ABSTRACT

WAN, CHAO. Vibration-based Damage Detection Using High-speed Camera. (Under the direction of Dr. Fuh-Gwo Yuan).

Structural health monitoring (SHM) has received increasing attention in the mechanical, aerospace and civil engineering fields over the past decades. There are growing challenges in monitoring and maintaining structures that are near or beyond their design lifetimes. Since the increasing age of existing large structures such as aircrafts, bridges and buildings make the cost of replacement and maintenance a growing concern, the SHM technology has played an important role in the society. A SHM system can facilitate the detection and characterization of damage in a structure at an early stage so that repairs can be implemented to minimize downtime and operational costs. It could ensure increased safety and reliability while reducing maintenance costs.

In this dissertation, a method for damage detection in beam structures using high-speed camera is presented. The existence of damage in a structure causes changes of the vibration parameters of the system, such as stiffness, natural frequencies and mode shapes. Therefore, the vibration measurement of a structure can be used to extract structural modal properties and thereby makes damage detection feasible. Traditional methods of vibration measurement in structures typically involve contact (i.e., accelerometer) or non-contact sensors (i.e., laser vibrometer or capacitive sensor) which can be costly and time consuming to inspect an entire structure. With the advances in digital camera and the development of computer vision technology, video cameras offer a viable capability of measurement including higher spatial resolution, remote sensing and low-cost. In the study, a vibration measurement system based on the computer vision is proposed. The system uses a high-speed camera to capture the vibration motion of cantilever beams. The cantilever beams with artificial cracks are excited by an impact
hammer. The Lucas–Kanade tracking method is used to extract the vibration displacement for the modal identification of the beams. In this study, different cantilever beam structures are tested in the proposed vision-based damage detection system. Finite element analysis and experiments are used to validate the proposed method. Noise effect analysis is also investigated. Suggestions for applications of this methodology and future work are discussed.
Vibration-based Damage Detection Using High-speed Camera

by
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DEDICATION

To my parents, parents-in-law

and my wife, Yanqian Wang
BIOGRAPHY

Chao Wan was born on March 17, 1987 in Kunming, Yunnan province, China. He attended Beijing Institute of Technology during 2005-2009 and obtained his B.S. degree in Automation. After graduation, he worked as a research assistant in the Robotics and Automation Laboratory of Tsinghua University for two years. In November 2011, he moved to Raleigh, Carolina and spent two years as a visiting scholar in the Smart Structures and Materials Laboratory of North Carolina State University. He enrolled at North Carolina State University as a doctoral student of Mechanical Engineering in the spring of 2014, working toward his Ph.D. degree in the area of structural health monitoring.
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1 Introduction

1.1 Structural Health Monitoring (SHM)

Structural health monitoring (SHM) refers to the process of implementing a damage identification strategy for aerospace, civil and mechanical engineering infrastructure. This process involves the observation of a structure or mechanical system over time using periodically spaced measurements, the extraction of damage-sensitive features from these measurements and the statistical analysis of these features to determine the current state of system health\(^1\).

Over the past decades, the safety of structures has been the main consideration for the aging bridges, buildings, aircrafts or other aging infrastructures. The development of structural health monitoring technology has provided the capability for monitoring structures in real time and long term. The application of SHM can help prolong the life of structure by detecting the damage at an early stage. And structural repair and improvement can be made in time.

Traditional method, such as visual inspection which is a common process in civil engineering, it is usually labor-intensive and time consuming. For some cases, disassembly of certain part of the structure is required. On the contrary, SHM system can be installed near the important load-bearing element of structure in advance and provide real-time feedback about the structural health status.

Farrar et al\(^2\). defined the SHM process in terms of a four-step statistical pattern recognition paradigm. The four-step process includes:

(1) Operational evaluation;

(2) Data acquisition, normalization and cleansing;

(3) Feature selection and information condensation;

(4) Statistical model development for feature discrimination;
Operational evaluation attempts to set the limitations on what will be monitored and how the monitoring will be accomplished. The data-acquisition step involves selecting the types of data to be acquired and defining the data to be used (or not used) in the feature selection process. Feature selection process attempts to identify the data features that allow one to distinguish between the undamaged and damaged structure. The best features for damage detection are typically application specific. Statistical model development step involves developing statistical models to enhance damage detection\(^2\). Figure 1.1 shows the flow chart for implementing a structural health monitoring process\(^2\).

Figure 1.1. The flow chart for implementing a structural health monitoring process\(^2\)
The definition of damage will be limited to changes to the material and/or geometric properties of the system, including changes to the boundary conditions and system connectivity, which can adversely affect the current or future performance of the system\textsuperscript{1}. The effects of damage on a structure can be classified as linear or nonlinear. For linear damage situation, the structure remains linear-elastic after damage. There are changes in the modal properties caused by the changes in the geometry or the material properties of the structure. But the structural response can still be modeled using linear model. Linear methods can be further classified as model-based and non-model based. The model-based method can use finite element analysis or analytic model to provide accurate structural response.

For the nonlinear damage, the structure behaves in a nonlinear manner after the damage has been introduced. The formation of a fatigue crack in the structure under normal operating environment is one example of nonlinear damage. Other examples include loose connections and nonlinear material behavior that exhibited by polymer. The majority of the studies focus on the problem of linear damage detection\textsuperscript{3}.

For the different types of damage identification, Rytter\textsuperscript{4} has proposed a four-level classification method as shown in Figure 1.2. Level I gives the information that whether the damage is existent in the system, Level II provides the locations of the damage and Level III can evaluate the damage extent, such as damage size, stiffness reduction or crack length. Level IV is the most challenging one and needs to predict the remaining lifetime of the structure. It requires the combination of global structural model with local fracture mechanics model to prognosis the evolution of the damage.
Figure 1.2. The four diagnostic level for damage assessment\textsuperscript{4}
1.2 Vibration-based Damage Detection

Damage identification can be divided into two groups: local and global methods. Local methods usually detect the damage in a small and local region. They usually have high spatial resolution for the damage. The typical local methods include the sensor using ultrasonic waves, magnetic fields, eddy-current and thermal fields. For this method, dense sensor network is usually required. However, the local methods are very sensitive to small defects.

On the contrary, the global method is based on the global behavior of the structure. When a reduction of stiffness happened in the structure, it can cause a decrease of natural frequencies, which is a global property of the structure. These methods are usually based on the vibration measurement. The changes in the natural frequencies, modal damping ratios and mode shapes can be the indicators for the damage identification. For the global methods, the sensor network can be coarser than the local method. And the sensor does not need to be placed near the damage location. But the sensitivity of the global methods is usually lower than the local methods, they have lower spatial resolution.

Vibration based damage identification methods that do not combine with the structural model primarily provide Level I and Level II damage identification. When the structural model is used, Level III damage detection can be achieved. Level IV identification must be combined with fracture mechanics and fatigue-life analysis.

For the vibration-based damage detection, there are many different methods based on different dynamic parameters, such as natural frequency, mode shape, mode shape curvature, frequency response function curvature, flexibility and model strain energy.
1.3 Vision-based Vibration Measurement

With the rapid development of computer vision technology, computer vision has been receiving more and more attention in different fields. Computer vision-based technology have been proved to be a potential tool for the dynamic and static displacement measurement.

Compared with traditional methods, vision-based methods have the advantages of non-contact and low-cost. At the same time, they also offer the potential of both high spatial and temporal measurement resolution. Over the past decades, vision-based methods have been developed for vibration displacement measurement for the structure. For example, the technique called digital image correlation (DIC)\(^22,23\) has been commonly used as a practical and effective tool in the field of experimental mechanics such as material mechanical testing and structural stress analysis. This method can provide full-field displacement and strain measurement with sub-pixel accuracy. But in order to achieve reliable and accurate analysis, artificial speckle or texture patterns are often required to be applied on the specimen surface\(^24,25\). Wahbeh et al.\(^26\) developed a vision-based approach which combined a camera with two high-resolution light-emitting diodes (LEDs) on the target for the displacement measurement of a bridge. Lee et al.\(^27\) proposed a vision-based system for remote sensing of bridge displacement. In the system, target panel was mounted on the measurement point. By measuring the movement of the target panel in the video images, the displacement of the measurement point can be extracted in real-time. Guo and Zhu\(^28\) proposed a modified Lucas–Kanade tracking algorithm for the vibration displacement measurement. The method can achieve vibration measurement without pre-designed target. By replacing the interpolation step with a rounding-off operation in the original Lucas–Kanade algorithm, the proposed method can measure vibration displacement with high time efficiency. Feng et al.\(^29\) measured the structural displacements based on a subpixel template matching
method which can improve the measurement accuracy. Chen et al.\textsuperscript{30} used phase-based optical flow method to extract different operational deflection shapes of the measured structure. These researches have demonstrated the potential for measuring structural vibrations using vision-based methods.

1.4 Overview of This Research

The objective of this research is to develop an experimental system based on computer vision algorithm for structural vibration measurement for the purpose of damage detection on the structure. This system has several advantages such as non-contact, low cost, sub-pixel resolution, efficient and high spatial (multiple points measurement) and temporal resolution.

This thesis chapters are organized as follows:

**Chapter 1:** General background information about structural health monitoring, vibration-based damage detection and vision-based vibration measurement, research objective and thesis organization.

**Chapter 2:** An introduction and description of the traditional vibration measurement methods. These methods can be divided into two groups: contact methods and non-contact methods. For contact methods, they include potentiometric sensor, piezoelectric accelerometer and linear variable differential transformer (LVDT). For non-contact methods, they contain capacitive displacement sensor, inductive displacement sensor and laser doppler vibrometer (LDV). For each method, its working principle, applied environment and pros and cons will be discussed.
Chapter 3: An introduction and discussion of the computer vision-based displacement measurement algorithms. These algorithms include optical flow method, digital image correlation method, Lucas-Kanade method and Lucas-Kanade inverse compositional method. Their working principles, advantages and limitations will be investigated.

Chapter 4: Deals with the transverse vibration of the cantilever beam. The derivation for the equation of motion of the beam based on Euler–Bernoulli theory will be introduced. The natural frequencies and mode shapes of the beam are calculated. The modeling of a cantilever beam with single crack is give. Finally, the finite element method for the beam will be discussed.

Chapter 5: Several vibration-based damage detection methods will be reviewed and discussed. Including natural frequency method, mode shape method, mode shape curvature method, flexibility matrix method, frequency response function (FRF) curvature method, model strain energy method and model strain-based method.

Chapter 6: Different cantilever beam structures will be tested in the proposed vision-based damage detection system. The system setup, finite element analysis and experiment results for three different cases (aluminum cantilever beam with single damage, aluminum cantilever beam with multiple damages, wood cantilever beam with single damage) will be discussed. Noise effect analysis will also be given.

Chapter 7: Summary and conclusion of the presented work. Discuss possible directions for the future research on the vision-based damage detection.
2 Traditional Vibration Measurement Methods

Traditional vibration measurement methods can be divided into two groups: contact methods and non-contact methods. For contact methods, they include potentiometric sensor, piezoelectric accelerometer and linear variable differential transformer (LVDT). For non-contact methods, they contain capacitive displacement sensor, inductive displacement sensor and laser doppler vibrometer (LDV). In this chapter, these methods' working principle, applied environment, advantages and limitations will be introduced and investigated.

2.1 Contact Methods

2.1.1 Potentiometric Sensor

Potentiometric sensor or resistive position sensor is a very popular, relatively cheap sensor which is used in many industry applications. The sensor is widely used in the nondigital versions of the volume and tone controls on many audio and video electronic devices.

The structure of potentiometric sensor is simple and easy to understand. As shown in Figure 2.1. A voltage $V_{in}$ is applied across on the potentiometric sensor which consists of a resistive element. An electrically conductive wiper can slide along the resistive element. When the position of the wiper changes, it can cause the change of the output voltage $V_{out}$. Because there is a linear relationship between the position of the wiper and the output voltage $V_{out}$. The output voltage $V_{out}$ can be used to indicate the displacement of the wiper$^{31,32}$. 

The output voltage $V_{out}$ is defined in Eq. (2.1-1). The resistance below the wiper connection is $R_a$ and that above the wiper is $R_b$. And $d$ is the displacement of the wiper relative to the lower end of the resistive element. $D$ is the total length of the resistive element.

$$V_{out} = \frac{R_a}{R_a + R_b}V_{in} = \frac{d}{D}V_{in}$$ (2.1-1)

Although the potentiometric sensor is quite useful in many applications, it still has several disadvantages: (1) The mechanical friction between the wiper and the resistive element is large, which can change the dynamics of the measured object; (2) Need physical coupling with the measured object; (3) Low measurement resolution; (4) Low dynamics measurement range; (5) The mechanical friction and the current flow through the resistive element can cause the heating of the sensor; (6) The wiper and resistive element will suffer mechanical abrasion.
2.1.2 Piezoelectric Accelerometer

Accelerometer is a widely used sensor for measuring object vibration in many different industries, environments and applications. It can be used for monitoring the vehicle acceleration, navigating aircraft and measuring structure vibration. There are several different types of accelerometers, such as capacitive accelerometer, Hall-effect accelerometer, MEMS accelerometer and piezoelectric accelerometer. And the piezoelectric accelerometer is the most commonly used one because its wide dynamic range and good linearity of measurement. It is relatively robust and reliable. Its characters can remain stable over a long period.

The piezoelectric accelerometer is designed based on the piezoelectric effect. The main part of a piezoelectric accelerometer is a slice of piezoelectric material which is a kind of polarized ferroelectric ceramic. A typical design of piezoelectric accelerometer is shown in Figure 2.2. When the accelerometer is moving up, the mass will exert a compression force on the piezoelectric material according to Newton’s second law of motion. Because of the piezoelectric effect, the compression force can cause the piezoelectric material to generate electrical charge. The amount of generated charge is proportional to the amplitude of the force and the acceleration of the mass. By measuring the amplitude of charge, the acceleration can be obtained.

![Figure 2.2. The structure of compression type piezoelectric accelerometer](image)
There are two commonly used structure configurations for the piezoelectric accelerometers: compression type and shear type. For the compression type, the mass exerts a compression force on the piezoelectric material. For the shear type, the mass exerts a shear force on the piezoelectric material. The piezoelectric materials are usually more sensitive to shear force than compression force. Thus, the shear type sensor usually has higher sensitivity than the compression type sensor with equal seismic mass. The shear type accelerometer also exhibits less sensitivity to base strain\cite{33,34}.

When choosing piezoelectric accelerometers, there are several characters need to consider. The first one is the sensitivity of the accelerometers (Unit: pc/ms$^2$). Usually, if we want to obtain high sensitivity of the measurement, a larger seismic mass is needed which usually means larger assembly size and mass of the sensor. In the vibration measurement of small and delicate structures, the weight of the sensor can alter the structure dynamics. For the accelerometer with small size and mass, they usually have lower sensitivity, but can be used for high frequency measurement in small structure.

The second character of the accelerometer is the dynamic range (Unit: Hz) which indicates the normal working range of the sensor. It has a lower frequency limit and a higher frequency limit. The lower frequency limit is mainly affected by the ambient temperature fluctuations and the upper frequency limit is determined by the accelerometer’s resonant frequency.

Another important character of the accelerometer is its mounted resonance frequency $f_m$ (Unit: Hz) as shown in Eq. (2.1-2).

\[
f_m = \frac{1}{2\pi} \sqrt{\frac{k_c}{m_s}}
\]  

(2.1-2)
And $k_c$ is the stiffness of the piezoelectric material and $m_s$ is the seismic mass of the sensor.

When the vibration frequency approaches the mounted resonance frequency, the output accelerometer signal will rise sharply. So, the upper frequency limit of the accelerometer is lower than its mounted resonance frequency.

The final character of the accelerometer is the temperature sensitivity (Unit: %/°C) which measures the change of the signal caused by the temperature fluctuation. Usually the shear type accelerometer has a lower temperature sensitivity than the compression type\textsuperscript{34}.

The advantages of the piezoelectricity accelerometer are its simple structure, compact design, low cost, high sensitivity and wide dynamic range. But it also suffers from some drawbacks. Such as time-consuming installation, the need for additional preconditioning circuit and low spatial measurement resolution.
2.1.3 Linear Variable Differential Transformer (LVDT)

Linear variable differential transformer (LVDT) is a kind of displacement sensor that based on electromagnetic induction. The basic structure of LVDT consists of at least two coils: primary and secondary. An AC voltage $V_{ref}$ is applied on the primary coil. The excitation voltage $V_{ref}$ introduces a steady AC voltage $V_{out}$ in the secondary coil. The amplitude of $V_{out}$ depends on the flux coupling between the coils. And the coupling is determined by the position of the ferromagnetic core. As shown in Figure 2.3, a ferromagnetic core is inserted into the middle of the transformer without touching the coils. The two secondary coils are connected in the opposed phase. If the ferromagnetic core is placed in the center of the transformer, the output voltage of the secondary coil $V_{out}$ will be zero. And if the core is moved away from the central position, the reluctance of the flux path will change. There will be an output voltage in the secondary coil. And the output voltage is proportional to the core displacement. The phase angle between the primary voltage $V_{ref}$ and the output voltage $V_{out}$ can determine the direction of the displacement.\(^\dagger\)

![Figure 2.3. The structure diagram of LVDT](image)

\(^\dagger\) Figure 2.3. The structure diagram of LVDT
LVDT can be used to measure transient motion and slow motion accurately. For transient motion, the frequency of the primary voltage should be at least ten times higher than the highest frequency of the motion. For slow motion, the frequency of primary voltage can be 60 or 50 Hz\textsuperscript{31}.

LVDT has several advantages: (1) The ferromagnetic core of the sensor doesn’t connect to the sensor which means there is little friction resistance for the motion; (2) The magnetic and mechanical hysteresis of the sensor can be negligible; (3) Low sensitivity to environmental noise; (4) High measurement resolution (It is determined by the voltage measurement resolution). But it also has several disadvantages: (1) A fixed installation location is needed; (2) Shielding is required since it is sensitive to stray magnetic field; (2) the performance of the sensor can be affected by vibrations and temperature changes\textsuperscript{31,32}.

LVDT has been widely used as displacement sensor in civil and aerospace engineering. It is a contact type sensor which can measure large displacement stroke. It has good resolution and dynamic performance in large structure displacement measurement.
2.2 Non-contact Methods

2.2.1 Capacitive Displacement Sensor

Capacitive displacement sensor is a non-contact device based on capacitance sensing which can achieve high precision displacement measurement. It provides a relatively simple technique to implement non-contact position measurement. The sensors are used widely in a variety of applications in the industrial world. They can be used for the measurement of a conductive target’s position or a nonconductive material’s thickness or density. Because of their high measurement accuracy and high frequency response, they have been widely applied in the semiconductor, disk drive and precision manufacturing industries\(^3\).

The measurement resolution of capacitive displacement sensor can reach to nanometer level. And the frequency response can be up to 20 kHz or higher. It also has good temperature stability. The typical measurement range is from 10 μm to 10 mm. In some applications, smaller or larger measurement ranges can also be achieved\(^3\).

Because capacitive displacement sensor doesn’t produce direct displacement output like potentiometric sensor. It usually requires driving and conditioning circuit to convert the capacitance measurement to the displacement signal. A typical capacitive displacement sensor is shown in Figure 2.4.

![Figure 2.4. Capacitive displacement sensor](image)
To understand the principle of the capacitive displacement sensor, we first need to introduce the concept of capacitance. Considering two parallel metal plates, capacitance $C$ is defined as the ratio of the electrical charge $Q$ on the plate to the voltage $V$ applied on the plates. The relationship between capacitance and the electrical charge and applied voltage is shown in Eq. (2.2-1).

$$ C = \frac{Q}{V} \quad (2.2-1) $$

For a simple capacitor formed by two close parallel metal plates, the distance between the two plates is $d$, the effective area of the two plates is $A$, $\varepsilon_0$ is the permittivity of free space, $\varepsilon_r$ is the relative permittivity of the dielectric material between the two plates. Then the capacitance of the two plates would be given by Eq. (2.2-2). Once the capacitance $C$ is determined, the distance between the two metal plates can be calculated\(^\text{32}\).

$$ C = \frac{\varepsilon_0 \varepsilon_r A}{d} \quad (2.2-2) $$

Because the capacitance is sensitive to the material in the gap between the two plates. The capacitive displacement sensor cannot work in dirty or dusty environment. Generally, the material in the gap is air. When used with a conductive target, the capacitive sensor is usually factory calibrated. And when using the capacitive sensor with nonconductive materials, experimentations are needed to determine the sensor’s sensitivity to the material\(^\text{35}\).

Capacitive displacement sensor has several advantages: (1) Non-Contact measurement; (2) High measurement resolution; (3) High frequency response; (4) Less expensive and more compact than optical sensor; But is also suffers from some drawbacks: (1) Cannot work in dirty or wet environment; (2) Need to be placed very close to the target.
2.2.2 Inductive Displacement Sensor

Inductive displacement sensor is also known as eddy current sensor which is a non-contact device for the measurement of a target’s position. The principle of inductive sensor is based on the Faraday's law of induction. It requires that the target’s material must be conductive. Compare to capacitive displacement sensor, the nonconductive material between the gap of the probe and target will not affect the measurement of inductive sensor. So the sensor can be used in dirty and wet environments where oil, coolants or other liquid materials can appear. For the different conductive materials like copper, aluminum and steel, the inductive sensor will have different measurement response. So careful calibration for the target material is needed for the high precision measurement\textsuperscript{35}.

Inductive displacement sensor can also have high measurement resolution up to nanometer level. And its frequency response can be up to 80 kHz or higher. The typical measurement range of the inductive sensor is from 0.5 mm to 15 mm. In special applications, smaller or larger measurement range can be achieved. Because the inductive sensor is not sensitive to the material between the probe and target, it can be used in the hostile environment, or even in the liquid\textsuperscript{35}. A typical inductive displacement sensor is shown in Figure 2.5.

![Figure 2.5. Inductive displacement sensor](image-url)
To understand the principle of the inductive displacement sensor, we need to first introduce the concept of inductance. Inductance is an important property for all the electrical conductors. When a current is flowing in a conductor, it will generate a magnetic field according to Ampere's Law. The inductance is a measure of the capability the conductor can oppose a change in the electric current through it. When a current flows through a conductor, it creates a magnetic field around the conductor. A changing current can create a changing magnetic field. In order to oppose the change in current, the conductor will generate a back electromotive force (back EMF) to maintain the current in the conductor. The unit of EMF is volt (V). Consider a coil with current $i$ through it, the EMF $V$ across the coil is proportional to the rate of change of current as shown in Eq. (2.2-3). $L$ is the inductance of the coil, and the unit of inductance is henry (H).

$$V = L \frac{di}{dt}$$  \hspace{1cm} (2.2-3)

Inductive sensor uses the electromagnetic field that penetrates the surface of the conductive target to sense the position change of the target. At the end of the inductive sensor’s probe, there is a coil with alternating current. The alternating current creates alternating electromagnetic field around the probe. When the probe approaches the target, the alternating electromagnetic field can induce small eddy current in the target’s material. These small eddy current also can create small electromagnetic fields around them. Then these small electromagnetic fields can react with the probe. By sensing the amplitude of the coil voltage in the probe, the position of the target can be measured.

The inductive sensor can be affected by the size of the probe and target, target material and the distance between the probe and the target. When the sensor is used for displacement measurement, calibration for the target material is needed. During the measurement, the target
material and the sizes of probe and target remain constant. The distance between the probe and the target is the only variable changes the voltage output of the probe. Because the inductive sensor is very sensitive to material changes, similar sensor can also be applied for flaws, cracks and weld seams detection in conductive materials\textsuperscript{35}.

Inductive sensor can be used for two basic types of materials: ferrous and nonferrous. Ferrous materials include iron and most steels. And nonferrous materials include copper, aluminum, brass, zinc and other conductive materials. Some specialized inductive sensor can work with both types of materials.

When the inductive displacement sensor is working, the electromagnetic field that the probe creates is usually larger than the size of the probe. There is a minimum size requirement for the target. The target’s size must be at least three times the probe’s diameter\textsuperscript{35}.

Because the electromagnetic field can penetrate the target material, there is a minimum thickness requirement for the target. The minimum thickness of the target can be defined by Eq. (2.2-4). $\delta$ is the minimum thickness of the target. $f$ is the operating frequency of the sensor. $\mu$ is the magnetic permeability of the target and $\sigma$ is the electrical conductivity of the target\textsuperscript{31}.

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad (2.2-4)$$

Inductive displacement sensor has several advantages: (1) Non-Contact measurement; (2) High measurement resolution; (3) High frequency response; (4) Less expensive than capacitive displacement sensor; (5) Can work in dirty or wet environment; But it also has some disadvantages: (1) Only work for the conductive material; (2) Has minimum size and thickness requirement for the target.
2.2.3 Laser Doppler Vibrometer (LDV)

Laser Doppler Vibrometer (LDV) is a widely used optical instrument based on the principle of Doppler effect. It is a non-contact method for the velocity or displacement measurement. LDV has been used in a wide variety of scientific and industrial applications. For example, LDV is being used extensively in the automotive applications, such as structural dynamics analysis, noise quantification and diagnosis. In the aerospace industry, LDV has been used for the non-destructive damage detection in the aerospace structure. In the civil engineering, it has been used for the vibration measurement of bridges and large buildings.

LDV has several different types. Such as single-point vibrometer, scanning vibrometer, rotational vibrometer and 3-D vibrometer. Single-point vibrometer is the most common type of LDV. It is used to measure the out of plane motion of one point. The scanning vibrometer has a scanning head in it which allows the LDV to scan the target surface in vertical and horizontal directions. The rotational vibrometer can be used to measure the angular velocity of target. 3-D vibrometer consists of three single-point vibrometers. It uses three laser beams to measure one location. The complete in plane and out of plane velocity of the target can be determined.

![Figure 2.6. Basic schematic of laser Doppler vibrometer](image-url)
A typical schematic of laser Doppler vibrometer is shown in Figure 2.6. A laser beam with frequency \( f_0 \) is generated from the laser source. The beam is divided into a reference beam and a test beam by a beam splitter. Then the test beam passes through a Bragg cell which is also called acousto-optic modulator. The Bragg cell uses the acousto-optic effect to diffract and shift the frequency of incident light. When the test beam passes through the Bragg cell, a shift frequency \( f_b \) is added to the test beam. And the beam is directed to the target. The movement of target will add a Doppler frequency shift \( f_d \) to the beam. The shift frequency \( f_d \) is determined by the Eq. (2.2-5). \( v(t) \) is the velocity of the target as a function of time, \( \alpha \) is the angle between the velocity direction and the beam direction. And \( \lambda \) is the wavelength of the light.

\[
f_d = \frac{2v(t) \cos \alpha}{\lambda}
\]  

(2.2-5)

When the test beam is reflected from the target, it will be scattered in all directions. Some portion of the scattered light can be collected by the LDV. Then the received beam will be reflected into a photodetector by two beam splitters. Now the beam has a frequency equal to \( f_0 + f_b + f_d \). This beam will interfere with the reference beam at the photodetector. The original frequency of the reference beam is very high (> 10^{14} \text{ Hz}). The photodetector is not able to respond. But after the interference, the beat frequency of the two beams is \( f_b + f_d \) which is usually in the range of tens of MHz. And then the photodetector can respond to the signal.

The output of the photodetector is a frequency modulated signal. The frequency of the Bragg cell is the carrier frequency, and the Doppler shift frequency is the modulation frequency. The output signal can be demodulated to derive the velocity of the moving target^{36}.

Laser Doppler vibrometer has several advantages: (1) Non-Contact measurement; (2) High measurement resolution; (3) High frequency response; (4) Long measurement range; (5)
High measurement efficiency; But it also has some drawbacks: (1) High cost; (2) Retroreflective tape is needed for some material.

The pros and cons of several different contact and non-contact sensors used for vibration measurement are summarized in the Table 2.1.

**Table 2.1. Pros and cons of several different sensors for vibration measurement**

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Pros</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Contact Sensor</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Potentiometric Sensor</td>
<td>• simple and compact structure</td>
<td>• large mechanical friction</td>
</tr>
<tr>
<td></td>
<td>• low cost</td>
<td>• low measurement resolution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• low dynamics measurement range</td>
</tr>
<tr>
<td>Piezoelectricity Accelerometer</td>
<td>• simple and compact structure</td>
<td>• preconditioning circuit is needed</td>
</tr>
<tr>
<td></td>
<td>• low cost</td>
<td>• low spatial measurement resolution</td>
</tr>
<tr>
<td>Linear Variable Differential Transformer (LVDT)</td>
<td>• little mechanical friction</td>
<td>• performance of the sensor can be affected by temperature changes</td>
</tr>
<tr>
<td></td>
<td>• high measurement resolution</td>
<td>• installation is time consuming</td>
</tr>
<tr>
<td></td>
<td>• low sensitivity to environmental noise</td>
<td></td>
</tr>
<tr>
<td><strong>Non-contact Sensor</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capacitive Displacement Sensor</td>
<td>• high measurement resolution</td>
<td>• cannot work in dirty or wet environment</td>
</tr>
<tr>
<td></td>
<td>• high dynamics measurement range</td>
<td>• need to be placed close to the target</td>
</tr>
</tbody>
</table>
Table 2.1. (continued)

<table>
<thead>
<tr>
<th></th>
<th>Inductive Displacement Sensor</th>
<th>Laser Doppler Vibrometer (LDV)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>• high measurement resolution</td>
<td>• highest measurement resolution</td>
</tr>
<tr>
<td></td>
<td>• high dynamics measurement range</td>
<td>• high dynamics measurement range</td>
</tr>
<tr>
<td></td>
<td>• can work in dirty or wet environment</td>
<td>• long range measurement</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• retroreflective tape is needed for some material</td>
</tr>
<tr>
<td></td>
<td><strong>only work for the conductive material</strong></td>
<td><strong>high cost</strong></td>
</tr>
<tr>
<td></td>
<td><strong>has minimum size and thickness requirement for the target</strong></td>
<td><strong>retroreflective tape is needed for some material</strong></td>
</tr>
</tbody>
</table>
3 Vision-based Measurement Methods

In this chapter, we will introduce and discuss several classical computer vision-based algorithms for the motion estimation. These algorithms include optical flow method, digital image correlation method, Lucas-Kanade method and Lucas-Kanade inverse compositional method. Their working principles, advantages and limitations will also be investigated.

3.1 Optical Flow Method

Optical flow is the apparent motion of brightness patterns in the image. It is an important concept in computer vision, and has been widely used in motion detection, object segmentation, image registration and stereo disparity measurement\(^{38}\).

To illustrate this concept, consider an example depicted in Figure 3.1. The image brightness at the point \((x, y)\) in the image at time \(t\) is denoted by \(I(x, y, t)\). When the point moves in the image, we make two assumptions: (1) The brightness of the point in the image remains constant. (2) The displacement of the point between two consecutive images is very small.

![Figure 3.1. The movement of point in the image sequence](image)

\[I(x, y, t) \quad I(x + u\delta t, y + v\delta t)\]

Figure 3.1. The movement of point in the image sequence
From these two assumptions, Eq. (3.1-1) can be obtained. Where \( \delta x \) and \( \delta y \) are the point displacements in \( x \) and \( y \) directions. \( u \) and \( v \) are the point velocities in \( x \) and \( y \) directions. \( \delta t \) is the time difference between the two images.

\[
I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t) = I(x + u \delta t, y + v \delta t, t + \delta t)
\]

(3.1-1)

Expanding \( I(x + \delta x, y + \delta y, t + \delta t) \) about the point \((x, y, t)\) using Taylor expansion, Eq. (3.1-1) can be rewritten as:

\[
I(x, y, t) = I(x, y, t) + \delta x \frac{\partial I}{\partial x} + \delta y \frac{\partial I}{\partial y} + \delta t \frac{\partial I}{\partial t} + \varepsilon
\]

(3.1-2)

Where \( \varepsilon \) is the higher order term.

Omit higher order term and rearrange Eq. (3.1-2) yields:

\[
\frac{\delta x}{\delta t} \frac{\partial I}{\partial x} + \frac{\delta y}{\delta t} \frac{\partial I}{\partial y} + \frac{\partial I}{\partial t} = 0
\]

(3.1-3)

Or:

\[
\frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} = 0
\]

(3.1-4)

In Eq. (3.1-4), \( u \) and \( v \) are the only two unknowns, they represent the velocities at point \((x, y)\) in \( x \) and \( y \) directions. The measure of brightness constancy is defined in Eq. (3.1-5).

\[
E_c = \left( \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} \right)^2
\]

(3.1-5)

Since two unknowns cannot be determined by just one equation, other constraint is needed.

In the real world, most objects are rigid or deform elastically, so it is reasonable to assume the points on the object are moving coherently. The measure of smoothness of the velocity can be defined in Eq. (3.1-6).

\[
E_s = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2
\]

(3.1-6)
Due to quantization error and noise in the image, $E_c$ cannot be identically zero. So, the problem becomes an optimization problem. The object is to minimize Eq. (3.1-7), where $\lambda$ is a positive weighting factor.

$$E = \iint (E_c + \lambda E_x) dx dy$$

$$= \iint \left[ \left( \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} \right)^2 + \lambda \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right] dx dy$$

(3.1-7)

Apply the calculus of variation to Eq. (3.1-7), the following equations are derived:

$$I_x (I_x u + I_y v + I_t) - \lambda (u_{xx} + u_{yy}) = I_x (I_x u + I_y v + I_t) - \lambda \Delta u = 0$$

$$I_y (I_x u + I_y v + I_t) - \lambda (v_{xx} + v_{yy}) = I_y (I_x u + I_y v + I_t) - \lambda \Delta v = 0$$

(3.1-8)

The approximate Laplacians of $u$ and $v$ takes the following form:

$$\Delta u = \bar{u}_{i,j} - u_{i,j}$$

$$\Delta v = \bar{v}_{i,j} - v_{i,j}$$

(3.1-9)

Where $\bar{u}_{i,j}$ and $\bar{v}_{i,j}$ are the local average defined as follow:

$$\bar{u}_{i,j} = \frac{1}{6} (u_{i-1,j} + u_{i,j+1} + u_{i+1,j} + u_{i,j-1}) + \frac{1}{12} (u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} + u_{i+1,j-1})$$

$$\bar{v}_{i,j} = \frac{1}{6} (v_{i-1,j} + v_{i,j+1} + v_{i+1,j} + v_{i,j-1}) + \frac{1}{12} (v_{i-1,j-1} + v_{i-1,j+1} + v_{i+1,j+1} + v_{i+1,j-1})$$

(3.1-10)

Figure 3.2 illustrates the kernel window of the approximate Laplacian operation.

<table>
<thead>
<tr>
<th>$\frac{1}{12}$</th>
<th>$\frac{1}{6}$</th>
<th>$\frac{1}{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{6}$</td>
<td>$-1$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

Figure 3.2. The kernel window of the approximate Laplacian operation
Rearrange Eq. (3.1-8), yields:

\[
\left(\lambda + I_x^2\right)u + I_x I_y v = \lambda \bar{u} - I_x I_t,
\]
\[
I_x I_t u + \left(\lambda + I_y^2\right)v = \lambda \bar{v} - I_y I_t,
\]

Eq. (3.1-11)

Solving for \(u\) and \(v\) yields:

\[
\left(\lambda + I_x^2 + I_y^2\right)u = \left(\lambda + I_y^2\right)\bar{u} - I_x I_y \bar{v} - I_x I_t,
\]
\[
\left(\lambda + I_x^2 + I_y^2\right)v = -I_x I_y \bar{u} + \left(\lambda + I_x^2\right)\bar{v} - I_y I_t,
\]

Eq. (3.1-12)

The above method is called Horn–Schunck method\(^{39}\). It was first introduced by Horn and Schunck in 1981.

Fleet and Jepson\(^{40}\) has showed that the temporal evolution of contours of constant phase can provide better approximation to the local velocity than do contours of constant amplitude. They found that phase contours are more robust with respect to lighting variations, can produce more accurate results.

Based on Fleet and Jepson’s method, Chen et al.\(^{41}\) introduced a phase-based measurement method to extract the vibration displacement of structure.
For an image $I(x,y,t)$ at time $t$, its spatial amplitude and phase information can be calculated by:

$$(G_2^\theta + iH_2^\theta) \otimes I(x,y,t) = A_\theta(x,y,t)e^{i\phi_\theta(x,y,t)} \quad (3.1-15)$$

Where $A_\theta(x,y,t)$ and $\phi_\theta(x,y,t)$ are the local amplitude and phase of the image at orientation $\theta$. $G_2^\theta$ is the second derivative of a Gaussian function at the direction of $\theta$. $H_2^\theta$ is the Hilbert transform of $G_2^\theta$. $G_2^\theta$ and $H_2^\theta$ are quadrature pair filters, they have same amplitude of Fourier transform but $90^\circ$ phase difference. $\otimes$ is the convolution operator. The quadrature pair filters used to compute local amplitude and phase are shown in Figure 3.3.
Figure 3.3. Filters used to compute local phase and local amplitude: (a) real horizontal ($G_2^0$), (b) imaginary horizontal ($H_2^0$), (c) real vertical ($G_2^{\pi/2}$), (d) imaginary vertical ($H_2^{\pi/2}$).
Fleet and Jepson\textsuperscript{40} demonstrated that a point moves from \((x, y)\) to \((x + u\delta t, y + v\delta t)\) satisfies Eq. (3.1-16):

\[
\phi_0(x, y, t) = \phi_0(x + u\delta t, y + v\delta t, t + \delta t) = c
\] (3.1-16)

Where \(u\) and \(v\) are the velocity in the \(x\) and \(y\) directions, and \(c\) is a constant.

Apply the same method used to derive Eq. (3.1-4) we can get Eq. (3.1-17):

\[
\frac{\partial \phi_0(x, y, t)}{\partial x} u + \frac{\partial \phi_0(x, y, t)}{\partial y} v + \frac{\partial \phi_0(x, y, t)}{\partial t} = 0
\] (3.1-17)

Since the oriented filter can only measure phase changes in the filter’s direction, so we can get:

\[
\frac{\partial \phi_0(x, y, t)}{\partial y} \approx 0
\]

\[
\frac{\partial \phi_{\pi/2}(x, y, t)}{\partial x} \approx 0
\] (3.1-18)

Substituting Eq. (3.1-18) into Eq. (3.1-17) yields:

\[
u = -\left(\frac{\partial \phi_{\pi/2}(x, y, t)}{\partial y}\right)^{-1} \frac{\partial \phi_{\pi/2}(x, y, t)}{\partial t}
\]

\[
u = -\left(\frac{\partial \phi_{\pi/2}(x, y, t)}{\partial x}\right)^{-1} \frac{\partial \phi_{\pi/2}(x, y, t)}{\partial t}
\] (3.1-19)

Integrate both sides of Eq. (3.1-19), the horizontal and vertical displacements \(d_x\) and \(d_y\) at time \(t\) can be derived as:

\[
d_x(t) = -\left(\frac{\partial \phi_0(x, y, t)}{\partial x}\right)^{-1} \left(\phi_0(x, y, t) - \phi_0(x, y, 0)\right)
\]

\[
d_y(t) = -\left(\frac{\partial \phi_{\pi/2}(x, y, t)}{\partial y}\right)^{-1} \left(\phi_{\pi/2}(x, y, t) - \phi_{\pi/2}(x, y, 0)\right)
\] (3.1-20)
3.2 Digital Image Correlation Method

Digital image correlation (DIC) is an effective tool for quantitative deformation measurement of an object surface. It has been commonly used in the field of experimental mechanics. It can provide the full-field displacement and strain information by comparing the digital images of the object surface before and after deformation.

Compare to other full-field measurement methods, such as the methods based on the interferometry, DIC offers several attractive advantages: (1) Simple experimental setup and specimen preparation: For 2D DIC, only one camera is needed. Specimen’s surface pattern can be made by spraying paints. (2) Low requirements in measurement environment: Do not need a laser source. A white light source can be used for the experiment. (3) Wide range of measurement sensitivity and resolution: DIC method can be applied with different optical system to achieve microscale to nanoscale deformation measurement\(^\text{42}\).

Figure 3.4. A typical 2D DIC measurement system
Nevertheless, the DIC method also suffers some disadvantages: (1) The surface of the test object must have a random gray intensity pattern; (2) The measurement quality depend heavily on the quality of the imaging system; (3) The strain measurement accuracy of the DIC method is lower than that of interferometric methods\textsuperscript{42}.

Figure 3.4 shows the schematic diagram for a typical 2D DIC measurement system. The specimen surface should have a random speckle pattern. The surface of the specimen must be flat and remain normal to the CCD camera’s axis. The white light source is used to provide appropriate light intensity contrast for the CCD camera. After the image acquisition of the experimental process, the images are sent to the computer for analyzing. The full-field displacement or strain information can then be obtained by DIC method.

To implement the 2D DIC method, the measurement area (i.e. region of interest, ROI) in the reference image must be defined at first. The region then will be divided evenly into small subsets (red squares) as shown in Figure 3.5. The displacement of the center point of each red square is calculated to obtain the full-field deformation.

![Figure 3.5. Measurement subsets on the reference image](image-url)
The basic principle of 2D DIC is the tracking of the same points between the two images recorded before and after the deformation. As shown in Figure 3.6, a square reference subset centered at point \( P(x_0, y_0) \) is chosen and used to track its corresponding location in the deformed image. The size of the subset square in the reference image is \( 2M + 1 \) by \( 2M + 1 \). The point \( P'(x'_0, y'_0) \) is the center of the target subset in the deformed image. \( Q(x_i, y_j) \) is the point around the center point \( P(x_0, y_0) \) in the reference subset. And \( Q'(x'_i, y'_j) \) is its corresponding point in the target subset.

![Figure 3.6. Square subset before and after deformation](image)

The coordinates of point \( Q(x_i, y_j) \) and \( Q'(x'_i, y'_j) \) have the following relationship:

\[
x'_i = x_i + \xi(x_i, y_j) \\
y'_j = y_j + \eta(x_i, y_j)
\]  

Where \( \xi(x_i, y_j) \) and \( \eta(x_i, y_j) \) are called shape function or displacement mapping function. If only rigid body translation exists between the reference and target image, the displacements of each point in the subset are same. Then the shape function is zero-order, and is given by:

\[
\xi_0(x_i, y_j) = u \\
\eta_0(x_i, y_j) = v
\]  

Where \( u \) and \( v \) are the translation in the horizontal and vertical directions, respectively.
Where $u$ and $v$ are the displacements of the reference subset center in the $x$ and $y$ directions.

If the shape of the reference subset changes, the first-order shape function is needed. It allows translation, rotation, shear, normal strains and their combinations. The first-order shape function is defined by:

$$
\xi_i(x_i, y_i) = u + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \\
\eta_i(x_i, y_i) = v + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y
$$

For more complicated deformation, the second-order function is needed. The function is defined by:

$$
\xi_i(x_i, y_i) = u + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (\delta x)^2 + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} (\delta y)^2 + \frac{\partial^2 u}{\partial x \partial y} \delta x \delta y \\
\eta_i(x_i, y_i) = v + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} (\delta x)^2 + \frac{1}{2} \frac{\partial^2 v}{\partial y^2} (\delta y)^2 + \frac{\partial^2 v}{\partial x \partial y} \delta x \delta y
$$

For the Eq. (3.2-3) and Eq. (3.2-4), we have:

$$
\delta x = x_i - x_0 \\
\delta y = y_i - y_0
$$

To evaluate the similarity degree between the reference subset and the deformed subset, several correlation criterions can be used. Two of the mostly used criterions are the cross-correlation (CC) criterion and sum-squared difference (SSD) correlation criterion. They are defined by:

$$
C_{CC} = \sum_{i=M}^{M} \sum_{j=M}^{M} \left[ f(x_i, y_i) g(x'_j, y'_j) \right] \\
C_{NCC} = \sum_{i=M}^{M} \sum_{j=M}^{M} \left[ \frac{f(x_i, y_i) g(x'_j, y'_j)}{\bar{f} \bar{g}} \right]
$$

Where $\bar{f}$ and $\bar{g}$ are given by:
\[
\bar{f} = \sqrt{\sum_{i=-M}^{M} \sum_{j=-M}^{M} \left[ f(x_i, y_j) \right]^2}
\]
\[
\bar{g} = \sqrt{\sum_{i=-M}^{M} \sum_{j=-M}^{M} \left[ g(x'_i, y'_j) \right]^2}
\]

(3.2-7)

From Eq. (3.2-1) and Eq. (3.2-3), we can find that the correlation criterions \( C_{cc} \) and \( C_{ncc} \) are the functions of displacement components \( u \) and \( v \), and the displacement gradients \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \). The peak position of the correlation coefficient also indicates the best matching for the reference subset and the target subset. Therefore, the peak position can be used to calculate the displacement information of the target subset\(^{42}\).

There are two steps to estimate the displacement information. First, an initial displacement estimation with one-pixel resolution is used. This step can be implemented either in spatial domain or in frequency domain. In spatial domain, the target subset can be determined by a pixel by pixel search within a specified range in the target image. For the frequency domain, the Fourier transforms of the reference and target subset will be used. By using the Fast Fourier Transform (FFT) method, the frequency domain method can be achieved with high speed. After the first step, gray level interpolation of the image is needed. To find the maximum value position of the correlation. We can differentiate Eq. (3.2-6) by the displacements and the displacement gradients and set them to zero to find the position of the maximum correlation coefficient. The equations are given by:
\[
\begin{aligned}
\frac{\partial C}{\partial u} &= 0 \\
\frac{\partial C}{\partial v} &= 0 \\
\frac{\partial C}{\partial (\frac{\partial u}{\partial x})} &= 0 \\
\frac{\partial C}{\partial (\frac{\partial u}{\partial y})} &= 0 \\
\frac{\partial C}{\partial (\frac{\partial v}{\partial x})} &= 0 \\
\frac{\partial C}{\partial (\frac{\partial v}{\partial y})} &= 0
\end{aligned}
\] (3.2-8)

A numerical technique such as a Newton-Raphson method can be used to solve Eq. (3.2-8). Then the final displacements and displacement gradients can be obtained\textsuperscript{43}.
3.3 Lucas-Kanade Tracking Method

Lucas-Kanade Tracking method or Lucas-Kanade optical flow is one of the most widely used techniques in computer vision. The method was proposed by Lucas and Kanade in 1981\textsuperscript{44}. It has been used in different applications range from motion estimation, image alignment, video stabilization etc\textsuperscript{38}.

3.3.1 Lucas-Kanade Algorithm

The goal of original Lucas-Kanade algorithm is to align a template image $T(x)$ to an input image $I(x)$, where $x = (x, y)^T$ is a column vector containing the pixel coordinates. As shown in Figure 3.7, the template $T(x)$ is an extracted sub-region of the image at $t = N$ and $I(x)$ is the image at $t = N + 1$.

Let $W(x; p)$ denote the parameterized set of warps, where $p = (p_1, ..., p_n)^T$ is a vector of the warp parameters. The warp $W(x; p)$ will map the pixel $x$ in the coordinate frame of the template $T(x)$ to the sub-pixel location $W(x; p)$ in the coordinate frame of the image $I(x)$. If we consider a two-dimensional translation, then the warps $W(x; p)$ is given by:

![Template tracking diagram](image)

Figure 3.7. Template tracking diagram
\[ W(x; p) = \begin{pmatrix} x + p_1 \\ y + p_2 \end{pmatrix} \]  

(3.3-1)

Where the vector of parameters \( p = (p_1, p_2)^T \) is the translation vector. If the template is moving in three dimensions, then the affine warps can be used:

\[
W(x; p) = \begin{pmatrix} (1 + p_1)x + p_3y + p_5 \\ p_2x + (1 + p_4)y + p_6 \end{pmatrix} = \begin{pmatrix} 1 + p_1 & p_3 & p_5 \\ p_2 & 1 + p_4 & p_6 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]

(3.3-2)

Where the vector of the warp \( p = (p_1, p_2, p_3, p_4, p_5, p_6)^T \) has six parameters.

To align the template image \( T(x) \) to an input image \( I(x) \) is equivalent to minimize the sum of squared error between two images, the template image \( T(x) \) and the input image \( I(x) \) warped back onto the coordinate frame of the template. It yields:

\[
\sum_x [I(W(x; p)) - T(x)]^2
\]

(3.3-3)

To minimize the Eq. (3.3-3), we assume that a initial estimate of \( p \) is known, then the increments to the parameters \( \Delta p \) will be iteratively solved. The Eq. (3.3-3) can be rewritten in the following form:

\[
\sum_x [I(W(x; p + \Delta p)) - T(x)]^2
\]

(3.3-4)

The Eq. (3.3-4) is minimized with respect to \( \Delta p \), and the warp vector \( p \) is updated until the parameters of \( p \) converge (\( \|\Delta p\| \leq \varepsilon \)).

\[
p \leftarrow p + \Delta p
\]

(3.3-5)

The Lucas-Kanade method can be thought as a Gauss-Newton gradient descent non-linear optimization problem. The Eq. (3.3-4) can be linearized by applying a Taylor expansion on \( I(W(x; p + \Delta p)) \):
\[
\sum_x [I(W(x; p + \Delta p)) - T(x)]^2
\]
\[
= \sum_x [I(W(x; p)) + \frac{\partial I(W(x; p))}{\partial p} \Delta p - T(x)]^2
\]
\[
= \sum_x [I(W(x; p)) + \frac{\partial I(W(x; p))}{\partial W(x; p)} \frac{\partial W(x; p)}{\partial p} \Delta p - T(x)]^2
\]
\[
= \sum_x [I(W(x; p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x)]^2
\]

Where \( \nabla I \) is the gradient of image \( I \) evaluated at \( W(x; p) \); \( \nabla I \) is computed in the coordinate frame of \( I \) and then warped back onto the coordinate frame of \( T \) using the warp \( W(x; p) \). The term \( \frac{\partial W}{\partial p} \) is the Jacobian matrix of the warp. If we have \( W(x; p) = \begin{pmatrix} W_1(x; p) \\ W_2(x; p) \end{pmatrix} \), then the Jacobian matrix is given by:

\[
\frac{\partial W}{\partial p} = \begin{pmatrix} \frac{\partial W_1}{\partial p_1} & \frac{\partial W_1}{\partial p_2} & \cdots & \frac{\partial W_1}{\partial p_n} \\ \frac{\partial W_2}{\partial p_1} & \frac{\partial W_2}{\partial p_2} & \cdots & \frac{\partial W_2}{\partial p_n} \end{pmatrix}
\]

For example, the Jacobian matrix of the affine warp is given by:

\[
\frac{\partial W}{\partial p} = \begin{pmatrix} x & 0 & y & 0 & 1 & 0 \\ 0 & x & 0 & y & 0 & 1 \end{pmatrix}
\]

Minimizing the expression in Eq. (3.3-6) is a least squares problem, its solution has a closed form. The partial derivative of the expression in Eq. (3.3-6) with respect to \( \Delta p \) is:

\[
2 \sum_x \left[ \nabla I \frac{\partial W}{\partial p} \right]^T [I(W(x; p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x)]
\]

Where \( \nabla I \frac{\partial W}{\partial p} \) is called the steepest descent images. Let the expression in Eq. (3.3-9) equal to zero, we can get the closed form solution for the least squares problem:

\[
\Delta p = H^{-1} \sum_x \left[ \nabla I \frac{\partial W}{\partial p} \right]^T [T(x) - I(W(x; p))]
\]
Where $H$ is the $n \times n$ Hessian matrix which is defined by:

$$H = \sum_x \left[ \nabla I \left( \frac{\partial W}{\partial p} \right) \right]^T \left[ \nabla I \left( \frac{\partial W}{\partial p} \right) \right]$$

(3.3-11)

The procedures of the Lucas-Kanade method is shown in the Figure 3.8.

**While** $\|\Delta p\| > \varepsilon$ **Do**

1. Warp image $I(W(x; p))$
2. Compute the error image $T(x) - I(W(x; p))$
3. Compute the gradient of the warped image $\nabla I(W(x; p))$
4. Evaluate the Jacobian matrix $\frac{\partial W}{\partial p}$
5. Compute the steepest descent images $\nabla I \frac{\partial W}{\partial p}$
6. Compute the Hessian matrix $H = \sum_x \left[ \nabla I \left( \frac{\partial W}{\partial p} \right) \right]^T \left[ \nabla I \left( \frac{\partial W}{\partial p} \right) \right]$
7. Compute $\Delta p$ $\Delta p = -H^{-1} \sum_x \left[ \nabla I \left( \frac{\partial W}{\partial p} \right) \right]^T [T(x) - I(W(x; p))]$
8. Update parameters $p \leftarrow p + \Delta p$

**End**

Figure 3.8. The Lucas-Kanade algorithm

If $n$ is the number of warp parameters and $N$ is the number of pixels in the template $T(x)$, then the computation cost of each step in Figure 3.8 can be given in Table 3.1:

Table 3.1. The computation cost of one iteration of the Lucas-Kanade Algorithm

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
<th>Step 5</th>
<th>Step 6</th>
<th>Step 7</th>
<th>Step 8</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(nN)$</td>
<td>$O(N)$</td>
<td>$O(nN)$</td>
<td>$O(nN)$</td>
<td>$O(nN)$</td>
<td>$O(n^2N)$</td>
<td>$O(nN + n^3)$</td>
<td>$O(n)$</td>
<td>$O(n^2N + n^3)$</td>
</tr>
</tbody>
</table>
3.3.2 Lucas-Kanade Inverse Compositional Algorithm

As shown in Figure 3.8, we know that the evaluation of the Hessian matrix $H$ is needed in each iteration for the Lucas-Kanade algorithm. Several researchers have found that there is a huge computational cost for calculating Hessian matrix in each iteration$^{44}$. If the Hessian matrix was constant and could be precomputed and re-used, then the efficiency of the algorithm could be improved significantly. Based on this idea, a new algorithm called inverse compositional algorithm is proposed. The minimization problem of Eq. (3.3-3) can be reformulated in an equivalent way, but with constant Hessian matrix. The key of the inverse compositional algorithm is to switch the role of the image and the template. The result is that the inverse compositional algorithm minimizes:

$$\sum_{x} [T(W(x; \Delta p)) - I(W(x; p))]^2$$  \hspace{1cm} (3.3-12)

With respect to $\Delta p$.

Compare with Eq. (3.3-4), the roles of $I$ and $T$ are reversed in Eq. (3.3-12), and the warp is updated by:

$$W(x; p) \leftarrow W(x; p) \circ W(x; \Delta p)^{-1}$$  \hspace{1cm} (3.3-13)

Where $W(x; \Delta p)^{-1}$ is the inverse of the affine warp $W(x; \Delta p)$. The parameters of $W(x; \Delta p)^{-1}$ can be calculated by:

$$\frac{1}{(1 + \Delta p_1)(1 + \Delta p_4) - \Delta p_2 \Delta p_3} \begin{pmatrix} -\Delta p_1 - \Delta p_4 \Delta p_4 + \Delta p_2 \Delta p_3 \\ -\Delta p_2 \\ -\Delta p_3 \\ -\Delta p_4 - \Delta p_3 \Delta p_3 + \Delta p_2 \Delta p_6 \\ -\Delta p_5 - \Delta p_4 \Delta p_5 + \Delta p_3 \Delta p_6 \\ -\Delta p_6 - \Delta p_2 \Delta p_6 + \Delta p_2 \Delta p_5 \end{pmatrix}$$  \hspace{1cm} (3.3-14)

Note that if $(1 + \Delta p_1)(1 + \Delta p_4) - \Delta p_2 \Delta p_3 = 0$, then the affine warp is not invertible, we exclude all such affine warps from consideration.
Similarly, the Eq. (3.3-12) can be linearized by applying a Taylor expansion on

$$T(W(x; \Delta p)) :$$

$$\sum_x [T(W(x; \Delta p)) - I(W(x; p))]^2$$

$$= \sum_x [T(W(x; 0 + \Delta p)) - I(W(x; p))]^2$$

$$= \sum_x [T(W(x; 0)) + \frac{\partial T(W(x; 0))}{\partial W(x; 0)} \frac{\partial W(x; 0)}{\partial p} \Delta p - I(W(x; p))]^2$$

$$= \sum_x [T(W(x; 0)) + \nabla T \frac{\partial W}{\partial p} \Delta p - I(W(x; p))]^2$$

(3.3-15)

Where $W(x; 0)$ is the identity warp, $\nabla T$ is the gradient of template $T$. And the Jacobian matrix $\frac{\partial W}{\partial p}$ is evaluated at $(x; 0)$.

The partial derivative of the expression in Eq. (3.3-15) with respect to $\Delta p$ is:

$$2 \sum_x \left[ \nabla T \frac{\partial W}{\partial p} \right]^T [T(W(x; 0)) + \nabla T \frac{\partial W}{\partial p} \Delta p - I(W(x; p))]$$

(3.3-16)

Let the expression in Eq. (3.3-16) equal to zero, we can get the closed form solution for the least squares problem:

$$\Delta p = H^{-1} \sum_x \left[ \nabla T \frac{\partial W}{\partial p} \right]^T [I(W(x; p)) - T(W(x; 0))]$$

(3.3-17)

$$= H^{-1} \sum_x \left[ \nabla T \frac{\partial W}{\partial p} \right]^T [I(W(x; p)) - T(x)]$$

Where $H$ is the $n \times n$ Hessian matrix which is given by:

$$H = \sum_x \left[ \nabla T \frac{\partial W}{\partial p} \right]^T \left[ \nabla T \frac{\partial W}{\partial p} \right]$$

(3.3-18)

From Eq. (3.3-18), we can find that the gradient of the template $\nabla T$ and the Jacobian matrix $\frac{\partial W}{\partial p}$ are both evaluated at $(x; 0)$, which means the Hessian matrix do not depend on the warp parameter vector $p$ and can be pre-computed.
Compare with the original Lucas-Kanade algorithm, the inverse compositional algorithm has several steps can be pre-computed. For example, in the step 3, the gradient $\nabla T$ of the template $T(x)$ can be pre-computed. In the step 4, the Jacobian matrix $\frac{\partial W}{\partial p}$ are evaluated at $(x; 0)$, then the steepest descent images $\nabla T \frac{\partial W}{\partial p}$ and the Hessian matrix can also be pre-calculated. The procedures of the inverse compositional algorithm can be shown in the Figure 3.9.

**Pre-compute:**

3. Compute the gradient of the template $\nabla T$

4. Compute the Jacobian matrix $\frac{\partial W}{\partial p}$

5. Evaluate the steepest descent images $\nabla T \frac{\partial W}{\partial p}$

6. Evaluate the Hessian matrix $H = \sum \frac{\nabla T \frac{\partial W}{\partial p}^T \nabla T \frac{\partial W}{\partial p}}{\partial p}$

**While** $||\Delta p|| > \epsilon$ **Do**

1. Warp image $I(W(x; p))$

2. Compute the error image $I(W(x; p)) - T(x)$

7. Compute $\Delta p$ $\Delta p = H^{-1} \sum \frac{\nabla T \frac{\partial W}{\partial p}^T}{\partial p} [I(W(x; p)) - T(x)]$

8. Update the warp $W(x; p) \leftarrow W(x; p) \circ W(x; \Delta p)^{-1}$

**End**

Figure 3.9. The Lucas-Kanade inverse compositional algorithm

Similarly, if $n$ is the number of warp parameters and $N$ is the number of pixels in the template $T(x)$, the computation cost of each step of the Lucas-Kanade inverse compositional algorithm can be summarized in Table 3.2."
Table 3.2. The computation cost of the Lucas-Kanade inverse compositional algorithm

<table>
<thead>
<tr>
<th></th>
<th>Step 3</th>
<th>Step 4</th>
<th>Step 5</th>
<th>Step 6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-computation</td>
<td>$O(N)$</td>
<td>$O(nN)$</td>
<td>$O(nN)$</td>
<td>$O(n^2N)$</td>
<td>$O(n^2N)$</td>
</tr>
<tr>
<td>Per iteration</td>
<td>$O(nN)$</td>
<td>$O(N)$</td>
<td>$O(nN + n^3)$</td>
<td>$O(n^2)$</td>
<td>$O(nN + n^3)$</td>
</tr>
</tbody>
</table>

As shown in Table 3.2, the calculation of the Hessian matrix spends the most time. Now since it can be pre-calculated, the Lucas-Kanade inverse compositional algorithm can save a large amount of computational cost. This is the main advantage of this algorithm.
4 Beam Theory

This chapter deals with mathematical modeling of the transverse vibration for the cantilever beam. The derivation for the equation of motion of the beam based on Euler–Bernoulli theory will be introduced. The theoretical modeling of a cantilever beam with single crack is given. And the finite element analysis for the beam will also be investigated.

4.1 Euler-Bernoulli Beam Theory

Consider a simple beam as shown in Figure. 4.1, the beam vibrates in the direction perpendicular to its length direction. It has a rectangular cross section area $A$ and length $L$. The bending stiffness of the beam is $EI$, where $E$ is the Young’s modulus for the beam and $I$ is the cross-sectional area moment of inertia. From mechanics of material, the bending moment $M(x,t)$ in the beam can be related to the beam deflection $w(x,t)$ by Eq. (4.1-1)\(^5\).

$$M(x,t) = EI \frac{\partial^3 w(x,t)}{\partial x^2} \quad (4.1-1)$$

The model of the bending beam can be derived from examining the free body diagram of an infinitesimal element of the beam as shown in Figure 4.1. For the Euler-Bernoulli beam, we assume the shear deformation is much smaller than the beam deflection $w(x,t)$. $f(x,t)$ is the
Figure 4.1. The diagram of a simple beam

distributed force-per-unit length on the beam. Sum all the forces in the $y$ direction yields Eq. (4.1-2). $V(x,t)$ is the shear force at the left side of the element and $V(x,t) + \frac{\partial V(x,t)}{\partial x} dx$ at the right side of the element. $f(x,t) dx$ is the total external force applied on the beam and $\rho$ is the density of the beam.

$$
\left[ V(x,t) + \frac{\partial V(x,t)}{\partial x} dx \right] - V(x,t) + f(x,t) dx = \rho Adx \frac{\partial^2 w(x,t)}{\partial t^2} \quad (4.1-2)
$$

Sum all the moments acting on the element yields:

$$
\left[ M(x,t) + \frac{\partial M(x,t)}{\partial x} dx \right] - M(x,t) + \left[ V(x,t) + \frac{\partial V(x,t)}{\partial x} dx \right] dx + \left[ f(x,t) dx \right] \frac{dx}{2} = 0 \quad (4.1-3)
$$

The right-hand side of Eq. (4.1-3) is equal to zero, because the rotary inertia of the small element can be negligible. Eq. (4.1-3) can be further simplified to:
\[
\left[ V(x,t) + \frac{\partial M(x,t)}{\partial x} \right] dx + \left[ \frac{\partial V(x,t)}{\partial x} + \frac{f(x,t)}{2} \right] (dx)^2 = 0 \quad (4.1-4)
\]

Since \( dx \) is assumed to be very small, \((dx)^2\) can be assumed to be zero. So, Eq. (4.1-4) can be led to:

\[
V(x,t) = -\frac{\partial M(x,t)}{\partial x} \quad (4.1-5)
\]

Eq. (4.1-5) means the shear force is proportional to the spatial change in the bending moment. Substitute Eq. (4.1-5) into Eq. (4.1-2), we can get:

\[
-\frac{\partial^2 \left[ M(x,t) \right]}{\partial x^2} dx + f(x,t) dx = \rho A dx \frac{\partial^2 w(x,t)}{\partial t^2} \quad (4.1-6)
\]

Substitute Eq. (4.1-1) into Eq. (4.1-6) and divide two sides by \( dx \) yields:

\[
\frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 w(x,t)}{\partial x^2} \right] + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t) \quad (4.1-7)
\]

If there is no external force applied on the beam, we can get the governing equation of free vibration beam:\(^45\):

\[
c^2 \frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad c = \sqrt{\frac{EI}{\rho A}} \quad (4.1-8)
\]

Since Eq. (4.1-8) contains four spatial derivatives, four boundary conditions are needed. The presence of the second derivative requires two initial conditions need to be specified. One is for the displacement and one is for the velocity.

For a simple beam, there are several different boundary conditions:\(^45\).

If the beam is free at one end, the deflection and slope at the end are unrestricted. The bending moment and shear force must be zero:

\[
EI \frac{\partial^2 w}{\partial x^2} = 0
\]
\[
\frac{\partial}{\partial x} \left[ EI \frac{\partial^2 w}{\partial x^2} \right] = 0 \quad (4.1-9)
\]
If one end of the beam is fixed or clamped, the bending moment and shear force at the end are unrestricted. The deflection and slope at that end are restricted to be zero:

\[ w = 0 \]
\[ \frac{\partial w}{\partial x} = 0 \] (4.1-10)

For a simply supported or pinned end, the slope and shear force at the end are unrestricted, the deflection and moment must be zero:

\[ w = 0 \]
\[ EI \frac{\partial^2 w}{\partial x^2} = 0 \] (4.1-11)

For a sliding end, the deflection and moment at the end are unrestricted, the slope and shear force must be zero:

\[ \frac{\partial w}{\partial x} = 0 \]
\[ \frac{\partial}{\partial x} \left[ EI \frac{\partial^2 w}{\partial x^2} \right] = 0 \] (4.1-12)

In addition to the four spatial boundary conditions, the two initial conditions in time also must be specified:

\[ w(x, 0) = w_0(x) \]
\[ w'_x(x, 0) = \dot{w}_0(x) \] (4.1-13)

\( w_0(x) \) and \( \dot{w}_0(x) \) are the initial deflection and velocity of the beam at location \( x \).

Using the method of separation of variables, we can assume \( w(x, t) = X(x)T(t) \).

Substitute it into the Eq. (4.1-8) yields:

\[ e^2 \frac{d^4 X(x)}{dx^4} = - \frac{d^2 T(t)}{dt^2} = \omega^2 \] (4.1-14)

\( \omega^2 \) is the separation constant. It comes from the temporal equation:

\[ \ddot{T}(t) + \omega^2 T(t) = 0 \] (4.1-15)
The temporal equation has a general solution:

\[ T(t) = A \sin \omega t + B \cos \omega t \]  \hspace{1cm} (4.1-16)

\( A \) and \( B \) are the constants determined by the initial displacement and velocity conditions.

Rearranging Eq. (4.1-14), we can get:

\[ \frac{d^4 X(x)}{dx^4} - \beta^4 X(x) = 0 \]  \hspace{1cm} (4.1-17)

Where \( \beta \) is a constant defined by \( \beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \).

The general solution of Eq. (4.1-17) can be shown to be:

\[ X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x \]  \hspace{1cm} (4.1-18)

Where \( \beta \) and the four constants \( a_1, a_2, a_3, \) and \( a_4 \) can be determined by the four boundary conditions.

The final solution for the transverse vibration beam without external force is given by \(^{45}\):

\[ w(x,t) = X(x)T(t) = (a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x)(A \sin \omega t + B \cos \omega t) \]  \hspace{1cm} (4.1-19)

Considering the transverse vibration of a cantilever beam as shown in Figure 4.2, the beam is uniform. One end of the beam is clamped and the other is free.

Figure 4.2. A cantilever beam diagram
From the boundary conditions, we can get:

\[
\begin{align*}
\text{At } x &= 0 \\
& \begin{cases} 
X = 0 \\
\frac{dX}{dx} = 0 
\end{cases} \\
\text{At } x &= L \\
& \begin{cases} 
EI \frac{d^2X}{dx^2} = 0 \\
EI \frac{d^3X}{dx^3} = 0 
\end{cases}
\end{align*}
\]

(4.1-20)

Substituting the boundary conditions into Eq. (4.1-18), we obtain:

\[
\begin{align*}
\text{At } x &= 0 \\
& \begin{cases} 
X = a_1 \sin 0 + a_2 \cos 0 + a_3 \sinh 0 + a_4 \cosh 0 = 0 \\
\frac{dX}{dx} = a_1 \cos 0 - a_2 \sin 0 + a_3 \cosh 0 + a_4 \sinh 0 = 0 
\end{cases} \\
\text{At } x &= L \\
& \begin{cases} 
\frac{d^2X}{dx^2} = \beta^2 (-a_1 \sin \beta L - a_2 \cos \beta L + a_3 \sinh \beta L + a_4 \cosh \beta L) = 0 \\
\frac{d^3X}{dx^3} = \beta^3 (-a_1 \cos \beta L + a_2 \sin \beta L + a_3 \cosh \beta L + a_4 \sinh \beta L) = 0 
\end{cases}
\end{align*}
\]

(4.1-21)

Simplify and rearrange Eq. (4.1-21) gives:

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & a_1 \\
1 & 0 & 1 & 0 & a_2 \\
-\sin \beta L & - \cos \beta L & \sinh \beta L & \cosh \beta L & a_3 \\
- \cos \beta L & \sin \beta L & \cosh \beta L & \sinh \beta L & a_4
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(4.1-21)

In order to get nontrivial solution for \(a_1\), \(a_2\), \(a_3\) and \(a_4\), the determinant of the square matrix in Eq. (4.1-21) must be zero. Then yields:

\[
\cos \beta L = - \frac{1}{\cosh \beta L}
\]

(4.1-22)

To solve Eq. (4.1-22), we can draw the plot of function \(\cos \beta L\) and \(- \frac{1}{\cosh \beta L}\) as shown in Figure 4.3. The crossing points of the two functions give the value \(\beta L\) which satisfy Eq. (4.1-22).
From Figure 4.3 we can find the function $-\frac{1}{\cosh \beta L}$ approaches zero asymptotically, so the fourth and higher roots of Eq. (4.1-22) can be determined by $\cos \beta L = 0$. Then we can get the approximate relationship:

$$\beta L_n = \frac{\pi}{2}(2n-1) \quad n = 4, 5, 6, \ldots \quad (4.1-23)$$

The first three roots of Eq. (4.1-22) can be determined from the plot as:

$$\beta L_1 = 1.875$$
$$\beta L_2 = 4.694$$
$$\beta L_3 = 7.855 \quad (4.1-24)$$

From the definition of the constant $\beta$, we have $\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$, it gives:

$$\omega_n = (\beta L_n^2) \sqrt{\frac{EI}{\rho AL^4}} \quad (4.1-25)$$
Then the natural circular frequencies for the first six modes of the cantilever beam can be determined as shown in Table 4.1.

Table 4.1. Natural circular frequencies of cantilever beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Circular Frequency $\omega_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\omega = (1.875)^2 \frac{EI}{\rho AL^3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\omega = (4.694)^2 \frac{EI}{\rho AL^3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\omega = (7.855)^2 \frac{EI}{\rho AL^3}$</td>
</tr>
<tr>
<td>4</td>
<td>$\omega = (10.996)^2 \frac{EI}{\rho AL^4}$</td>
</tr>
<tr>
<td>5</td>
<td>$\omega = (14.137)^2 \frac{EI}{\rho AL^4}$</td>
</tr>
<tr>
<td>6</td>
<td>$\omega = (17.279)^2 \frac{EI}{\rho AL^4}$</td>
</tr>
</tbody>
</table>
Solve Eq. (4.1-21), we can get:

\[
\begin{aligned}
    a_1 &= a_i \\
    a_2 &= \frac{\cos \beta L + \cosh \beta L}{\sin \beta L - \sinh \beta L} a_i \\
    a_3 &= -a_i \\
    a_4 &= -\frac{\cos \beta L + \cosh \beta L}{\sin \beta L - \sinh \beta L} a_i
\end{aligned}
\]  

(4.1-26)

Substitute it into Eq. (4.1-18) yields:

\[
X(x) = a_{1} \left[ \sin \beta x - \sinh \beta x + \frac{\cos \beta L + \cosh \beta L}{\sin \beta L - \sinh \beta L} (\cos \beta x - \cosh \beta x) \right]
\]  

(4.1-27)

Substituting different \((\beta L)_n\) into Eq. (4.1-27), it gives the different mode shape functions of the cantilever beam. The first four mode shapes of the beam are plotted in the Figure 4.4.

![Graphs showing the first four mode shapes of a cantilever beam](image-url)
4.2 Cracked Beam Theory

Consider a cantilever beam with a crack as shown in Figure 4.5. The beam has a uniform cross section with thickness $h$ and a crack located at $L_c$. The crack has a uniform depth $d$ and is assumed to be open. Because of the localized crack, the beam can be simulated as two small beams connected by a massless spring\(^7\) (Figure 4.6).

![Figure 4.5. A cantilever beam with single crack](image)

For general loading, a local flexibility matrix relates displacement to forces. In the present analysis, the rotational crack compliance is assumed to be dominant in the local flexibility, since only bending vibration of the beam is considered. The bending spring constant $k_r$ in the vicinity of the cracked section is given by\(^7\):

$$k_r = \frac{1}{c}$$

$$c = 5.346 \frac{h}{EI J\left(\frac{d}{h}\right)}$$  \hspace{1cm} (4.2-1)

Where $E$ is the Young’s modulus for the beam, $I$ is the cross-sectional area moment of inertia and $J\left(\frac{d}{h}\right)$ is the dimensionless local compliance function given by:
\[ J\left( \frac{d}{h} \right) = 1.8624\left( \frac{d}{h} \right)^2 - 3.95\left( \frac{d}{h} \right)^3 + 16.37\left( \frac{d}{h} \right)^4 - 37.226\left( \frac{d}{h} \right)^5 \]
\[ + 76.81\left( \frac{d}{h} \right)^6 - 126.9\left( \frac{d}{h} \right)^7 + 172\left( \frac{d}{h} \right)^8 - 43.97\left( \frac{d}{h} \right)^9 + 66.56\left( \frac{d}{h} \right)^{10} \]  
(4.2-2)

The displacement in each part of the beam is:

\[ X_1(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x \]
\[ X_2(x) = a_5 \sin \beta x + a_6 \cos \beta x + a_7 \sinh \beta x + a_8 \cosh \beta x \]  
(4.2-3)

Where \( \beta \) is a constant defined by \( \beta^4 = \frac{\rho A \omega^2}{EI} \). \( A \) is the cross section area. \( \rho \) is the material density and \( \omega \) is the vibration angular frequency. \( a_i, i = 1, 2, \ldots, 8 \), are constants that can be determined from the boundary conditions:

\[
\begin{align*}
\text{At } x = 0 & \quad \begin{cases} X_1 = 0 \\ \frac{dX_1}{dx} = 0 \end{cases} \\
\text{At } x = L & \quad \begin{cases} EI \frac{d^2X_2}{dx^2} = 0 \\ EI \frac{d^3X_2}{dx^3} = 0 \end{cases}
\end{align*}
\]  
(4.2-4)

Figure 4.6. Cracked cantilever beam model

For the connection part between the two beams, additional conditions are introduced. The continuity of displacement, bending moment and shear force at the connection needs to be guaranteed. Also, the equilibrium between the bending moment and rotation of the spring at the crack needs to be satisfied. So, the boundary conditions at the crack can be given by:
At $x = L$,

$$\begin{cases} 
X_1 = X_2 \\
EI \frac{d^2X_1}{dx^2} = EI \frac{d^2X_2}{dx^2} \\
EI \frac{d^3X_1}{dx^3} = EI \frac{d^3X_2}{dx^3} \\
- EI \frac{d^2X_1}{dx^2} = k_r \left[ \frac{dX_1}{dx} - \frac{dX_2}{dx} \right]
\end{cases} \tag{4.2-5}$$

Substitute Eq. (4.2-4) and Eq. (4.2-5) into Eq. (4.2-3), the characteristic equation for the cracked beam with single crack can be solved.
4.3 Finite Element Method for Beam

This section introduces finite element method for beam vibration analysis. Finite element method also refers to finite element analysis (FEA) which is a powerful tool for modeling complicated structures and machines in real world. It uses variational and interpolation methods for modeling and solving boundary-value problems. The FEA is very useful for complicated structures with unusual geometric shapes. It can be easily implemented on a digital computer to solve a wide range of statics and dynamics problems.

The finite element method approximates a structure in two steps. First, it divides the structure into a number of small and simple parts. These small parts are called finite elements. Second, the solution of the equation for each element is approximated by a linear combination of low-order polynomials. The solutions of all elements are brought together in an assembly procedure, resulting in global mass and stiffness matrices, which describe the dynamic characteristics of the whole structure.

In this section, the finite element method for the transverse vibration of a beam will be discussed. As shown in Figure 4.7, a single finite element model of beam has two transverse coordinates $u_1(t), u_3(t)$ and two rotational coordinates $u_2(t), u_4(t)$ to describe the vibration of this element.

![Figure 4.7. A single finite element model of beam](image)
For the single beam element, each node has two degree of freedom, one accounts for the transverse motion and the other for the slope. When the beam is free of load, the transverse static displacement satisfies:

$$\frac{\partial^2}{\partial x^2}\left[ EI \frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0 \quad (4.3-1)$$

For the beam element with constant $EI$, it leads to $\frac{\partial^4 u(x,t)}{\partial x^4} = 0$. So $u(x,t)$ can be given by:

$$u(x,t) = c_1(t)x^3 + c_2(t)x^2 + c_3(t)x + c_4(t) \quad (4.3-2)$$

$u(x,t)$ is the approximated transverse displacement within the element.

As shown in Figure 4.7, the beam element must satisfy the boundary conditions:

$$u(0,t) = u_i(t)$$
$$\frac{\partial u(0,t)}{\partial x} = u_2(t)$$
$$u(l,t) = u_3(t)$$
$$\frac{\partial u(l,t)}{\partial x} = u_4(t) \quad (4.3-3)$$

Substitute Eq. (4.3-2) into these boundary conditions, yields:

$$c_1(t) = \frac{1}{l^5} \left\{2\left[u_i(t) - u_3(t)\right] + l\left[u_2(t) + u_4(t)\right]\right\}$$
$$c_2(t) = \frac{1}{l^5} \left\{3\left[u_3(t) - u_1(t)\right] - l\left[2u_2(t) + u_4(t)\right]\right\} \quad (4.3-4)$$
$$c_3(t) = u_3(t)$$
$$c_4(t) = u_4(t)$$

Substitute Eq. (4.3-4) into Eq. (4.3-2) and rearrange, the approximate displacement $u(x,t)$ for the element can be given as:

$$u(x,t) = \left(1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right)u_i(t) + l\left(\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right)u_2(t)$$
$$+\left(3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right)u_3(t) + l\left(-\frac{x^2}{l^2} + \frac{x^3}{l^3}\right)u_4(t) \quad (4.3-5)$$
The coefficient functions $1 - 3 \frac{x^2}{l^2} + 2 \frac{x^3}{l^3}$, $l \left( \frac{x}{l} - 2 \frac{x^2}{l^2} + \frac{x^3}{l^3} \right)$, $\left( \frac{3x^2}{l^2} - 2 \frac{x^3}{l^3} \right)$ and $l \left( -\frac{x^2}{l^2} + \frac{x^3}{l^3} \right)$ are the shape functions for the transverse beam element.

The kinetic energy of the element can be obtained from:

$$
T(t) = \frac{1}{2} \int_0^l A \rho \left[ \frac{\partial u(x,t)}{\partial t} \right]^2 \, dx
$$

(4.3-6)

And it also can be written in the form:

$$
T(t) = \frac{1}{2} \mathbf{u}^T M \mathbf{u}
$$

(4.3-7)

$M$ is the mass matrix of the element. The vector $\mathbf{u}$ is the time derivative of the vector $\mathbf{u}(t)$ which is the nodal displacement vector defined by:

$$
\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}
$$

(4.3-8)

Substituting Eq. (4.3-5) into Eq. (4.3-6) and rearranging the equation in the form of Eq. (4.3-7), we can get the mass matrix for the beam element:

$$
M = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}
$$

(4.3-9)

The strain energy of the element can be given by:

$$
V(t) = \frac{1}{2} \int_0^l EI \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right]^2 \, dx
$$

(4.3-10)

And it also can be written in the form:

$$
V(t) = \frac{1}{2} \mathbf{u}^T K \mathbf{u}
$$

(4.3-11)

$K$ is the stiffness matrix of the element which can be defined by:

60
Consider an undamped cantilever beam with two elements as shown in Figure 4.8.

For the first element of the beam, from Eq. (4.3-9) and Eq. (4.3-12), the mass and stiffness matrices are:

\[
K = \frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix}
\]

\[\text{(4.3-12)}\]

\[
M_1 = \frac{\rho Al}{840} \begin{bmatrix}
156 & 11l & 54 & -\frac{13}{2}l \\
11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\
54 & \frac{13}{2}l & 156 & -11l \\
-\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2
\end{bmatrix}
\]

\[
K_1 = \frac{8EI}{l^3} \begin{bmatrix}
12 & 3l & -12 & 3l \\
3l & l^2 & -3l & \frac{1}{2}l^2 \\
-12 & -3l & 12 & -3l \\
3l & \frac{1}{2}l^2 & -3l & l^2
\end{bmatrix}
\]

\[\text{(4.3-13)}\]

Similarly, for element 2:

\[
M_2 = \frac{\rho Al}{840} \begin{bmatrix}
156 & 11l & 54 & \frac{13}{2}l \\
11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\
54 & \frac{13}{2}l & 156 & -11l \\
-\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2
\end{bmatrix}
\]

\[
K_2 = \frac{8EI}{l^3} \begin{bmatrix}
12 & 3l & -12 & 3l \\
3l & l^2 & -3l & \frac{1}{2}l^2 \\
-12 & -3l & 12 & -3l \\
3l & \frac{1}{2}l^2 & -3l & l^2
\end{bmatrix}
\]

\[\text{(4.3-14)}\]
For the first element, we can apply the boundary conditions $u_1 = u_2 = 0$ to the dynamic equation, then yields:

$$\frac{\rho A l}{840} \begin{bmatrix} 156 & -11l^3 & 0 \\ -11l^3 & l^2 & -11l^3 \\ 0 & -11l^3 & l^2 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \frac{8EI}{l} \begin{bmatrix} 12 & -3l & 0 \\ -3l & l^2 & -3l \\ 0 & -3l & l^2 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(4.3-15)

For the second element, the dynamic equation is given by:

$$\frac{\rho A l}{840} \begin{bmatrix} 156 & 11l & 54 & -13l^2 \\ 11l & l^2 & 13 & -3l^2 \\ 54 & 13 & 156 & -11l \\ -13 & -3 & l^2 & -11l^3 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} + \frac{8EI}{l^3} \begin{bmatrix} 12 & 3l & -3l & 1 \frac{l^2}{2} \\ 3l & l^2 & -3l & 2 \frac{l^2}{2} \\ -12 & -3l & 12 & -3l \\ 3l & 1 \frac{l^2}{2} & -3l & l^2 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(4.3-16)

Combine the Eq. (4.3-15) and Eq. (4.3-16), the global equation for the beam with two elements can be obtained:

$$\begin{bmatrix} 312 & 0 & 54 & -13l^2 \\ 0 & 2l^2 & 13 & -3l^2 \\ 54 & 13 & 156 & -11l \\ -13 & -3 & l^2 & -11l^3 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} + \frac{8EI}{l^3} \begin{bmatrix} 12 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & 1 \frac{l^2}{2} \\ -12 & -3l & 12 & -3l \\ 3l & 1 \frac{l^2}{2} & -3l & l^2 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(4.3-17)

Eq. (4.3-17) can be written in the form:

$$M\ddot{u}(t) + Ku(t) = 0$$

(4.3-18)

Where $M$ and $K$ are the global mass and stiffness matrices for the beam. We can assume $u(t) = xe^{j\omega t}$ and substitute it into Eq. (4.3-18), yields:

$$(K - \omega^2 M)x = 0$$

(4.3-19)

Eq. (4.3-19) is a typical eigenvalue problem, the nontrivial solutions $(x \neq 0)$ exist only if:

$$\det(K - \omega^2 M) = 0$$

(4.3-20)
For a beam with \( n \) elements, the left side of Eq. (4.3-20) is an \( n \)-th order polynomial of \( \omega^2 \). We can find \( n \) solutions or eigenvalues \( \omega_i \ (i = 1, \ 2, \ ..., \ n) \) for the equation. These eigenvalues are the natural frequencies of the beam.

For each eigenvalue \( \omega_i \), it gives one solution:

\[
(K - \omega^2 M)\Phi_i = 0
\]  
(4.3-21)

Where \( \phi_i \ (i = 1, \ 2, \ ..., \ n) \) are the normal modes (or mode shapes) of the beam. They also satisfy these properties (For \( i \neq j \)):

\[
\Phi_i^T M \Phi_j = 0
\]
\[
\Phi_i^T K \Phi_j = 0
\]
(4.3-22)

Normal modes are usually normalized to satisfy:

\[
\tilde{\Phi}_i^T M \tilde{\Phi}_i = 1
\]
\[
\tilde{\Phi}_i^T K \tilde{\Phi}_i = \omega_i^2
\]
(4.3-23)

Consider a cantilever beam with the following properties:

<table>
<thead>
<tr>
<th>Table 4.2. Material properties of a cantilever beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (GPa)</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
</tr>
<tr>
<td>Length (m)</td>
</tr>
<tr>
<td>Width (m)</td>
</tr>
<tr>
<td>Height (m)</td>
</tr>
</tbody>
</table>

The natural frequencies of the beam calculated by finite element method and theoretical method are shown in Table 4.3.
Table 4.3. Natural frequencies of cantilever beam calculated by finite element method and theoretical method

<table>
<thead>
<tr>
<th>FEA Method Number of Elements</th>
<th>( \omega_1 ) (Hz)</th>
<th>Relative Error</th>
<th>( \omega_2 ) (Hz)</th>
<th>Relative Error</th>
<th>( \omega_3 ) (Hz)</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8.7974</td>
<td>0.059%</td>
<td>55.5732</td>
<td>0.853%</td>
<td>187.9587</td>
<td>21.808%</td>
</tr>
<tr>
<td>3</td>
<td>8.7940</td>
<td>0.021%</td>
<td>55.2866</td>
<td>0.332%</td>
<td>156.2198</td>
<td>1.240%</td>
</tr>
<tr>
<td>4</td>
<td>8.7934</td>
<td>0.014%</td>
<td>55.1698</td>
<td>0.120%</td>
<td>155.4919</td>
<td>0.768%</td>
</tr>
<tr>
<td>6</td>
<td>8.7932</td>
<td>0.012%</td>
<td>55.1192</td>
<td>0.029%</td>
<td>154.5796</td>
<td>0.177%</td>
</tr>
<tr>
<td>8</td>
<td>8.7931</td>
<td>0.011%</td>
<td>55.1100</td>
<td>0.012%</td>
<td>154.3911</td>
<td>0.055%</td>
</tr>
<tr>
<td>10</td>
<td>8.7931</td>
<td>0.011%</td>
<td>55.1074</td>
<td>0.007%</td>
<td>154.3365</td>
<td>0.019%</td>
</tr>
<tr>
<td>15</td>
<td>8.7931</td>
<td>0.011%</td>
<td>55.1060</td>
<td>0.005%</td>
<td>154.3052</td>
<td>0.001%</td>
</tr>
<tr>
<td>30</td>
<td>8.7931</td>
<td>0.011%</td>
<td>55.1056</td>
<td>0.004%</td>
<td>154.2977</td>
<td>0.006%</td>
</tr>
<tr>
<td>Theoretical Method</td>
<td>8.7922</td>
<td></td>
<td>55.1034</td>
<td></td>
<td>154.3068</td>
<td></td>
</tr>
</tbody>
</table>

The first three mode shapes for the beam calculated by finite element method and theoretical method are shown in Figure 4.9 to Figure 4.11.

Figure 4.9. First mode shape for the cantilever beam
Figure 4.10. Second mode shape for the cantilever beam

Figure 4.11. Third mode shape for the cantilever beam
5 Vibration-based Damage Detection

5.1 Introduction

In structural health monitoring, damage identification can be divided into two groups: local and global methods. Local methods usually detect the damage in a small and local region. They usually have high spatial resolution for the damage. The typical local methods include the sensor using ultrasonic waves, magnetic fields, eddy-current and thermal fields.

On the contrary, the global method is based on the global behavior of the structure. When a reduction of stiffness happened in the structure, it can cause a decrease of natural frequencies, which is a global property of the structure. These methods are usually based on the vibration measurement. The changes in the natural frequencies, modal damping ratios and mode shapes can be the indicators for the damage identification. For the global methods, the sensor network can be coarser than the local method. And the sensor does not need to be placed near the damage location. But the sensitivity of the global methods is usually lower than the local methods, they have lower spatial resolution.

Vibration based damage identification methods that do not combine with the structural model primarily provide Level I and Level II damage identification. When the structural model is used, Level III damage detection can be achieved. Level IV identification must be combined with fracture mechanics and fatigue-life analysis.

In this chapter, several vibration-based damage detection methods will be introduced and discussed. Including natural frequency method, mode shape method, mode shape curvature method, flexibility matrix method, frequency response function (FRF) curvature method, model strain energy method and model strain-based method.
5.2 Natural Frequency Method

In modal testing, natural frequencies are easy to measure compare to other modal parameters. They are independent to the location of measurement. Since the existence of structural damage can cause the stiffness reduction in a structure, and the stiffness change will lead to changes in the natural frequencies of structure. The natural frequency can be a good indicator for the structural damage evaluation.

Back to 1979, Cawley and Adams \(^7\) gave a formulation to detect damage in structure using natural frequency shifts. They assumed that the change in the natural frequency of mode \(i\) of a structure due to localized damage is a function of the position vector of the damage \(\mathbf{r}\) and the stiffness reduction caused by the damage \(\Delta K\). Thus:

\[
\Delta \omega_i = f(\Delta K, \mathbf{r}) \tag{5.2-1}
\]

Applying Taylor series expansion to Eq. (5.2-1) about the undamaged state \((\Delta K = 0)\) and ignoring second and higher order terms, yields:

\[
\Delta \omega_i = f(0, \mathbf{r}) + \Delta K \frac{\partial f}{\partial (\Delta K)} (0, \mathbf{r}) \tag{5.2-2}
\]

Since there is no frequency change for structure without damage, we have \(f(0, \mathbf{r}) = 0\), then Eq. (5.2-2) can be written as:

\[
\Delta \omega_i = \Delta K \frac{\partial f}{\partial (\Delta K)} (0, \mathbf{r}) = \Delta K g_i(\mathbf{r}) \tag{5.2-3}
\]

Where \(g_i(\mathbf{r})\) is a function of the position vector of the damage \(\mathbf{r}\) correspondent to the \(i\)th mode of the structure. We can get similar relationship for other modes:

\[
\Delta \omega_j = \Delta K g_j(\mathbf{r}) \tag{5.2-4}
\]

Combine Eq. (5.2-3) and Eq. (5.2-4), yields:

\[
\frac{\Delta \omega_i}{\Delta \omega_j} = \frac{g_i(\mathbf{r})}{g_j(\mathbf{r})} \tag{5.2-5}
\]
From Eq. (5.2-5), we can find that the ratio of the frequencies changes in two modes is independent of stiffness change. The ratio is only a function of the damage location. Based on this, we can use the measured frequency change ratio and compare it to the theoretically determined ratios of different damage location. Then the damage location of the structure can be determined. For the complex structure, it is difficult to get the theoretical solution. Finite element method can be used to calculate the ratios for different damage location.

An alternative way to compute the frequency change ratio due to localized damage is to use sensitivity analysis. By this method, the sensitivities of the natural frequencies of a structure to small changes in the stiffness are calculated from the mode shapes of the undamaged structure.

The basic dynamic equation for an undamped system is:

\[(K - \lambda M)\Phi = 0\]  
(5.2-6)

Consider a small change \(\Delta K\) in the stiffness matrix caused by the damage, then Eq. (5.2-6) becomes:

\[\{(K + \Delta K) - (M + \Delta M)(\lambda + \Delta \lambda)\}(\Phi + \Delta \Phi) = 0\]  
(5.2-7)

Expand the Eq. (5.2-7) and neglect the second and higher order terms yields:

\[K\Phi - \lambda M\Phi - \lambda \Delta M\Phi + \Delta K\Phi - \Delta \lambda M\Phi + K\Delta \Phi - \lambda M\Delta \Phi = 0\]  
(5.2-8)

Assume the damage does not affect the mass matrix, we have \(\Delta M = 0\) and substitute Eq. (5.2-6) into Eq. (5.2-8), gives:

\[\Delta K\Phi - \lambda \Delta M\Phi + K\Delta \Phi - \lambda M\Delta \Phi = 0\]  
(5.2-9)

Multiply Eq. (5.2-9) by \(\Phi^T\), we have:

\[\Phi^T\Delta K\Phi - \lambda \Phi^T M\Phi + (\Phi^T K - \lambda \Phi^T M)\Delta \Phi = 0\]  
(5.2-10)

Since \(K\) and \(M\) are symmetric matrices, then we have \(\Phi^T (K - \lambda M) = 0\), substitute it into Eq. (5.2-10) yields:
From Eq. (5.2-11), the sensitivities of the natural frequencies to stiffness change $\Delta K$ can be calculated from the mode shape vectors and the mass matrix of the undamaged structure.

Due to its assumption, this method can only be applied for detecting single damage. For the symmetric structure, this method may lead to two symmetric damage locations.

Another frequency-based method is proposed by Williams et al. They defined a damage location assurance criterion (DLAC) using the frequency changes in several modes. This method is based on the correlation approach which is very similar to the modal assurance criterion. The DLAC is defined as:

$$DLAC(j) = \frac{[(\Delta \delta f)^T (\delta f_j)]^2}{[(\Delta \delta f)^T (\Delta \delta f)][(\delta f_j)^T (\delta f_j)]}$$  \hspace{1cm} (5.2-12)

Where $j$ is the location, $\Delta \delta f$ is the measured frequency change vector and $\delta f_j$ is the analytical frequency change vector for damage at location $j$. From an analytical or finite element model, $\delta f_j$ can be precomputed for different location $j$. DLAC values range from 0 to 1, with 0 indicating no correlation and 1 indicating an exact match between the patterns of frequency change. The location $j$ giving the highest DLAC value indicates the possible damage location.

In their study, the authors show that the method is capable of giving accurate predictions of the damage site with large frequency measurement error. The method can provide good result when the damage is larger than 20%. Prediction errors may occur when the level of the damage is low.
5.3 Mode Shape Method

Mode shape is another important vibration parameter that can be used for damage identification. Compare to the natural frequency change methods, mode shapes are more sensitive to damage, especially for higher order modes. The damage in the structure will alter the local mode shape, therefore the mode shapes can be used for locating the damage in a structure.

Two of the commonly used damage detection methods based on mode shapes are the modal assurance criterion (MAC) and the coordinate modal assurance criterion (COMAC)\(^49\).

The MAC can be used to determine the existence of the structural damage. It is defined by:

\[
MAC(\Phi_i^u, \Phi_j^d) = \frac{\left(\Phi_i^{uT}\Phi_j^d\right)^2}{\left(\Phi_i^{uT}\Phi_i^u\right)\left(\Phi_j^{dT}\Phi_j^d\right)}
\]  \hspace{1cm} (5.3-1)

Where \(\Phi_i^u\) is the \(i\)th mode shape of the undamaged structure and \(\Phi_j^d\) is the \(j\)th mode shape of the damaged structure. The value of MAC ranges from 0 to 1. 1 represents the two mode shapes are highly correlated and 0 means completely uncorrelated. The low value of MAC indicates the possible damage\(^50\).

The COMAC is similar to MAC and is also based on the correlation method. But it is a pointwise measure of the difference between the two sets of mode shapes. The COMAC is defined by:

\[
COMAC(j) = \frac{\sum_{i=1}^{N} \left[\Phi_i^u(j)\Phi_i^d(j)\right]^2}{\sum_{i=1}^{N} \left[\Phi_i^u(j)\Phi_i^u(j)\right]\sum_{i=1}^{N} \left[\Phi_i^d(j)\Phi_i^d(j)\right]} \hspace{1cm} (5.3-2)
\]

Where \(\Phi_i^u(j)\) and \(\Phi_i^d(j)\) are the \(i\)th mode shapes of the undamaged and damaged structure at location \(j\). \(N\) is the number of selected modes. A low value of COMAC will indicate a possible damage location\(^51\).
5.4 Mode Shape Curvature Method

The mode shape curvatures are related to the bending stiffness of a structure cross-sections. The curvature at a point has the following form\textsuperscript{13}:

\[ \Phi'' = \frac{M}{EI} \]  

(5.4-1)

Where \( \Phi'' \) is the mode shape curvature at the point, \( M \) is the bending moment, \( E \) is the Young’s modulus and \( I \) is the cross-sectional area moment of inertia.

If there is a crack in the structure, it can reduce the \( EI \) of the structure at the damage location. Then the curvature of the damage region will be increased. Because the change of curvature is local, it can be used for detecting and locating damage. And the amplitude of the change is related to the change of \( EI \), so the mode shape curvatures can also indicate the severity of the damage.

Pandey et al.\textsuperscript{13} used central difference approximation to calculate the mode shape curvature, it is defined as:

\[ \Phi''_i(j) = \frac{\Phi''_i(j+1) - 2\Phi''_i(j) + \Phi''_i(j-1)}{L^2} \]  

(5.4-2)

\[ \Phi''_d(j) = \frac{\Phi''_d(j+1) - 2\Phi''_d(j) + \Phi''_d(j-1)}{L^2} \]

Where \( \Phi''_i(j) \) and \( \Phi''_d(j) \) are the \( i \)th mode shape curvatures of the undamaged and damaged structure at location \( j \). \( \Phi_i(j-1), \Phi_i(j) \) and \( \Phi_i(j+1) \) are the \( i \)th mode shape values at the \((j-1)\)th, \( j \)th and \((j+1)\)th locations, respectively. And \( L \) is the distance between each location.

It is shown that the difference between \( \Phi''_i(j) \) and \( \Phi''_d(j) \) is a good indicator for the damage location. The method is more sensitive than the methods based on the natural frequency changes and mode shape.
5.5 Flexibility Matrix Method

Damage in a structure can cause a reduction in stiffness or an increase in flexibility. So, it is intuitive to measure the flexibility matrix of the structure and use it to find the position of the damage in a structure. From the chapter 3, we have the following relationship for a structure:

\[(K - \omega^2 M)\Phi_i = 0\]  \hspace{1cm} (5.5-1)

Where \(\Phi_i\) \((i = 1, 2, \ldots, n)\) are the normal modes (or mode shapes) of the structure. Substitute \(\lambda_i = \omega_i^2\) into Eq. (5.5-1) and rearrange, we can get:

\[K\Phi_i = \lambda_i M\Phi_i\]  \hspace{1cm} (5.5-2)

Multiplying \(\Phi_j^T\) to the Eq. (5.5-2), yields:

\[\Phi_j^T K\Phi_i = \lambda_i \Phi_j^T M\Phi_i\]  \hspace{1cm} (5.5-3)

A similar relationship can be obtained by exchanging the subscripts \(i\) and \(j\).

\[\Phi_i^T K\Phi_j = \lambda_j \Phi_i^T M\Phi_j\]  \hspace{1cm} (5.5-4)

Since the mass matrix \(M\) and stiffness matrix \(K\) are symmetric, it gives:

\[\Phi_j^T K\Phi_i = \Phi_i^T K\Phi_j\]
\[\Phi_j^T M\Phi_i = \Phi_i^T M\Phi_j\]  \hspace{1cm} (5.5-5)

Substitute the Eq. (5.5-5) and Eq. (5.5-4) into Eq. (5.5-3), we have:

\[\lambda_i - \lambda_j \Phi_i^T M\Phi_j = 0\]
\[\lambda_i - \lambda_j \Phi_i^T K\Phi_j = 0\]  \hspace{1cm} (5.5-6)

If \(i \neq j\), we obtain:

\[\Phi_i^T M\Phi_j = 0\]
\[\Phi_i^T K\Phi_j = 0\]  \hspace{1cm} (5.5-7)

If \(i = j\), it gives:

\[\Phi_i^T M\Phi_i = M_{ii}\]
\[\Phi_i^T K\Phi_i = K_{ii}\]  \hspace{1cm} (5.5-8)

Where \(M_{ii}\) and \(K_{ii}\) are called the generalized mass and the generalized stiffness\(^\text{52}\).
If each of the normal modes $\Phi_i$ is divided by the square root of the generalized mass $M_i$, then the new normal mode is called orthonormal mode and designated as $\tilde{\Phi}_i$. The Eq. (5.5-8) will become:

$$
\tilde{\Phi}_i^T M \tilde{\Phi}_i = 1 \\
\tilde{\Phi}_i^T K \tilde{\Phi}_i = \lambda_i 
$$

(5.5-9)

Consider a N-DOF modal matrix $\Phi$, which is defined as:

$$
\Phi = [\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_n] 
$$

(5.5-10)

Then we can obtain:

$$
\tilde{\Phi}_i^T M \tilde{\Phi}_i = [\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_n]^T M [\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_n] = I \\
\tilde{\Phi}_i^T K \tilde{\Phi}_i = [\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_n]^T K [\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_n] = \Lambda 
$$

(5.5-11)

Where $I$ is the identity matrix and $\Lambda$ is the diagonal matrix of the eigenvalue $\lambda_i$ ($i = 1, 2, \ldots, n$).

From Eq. (5.5-11), we also can get:

$$
K = \Phi \Lambda \Phi^T \\
F = \Phi \Lambda^{-1} \Phi^T 
$$

(5.5-12)

Where $F$ is the flexibility matrix which is the inverse matrix of $K$. $F$ can also be written as the following form:\textsuperscript{19}

$$
F = \sum_{i=1}^{n} \frac{1}{\omega_i^2} \tilde{\Phi}_i \tilde{\Phi}_i^T 
$$

(5.5-13)

Pandey and Biswas\textsuperscript{19} used the flexibility matrix change to locate the damage in a structure, the flexibility change is obtained as:

$$
\Delta F = F^u - F^d 
$$

(5.5-14)

Where $F^u$ and $F^d$ are the flexibility matrices for the undamaged and the damaged structure. For each DOF $j$, let $\overline{\delta}_j$ be the maximum absolute value of the elements in the $j$th column of $\Delta F$. Which is defined as:
\[ \bar{\delta}_j = \max_i |\Delta F_{ij}| \]  \hspace{1cm} (5.5-15)

The quantity \( \bar{\delta}_j \) is used as the change of flexibility for each measurement location.

The advantage of this method is that it can use only a few of lower frequency modes to have a good estimation of the flexibility matrix. From the Eq. (5.5-13), we know that the higher frequency modes have smaller contributions to the flexibility matrix. The matrix is dominated by the lower frequency modes. So, the flexibility matrix method can be used to detect the damage with only a few of lower frequency modes which is more practical in the real experimental environment.

Based on the flexibility matrix, Lu et al.\(^5^3\) proposed a method which uses the flexibility curvature to locate multiple damages in a structure. In practice, the model parameters of the undamaged structure are usually not available. So, the motivation of this method is to only use the flexibility matrix of the damaged structure to locate the damage. This method introduced a new parameter called flexibility curvature, which is defined as:

\[ F'_i^{(c)} = \frac{F_{i-1,j-1} - 2F_{i,j} + F_{i+1,j+1}}{L} \]  \hspace{1cm} (5.5-16)

Where \( F_{i,j} \) and \( F'_i^{(c)} \) are the \( i \)th diagonal element of the flexibility matrix and the \( i \)th item of the flexibility curvature vector, respectively. And \( L \) is the distance between each measurement location.

For an undamaged continuous structure, the flexibility curvature vector of it will have a smooth curve shape. If a damage is introduced into the structure, it can cause a local peak on the flexibility curvature. Thus, the local peak position can be used to localize the damage.
5.6 Frequency Response Function (FRF) Curvature Method

A damage in the structure will cause change in the structure’s frequency response function (FRF). Thus, by analyzing the change of frequency response function, damage location can be obtained.

Sampaio\textsuperscript{16} first used the frequency response function curvature method to locate damage in a structure. This method is similar to the mode shape curvature method. The difference is that it uses all frequencies’ response in a certain measurement range and not just uses the modal shapes of the structure. In fact, this method uses the “operational mode shape” at different frequencies.

The curvature for each frequency is defined as:

$$H^*(\omega)_{i,j} = \frac{H(\omega)_{i+1,j} - 2H(\omega)_{i,j} + H(\omega)_{i-1,j}}{L^2}$$

(5.6-1)

Where $H(\omega)_{i,j}$ is the receptance FRF measured at location $i$ for a force applied at location $j$. $L$ is the distance between each measurement location. The absolute difference between the FRF curvature of undamaged and damaged structure along the chosen frequency range is calculated by:

$$\Delta H^*_{i,j} = \sum_\omega \left| H^*(\omega)_{i,j}^u - H^*(\omega)_{i,j}^d \right|$$

(5.6-2)

$H^*(\omega)_{i,j}^u$ and $H^*(\omega)_{i,j}^d$ are the frequency response function curvature for undamaged and damaged structure.

Sampaio\textsuperscript{16} found that this method worked well for the frequency range before the first anti-resonance or resonance frequency (whichever comes first). The method is also quite insensitive to noise. It has better performance compare to the mode shape curvature method, although the mode shape curvature method has better results for high order mode shapes.
5.7 Model Strain Energy Method

When a particular vibration mode stores a large amount of strain energy in a particular structural load path, the modal data of that mode will be sensitive to changes in the load path. Thus, changes in the structure’s modal strain energy can be used as a useful indicator for damage identification\textsuperscript{49}.

Kim and Stubbs developed a damage detection method based on the modal strain energy to locate and quantify damage in a plate girder\textsuperscript{54}. Consider a linear, undamaged structure with $NE$ elements and $N$ nodes. The $i$th modal stiffness of the structure $KM_i^u$ is given by:

$$KM_i^u = \Phi_i^u K^u \Phi_i^u$$

(5.7-1)

Where $\Phi_i^u$ is the $i$th modal shape vector and $K^u$ is the stiffness matrix of the structure. The contribution of the $j$th element to the $i$th modal stiffness $KM_{i,j}$ is obtained by:

$$KM_{i,j}^u = \Phi_i^u K_j^u \Phi_i^u$$

(5.7-2)

Where $K_j^u$ is the contribution of the $j$th element to the structure stiffness matrix. Then the fraction of modal energy of $i$th mode and $j$th element for the undamaged structure can be given by:

$$F_{i,j}^u = \frac{KM_{i,j}^u}{KM_i^u}$$

(5.7-3)

Similarly, for the damaged structure, we have the following relationships:

$$F_{i,j}^d = \frac{KM_{i,j}^d}{KM_i^d}$$

$$KM_{i,j}^d = \Phi_i^d K_j^d \Phi_i^d$$

(5.7-4)

$$KM_i^d = \Phi_i^d K^d \Phi_i^d$$

Divide Eq. (5.7-4) by Eq. (5.7-3) yields:

$$\frac{F_{i,j}^d}{F_{i,j}^u} = \left( \frac{KM_{i,j}^d}{KM_{i,j}^u} \right) \left( \frac{KM_i^u}{KM_i^d} \right)$$

(5.7-5)
The quantities $K^u_j$ and $K^d_j$ in Eq. (5.7-2) and Eq. (5.7-4) can be written as:

$$
K^u_j = E^u_j C_j \\
K^d_j = E^d_j C_j
$$

(5.7-6)

Where $E^u_j$ and $E^d_j$ are the material stiffness parameters of undamaged and damaged structure, they are related to the effective modulus of elasticity of the element. And the matrix $C_j$ involves only geometric quantities. Suppose we make the approximation that the fraction of modal energy is same for the undamaged and damaged structures. Substituting Eq. (5.7-6) into Eq. (5.7-5) and rearranging, then a damage index $\beta_{i,j}$ of $i$th mode and $j$th element can be given by:

$$
\beta_{i,j} = \frac{E^u_j}{E^d_j} = \frac{\Phi_i^{dT} C_j \Phi_i^d}{\Phi_i^{dT} C_j \Phi_i^u} \frac{K^u_i}{K^d_i}
$$

(5.7-7)

For $NM$ vibrational modes, the damage index $\beta_j$ of $j$th element can be given by:

$$
\beta_j = \sum_{i=1}^{NM} \frac{\Phi_i^{dT} C_j \Phi_i^d K^u_i}{\sum_{i=1}^{NM} \Phi_i^{dT} C_j \Phi_i^u K^d_i}
$$

(5.7-8)

Where $\beta_j \geq 0$, if $\beta_j > 1$, it indicates a damage at the $j$th element of the structure.
5.8 Model Strain-Based Method

With the development of fiber optical sensor, it will be efficient and economical to collect structural modal strain with optical fiber strain gauge in the field. Serna and Stubbs\textsuperscript{55} proposed a damage detection method based on the direct strain measurements.

Considering a linear, undamaged structure with $NE$ elements. The strain energy of a single element in $i$th mode can be obtained by:

$$W_{ij}^u = E_j^u u_{ij}^2 V_j$$  \hspace{1cm} (5.8-1)

Where $E_j^u$ is the Young’s modulus, $u_{ij}^u$ is the strain and $V_j$ is the volume of the $j$th element. The total strain energy of the structure in $i$th mode can be defined as:

$$W_T^u = \sum_{j=1}^{NE} E_j^u u_{ij}^2 V_j$$  \hspace{1cm} (5.8-2)

The fraction of strain energy contributed by the $j$th element in the $i$th mode is defined as:

$$F_{ij}^u = \frac{W_{ij}^u}{W_T^u} = \frac{E_j^u u_{ij}^2 V_j}{\sum_{j=1}^{NE} E_j^u u_{ij}^2 V_j}$$  \hspace{1cm} (5.8-3)

Similarly, the fraction of strain energy contributed by the $j$th element in the $i$th mode for the damaged structure can be defined as:

$$F_{ij}^d = \frac{E_j^d u_{ij}^d V_j}{\sum_{j=1}^{NE} E_j^d u_{ij}^d V_j}$$  \hspace{1cm} (5.8-4)

The quantities $F_{ij}^u$ and $F_{ij}^d$ are related by Eq. (5.8-5):

$$F_{ij}^d = F_{ij}^u + dF_{ij}^u$$  \hspace{1cm} (5.8-5)
Where \( dF^u_{ij} \) is related to the change in the fraction of strain energy of the \( j \)th element for the \( i \)th mode. Assuming the damage is at the \( j \)th element and the resulting change in \( F^u_{ij} \) is only a function of \( E_j^u \), then the first order approximation of \( dF^u_{ij} \) can be obtained by:

\[
dF^u_{ij} = \frac{\partial F^u_{ij}}{\partial E_j^u} dE_j^u + \frac{\partial F^u_{ij}}{\partial \epsilon_{ij}^2} \frac{\partial \epsilon_{ij}^2}{\partial E_j^u} dE_j^u
\]

(5.8-6)

Using the relationship \( E_j^u \epsilon_{ij}^u = \sigma_{ij}^u \), where \( \sigma_{ij}^u \) is the stress of the \( j \)th element for the \( i \)th mode.

Then we can get:

\[
dF^u_{ij} = -F^u_{ij} \alpha_j
\]

(5.8-7)

\[
F^d_{ij} = F^u_{ij} - F^u_{ij} \alpha_j
\]

(5.8-8)

Where \( \alpha_j \) can be represented as:

\[
\alpha_j = \frac{dE_j^u}{E_j^u} = \frac{E_j^d - E_j^u}{E_j^u}
\]

(5.8-9)

Substituting Eq. (5.8-3), Eq. (5.8-4) and Eq. (5.8-9) into Eq. (5.8-8) and rearranging, the following relationship can be obtained:

\[
\beta_j = \frac{E_j^u}{E_j^d} = \frac{\epsilon_{ij}^d W^u_{ij}}{2 \epsilon_{ij}^2 W^u_{ij} + \frac{1}{2}}
\]

(5.8-10)

Where \( \beta_j \) is the damage indicator in the \( j \)th element using the \( i \)th mode. To avoid the problem of division-by-zero in Eq. (5.8-10), the fundamental damage indicator can be rewritten as:

\[
\beta_j = \frac{\epsilon_{ij}^d + 1}{\epsilon_{ij}^2 + 1}
\]

(5.8-11)

When several modes are available, \( \beta_j \) is first normalized for each mode by using the normalization:

\[
Z_{ij} = \frac{\beta_j - \mu_{\beta_j}}{\sigma_{\beta_j}}
\]

(5.8-12)
Where $\mu_{\beta_{ij}}$ and $\sigma_{\beta_{ij}}$ are the mean and standard deviations of the $\beta_{ij}$ for $NE$ elements. Then the damage location can be identified by the magnitude of the normalized damage indicators.
6 Laboratory Experiments

In this chapter, several different cantilever beam structures will be tested in the laboratory. The vision-based displacement extraction algorithms and vibration-based damage detection methods discussed in the chapter 3 and chapter 5 will be applied in these experiments for the damage localization. Aluminum cantilever beams with single and multiple damages will be first tested to demonstrate the effectiveness of the proposed algorithm. Then the experiments of wood cantilever beam with single notch will be introduced. Additionally, noise analysis of the experiments will also be discussed.

6.1 Aluminum Cantilever Beam with Single Damage

6.1.1 Finite Element Analysis

In this experiment, an aluminum cantilever beam with uniform rectangular cross-section was used. The dimensions of the beam are 300 × 12.7 × 1.59 mm, A drill-through hole which locates 100 mm away from the fixed end was used to simulate the damage. The detailed dimensions of the beam are shown in Figure 6.1.

Figure 6.1. The dimensions of the aluminum cantilever beam
The material of the beam is 6061 aluminum. And the finite element model of the beam is established using ANSYS Workbench 18.1. The parameters of the finite element analysis are shown in Table 6.1.

Table 6.1. Parameters of the finite element analysis

<table>
<thead>
<tr>
<th>Finite Element Analysis Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Software</td>
<td>Ansys Workbench 18.1</td>
</tr>
<tr>
<td>Material</td>
<td>6061 Aluminum</td>
</tr>
<tr>
<td>Node Number</td>
<td>92949</td>
</tr>
<tr>
<td>Element Number</td>
<td>54835</td>
</tr>
</tbody>
</table>

Using the model analysis module in the Ansys Workbench 18.1, the first ten natural frequencies of the beam can be calculated. The results are shown in Table 6.2.

Table 6.2. First ten natural frequencies of the beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.052</td>
</tr>
<tr>
<td>2</td>
<td>89.178</td>
</tr>
<tr>
<td>3</td>
<td>113.14</td>
</tr>
<tr>
<td>4</td>
<td>247.91</td>
</tr>
<tr>
<td>5</td>
<td>489.77</td>
</tr>
<tr>
<td>6</td>
<td>606.25</td>
</tr>
<tr>
<td>7</td>
<td>707.78</td>
</tr>
<tr>
<td>8</td>
<td>805.22</td>
</tr>
<tr>
<td>9</td>
<td>1198.2</td>
</tr>
<tr>
<td>10</td>
<td>1689.2</td>
</tr>
</tbody>
</table>
By checking the mode shape at each natural frequency, we can find that the 3rd, 6th and 7th modes are not the bending modes of the beam.

(a) 1st Bending mode

(b) 2nd Bending mode
The 3D mode shapes of the first four bending modes are shown in the Figure 6.2. After extracting the transverse displacement of the mode shapes and normalization, the normalized mode shapes of the beam are shown in Figure 6.3.
Figure 6.3. The first four normalized bending mode shapes of the aluminum beam
Figure 6.4. The modal curvatures of the first four normalized bending mode shapes
Using the central difference approximation method discussed in Chapter 5 (Eq. (5.4-2)), the modal curvatures of the first four normalized bending mode shapes are shown in Figure 6.4. From Figure 6.4, we can find that the discontinuity in each modal curvature is able to indicate the location of the damage. Since the damage is located near the nodal point of the fourth bending mode shape, the discontinuity in the fourth modal curvature is not noticeable.

### 6.1.2 Experiment Results

An experiment is designed to demonstrate the feasibility of the vision-based damage detection method. The experiment setup is shown in Figure 6.5. An aluminum cantilever beam was fixed by a clamp. The impact hammer was used to excite the vibration of the beam. One high-speed camera (Photron FASTCAM Mini AX200) was used to capture the beam vibration. A LED light with adjustable light intensity is used to illuminate the beam.

![The cantilever beam experimental setup](image)

*Figure 6.5. The cantilever beam experimental setup*
Figure 6.6 shows the screenshot of the captured video. To reduce the glare reflection from the aluminum surface, the beam was painted with white spray paint. Thirty cross markers were drawn manually on the lateral surface of the beam. Each of the marker was used as a target for the tracking algorithm.

In order to increase the measurement accuracy for each marker, the lens was zoomed-in to cover only one third of the entire beam. As shown in Figure 6.6, the damage of the beam is located near to the sixth marker from the bottom up.

The resolution of the measurement video is $1024 \times 128$ pixels, the frame rate is 1000 fps, the shutter speed is $1/1000$ s and the total measurement length is 4000 ms. For each marker, the size is $14 \times 14$ pixels in the video. Detailed experiment parameters are shown in Table 6.3.
Table 6.3. Experiment parameters

<table>
<thead>
<tr>
<th>Experiment Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Camera</strong></td>
<td>Photon FASTCAM Mini AX200</td>
</tr>
<tr>
<td><strong>Video</strong></td>
<td></td>
</tr>
<tr>
<td>Height</td>
<td>1024 pixels</td>
</tr>
<tr>
<td>Width</td>
<td>128 pixels</td>
</tr>
<tr>
<td>Frame Rate</td>
<td>1000 fps</td>
</tr>
<tr>
<td>Shutter Speed</td>
<td>1/1000 s</td>
</tr>
<tr>
<td>Length</td>
<td>4000 ms</td>
</tr>
<tr>
<td><strong>Marker</strong></td>
<td></td>
</tr>
<tr>
<td>Size</td>
<td>14 × 14 pixels</td>
</tr>
<tr>
<td>Number</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 6.7. The estimated displacement of the sixth marker
Apply the Lucas-Kanade inverse compositional algorithm discussed in Chapter 3 to the video, the displacement of the markers can be estimated. Figure 6.7 shows the estimated displacement of the sixth marker from the bottom up in the video.

After extracting the displacement of all the markers, fast Fourier transform (FFT) was applied to each displacement. Then an averaged FFT result for all the markers can be calculated. The averaged FFT result is shown in Figure 6.8. From Figure 6.8, we can find several dominant peaks in the frequency domain which represent the first few natural frequencies of the aluminum beam.

Figure 6.8. The averaged FFT result from ten markers
The natural frequency comparison between FEA and experiment results are shown in Table 6.4. From the table, we can find that the camera results are consistent with the FEA results. Except for the first mode, the relative frequency error is less than 1%.

### Table 6.4. Natural frequency comparison between FEA and Camera results

<table>
<thead>
<tr>
<th>Vibration Mode</th>
<th>Natural Frequency (Hz)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEA Result</td>
<td>Camera Result</td>
</tr>
<tr>
<td>1st</td>
<td>14.05</td>
<td>13.75</td>
</tr>
<tr>
<td>2nd</td>
<td>89.18</td>
<td>88.52</td>
</tr>
<tr>
<td>3rd</td>
<td>247.91</td>
<td>246.3</td>
</tr>
<tr>
<td>4th</td>
<td>489.77</td>
<td>486.6</td>
</tr>
</tbody>
</table>

Choosing the first target’s displacement as the reference, then we can calculate the cross power spectrum density (CPSD) for all the extracted displacement signals. From Figure 6.8, we can notice that the tested structure is lightly damped with well separated modes. Thus, the simple peak-picking method can be applied on the cross power spectrum density result to estimate the mode shapes of the structure\textsuperscript{56–58}. Figure 6.9 shows the normalized mode shapes result from the camera experiment and the finite element analysis. It shows that the experimental results are consistent with the finite element analysis results.
Figure 6.9. The mode shapes from the camera experiment and finite element analysis

Figure 6.10. The modal curvatures from the experiment and finite element analysis
Using the central difference approximation method discussed in Chapter 5 (Eq. (5.4-2)),
the modal curvatures of the first four modes are shown in Figure 6.10. In this figure, the grey bar
indicates the location of the damage. We can notice that the modal curvatures match the finite
element analysis results well in the first, second and third modes and the peak location in these
experimental modal curvatures can be used to indicate the damage. However, for the fourth
modal curvatures, the damage location is hard to be observed. There are two possible reasons:
(1) The damage location is very close to one of the nodal points in the fourth mode shape, the
damage has very limited influence to the modal curvature of that mode; (2) The displacement
amplitude of the fourth mode is very small, the signal to noise ratio (SNR) for this mode is low.
The modal curvature result can be easily affected by the measurement noise.
6.2 Aluminum Cantilever Beam with Multiple Damages

6.2.1 Finite Element Analysis

In this experiment, an aluminum cantilever beam with uniform rectangular cross-section was used. The dimensions of the beam are $300 \times 12.7 \times 1.59$ mm. Two different cuts on the beam was used to simulate the multiple damages. From left to right, the first cut has 50% depth in the width direction. The second cut has 70% depth in the width direction. These two cuts were used to simulate 50% and 70% stiffness reduction in the beam. The detailed dimensions of the beam are shown in Figure 6.11.

![Diagram of aluminum cantilever beam with two cuts](image)

Figure 6.11. The dimensions of the aluminum cantilever beam with two cuts
The material of the beam is 6061 aluminum. And the finite element analysis of the beam was also conducted. The parameters of the finite element analysis are shown in Table 6.5.

Table 6.5. Parameters of the finite element analysis

<table>
<thead>
<tr>
<th>Finite Element Analysis Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Node Number</td>
</tr>
<tr>
<td>Element Number</td>
</tr>
</tbody>
</table>

The first ten natural frequencies of the beam were calculated. The results are shown in Table 6.6. The normalized mode shapes of the first four bending modes are shown in the Figure 6.12.

Table 6.6. First ten natural frequencies of the beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.677</td>
</tr>
<tr>
<td>2</td>
<td>89.931</td>
</tr>
<tr>
<td>3</td>
<td>251.019</td>
</tr>
<tr>
<td>4</td>
<td>494.736</td>
</tr>
<tr>
<td>5</td>
<td>815.750</td>
</tr>
<tr>
<td>6</td>
<td>1215.5</td>
</tr>
<tr>
<td>7</td>
<td>1706.4</td>
</tr>
<tr>
<td>8</td>
<td>2266.1</td>
</tr>
<tr>
<td>9</td>
<td>2903.6</td>
</tr>
<tr>
<td>10</td>
<td>3645.1</td>
</tr>
</tbody>
</table>
Figure 6.12. The first four normalized bending mode shapes of the aluminum beam
Figure 6.13. The modal curvatures of the first four normalized bending mode shapes
Similarly, by using the central difference approximation method, the modal curvatures of the first four normalized bending mode shapes are shown in Figure 6.13. From Figure 6.13, we can find that the discontinuity in each modal curvature is able to indicate the location of the damages. Since the two damages are located near the nodal points of the fourth bending mode shape, the discontinuities in the fourth modal curvature are not noticeable.

6.2.2 Experiment Results

For this experiment, the setup is same as shown in Figure 6.5. An aluminum cantilever beam with two cuts was fixed by a clamp. The impact hammer was used to excite the vibration of the beam. One high-speed camera (Photron FASTCAM Mini AX200) was used to capture the beam vibration. A LED light with adjustable light intensity is used to illuminate the beam.

Figure 6.14 shows the screenshots of the captured video. To reduce the glare reflection from the aluminum surface, the beam was painted with white spray paint. Sixty cross markers were drawn manually on the lateral surface of the beam. Each of the marker was used as a target for the tracking algorithm.

In order to increase the measurement accuracy for each marker, the lens was zoomed-in to cover only one third of the entire beam for each measurement. To measure the vibration of the beam near the two damages, two different measurement videos were recorded. The screenshots of the two videos are shown in Figure 6.14. The damages are both located near to the tenth marker from the bottom up in each video.

The resolution of the measurement video is $1024 \times 128$ pixels, the frame rate is 1000 fps, the shutter speed is $1/4000$ s and the total measurement length is 1000 ms. For each marker, the size is $17 \times 17$ pixels in the video. Detailed experiment parameters are shown in Table 6.7.
Figure 6.14. The screenshot of the measurement video
Table 6.7. Experiment parameters

<table>
<thead>
<tr>
<th>Experiment Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Camera</strong></td>
</tr>
<tr>
<td>Photron FASTCAM Mini AX200</td>
</tr>
<tr>
<td><strong>Video</strong></td>
</tr>
<tr>
<td>Height</td>
</tr>
<tr>
<td>Width</td>
</tr>
<tr>
<td>Frame Rate</td>
</tr>
<tr>
<td>Shutter Speed</td>
</tr>
<tr>
<td>Length</td>
</tr>
<tr>
<td><strong>Marker</strong></td>
</tr>
<tr>
<td>Size</td>
</tr>
<tr>
<td>Number</td>
</tr>
</tbody>
</table>

Apply the Lucas-Kanade inverse compositional algorithm discussed in Chapter 3 to the video, the displacement of the markers can be estimated. Then the fast Fourier transform (FFT) was applied to each displacement. An averaged FFT result for all the markers can be calculated. The averaged FFT result is shown in Figure 6.15. From Figure 6.15, we can find several dominant peaks in the frequency domain which represent the first few natural frequencies of the aluminum beam.
Figure 6.15. The averaged FFT result from twenty markers

The natural frequency comparison between FEA and experiment results are shown in Table 6.8.

<table>
<thead>
<tr>
<th>Vibration Mode</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEA Result</td>
</tr>
<tr>
<td>1st</td>
<td>15.68</td>
</tr>
<tr>
<td>2nd</td>
<td>89.93</td>
</tr>
<tr>
<td>3rd</td>
<td>251.02</td>
</tr>
<tr>
<td>4th</td>
<td>494.74</td>
</tr>
</tbody>
</table>
We also chose the first target’s displacement in each video as the reference to calculate the cross power spectrum density (CPSD) for all the extracted displacement signals. From Figure 6.15, we can note that the tested structure is lightly damped with well separated modes. Thus, the simple peak-picking method can be applied on the cross power spectrum density result to estimate the mode shapes of the structure\textsuperscript{56–58}. Figure 6.16 and 6.17 show the normalized mode shapes result from the first and second experimental videos with the finite element analysis results. It shows that the experimental results are consistent with the finite element analysis.

![Mode Shape](image)

Figure 6.16. The mode shapes of the first section of the beam from the camera experiment and finite element analysis
Figure 6.17. The mode shapes of the second section of the beam from the camera experiment and finite element analysis

Using the central difference approximation method discussed in Chapter 5 (Eq. (5.4-2)), the modal curvatures of the first four modes for the two sections of the beam are shown in Figure 6.18 and 6.19. In order to increase the accuracy of the modal curvature estimation, we used ten points with the same interval from the original modal shapes. In these figures, the grey bar indicates the location of the damage.
Figure 6.18. The modal curvatures of the first section of the beam from the finite element analysis (Blue) and camera experiment (Red)
Figure 6.19. The modal curvatures of the second section of the beam from the finite element analysis (Blue) and camera experiment (Red)
From Figure 6.18, we can note that in the first three modes, there are modal curvature discontinuities near the damage location. And the modal curvatures estimated from the camera experiment match the finite element analysis results well in the first four modes of the beam. The location of the discontinuity in these experimental modal curvatures can be used to indicate the damage. For the fourth modal curvatures, the damage location is hard to be observed. Since the damage location is very close to one of the nodal points in the fourth mode shape, the damage has very limited influence on the modal curvature of the fourth mode.

From Figure 6.19, we can note that in the first three modes, there are modal curvature discontinuities near the damage location. However, for the first experimental modal curvature, we can also observe false damage detection. Except for the first mode, the modal curvatures estimated from the camera experiment match the finite element analysis results well. In this experiment, the location of the discontinuity in the second and third modal curvatures can be used to indicate the damage. One possible reason for the false damage detection in the first modal curvature is that the damage in this experiment is very close to the free end of the cantilever beam. The amplitude of the curvature near the free end is very small and can be easily affected by the measurement noise.
6.3 Wood Cantilever Beam with Single Damage

In this experiment, a wood cantilever beam with uniform rectangular cross-section was used. The dimensions of the beam are $135 \times 6.65 \times 1.96$ mm. A cut which locates 45 mm away from the fixed end was used to simulate the damage. The detailed dimensions of the beam are shown in Figure 6.20.

![Diagram of wood cantilever beam with dimensions](image)

Figure 6.20. The dimensions of the wood cantilever beam

For this experiment, the setup is same as shown in Figure 6.5. The wood cantilever beam was fixed by a clamp. The impact hammer was used to excite the vibration of the beam. One high-speed camera (Photron FASTCAM Mini AX200) was used to capture the beam vibration. A LED light with adjustable light intensity is used to illuminate the beam.

Different from the previous experiments. The measurement video of this experiment covers the whole length of the beam. Figure 6.21 shows the screenshot of the captured video. Twenty-six dot markers were drawn manually on the lateral surface of the beam. Each of the marker was used as a target for the tracking algorithm.
Figure 6.21. The screenshot of the measurement video
The resolution of the measurement video is $1024 \times 256$ pixels, the frame rate is 1000 fps, the shutter speed is $1/3000$ s and the total measurement length is 1000 ms. For each marker, the size is $20 \times 20$ pixels in the video. Detailed experiment parameters are shown in Table 6.9.

Table 6.9. Experiment parameters

<table>
<thead>
<tr>
<th>Experiment Parameters</th>
<th>Camera</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>1024 pixels</td>
</tr>
<tr>
<td>Width</td>
<td>256 pixels</td>
</tr>
<tr>
<td>Frame Rate</td>
<td>1000 fps</td>
</tr>
<tr>
<td>Shutter Speed</td>
<td>$1/3000$ s</td>
</tr>
<tr>
<td>Length</td>
<td>1000 ms</td>
</tr>
<tr>
<td>Marker Number</td>
<td>26</td>
</tr>
<tr>
<td>Marker Size</td>
<td>$20 \times 20$ pixels</td>
</tr>
</tbody>
</table>
Apply the Lucas-Kanade inverse compositional algorithm to the video, the displacement of all the markers can be estimated. Then the fast Fourier transform (FFT) was applied to each displacement. An averaged FFT result for all the markers can be calculated. The averaged FFT result is shown in Figure 6.22. From Figure 6.22, we can find that there is only one dominant peak in the frequency domain which represent the first natural frequency of the wood cantilever beam.

Figure 6.22. The averaged FFT result
Same as the previous experiment, we chose the first target’s displacement in each video as the reference to calculate the cross power spectrum density (CPSD) for all the extracted displacement signals. From Figure 6.22, we can note that the tested structure is lightly damped with only one dominant mode. Thus, the simple peak-picking method can be applied on the cross power spectrum density result to estimate the mode shapes of the structure\textsuperscript{56–58}. Figure 6.23 shows the normalized mode shape result from the experimental video.

![1st Mode Shape](image)

Figure 6.23. The first mode shape of the beam from the camera experiment
Using the central difference approximation method, the modal curvature of the first mode for the beam can be shown in Figure 6.24. In order to increase the accuracy of the modal curvature estimation, we used ten points with the same interval from the original modal shape\textsuperscript{59}. In this figure, the grey bar indicates the location of the damage.

![1st Modal Curvature](image)

**Figure 6.24.** The first modal curvature of the beam from the camera experiment

From Figure 6.24, we can find in the first modal curvature, there is a modal curvature discontinuity near the damage location. Thus, the location of the discontinuity in the experimental modal curvature can be used to indicate the damage.
6.4 Noise Effect Analysis

In this section, we will discuss the effect of measurement noise on the damage localization. We will first discuss the influence of video image noise on the displacement measurement accuracy. Second, the effect of the measured noisy mode shapes on the damage localization will be presented.

6.4.1 Video Image Noise

In order to study the effect of the video image noise on the displacement measurement accuracy, we built a ground-truth video which has two white crosses moving in the vertical direction with predefined sine trajectories. The amplitude of the vertical displacement for the two crosses are 20 pixels and 0.0001 pixels respectively. The horizontal displacements of the two crosses are both equal to zero. For the original video, there is no noise in the video images. Figure 6.25 shows the first frame of the video. The total length of the video is 100 frames. And the first frame to the twenty-first frame of the video are shown in the Figure 6.26.

![The first frame of the ground-truth video](image)

Figure 6.25. The first frame of the ground-truth video
Figure 6.26. The frame sequence of the ground-truth video

Since the amplitude of the vertical displacement for the right cross is under sub-pixel level. Its movement cannot be observed from the video.
Apply the Lucas-Kanade inverse compositional algorithm to the video, the vertical displacement of the two crosses can be estimated. The real and measured vertical displacement of the two crosses are shown in Figure 6.27. The figure also shows the displacement error between the real and measured value. From the figure, we can find that the Lucas-Kanade inverse compositional algorithm can achieve very high measurement accuracy ($10^{-5}$ pixels) when the video image is noise free.

Figure 6.27. The real and measured vertical displacement of the two crosses (Top) and the displacement error between the real and measured value (Bottom) for the noise free video
If we add Gaussian white noise (Mean = 0 and Variance = 0.001) to the original video and apply the Lucas-Kanade inverse compositional algorithm again, we can get the displacement and displacement error results as shown in Figure 6.28.

![Displacement graphs](image1)

Figure 6.28. The real and measured vertical displacement of the two crosses (Top) and the displacement error between the real and measured value (Bottom) for the video with Gaussian white noise (Mean = 0 and Variance = 0.001)
Similarly, we add Gaussian white noise (Mean = 0 and Variance = 0.01) to the original video and extract the displacement again, we can get the displacement and displacement error results as shown in Figure 6.29.

![Displacement graphs](image)

Figure 6.29. The real and measured vertical displacement of the two crosses (Top) and the displacement error between the real and measured value (Bottom) for the video with Gaussian white noise (Mean = 0 and Variance = 0.01)

From Figure 6.27 to 6.29, we can have the following conclusions: (1) The Lucas-Kanade inverse compositional algorithm can achieve high measurement resolution for the noise free video; (2) For the small displacement measurement, the Lucas-Kanade algorithm can be affected seriously by the noise in the video images. For the large displacement, the algorithm still has
good measurement accuracy; (3) The overall measurement accuracy is inversely proportional to the noise level of the video images.

6.4.2 Mode Shape Noise

Since in the real experimental environment, the measurement noise cannot be avoided. The noise of displacement measurement can be introduced into the mode shapes’ estimation. And it will further affect the accuracy of the damage localization in the structure. In this section, we will discuss the effect of the mode shape noise on the damage localization.

Let’s consider the beam case which is shown in Figure 6.1. The first four mode shapes and modal curvatures of the beam from finite element analysis is shown in Figure 6.30. We assume this result is noise free. From this figure, we can find that the discontinuity in the modal curvature can indicate the damage location accurately. If we add normally distributed random noise (With mean = 0 and standard deviation = 0.001) to the original mode shapes, and calculate the modal curvatures again, we can get the results as shown in Figure 6.31. From Figure 6.31, we can find that the change in the mode shapes are not noticeable, but the noise in the mode shapes has serious impact on the modal curvature estimation which can lead to false damage localization. The reason for this problem is that the central difference approximation method for the modal curvature estimation is very sensitive to the high frequency noise. A better solution for this problem is to filter the mode shapes before using the central difference approximation.
Figure 6.30. The first four normalized mode shapes (Without noise) and modal curvatures of the cantilever beam
Figure 6.31. The first four normalized mode shapes (With normally distributed random noise, mean = 0 and standard deviation = 0.001) and modal curvatures of the cantilever beam.
7 Conclusions and Future Work

7.1 Conclusions

The main objective of this research is to develop an experimental system based on computer vision method for damage detection in structure using measured vibration data. We first reviewed the working principles of several traditional vibration measurement methods. Such as piezoelectric accelerometer, linear variable differential transformer (LVDT) and laser doppler vibrometer (LDV). Discussed and compared their advantages and limitations. Then the principles of vision-based measurement methods were introduced. Including the optical flow method, digital image correlation (DIC) method and Lucas-Kanade tracking method. A detailed description was focused on the Lucas-Kanade algorithm and Lucas-Kanade inverse compositional algorithm. To introduce the vibration-based damage detection, we first presented the Euler-Bouroulli beam theory and the cracked beam theory. Then we discussed several vibration-based damage detection methods based on different dynamic parameters, such as natural frequency, mode shape, mode shape curvature, frequency response function curvature, flexibility and model strain energy. Finally, an experimental system for damage detection using high speed camera had been designed and tested. Experimental studies were conducted on several different cantilever beams (aluminum cantilever beam with single damage, aluminum cantilever beam with multiple damages, wood cantilever beam with single damage) where natural frequencies and mode shapes were measured. The artificial damage made by drilling or cutting could be localized by the camera measurement. The experimental results were compared and verified with the finite element analysis.

The specific contributions made in this study are summarized as follows:
(a) An experimental system using high speed camera for damage detection based on structural vibration measurement is proposed and tested. This system aims at providing an efficient method for damage detection in structure. Except for the damage detection, the proposed system can also be used for vibration monitoring, experimental modal analysis and operational modal analysis for small and large structures.

(b) Lucas-Kanade inverse compositional tracking algorithm is applied to the measurement video to obtain vibration displacement of structure. In comparison with traditional vibration sensors, the proposed system has demonstrated its capability for high spatial measurement resolution. It can extract the displacements of multiple targets simultaneously.

(c) The vibration measurement of lightweight structure by using the proposed system has been demonstrated. Since the proposed system is non-contact. It will not affect the vibration parameters of the measured structure. Which means it is very suitable for the vibration measurement of small or lightweight structures. And it also can be used for the vibration estimation for the remote structures or the structures that are difficult to access.

(d) Damage detection experiments were conducted on several different cantilever beams (aluminum cantilever beam with single damage, aluminum cantilever beam with multiple damages, wood cantilever beam with single damage) where natural frequencies and mode shapes were measured. Applying the modal curvature method to the extracted mode shapes, the artificial damages made by drilling or cutting could be localized in the test beams. The experimental results from the proposed system
were compared and verified with the finite element analysis. It shows that the proposed system has the ability for damage localization in structure.

(e) The noise analysis which includes the video image noise and the mode shape noise was discussed and investigated.

7.2 Future Work

The current study is mainly focused on the damage detection of small and simple structure. But it also can be applied for the large and complex structures. To further improve and explore the present methodology, there are several future research directions can be considered:

(a) In this study, the damage detection is based on the out-of-plane vibration measurement of a simple cantilever beam. The boundary condition of the tested structure can be extended to any other types. And the present method and system can be easily adapted for the in-plane vibration measurement of a plate. Which means it has the potential for the damage detection on the 2D plate.

(b) Combine with a second high speed camera, the current system can be integrated into a 3D vibration measurement system. It can be used for the in-plane and out-of-plane vibration measurement of complex structures.

(c) In order to improve the measurement accuracy, hand painted targets were used in the present study. To improve the practicability of this method, natural targets such as edges, textures and special patterns of the tested structure can be used and explored for the vibration measurement.

(d) Measurement noise is a main limitation for the vision-based damage detection system. To alleviate the measurement noise effect, advanced noise filtering method or sensor fusion from multiple sensors can be further investigated and explored.
8 References


