results of the previous sections using the language of permutation statistics.

Sections 2.4, 2.5, and 2.6 follow the same general strategy. We start with a map from all permutations to lattice paths (Dyck paths for Section 2.4 and restricted Schröder paths for Sections 2.5, and 2.6) .

1. For each lattice path in the codomain of the map, construct a directed graph.

2. Show that the directed graph is acyclic.

3. Consider the poset corresponding to the transitive closure of the acyclic graph.

4. Show that the linear extensions of the poset are precisely the fiber of the lattice path. (In particular, this shows the map is surjective)

5. Show that each poset is regular labeled (See Section 2.2.3 for details) to conclude that the fibers are intervals on the weak order.

6. Show that the fibers are equivalence classes of an equivalence relation on the weak order based on pattern avoidance.
7. Use the equivalence relation to identify the top and bottom elements of each interval as a pattern avoiding permutation.
8. Show that the number pattern avoiding permutations is the same as the number of lattice paths.
9. Conclude that the map restricted to the top or bottom elements gives a bijection from permutations to lattice paths.

2.2 Background

2.2.1 Pattern Avoidance

Definition 2.2.1. A permutation, or *pattern*, π is said to be contained in another permutation σ, if σ has a subsequence whose terms are order isomorphic to (i.e. have the same relative ordering as) π. If σ does not contain π, we say that σ *avoids* π.

Example 2.2.2. The permutation 314592687 contains 1423 because the subsequence 4968 (among others) is ordered in the same way as 1423. The permutation 314592687 avoids 3241 since it has no subsequence ordered in the same way as 3241.

Definition 2.2.3. A *vincular* pattern specifies adjacency conditions. A vincular pattern is written as a permutation where terms that must be adjacent are underlined.

Example 2.2.4. The permutation 314265 contains two occurrences of $2\underline{31}4$ (3426 and 3425) and a single occurrence of $2\underline{314}$ (3426), but avoids $\underline{23}\,\underline{14}$ (3425 and 3425 are non-examples because the 3 and 4 are not adjacent in 314265).

Definition 2.2.5. The *weak order* on S_n is the partial order on permutations where $x \leq y$ if and only if the set of inversions of x is a subset of the inversions of y. Cover relations in the weak order are the relations $x \lessdot y$ such that $x = x_1 \cdots x_n$ and $y = x_1 \cdots x_{j+1} x_j \cdots x_n$ and $x_{j+1} > x_j$ (That is, y is the same as x except for introducing exactly one more inversion via transposing adjacent terms).

2.2.2 Equivalence Relations on Posets using Moves on the Weak Order

Definition 2.2.6. Fix a vincular pattern π with exactly one adjacency condition on $\pi_j\pi_{j+1}$ where $\pi_j < \pi_{j+1}$. Let the *partner vincular* pattern of π, π' be the same pattern as π but with π_j and π_{j+1} swapped. Let $\sigma \lessdot \sigma'$ or $\sigma \gtrdot \sigma'$ in the weak order with $\sigma' = \sigma_1 \cdots \sigma_i \sigma_{i+1}, \cdots \sigma_n$. We say we apply a $\pi \to \pi'$ *move* on σ at π if $\sigma_i = \pi_j$ and $\sigma_{i+1} = \pi_{j+1}$ for a pattern π in σ and a pattern π' in σ'.

Definition 2.2.7. Let Θ_π be an equivalence relation on S_n defined as follows. Two permutations $x, y \in S_n$ are equivalent if and only if there exists a sequence of $\pi \to \pi'$ and $\pi' \to \pi$ moves that transform x into y.

Example 2.2.8. Consider $\Theta_{\underline{13}2}$. Then $341265 \equiv 134625$ since $3\mathbf{41}265 \equiv \mathbf{31}4265 \equiv 134\mathbf{26}5 \equiv 134625$.

Definition 2.2.9. Let $\Theta_{\pi_1,\cdots,\pi_k}$ be the equivalence relation on S_n where two permutations are equivalent if we can perform a sequence of $\pi_i' \to \pi_i$ or $\pi_i \to \pi_i'$ moves for any choices of $1 \le i \le k$. (The index i is allowed to vary within the sequence).

2.2.3 Regular Labelings

Many posets in this paper have base set $[n] = \{1, 2, \cdots, n\}$. To minimize confusion, $<$ is the standard total order on the integers and $\prec$ will refer to partial orders.

Definition 2.2.10. Suppose that $P = ([n], \prec)$ is a poset. P is *regularly labeled* if for all $x \prec z$ and $y \in P$, if $x < y < z$ or $z < y < x$, then $y \prec z$ or $x \prec y$.

Definition 2.2.11. Let $\mathcal{L}(P)$ be the set of linear extensions of a poset P.

The following is a rephrasing of a result of Björner and Wachs [7, Theorem 6.8].

Theorem 2.2.12. *Let $U \subseteq S_n$ and $|U| \ge 1$. The following conditions are equivalent:*

- *U is an interval in the weak order.*
- *$U = \mathcal{L}(P)$ for some regularly labeled poset P.*

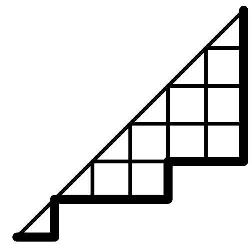


Figure 2.1: An example of a Dyck path

2.2.4 Lattice Paths

Definition 2.2.13. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, that is, underdiagonal paths from $(0,0)$ to (n,n) that use north and east steps of length 1.

Definition 2.2.14. Let $\mathcal{SP}_n$ be the set of Schröder paths of semilength n, that is, underdiagonal paths from $(0,0)$ to (n,n) that use north and east steps of length 1 and and diagonal northeast (length $\sqrt{2}$) steps.

Definition 2.2.15. Let $\mathcal{RSP}_n$ be the set of Schröder paths of semilength n with no diagonal steps on the main diagonal, and such that every diagonal step is followed by an east step.

Definition 2.2.16. Each of these lattice paths can be transformed into a *word* using the alphabet $\{N, E, D\}$ where we record the type of each step as we transverse the path from $(0,0)$ to (n,n).

Definition 2.2.17. A *factor* of a lattice path is a subword of adjacent steps that appears in the word of the path. A NEE factor of a lattice path is *centered* at column i, if the left east step of the factor lies in column i.

Example 2.2.18. The word for the lattice path in Figure 2.1 is $ENEEENEENNNN$. It has NEE factors centered at columns 2 and 5.

Definition 2.2.19. For fixed $p \in \mathcal{RSP}_n$, we define the *inner Dyck path* of p to be the following Dyck path. We start with the word for p and transform each DE factor into a NEE factor. The resulting word will correspond to the desired Dyck path.

2.2.5 Inversion Sequences

Recent progress has been made in problems in permutation patterns by analyzing the inversion sequences of permutations [11, 21].

Definition 2.2.20. The set of inversion sequences of length $n \geq 0$ is

$$I_n = \{(e_1, e_2, \cdots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i \leq i-1\}.$$

Definition 2.2.21. Inversion sequences are naturally in bijection with permutations. One bijection is to use the tuple $e_i = |\{j \mid j < i \text{ and } \pi_j > \pi_i\}|$, which is called the *inversion code* of a permutation. Another bijection is to take the tuple $e_i = |\{j \mid j > i \text{ and } \pi_j < \pi_i\}|$ and reverse it, which is called the *reverse Lehmer code* of a permutation.

Definition 2.2.22. Patterns can be defined on inversion sequences using binary relations. The binary relations can come from the set $\{<, >, \leq, \geq, =, \neq\}$. A pattern in an inversion sequence is a subsequence of length m that satisfies $m-1$ binary relations. Given one binary relation ρ, one can define $I_n(e_i \rho e_j)$ as the set of inversion sequences of length n that have no subsequence of the form $e_i \rho e_j$ for all $1 \leq i < j \leq n$. Given two binary relations, ρ_1, ρ_2, one can define $I_n(e_i \rho_1 e_j, e_j \rho_2 e_k)$ as the set of inversion sequences of length n that have no subsequence of the form (e_i, e_j, e_k) satisfying $e_i \rho_1 e_j$ and $e_j \rho_2 e_k$ for all $1 \leq i < j < k \leq n$.

Example 2.2.23. In the inversion sequence $(0,1,2,0,3,5)$, the subsequence $(2,0,5)$ (among others) is an instance of the pattern $e_i > e_j \leq e_k$. The inversion sequence $(0,0,2,3,3,5)$ avoids the pattern $e_i > e_j \leq e_k$ since no subsequence of the tuple has entries that satisfy the desired inequalities.

The main families of pattern avoiding inversion sequences in this paper are the following.

Definition 2.2.24. $I_n(e_i > e_j)$ is the set of weakly increasing inversion sequences of length n.

Definition 2.2.25. $I_n(e_i > e_j, e_j \leq e_k)$, (which we abbreviate $I_n(e_i > e_j \leq e_k)$) is the set of inversion sequences of length n that avoid the pattern $e_i > e_j \leq e_k$.

2.3 Enumerative Combinatorics

2.3.1 Generating Function for $\mathcal{RSP}_n$

The set $\mathcal{RSP}_n$ is considered by Aguiar and Moreira in [1]. They find that elements of $\mathcal{RSP}_n$ are in bijection with rooted ordered trees with $n+1$ leaves such that each internal node has at least two children and only the left and right children of a node are allowed to be leaves [1, Appendix A]. Let us call such trees AM-trees. Aguiar noted that Equation (2.1) is the ordinary generating function for AM-trees. We now give a generating function proof of this fact.

Proposition 2.3.1. *Equation* (2.1) *is the ordinary generating function for AM-trees.*

Proof. Let s_n be the number of AM-trees with $n+1$ leaves. We desire to find $S(x) = \sum_{n=0}^{\infty} s_n x^n$. We note that $xS(x)$ is the generating function for AM-trees weighted by the number of leaves. Furthermore, an AM-tree with at least two leaves has a composite structure. Given an AM-tree such that the root has degree d, we can create a d-tuple of AM-trees where the ith tree in the tuple is the subtree rooted by the ith child of the root of the original tree. The first and last elements of the tuple can be the one-element tree, while all other entries can be arbitrary AM-trees with more than one node. Let $S_d(x)$ be the generating function for AM-trees whose root has degree d.

$$\begin{aligned}
xS(x) &= \sum_{d=0}^{\infty} S_d(x) \\
&= S_0(x) + S_1(x) + \sum_{i=2}^{\infty} S_d(x) \\
&= x + 0 + \sum_{d=2}^{\infty} (xS(x))[x(S(x)-1)]^{d-2}(xS(x)) \\
&= x + (xS(x))^2 \sum_{j=0}^{\infty} [x(S(x)-1)]^j \\
&= x + \frac{(xS(x))^2}{1-[x(S(x)-1)]}
\end{aligned}$$

Solving for $S(x)$ and noting that $xS(x)$ should be 0 at $x=0$, we find that:

$$S(x) = \frac{1+2x-\sqrt{1-4x-4x^2}}{4x}$$

$\square$

By the bijection given in [1, Appendix A] we can produce a restricted Schröder path from any AM-tree. Start with a AM-tree and do a depth first search of the tree. Produce a word during the depth first search, by recording E for a left child, D for an internal child, and N for a right child. The resulting word will correspond with a restricted Schröder path [1]. We conclude that Equation (2.1) is the ordinary generating function for $\mathcal{RSP}_n$.

Corollary 2.3.2. *The ordinary generating function for $\mathcal{RSP}_n$ is Equation* (2.1)*.*

For another proof of 2.3.2, see Corollary 2.7.4.

2.3.2 Generating Function for Pattern Avoiding Permutations

Consider the set $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32) = \mathrm{Av}_n$. We show in this section that the coefficient sequence $|\mathrm{Av}_n|$ has ordinary generating function given by Equation (2.1). To prove this we make use of certain noncrossing arc diagrams that are in bijection with permutations.

Definition 2.3.3. A *noncrossing (arc) diagram* is a certain configuration of points and arcs. Start with n distinct points on a vertical line. Label them with the points $[n]$ with 1 at the bottom. Each diagram consists of some (or no) curves called arcs connecting the points. Each arc must satisfy the following requirements.

1. The arc connects a point p to a strictly higher point q, moving monotone upwards from p to q and passing to the left or to the right of each point between p and q. The arc may pass to the left of some points and to the right of others.

2. No two arcs intersect, except possibly at their endpoints.

3. No two arcs share the same upper endpoint or the same lower endpoint.

Construction 2.3.4. Permutations are in bijection [27, Theorem 3.1] with noncrossing diagrams. The bijection from permutations to noncrossing diagrams is given by the following construction. For a permutation $x \in S_n$, start by plotting the points (i, x_i) in $\mathbb{R}^2$ for all $1 \leq i \leq n$. Draw a line between (i, x_i) and $(i+1, x_{i+1})$ if $x_i > x_{i+1}$ for all $1 \leq i < n$. Homotope the lines and points so that the points form a vertical line. By definition, if the permutation has k descents, than the noncrossing diagram will have k arcs. See Figure 2.2 for an example.

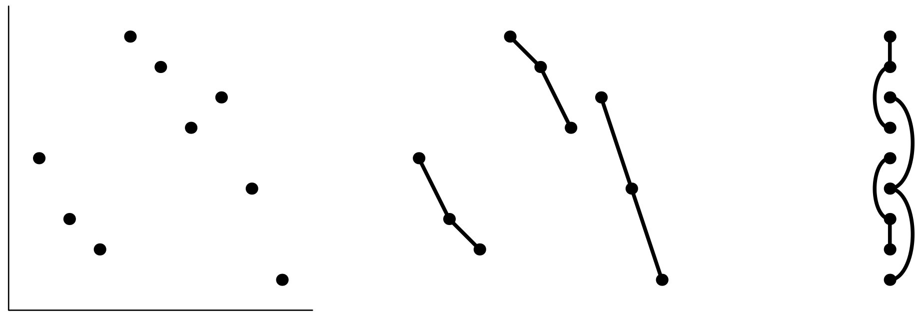


Figure 2.2: Building the noncrossing diagram corresponding to 532986741

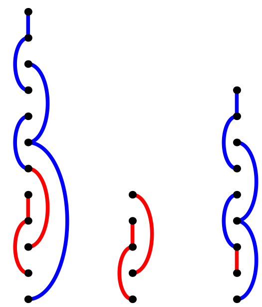
Let A_n be the set of noncrossing diagrams corresponding to Av_n and $a_n = |\mathrm{Av}_n| = |A_n|$. Converting the forbidden patterns into forbidden subdiagrams, we see that A_n is the set of noncrossing diagrams such that any arc that passes two or more vertices must pass to the right of all of them. Let B_n be the subset of A_n such that 3 and 2 are endpoints of an arc and 4 and 1 are endpoints of an arc and $b_n = |B_n|$. Let $C_{n,k}$ be the subset of A_n such that k and 1 are endpoints of an arc for $n \geq k > 1$ and $|C_{n,k}| = c_{n,k}$. Let $C_{n,1} = \{x \in A_n \mid x(1) = 1\}$.

Proposition 2.3.5.

$$c_{n,k} = \begin{cases} a_{n-1} & \textit{if} \quad k \leq 3 \\ a_{k-2} \cdot b_{n-(k-4)} & \textit{if} \quad k \geq 4 \end{cases}$$

Proof. Let $\alpha_{n,1} : C_{n,1} \to \mathrm{Av}_{n-1}$ be the map that takes a permutation in $C_{n,1}$, removes the first entry and subtracts 1 from the remaining entries. Let $\beta_{n,1} : \mathrm{Av}_{n-1} \to C_{n,1}$ be the map that takes a permutation y in Av_{n-1} adds 1 to every entry and inserts a 1 at the beginning. Let $\alpha_{n,2} : C_{n,2} \to \mathrm{Av}_{n-1}$ be the map that takes a permutation in $C_{n,2}$, removes the 1 and subtracts 1 from the remaining entries. Let $\beta_{n,2} : \mathrm{Av}_{n-1} \to C_{n,2}$ be the map that takes a permutation y in Av_{n-1} adds 1 to every entry and inserts a 1 immediately after the new 2. Let $\alpha_{n,3} : C_{n,3} \to \mathrm{Av}_{n-1}$ be the map that takes a permutation in $C_{n,3}$, removes the 1 and subtracts 1 from the remaining entries. Let $\beta_{n,3} : \mathrm{Av}_{n-1} \to C_{n,3}$ be the map that takes a permutation y in Av_{n-1} adds 1 to every entry and inserts a 1 immediately after the new 3. The map $\alpha_{n,k} \circ \beta_{n,k}$ is the identity map on Av_{n-1} and $\beta_{n,k} \circ \alpha_{n,k}$ is the identity map on $C_{n,k}$ for $1 \leq k \leq 3$. We conclude that $|C_{n,k}| = |\mathrm{Av}_{n-1}|$ for $1 \leq k \leq 3$.

Now suppose $k \geq 4$. We consider the following two maps. We define a map $\alpha_{n,k} : C_{n,k} \to A_{k-2} \times B_{n-k+4}$. Given an element z of $C_{n,k}$, let x be the subdiagram of z induced on the points 2 to $k-1$. Let y be the noncrossing diagram obtained by collapsing x to



$$C_{12,7} \longleftrightarrow A_5 \times B_9$$
$$5\ 4\ 2\ 8\ 6\ 3\ 12\ 11\ 9\ 10\ 7\ 1 \longleftrightarrow (43152, 532986741)$$

Figure 2.3: An example of one correspondence under the bijection specified in Proposition 2.3.5.

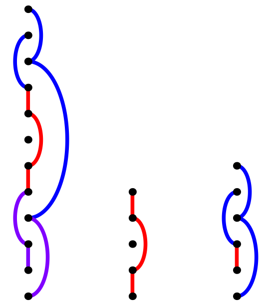
two adjacent points with an arc between them. Define $\alpha_{n,k}(z)$ to be (x, y). See Figure 2.3 for an example. Let $\beta_{n,k} : A_{k-2} \times B_{n-k+4} \to C_{n,k}$ be the map that takes a tuple (x, y) and maps to the following noncrossing diagram z. The noncrossing diagram z is built by starting with y, deleting the arc from 2 to 3 and those points, and placing the arcs of x onto $k-2$ new points replacing the deleted points. We find that $\alpha_{n,k} \circ \beta_{n,k}$ is the identity mapping on $Av_{k-2} \times B_{n-k-4}$ and $\beta_{n,k} \circ \alpha_{n,k}$ is the identity mapping on $C_{n,k}$. Thus, $|C_{n,k}| = |A_{k-2}| \cdot |B_{n-k-4}|$ for $n \geq k \geq 4$. □

Let $D_{n,k}$ be the subset of B_n such that 5 and 3 are endpoints of an arc and k and 4 are endpoints of an arc for $n \geq k \geq 5$. Let $D_{n,4}$ be the subset of B_n such that 5 and 3 are endpoints of an arc and 4 is not the bottom endpoint of any arc for $n \geq 5$. Let $D_{n,3}$ be the subset of B_n such that no arc connects 5 to 3 for $n \geq 4$. Let $|D_{n,k}| = d_{n,k}$.

Proposition 2.3.6.

$$d_{n,k} = \begin{cases} a_{n-3} & \textit{if} \quad k = 3 \\ a_{n-4} & \textit{if} \quad k = 4 \\ 0 & \textit{if} \quad k = 5 \\ a_{k-5} \cdot b_{n-(k-4)} & \textit{if} \quad k \geq 6 \end{cases}$$

Proof. Given $x \in D_{n,3}$, remove the arcs below 4 and obtain a noncrossing diagram corresponding to an avoider on $n-3$ dots. Given $x \in D_{n,4}$, remove the arcs below 5, and obtain a noncrossing diagram corresponding to an avoider on $n-4$ dots. The set $D_{n,5}$ is empty, because no two arcs can share the same upper endpoint by definition. We now consider



$$D_{12,10} \longleftrightarrow A_5 \times B_6$$
$$7\ 11\ 9\ 8\ 6\ 5\ 3\ 2\ 12\ 10\ 4\ 1 \longleftrightarrow (35421, 532641)$$

Figure 2.4: An example of the correspondence given in Proposition 2.3.6

$D_{n,k}$ for $n \geq k \geq 6$. We consider the following two maps. Let $\alpha_{n,k} : D_{n,k} \to A_{k-5} \times B_{n-(k-4)}$ be defined as follows. Given $z \in D_{n,k}$, let x be the subdiagram of z induced on the points 5 to $k-1$. Let y be the noncrossing diagram obtained by collapsing x to two adjacent points with an arc between them and deleting the arcs for 53, 41, 32 and deleting the bottom 3 points. Then $\alpha_{n,k}$ sends z to (x, y). See Figure 2.4 for an example. Let $\beta_{n,k} : A_{k-5} \times B_{n-k+4} \to D_{n,k}$ be defined as the following map. For fixed $(x, y) \in A_{k-5} \times B_{n-k+4}$, we build a noncrossing diagram $z \in D_{n,k}$ as follows. We begin with y, delete the arc from 2 to 3, replace these two points with $k-5$ new points, place x onto the $k-5$ points, add 3 points below, and add arcs for 53, 41, 32. We find that $\alpha_{n,k} \circ \beta_{n,k}$ is the identity map on $\mathrm{Av}_{k-5} \times B_{n-k-4}$ and $\beta_{n,k} \circ \alpha_{n,k}$ is the identity map on $D_{n,k}$. Thus, $|D_{n,k}| = |A_{k-5}| \cdot |B_{n-k-4}|$ for $n \geq k \geq 6$. □

Proposition 2.3.7. *The ordinary generating function for* $S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)$ *is Equation* (2.1).

Proof. As before, let $a_n = |\mathrm{Av}_n|$. Write $A(x)$ for $\sum_{n=0}^{\infty} a_n x^n$. By Proposition 2.3.5,

$$a_n = \sum_{k=1}^{n} c_{n,k} = 3a_{n-1} + \sum_{k=4}^{n} a_{k-2} b_{n-k+4} = 3a_{n-1} + \sum_{k=2}^{n-2} a_k b_{n+2-k}.$$

Thus,

$$\begin{aligned}
\sum_{n=4}^{\infty} a_n x^n &= 3x \sum_{n=4}^{\infty} a_{n-1} x^{n-1} + x^{-2} \sum_{n=4}^{\infty} \sum_{k=2}^{n-2} a_k x^k b_{n+2-k} x^{n+2-k} \\
&= 3x \sum_{n=3}^{\infty} a_n x^n + x^{-2} \sum_{n=4}^{\infty} \sum_{k+j=n+2} a_k x^k b_j x^j \\
&= 3x \sum_{n=3}^{\infty} a_n x^n + x^{-2} \sum_{k=2}^{\infty} a_k x^k \sum_{j=4}^{\infty} b_j x^j
\end{aligned}$$

Let $B(x) = \sum_{n=4}^{\infty} b_n x^n$. Since $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 6$, our calculation above becomes:

$$A(x) - 1 - x - 2x^2 - 6x^3 = 3x(A(x) - 1 - x - 2x^2) + x^{-2}(A(x) - 1 - x)B(x)$$

We solve for $B(x)$ to obtain

$$B(x) = \frac{x^2 A(x) - 3x^3 A(x) - x^2 + 2x^3 + x^4}{A(x) - 1 - x} \tag{2.2}$$

By Proposition 2.3.6, we find that:

$$b_n = \sum_{k=3}^{n} d_{n,k} = \begin{cases} a_1 & \text{if} \quad n = 4 \\ a_2 + a_1 & \text{if} \quad n = 5 \\ a_{n-3} + a_{n-4} + \sum_{k=6}^{n} a_{k-5} b_{n-(k-4)} & \text{if} \quad n \geq 6 \end{cases}$$

Thus,

$$\begin{aligned}
\sum_{n=4}^{\infty} b_n x^n &= x^3 \sum_{n=4}^{\infty} a_{n-3} x^{n-3} + x^4 \sum_{n=5}^{\infty} a_{n-4} x^{n-4} + x \sum_{n=6}^{\infty} \sum_{k=6}^{n} a_{k-5} x^{k-5} b_{n-(k-4)} x^{n-(k-4)} \\
&= x^3 \sum_{n=1}^{\infty} a_n x^n + x^4 \sum_{n=1}^{\infty} a_n x^n + x \sum_{n=6}^{\infty} \sum_{i+j=n-1} a_i x^i b_j x^j \\
&= (x^3 + x^4) \sum_{n=1}^{\infty} a_n x^n + x \sum_{i=1}^{\infty} a_i x^i \sum_{j=4}^{\infty} b_j x^j
\end{aligned}$$

We have shown that

$$B(x) = (x^3 + x^4)(A(x) - 1) + x(A(x) - 1)B(x)$$

and we solve to obtain

$$B(x) = \frac{(x^3 + x^4)(A(x) - 1)}{1 - x(A(x) - 1)}$$

By equating this to (2.2), and solving for $A(x)$, we obtain:

$$A(x) = \frac{1 + 2x \pm \sqrt{1 - 4x - 4x^2}}{4x}$$

Given that $4x(A(x) - 1)$ should equal 0 when $x = 0$, we choose the minus sign. □

2.3.3 Generating Function for $I_n(e_i > e_j \leq e_k)$

We now show that $I_n(e_i > e_j \leq e_k)$ has ordinary generating function given by Equation (2.1).

Definition 2.3.8. Let $C_s = I_s(e_i > e_j) \subseteq I_s$ be weakly increasing inversion sequences, i.e of the form:

$$e_1 \leq e_2 \leq \cdots \leq e_s$$

for $s \geq 1$ such that $0 \leq e_i \leq i - 1$.

It is not hard to see that the reverse Lehmer codes of weakly increasing inversion sequences are elements of $S_n(132)$, thus are counted by the Catalan numbers c_n (Using the convention $c_0 = c_1 = 1, c_2 = 2$).

Definition 2.3.9. Let $I_n(e_i > e_j \leq e_k) \subseteq I_n$ be the inversion sequences of the following form for some $t \geq 1$.

$$e_1 \leq e_2 \leq \cdots \leq e_t > e_{t+1} > \cdots > e_n$$

Definition 2.3.10. For fixed $(e_1, \cdots, e_n) \in I_n(e_i > e_j \leq e_k)$, let its *peak* be t as in the previous definition. That is, the peak is the index where the sequence switches from weakly increasing to strictly decreasing.

Lemma 2.3.11. *For fixed* $(e_1, \cdots, e_n) \in I_n(e_i > e_j \leq e_k)$, *if* $n \geq 2$, *then its peak* t *satisfies* $t \geq 2$.

Proof. If $t = 1$, then $0 = e_1 > e_2$, which is impossible. □

For $s, k \geq 1$, let

$$I_{s,k} = \{((e_1, \ldots, e_s), (e_{s+1}, \ldots, e_{s+k})) \mid (e_1, \ldots, e_s) \in C_s,\ s \geq e_{s+1} > \ldots > e_{s+k}\}$$

Let $I_{s,0} = C_s$.
Note that if $((e_1, \cdots, e_s), (e_{s+1}, \cdots, e_{s+k})) \in I_{s,k}$, then $(e_1, \cdots, e_{s+k}) \in I_{s+k}$.

Let

$$I^{>}_{s,k} = \{((e_1, \ldots, e_s), (e_{s+1}, \ldots, e_{s+k})) \in I_{s,k} \mid e_s > e_{s+1}\}$$

for $s \geq 1$ and $k \geq 1$. Let $I^{>}_{s,0} = C_s$.
Let

$$I^{\leq}_{s,k} = \{((e_1, \ldots, e_s), (e_{s+1}, \ldots, e_{s+k})) \in I_{s,k} \mid e_s \leq e_{s+1}\}$$

for $s \geq 1$ and $k \geq 1$.

Lemma 2.3.12. *$I_{s,k}$ is the disjoint union of $I^{>}_{s,k}$ and $I^{\leq}_{s,k}$ for $s \geq 2$ and $k \geq 1$.*

We can count the size of $I_{s,n-s}$.

Lemma 2.3.13. $|I_{s,k}| = c_s \binom{s+1}{k}$

Proof. Since the left half of the sequence is a weakly increasing sequence of length s, they are counted by the Catalan numbers. Since the right half is a strictly decreasing sequence of length k with largest part bounded above by s, they are counted by subsets of $[0, s]$ of size k. □

Theorem 2.3.14. *Equation* (2.1) *is the ordinary generating function for $I_n(e_i > e_j \leq e_k)$.*

Proof. Given an inversion sequence of length $n \geq 2$ and peak t, we can break the sequence into two pieces at the right of the peak so that the left piece is weakly increasing and the right piece is strictly decreasing. In other words, $a_n = \sum_{t=1}^{n} |I^{>}_{t,n-t}|$. But by Lemma 2.3.11, $a_n = \sum_{t=2}^{n} |I^{>}_{t,n-t}|$. Similarly, we can break the sequence at the left of the peak, which yields $a_n = \sum_{t=2}^{n} |I^{\leq}_{t-1,n-t+1}|$. Thus, for $n \geq 2$:

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} \sum_{t=2}^{n} |I^{>}_{t,n-t}| x^n + \sum_{n=2}^{\infty} \sum_{t=2}^{n} |I^{\leq}_{t-1,n-t+1}| x^n \\
&= \sum_{t=2}^{\infty} \sum_{n=t}^{\infty} |I^{>}_{t,n-t}| x^n + \sum_{t=2}^{\infty} \sum_{n=t}^{\infty} |I^{\leq}_{t-1,n-t+1}| x^n.
\end{aligned}$$

Reindexing, we obtain:

$$
\begin{aligned}
2\sum_{n=2}^{\infty} a_n x^n &= \sum_{s=2}^{\infty}\sum_{n=s}^{\infty} |I^{>}_{s,n-s}| x^n + \sum_{s=1}^{\infty}\sum_{n=s+1}^{\infty} |I^{\leq}_{s,n-s}| x^n \\
&= \sum_{s=2}^{\infty} |I^{>}_{s,0}| x^s + \sum_{s=2}^{\infty}\left(\sum_{n=s+1}^{\infty}\left(|I^{>}_{s,n-s}| + |I^{\leq}_{s,n-s}|\right) x^n\right) + \sum_{n=2}^{\infty} |I^{\leq}_{1,n-1}| x^n.
\end{aligned}
$$

By Lemma 2.3.12,

$$
2\sum_{n=2}^{\infty} a_n x^n = \sum_{s=2}^{\infty} |I^{>}_{s,0}| x^s + \sum_{s=2}^{\infty}\sum_{n=s+1}^{\infty} |I_{s,n-s}| x^n + \sum_{n=2}^{\infty} |I^{\leq}_{1,n-1}| x^n
$$

Since an element $((e_1),(e_2,\cdots,e_n))$ of $I^{\leq}_{1,n-1}$ has $1 \geq e_2 > e_3 > \cdots > e_n \geq 0$, $\sum_{n=2}^{\infty} |I^{\leq}_{1,n-1}| x^n = \sum_{n=2}^{3} |I^{\leq}_{1,n-1}| x^n = 2x^2 + x^3$, so

$$
2\sum_{n=2}^{\infty} a_n x^n = 2x^2 + x^3 + \sum_{s=2}^{\infty} |I^{>}_{s,0}| x^s + \sum_{s=2}^{\infty}\sum_{n=s+1}^{\infty} |I_{s,n-s}| x^n
$$

By Lemma 2.3.13 and because $I^{>}_{s,0} = C_s$,

$$
\begin{aligned}
2\sum_{n=2}^{\infty} a_n x^n &= 2x^2 + x^3 + \sum_{s=2}^{\infty} c_s x^s + \sum_{s=2}^{\infty}\sum_{n=s+1}^{\infty} c_s \binom{s+1}{n-s} x^n \\
&= 2x^2 + x^3 + \sum_{s=2}^{\infty} c_s x^s + \sum_{s=2}^{\infty}\sum_{k=1}^{\infty} c_s \binom{s+1}{k} x^{s+k} \\
&= 2x^2 + x^3 + \sum_{s=2}^{\infty} c_s x^s + \sum_{s=2}^{\infty} c_s x^s \sum_{k=1}^{\infty} \binom{s+1}{k} x^k \\
&= 2x^2 + x^3 + \sum_{s=2}^{\infty} c_s x^s \sum_{k=0}^{\infty} \binom{s+1}{k} x^k \\
&= 2x^2 + x^3 + \sum_{s=2}^{\infty} c_s x^s (x+1)^{s+1} \\
&= \left(\sum_{s=0}^{\infty} c_s x^s (x+1)^{s+1}\right) - 2x - 1
\end{aligned}
$$

Since $a_0 = a_1 = 1$ and $\sum_{s=0}^{\infty} c_s y^s$ is well known to be $\frac{1-\sqrt{1-4y}}{2}$

$$2(A(x) - 1 - x) = (x+1)\frac{1-\sqrt{1-4x(x+1)}}{2x(x+1)} - 2x - 1$$

Rearranging, we obtain:

$$A(x) = \frac{1+2x-\sqrt{1-4x-4x^2}}{4x}$$

□

2.4 Permutations and Dyck paths

We construct a map from all permutations to Dyck paths that will be vital to constructing explicit bijections alluded to by the previous section. In this section, for a permutation $x \in S_n$, let $x_1 \cdots x_n$ be its one-line notation.

Definition 2.4.1. We call a pair (x_i, x_j) an *inversion* of x if $i < j$ and $x_i > x_j$.

Definition 2.4.2. A *non-Dyck inversion* of a permutation $x \in S_n$ is an inversion (x_i, x_j) such that there exists some x_k where $i < j < k$ and $x_j < x_k < x_i$. A *Dyck inversion* of a permutation $x \in S_n$ is an inversion (x_i, x_j) that is not a non-Dyck inversion.

Definition 2.4.3. The *height* of an east step of a Dyck path $p \in \mathcal{D}_n$ is its distance from the southern boundary of the $n \times n$ box.

Definition 2.4.4. Let $\tau : S_n \to \mathcal{D}_n$ be the following map. For $1 \le i \le n$, define

$$d_i = |\{x_j \mid (x_i, x_j) \text{ is a Dyck inversion}\}|$$

We define $\tau(x)$ to be the unique Dyck path where the east step in the ith column occurs at height $i - 1 - d_i$. See Figure 2.5 for an example.

Remark 2.4.5. *The map τ is inspired by [2]. In fact, when τ is restricted to $S_n(312)$, we recover the main bijection of [2].*

Definition 2.4.6. Let $x \in S_n$. For fixed x_i, define $B_{x_i} = \{x_j \mid (x_i, x_j) \text{ is a Dyck inversion}\}$.

Definition 2.4.7. Let $x \in S_n$. For fixed x_j, define $D_{x_j} = \{x_i \mid (x_i, x_j) \text{ is a Dyck inversion}\}$.

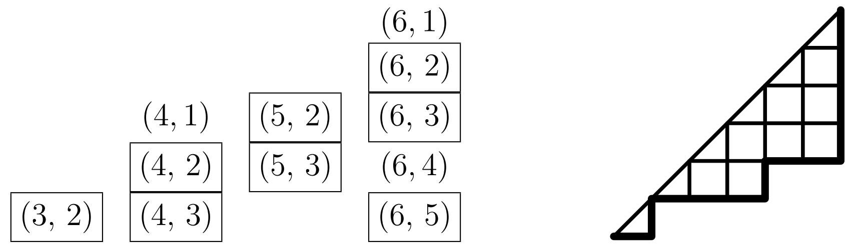


Figure 2.5: Left: The inversions of 641532 (with Dyck inversions boxed). Right: $\tau(641532)$.

Proposition 2.4.8. *The map τ is well-defined.*

Proof. We need to verify for fixed $x \in S_n$, that $\tau(x)$ is a legitimate Dyck path. Since x_i is a part of at most $x_i - 1$ inversions of the form (x_i, x_j), $i - 1 - d_i \geq 0$. What remains to show is that the height of $x_i + 1$ is greater than or equal to the height of x_i for all $1 \leq i < n$. There are two cases to consider.

Case 1: Suppose that x_i occurs to the left of $x_i + 1$ in the one-line notation of x. We claim that $B_{x_i+1} \subseteq B_{x_i}$. If $(x_i + 1, x_j)$ is a Dyck inversion, all intermediate values between x_j and $x_i + 1$ occur to the left of x_j. Therefore, all intermediate values between x_j and x_i are to the left of x_j, i.e (x_i, x_j) is a Dyck inversion, which proves the claim. Since $B_{x_i+1} \subseteq B_{x_i}$, the height of $x_i + 1$ is greater than the height of x_i (the heights cannot be equal in Case 1 because $(x_i + 1, x_i)$ is not a Dyck inversion).

Case 2: Similarly, in the case that x_i occurs to the right of $x_i + 1$, we get $B_{x_i} \subseteq B_{x_i+1}$. However in this case, we claim that $(B_{x_i+1} \setminus \{x_i\}) \subseteq B_{x_i}$. We see that no x_j located between $x_i + 1$ and x_i can form a Dyck inversion with $x_i + 1$ because then $(x_i + 1)x_j x_i$ would be a 312 pattern. Therefore, $(B_{x_i+1} \setminus \{x_i\}) = B_{x_i}$. We conclude that in this case, $x_i + 1$ and x_i have the same height. $\square$

Definition 2.4.9. Let $\text{left}(a, b)$ be whichever of a and b appears first in x in the one-line notation of x.

The proof of Proposition 2.4.8 also establishes the following Lemma.

Lemma 2.4.10. *Suppose $x \in S_n$ and $q \in \mathcal{D}_n$ satisfy $\tau(x) = q$. Then $\text{left}(x_j, x_j + 1) = x_j$ if and only if $x_j + 1$ has height strictly greater than x_j in $\tau(x)$. Also, $\text{left}(x_j, x_j + 1) = x_j + 1$ if and only if $x_j + 1$ has the same height as x_j in $\tau(x)$.*

Dyck inversions satisfy the following useful properties.

Proposition 2.4.11. *Let $x \in S_n$. For fixed x_j, if D_{x_j} is nonempty, then $D_{x_j} = \{x_j + 1, x_j + 2, \cdots, M\}$ for some M.*

Proof. If D_{x_j} is nonempty, its maximum exists which we call M. Let x_i be a number such that $x_j + 1 \leq x_i \leq M$. We must show that (x_i, x_j) is a Dyck inversion. In the one-line notation of x, x_i must lie to the left of x_j because otherwise x_i would serve as a witness for (M, x_j) not being a Dyck inversion. Thus, (x_i, x_j) is an inversion. Also, there can exist no witness x_k for (x_i, x_j) to not be a Dyck inversion, because again, x_k would serve as a witness for (M, x_j) not being a Dyck inversion. □

Definition 2.4.12. Each Dyck path has a natural *edge labeling* as follows. We label the unique east edge in the ith column with i. To label a north edge, travel southwest from the center of the edge until we hit the path again, necessarily on an east edge. Label the north edge to agree with that east edge. See Figure 2.6 for an example.

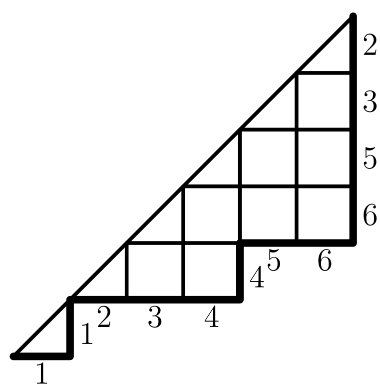


Figure 2.6: The edge labeling of a Dyck path

We note that a natural statistic on Dyck paths is area.

Definition 2.4.13. The *area* of a Dyck path is the number of boxes (unit squares) that can be packed below the diagonal and above the path.

We note that if an east step in the ith column has height $i - 1 - d_i$, then there are d_i boxes above the step in the column. Equivalently, the map τ is statistic preserving, sending the number of Dyck inversions to area. Furthermore, if $\tau(x) = q$, we can specify

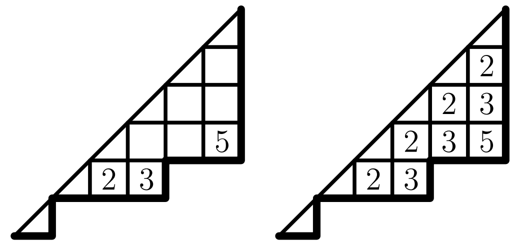


Figure 2.7: The base boxes and box labeling of a Dyck path

which box in the area of q corresponds to which Dyck inversion of x using the following *box labeling*, which is closely related to its edge labeling.

Definition 2.4.14. Suppose that $q \in \mathcal{DP}_n$ with edge labeling as in Definition 2.4.17. We define the following *box labeling* recursively. Let a box making up the area of q be a *base box* if its southern edge is the right E step of a EE factor of q. For each EE factor of q, suppose that the left E step is labeled with d. Label the base box whose southern edge coincides with the right E step with d. We now describe recursively how to label the boxes making up the area of q that are non-base boxes. For non-base box $\mathcal{B}$, there will be another *neighbor box* $\mathcal{A}$ whose northeast corner coincides with the southwest corner of $\mathcal{B}$. Label $\mathcal{B}$ to match the label of its neighbor box. If we continually take neighbor boxes of neighbor boxes of a non-base box, we will eventually hit a base box. Thus, this labeling is well-defined and constant on diagonals of neighbor boxes. See Figure 2.7 for an example.

Proposition 2.4.15. *Suppose that $q \in \mathcal{DP}_n$ is edge and box labeled by the above definitions. For a fixed column, the box labels decrease going up.*

Proof. Assume for the sake of contradiction there exists a column c with a box labeled a above a box labeled b with $a > b$. Then $a + 1 > b + 1$, so the leftmost column with a box labeled with a is to the left of the leftmost column with a box labeled with b. Let M_b be the rightmost column with a box labeled with b. Then by construction, Definition 2.4.14 tells us there is a box labeled with b in the column a, since $b + 1 \leq a < c \leq M_b$. But if $a > b$, then the east step labeled a is above the box labeled with b, which is impossible for a Dyck path, since the boxes making up the area of a Dyck path of a fixed column are supposed to be above the east step of that column. $\square$

This box labeling plays well with Dyck inversions.

Proposition 2.4.16. *Let $x \in S_n$ and $q \in \mathcal{D}_n$ with $\tau(x) = q$. Then (c, d) is a Dyck inversion of x if and only if there is a box labeled with d in the cth column of q.*

Proof. We write $\tau_n : S_n \to \mathcal{D}_n$ to emphasize the dependence of τ on n. We will prove the forward direction by induction on n. The statement is clear for $n = 1$. Let $x \in S_{k+1}$. Consider the permutation $x' \in S_k$ obtained by removing the $k+1$ from x. We note that $\tau_k(x')$ is the Dyck path formed by deleting the $(k+1)$th column of $\tau_{k+1}(x)$. The statement is true for x' and $\tau(x')$ by induction. Thus, the statement is true for Dyck inversions of x of the form $(e, .)$ for $e < k+1$. Consider a Dyck inversion $(k+1, d)$ of x (It may be useful to look back at Definition 2.4.6 and the proof of Proposition 2.4.8 before reading the remaining paragraphs).

If $d < k$, since $B_{k+1} \setminus \{k\} \subseteq B_k$, there will be a Dyck inversion (k, d) of x'. Furthermore, $\{y \in B_{k+1} \mid y \leq d\} = \{y \in B_k \mid y \leq d\}$. Let $u = |\{y \in B_k \mid y \leq d\}|$. By the inductive hypothesis and Proposition 2.4.15, the highest u boxes in the kth column of $\tau_k(x')$ will be labeled with the elements of $\{y \in B_k \mid y \leq d\}$ in increasing order from top to bottom. By the definition of τ there are at least u boxes in the $(k+1)$th column of $\tau_{k+1}(x)$. By Definition 2.4.14 and the fact that $\{y \in B_{k+1} \mid y \leq d\} = \{y \in B_k \mid y \leq d\}$, there will be a box labeled with d in the $(k+1)$th column.

If $d = k$, then k and $k+1$ are labels of an EE factor of q, thus there will be a base box in the $(k+1)$th column labeled with k. This completes the induction.

We will prove the backward direction by induction on n. The statement is clear for $n = 1$. Let $q \in \mathcal{D}_{k+1}$. Let $q' \in \mathcal{D}_k$ be the Dyck path created by deleting the $(k+1)$th column of q. Let $x' \in S_k$ be as previously defined. Clearly, $\tau(x') = q'$. The statement is true for boxes of q not in the $(k+1)$th column by the inductive hypothesis on q' and x'. Let $\mathcal{A}$ be a box in the $(k+1)$th column of q.

If $\mathcal{A}$ is labeled with a number $d < k$, then by Definition 2.4.14, there is a box labeled with d in the kth column. By the inductive hypothesis, (k, d) is a Dyck inversion of x' and x. We claim that $(k+1, d)$ is a Dyck inversion of x. It suffices to show that $(k+1, d)$ is an inversion, because then $\{y \in B_{k+1} \mid y \leq d\} = \{y \in B_k \mid y \leq d\}$. Let $u = |\{y \in B_k \mid y \leq d\}|$. If $(k+1, d)$ were not an inversion, then since $B_{k+1} \subseteq B_k$, k would have fewer than u Dyck inversions. But this would contradict the existence of the box $\mathcal{A}$ labeled with d. This gives the desired result.

If $\mathcal{A}$ is labeled with k, then it is a base box, which implies that $(k+1)$ has k Dyck inversions, which can only happen if $k+1$ is to the left of k. In other words, $(k+1, k)$ is a Dyck inversion of x. This completes the induction. □

Definition 2.4.17. Let $q \in \mathcal{DP}_n$ with edge labeling as defined in Definition 2.4.12. For each instance of NN, suppose the lower north step is labeled with c and the upper north step is labeled with d. Let $c \prec_{NN} d$ be called a *NN relation*. Similarly, for each instance of NE, let the north step be labeled with a and the east step be labeled with b. Let $a \prec_{NE} b$ be called a *NE relation*. Let the label of the highest north step of q be called the *root*.

Proposition 2.4.18. *For fixed $q \in \mathcal{DP}_n$, the transitive closure of the NN and NE relations form a poset, which we call $P_\tau(q)$.*

Proof. Let $G_\tau(q)$ be the directed graph on $[n]$ with directed edges for each NN and NE relation (as described in Definition 2.4.12) with a directed to b if and only if $a \prec_{NN} b$ or $a \prec_{NE} b$. There are $n-1$ north steps with a step after them, so the graph has $n-1$ edges. We claim that $G_\tau(q)$ is connected. More specifically, if the root is labeled with r, every $s \in [n] \smallsetminus r$ has a directed path to r. We note that the outdegree of every $s \in [n] \smallsetminus r$ is 1 using whichever relation corresponds to the north step labeled with s. If we start with s and travel in the graph along the unique directed path out of s, we always run into labels whose corresponding north steps in the Dyck path have larger heights, so eventually we will hit the north step labeled with r. This proves that the undirected $G_\tau(q)$ is connected. Since $G_\tau(q)$ is a connected graph with n vertices and $n-1$ edges, it is a tree, and thus is acyclic. □

Remark 2.4.19. *$G_\tau(q)$ is the Hasse diagram of $P_\tau(q)$ (This will not be true for other graphs/posets in later sections). $G_\tau(q)$ is a binary tree on $[n]$, and can be embedded in the plane in a nice way. See Figure 2.8 for an example. See [20, Section 3.1] for more details.*

We note that a permutation $x \in S_n$ can be viewed as a total order on $[n]$ where x_i is lower than x_j if and only if x_i precedes x_j in x.

Lemma 2.4.20. *Let $q \in \mathcal{DP}_n$. Let $x \in S_n$ with x viewed as a total order $x_1 \prec \cdots \prec x_n$ on $[n]$. Fix a NN factor of q and consider its corresponding NN relation, $c \prec_{NN} d$ of $P_\tau(q)$. Suppose that the vertical line containing the north steps labeled by c and d is contained in the western boundary of the bth column. We claim that if x is a linear extension of $P_\tau(q)$, then $b-1$ is the largest entry i of x such that (i,d) is a Dyck inversion of x.*

Proof. By examining the east step labeled by b, we find that $d \lessdot_{NN} \cdots \lessdot_{NN} w \lessdot_{NE} b$, where w is the highest north step vertically above d. Since $b > d$, this shows that (b,d) is not an inversion of x, so cannot be a Dyck inversion of x. By definition of edge labeling, the lowest north step vertically below c is labeled $b-1$. We know that $b-1 \lessdot_{NN} \cdots \lessdot_{NN} c \lessdot_{NN} d$, i.e. $(b-1,d)$ is an inversion of x.

Assume by way of contradiction that $(b-1,d)$ is not a Dyck inversion of x. That is, there exists an e such that $d < e < b-1$ and $d \lessdot e$. If we consider the unique directed path from e to the root in $G_\tau(q)$, we see that $e \to \cdots \to c \to d$ (That is if q is a valid Dyck path, going up from e will eventually hit a north step in the western boundary of column b). But then, $e \lessdot d$, which contradicts $d \lessdot e$. Thus, $(b-1,d)$ is a Dyck inversion of x. By Proposition 2.4.11, we conclude that the set of entires that form Dyck inversions of the form $(\cdot, d)$ for x is $\{d+1, d+2, \cdots, b-1\}$.

□

Theorem 2.4.21. *Let $q \in \mathcal{DP}_n$. Let $x \in S_n$ with x viewed as a total order $x_1 \lessdot \cdots \lessdot x_n$ on $[n]$. Then x is a linear extension of $P_\tau(q)$ if and only if $\tau(x) = q$.*

Proof. We first prove the forward direction. This amounts to showing that, if x is a linear extension of $P_\tau(q)$, then $\tau(x)$ and q have the same set of box labels in each column (but a priori we don't assume these two Dyck paths are the same, that's what this paragraph is trying to prove). Fix a NN factor of q, and consider its corresponding NN relation, $c \lessdot_{NN} d$ of $P_\tau(q)$. By the Lemma 2.4.20, the columns of $\tau(x)$ that have boxes labeled with d are $\{d+1, \cdots, b-1\}$ (so in particular, the set of columns is nonempty). We note that by definition of edge and box labelings, the set of boxes labeled with d will be nonempty and is $\{d+1, \cdots, b-1\}$. Fix a EN factor of q where the E and N are labeled a. By definition of box labeling, there are no boxes labeled a in q. Assume for the sake of contradiction that x has a Dyck inversion of the form (g,a). It is impossible for $a = n$. If $g = a+1$, then we would have a contradiction since $a \lessdot_{NN} \cdots \lessdot_{NN} f \lessdot_{NE} a+1 = g$. If $a \neq n$ and $g > a+1$, then in x we would still have $a \lessdot_{NN} \cdots \lessdot_{NN} f \lessdot_{NE} a+1$. But then $a+1$ would be a witness for (g,a) not being a Dyck inversion. Thus, we conclude that $\tau(x)$ also has no boxes labeled with a. Since we have systematically gone through every north step label (The NN factors give rise to the same sets of nonempty boxes, and the EN factors give rise to no boxes), the two Dyck paths have the same box labelings, and we conclude that $\tau(x) = q$.

Now we prove the backward direction. Suppose $x \in S_n$ with $\tau(x) = q$. Fix a NN factor

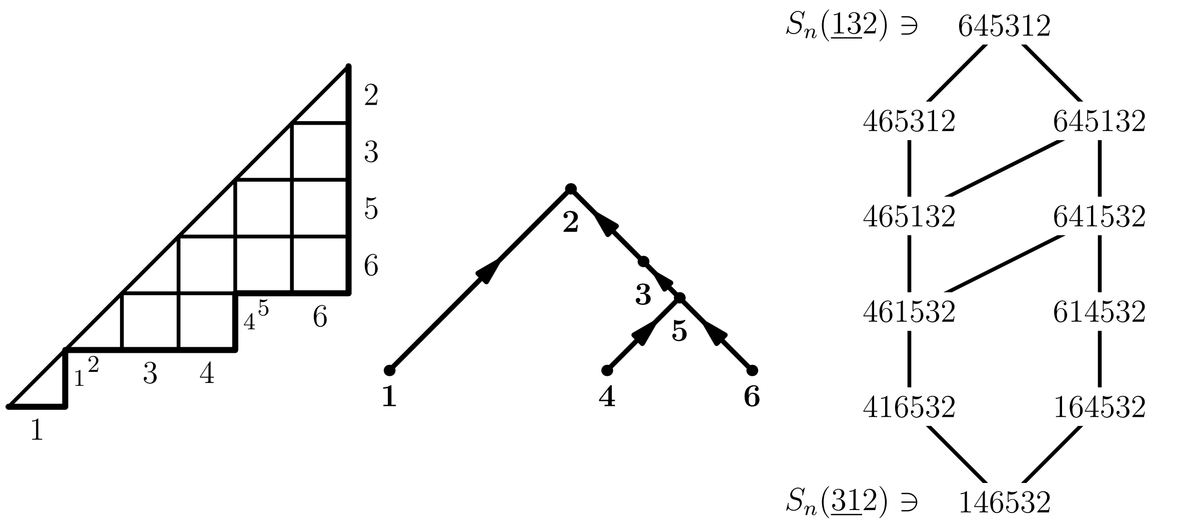


Figure 2.8: A Dyck path q, $G_\tau(q)$, linear extensions of $P_\tau(q)$ or equivalently, $\tau^{-1}(q)$

of q and consider its corresponding NN relation, $c \lessdot_{NN} d$, in $P_\tau(q)$. By definition of box labeling, we know that the lowest box in the cth column is labeled with a d, i.e. (c,d) is a Dyck inversion, so in particular is an inversion. That is, $\text{left}(c,d) = c$. Fix a NE relation, $a \lessdot_{NE} b$, in $P_\tau(q)$. We note that if the lowest box in the bth column is labeled with f, then by definition of box labels, the lowest box in the ath column is also labeled with f. Therefore, $f \neq a$, i.e. (b,a) is not a Dyck inversion. We claim that (b,a) is not a non-Dyck inversion. Assume for the sake of contradiction that the set $\{u \mid a < u < b, \text{left}(a,u) = a\}$ is nonempty. Since this set is finite it has a maximum, u_{max}. But then (b, u_{max}) would have to be a Dyck inversion. But since $u_{max} > a > f$, this contradicts f being the largest Dyck inversion of b. Thus the set of witnesses for (b,a) being an inversion but not a Dyck inversion is empty. Thus, we conclude that (b,a) is not an inversion, i.e $\text{left}(a,b) = a$. We have shown that x satisfies all generating relations of $P_\tau(q)$, thus is a linear extension of it. This completes the proof of the backwards direction. $\square$

Since every poset has at least one linear extension, we have the following corollary of Theorem 2.4.21.

Corollary 2.4.22. *The map $\tau : S_n \to \mathcal{DP}_n$ is surjective.*

We can describe the principal order ideals of $P_\tau(q)$ directly from q.

Proposition 2.4.23. *Fix $q \in \mathcal{DP}_n$. Suppose $b \in [n]$ and the N step in q labeled with b is part of the eastern boundary of the Mth column. Suppose the lowest box in the bth column is labeled with d. Then the principal order ideal of b in $P_\tau(q)$, is $\{d+1, \cdots, M\}$.*

Proof. By definition of box and edge labelings, $d+1 \leq b \leq M$. Let $e \in [n]$. Clearly if $e = b$, e is in the principal order ideal of b. It may be helpful to refer to Figure 2.9 while reading the rest of the proof.

Suppose that $d+1 \leq e < b$. Since $d+1 \leq e < b$, by definition of box labeling, there is a box labeled with d in column $b-1$. There is necessarily a relation of the form $w_l \prec_{NE} b$, where w_l is the label on the eastern side of the box labeled with d in column $b-1$ (w_l is also the north step of the NE factor whose east step is labeled with b). We claim that e is in the principal order ideal of b. We prove this by strong induction on $b-e$. If $b-e=1$, then the claim is true since by examining the north step labeled with e, we see that $e \prec_{NN} \cdots \prec_{NN} w_l \prec_{NE} b$. Suppose that $b-e>1$. If we consider the east step labeled by e, by definition of edge labeling, the vertical step labeled by e will be in the western boundary of a column f with $e < f \leq b$. By examining the vertical step labeled by e, we see that $e \prec_{NN} \cdots \prec_{NN} w_f \prec_{NE} f$ where w_f is the label of the north step that forms a NE factor with the east step labeled with f. If $f=b$, we see that $e \prec f = b$. Otherwise, by definition of edge labeling, f satisfies $b-f < b-e$, so by the inductive hypothesis is in the order ideal of b. But then $e \prec f \prec b$, and we conclude that e is in the order ideal of b.

Similarly, suppose that $b < e \leq M$. By definition of edge labeling, there will be an east step in column $b+1$ at the same height as the east step in column b. Furthermore, the east step in column $b+1$ will be the southern side of the base box labeled by b. By definition of box labeling, there will be a box labeled by b directly below the box labeled by d in column M. Let w_r the label of the north step that forms the eastern side of this box labeled by b in column M. Then we know that $w_r \prec_{NN} b$. We can use a similar strong induction argument to show that $e \prec w_r \prec_{NN} b$.

Suppose that $e \in [n]$ and $e \notin [d+1, M]$. Assume for the sake of contradiction that $e \prec b$ in $P_\tau(q)$. Then there will exist a chain consisting of NN and NE relations $e \prec y_1 \prec y_2 \prec \cdots y_n \prec b$. Let i be the largest index such that $y_i \notin [d+1, M]$ and $y_j \in [d+1, M]$ for all $j > i$. Consider the relation $y_i \prec y_{i+1}$.

If $y_i \prec_{NE} y_{i+1}$, then by definition of edge labeling, $y_i < y_{i+1} \leq M$. Let a be the label of the lowest box in column y_{i+1}. By definition of box labeling, the lowest box in column y_i will also be labeled with a, so in particular, $a < y_i$. By definition of i, $y_i \leq d$, so we find that

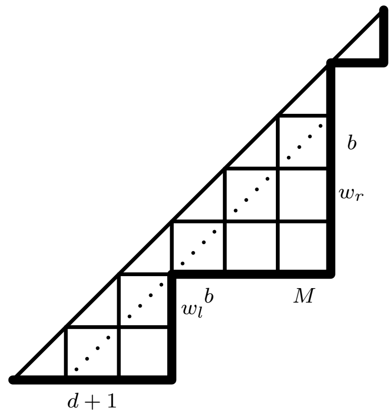


Figure 2.9: The principal order ideal of b in $P_\tau(q)$. Every east and north step label below the dotted line is an element of the ideal.

$a < d$. But then in column y_{i+1}, we have a box labeled with d above a box labeled with a, contradicting Proposition 2.4.15.
If $y_i \prec_{NN} y_{i+1}$, then by definition of edge labeling, $y_i > y_{i+1} \geq d+1$. We note that $y_{i+1} \in [d+1, M]$, there is a chain $y_{i+1} \prec_{NN} \cdots \prec b$. So in particular, the north step labeled by y_{i+1} is in the eastern boundary of a column weakly to the left of column M. But this means the north step labeled by y_i is also in a column weakly to the left of column M, which implies that the east step labeled by y_i is weakly to the left of column M. But then, $d+1 < y_i \leq M$, contradicting the definition of i. We conclude that no such i can exist, thus for all $e \notin [d+1, M]$, e is not in the principal order ideal of b. □

Corollary 2.4.24. *Fix $q \in \mathcal{DP}_n$. Then $P_\tau(q)$ is regularly labeled.*

Proof. Fix $z \in [n]$. Suppose the principal order ideal of z in $P_\tau(q)$ is $\{d+1, \cdots, M\}$. Suppose that $x \prec z$ in $P_\tau(q)$. If $x < z$, suppose y is such that $x < y < z$. But then $d+1 \leq x < y < z \leq M$, so y is in the principal order ideal of z in $P_\tau(q)$, or in other words, $y \prec z$. If $x > z$, suppose y is such that $x > y > z$. But then $M > y > x > z$, so y is in the principal order ideal of z in $P_\tau(q)$, or in other words, $y \prec z$. This proves the proposition. □

Remark 2.4.25. *Since we have shown that for any y numerically between x and z, $y \prec z$, we have also shown that $P_\tau(q)$ has what Björner and Wachs call a recursive labeling [7].*

Recalling Theorem 2.2.12, we now get the following corollary.

Corollary 2.4.26. *The fibers of τ are intervals in the weak order.*

The rest of this section is closely related to [8, Section 9].

Proposition 2.4.27. *Let $x \lessdot y$ in the weak order. Then $\tau(x) = \tau(y)$ if and only if y is obtained from x by a $\underline{13}2$ to $\underline{31}2$ move.*

Proof. If $x \lessdot y$ in the weak order, $y = x_1 \cdots x_{j+1} x_j \cdots x_n$ for some $1 \le j \le n$. Since $\tau(x) = \tau(y)$, (x_{j+1}, x_j) is a non-Dyck inversion, so there must exist some witness x_k with $k > j+1$ and $x_j < x_k < x_{j+1}$. But then y is obtained from x by the desired move since where x_j, x_k, x_{j+1} play the roles of the 1, 2, 3 of the move. Conversely, performing a $132 \to 312$ move creates exactly one non-Dyck inversion. Since the move does not destroy Dyck inversions, $\tau(x) = \tau(y)$. □

By applying transitivity to the cover relations within a fiber, we get the following corollary. We remind the reader of Definition 2.2.7.

Corollary 2.4.28. *τ is constant on classes of $\Theta_{\underline{13}2}$.*

Corollary 2.4.29. *A permutation is a bottom element of a fiber of τ if and only if it is an element of $S_n(\underline{31}2)$. A permutation is a top element of a fiber of τ if and only if it is an element of $S_n(\underline{13}2)$.*

Proof. The forward directions follow from Corollary 2.4.26 and Corollary 2.4.28. To prove the backward directions, we first note that by Corollary 2.4.28, each fiber is a union of equivalence classes. It is well known that $S_n(\underline{31}2)$, $S_n(\underline{13}2)$, and $\mathcal{DP}_n$ are counted by the Catalan numbers. In other words, the number of classes is equal to the number of fibers, so the classes and fibers coincide. □

The previous propositions give us the desired bijection.

Corollary 2.4.30. *The restriction of τ to the bottom elements of its fibers gives a bijection between the elements of $S_n(\underline{31}2)$ and $\mathcal{DP}_n$. The restriction of τ to the top elements of its fibers gives a bijection between the elements of $S_n(\underline{13}2)$ and $\mathcal{DP}_n$.*

2.5 Bijective Combinatorics of Pattern Avoiding Permutations

We now mimic the strategy used in the previous section to construct a bijection between $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $\mathcal{RSP}_n$. We first provide the background necessary to define a map $\rho : S_n \to \mathcal{RSP}_n$. We then prove that the fibers of ρ are the linear extensions of regularly labeled posets. Finally, we show that the fibers of ρ are the congruence classes of $\Theta_{2\underline{14}3,3\underline{14}2,\underline{14}23,\underline{14}32}$ to prove that certain restrictions of ρ give us the desired bijections.

Definition 2.5.1. Let $x \in S_n$ be fixed. A *strong 312 pattern* of x is a subsequence $x_i x_j x_k$ where $i < j < k$ and $x_j + 1 = x_k = x_i - 1$. A *strong 132 pattern* of x is defined analogously.

Proposition 2.5.2. *Let $x \in S_n$ be fixed. If x_j is the '2' of a strong 312 or strong 132 pattern of x, then the east step of $\tau(x)$ labeled with x_j is the middle east step of a NEE factor of $\tau(x)$.*

Proof. This follows from applying Lemma 2.4.10 to the non-inversion $(x_j - 1, x_j)$ and the inversion $(x_j + 1, x_j)$ and the east steps labeled with $x_j - 1$, x_j, and $x_j + 1$. □

Definition 2.5.3. Fix $q \in \mathcal{DP}_n$. We say a NEE factor of q is *centered* at i, if the middle E step of the factor is labeled with i.

Definition 2.5.4. Fix $p \in \mathcal{RSP}_n$. If a column has a D, we say that the column has a *triangle* and the column is *triangle-positive*. If a column does not have a D, we say the column is *triangle-free*

Definition 2.5.5. Fix $x = x_1 \cdots x_n \in S_n$. If x_i is the 2 of a strong 312 pattern, we say x_i is *triangle-positive*. If x_i is not the 2 of a strong 312 pattern, we say x_i is *triangle-free*.

Definition 2.5.6. Let $\rho : S_n \to \mathcal{RSP}_n$ be the following map. Let $x \in S_n$. Start with $\tau(x)$. If x_j is the '2' of a strong 312 pattern, then Proposition 2.5.2 says that $\tau(x)$ has a NEE factor centered at x_j. Replace that NEE factor by a DE factor. The resulting path is $\rho(x)$. See Figure 2.10 for an example.

Proposition 2.5.7. *The map ρ is well-defined.*

Proof. Fix $x \in S_n$. Since $\tau(x)$ is underdiagonal, $\rho(x)$ will have no diagonal steps on the main diagonal. Each diagonal step is followed by an east step by construction. □

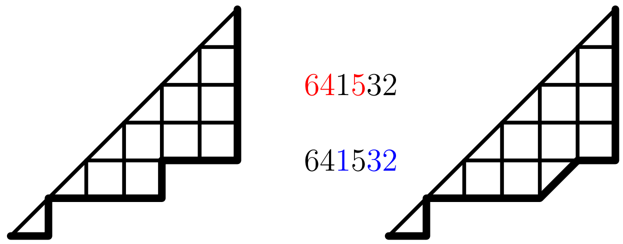


Figure 2.10: $\tau(641532)$ and $\rho(641532)$

Definition 2.5.8. Let $p \in \mathcal{RSP}(n)$. For each fixed instance of NEE or DE in the inner Dyck path, suppose the right E step is labeled with e. For each NEE factor of p, let $e-2 \prec_{NEE} e$ be called a ρ-NEE relation. For each DE factor of p, let $e \prec_{DE} e-2$ be called a ρ-DE relation.

Proposition 2.5.9. *For fixed $p \in \mathcal{RSP}_n$, let $q \in \mathcal{DP}_n$ be its inner Dyck path. Then the transitive closure of the NN and NE relations of q and the ρ-NEE and ρ-DE relations of p is a poset, which we call $P_\rho(p)$.*

Proof. Let $G_\rho(p)$ be the directed graph on $[n]$ with directed edges for each of the relations above. We claim that $G_\rho(p)$ is acyclic. Suppose we have a cycle $y_1 \to y_2 \to \cdots \to y_k \to y_1$ in $G_\rho(p)$. The cycle cannot consist of entirely of NN and NE relations because that would contradict the acyclicity of $G_\tau(q)$. The cycle cannot consist of entirely of NEE relations since the elements would be monotonically decreasing numerically. Similarly, the cycle cannot consist of entirely DE relations.

We note that by Definition 2.5.8, if $a \prec_{NEE} b$, then $a < b$, and if $a \prec_{DE} b$, then $a > b$. Suppose the cycle has at least one DE relation. Without loss of generality, let y_1 be the largest among all such DE relations $y_1 \prec_{DE} y_2$. (or equivalently the label y_1 of the E of the rightmost DE factor of p). We can color each vertex w of the cycle with L or R based on whether w is in the principal order ideal of y_1 in $P_\tau(q)$ or not. Since y_2 is colored R and y_1 is colored L, there exists some edge of the cycle $y_i \to y_{i+1}$ (or possibly the edge $y_k \to y_1$) such that y_i is colored R and y_{i+1} is colored L. We note that $y_i \to y_{i+1}$ cannot come from a NN or NE relation since y_i and y_{i+1} are colored differently. We see that $y_i \to y_{i+1}$ cannot come from a NEE relation because by Proposition 2.4.23, the only possible NEE relation that satisfies the necessary colors would be $y_2 \prec_{NEE} y_1$, which is impossible by the original assumption that $y_1 \prec_{DE} y_2$. Let M_{y_1} be the numerically largest element of the principal order ideal of y_1 in $P_\tau(q)$. We see that $y_i \to y_{i+1}$ cannot come from a DE relation because by Proposition 2.4.23, the only possible DE relation that satisfies the necessary colors would be $M_{y_1} + 2 \prec_{DE} M_{y_1}$. But $M_{y_1} + 2$ contradicts the maximality of y_1 being the largest among all DE relations. Therefore there exist no DE relations in the cycle.

Similarly, we can rule out cycles that have at least one NEE relation using an analogous coloring argument. This rules out any possible cycle, and we conclude the graph $G_\rho(p)$ is acyclic. $\square$

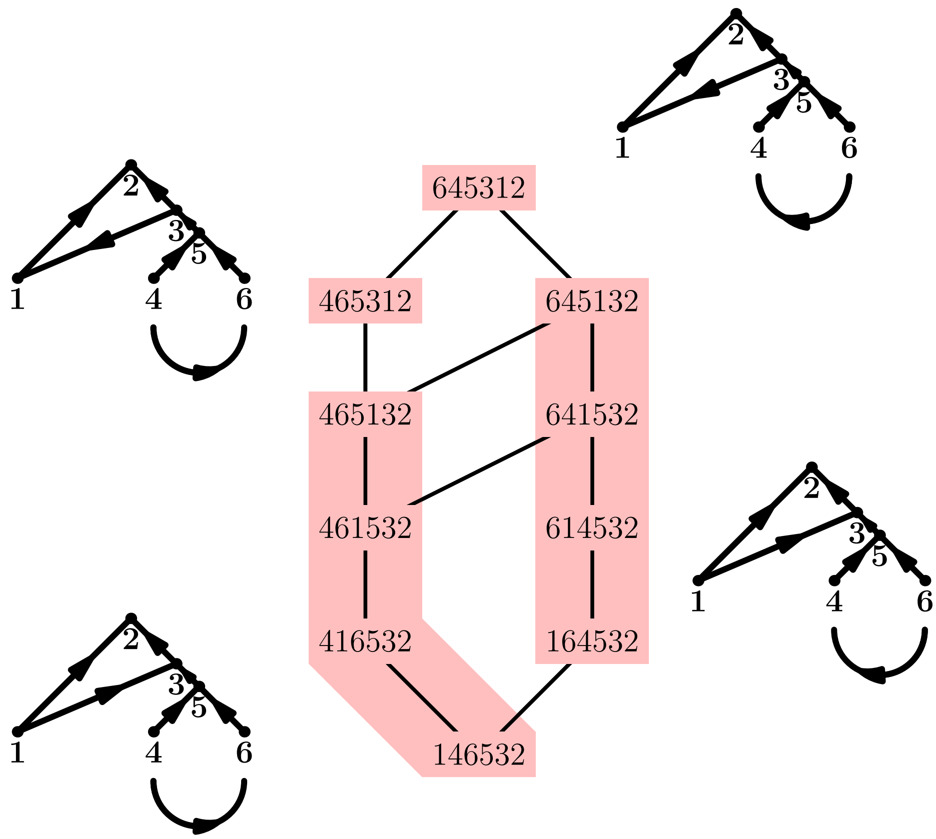


Figure 2.11: $G_\rho(p)$ for the all p that have the inner Dyck path as in the previous figures, and their linear extensions

Theorem 2.5.10. *Let $p \in \mathcal{RSP}(n)$. Let $x \in S_n$ with x viewed as a total order $x_1 \prec \cdots \prec x_n$ on $[n]$. Then x is a linear extension of $P_\rho(p)$ if and only if $\rho(x) = p$.*

Proof. We first prove the forward direction. Let x be a linear extension of $P_\rho(p)$. Let q

be the inner Dyck path of p. Since $P_\rho(p)$ is an extension of $P_\tau(q)$, by Theorem 2.4.21, the inner Dyck path of $\rho(x)$ will be q. If p has a NEE factor centered at column $e+1$, then $e \prec_{NEE} e+2$, so e precedes $e+2$ in x. We know that $e+2$ precedes $e+1$ (since $e+2$ is in the principal order ideal of $e+1$), so we conclude that $(e)(e+2)(e+1)$ is a strong 132 pattern, which implies that $\rho(x)$ also has a NEE factor centered at $e+1$. Similarly, if p has a DE factor centered at column $e+1$, then $e+2 \prec_{DE} e$, so $e+2$ precedes e in x. We know that e precedes $e+1$ through NE and NN relations, so we conclude that $(e+2)(e)(e+1)$ is a strong 312 pattern, which implies that $\rho(x)$ also has a DE factor centered at $e+1$. Since $\rho(x)$ and p have the same inner Dyck path q and the same NEE and DE factors, we conclude that $\rho(x) = p$.

We now prove the backward direction. Suppose that $x \in S_n$ satisfies $\rho(x) = p$. By Theorem 2.4.21, since the inner Dyck path of $\rho(x)$ and p are both q, x will satisfy all NE and NN relations of $P_\rho(p)$. Suppose that $e \prec_{NEE} e+2$ is a NEE relation of $P_\rho(p)$. Then there will be a NEE factor of p centered at $e+1$ by Proposition 2.5.2. But by definition of ρ, $(e)(e+2)(e+1)$ form a strong 132 pattern, so e precedes $e+2$ in x. Suppose that $e+2 \prec_{DE} e$ is a DE relation of $P_\rho(p)$. Then there will be a DE factor of p whose corresponding NEE factor is centered at $e+1$. But by definition of ρ, $(e+2)(e)(e+1)$ form a strong 312 pattern, so $e+2$ precedes e in x. Thus, x satisfies every generating relation for $P_\rho(p)$, so x is a linear extension of $P_\rho(p)$. □

Since every poset has at least one linear extension, we have the following corollary of Theorem 2.5.10.

Proposition 2.5.11. *ρ is surjective.*

Definition 2.5.12. Fix $p \in \mathcal{RSP}_n$. Let q be the inner Dyck path of p. Let $g_i \in [n]$ be called *DE-open* if the east step labeled with g_i is the middle E of a NEE factor of q and there is a triangle in column g_i. Let $g_i \in [n]$ be called *NEE-open* if the east step labeled with g_i is the middle E of a NEE factor of q and there is no triangle in column g_i.

Fix $p \in \mathcal{RSP}_n$. Let q be the inner Dyck path of p. Suppose that the principal order ideal of b in $P_\tau(q)$ is $[m_0, M_0]$. If M_0+1 is DE-open, let $g_1 = M_0+1$, and consider the principal order ideal of M_0+2, which we label as the interval $[m_1, M_1]$ (necessarily, $m_1 = M_0+2$). If m_0-1 is NEE-open, let $g_0 = m_0-1$, and consider the principal order ideal of m_0-2, which we label as the interval $[m_{-1}, M_{-1}]$ (necessarily, $M_{-1} = m_0-2$). We can continue in each direction naming intervals as long as the g_i are DE-open on the

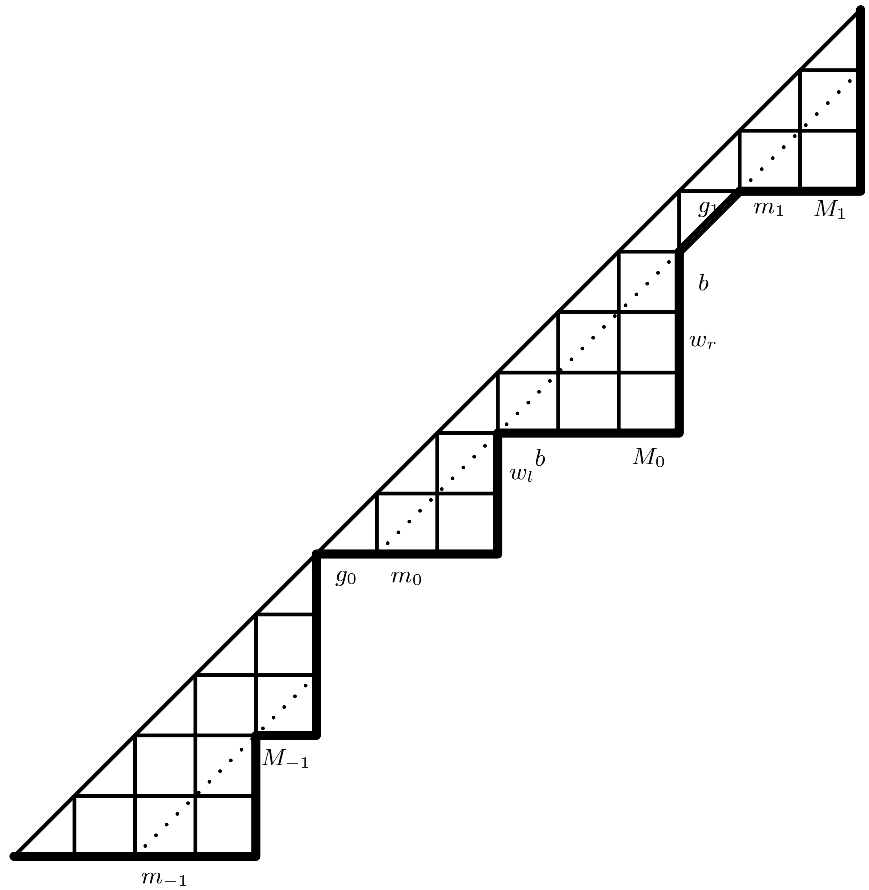


Figure 2.12: A principal order ideal of b in $P_\tau(p)$. Every north and east step on the inner Dyck path q below the dotted lines lies in the ideal.

right and NEE-open on the left. We choose the unique $k \geq 0$ and $-\ell \leq 0$ such that that g_i is DE-open for $1 \leq i \leq k$, $M_k + 1$ is not DE-open, g_i is NEE-open for $-\ell \leq i \leq 0$, and $m_{-\ell} - 1$ is not NEE-open. See Figure 2.12 for an example.

Proposition 2.5.13. *Fix $p \in \mathcal{RSP}_n$. Let q be the inner Dyck path of p. Suppose that the principal order ideal of b in $P_\tau(q)$ is $[m_0, M_0]$. The principal order ideal of b in $P_\rho(p)$ is $[m_{-\ell}, M_{-\ell}] \cup \cdots \cup [m_0, M_0] \cup \cdots \cup [m_k, M_k]$. See Figure 2.12 for an example.*

Proof. Every element e in $[m_0, M_0]$ satisfies $e \lessdot b$ in $P_\rho(p)$ by Proposition 2.4.23. Suppose that e in $[m_i, M_i]$ for $i > 0$. We note that $[m_i, M_i]$ is the principal order ideal of m_i in $P_\tau(q)$ for $i > 0$. We claim that $e \lessdot b$ in $P_\rho(p)$. We prove this by induction on i. For $i = 1$, we see that for any e in $[m_1, M_1]$, $e \leq m_1 \prec_{DE} M_0 \prec b$. For $j > 1$ and e in $[m_j, M_j]$, we see that $e \leq m_j \prec_{DE} M_{j-1} \prec m_{j-1}$. But, m_{j-1} is in an interval $[m_{j-1}, M_{j-1}]$, so by the inductive hypothesis, $m_{j-1} \prec b$, and by transitivity conclude that $e \prec b$. We can use a similar induction argument to show that e in $[m_j, M_j]$ for $-\ell \leq j \leq 0$ using NEE relations and the fact that $[m_i, M_i]$ is the principal order ideal of M_i in $P_\tau(q)$ for $i < 0$.

Suppose that $e \in [n]$ and $e \notin [m_\ell, M_\ell] \cup \cdots \cup [m_0, M_0] \cup \cdots \cup [m_k, M_k] = POI_\rho(b)$. Assume for the sake of contradiction that $e \prec b$ in $P_\rho(q)$. Then there will exist a chain consisting of NN, NE, NEE,and DE relations $e \prec y_1 \prec y_2 \prec \cdots y_n \prec b$. Let i be the largest index such that $y_i \notin POI_\rho(b)$ and $y_j \in POI_\rho(b)$ for all $j > i$. Consider the relation $y_i \prec y_{i+1}$. Suppose that $y_{i+1} \in [m_j, M_j]$.
If $y_i \prec_{NE} y_{i+1}$ or $y_i \prec_{NN} y_{i+1}$, we have a contradiction since $y_i \in [m_j, M_j] \subseteq POI_\rho(b)$ by Proposition 2.4.23.
If $y_i \prec_{DE} y_{i+1}$, we know that $y_i = y_{i+1} - 2$. But $y_i = y_{i+1} - 2$ implies that $y_i \in POI_\rho(b)$ unless $y_i = g_j$. But then m_j would have to be DE-open, which is impossible because g_i and m_i form an EE factor (that is, there is no N to form the NEE factor with middle E step labeled by m_i).

If $y_i \prec_{NEE} y_{i+1}$, we know that $y_i = y_{i+1} + 2$. But $y_i = y_{i+1} + 2$ implies that $y_i \in POI_\rho(b)$ unless $y_i = g_{j+1}$. But then M_j would have to be NEE-open, which is impossible because M_j and g_{j+1} are at different heights. We conclude that no such i can exist, thus for all $e \notin POI_\rho(b)$, e is not in the principal order ideal of b in $P_\rho(p)$. □

Proposition 2.5.14. *Fix $p \in \mathcal{RSP}_n$. Then $P_\rho(p)$ is regularly labeled.*

Proof. Let q be the inner Dyck path of p. Fix $z \in [n]$, and consider the principal order ideal $POI_\rho(z) = [m_{-\ell}, M_\ell] \cup \cdots \cup [m_k, M_k]$ of z in $P_\rho(p)$. Suppose $x \prec z$ in $P_\rho(p)$. Then $x \in [m_j, M_j]$ for some j.

If $x > z$, then $j > 0$. If y satisfies $x > y > z$, and $y \in POI_\rho(z)$, then $y \prec z$ and we are done. The only way that y satisfies $x > y > z$, and $y \notin POI_\rho(z)$ is if $y = g_s$ for $0 < s \leq j$ using the notation of the previous proposition. Since g_s and m_s form an EE factor, m_s is in the principal order ideal of g_s in $P_\tau(q)$, or in other words, $m_s \prec g_s$. We claim that $x \prec g_s$ for all $0 < s \leq j$. To prove this, we will prove the stronger condition $x \prec m_s$ by induction on $j - s$. The base case for $j - s = 1$, follows from Proposition 2.4.23. For $j - s > 1$, since $s > 0$, $[m_s, M_s]$ is the principal order ideal of m_s in $P_\tau(q)$. We note that M_s and m_{s+1} differ by 2 and that the number between them is g_s, which labels the middle E step of a NEE of q. We find that $M_s \prec m_s$. But then $m_{s+1} \prec_{DE} M_s$. By inductive hypothesis,

$x \lessdot m_{s+1}$, and by transitivity, we conclude that $x \lessdot m_s$. This completes the induction and the proof of regular labeling for the case $x > z$.

If $x < z$, then $j < 0$. If y satisfies $x < y < z$, and $y \in POI_\rho(z)$, then $y \lessdot z$ and we are done. The only way that y satisfies $x < y < z$, and $y \notin POI_\rho(z)$ is if $y = g_s$ for $j < s \leq 0$ using the notation of the previous proposition. By examining q, we see that $M_{s-1} \lessdot g_s$ since $M_{s-1} \lessdot_{NN} \cdots \lessdot_{NN} w_{g_s} \lessdot_{NE} g_s$ where w_{g_s} is the label on the north step of the NE factor with east step labeled by g_s. We claim that $x \lessdot g_s$ for all $j < s \leq 0$. To prove this, we will prove the stronger condition $x \lessdot M_{s-1}$ by induction on $s - j$. The base case $s - j = 1$ follows from Proposition 2.4.23. For $s - j > 1$, since $s - j > 0$, $[m_{s-1}, M_{s-1}]$ is the principal order ideal of M_{s-1} in $P_\tau(q)$. We find that $m_{s-1} \lessdot M_{s-1}$. We note that M_{s-2} and m_{s-1} differ by 2 and that the number between them is g_{s-1}, which labels the middle E step of a NEE of q. From this, we conclude that $M_{s-2} \lessdot_{NEE} m_{s-1}$. By inductive hypothesis, $x \lessdot M_{s-2}$, and by transitivity, we conclude that $x \lessdot M_{s-1}$. This completes the induction and the proof of regular labeling for the case $x < z$. □

In light of Theorem 2.2.12, we get the following corollary.

Corollary 2.5.15. *The fibers of ρ are intervals in the weak order.*

See Figure 2.11 for examples of fibers.

Proposition 2.5.16. *Let $x \lessdot y$ in the weak order. Then $\rho(x) = \rho(y)$ if and only if y is obtained from x by one of the following moves*

1. $\underline{14}23$ *to* $\underline{41}23$ *move*
2. $\underline{14}32$ *to* $\underline{41}32$ *move*
3. $2\underline{14}3$ *to* $2\underline{41}3$ *move*
4. $3\underline{14}2$ *to* $3\underline{41}2$ *move .*

Proof. Suppose that $\rho(x) = \rho(y)$. If $x \lessdot y$ in the weak order, $y = x_1 \cdot x_{j+1} x_j \cdot x_n$ for some $1 \leq j \leq n$. Since $\rho(x) = \rho(y)$, we know that $\tau(x) = \tau(y)$, so (x_{j+1}, x_j) is a non-Dyck inversion, so there must exist some witness x_k with $k > j+1$ and $x_j < x_k < x_{j+1}$. But we also know that if $\rho(x) = \rho(y)$, then x_{j+1} and x_j cannot be the 3 and the 1 of a strong 312 pattern. Therefore, $x_{j+1} - x_j \geq 3$. There must therefore exist some x_m with $x_j < x_m < x_{j+1}$.

If $m < j$, then $x_m\underline{x_{j+1}x_j}x_k$ will form a $2\underline{41}3$ or $3\underline{41}2$ pattern in y. If $j+1 < m$, then $x_m\underline{x_{j+1}x_j}x_k$ will form a $\underline{41}23$ or $\underline{41}32$ pattern in y. This proves the forward direction.

Conversely, performing one of the four moves creates exactly one inversion and it is not a Dyck inversion since they are all special cases of $\underline{13}2 \to \underline{31}2$ moves. Since the move does not destroy Dyck inversions, $\tau(x) = \tau(y)$. Since the 1 and the 4 differ by more than 2, we do not create nor destroy strong 132 patterns nor strong 312 patterns. Therefore, $\rho(x) = \rho(y)$ as desired. □

By applying transitivity to the cover relations within a fiber, we get the following corollary.

Proposition 2.5.17. *ρ is constant on classes of $\Theta_{2\underline{14}3,3\underline{14}2,\underline{14}23,\underline{14}32}$.*

Proposition 2.5.18. *The congruence classes of $\Theta_{2\underline{14}3,3\underline{14}2,\underline{14}23,\underline{14}32}$ correspond with the fibers of ρ.*

Proof. By the previous proposition, the fibers are unions of congruence classes. By Theorem 2.3.2, Theorem 2.5.11, and Proposition 2.3.7, the number of classes is equal to the number of fibers, thus we conclude that the classes and fibers coincide. □

Corollary 2.5.19. *A permutation is a bottom element of a fiber of ρ if and only if it is an element of $S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)$. A permutation is a top element of fiber of ρ if and only if it is an element of $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$.*

Proof. This is an immediate consequence of Corollary 2.5.15, Proposition 2.5.16, and Proposition 2.5.18. □

The previous propositions give us the desired bijection.

Corollary 2.5.20. *The restriction of ρ to the bottom elements of its fibers is a bijection between the elements of $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $\mathcal{RSP}_n$. The restriction of ρ to the top elements of its fibers is a bijection between the elements of $S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)$ and $\mathcal{RSP}_n$.*

2.6 Bijective Combinatorics of Inversion Sequences

We now mimic the strategy used in the previous section to construct a bijection between a different family of pattern avoiding permutations and $\mathcal{RSP}_n$. We first provide the background necessary to define a map $\omega : S_n \to \mathcal{RSP}_n$. We then prove that the fibers of ω are the linear extensions of regularly labeled posets. We then show that the fibers of ω are the equivalence classes of the family of pattern avoiding permutations. Finally, we show that the reverse Lehmer codes of these permutations are precisely the pattern avoiding inversion sequences we seek to count.

Definition 2.6.1. Fix $x \in S_n$ and a with $1 < a < n$. We define a to be *NEE-positive* (abbreviated NEE^{+}) if $a-1$ and $a+1$ both precede a in x. We define a to be *atomic* if $a-1$ and $a+1$ both follow a in x. Additionally, 1 is atomic if 2 follows 1 in x, and n is atomic if $n-1$ follows n in x. We will refer to atomic labels as *atoms*. We say that a is *increasing* if $a-1$ precedes a and $a+1$ follows a in x. We say that a is *decreasing* if $a-1$ follows a and $a+1$ precedes a in x. Additionally, n is increasing if $n-1$ precedes n in x and 1 is decreasing if 2 precedes 1 in x. See Figure 2.13 for an example.

Definition 2.6.2. Fix $q \in \mathcal{D}_n$, and let a with $1 < a < n$ be the label of a fixed east step. We define a to be *NEE-positive* if the east step labeled with a is the middle E step of a NEE factor of q. We define a to be *atomic* if the east step labeled with a is the middle E step of a EEN factor of q. Additionally, 1 is atomic if the east step labeled 1 is followed by a north step and n is atomic if the east step labeled n is preceded by an east step. We will refer to atomic edge labels as *atoms*. We say that a is *increasing* if the east step labeled with a is the east step of a NEN factor of q. We say that a is *decreasing* if the east step labeled with a is the east step of a EEE factor of q. Additionally, n is increasing if n labels the E of a NE factor and 1 is decreasing if it is the left E step of an EE factor. See Figure 2.13 for an example.

By applying Lemma 2.4.10 to the pairs of adjacent entries $(a-1, a)$ and $(a, a+1)$ for $1 < a < n$, we get the following proposition. See Figure 2.13 for an example.

Proposition 2.6.3. *Suppose $x \in S_n$ and $q \in \mathcal{DP}_n$ satisfy $\tau(x) = q$. Then a is NEE-positive, atomic, increasing, or decreasing in x if and only if the east step labeled with a in $\tau(x)$ is NEE-positive, atomic, increasing, or decreasing.*

Proposition 2.6.4. *Fix $x \in S_n$. Let b_i and b_{i+1} be atomic such that $b_i < b_{i+1}$ and there is no atomic b_j with $b_i < b_j < b_{i+1}$. Then there exists exactly one element a_{i+1} between b_i and b_{i+1} that is NEE-positive.*

641532

NEE^+ 1 3**2**
64 **5**

atomic **1** 2
4 53
6 5

decreasing 4 **3**2

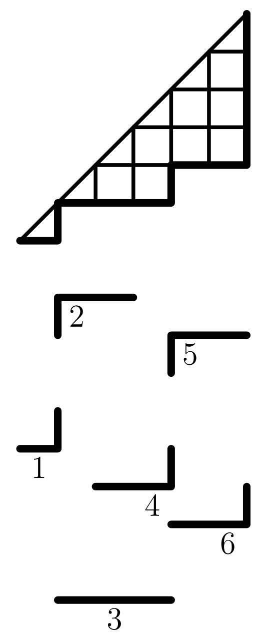


Figure 2.13: The NEE^+, atomic, and decreasing elements of 641532 and $\tau(641532)$

Proof. By the previous proposition, it suffices to look at the east steps of $\tau(x)$ labeled with b_i and b_{i+1}. There is at least one NEE^+ element between b_i and b_{i+1}, namely the label of the leftmost east step at the same height as b_{i+1}. If there were two NEE^+ elements $a_k < a_m$, then the label of the rightmost east step at the same height as a_k would be atomic, contracting the fact that there are no atoms between b_i and $b_i + 1$. $\square$

Corollary 2.6.5. *Let $x \in S_n$ be fixed. If x has k NEE-positive instances, then it has $k+1$ atoms. Moreover, if we arrange the NEE-positive and atoms in increasing order as a tuple, they will alternate.*

Proof. We note that the label of the rightmost east step at the same height as 1 will be the leftmost atom, so there are no NEE^+ elements to the left of the leftmost atom. There are also no NEE^+ elements to the right of the rightmost atom, because then the rightmost east step at the same height of the E steps of the NEE instance would be an atom, contradicting the claim that the original atom is rightmost. $\square$

Definition 2.6.6. Fix $x \in S_n$. Let $(b_0, a_1, b_1, a_2 \dots, b_{k-1}, a_k, b_k)$ be its tuple as in the previous corollary. We recursively define the *leader atom function* $\gamma_x : \{b_0, b_1, \dots, b_{k-1}\} \to \{b_1, \dots, b_k\}$ as follows. Our base case is $\gamma_x(b_{k-1}) = b_k$. For $0 \leq m < k-1$, $\gamma_x(b_m) =$

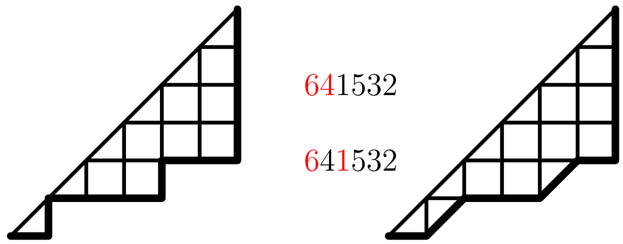


Figure 2.14: $\tau(641532)$ and $\omega(641532)$, note that $(b_0, a_1, b_1, a_2, b_3) = (1, 2, 4, 5, 6)$, $\gamma(4) = 6$, $\gamma(1) = 6$

left$(b_{m+1}, \gamma_x(b_{m+1}))$. If $\gamma_x(b_i) = b_j$, we refer to b_j as the *leader atom* of b_i. If x is clear from the context, we will omit the x subscript.

Definition 2.6.7. Fix $p \in \mathcal{RSP}_n$ with inner Dyck path $q \in \mathcal{DP}_n$. Suppose q has k NEE positive elements, $a_1, \cdots, a_k$ (listed in increasing order). Then q has $k+1$ atoms, $b_0, \cdots, b_k$ with $b_{i-1} < a_i < b_i$ for $1 \le i \le k$. We recursively define the *leader atom function* γ_p : $\{b_0, b_1, \ldots, b_{k-1}\} \to \{b_1, \ldots, b_k\}$ as follows. Our base case is $\gamma_p(b_{k-1}) = b_k$. For $0 \le m < k-1$, $\gamma_p(b_m) = b_{m+1}$ if there is no triangle in column a_{m+1} of p and $\gamma_p(b_m) = \gamma_p(b_{m+1})$ if there is a triangle in column a_{m+1} of p. If $\gamma_p(b_i) = b_j$, we refer to b_j as the *leader atom* of b_i. If p is clear from the context, we will omit the p subscript.

Definition 2.6.8. Let $\omega : S_n \to \mathcal{RSP}_n$ be the following map. Let $x \in S_n$. Start with the Dyck path $\tau(x)$. For $0 \le j \le k-1$ (with k as defined in Definition 2.6.7), if left$(b_j, \gamma_x(b_j)) = \gamma_x(b_j)$, transform the NEE factor centered at a_{j+1} to a DE factor. See Figure 2.14 for an example.

From the definition of ω, we immediately get the following lemma.

Lemma 2.6.9. *Suppose $x \in S_n$ and $p \in \mathcal{RSP}_n$ satisfy $\omega(x) = p$. Then* left$(b_m, \gamma_x(b_m)) = b_m$ *if and only if there is no triangle in column a_{m+1}. Similarly,* left$(b_m, \gamma_x(b_m)) = \gamma_x(b_m)$ *if and only if there is a triangle in column a_{m+1}.*

Proposition 2.6.10. *Suppose $x \in S_n$ and $p \in \mathcal{RSP}_n$ satisfy $\omega(x) = p$. For atoms b_i and b_j, $\gamma_x(b_i) = b_j$ if and only if $\gamma_p(b_i) = b_j$.*

Proof. Suppose x and p have $k+1$ atoms. We prove the proposition by induction on $k+1-m$. The base case for $m = 1$ is clear by the base cases for the recursive definitions of the leader atom functions. The inductive step follows from Lemma 2.6.9 and the leader atom function definitions. $\square$

Definition 2.6.11. Let $p \in \mathcal{RSP}_n$. Suppose there is a NEE factor of q centered at a_{j+1}. If there is also a NEE factor of p centered at a_{j+1}, let $b_j \prec_{NEE} \gamma(b_j)$ be called a ω-NEE *relation*. Otherwise, if there is a DE factor of p centered at a_{j+1}, let $\gamma(b_j) \prec_{DE} b_j$ be called a ω-DE relation.

Theorem 2.6.12. *For fixed $p \in \mathcal{RSP}_n$, let $q \in \mathcal{DP}_n$ be its inner Dyck path. Then the transitive closure of the NN and NE relations of q and the ω-NEE and ω-DE relations of p is a poset, which we call $P_\omega(p)$.*

Proof. We note it is not hard to see that an element of $P_\tau(q)$ is minimal if and only if it is an atom. Let $G_\omega(p)$ be the directed graph on $[n]$ with directed edges for each of the relations above. We claim that $G_\omega(p)$ is acyclic. Suppose we have a cycle $y_1 \to y_2 \to \ldots \to y_k \to y_1$ in $G_\omega(p)$. The cycle cannot consist of entirely of NN and NE relations because that would contradict the acyclicity of $G_\tau(q)$. The cycle cannot consist entirely of ω-NEE and ω-DE relations, because the subgraph of those relations form a tree (there are $k+1$ atoms/vertices, we add k relations/edges, and every atom is connected to the root $\text{right}(b_k, b_{k-1})$).
Thus, without loss of generality, $y_1 \prec_{NEE} y_2$ or $y_1 \prec_{DE} y_2$ and neither $y_k \nprec_{NEE} y_1$ nor $y_k \nprec_{DE} y_1$. However y_1 is an atom, meaning $y_k \nprec_{NN} y_1$ and $y_k \nprec_{NE} y_1$. Since there is no possible relation that gives rise to the edge $y_k \to y_1$, no such cycle can exist. This rules out any possible cycle, and we conclude the graph $G_\omega(p)$ is acyclic. □

Theorem 2.6.13. *Let $p \in \mathcal{RSP}_n$. Let $x \in S_n$ with x viewed as a total order $x_1 \prec \cdots \prec x_n$ on $[n]$. Then x is a linear extension of $P_\omega(p)$ if and only if $\omega(x) = p$.*

Proof. We first prove the forward direction. Let x be a linear extension of $P_\omega(p)$. Let q be the inner Dyck path of p. By Theorem 2.4.21, the inner Dyck path of $\omega(x)$ will be q. If p has a NEE factor centered at column a_{j+1}, then $b_j \prec_{NEE} \gamma_p(b_j)$. Since x is a linear extension of $P_\omega(p)$, b_j precedes $\gamma_x(b_j)$ in x. But by definition of ω, this implies that $\omega(x)$ also has a NEE factor centered at a_{j+1}. Similarly, if p has a DE factor centered at column a_{j+1}, then $\gamma_p(b_j) \prec_{DE} b_j$. Since x is a linear extension of $P_\omega(p)$, $\gamma_x(b_j)$ precedes b_j in x. By definition of ω, this implies that $\omega(x)$ also has a DE factor centered at a_{j+1}. Since $\omega(x)$ and p have the same inner Dyck path q and the same NEE and DE factors, we conclude that $\omega(x) = p$.

We now prove the backward direction. Suppose that $x \in S_n$ satisfies $\omega(x) = p$. By

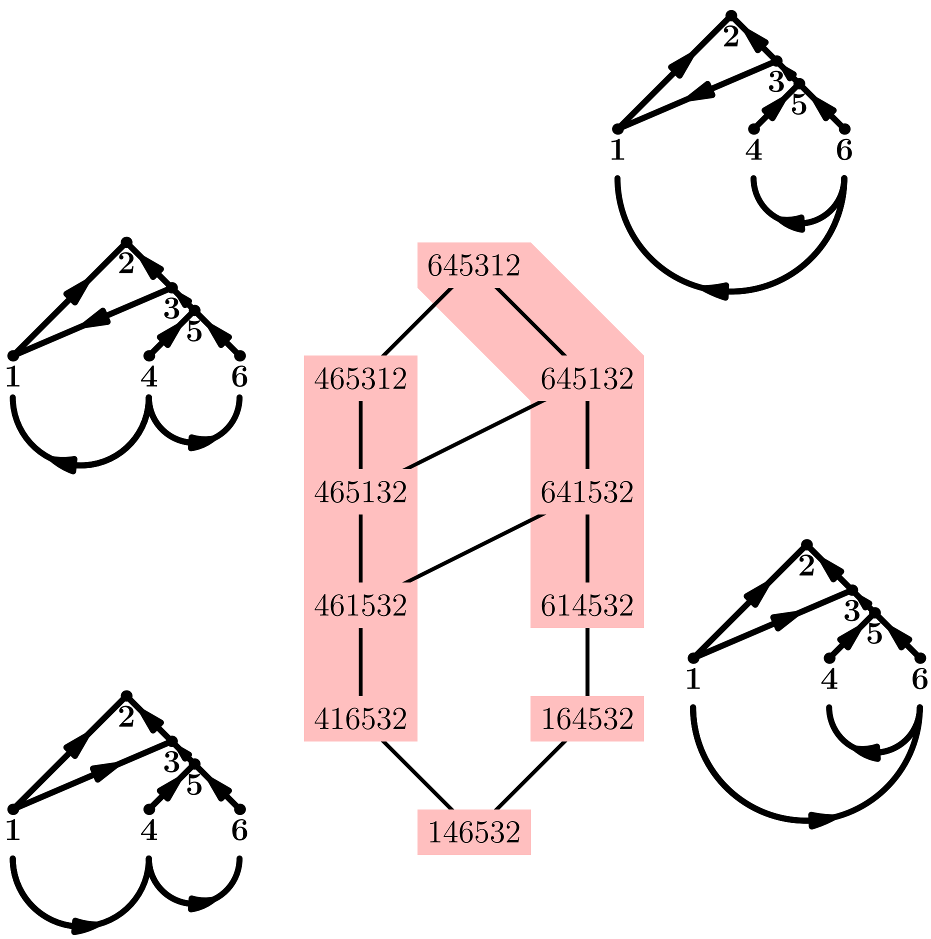


Figure 2.15: $G_\omega(p)$ for the all p that have the inner Dyck path as in the previous figures, and their linear extensions

Theorem 2.4.21, since inner Dyck path of $\omega(x)$ and p are both q, x will satisfy all NE and NN relations of $P_\omega(p)$. Suppose that $b_j \lessdot_{NEE} \gamma_p(b_j)$ is a NEE relation of $P_\omega(p)$. Then there will be a NEE factor of p centered at a_{j+1}. By definition of ω, $\text{left}(b_j, \gamma(b_j)) = b_j$ (Otherwise, there would be a DE factor centered at a_{j+1}). In other words, b_j precedes $\gamma(b_j)$ in x. Suppose that $\gamma(b_j) \lessdot_{DE} b_j$ is a DE relation of $P_\omega(p)$. Then there will be a DE factor of p whose corresponding NEE factor of q is centered at a_{j+1}. But by definition of ω, $\text{left}(b_j, \gamma(b_j)) = \gamma(b_j)$, so $\gamma(b_j)$ precedes b_j in x. Thus, x satisfies every generating relation for $P_\omega(p)$, so x is a linear extension of $P_\omega(p)$. □

Since every poset has at least one linear extension, we have the following corollary of Theorem 2.6.13.

Proposition 2.6.14. *ω is surjective.*

We seek to understand the fibers of ω. We first prove the following lemmas that will help us do this.

Lemma 2.6.15. *Let $x \in S_n$. If $\gamma(b_i) = b_j$, then $\gamma(b_u) = b_j$ for all $i < u < j$.*

Proof. If $j = i + 1$, the statement is vacuously true. Assume by way of contradiction there exists a minimal index v such that $i < v < j$, $\gamma(b_v) \neq b_j$ and $\gamma(b_{v-1}) = b_j$. By definition of γ, $\gamma(b_{v-1}) = \text{left}(b_v, \gamma(b_v))$. In particular, $\gamma(b_{v-1})$ is either b_v or $\gamma(b_v)$. But since $v < j$, $b_v \neq b_j$, and by definition, $\gamma(b_v) \neq b_j$. Thus, we have a contradiction. We conclude that no such index can exist and that $\gamma(b_u) = b_j$ for all $i < u < j$. □

Lemma 2.6.16. *Let $x \in S_n$. If $\gamma(b_i) = b_j$, then $\text{left}(b_u, \gamma(b_u)) = \gamma(b_u)$ for all $i < u < j$.*

Proof. If $j = i + 1$, the statement is vacuously true. Assume by way of contradiction there exists an index v such that $i < v < j$ and $\text{left}(b_v, \gamma(b_v)) = b_v$. But by the previous lemma, $\text{left}(b_v, \gamma(b_v)) = \gamma(b_{v-1}) = b_j$, which is a contradiction since $v < j$. □

Definition 2.6.17. Let $p \in \mathcal{RSP}_n$. For $j > 0$, let b_j be called a *range atom* if $\gamma_p(b_i) = b_j$ for some atom b_i.

Lemma 2.6.18. *Let $x \in S_n$. For $k > j > 0$, b_j is a range atom if and only if $\text{left}(b_j, \gamma(b_j)) = b_j$. If $k > 0$, then b_k is also a range atom.*

Proof. The backward direction is clear because by definition of γ, $\gamma(b_{j-1}) = \text{left}(b_j, \gamma(b_j)) = b_j$. The forward direction follows from Lemma 2.6.15. □

Lemma 2.6.19. *Let $x \in S_n$. If b_i is a range atom, then b_i precedes b_w in the one-line notation of x for all atoms b_w such that $w > i$.*

Proof. Consider the ith largest range atom. We prove this by induction on $k - i$. For the base case of $k - i = 0$, the statement is vacuously true for the largest range atom b_k. The inductive step follows from Lemma 2.6.16, Lemma 2.6.18, and transitivity applied to all atoms whose leader atom is the $(k - i)$th largest leader atom. □

Lemma 2.6.20. *The indices in this statement refer to the one-line notation of x as opposed to the indices on the set of atoms of x. Let $x \in S_n$ be fixed with $x = x_1 \dots x_j x_{j+1} \dots x_n$ where $x_j x_{j+1}$ forms an ascent. If x_j and x_{j+1} are both atomic, and $\gamma(x_j)$ is to the right of x_j, then $x_{j+1} = \gamma(x_j)$.*

Proof. Assume for the sake of contradiction that $x_{j+1} \neq \gamma(x_j)$. If $x_j < x_{j+1} < \gamma(x_j)$, then by Lemma 2.6.15, $\gamma(x_{j+1}) = \gamma(x_j)$. By assumption, x_j precedes $\gamma(x_j)$ and by Lemma 2.6.16, $\gamma(x_j)$ precedes x_{j+1}. If $x_j < \gamma(x_j) < x_{j+1}$, then by Lemma 2.6.19, $\gamma(x_j)$ precedes x_{j+1}. But x_j precedes $\gamma(x_j)$. Thus, in either case, we have that $\gamma(x_j)$ lies between x_j and x_{j+1}, contradicting the fact that they are adjacent. Thus, the only remaining possibility is that $x_{j+1} = \gamma(x_j)$. □

Lemma 2.6.21. *The indices in this statement refer to the one-line notation of x. Let $x \in S_n$ be fixed with $x = x_1 \dots x_j x_{j+1} \dots x_n$ where $x_j x_{j+1}$ forms an ascent. If x_j and x_{j+1} are both atomic, and $\gamma(x_j)$ is to the right of x_j, then $x_i < x_j$ for all $1 \leq i < j$.*

Proof. Assume for the sake of contradiction that there exists an i such that $x_i > x_j$. Without loss of generality, let x_i be the smallest of such witnesses. Suppose x_i is an atom. By Lemma 2.6.18, x_j is a range atom. By Lemma 2.6.19, x_j precedes x_i. But this contradicts x_i preceding x_j.

Suppose that x_i is not an atom. If x_i were decreasing or NEE^+, we can consider the principal order ideal of x_i in $P_\tau(\tau(x))$. By Proposition 2.4.23, there would exist an atom (the label on the rightmost east step at the same height as x_i) that precedes it that is numerically bigger. But the previous paragraph shows that such an atom cannot exist. Thus x_i must be increasing. This implies that $x_i - 1$ is to the left of x_i. But then, $x_i - 1$ is an element bigger than x_j and preceding x_j contradicting the minimality of the choice of x_i. Thus no such x_i can exist. This completes the proof of the lemma. □

Definition 2.6.22. Fix $x \in S_n$. Suppose that $x_i\underline{x_jx_k}x_m$ is an instance of a $1\underline{24}3$ pattern or $x_i\underline{x_kx_j}x_m$ is an instance of a $1\underline{42}3$ pattern. We call such a pattern *left inversion dense* or *LID* if

$$|\{x_u \mid i < u, x_i > x_u\}| \geq |\{x_u \mid j < u, x_j > x_u\}|.$$

Definition 2.6.23. Fix $x \in S_n$. Suppose that $x_i\underline{x_jx_k}x_m$ is an instance of a LID $1\underline{24}3$ pattern or $x_i\underline{x_kx_j}x_m$ is an instance of a LID $1\underline{42}3$ pattern. We say that such a pattern is *leader atom neutral* or *LAN* if either

- at least one of x_j and x_k is not atomic
- both x_j and x_k are atomic and $\gamma(x_j) \neq x_k$

Proposition 2.6.24. *Let $x \lessdot y$ in the weak order. Then $\omega(x) = \omega(y)$ if and only if y is obtained from x by one of the following moves.*

- *A $2\underline{14}3$ to $2\underline{41}3$ move*
- *A $3\underline{14}2$ to $3\underline{41}2$ move*
- *A $4\underline{13}2$ to $4\underline{31}2$ move*
- *A $1\underline{24}3$ to $1\underline{42}3$ move where $1\underline{24}3$ is LAN in x and $1\underline{42}3$ is LAN in y*

Proof. Let $x = x_1 \dots x_jx_{j+1} \dots x_n$ and $y = x_1 \dots x_{j+1}x_j \dots x_n$.

Suppose that $\omega(x) = \omega(y)$. In particular this implies that $\tau(x) = \tau(y)$, so we know that there will always exist some x_k with $k > j + 1$ and $x_j < x_k < x_{j+1}$. There are sixteen cases for what the ascent x_jx_{j+1} looks like based on the four possibilities in Definition 2.6.1 applied to x_j and x_{j+1}. See Figure 2.16 for more details. If x_j is NEE^+ or decreasing, then $(x_j + 1)\underline{x_jx_{j+1}}x_k$ forms a $2\underline{14}3$ pattern. If x_{j+1} is NEE^+ or increasing, then $(x_{j+1} - 1)\underline{x_jx_{j+1}}x_k$ forms a $3\underline{14}2$ pattern. If x_{j+1} is decreasing, then $(x_{j+1} + 1)\underline{x_jx_{j+1}}x_k$ forms a $4\underline{13}2$ pattern. If x_j is increasing, then $(x_j-1)\underline{x_jx_{j+1}}x_k$ forms a $1\underline{24}3$ pattern. Let $x_i = x_j-1$, and consider the sets $\{x_u \mid i < u, x_i > x_u\}$ and $\{x_u \mid j < u, x_j > x_u\}$. Since x_i and x_j differ by exactly 1, $\{x_u \mid i < u, x_i > x_u\} \supseteq \{x_u \mid j < u, x_j > x_u\}$. Therefore, $(x_j - 1)\underline{x_jx_{j+1}}x_k$ forms a $1\underline{24}3_{LID}$ pattern. Since x_j is not atomic, $(x_j-1)\underline{x_jx_{j+1}}x_k$ forms a $1\underline{24}3_{LAN}$ pattern. This shows that if at least one one of x_j and x_{j+1} is not atomic, then we can find a witness to the left of x_j that forms one of the four desired moves. Suppose that x_j and x_{j+1} are both atomic. If $\gamma(x_j)$ is to the left of x_j, since $\gamma(x_j) > b_j$, we find that $\gamma(x_j)\underline{x_jx_{j+1}}x_k$ will

	x_{j+1}			
x_j	atomic	NEE^+	increasing	decreasing
atomic	vincular	$3\underline{14}2$	$3\underline{14}2$	$4\underline{13}2$
NEE^+	$2\underline{14}3$	$2\underline{14}3$	$2\underline{14}3$	$2\underline{14}3$
increasing	$1\underline{24}3_{LAN}$	$3\underline{14}2$	$3\underline{14}2$	$4\underline{13}2$
decreasing	$2\underline{14}3$	$2\underline{14}3$	$2\underline{14}3$	$2\underline{14}3$

Figure 2.16: A list describing at least one witness type in all sixteen cases

either form a $2\underline{14}3$, $3\underline{14}2$, or a $4\underline{13}2$. Assume for the sake of contradiction, that $\gamma(x_j)$ is to the right of x_j. By Lemma 2.6.20, $x_{j+1} = \gamma(x_j)$. But then, if $x_j = b_w$, $\omega(x)$ will have no triangle in column a_{w+1} and $\omega(y)$ will have a triangle in column a_{w+1}, contradicting $\omega(x) = \omega(y)$. This completes the proof of the forward direction in all possible cases.

Conversely, suppose y is obtained from x by one of the above four moves. Clearly all moves preserve Dyck inversions, so $\tau(x) = \tau(y)$. Suppose x and y differ by a $1\underline{24}3_{LAN}$ to $1\underline{42}3_{LAN}$ move. Since such a move is LAN, if x_i and x_{i+1} are both atomic, $\text{left}(x_i, \gamma(x_i))$ and $\text{left}(x_{i+1}, \gamma(x_{i+1}))$ do not change when y is obtained from x. We conclude that $\omega(x) = \omega(y)$. If x and y differ by a $2\underline{14}3$ to $2\underline{41}3$ move or a $3\underline{14}2$ to $3\underline{41}2$ move, then the 4 cannot be the leader atom of the 1 since the left witness cannot exist by Lemma 2.6.21. Similarly, if x and y differ by a $4\underline{13}2$ to $4\underline{31}2$ move, the 3 cannot be the leader atom of the 1 since the left witness (the 4) cannot exist by Lemma 2.6.21. Thus all moves preserve $\text{left}(b_j, \gamma(b_j))$ for $0 \leq j \leq k$, i.e. $\omega(x) = \omega(y)$. This completes the proof of the backwards direction. □

Proposition 2.6.25. *Fix $p \in \mathcal{RSP}_n$. Let q be the inner Dyck path of p. Suppose that the principal order ideal of z in $P_\tau(q)$ is $[m_0, M_0]$. If $[m_0, M_0]$ contains at least one range atom, then let A be the set of all range atoms numerically less than m_0 and let A be empty if $[m_0, M_0]$ contains no range atoms. Let b_w be the (numerically) largest atom in $[m_0, M_0]$. Let B be $\{\gamma_p(b_w)\}$ if b_w is not a range atom and let B be empty if b_w is a range atom. The principal order ideal of z in $P_\omega(p)$ is $A \cup [m_0, M_0] \cup B$.*

Proof. Every element e in $[m_0, M_0]$ satisfies $e \prec z$ in $P_\omega(p)$ by Proposition 2.4.23. If b_w is not a range atom, then by Lemma 2.6.18, $\text{left}(b_w, \gamma(b_w)) = \gamma(b_w)$. By Lemma 2.6.9, there is a triangle in column a_{w+1}. By Definition 2.6.11, we conclude that $\gamma_p(b_w) \prec_{DE} b_w$. In other words, every element of B is in the principal order ideal of z in $P_\omega(p)$.

Suppose that $[m_0, M_0]$ contains at least one range atom. Let b_{u_0} be the smallest such range atom. Let A consist of the atoms $b_{u_0} > b_{u_1} > \cdots > b_{u_{|A|-1}}$. By Lemma 2.6.19, $b_{u_i} \prec b_{u_0}$ for all $0 < i \leq |A| - 1$. In other words, every element of A is in the principal order ideal of z in $P_\omega(p)$.

Suppose that $e \in [n]$ and $e \notin A \cup [m_0, M_0] \cup B$. Assume for the sake of contradiction that $e \prec z$ in $P_\omega(p)$. Then there will exist a chain consisting of NN, NE, NEE, and DE relations $e \prec y_1 \prec y_2 \prec \cdots y_n \prec z$. Let i be the largest index such that $y_i \notin A \cup [m_0, M_0] \cup B$. If $y_i \prec_{NE} y_{i+1}$ or $y_i \prec_{NN} y_{i+1}$, we have a contradiction since $y_i \prec z$ by Proposition 2.4.23. Therefore, y_i and y_{i+1} have to be atoms.

If $y_i \prec_{DE} y_{i+1}$, we know that $\gamma_p(y_{i+1}) = y_i$ and $y_i > y_{i+1}$. By Lemma 2.6.9 and Lemma 2.6.18, y_{i+1} is not a range atom and y_i is a range atom. Since A and B contain only range atoms, y_{i+1} must lie in $[m_0, M_0]$. We note that $y_{i+1} \leq b_w$. If $y_i > b_w$, then by Lemma 2.6.15, $\gamma_p(b_w) = y_i$. But then $y_i = \gamma_p(b_w) \in B$, which contradicts the assumption that $y_i \notin A \cup [m_0, M_0] \cup B$. If $y_i \leq b_w$, then $y_i \in [m_0, M_0]$, which contradicts the assumption that $y_i \notin A \cup [m_0, M_0] \cup B$.

If $y_i \prec_{NEE} y_{i+1}$, we know that $\gamma_p(y_i) = y_{i+1}$ and $y_i < y_{i+1}$. By Lemma 2.6.9 and Lemma 2.6.18, y_i is a range atom and y_{i+1} is a range atom. If y_{i+1} is in A or $[m_0, M_0]$, we have a contradiction because then y_i is in A or $[m_0, M_0]$. If y_{i+1} is in B, we have a contradiction because then y_i is in $[m_0, M_0]$.

We conclude that no such i can exist, thus for all $e \notin A \cup [m_0, M_0] \cup B$, e is not in the principal order ideal of z in $P_\rho(p)$. □

Theorem 2.6.26. *Fix $p \in \mathcal{RSP}_n$. Then $P_\omega(p)$ is regularly labeled.*

Proof. Suppose that $x \prec z$ in $P_\omega(p)$. In other words (by Proposition 2.6.25), suppose $x \in A \cup [m_0, M_0] \cup B$.

If $x \in [m_0, M_0]$ then any y numerically between x and z will satisfy $y \prec z$ by Remark 2.4.25.

Suppose B is nonempty, and let b_w be the largest atom in $[m_0, M_0]$, necessarily a non-range atom. If $x \in B$, then $x > z$. Suppose y satisfies $x > y > z$. If y is atomic, then by Lemma 2.6.15, $\gamma_p(y) = x$. By Lemma 2.6.16, $x \prec_{DE} y$. If y is decreasing or NEE^+, then the principal order ideal of y in $P_\tau(q)$ contains at least one atom numerically bigger than y. If this atom happens to be x, we are done since $x \prec y$. Otherwise, call this atom b_y. But then, $b_y \prec y$, and by Lemma 2.6.15 and Lemma 2.6.16, $x \prec_{DE} b_y$, so we conclude that $x \prec y$ by transitivity. If y is increasing, then the principal order ideal of y in $P_\tau(q)$

contains at least one atom numerically smaller than y. If this atom happens to be b_w, we are done because $x \lessdot_{DE} b_w \lessdot y$. Otherwise, call this atom b_y. But then, $b_y \lessdot y$, and by Lemma 2.6.15 and Lemma 2.6.16, $x \lessdot_{DE} b_y$, so we conclude that $x \lessdot y$. Thus, in all cases, y will satisfy $x \lessdot y$.

Suppose A is nonempty. If $x \in A$, then $x < z$ and x is a range atom. Suppose y satisfies $x < y < z$. If y is a range atom, then we are done because then $y \in A$ and therefore $y \lessdot z$ by Proposition 2.6.25. If y is a non-range atom, then by Lemma 2.6.15, its leader atom will be a range atom numerically bigger than x and in A. By Lemma 2.6.19, we see that $x \lessdot y$. If y is decreasing or NEE^+, then the principal order ideal of y in $P_\tau(q)$ contains at least one atom numerically bigger than y. Either this atom or its leader atom is a range atom bigger than x. By Lemma 2.6.19 and transitivity, we find that $x \lessdot y$. If y is increasing, then the principal order ideal of y in $P_\tau(q)$ contains at least one atom numerically smaller than y. If the smallest such atom happens to be x, we are done. Otherwise, call this atom b_y. But then, $b_y \lessdot y$, and by Lemma 2.6.19, $x \lessdot b_y$, so we conclude that $x \lessdot y$. Thus, in all cases, y will satisfy $x \lessdot y$. $\square$

In light of Theorem 2.2.12, we get the following corollary.

Corollary 2.6.27. *The fibers of ω are intervals in the weak order.*

See Figure 2.15 for an example of fibers. By applying transitivity to the cover relations within a fiber, we get the following corollary.

Proposition 2.6.28. *ω is constant on equivalence classes of $\Theta_{2\underline{14}3,3\underline{14}2,4\underline{13}2,1\underline{24}3_{LAN}}$.*

Proposition 2.6.29. *The equivalence classes of $\Theta_{2\underline{14}3,3\underline{14}2,4\underline{13}2,1\underline{24}3_{LAN}}$ correspond with the fibers of ω.*

Proof. By the previous proposition, the fibers are unions of equivalence classes. By Theorem 2.3.2, Theorem 2.6.14, and Proposition 2.3.14, the number of classes is equal to the number of fibers, thus we conclude that the classes and fibers coincide. $\square$

Corollary 2.6.30. *Fix a fiber of ω. A permutation is the top element of a fiber of ω if and only if it is an element of $S_n(2\underline{14}3, 3\underline{14}2, 4\underline{13}2, 1\underline{24}3_{LAN})$ patterns. A permutation is the bottom element of a fiber of ω if and only if it is an element of $S_n(2\underline{41}3, 3\underline{41}2, 4\underline{31}2, 1\underline{42}3_{LAN})$ patterns.*

Proof. This is an immediate consequence of Corollary 2.6.27, Proposition 2.6.24, and Proposition 2.6.29. □

The previous propositions give us a bijection.

Corollary 2.6.31. *The restriction of ω to the top elements of its fibers is a bijection between the elements of $S_n(4\underline{13}2, 3\underline{14}2, 2\underline{14}3, 1\underline{24}3_{LAN})$ and $\mathcal{RSP}_n$. The restriction of ω to the bottom elements of its fibers is a bijection between the elements of $S_n(4\underline{31}2, 3\underline{41}2, 2\underline{41}3, 1\underline{42}3_{LAN})$ and $\mathcal{RSP}_n$.*

Now we show that this bijection can be used to construct an explicit bijection between $I_n(e_i > e_j \leq e_k)$ and $\mathcal{RSP}_n$.

Definition 2.6.32. Let $\mathrm{MIAv}_n = S_n(4\underline{13}2, 3\underline{14}2, 2\underline{14}3, 1\underline{24}3_{LID})$ be the set of permutations avoiding the vincular patterns $4\underline{13}2, 3\underline{14}2, 2\underline{14}3$ and LID $1\underline{24}3$ patterns.

Definition 2.6.33. Let $\mathrm{IAv}_n = S_n\left(4\underline{13}2, 3\underline{14}2, 2\underline{14}3, 1\underline{24}3_{LAN}\right)$ be the set of permutations avoiding the vincular patterns $4\underline{13}2, 3\underline{14}2, 2\underline{14}3$ and LAN $1\underline{24}3$ patterns.

Since $1\underline{24}3_{LAN}$ patterns are by definition instances of $1\underline{24}3_{LID}$ patterns, we get the following lemma.

Lemma 2.6.34. $\mathrm{MIAv}_n \subseteq \mathrm{IAv}_n$.

Recalling Definition 2.2.21, let I_n^R be the set of reversed inversion sequences and $L : S_n \to I_n^R$ be the bijection between permutations and Lehmer codes.

Proposition 2.6.35. $L^{-1}(I_n^R(e_i \geq e_j < e_k)) \subseteq \mathrm{MIAv}_n$

Proof. Fix $\mathbf{e} = (e_1, \cdots, e_n) \in I_n^R(e_i \geq e_j < e_k)$ and let $x \in S_n$ be the unique permutation such that the Lehmer code of x is $\mathbf{e}$. Assume for the sake of contradiction that x has one of the forbidden four subpatterns of MIAv_n. Suppose the forbidden subpattern is $x_i\underline{x_jx_k}x_m$. Suppose that the forbidden subpattern is of type $4\underline{13}2, 3\underline{14}2$, or $2\underline{14}3$. We note that in all of these types, $x_i > x_j$. Since $x_i > x_j$, we see that $\{u \mid u > i \text{ and } x_u < x_i\} \supseteq \{u \mid u > j \text{ and } x_u < x_j\}$. Or in other words, by definition of Lehmer code, $e_i \geq e_j$. If the forbidden subpattern is of type $1\underline{24}3_{LID}$, then by definition, $e_i \geq e_j$.
Since x_j and x_k form an ascent, we find that $\{u \mid u > j \text{ and } x_u < x_j\} \subseteq \{u \mid u > k \text{ and } x_u < x_k\}$, or in other words, $e_j \leq e_k$. We note

that in all types, $x_j < x_m < x_k$. This implies that $m \in \{u \mid u > k \text{ and } x_u < x_k\}$ and $m \notin \{u \mid u > j \text{ and } x_u < x_j\}$, so we conclude that $e_j < e_k$. Thus in all four types of subpatterns, we have an instance of the pattern $e_i \geq e_j < e_k$, which contradicts $\mathbf{e} \in I_n^R(e_i \geq e_j < e_k)$. This gives the desired result. □

Corollary 2.6.36. $\mathrm{IAv}_n = L^{-1}(I_n^R(e_i \geq e_j < e_k))$

Proof. We note that $\mathrm{IAv}_n \supseteq L^{-1}(I_n^R(e_i \geq e_j < e_k))$ by Proposition 2.6.35 and Lemma 2.6.34. By Corollary 2.3.2, Corollary 2.6.31, and Theorem 2.3.14, the two sets have the same size so are equal. □

Corollary 2.6.37. *The map ω restricts to a bijection between $L^{-1}(I_n^R(e_i \geq e_j < e_k))$ and $\mathcal{RSP}_n$.*

Proof. This follows from Theorem 2.6.27, Corollary 2.6.30, and Corollary 2.6.36. □

Corollary 2.6.38. $L^{-1}(I_n^R(e_i \geq e_j < e_k)) = \mathrm{MIAv}_n = \mathrm{IAv}_n$

2.7 Permutation and Dyck Path Statistics

Bijections between permutations and lattice paths that are statistic preserving give useful interpretations for bivariate generating functions. For instance, the Narayana numbers (a refinement of the Catalan numbers), $\mathrm{nar}_{n,k}$, count the number of permutations in $S_n(231)$ with k descents and the number of Dyck paths of semilength n with k EN factors (also called peaks). For a Dyck path q, let $EN(q)$ be the number of EN factors in q.

Theorem 2.7.1 ([24], Chapter 2)**.**

$$\begin{aligned}\sum_{n\geq 0}\sum_{k\geq 0}\mathrm{nar}_{n,k}z^n t^k &= \sum_{n\geq 0}\sum_{w\in S_n(231)} z^n t^{\mathrm{des}(w)}\\ &= \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{EN(q)}\\ &= \frac{1+z(t-1)-\sqrt{1-2z(t+1)+z^2(t-1)^2}}{2z}\end{aligned}$$

Instead of considering EN factors of Dyck paths, one can consider the EEN and NEE factors. For a Dyck path q, let $EEN(q)$ be the number of EEN factors in q. Define $NEE(q)$ analogously. Work of Barnabei et al. in conjuction with work of Bukata et al. connects the joint distribution of semilength and nunber of EEN factors to the joint distribution of permutation length and descents of $S_n(321)$.

Theorem 2.7.2 ([3, 10])**.**

$$\begin{aligned}\sum_{n\geq 0}\sum_{w\in S_n(321)} z^n t^{\mathrm{des}(w)} &= \frac{1-\sqrt{1-4z(1-z+tz)}}{2z(1-z+tz)} \\ &= \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{EEN(q)}\end{aligned}$$

Petersen notes that $\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \sum_{n\geq 0}\sum_{w\in S_n(231)} z^n t^{\mathrm{pk}(w)}$, where a peak of a permutation is an index i such that $x_{i-1} < x_i > x_{i+1}$ [24, Chapter 4]. We can use Corollary 2.6.5 to give a different proof of the generating function for NEE factors. Every EEN factor of a Dyck path corresponds to an atomic label. The only other way to produce an atomic label is if the Dyck path begins with a EN factor.

Theorem 2.7.3.

$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \frac{1+2z(t-1)-\sqrt{1-4z(1-z+tz)}}{2zt}$$

Proof. Let $A(z,t)$ stand for the generating function in Theorem 2.7.2. Every Dyck path either begins with EN or EE, so we decompose $A(z,t)$ into these two parts.

$$\begin{aligned}A(z,t) &= \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{EEN(q)} \\ &= \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n, q=EN\ldots} z^n t^{EEN(q)} + \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n, q=EE\ldots} z^n t^{EEN(q)}\end{aligned}$$

By definition of atom, if the Dyck path begins with EN, 1 is an atom. For Dyck paths that begin with EE, the number of atoms is the number of EEN factors. Thus,

$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{\mathrm{atom}(q)} = t\cdot\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n, q=EN\ldots} z^n t^{EEN(q)} + \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n, q=EE\ldots} z^n t^{EEN(q)}$$

We note that $\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n, q=EN\cdots} z^n t^{EEN(q)} = 1 + zA(z,t)$ because adjoining a EN to the beginning of a Dyck path increases the semilength by 1 and does not change the number of EEN factors. Thus, we can rewrite the equation as:

$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{\mathrm{atom}(q)} = t(1+zA(z,t)) + (A(z,t) - (1+zA(z,t)))$$
$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{\mathrm{atom}(q)} = t - 1 + (1 - z + tz)A(z,t)$$

By Corollary 2.6.5, the number of atoms is one more than the number of NEE factors.

$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{\mathrm{atom}(q)-1}$$
$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \frac{1}{t}\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{\mathrm{atom}(q)}$$
$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \frac{t - 1 + (1 - z + tz)A(z,t)}{t}$$
$$\sum_{n\geq 0}\sum_{q\in\mathcal{D}_n} z^n t^{NEE(q)} = \frac{1 + 2z(t-1) - \sqrt{1 - 4z(1 - z + tz)}}{2zt} \qquad \square$$

Since we can build $\mathcal{RSP}_n$ by making a binary choice of transforming or not transforming each NEE factor into a DE factor, by plugging in $t = 2$, we get another proof of Corollary 2.3.2.

Corollary 2.7.4.

$$\sum_{n\geq 0}\sum_{p\in\mathcal{RSP}_n} z^n = \frac{1 + 2z - \sqrt{1 - 4z(1+z)}}{4z}$$

For $w \in S_n$, let $\mathrm{s132}(w)$ and $\mathrm{s312}(w)$ refer to the number of strong 132 patterns and strong 312 patterns respectively. For $w \in S_n$, let $\mathrm{al}(w)$ and $\mathrm{ar}(w)$ refer to the number of atoms (that have leader atoms) whose leader atoms are to their left and whose leader atoms are to their right respectively. The bijections of this paper then give a refinement of Theorem 2.7.3 with the substitution $t = x + y$.

Corollary 2.7.5.

$$\begin{aligned}\sum_{n\geq 0}\sum_{p\in\mathcal{RSP}_n} z^n x^{NEE(p)} y^{DE(p)} &= \sum_{n\geq 0}\sum_{w\in\mathrm{Av}_n} z^n x^{s132(w)} y^{s312(w)} \\ &= \sum_{n\geq 0}\sum_{w\in\mathrm{IAv}_n} z^n x^{\mathrm{al}(w)} y^{\mathrm{ar}(w)} \\ &= \frac{1+2z(x+y-1)-\sqrt{1-4z(1-z+(x+y)z)}}{2z(x+y)}\end{aligned}$$

We end by justifying the name Boolean-Catalan numbers. For fixed $q \in \mathcal{DP}_n$, suppose q has k NEE factors. Then there are 2^k elements of $\mathcal{RSP}_n$ that have q as their inner Dyck path. Based on the poset constructions, this means that we can partition the fiber of $\tau^{-1}(q)$ into 2^k intervals in two different ways by looking at the fibers of ρ or the fibers of τ. See Figures 2.8, 2.11, and 2.15 for an example for $k = 2$.

Chapter 3

The Hopf Algebra of Restricted Schröder Paths

3.1 Introduction

In this chapter, we describe the Hopf algebra structure on restricted Schröder paths that is induced from a known Hopf algebra structure on $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. We first define a Hopf algebra, describe the Malvenuto-Reutenauer Hopf algebra of permutations, and give some relevant lattice theory background.

Let $\mathbb{K}$ be a fixed field. All unlabeled arrows in Figures 3.1 - 3.4 are the obvious isomorphisms. The map I denotes the identity map. A unital associative algebra is a vector space A over a field $\mathbb{K}$ with a product map $m : A \otimes A \to A$ and another map $\mu : \mathbb{K} \to A$ that satisfies the commutative diagrams in Figure 3.1. A counital coassociative algebra is a vector space A over a field $\mathbb{K}$ with a coproduct map $\Delta : A \to A \otimes A$ and another map $\epsilon : A \to \mathbb{K}$ that satisfies the commutative diagrams in Figure 3.2. Let $T : A \otimes A \to A \otimes A$ be the map defined by sending basis elements $a \otimes b \mapsto b \otimes a$ and extending linearly. A bialgebra is a vector space A over a field $\mathbb{K}$ that is both a unital associative algebra and a counital coassociative algebra that satisfies the commutative diagrams in Figure 3.3. A Hopf algebra is a bialgebra A with an additional map $S : A \to A$ that satisfies the commutative diagrams in Figure 3.4. For our purposes, an vector space A is graded if it can be written as $A = \oplus_{n=0}^{\infty} A_n$ for a countably infinite family of vector spaces $\{A_i\}_{n=0}^{\infty}$. A graded vector space A can be turned into a graded algebra, by inducing a grading on $A \otimes A$ and a grading on $\mathbb{K}$ and requiring that all of the relevant maps be graded maps.

The nth graded piece of $A \otimes A$ is the direct sum, $\oplus_{p+q=n} A_p \otimes A_q$. The field $\mathbb{K}$ has the structure of a graded algebra with its 0th component being the one-dimensional $\mathbb{K}$ and trivial component for all other components. A graded algebra is connected if A_0 is isomorphic to $\mathbb{K}$ as an algebra. By [14, Proposotion 1.4.14], every graded, connected Hopf algebra has an antipode. In this chapter, we consider combinatorial Hopf algebras, which are graded, connected Hopf algebras where the basis elements are a family of combinatorial objects.

Let the permutations of S_n serve as the basis for an $n!$-dimensional vector space $\mathbb{K}[S_n]$. We consider the graded vector space $\bigoplus_{n\geq 0} \mathbb{K}[S_n]$. The vector space $\bigoplus_{n\geq 0} \mathbb{K}[S_n]$ has a Hopf algebra structure known as the Malvenuto-Reutenauer Hopf algebra of permutations (abbreviated as MR). The Hopf algebra MR is a connected, graded Hopf algebra. We now describe the necessary background to define the product and coproduct of MR.

Definition 3.1.1. Let $x = x_1 \cdots x_v \in S_v$. For fixed $u \in \mathbb{N}$, let $x_{[u]}{}' = (x_1 + u) \cdots (x_v + u)$ be the *shift of x by u*. A permutation $z \in S_n$ is a shifted shuffle of $x \in S_u$ and $y \in S_v$ if $u + v = n$ and x and $y'_{[u]}$ are subsequences of z.

Definition 3.1.2. We define the product on MR by describing how it acts on basis elements $x \otimes y$ for $x \in S_u$ and $y \in S_v$. The general product is then found by extending linearly. Sometimes, we abbreviate $m(x \otimes y)$ as $x \bullet_{MR} y$. The product $x \bullet_{MR} y$ is defined as the sum (formal linear combination) of all shifted shuffles of x and y.

To define the coproduct on MR, we make use of the following definition.

Definition 3.1.3. The map st takes a sequence $v_1, v_2, \cdots v_p$ of p distinct positive numbers and produces the unique permutation $w = w_1 = w_1 w_2 \cdots w_p \in S_p$ where $w_i < w_j$ iff $v_i < v_j$. The resulting permutation is called the *standardization* of the resulting sequence.

Definition 3.1.4. The *coproduct* in the Hopf algebra MR is then defined on basis elements $w \in S_n$ as

$$\Delta_{\mathrm{MR}}(w) = \sum_{i=0}^{n} \mathrm{st}(w_1, \cdots, w_i) \otimes \mathrm{st}(w_{i+1}, \cdots w_n)$$

To describe the product of our desired Hopf algebra, we will need some lattice theory. We recall that S_n has a poset structure known as the weak order, described in Definition 2.2.5. We now give definitions and results from lattice theory.

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes I"] \arrow[d, "I \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{K} \arrow[d, "I \otimes \mu"'] & A \arrow[l] \arrow[r] \arrow[d, "I"] & \mathbb{K} \otimes A \arrow[d, "\mu \otimes I"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}
$$

Figure 3.1: Unital associative algebra commutative diagrams

$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes I"'] \\
A \otimes A \arrow[u, "I \otimes \Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{K} \arrow[r] & A & \mathbb{K} \otimes A \arrow[l] \\
A \otimes A \arrow[u, "I \otimes \epsilon"] & A \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "I"'] & A \otimes A \arrow[u, "\epsilon \otimes I"']
\end{tikzcd}
$$

Figure 3.2: Counital coassociative algebra commutative diagrams

$$
\begin{tikzcd}
\mathbb{K} \arrow[r, "\mu"] \arrow[d] & A \arrow[d, "\Delta"] \\
\mathbb{K} \otimes \mathbb{K} \arrow[r, "\mu \otimes \mu"'] & A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} & A \arrow[l, "\epsilon"'] \\
\mathbb{K} \otimes \mathbb{K} \arrow[u] & A \otimes A \arrow[l, "\epsilon \otimes \epsilon"] \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \arrow[r, "\mu"] \arrow[dr, "I"'] & A \arrow[d, "\epsilon"] \\
 & \mathbb{K}
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "m"'] & A \otimes A \otimes A \otimes A \arrow[r, "I \otimes T \otimes I"] & A \otimes A \otimes A \otimes A \arrow[d, "m \otimes m"] \\
A \arrow[rr, "\Delta"'] & & A \otimes A
\end{tikzcd}
$$

Figure 3.3: Bialgebra commutative diagrams

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes I"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & \mathbb{K} \arrow[rr, "\mu"] & & A \\
 & A \otimes A \arrow[rr, "I \otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Figure 3.4: Hopf algebra commutative diagram

Definition 3.1.5. Given a subset C of a finite poset $(P, \leq)$, an *upper bound* of C is an element M such that $M \geq c$ for all $c \in C$. A *least upper bound* or *join* of a set C is a unique upper bound M of C such that $M \leq M'$ for any upper bound M' of C. The notions of *greatest lower bound* and *meet* are defined analogously. If the join and meet exist for every subset of P, then P is called a *lattice*. The join of one-element subsets $\{x\}$ and $\{y\}$ is usually notated $x \vee y$ and the meet as $x \wedge y$.

Definition 3.1.6. A *lattice congruence* is an equivalence relation Θ on L such that if $x_1 \equiv x_2\ (\Theta)$ and $y_1 \equiv y_2\ (\Theta)$, then $x_1 \vee y_1 \equiv x_2 \vee y_2\ (\Theta)$ and $x_1 \wedge y_1 \equiv x_2 \wedge y_2\ (\Theta)$.

Definition 3.1.7. A *quotient lattice* L/Θ is defined as the partial order on equivalence classes of Θ where one class C_1 is less than another class C_2 if and only if there exist representatives $x \in C_1$ and $y \in C_2$ such that $x \leq y$.

Theorem 3.1.8. *[25, Proposition 9-5.2] An equivalence relation Θ on a finite lattice L is a lattice congruence if and only if the following three conditions hold.*

- *Each equivalence class is an interval in L.*
- *The map $\pi_{\downarrow}^{\Theta}$ mapping each element to the bottom element of its equivalence class is order preserving.*
- *The map $\pi_{\uparrow}^{\Theta}$ mapping each element to the top element of its equivalence class is order preserving.*

Given a lattice congruence Θ on a lattice L, one can consider the induced subposet $\pi_\downarrow^\Theta(L)$ of bottom elements of congruence classes.

Theorem 3.1.9. *[25, Proposition 9-5.5] If L is a finite lattice and Θ is a congruence on L, then $\pi_\downarrow^\Theta(L)$ is a lattice isomorphic to the quotient lattice L/Θ.*

The weak order is known to be a lattice [29, Chapter 3]. Björner and Wachs showed that the set $S_n(\underline{31}2)$ is isomorphic to the famous Tamari lattice [8]. However, it can also be seen by applying Theorem 3.1.9 to the lattice congruence $\Theta_{\underline{31}2}$. The equivalence relation $\Theta_{2\underline{41}3,3\underline{41}2,\underline{41}23,\underline{41}32}$ is a lattice congruence. Thus, the weak order on its bottom elements is a lattice. In the next section, we work out the lattice structure on restricted Schröder paths induced by the bijection between $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $\mathcal{RSP}_n$. This lattice structure will help describe the Hopf algebra product on the set of restricted Schröder paths. The combinatorics surrounding τ, $S_n(\underline{31}2)$, and Dyck paths is similar. We now describe the analogous results for Dyck paths.

The Tamari lattice can be realized using Dyck paths. The cover relations for this realization of the Tamari lattice are given in [9, Definition 1]. We translate the results of [9] into our conventions. More specifically, we illustrate how the covers are suggested by the lattice theory of the quotient lattice S_n/Θ_{312}. The covers for the Tamari lattice are as follows. Let q be a fixed Dyck path. Consider a NN factor of the Dyck path. Draw a line segment of slope 1 starting at the south extremity of the upper N step and travel southwest until it touches q again. Construct q' from q by shifting up the portion of the path cut off by the segment by one (This portion of the path fits in a triangular envelope). See Figure 3.5 for an example. A Dyck path q' satisfies $q' \lessdot q$ iff there exists a NN factor and segment that transforms q into q' in this way.

We recall the following useful proposition [19, Proposition 1.2.2], which is a special case of [26, Proposition 2.2] for finding the cover relations for a lattice obtained from a congruence on the weak order. Let $\pi_\downarrow : S_n \to S_n$ be the map that takes a permutation to the unique avoider of its congruence class.

Proposition 3.1.10. *Given Θ, a congruence on the weak order of S_n, and $x \in S_n$, the map $y \to [y]_\Theta$ restricts to a one-to-one correspondence between elements of the quotient lattice S_n/Θ covered by $\pi_\downarrow(x)$ and elements of Av_n covered by $[x]_\Theta$.*

Since covers in the weak order are determined by descents, applying the previous proposition to $\Theta_{\underline{31}2}$ and the description of covers in the Tamari lattice yields the following

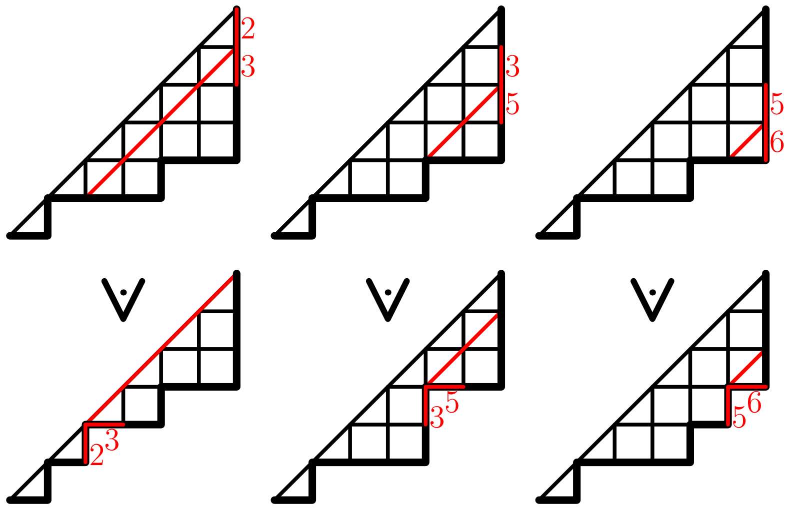


Figure 3.5: The elements covered by $\tau(146532)$ in the Tamari lattice, $\tau(1465\mathbf{23}) \lessdot \tau(1465\mathbf{32})$, $\tau(146\mathbf{35}2) \lessdot \tau(146\mathbf{53}2)$, $\tau(14\mathbf{56}32) \lessdot \tau(14\mathbf{65}32)$

geometric way to intrinsically "see" descents in a Dyck path.

Corollary 3.1.11. *Fix $q \in \mathcal{D}_n$. For a given Dyck path, let $\tau_{Av}^{-1}(q)$ be the unique* 312 *avoider that maps to it under τ. The descents of $\tau_{Av}^{-1}(q)$ are in one-to-one correspondence with the NN factors of q.*

Let $\bigoplus_{n\geq 0}\mathbb{K}[S_n(\underline{31}2)]$ be the vector space with basis $\bigcup_{n=0}^{\infty} S_n(\underline{31}2)$. In [26, Section 9], Reading shows that $\bigoplus_{n\geq 0}\mathbb{K}[S_n(\underline{31}2)]$ embeds as a sub-Hopf algebra of MR. Reading describes the product $\bullet_{312}$ (on basis elements) in this sub-Hopf algebra in a compact way. For $x \in S_u$ and $y \in S_v$, the product is

$$x \bullet_{312} y = \sum \left[x y_{[u]}{}', \pi_{\downarrow}^{\Theta_{312}}(y_{[u]}{}' x) \right]$$

where $xy'_{[u]}$ means concatenation of permutations and $\sum[,]$ means take the sum (formal linear combination) of all elements in the interval (of the lattice S_{u+v}). The coproduct of this sub-Hopf algebra is given by taking the ordinary MR coproduct on basis elements of $S_n(312)$ and then canceling terms that do not belong to $\mathbb{K}[S_n(312)] \otimes \mathbb{K}[S_n(312)]$.

One can use bijections between $S_n(312)$ and other families of combinatorial objects to produce isomorphic Hopf algebras that have cancellation-free formulas for the product and coproduct. Law and Reading do this in [20], by describing a Hopf algebra structure on the set of planar binary trees. Similarly, we can use the τ to help define a Hopf algebra structure on the set of Dyck paths. This Hopf algebra structure essentially appears in work of Lopez N. et. al [23]. Suppose that $q_1 \in \mathcal{DP}_u$ and $q_2 \in \mathcal{DP}_v$ with $\tau(x) = q_1$ and $\tau(y) = q_2$. Then,

$$q_1 \bullet_{DP} q_2 = \sum [\tau(xy_{[u]}'), \tau(y_{[u]}'x)].$$

To describe the product intrinsically, we borrow the following notation from [23, Definition 2.2].

Definition 3.1.12. Let $q_1 \in \mathcal{DP}_u$. And suppose that the word for q_1 ends with $E\underbrace{N\cdots N}_{k}$. Let $q_2 \in \mathcal{DP}_v$. The *ith concatenation* (for $0 \le i \le k$) of q_1 and q_2 is defined to be the path in $\mathcal{DP}_{u+v}$ corresponding to the word where we take the word for q_2 and insert it before the ith rightmost N of the word for q_1.

One can check that the path $\tau(xy_{[u]}')$ is the 0th concatenation of $\tau(x)$ and $\tau(y)$ and that the path $\tau(y_{[u]}'x)$ is the kth concatenation of $\tau(x)$ and $\tau(y)$.

Proposition 3.1.13. *Let q_1q_2' be the 0th concatenation of the paths. Let $q_2'q_1$ be the kth concatenation of the paths. The product of q_1 and q_2 then is the sum over all elements of the interval $[q_1q_2', q_2'q_1]$.*

In this chapter, we describe the analogous story for $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $\mathcal{RSP}_n$. In Section 3.2, we use ρ to impose a lattice structure on $\mathcal{RSP}_n$. The main tool to do this is Proposition 3.1.10. In Section 3.3, we use ρ to describe the product on the Hopf Algebra structure $\bigoplus_{n\ge 0} \mathbb{K}[\mathcal{RSP}_n]$.

3.2 Cover Relations

We recall from Section 2.5, that $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ can be realized as the lower elements of the congruence $\Theta_{2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32}$. By [26], $\Theta_{2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32}$ is a lattice congruence of the weak order. We use ρ to define a lattice structure on $\mathcal{RSP}_n$. Specifically, ρ gives a bijection between the quotient lattice S_n/Θ and $\mathcal{RSP}_n$. We seek to understand the cover relations in this lattice. We now make some definitions in order to state our main result describing the cover relations.

Definition 3.2.1. Fix an element $p \in \mathcal{RSP}_n$. Let q be the inner Dyck path of p as defined in Definition 2.2.19. Consider a fixed NN factor of q. Suppose the lower N step of the factor is labeled with c. We call the NN factor *stable* if the west extremity of the east step labeled with c is not the northeast extremity of a diagonal step in p.

Remark 3.2.2. *The phrase stable NN factor is a bit ambiguous because the NN factor is part of the inner Dyck path q, but to figure out if it is stable, we must look at p. For instance, the N of a ND factor of p is the lower N of a NN factor of q. For the rest of the section, we will use the phrase "stable NN factor of p" to mean a NN factor of q that is stable with respect to p.*

Theorem 3.2.3. *Fix $p_1 \in \mathcal{RSP}_n$. The elements p_2 that are covered by p_1 are governed by the stable NN factors and diagonal steps of p_1. Specifically, for a fixed stable NN factor labeled with (c, d), consider the segment connecting the upper extremity of the north step labeled by c and the western extremity of the east step labeled by c. Then there exists a path $p_2 \in \mathcal{RSP}_n$ with $p_2 \lessdot p_1$, where:*

- *(Triangle Neutral) If c is triangle-free and not the middle E of an EEE factor, construct p_2 from p_1 by shifting the portion of the path cut off by the segment up by one.*

- *(Triangle Create) If c is triangle-free and the middle E of an EEE factor, construct p_2 from p_1 by shifting the portion of the path cut off by the segment by one, and add a triangle in column c.*

- *(Triangle Shift) If c is triangle-positve, construct p_2 from p_1 by shifting up the portion of the path cut off by the segment by one and shift up the triangle in column c by one.*

For a fixed diagonal step of p_1,

- *(Triangle Destroy) Suppose there is a diagonal step in a column of p_1. Construct p_2 by transforming the DE factor to a NEE factor.*

See Figure 3.6 for an example. Triangle shift, triangle neutral, and triangle create covers will also lose boxes with the same label. Triangle destroy covers will not lose boxes. The logic in this section towards proving Theorem 3.2.3 also produces the following result.

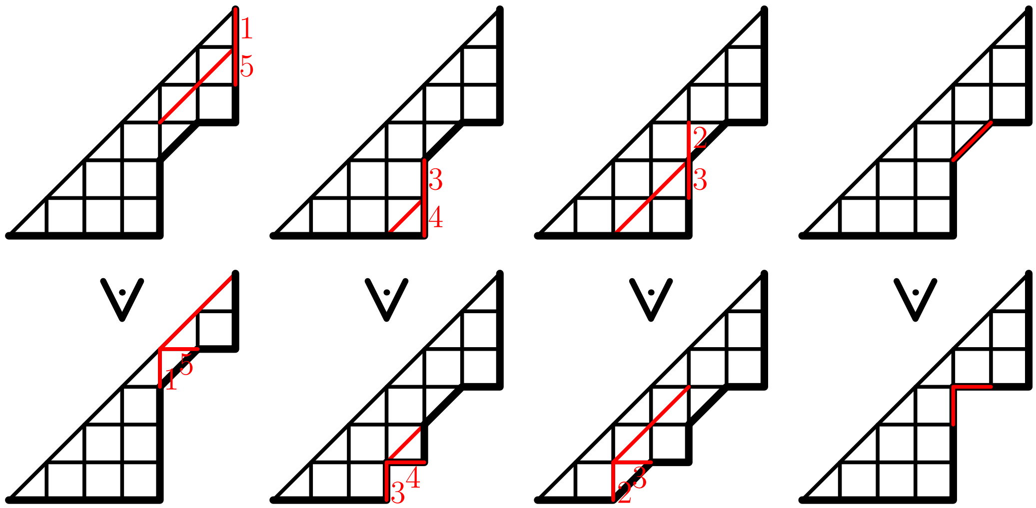


Figure 3.6: A triangle shift cover $\rho(6432\mathbf{15}) \lessdot \rho(6432\mathbf{51})$, a triangle neutral cover $\rho(6\mathbf{34}251) \lessdot \rho(6\mathbf{43}251)$, a triangle create cover $\rho(64\mathbf{23}51) \lessdot \rho(64\mathbf{32}51)$, and a triangle destroy cover $\rho(\mathbf{46}3251) \lessdot \rho(\mathbf{64}3251)$. Note that the NN factor labeled with $(6,5)$ is not stable, so does not correspond with a cover.

Theorem 3.2.4. *Fix $p \in \mathcal{RSP}_n$. Suppose $\rho(x) = p$ with $x \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. The descents of x are in one-to-one correspondence with the set consisting of the union of the stable NN factors and diagonal steps of p.*

We will use Proposition 3.1.10 to prove Theorem 3.2.3. Our first task is to describe the descents of elements of $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$.

For a fixed $w \in S_n$ written in one-line notation as $w = w_1 \cdots w_n$, descents are classically defined as indices i such that $w_i > w_{i+1}$ [29, Section 1.4]. In this section, we will find it useful to define descents as follows.

Definition 3.2.5. For a fixed $w = w_1 \cdots w_n \in S_n$, a *descent* is a pair (w_i, w_{i+1}) such that $w_i > w_{i+1}$.

Definition 3.2.6. For a fixed $w = w_1 \cdots w_n \in S_n$, a *Dyck descent* is a descent (w_i, w_{i+1}) such that (w_i, w_{i+1}) is also a Dyck inversion.

We remind the reader of the notation from Definition 2.4.7 and Proposition 2.4.11. For fixed y_{i+1}, let $M_{y_{i+1}} = M$ as in Proposition 2.4.11.

We see that there is one type of cover in the Tamari lattice. On the level of Dyck paths, we are losing boxes with the same label as we go down. The lattice $\mathcal{RSP}_n$ will turn out to have four types of covers as given in the Theorem 3.2.3. The proof of Theorem 3.2.3 appears at the end of this section.

In light of Proposition 3.1.10, we characterize the descents of elements in $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. We remind the reader of Definition 2.5.5.

Lemma 3.2.7. *Fix an avoider* $y \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. *Each descent* (y_i, y_{i+1}) *of* y *fits into exactly one of the following six cases. If* (y_i, y_{i+1}) *is a Dyck inversion, let* $M = M_{y_{i+1}}$ *be defined as in Proposition 2.4.11.*

- $y_i - y_{i+1} \geq 2$, (y_i, y_{i+1}) *is a Dyck inversion, and* y_i *is the 2 of a strong 312 pattern*
- $y_i - y_{i+1} \geq 2$, (y_i, y_{i+1}) *is a Dyck inversion,* y_i *is not the 2 of a strong 312 pattern,* $M > y_i$
- $y_i - y_{i+1} \geq 2$, (y_i, y_{i+1}) *is a Dyck inversion,* y_i *is not the 2 of a strong 312 pattern,* $M = y_i$
- $y_i - y_{i+1} = 2$ *and* (y_i, y_{i+1}) *is not a Dyck inversion*
- $y_i - y_{i+1} = 1$ *and* $(y_i + 1, y_i)$ *is an inversion*
- $y_i - y_{i+1} = 1$ *and* $(y_i + 1, y_i)$ *is not an inversion*

Proof. We note that if $y_i - y_{i+1} > 2$ and (y_i, y_{i+1}) is not a Dyck inversion, then we will have a forbidden pattern where the y_i is the 4, y_{i+1} is the 1, and the witness for not being a Dyck inversion is the 2 or 3. But this contradicts $y \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. The Lemma follows by considering the remaining possibilities of $y_i - y_{i+1}$. □

We organize the types of stable NN factors using in the following Lemma. We remind the reader of Definition 2.5.4.

Lemma 3.2.8. *Fix an element* $p \in \mathcal{RSP}_n$. *We can classify stable* NN *factors of* p *as follows. For a fixed* NN *factor, let* E_0 *be the corresponding east step in* q *as defined in Definiton 3.2.1. Suppose* E_0 *lies in the column* c. *Then in* q, E_0 *is preceded by a step* A *and succeeded by a step* B. *Each stable* NN *factor of* p *falls into exactly one of the following five cases.*

- $A = E, B = N$

- $A = E, B = E$
- $A = N, B = N$, *c is triangle-free*
- $A = N, B = E$, *c is triangle-free*
- $A = N, B = E$, *c is triangle-positive*

Proof. We note that $A = N, B = N$, c is triangle-positive is impossible since D steps must be followed by E steps. The lemma follows by considering the remaining cases. □

Theorem 3.2.9. *Let $x \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $p \in \mathcal{RSP}_n$ with $\rho(x) = p$. Then (c,d) is a Dyck descent of x if and only if there exists a stable NN factor of p labeled with (c,d)*

Proof. We note that by Proposition 2.4.15 and the definitions of box and edge labelings, that a stable NN factor of p is labeled by (c,d) if and only if the lowest box in column c is labeled with d.

Suppose (c,d) is a Dyck descent of x. By Proposition 2.4.16, there exists a box in column c labeled with d. Assume by way of contradiction that the north step labeled by c is succeeded by an east step (labeled by e). By definition of box labeling and Proposition 2.4.16, there will be a box labeled d in column e, i.e. (e,d) is a Dyck inversion, so in particular e is to the left of d. But since c and e label a NE factor, e should be to the right of c by Definition 2.4.17 and Proposition 2.4.21. But if (c,d) form a descent, then e is also to the right of d, which is a contradiction. Therefore, the north step labeled by c is succeeded by a north step, and by definition of edge labeling, will be labeled with d. The NN factor labeled by (c,d) is stable because otherwise $d = c-1$ and $(c)(c-2)(c-1)$ would form a strong 312 pattern contradicting the fact that c and $d = c-1$ are adjacent.

Suppose there exists a stable NN factor labeled by (c,d). By Proposition 2.4.16, (c,d) is a Dyck inversion. Suppose $y = y_1 \cdots y_n$. Let $c = y_i$ and $d = y_k$ with $i < k$. Assume for the sake of contradiction that (y_i, y_k) is not a descent of x. Then the set $F_{y_i} = \{y_j \mid i < j < k\}$ is nonempty.

Assume for the sake of contradiction that $y_{i+1} < y_i$. Suppose that (y_i, y_{i+1}) is a Dyck inversion. If $y_{i+1} < y_k$, then y_k serves as a witness for (y_i, y_{i+1}) not being a Dyck inversion, which is a contradiction. If $y_{i+1} > y_k$, there would be a box labeled y_{i+1} in column y_i by Proposition 2.4.16. However, then by Proposition 2.4.15, we contradict the assumption the lowest box in column $c = y_i$ is labeled with $d = y_k$. Therefore, (y_i, y_{i+1}) is a non-Dyck inversion, i.e there exists a y_l with $l > i+1$ and $y_{i+1} < y_l < y_i$. Since (y_i, y_{i+1}) is not a Dyck inversion, $y_i - y_{i+1} > 1$. If $y_i - y_{i+1} = 2$, then $y_{i+1} = y_i - 2$. Since (y_i, y_{i+1}) is not a Dyck inversion, $y_i - 1$ must occur to the right of $y_i - 2$. This implies that $y_i(y_i-2)(y_i-1)$ is a strong 312 pattern and that there must be a triangle in column $(y_i - 1)$. Since diagonal steps must be followed by east steps, the northeast corner of the diagonal step must coincide with the western corner of the step labeled with y_i. But this contradicts the stability of the considered NN factor. If $y_i - y_{i+1} \geq 3$, then there exists an element y_m such that $y_m \in \{y_j \in \mathbb{Z} \mid y_{i+1} < y_j < y_i\} \smallsetminus y_l$. No matter where y_m lies in the permutation, y_i will be the 4 and y_{i+1} will be the 1 of a forbidden pattern where the y_m, y_l play the roles of the $2,3$. Thus, we have a contradiction no matter what the difference $y_i - y_{i+1}$ is, thus we conclude that $y_{i+1} > y_i$.

Assume for the sake of contradiction that $y_{k-1} > y_k$. Suppose that $y_{k-1} > y_i$ and (y_{k-1}, y_k) is not a Dyck inversion. Then there exists a y_m with $m > k$ and $y_k < y_m < y_{k-1}$. Then the y_{k-1} will serve as the 4 and the y_k will serve as the 1 of a forbidden pattern where the y_m and y_i serve as the 2 and the 3. Suppose that $y_{k-1} > y_i$ and (y_{k-1}, y_k) is a Dyck inversion. Let E_0 be the east step labeled with y_i. Let B be the step immediately succeeding E_0 and M be the rightmost column left of the stable NN factor. If $B = N$, we have a contradiction because $M = y_i$ implies that the rightmost column with a box labeled y_k is y_i, so there can be no box labeled y_k in column y_{k-1}. If $B = E$, then Lemma 2.4.10 implies that $y_i + 1$ is to the left of y_i. However this implies that the lowest box in column (y_i+1) is labeled with y_i. Furthermore, every column between column (y_i+1) and column M must have a box labeled y_i underneath the box labeled y_k by definition of box and edge labeling. In particular, by Proposition 2.4.16, (y_{k-1}, y_i) should be a Dyck inversion. However, this is a contradiction because (y_{k-1}, y_i) is not even an inversion. Suppose that $y_k < y_{k-1} < y_i$. Then (y_i, y_{k-1}) must be an inversion but not a Dyck inversion (otherwise the lowest box in column y_i would not be labeled y_k). Thus, there exists a y_l with $l > k-1$ such that $y_{k-1} < y_l < y_i$. But then y_l also satisfies $y_k < y_l < y_i$, i.e. it serves as a witness for (y_i, y_k) not being a Dyck inversion, which is a contradiction. Thus, we conclude that $y_{k-1} < y_k$.

We know by the previous two paragraphs, that $y_{i+1} > y_i > y_k$ and $y_{k-1} < y_k$. Therefore, there is some rightmost descent (y_u, y_{u+1}) in F_{y_i} such that $y_u > y_k$ and $y_{u+1} < y_k$. Since $y_{u+1} < y_k < y_u$, $y_{k-2} - y_{k-1} > 1$. Suppose that $y_u - y_{u+1} \geq 3$. Then there exists a $y_l \in \{y_j \in$

$\mathbb{Z} \mid y_{u+1} < y_j < y_u\} \setminus y_k$. But then the y_u will serve as the 4 and the y_{u+1} will serve as the 1 of a forbidden pattern with the y_l, y_k serving the roles of the 2, 3. Thus it must be that $y_u - y_{u+1} = 2$. If this is true, then $y_u = y_k + 1$ and $y_{u+1} = y_k - 1$. We see that (y_i, y_u) is an inversion but not a Dyck inversion (otherwise a box with label $y_u = y_k + 1$ would contradict the lowest box in the column y_i is labeled with y_k), therefore there exists a y_l with $l > u$ and $y_u < y_l < y_i$. But then y_l also satisfies $y_k < y_l < y_i$, so in particular, $l > k$ since by definition we picked the rightmost descent with $y_u > y_k$. This means y_l would serve as a witness for (y_i, y_k) not being Dyck, which is a contradiction. Thus, we conclude that the set F_{y_i} is empty implying that (y_i, y_k) is a descent of x.

□

Proposition 3.2.10. *Let $y \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ and $p \in \mathcal{RSP}_n$ with $\rho(y) = p$. Let $M = M_{y_{i+1}}$ be defined as in Proposition 2.4.11.*

- *The tuple (c, d) is a Dyck descent of y with $c - d \geq 2$ and c is the 2 of a strong 312 pattern iff (c, d) labels a stable NN factor of p such that the east step labeled with c is the middle E of a NEE factor and there is a triangle in column c.*

- *The tuple (c, d) is a Dyck descent of y with $c - d \geq 2$, c is not the 2 of a strong 312 pattern, $M > c$ iff (c, d) labels a stable NN factor of p such that the east step labeled with c is the middle E of a NEE factor and there is no triangle in column c.*

- *The tuple (c, d) is a Dyck descent of y with $c - d \geq 2$, c is not the 2 of a strong 312 pattern, $M = c$ iff (c, d) labels a stable NN factor of p such that the east step labeled with c is the middle E of a NEN factor.*

- *The tuple (c, d) is a non-Dyck descent of y with $c - d = 2$ iff the east step labeled with c is the E of a DE factor of p.*

- *The tuple (c, d) is a Dyck descent of y with $c - d = 1$ and $(c+1, c)$ is an inversion iff (c, d) labels a stable NN factor of p such that the east step labeled with c is the middle E of a EEE factor.*

- *The tuple (c, d) is a Dyck descent of y with $c - d = 1$ and $(c+1, c)$ is not an inversion iff (c, d) labels a stable NN factor of p such that the east step labeled with c is the middle E of a EEN factor.*

Proof. Applying the definitions of edge and box labeling to the east steps labeled c and d, we conclude the following. We see that $c - d \geq 2$ if and only if the east step labeled with c is preceded by a north step. Equivalently, $c - d = 1$ if and only if the east step labeled with c is preceded by an east step. Applying the definitions of edge and box labeling to the north steps labeled c and d, we conclude the following. We see that $M > c$ if and only if the east step labeled with c is succeeded by an east step. Equivalently, $M = c$ if and only if the east step labeled with c is succeeded by a north step. Note that if c is triangle positive, since D steps must be followed be east steps, the east step labeled by c must be followed by an east step. Note that by Lemma 2.4.10, if $c - d = 1$, then $M > c$ if and only if $(c+1, c)$ is an inversion and $M = c$ if and only if $(c+1, c)$ is not an inversion. Every case except the fourth case now follows from Theorem 3.2.9.

Suppose (c, d) is a non-Dyck descent of y with $c - d = 2$. Then the witness for not being a Dyck inversion must be $c - 1$, meaning $c(c-2)(c-1)$ forms a strong 312 pattern. By definition ρ, we will have a DE factor with E step labeled by c.

Conversely, suppose we have a DE factor of p with E step labeled by c. By definition of ρ, we know that $c(c-2)(c-1)$ forms a strong 312 pattern. Letting $d = c - 2$, we see that (c, d) is a non-Dyck inversion. We claim that (c, d) is a descent. Let $c = y_i$ and $d = y_k$. We mimic the argument of the backwards direction of Theorem 3.2.9.

Assume for the sake of contradiction that the set $F_{y_i} = \{y_j \mid k < j < i\}$ is nonempty. We find that if $y_{i+1} < y_k$, then $\underline{cy_{i+1}}(c-2)(c-1)$ is a forbidden pattern, so $y_{i+1} > y_k$. We see that if $y_{k-1} > y_i$, $c\underline{y_{k-1}(c-2)}(c-1)$ is a forbidden pattern . We note that $y_k < y_{k-1} < y_i$ is impossible ($y_i - 1$ is the only number between y_k and y_i). Thus, we conclude that $y_{k-1} < y_k$.

We know that $y_{i+1} > y_k$ and $y_{k-1} < y_k$. Therefore, there is some rightmost descent (y_u, y_{u+1}) in F_{y_i} with $y_u > y_k$ and $y_{u+1} < y_k$. Since $y_{u+1} < y_k < y_u$, $y_{k-2} - y_{k-1} > 1$. Suppose that $y_u - y_{u+1} \geq 3$. Then there exists a $y_l \in \{y_j \in \mathbb{Z} \mid y_{u+1} < y_j < y_u\} \setminus y_k$. But then the y_u will serve as the 4 and the y_{u+1} will serve as the 1 of a forbidden pattern with the y_l, y_k serving the roles of the 2, 3. Therefore, $y_u - y_{u+1} = 2$. But then, $y_{u+1} < y_k < y_u$ implies that $y_u = y_k + 1 = c - 1$. But this is a contradiction since $c - 1$ is to the right of y_k. Thus, we have a contradiction and conclude that the set F_{y_i} is empty implying that (y_i, y_k) is a descent of x.

□

We now have everything we need to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. Let $y \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$. Suppose $x \in S_n$ is covered by y in the weak order. Let $x = y_1 \cdots y_{i+1} y_i \cdots y_n$. Specifically, $x_i = y_{i+1} < y_i = x_{i+1}$. Suppose $\rho(y) = p_1$ and $\rho(x) = p_2$. If $x \lessdot y$, we claim that p_2 and p_1 satisfy for one of the four types listed in Theorem 3.2.3. By Lemma 3.2.7 and Theorem 3.2.10, the descent (y_i, y_{i+1}) falls into one of six cases.

Suppose (y_i, y_{i+1}) is a Dyck inversion. By Proposition 2.4.16, there is a box labeled by y_{i+1} in columns y_i through M. If we compare y and x, we notice that in x, y_i serves as a witness for every element y_k with $y_i < y_k \leq M$ to not be part of a Dyck inversion of the form (y_k, y_{i+1}) in x. Additionally, (y_i, y_{i+1}) is no longer a Dyck inversion of x. In other words, $\rho(x)$ is missing the boxes labeled by y_{i+1} in columns y_i through M when compared to $\rho(y)$. We claim that with regards to boxes, this is the only difference between $\rho(x)$ and $\rho(y)$. This is because y_i is too big to serve as a witness for any element whose column is less than y_i to not be a Dyck inversion. Additionally the Dyck inversions with lower element y_i are the same for x and y since moving y_{i+1} to the left does not affect potential witnesses. In summary, for the five cases where (y_i, y_{i+1}) is a Dyck inversion, we are shifting up the inner Dyck path between columns y_i and M by 1. We now consider how triangles are affected in all cases.

If $y_i - y_{i+1} \geq 2$ and y_i is not the 2 of a strong 312 pattern, then x and y have NEE and DE factors in the same columns. This is because if y_i is not the 2 of a strong 312 pattern, then no triangles are destroyed when going from y to x. Additionally, we see that $y_i - y_{i+1} \geq 2$ implies that no triangle is created. Thus, the net effect of losing the boxes labeled by y_{i+1} is to shift up the portion of the path y beneath the y_i segment.

If $y_i - y_{i+1} \geq 2$ and y_i is the 2 of a strong 312 pattern, then x and y have NEE and DE factors in the same columns. This is because switching y_i and y_{i+1} preserves the respective strong 312 pattern when going from x to y. Additionally, $y_i - y_{i+1} \geq 2$ implies that no triangle is created. Thus, the net effect of losing the boxes labeled by y_{i+1} is to shift up the portion of the path y beneath the y_i segment. However in this case, the triangle corresponding to y_i is also shifted up one row in x.

If $y_i - y_{i+1} = 1$ and $(y_i + 1, y_i)$ is an inversion, then y_i will be the 2 of a strong 312 pattern in x and not the 2 of a strong 312 pattern in y. In all other columns, x and y will have the NEE and DE factors in the same columns. This implies that we add a triangle in column y_i when going from y to x. The net effect of losing the boxes labeled with y_{i+1} and y_i becoming triangle-positive is to shift up the portion of the path y beneath the y_i

segment, and add a triangle in column y_i.

If $y_i - y_{i+1} = 1$ and $(y_i + 1, y_i)$ is not an inversion, then x and y have NEE and DE factors in the same columns. Thus, the net effect of losing the boxes labeled by y_{i+1} is to shift up the portion of the path y beneath the y_i segment.

Suppose (y_i, y_{i+1}) is not a Dyck inversion. Then, $y_i - y_{i+1}$ is necessarily equal to 2 by Lemma 3.2.7. We see that x and y have the same Dyck inversions. We note that y_i will be the 2 of a strong 312 pattern in y and not the 2 of a strong 312 pattern in x. Thus, the only change is the removal of the triangle in column y_i when going from y to x.

This completes the proof of that if $x \lessdot y$, then $p_2 \lessdot p_1$ for one of the types listed in Theorem 3.2.3. In summary, cases 2, 3, and 6 of Theorem 3.2.10 correspond with case 1 of this theorem. Case 1 of Theorem 3.2.10 corresponds with case 2 of this theorem. Case 5 of Theorem 3.2.10 corresponds with case 3 of this theorem. Case 4 of Theorem 3.2.10 corresponds with case 4 of this theorem.

We now show that if p_2 and p_1 satisfy any of the four cases, that there exists a x such that $\rho(x) = p_2$ and $x \lessdot y$. Suppose $\rho(y) = p_1$ with $y \in S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle create cover. Then by definition, c is the middle E of a EEE factor of p_1. Let d be the label of the lowest box in column c. Then (c, d) labels a stable NN factor of p_1. By Proposition 3.2.10, y has a Dyck descent (c, d) with $c - d = 1$. Now let x be the permutation where we start with y and turn the descent (c, d) into an ascent (d, c). From the first half of the proof, we showed that $\rho(x) = p_2$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle shift cover. Then by definition, c is triangle positive. Let d be the label of the lowest box in column c. Then (c, d) labels a stable NN factor of p_1. By Proposition 3.2.10, y has a Dyck descent (c, d). Now let x be the permutation where we start with y and turn the descent (c, d) into an ascent (d, c). From the first half of the proof, we showed that $\rho(x) = p_2$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle destroy cover in column $c - 1$. Then c is the E of a DE factor of p_1. By Proposition 3.2.10, y has a non-Dyck descent $(c, c-2)$. Now let x be the permutation where we start with y and turn the descent $(c, c-2)$ into an ascent $(c-2, c)$. From the first half of the proof, we showed that $\rho(x) = p_2$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle neutral cover and c is the middle E of a NEE factor. Then by definition, c is triangle-free and $M > c$. Let d be the label of the lowest box in column c. By Proposition 3.2.10, y has a Dyck descent (c, d) with $c - d \geq 2$

and c is not the 2 of a strong 312 pattern. Now let x be the permutation where we start with y and turn the descent (c,d) into an ascent (d,c). By the first half of the proof, we see that $\rho(x) = p_1$. From the first half of the proof, we showed that $\rho(x) = p_2$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle neutral cover and c is the middle E of a NEN factor. Then by definition, c is triangle-free and $M = c$. Let d be the label of the lowest box in column c. By Proposition 3.2.10, y has a Dyck descent (c,d) with $c-d \geq 2$ and c is not the 2 of a strong 312 pattern. Now let x be the permutation where we start with y and turn the descent (c,d) into an ascent (d,c). From the first half of the proof, we showed that $\rho(x) = p_2$.

Suppose that $p_2 \lessdot p_1$ with respect to a triangle neutral cover and c is the middle E of a EEN factor. Then $(c+1,c)$ is not a Dyck inversion, so in particular, is not an inversion. Let d be the label of the lowest box in column c. By Proposition 3.2.10, y has a Dyck descent (c,d) with $c-d=1$ and c is not the 2 of a strong 312 pattern. Now let x be the permutation where we start with y and turn the descent (c,d) into an ascent (d,c). From the first half of the proof, we showed that $\rho(x) = p_2$. This completes the proof of the theorem.

$\square$

3.3 Product

By [26, Corollary 1.4], we see that the product in the sub-Hopf algebra $S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ for $x \in S_u$ and $y \in S_v$ is

$$x \bullet_{Av} y = \sum \left[x y_{[u]}', \pi_{\downarrow}(y_{[u]}' x) \right]$$

where $\pi_{\downarrow} : S_n \to S_n(2\underline{41}3, 3\underline{41}2, \underline{41}23, \underline{41}32)$ is the map taking a permutation to its unique avoider in $\Theta_{2\underline{41}3,3\underline{41}2,\underline{41}23,\underline{41}32}$ and $\sum[,]$ means take the sum (formal linear combination) of all elements in the interval (of the lattice S_{u+v}).

We can use the ρ to help define a product on the Hopf Algebra of Restricted Schröder paths. Suppose that $p_1 \in \mathcal{RSP}_u$ and $p_2 \in \mathcal{RSP}_v$ with $\rho(x) = p_1$ and $\rho(y) = p_2$. Then since $\pi_{\downarrow}$ is constant on congruence classes,

$$p_1 \bullet_{RSP} p_2 = \sum [\rho(x y_{[u]}'), \rho(\pi_{\downarrow}(y_{[u]}' x))] = \sum [\rho(x y_{[u]}'), \rho(y_{[u]}' x)].$$

Theorem 3.3.1. *Suppose we want to compute the product of $p_1 \in \mathcal{RSP}_u$ and $p_2 \in \mathcal{RSP}_v$. Let p_3 be the 0th concatenation of the paths p_1 and p_2. Let p_4 be the kth concatenation of the paths if the largest east step label of p_1 is not part of a NE factor. Let p_4 be the kth concatenation of the paths plus a triangle in the rightmost column of p_1 if the largest east step label of p_1 is part of a NE factor. The product of p_1 and p_2 then is the sum over all elements of the interval $[p_3, p_4]$ in the lattice of restricted Schröder paths.*

The proof of this theorem consists of proving the following two lemmas.

Lemma 3.3.2. *The path $\rho(xy_{[u]}')$ is the 0th concatenation of $\rho(x)$ and $\rho(y)$.*

Proof. Since every entry of x is to the left of every entry of y_u, there will be no box in columns $u+1, \cdots, v$ with label less than $u+1$. In other words, the boxes in columns $1, \cdots, u$ are exactly the boxes of $\rho(x)$ with the same box labels. The boxes in columns $u+1, \cdots, v$ are in bijection with the boxes of $\rho(y)$ but with labels shifted by u. A 0th concatenation (of the inner Dyck paths) cannot create a NEE factor centered at column u. If a NEE factor centered at column $u+1$ is created, then $u(u+2)(u+1)$ forms a strong 132 pattern, so we do not add a triangle. No other strong 132 or 312 patterns are created or destroyed, so any NEE and DE factors that existed in $\rho(x)$ and $\rho(y)$ will also appear in $\rho(xy_{[u]}')$. □

Lemma 3.3.3. *Suppose $\rho(x)$ ends with k north steps. If u is not the label of the east step of a NE factor of $\rho(x)$, the path $\rho(y_{[u]}'x)$ is the kth concatenation of $\rho(x)$ and $\rho(y)$. If u is the label of the east step of a NE factor of x, the path $\rho(y_{[u]}'x)$ is the kth concatenation of $\rho(x)$ and $\rho(y)$ with the NEE factor centered at u transformed into a DE factor.*

Proof. We see that if (u, x_i) is a Dyck inversion of x, then $(\cdot, x_i)$ will be a Dyck inversion for every element of $y_{[u]}'$. Therefore, a copy of all boxes in column u will exist in all columns to the right of u. This corresponds to performing a kth concatenation. A kth concatenation (of the inner Dyck paths) cannot produce a NEE factor centered at column $u+1$. A NEE factor centered at column u is created iff u is the label of the east step of a NE factor of $\rho(x)$. If there is a NEE factor centered at column u, we see that $(u+1)(u-1)u$ forms a strong 312 pattern. No other strong 132 or 312 patterns are created or destroyed, so any NEE and DE factors that existed in $\rho(x)$ and $\rho(y)$ will also appear in $\rho(xy_{[u]}')$. This completes the proof. □

Chapter 4

Towards a Partitioning of $\Delta(\Pi_n^B)/S_n^B$

4.1 Introduction

In [16], Hersh finds a partitioning of the quotient complex $\Delta(\Pi_n)/S_n$. She does this by using a certain chain labeling (similar to a CC labeling), and then modifying it into a partitioning. We consider the analogous question of whether $\Delta(\Pi_n^B)/S_n^B$ is partitionable where Π_n^B and S_n^B denote they type B analog of the partition lattice and type B analog of the symmetric group. One way to get a partitioning would be to find a shelling. However, we show that $\Delta(\Pi_n^B)/S_n^B$ is not shellable in Theorem 4.4.2. Therefore, another method is needed. In Section 4.3, we give a combinatorial model for the cells of the quotient complex. In Section 4.3, we also describe the cover relations in the face poset of $\Delta(\Pi_n^B)/S_n^B$ with respect to this model. We hope that this is a first step towards finding a partitioning of $\Delta(\Pi_n^B)/S_n^B$. A major motivation of finding this partitioning is that it could give useful information for representation stability.

4.2 Background

4.2.1 Shellability, Partitioning, Cohen-Macaulay Properties

In this section, we describe the background related to necessary conditions for shellabilty that will be violated by our counterexample.

Definition 4.2.1 ([16, Definition 1.2]). A simplicial complex is called *shellable* if there exists a total ordering $F_1, \ldots, F_k$ on its facets (maximal faces) such that the intersection of each fixed intermediate facet F_j with the union of the closures of the previous facets, $\overline{F}_j \cap \bigcup_{i<j} \overline{F}_i$, is pure of codimension one in the intermediate facet, $\overline{F}_j$.

Definition 4.2.2. The *face poset* $F(\Delta)$ of a simplicial complex Δ is the poset of nonempty faces ordered by inclusion.

Definition 4.2.3 ([28, Section II.4])**.** A pure simplicial complex Δ is partitionable if the face poset of Δ can be written as a disjoint union

$$F(\Delta) = [G_1, F_1] \cup \cdots \cup [G_s, F_s]$$

where each F_i is a facet of Δ and each G_i is a face of Δ.

One way to show that a complex is partitionable, is to find a shelling.

Theorem 4.2.4. *If a simplicial complex is shellable, then it is partitionable.*

Shellable complexes satisfy certain necessary conditions as described below.

Definition 4.2.5. [28, Section II.4] Let Δ be a simplicial complex, then for $F \in \Delta$, define the link

$$\mathrm{lk}_\Delta F = \{G \in \Delta \mid G \cup F \in \Delta, G \cap F = \varnothing\}$$

Unless otherwise stated, the remaining theorems of this section appear in [30].

Definition 4.2.6. A simplicial complex Δ is k-connected if $\pi_r(\Delta) = 0$ for all $r \leq k$.

Definition 4.2.7. A simplicial complex Δ is homotopy Cohen-Macaulay if $\mathrm{lk}_\Delta F$ is $(\dim \mathrm{lk}_\Delta F - 1)$-connected for all $F \in \Delta$.

Definition 4.2.8. A simplicial complex Δ is k-acyclic if $\tilde{H}_r(\Delta) = 0$ for all $r \leq k$.

Definition 4.2.9. A simplicial complex Δ is Cohen-Macaulay if $\mathrm{lk}_\Delta F$ is $(\dim \mathrm{lk}_\Delta F - 1)$-acyclic for all $F \in \Delta$.

Theorem 4.2.10. *If a simplicial complex is pure and shellable, then it is homotopy Cohen-Macaulay.*

Theorem 4.2.11. *If a simplicial complex Δ is homotopy Cohen-Macaulay, then it is pure and homotopy equivalent to a wedge of top dimensional spheres, $\bigvee \mathbb{S}^{\dim \Delta}$.*

Theorem 4.2.12 ([8, Proposition 10.14]). *The link of every face of a shellable simplicial complex is shellable.*

Corollary 4.2.13. *If a simplicial complex* Δ *is homotopy Cohen-Macaulay, then for every face* F, $\mathrm{lk}_\Delta(F)$ *is pure and homotopy equivalent to a wedge of top dimensional spheres.*

Theorem 4.2.14. *If a simplicial complex* Δ *is homotopy Cohen-Macaulay, then it is Cohen-Macaulay.*

Corollary 4.2.15. *If a simplicial complex* Δ *is pure and shellable, then it is Cohen-Macaulay.*

Remark 4.2.16. *The above background is technically about simplicial complexes. However, much of it also applies to boolean complexes.*

4.2.2 Lexicographic Shellability

This section describes background for future work that may lead to a certain labeling of the desired quotient complex.

Definition 4.2.17. [16, Definition 1.3] An *EL-labelling* of a finite poset P with $\hat{0}$ and $\hat{1}$ is a labeling λ of the edges of its Hasse diagram with positive integers such that

- Each interval $[u, v]$ has a unique saturated chain

  $$u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k = v$$

  whose labels

  $$\lambda(u_0, u_1) \leq \lambda(u_1, u_2) \leq \cdots \leq \lambda(u_{k-1}, u_k)$$

  are weakly increasing.

- The weakly ascending label sequence on a saturated chain of $[u, v]$ is lexicographically smaller than all other label sequences for saturated chains on $[u, v]$.

These two conditions together constitute the *increasing chain condition*.

The following theorem was first proved in [4], but we provide a self-contained proof for completeness.

Theorem 4.2.18. *If a finite poset P with $\hat{0}$ and $\hat{1}$ has an EL-labeling, then the order complex $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$ is shellable. Moreover, every total order extension of the lexicographic order on label sequences for saturated chains from $\hat{0}$ to $\hat{1}$ is a shelling order of the associated facets of $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$. If P is graded, $\mu_P(u,v) = (-1)^{rk(v)-rk(u)}$. for each $u \le v$ in P.*

Proof. Let $F_1, \dots, F_k$ be the facet order for $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$ induced by a total order extension of the lexicographic order on label sequences for saturated chains in $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$. For any fixed $1 < j \le k$, we consider the subcomplex $C = \overline{F}_j \cap \left(\bigcup_{i<j} \overline{F}_i \right)$. We claim that C is pure of codimension one in $\overline{F}_j$.

Let σ be a face in C. If σ has codimension one in $\overline{F}_j$, then we are done. Suppose otherwise. Since

$$\sigma \in C = \overline{F}_j \cap \left(\bigcup_{i<j} \overline{F}_i \right) = \bigcup_{i<j} \left(\overline{F}_j \cap \overline{F}_i \right),$$

there exists some i such that $\sigma \in \overline{F}_j \cap \overline{F}_i$. We first consider the case that F_i and F_j differ on more than one interval. We consider the subposet $F_j \cup F_i$ in the poset P. Let a be the greatest (with respect to the poset relation) vertex in F_i such that the saturated subchain $\hat{0} \lessdot \cdots \lessdot a$ in F_i is also shared by F_j. Let b be the smallest vertex in the saturated subchain $a \lessdot \cdots \lessdot \hat{1}$ in F_i strictly greater than a that is shared by F_i and F_j. Let $F_{i'}$ be the facet obtained by combining the saturated subchain $\hat{0} \lessdot \cdots \lessdot b$ in F_i with the saturated subchain $b \lessdot \cdots \lessdot \hat{1}$ of F_j. Let $a \lessdot c$ in F_i and $a \lessdot d$ in F_j. Since $F_i <_{lex} F_j$, by definition of a, $\lambda(a \lessdot c) < \lambda(a \lessdot d)$. Since a is also the first point that $F_{i'}$ and F_j differ, by construction of $F_{i'}$, $F_{i'} <_{lex} F_j$. Additionally, since F_i and F_j were assumed to differ on more than one interval, $F_i \cap F_j \subsetneq F_{i'} \cap F_j$ (the inclusion is strict since we add the vertices of the open interval(s) other than (a,b) where F_i and F_j differ). Then, $F_{i'}$ differs from F_j only on a single interval, namely (a,b). Thus, without loss of generality, for any face $\sigma \in C$ not of codimension one, we can find an i such that $\sigma \in F_i \cap F_j$, $F_i <_{lex} F_j$,and F_i differs from F_j only on a single interval.

We consider the interval $[a,b]$ in $F_i \cup F_j$ with F_i having the properties in the previous sentence. Since $[a,b]$ lies in an EL-labelable poset, and F_j is not lexicographically first, there exists a saturated subchain $u \lessdot u' \lessdot v$ in $(a,b) \cap F_j$ such that $\lambda(u,u') > \lambda(u',v)$. Since $[u,v]$ lies in an EL-labelable poset, there exists a unique saturated chain $u \lessdot u'' \lessdot v$ such that $\lambda(u,u'') < \lambda(u'',v)$. Moreover, the definition of EL-labeling also then implies $\lambda(u,u'') < \lambda(u,u')$ since the ascending chain on (u,v) is lexicographically smallest. We recall that $F_j = \hat{0} \lessdot \cdots \lessdot u \lessdot u' \lessdot v \cdots \lessdot \hat{1}$. If we consider $F_k = \hat{0} \lessdot \cdots \lessdot u \lessdot u'' \lessdot v \cdots \lessdot \hat{1}$, then $F_k \cap F_j = \dim F_j - 1$, i.e. $F_k \cap F_j$ is a facet of F_j. Thus, $\sigma \subseteq F_i \cap F_j \subseteq F_k \cap F_j$ with $F_k <_{lex} F_j$. Since the choice of σ is arbitrary, this shows that C is pure of codimension one in F_j. Thus, $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$ is shellable, as desired.

□

Definition 4.2.19. [16, p. 228] A *chain-labeling* is a labeling of poset covering relations such that the label assigned to the covering relation $u \lessdot v$ may depend on the choice of root, namely on the saturated chain $\hat{0} \lessdot u_1 \lessdot \cdots \lessdot u_k = u$ as well as on u and v.

Definition 4.2.20. [16, p. 228] A *CL-labeling* is a chain-labeling satisfying the increasing chain condition.

The following theorem was first proved in [6], but we provide a self-contained proof for completeness.

Theorem 4.2.21. *If a finite poset P with $\hat{0}$ and $\hat{1}$ has a CL-labeling, then the order complex $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$ is shellable. Moreover, every total order extension of the lexicographic order on label sequences for saturated chains from $\hat{0}$ to $\hat{1}$ is a shelling order of the associated facets of $\Delta(P \smallsetminus \{\hat{0}, \hat{1}\})$.*

Proof. We proceed as in the EL-labeling case, by trying to find two facets that differ on a single interval. Let F_i, F_j, F_i', a, b be defined as in the EL case. Let $a \lessdot c$ in F_i and $a \lessdot d$ in F_j. Since $F_i <_{lex} F_j$, by definition of a, $\lambda(a \lessdot c, 0 \lessdot \cdots \lessdot a) < \lambda(a \lessdot d, 0 \lessdot \cdots \lessdot a)$. Since a is also the first point that $F_{i'}$ and F_j differ (with respect to vertices and edge labels), by construction of $F_{i'}$, $F_{i'} <_{lex} F_j$. Thus, without loss of generality, for any face $\sigma \in C$ not of codimension one, we can find an i such that $\sigma \in F_i \cap F_j$, $F_i <_{lex} F_j$,and F_i differs from F_j only on a single interval.

The search for the codimension one differing facet proceeds essentially the same as in the EL-case, but with chains added to labels to assure that the labels are well-defined. We consider the interval $[a, b]$ in $F_i \cup F_j$ with F_i having the desired properties. Since $[a, b]$ lies in an CL-labelable poset, and F_j is not lexicographically first, there exists a saturated subchain $u \lessdot u' \lessdot v$ in $(a, b) \cap F_j$ such that $\lambda(u \lessdot u', 0 \lessdot \cdots \lessdot u) > \lambda(u' \lessdot v, 0 \lessdot \cdots \lessdot u \lessdot u')$. Since $[u, v]$ lies in an CL-labelable poset, there exists a unique saturated chain $u \lessdot u'' \lessdot v$ such that $\lambda(u \lessdot u'', 0 \lessdot \cdots \lessdot u) < \lambda(u'' \lessdot v, 0 \lessdot \cdots \lessdot u \lessdot u'')$. Moreover, the definition of CL-labeling also then implies that $\lambda(u \lessdot u'', 0 \lessdot \cdots \lessdot u) < \lambda(u \lessdot u', 0 \lessdot \cdots \lessdot u)$ since the ascending chain on (u,v) is lexicographically smallest. We recall that $F_j = \hat{0} \lessdot \cdots \lessdot u \lessdot u' \lessdot v \cdots \lessdot \hat{1}$. If we consider $F_k = \hat{0} \lessdot \cdots \lessdot u \lessdot u'' \lessdot v \cdots \lessdot \hat{1}$, then $F_k \cap F_j = \dim F_j - 1$, i.e. $F_k \cap F_j$ is a facet of F_j. Thus, $\sigma \subseteq F_i \cap F_j \subseteq F_k \cap F_j$ with $F_k <_{lex} F_j$. Since the choice of σ is arbitrary, this shows that C is pure of codimension one in F_j. Thus, $\Delta(P \setminus \{\hat{0},\hat{1}\})$ is shellable, as desired.

$\square$

Definition 4.2.22. [16, Definition 2.3] Fix a poset chain-labeling or edge-labeling. A pair of (rooted) edges $u \lessdot v$ and $v \lessdot w$ constitute a *topological ascent* if the word consisting of two consecutive labels $\lambda(u,v)$ and $\lambda(v,w)$ is lexicographically smallest on the interval from u to w. Otherwise, the pair of covering relations comprises a *topological descent*.

Definition 4.2.23. Topological ascents with increasing consecutive labels are called *honest ascents* and those with decreasing consecutive labels are called *swap descents*. Similarly, topological descents with decreasing labels are *honest descents* and all others are called *swap ascents*.

Definition 4.2.24. [16, p. 228] A poset is *EC-shellable* (resp. *CC-shellable*) if each interval (resp. rooted interval) has a unique topologically increasing chain (i.e. a chain consisting entirely of topological ascents) and this chain is lexicographically smallest on the (rooted) interval.

The following theorem was first proved in [18].

Theorem 4.2.25. *If a finite poset P with $\hat{0}$ and $\hat{1}$ has an EC-labeling, then the order complex $\Delta(P \setminus \{\hat{0},\hat{1}\})$ is shellable. Moreover, every total order extension of the lexicographic order on label sequences for saturated chains from $\hat{0}$ to $\hat{1}$ is a shelling order of the associated facets of $\Delta(P \setminus \{\hat{0},\hat{1}\})$.*

4.2.3 Type B Specifics

Definition 4.2.26. The type B analog of the symmetric group, also called the hyperoctahedral group, is the group of permutations of $[-n,n]$ such that $\sigma(-i) = -\sigma(i)$ for all $-n \leq i \leq n$.

Definition 4.2.27. A *signed set partition* is a set partition of $[-n,n]$ such that B is a block if and only if $-B$ $(\{-x \mid x \in B\})$ is also a block.

Definition 4.2.28. The set of all signed set partitions for fixed n forms a lattice ordered by refinement (that is, $x < y$ if and only if every block of x is contained in a block of y), called the type B partition lattice, Π_n^B. The partition lattice Π_n^B is ranked with rank function n - the number of blocks.

Definition 4.2.29. Elements of Π_n^B can be compactly represented as set partitions of $[0, n]$ where elements of $[1, n]$ can have bars except elements of the block that contains 0 and the smallest element of each block. Using this convention, the cover relations of Π_n^B are described by the following three cases. Suppose that $\pi_1 \lessdot \pi_2$. Then π_2 is obtained from π_1 by merging two blocks B_1 and B_2 into a single block B. Without loss of generality, suppose $\min B_1 < \min B_2$. In the first case, if $0 \in B$, then B is the union of B_1 and B_2 with all bars removed. If $0 \notin B$, then there are two cases. The second case is that B is the union of B_1 and B_2 with all bars intact. The last case (when $0 \notin B$) is that B is the union of B_1 and $\overline{B_2}$ (that is the block obtained by barring all unbarred elements and unbarring all barred elements). The hyperoctahedral group S_n^B of signed permutations acts naturally on the elements of Π_n^B.

Definition 4.2.30. The *type B braid arrangement* (or *type B Coxeter arrangement*) consists of the hyperplanes,

$$H_{i,j}^{+} = \{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}$$

and

$$H_{i,j}^{-} = \{\mathbf{x} \in \mathbb{R}^n \mid x_i = -x_j\}$$

and

$$H_i = \{\mathbf{x} \in \mathbb{R}^n \mid x_i = 0\}$$

for $1 \leq i < j \leq n$.

The following appears in [30].

Theorem 4.2.31. *The intersection lattice of the type B braid arrangement is isomorphic to the type B partition lattice.*

Fact 4.2.32. The action of S_n^B is rank-preserving and order-preserving.

Proof. Since signed permutations are in particular bijections, they will preserve block sizes, so in particular preserve the number of blocks, which gives rank-preservation. We verify order-preservation by going through the three cases of cover relations listed above. Let $\pi_1 \lessdot \pi_2$ with B, B_1, B_2 using the notation above. If $0 \in B$, then one of B_1, B_2 had to contain 0. Since every signed permutation sends 0 to 0, $g(B)$ will be the zero block of $g(\pi_2)$ (created by merging $g(B_1)$ and $g(B_2)$), thus the comparability is preserved in this case. If B is created by a simple merge of B_1, B_2, then $g(B)$ will be a simple merge of B_1 and B_2. Thus the comparability is preserved in this case. Similarly, if B is created by a merge of B_1 and $\overline{B_2}$, then $g(B)$ will be a merge of $g(B_1)$ and $g(\overline{B_2})$. Thus the comparability is preserved in this case. □

Definition 4.2.33. Let $\pi_1 < \pi_2 < \cdots < \pi_k$ be an element of the kth chain group of $\Delta(\Pi_n^B)$. We recall that the simplicial boundary map ∂ takes a k-chain and produces the alternating sum of the $(k-1)$-chains obtained by removing exactly one element:

$$\partial_k(\pi_1 < \pi_2 < \cdots < \pi_k) = \sum_i (-1)^i (\pi_1 < \pi_2 < \cdots < \hat{\pi}_i < \cdots < \pi_k)$$

Fact 4.2.34. The S_n^B-action on chains in Π_n^B commutes with the simplicial boundary map.

Proof. Let $\pi_1 < \pi_2 < \cdots < \pi_k$ be a generator of the kth chain group of $\Delta(\Pi_n^B)$. Let $g \in S_n^B$. Then, we find that

$$\begin{aligned}
\partial_k(\pi_1 < \pi_2 < \cdots < \pi_k) &= \sum_i (-1)^i (\pi_1 < \pi_2 < \cdots < \hat{\pi}_i < \cdots < \pi_k) \\
g(\partial_k(\pi_1 < \pi_2 < \cdots < \pi_k)) &= g\left(\sum_i (-1)^i (\pi_1 < \pi_2 < \cdots < \hat{\pi}_i < \cdots < \pi_k)\right) \\
&= \sum_i (-1)^i (g(\pi_1) < g(\pi_2) < \cdots < \widehat{g(\pi_i)} < \cdots < g(\pi_k)) \\
&= \partial_k(g(\pi_1 < \pi_2 < \cdots < \pi_k))
\end{aligned}$$

The result follows by extending linearly on a basis of generators. □

Example 4.2.35. Consider $\{0,2,5,-2,-5\}\{1,-7,9\}\{-1,7,-9\}\{3,4,-6,-8\}\{-3,-4,6,8\}$, which is an element of Π_9^B. It's corresponding intersection of hyperplanes is $\{\mathbf{x} \in \mathbb{R}^9 \mid x_2 = x_5 = 0, x_1 = -x_7 = x_9, x_3 = x_4 = -x_6 = -x_8\}$. It's compact representation is $025 \mid 1\bar{7}9 \mid 34\bar{6}\bar{8}$.

4.3 A combinatorial model of the elements of $\Delta(\Pi_n^B)/S_n^B$

The order complex $\Delta(\Pi_n^B)$ as defined in the previous section, consists of chains of successively coarser signed set partitions. We denote the dual poset to Π_n^B by $(\Pi_n^B)^*$. We utilize the fact that $\Delta(\Pi_n^B) = \Delta((\Pi_n^B)^*)$. We find the description of faces resulting from the $\Delta((\Pi_n^B)^*)$ viewpoint to be more convenient. We recall that $\Delta((\Pi_n^B)^*)$ consists of successively finer signed set partitions.

We will find it easier to consider $\Delta\left(\Pi_n^B \smallsetminus \{\hat{0}, \hat{1}\}\right)$. Dropping the order complex cone points of $\hat{0}$ and $\hat{1}$ will not affect whether or not a particular order of the facets is a shelling order. Therefore, we can figure out if $\Delta(\Pi_n^B)$ is shellable by figuring out if $\Delta(\Pi_n^B) \smallsetminus \{\hat{0}, \hat{1}\}$ is shellable. In order to prove non-shellability, we will need a model for the chain orbits of $\Delta(\Pi_n^B)/S_n^B$. Since signed set partitions are sign-symmetric, we will model the vertices as simply set partitions of $n+1$ elements with one block designed the zero block by using the appropriate elements of S_n^B to remove any bars. Hultman develops a model for chain orbits of $\Delta(\Pi_n^B)/S_n^B$ using certain labeled trees [17].

Definition 4.3.1. Let $\hat{0} < \pi_1 < \cdots < \pi_k$ be an element of $\Delta(\Pi_n^B)$. We recall each vertex π_i of this chain is modeled by a set partition of $n+1$ elements with a special zero block (that we mark with a dot). We can consider the *type* of each vertex to be the (integer) partition consisting of the cardinalities of the blocks. By convention, we mark the zero block with a dot above it, and record its cardinality as 1 less than the number of elements in it.

Definition 4.3.2. The *Hultman tree* representing the orbit of the chain $\hat{0} < \pi_1 < \cdots < \pi_k$ is built in the following way. We start with the root labeled with $\text{type}(\hat{0}) = \dot{n}$. The nodes of the tree at depth i will be labeled by the cardinalities of $\text{type}(\pi_i)$. We connect nodes at depth i to nodes at depth $i-1$ based on if the corresponding blocks in π_i are built by splitting corresponding blocks in π_{i-1}.

Hultman's above description creates many equivalent tree representatives for the same chain orbit. We will find it convenient to order the children of each tree node in a systematic way to create a unique representative. We note that the cardinality of a block is the number of balls in that block. If a node is not the block with the dotted ball, simply order the children from smallest to largest cardinality from left to right. If the node is

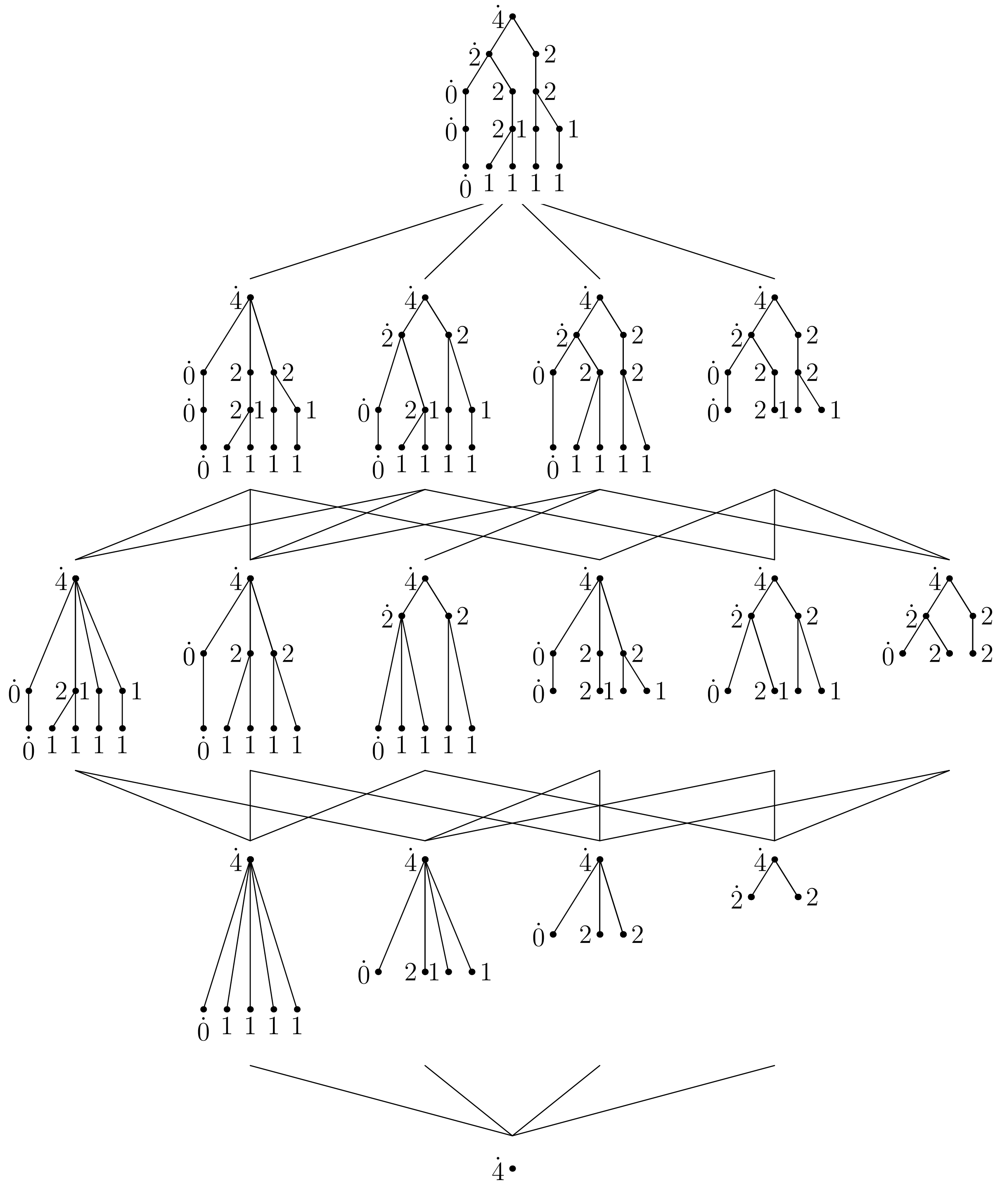


Figure 4.1: The faces in the closure of a facet of $\Delta(S_4^B)/\Pi_4^B$ in the tree model convention.

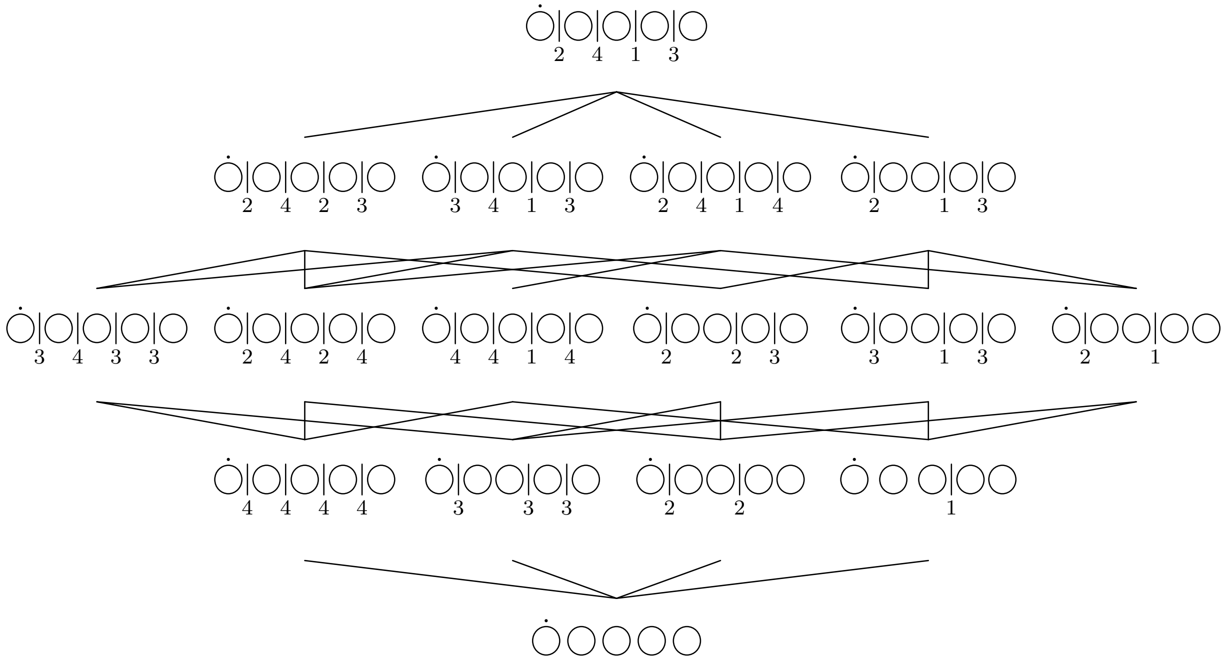


Figure 4.2: The faces in the closure of a facet of $\Delta(S_4^B)/\Pi_4^B$ in the ball and splitter model convention.

part of a zero block, order its unique child that is also part of a zero block leftmost, and then order the remaining children from smallest to largest cardinality from left to right. Hultman shows that these planar embeddings of trees are in bijection with the chain orbits of $\Delta(\Pi_n^B)/S_n^B$ [17, Proposition 5.2]. For each tree, we can build a certain arrangement of balls and labeled splitters as follows.

Definition 4.3.3. Let $\hat{0} < \pi_1 < \cdots < \pi_k$ be an element of $\Delta(\Pi_n^B)$. The ball and splitter model representing the orbit of the chain $\hat{0} < \pi_1 < \cdots < \pi_k$ is built in the following way. We start with $n+1$ balls in a row. We note that each element of π_i can be modeled using a set partition of $[0,n]$. We recall that the poset rank of an element of $\pi_i \in \Pi_n^B$ is the number of set partition blocks minus one. For $1 \le i \le k$, if the rank of π_i is r, we add new splitters such that the cardinalities of the blocks of π_i are equal to the number of balls in each block. Additionally, we label the splitters inserted at rank r to create these blocks with r.

There is a map giving a natural correspondence between Hultman's tree model and the ball and splitter model. These maps will allow us to deduce the equivalence of the two different models for quotient cells.

Definition 4.3.4. Let f be the following map from the set of tree model representatives to the set of ball and splitter model representatives. Start with one of Hultman's trees that is a representative of one of the cells of $\Delta(\Pi_n^B)/S_n^B$ as in the Definition 4.3.4. Let the *rank* of a node in the tree be one less than the number of nodes at the same depth as the given node. Let d be the maximum depth among nodes in the tree.

- If the root is labeled with $\dot{n}$, start with $n+1$ balls drawn in a row.
- For depth i from 1 to d:
  - For each node with label m at depth i:
    1. Go to the corresponding block of m balls (or $m+1$ balls if m is dotted).
    2. For each child (with label ℓ) of the node, create a new block of size ℓ (or $\ell+1$ if the block is dotted).
    3. Add a new splitter labeled with the rank of the child to the right of the new block.

See Figures 4.1 and 4.2 for many examples.

Definition 4.3.5. Let g be the following map from the set of ball and splitter model representatives to the set of tree model representatives. Start with a representative of a cell encoded using the ball and splitter model. Suppose r_i is the ith smallest splitter label for label set $r_1 < \cdots r_i \cdots r_d$.

- If there are $n+1$ balls, create a node and label it with $\dot{n}$
- For i from 1 to d:
  - For each block B of size m (or $m+1$ if B has the dotted ball) between splitters labeled with r_i:
    1. Construct a node N labeled with m (or $m+1$ if B has the dotted ball) at depth i of the tree.
    2. Identify the unique block B' at depth $i-1$ that refines B, and its corresponding node N' in the tree.
    3. Let the parent of N be N'.

See Figures 4.2 and 4.1 for many examples.

By construction, we see that $f \circ g$ is the identity map on the set of distinguished representatives of ball and splitter models and $g \circ f$ is the identity map on representatives of tree models. By constructing the ball and labeled splitter arrangement corresponding to the unique tree representative of a given chain orbit, we get the following theorem.

Theorem 4.3.6. *For each chain orbit arising as a cell of $\Delta(\Pi_n^B)/S_n^B$, there exists a well-defined chain orbit representative using balls and marked splitters obtained as follows. The labels on the splitters are the ranks at which we split blocks as we go up a chain in the orbit. By convention, we put the zero block is as far left as possible and each splitter as far left as possible among equivalent choices (in this chain orbit) as we go up the chain.*

In order to describe closure relations on cells, it will be necessary to consider other representatives of chain orbits. Hultman describes the closure relations on cells in terms of maps δ_i that comprise the unsigned components of the boundary map. Specifically, give a Hultman tree representative, the map δ_i deletes all the nodes at depth i, and connects the parents of the deleted nodes to their grandchildren [17, Proposition 5.2]. A cover relation $c_1 \lessdot c_2$ in the face poset is satisfied if and only if there is a tree representative t_1 of c_1 and a tree representative t_2 of c_2 such that $\delta_i(t_2) = t_1$ for some i. Compare nodes and their parents in Figure 4.1 for many examples. By translating [17, Proposition 5.2] to the splitter model, we get the following theorem.

Theorem 4.3.7. *Consider the distinguished representative of a fixed chain orbit $c \in \Delta(\Pi_n^B)/S_n^B$ as in Theorem 4.3.6. Suppose the splitters of c have labels from exactly the set $\{r_1, \cdots, r_k\}$ with $r_1 < \cdots < r_k$. Then c covers k other chain orbits, one orbit for each possible vertex that can be eliminated from a fixed chain in the orbit. More specifically, if we omit the vertex at rank r_k, c covers the orbit representative modeled by starting with the representative of c, and removing all splitters labeled with r_k. The chain orbit c also covers the chain orbits modeled by starting with the representative c, fixing a nonmaximal rank r_i, and relabeling all splitters labeled r_i with r_{i+1}.*

All closure relations on cells are found by taking the transitive closure of the cover relations specified in the previous theorem. In other words, f and g (as in Definitions 4.3.4 and 4.3.5) are order preserving. See Figures 4.1 and 4.2 for an example.

4.4 An obstruction to shelling $\Delta(\Pi_n^B)/S_n^B$

We now construct the promised counterexample.

Example 4.4.1. This is inspired by a similar example in type A appearing as Example 4.3 in [16]. We consider the following chain in Π_8^B (written using the notation of Definition 4.2.29),

$$012345678 > 0|12345678 > 0|1234|5678 > 0|1|234|5|678 > 0|1|2|34|5|6|78 > 0|1|2|3|4|5|6|7|8$$

The link of this face in $\Delta(\Pi_8^B)$ consists of the following six vertices

$$0 \underset{3}{|} 1234 \underset{3}{|} 5 \underset{3}{|} 678, \quad 0 \underset{3}{|} 1 \underset{3}{|} 234 \underset{3}{|} 5678, \quad 0 \underset{5}{|} 1 \underset{5}{|} 234 \underset{5}{|} 5 \underset{5}{|} 6 \underset{5}{|} 78,$$

$$0 \underset{5}{|} 1 \underset{5}{|} 2 \underset{5}{|} 34 \underset{5}{|} 5 \underset{5}{|} 678, \quad 0 \underset{7}{|} 1 \underset{7}{|} 2 \underset{7}{|} 3 \underset{7}{|} 4 \underset{7}{|} 5 \underset{7}{|} 6 \underset{7}{|} 78, \quad 0 \underset{7}{|} 1 \underset{7}{|} 2 \underset{7}{|} 34 \underset{7}{|} 5 \underset{7}{|} 6 \underset{7}{|} 7 \underset{7}{|} 8$$

with the following eight facets:

$$0 \underset{3}{|} 1 \underset{3}{|} 2 \underset{7}{|} 34 \underset{3}{|} 5 \underset{5}{|} 6 \underset{5}{|} 7 \underset{7}{|} 8, \quad 0 \underset{3}{|} 1 \underset{3}{|} 2 \underset{7}{|} 3 \underset{7}{|} 4 \underset{3}{|} 5 \underset{5}{|} 6 \underset{5}{|} 78,$$

$$0 \underset{3}{|} 1 \underset{3}{|} 2 \underset{5}{|} 3 \underset{7}{|} 4 \underset{3}{|} 5 \underset{5}{|} 6 \underset{7}{|} 78, \quad 0 \underset{3}{|} 1 \underset{3}{|} 2 \underset{5}{|} 34 \underset{3}{|} 5 \underset{5}{|} 6 \underset{7}{|} 7 \underset{7}{|} 8,$$

$$0 \underset{3}{|} 1 \underset{5}{|} 2 \underset{5}{|} 34 \underset{3}{|} 5 \underset{3}{|} 6 \underset{7}{|} 7 \underset{7}{|} 8, \quad 0 \underset{3}{|} 1 \underset{5}{|} 2 \underset{5}{|} 3 \underset{7}{|} 4 \underset{3}{|} 5 \underset{3}{|} 6 \underset{7}{|} 78,$$

$$0 \underset{3}{|} 1 \underset{5}{|} 2 \underset{7}{|} 3 \underset{7}{|} 4 \underset{3}{|} 5 \underset{3}{|} 6 \underset{5}{|} 78, \quad 0 \underset{3}{|} 1 \underset{5}{|} 2 \underset{7}{|} 34 \underset{3}{|} 5 \underset{3}{|} 6 \underset{5}{|} 7 \underset{7}{|} 8,$$

and with the addition of 12 1-faces, we see that the link is the sphere S^2. That is, if we arrange the faces into a two dimensional CW-complex in the right way, we see it is homeomorphic to S^2 as in Figure 4.3. Now we consider looking at one half of the sphere with center being the vertex $0 \underset{7}{|} 1 \underset{7}{|} 2 \underset{7}{|} 3 \underset{7}{|} 4 \underset{7}{|} 5 \underset{7}{|} 6 \underset{7}{|} 78$ as in Figure 4.4. Similarly, we can look at the opposite half of the sphere with center being the vertex $0 \underset{7}{|} 1 \underset{7}{|} 2 \underset{7}{|} 34 \underset{7}{|} 5 \underset{7}{|} 6 \underset{7}{|} 7 \underset{7}{|} 8$ as in Figure 4.5. We notice that under the group action by S_8^B, that these two center vertices are in the same orbit. Specifically, the group element $(3\ 7)(4\ 8)(1\ 5)(2\ 6)(-3\ -7)(-4\ -8)(-1\ -5)(-2\ -6)$ sends one center to the other. If we consider each face in the Figure 4.4, we find that the corresponding antipodal face in

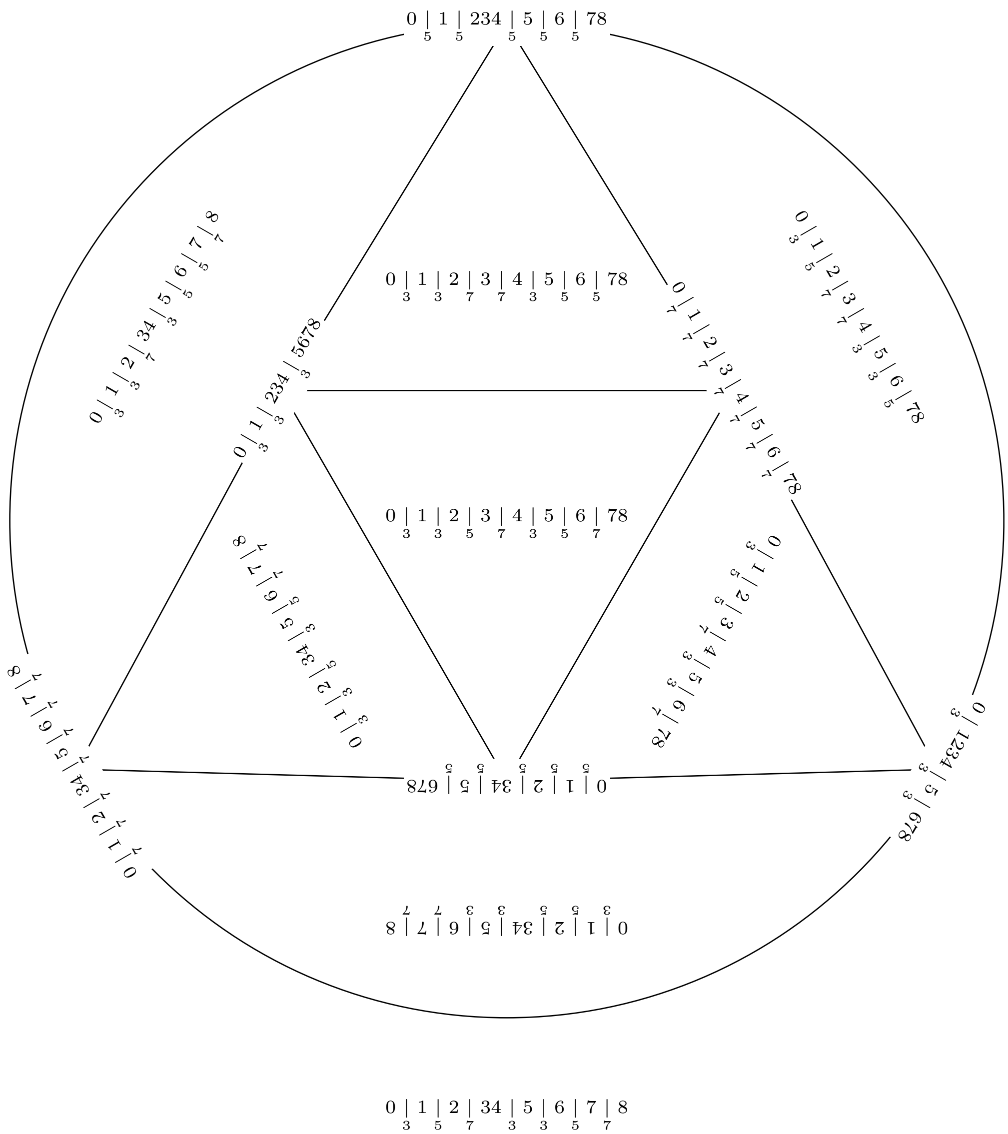


Figure 4.3: The link of the chain of Example 4.4.1.

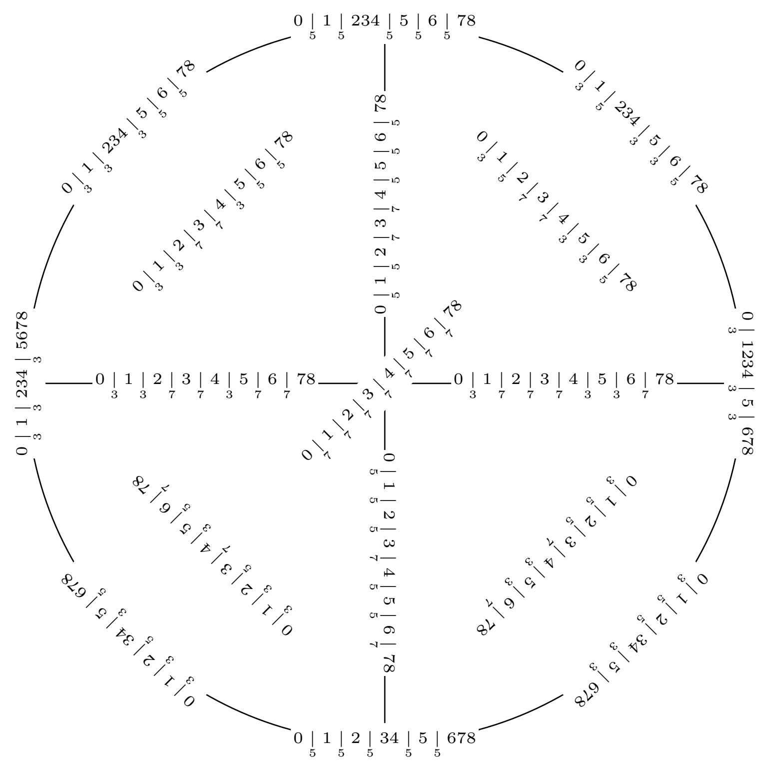


Figure 4.4: A perspective of the link of the face of Example 4.4.1 where examine the half of S_2 with center given by vertex $0 \underset{7}{\Big|} 1 \underset{7}{\Big|} 2 \underset{7}{\Big|} 3 \underset{7}{\Big|} 4 \underset{7}{\Big|} 5 \underset{7}{\Big|} 6 \underset{7}{\Big|} 78$.

Figure 4.5 is equivalent under the quotient by the group action (in fact, by the earlier group element). In other words, the net effect of acting on this link by S_8^B, is to apply the antipodal map to S_2. Thus, the link of this face in $\Delta(\Pi_8^B)/S_8^B$ is $\mathbb{RP}_2$.

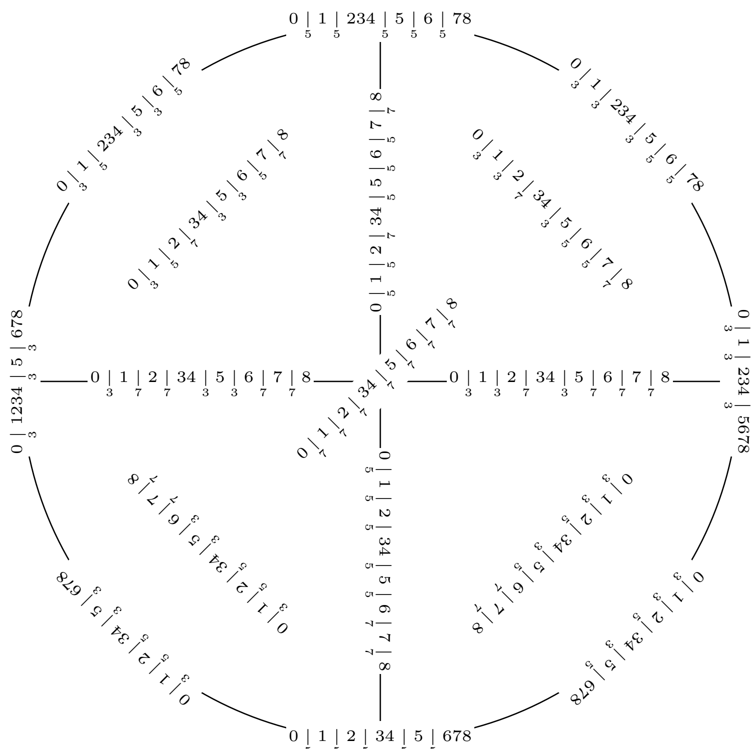


Figure 4.5: A perspective of the link of the face of Example 4.4.1 where we look at the half of S_2 with center given by vertex $0 \underset{7}{|} 1 \underset{7}{|} 2 \underset{7}{|} 34 \underset{7}{|} 5 \underset{7}{|} 6 \underset{7}{|} 7 \underset{7}{|} 8$

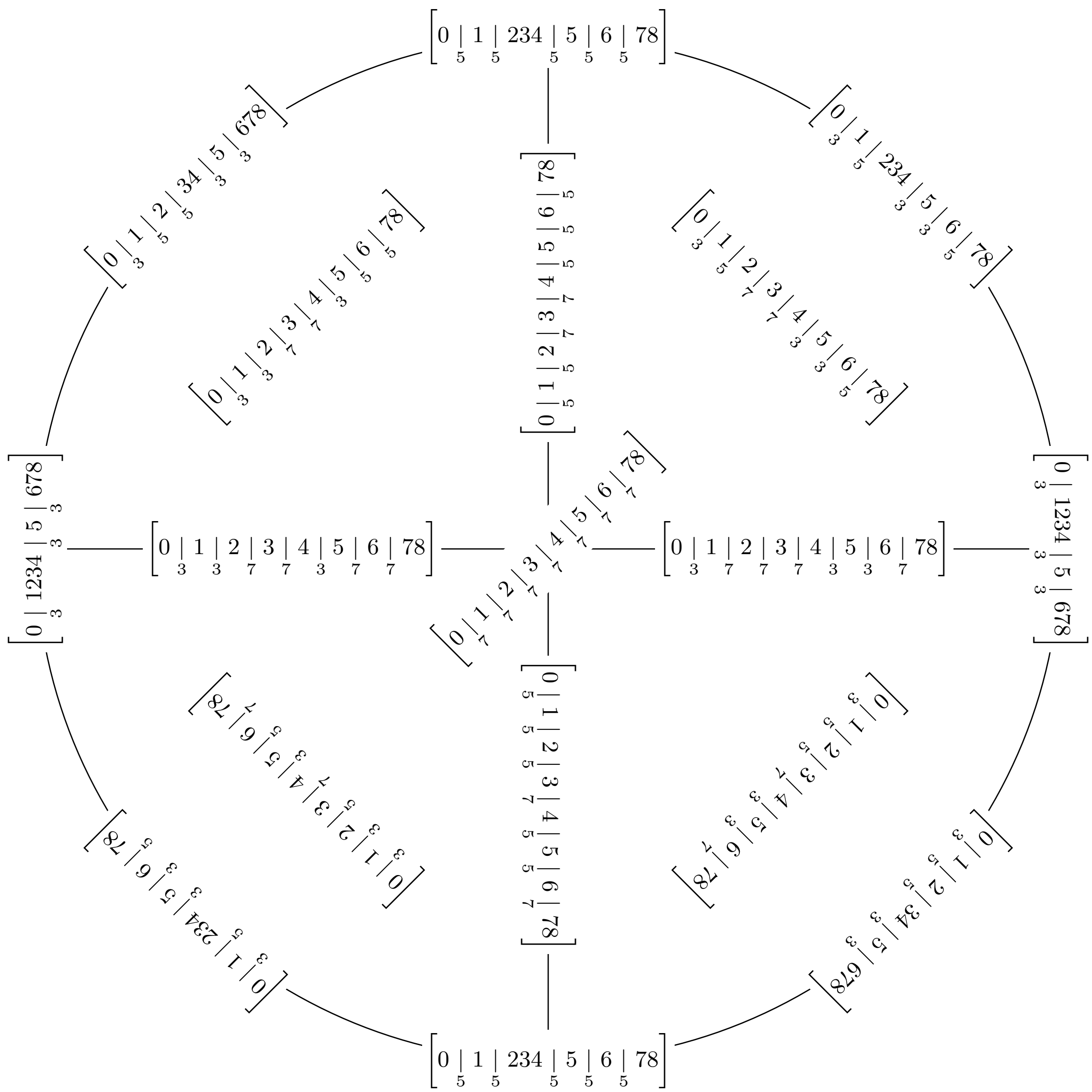


Figure 4.6: The chain orbit of the action of S_8^B on the link of the face from Example 4.4.1. Close analysis of the picture reveals the chain orbit is homeomorphic to $\mathbb{RP}_2$.

Theorem 4.4.2. *The quotient complex $\Delta(\Pi_n^B)/S_n^B$ is not shellable for $n \geq 8$.*

Proof. Consider Example 4.4.1. Since $\mathbb{RP}_2$ has nontrivial homology at dimension 1 (see e.g. [15]), the link of this face is not 1-acyclic, thus we conclude that $\Delta(\Pi_8)/S_8^B$ is not Cohen-Macaulay. By Theorems 4.2.10 and 4.2.14, this implies that $\Delta(\Pi_8)/S_8^B$ is not shellable.

We can extend Example 4.4.1 to an example for $n > 8$. We construct the example by doing the following. First, for each set partition of Π_8^B in the example, we can build a set partition of Π_n^B, by letting i be in the same zero block for all $n \geq i > 8$. Then we iteratively split off the smallest element in the zero block. That is, the chain will look like:

$$012345678\cdots n > 09\cdots n|12345678 > 09\cdots n|1234|5678 > 09\cdots n|1|234|5|678 > 09\cdots n|1|2|34|5|6|78$$

$$> 09\cdots n|1|2|3|4|5|6|7|8 > 0|9\cdots n|1|2|3|4|5|6|7|8|9 > \cdots > 0|9|10\cdots n|1|2|3|4|5|6|7|8|$$

$$> 0|9|10|\cdots|n|1|2|3|4|5|6|7|8|$$

Thus, we get a face of codimension 3 in $\Delta(\Pi_n^B)$. Observe that this face has the same link as the face of Example 4.4.1 because the chain also skips six vertices, twelve edges, and eight faces at the right ranks. These can be arranged in a CW-complex homeomorphic to S^2 as in Figure 4.3. The group element $(3\ 7)(4\ 8)(1\ 5)(2\ 6)(-3\ -7)(-4\ -8)(-1\ -5)(-2\ -6)$ considered as an element of S_n^B for any $n \geq 8$ still plays the role of identifying antipodal faces as equivalent under the group action of S_n^B.

We consider the permutations of S_n^B written in disjoint cycle notation (see [5, Chapter 8] for details). We see that among faces in the link, that $\{1, 2, \cdots, 8\}$ are the only nonnegative elements in nonzero blocks, and $\{0, 9, \cdots, n\}$ are the only nonnegative elements in zero blocks. Therefore, any nontrivial cycle that is part of a permutation sending one link face to another cannot contain both nonnegative elements from zero blocks and nonzero blocks. Therefore, we see that the only potential permutations that can identify two faces as equivalent in the link are permutations of the set $[-8, 8] \setminus \{0\}$. But all permutations of the set $[-8, 8] \setminus \{0\}$ that identify equivalent faces have already been accounted for in Example 4.4.1. Therefore, every quotient face in this link also has only two elements.

Thus the action of S_n^B on the link still produces $\mathbb{RP}_2$. Therefore, we have constructed a copy of a face whose link is $\mathbb{RP}_2$ in $\Delta(\Pi_n^B)/S_n^B$ for $n \geq 8$. $\square$

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