ABSTRACT

STRICKLAND, JAMES HERBERT. The Algebraic Habits of Mind of Community College Students Enrolled in Developmental Mathematics. (Under the direction of Dr. Allison McCulloch and Dr. Karen Hollebrands).

A cross-case analysis examined the algebraic habits of mind of community college students enrolled in a developmental mathematics program. Traditionally, algebraic habits of mind studies work with subjects that are labeled as advanced mathematics students, mathematicians, or educators. This study was the first of its kind to work with a population that does not fit this description. The goal of the study was to see if this population would exhibit the same algebraic habits of mind as those traditionally researched. Working with six tasks used in previous studies, the researcher observed the elicitation of algebraic habits from six community college students enrolled in developmental mathematics as they made sense of the tasks via task-based interviews.

The researcher found the participants demonstrated the same algebraic habits of mind as with other studies apart from one habit, equivalent expressions. The algebraic habit of mind known as equivalent expressions was not observed throughout the six tasks from any of the participants.

In addition to the established algebraic habits of mind, the researcher also noticed other productive behaviors that appeared to support sense-making of the algebraic tasks. Two of these behaviors, labeled as brute force and refinement, were behaviors determined by the researcher to not fall into the guidelines of the algebraic habits of mind framework (Driscoll, 1999) and were classified as arithmetic and general respectively. However, four potentially new algebraic habits of mind were uncovered from this study. How these habits fit into the algebraic habits of mind framework was argued in accordance to the guidelines established by Driscoll (1999). The
uncovering of these new algebraic habits of mind would impact researchers, educators, and curriculum developers.

Keywords: algebraic habits of mind, developmental student, algebra, habits, thinking behavior
The Algebraic Habits of Mind of Community College Students Enrolled in Developmental Mathematics

by

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A dissertation submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics Education

Raleigh, North Carolina 2019

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DEDICATION

This dissertation is dedicated to my lovely and patient wife, Sabrina Chappell-Strickland.
BIOGRAPHY

James Strickland attended East Carolina University on scholarship as a teaching fellow and subsequently began his career as an educator in 2002. In 2006, James received a scholarship from the North Carolina Principal Fellows program and attended North Carolina State University. Finding little satisfaction in his work as a school principal, James returned to teaching and pursued a doctor of philosophy in math education. James currently works at a community college as a center coordinator and lead tutor of a learning center. He resides in Cary, North Carolina with his lovely wife, Sabrina. Some of his notable achievements while in education include, North Carolina teaching fellow recipient, North Carolina principal fellow recipient, North Carolina Partnership in Mathematics and Science Trainer, North Carolina WISE facilitator, Phi Kappa Phi Honoree, and educator of the year.
ACKNOWLEDGMENTS

Throughout writing this dissertation, I have received a great deal of support and assistance. I would first like to thank my co-chairs, Allison McCulloch and Karen Hollebrands, for meeting with me weekly to discuss my dissertation and provide me guidance and peer review. Their expertise was invaluable in the overall process of writing about and completing my research. I would also like to thank my wife Sabrina Chappell for being patient and supportive of this endeavor that took far longer than it needed to take. As above, I would also like to thank my other chair members for being patiently and providing me the time that I need to have a completed quality dissertation. Lastly, I would like to thank Aaron Donaldson, a writing specialist, who assisted me with my grammar and word choice as I sought to complete this dissertation.
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Chapter 1: Introduction

Background

Nationally, 59% of community college students enroll in a developmental math course before taking their first college-level mathematics course (Hodara, 2013). Of this group, Complete College America (2012) found that only 22% successfully complete their first college algebra course within two years of starting college. Alarmingly, once students who initially enrolled in a developmental math course attempt their first college-level algebra class, 70% cease their college education altogether (Complete College America, 2012).

Frustration with understanding algebraic concepts is the most commonly cited reason provided for quitting by students who enroll in developmental math courses (Perin, 2018). One means to improve student understanding of algebraic concepts is to address an individual’s algebraic habits of mind (Papadopoulos, 2019; Eroğlu & Tanışlı, 2017; Goldenberg, Mark, & Cuoco, 2010). However, this researcher could not locate any research on the algebraic habits of mind of students enrolled in developmental mathematics programs and therefore, there is limited research on this population at the community college level.

What Are the Algebraic Habits of Mind?

Mathematical habits of mind are “more than just understanding important mathematical ideas and learning to apply useful methods and procedures…[h]abits of mind are useful for reasoning about the world from a quantitative or spatial perspective and for reasoning about the mathematical content itself” (Levasseur & Cuoco, 2009, p. 27). A habit of mind is not simply a strategy. A strategy is a behavior that an individual is taught to use in a specific circumstance. A habit of mind, on the other hand, is a thinking behavior that enables productive thinking about a concept and has become habituated to the extent that it used without prompting; therefore, a
mathematical habit of mind is a habit of mind that enables productively think about mathematics. Mathematical habits of mind represent a set of habits that transcend all mathematical disciplines, but there are also subject-specific (e.g. algebraic) habits of mind.

Algebraic habits of mind were first described by Driscoll (1999). In his categorization, Driscoll (1999) divides the construct of algebraic habits of mind into three categories: building rules to represent functions, doing and undoing, and abstracting from computation. Each category broadly defines a set of habits that, when used, encourage productive thinking about algebraic concepts and promote sense-making of algebraic problems.

Building rules to represent functions consists of seven habits: organizing information, predicting patterns, chunking the information, describing a rule, different representations, describing change, and justifying a rule. Students using habits from this category utilize habits that seek to identify relationships. Whereas building rules to represent functions relies heavily on the user to seek and identify functions, abstracting from computation relies on their use of structure.

Abstracting from computation includes the habits of computational shortcuts, calculating without computing, generalizing beyond examples, equivalent expressions, symbolic expressions, and justifying shortcuts. The user of these habits employ their understanding of algebraic structure to create shortcuts and generalizations about observed computations. For example, someone solving \((4x - 3)(2x + 1) = 0\) may know that the answers of a factored quadratic \((ax - c)(bx + d)\) when set to zero are always \(x = \frac{c}{a}, \frac{-d}{b}\). This sort of thinking - about the calculation independently of that particular numbers used - is an example of calculating without computing. Abstracting from computation requires the capacity to consider computations independent from the numbers used, “...thinking algebraically involves being able
to think about computations freed from the particular numbers to which they are tied in arithmetic” (Driscoll, 1999, p. 15). The two aforementioned categories of habits, building rules to represent functions and abstracting from computation, contain habituated acts focusing on related aspects of a concept that represent different ways of thinking; however, the inclusion of the third category of habits, doing and undoing, allows for natural movement of thought between these habits, which is why doing and undoing is often referred to as the bridging habit.

In the past, doing algebra most closely resembled the group of habits known as doing and undoing (Stigler, 2010). Doing and undoing is the symbolic manipulation one uses to solve, write, or rewrite mathematical sentences. For example, if given the equation $4x + 2 = 5x$, one would undo the given operations in this equation, working the operation backwards to determine the value of $x$ that would make this equation true. This category of habits (which includes input from output and working backwards) is often used when the participant transitions their thinking between the other two categories of habits (Driscoll, 1999).

**Why Focus on Algebraic Habits?**

Researchers have argued that productive habits of mind enable students to not only achieve success in school but also achieve success in their lives (Cuoco et al., 1996; Driscoll, 1999; Driscoll & Moyer, 2001; Matsuura et al., 2013). According to Cuoco et al. (1996), mathematical habits of mind are the habituated thinking behaviors that are classified as ‘good thinking’ or productive thinking in mathematical situations, “…it is to help high school students learn and adopt some of the ways that mathematicians think about problems…even more important is to give students the tools they will need in order to use, understand, and even make mathematics that does not yet exist (p. 376).” While evidence shows that many students who do well in mathematics draw upon the algebraic habits of mind to productively think about
problems (Driscoll et al., 2007; Matsuura et al., 2013; Lim, 2009), there is a significant gap in the research on elicited algebraic habits of mind of students enrolled in developmental mathematics courses.

Though no study was found that explicitly studied the algebraic habits of mind of students enrolled in developmental math programs, this researcher did discover some research that theorized about the habits of those who struggle to learn mathematical concepts. Though the research agrees that the habits of those who struggle to learn mathematical concepts are typically poor relative to the idealized mathematical habits of mind, the theories about ‘what exactly poor mathematical habits are’ can be divided into two separate groups. Some claim that poor mathematical habits (i.e. bad mathematical habits of mind) are unique and different from the idealized mathematical habits of mind, (Oesterle et al, 2016). On the other hand, some researchers like Goldenberg (1996), argue that 'bad habits' are simply underdeveloped mathematical habits of mind and are not distinct behaviors. Whether bad habits are distinct behaviors or underdeveloped could have significant pedagogical implications as there may be a difference in teaching one to change their habits as opposed to improving an already existing one. It is only with formal study of this population that one can more clearly see if students enrolled in developmental algebra courses draw upon the algebraic habits of mind or utilize a different set of habits as they think about algebraic concepts.

**Purpose of the Study**

This study sought to fill the void as to the specific habits that community college students enrolled in developmental mathematics courses utilize while engaged with algebraic tasks. Though algebraic habits of mind have been an area of focus for over twenty years, the research reported to date either emphasizes how educators can identify and improve algebraic habits of
mind of students (e.g. Driscoll & Moyer, 2001; Matsuura et al., 2013; Magiera, van den Kieboom, & Moyer, 2013) or examines examples of algebraic habits of mind of advanced mathematical students (e.g. Driscoll et al., 2007; Goldenberg, 1996; and Levasseur and Cuoco, 2009). The researcher could only find two studies that directly mentioned the algebraic habits of mind of students who struggled to learn mathematical concepts (Eroğlu & Tanışlı, 2017; Oesterle et al, 2016), but both were limited in that they discussed this population anecdotally as neither had intended to focus on this population exclusively.

Moreover, both studies characterized the thinking habits in oppositional ways. Eroğlu & Tanışlı (2017) found that all students, whether advanced or struggling, drew upon the same algebraic habits of mind when considering algebraic problems yet students who struggled with algebraic concepts tended to draw upon procedural/arithmetic skills, lacking the conceptual understanding to know how to best apply their procedural understanding. On the other hand, Oesterle et al. (2016) noted different dispositions between struggling students and advanced students. They noted that the struggling student tended to demonstrate 'bad habits' (e.g. looking for a quick answer, lacking persistence, used memorization of procedures and situations, guessing at what they think is expected). This study hopes to assist in resolving the debate as to ‘whether students enrolled in community college developmental mathematics courses use the algebraic habits of mind’ and to identify any other productive habit that this population may use as they make sense of algebraic concepts. Specifically, this study seeks to answer the following research questions:

- While solving algebraic problems, what algebraic habits of mind are drawn upon by community college students enrolled in a developmental mathematics course?
While solving algebraic problems, what other productive mathematical habits do community college students enrolled in a developmental mathematics course use?

**Overview of Methodological Approach**

This study design was an exploratory multiple-case study. An exploratory case study will be used as there is limited scholarship on algebraic habits of mind of students enrolled in developmental mathematics courses. Six tasks were chosen to elicit the algebraic habits of mind of students in developmental mathematics courses at a community college. The tasks were used in prior research and have been shown as effective at eliciting the habits of mind of mathematically advanced students. The researcher captured the habits observed within each task using video recording devices, written artifacts, and journaling.

**Significance of the Study**

It is argued that intervention is needed at both the secondary and developmental math levels to address the gaps in student thinking towards algebraic tasks (Complete College America, 2012). Calls for curricula organized around mathematical habits of mind have been made for the past two decades to improve student thinking about mathematics (Cuoco et al., 1996), and despite national efforts to incorporate habits of mind into the national curricula (Standard for Mathematical Practices, CCSS), first-year college students continue to enroll in developmental mathematics courses. This study will assist the research community in understanding exactly what habits students in developmental math courses use when thinking about algebraic problems. With these findings, educators, researchers, and other stakeholders can better math education curricula and classroom practices to ensure all students develop productive algebraic habits of mind.
**Definition of Terms**

Developmental Mathematics Student: A student who enrolled in a developmental mathematics program at a college before beginning their college-level mathematics.

Developmental Mathematics Programs: Programs designed to fill in gaps of understanding and procedural skills in college students as they prepare to take their first college-level course.

Algebraic Habits of Mind: Productive thinking habits that allow one to make sense of algebraic concepts.

Arithmetic Habit Approach: Using only numbers in computations to make sense of a mathematical situation.

Community College: A post-secondary public school that offers advanced training in multiple subjects and accepts enrollment of students from all backgrounds with a minimum of a high school equivalent.

**Organization of the Dissertation**

The next chapter will review the literature relevant to this study, chapter three will outline the methodology used in this study, chapter four will present the results, chapter five will discuss the results of a cross-case analysis, and chapter six will elucidate the implications and limitations.
Chapter 2: Literature Review

Introduction

Community colleges play a prominent role in our nation’s post-secondary educational system. Across the United States there are over 1,400 community colleges making up 46% (six million) of all college students in the nation (Juszkiewicz, 2015). And beyond sheer volume, the community college’s role in terms of positive social mobility is attested by the fact that many of the students educated in such institutions are those who may otherwise not have pursued their education beyond high school (McGrath & Spear, 1991). Community colleges provide access to higher education for many nontraditional students, minority students, first generation college students, and students of low socioeconomic status (National Center for Educational Statistics, 2008). Additionally, these institutions enroll the largest number of low income and first-generation college students (Bailey, Jenkins, & Leinbach, 2005). Community colleges accept students from all backgrounds and abilities as long as they have the minimum high school equivalency credit (Juszkiewicz, 2015). As a result, many students who enroll at a community college also enroll in developmental courses.

Fifty-nine percent of students who enroll at community colleges also enroll in developmental mathematics programs. This is compared to 33% of community college students who enroll in developmental English programs (Hodara, 2013, pg. 1). Difficulty with algebra is the most common reason students say that they enroll in a developmental mathematics program (Complete College America, 2012). As a result, developmental math programs design curricula that focuses on knowledge and skills linked to supporting algebra (Bailey, Jeong, & Cho, 2010). Yet, despite focusing on algebraic knowledge and skills in developmental math programs, 70%
of developmental students still cease their education after taking their first college-level math
course (Oesterle et al, 2016).

Driscoll and Moyer (2001) argue that those who demonstrate algebraic habits of mind can productively make sense of algebraic concepts. Given that students enroll in developmental mathematic programs at community colleges do so because they typically have difficulty with algebraic content, it would appear that Driscoll and Moyer (2001) would argue that this population may not utilize their habits productively or to the fullest extent.

In this chapter, the literature on community college mathematics and developmental algebra courses will be discussed, as well as a detailed review of the literature on habits of mind and algebraic habits of mind.

The Role of Community College in Mathematics Education

Historically, community college enrollment has outpaced that of four-year universities (Bulger & Watson, 2006). Given that community colleges have less academic requirements to enroll than four-year universities they also have a significantly higher percentage of enrollees into developmental mathematics (Hodara, 2013). This high percentage of students enrolled in developmental mathematics in community colleges leads to a need for understanding the educational needs of this population. Limited literature exists, that focuses on success of students enrolled in developmental programs with even less focused explicitly on mathematics and none, found by this researcher, on community college students enrolled in developmental mathematics programs.

In this section, the role of community colleges in mathematics education will be discussed. First there will be an examination of the important role that community colleges serve,
then there will be a deeper examination on the developmental math programs, and lastly, there will be a discussion on the particular issues students enrolled in developmental mathematics course have that act as a barrier to their success.

**The importance of community colleges and developmental courses.** As stated before, over six million students attend community college. Students attend community colleges for multiple purposes (e.g. workforce training, community enhancement, and developmental education). Sixty percent of community college students enroll for a degree while the remaining are seeking to transfer to finish their education (Juszkiewicz, 2015). Without community colleges, many at-risk and marginalized populations (e.g., special education, minority, low socioeconomic status) would not have an opportunity to attend college (National Center for Educational Statistics, 2008). In order to ensure its diverse population is prepared for college level courses, community colleges offer developmental courses in math, reading, and writing.

Seventy-nine percent of students entering community colleges need these developmental courses (Bailey, Jeong, & Cho, 2010), with 59% enrolling in developmental math courses (Hodara, 2013). While the number of students needing developmental coursework continues to grow, research on this population and their success rate, is limited (Barnett, 2008; Esch, 2009).

To clarify terms, it is important to note that remedial education and developmental education are not treated synonymously in the literature (Rings, 2001; Roueche & Roueche, 1999) with remedial having a deficit connotation (Boylan, 1999), although the term "remedial" has sometimes been used interchangeably with "developmental" when referring to developmental students or courses (Boylan & Saxon, 1999), for the sake of clarity and uniformity, this study will refer to the courses designed to prepare students for college-level material as "developmental".
**Developmental math programs.** Boylan and Saxon (1999) found two general trends in developmental education program research: (1) Methods and techniques characterizing effective instructional strategies (best practices) and (2) components and structure of developmental programs. Developmental education has often been considered unworthy of research in its own right (Boylan & Saxon, 1999; Grubb & Badway, 1998). As a result, there have been few attempts to identify best practices in postsecondary developmental education (Boylan & Saxon, 1999).

Between 1968 and 1978, Roueche and his colleagues were the most published developmental education researchers (Boylan, 1999). Initially, Roueche investigated the how learning theory could be applied to developmental courses (Boylan, 1999; Roueche, 1968; Roueche & Wheeler, 1973). In subsequent years, other studies found that to improve student performance, developmental courses should include clear goals and objectives (Donovan, 1974; Cross, 1976; Kulik & Kulik, 1991; Boylan, Bonham, Claxton, Bliss, 1992; and Boylan & Saxon, 1999), ensure the clarity of goals and objectives facilitated a "clear course structure" (Boylan & Saxon, 1999), and have well-defined philosophy forming the basis of any developmental programs (Casazza and Silverman 1996; Maxwell, 1997; Boylan & Saxon, 1998; and Boylan & Saxon, 1999).

As a result, developmental math education programs have undergone significant redesign in the past decade, along with other developmental programs with a shift from deficit thinking models to enhancement models (Boylan, 1999). More recently, developmental mathematics education programs began emphasizing skill enhancement and conceptual development as opposed to the global assumption of a lack of specific traits (Bahr, 2008). For example, one developmental mathematics program in a large southeastern community college uses their
entrance exam to assign development mathematics students into distinct curricula to address each individual’s unique needs creating specialized program paths to address each students’ strengths and weaknesses.

**Students in developmental mathematics programs.** In general, students who enroll in developmental programs begin college at an older age than other beginning college students (Burley, Butner, & Cejda, 2001), have more societal pressures causing them to share time between work, family, and school (Edgecombe, 2011; Rutschow et al, 2011), tend to have higher incidence of learning deficiencies (Burley et al., 2001; Rutschow et al., 2011) compared to the university student population, and tend to have lower self-efficacy than their peers (Bandura, 2005).

In addition, Stigler et al (2010) found that students enrolled in developmental mathematics programs rely on procedures and rules over number sense and reasoning (sense-making). As a result, Stigler (2010) furthers that since sense-making is under-utilized, students enrolled in developmental mathematics programs will often incorrectly and/or inappropriately use their procedural knowledge in mathematical situations and often lack understanding of the reasonableness of their answers.

Stigler et al. (2010) examined incorrect answers on four community college placement tests from 5,830 participants. Students who performed poorly on these exams were later enrolled in developmental mathematics programs. In this study, Stigler et al. (2010) wanted to see what common misunderstandings students had when they took these entrance exams. Specifically, they were trying to answer the research question: What do students actually understand about mathematics concepts that underlie the topics they’ve been taught? (p.4).
From their study, they found four themes. The first two suggested that students who struggled on the exam had the most difficulties with simplifying fractions and algebraic expressions and converting between decimals and fractions. They concluded that most students who did poorly on the entrance exam had an overreliance on rules and procedures rather than on concepts and number sense, “These errors reveal that rather than using number sense, students rely on memorized procedure, only to carry out the procedure incorrectly, or inappropriately (p. 10-11).

The third and fourth themes relate to problem solving. For the third theme, they noted that students who struggled on the assessments had a habit of “stopping short”. That is, on multiple step problems, they would correctly identify and carry out the first step but then abandon the problem. The researchers were unclear as to why the participants would stop short but hypothesized that they may have either been too reliant on procedure and forgot the next step or the student assumed that after the first step they must have completed the problem.

The fourth theme identified an overreliance on ‘math experience’. In some cases, students would use the structure of a problem, or answer set, and infer what the problem wants them to do rather than carefully consider what the problem is actually asking. For example, as explained by Stigler et al. (2010), when students were asked to find the LCM of 18 and 20, many recorded the answer of two which is the GCF. They found that this happened because when students practiced LCM and GCF problems, almost always the numbers chosen for practice for LCM problems had numbers that were coprime while GCF problems had almost always had numbers that shared common factors. Given that the numbers 18 and 20 had a common factor, the students chose to find the GCF even though the problem explicitly stated to find the least common multiple.
**Algebra in developmental mathematics courses.** Most developmental mathematics programs have an algebraic focus (Stigler et al, 2010). That is, most of these programs teach students the skills and concepts that would make them successful in a college algebra course instead of the skills and concepts needed to be successful in say a geometry course. The emphasis on algebra readiness of all students before beginning college-level curriculum makes it a clear expectation for all community college students. In developmental math courses, algebra is typically learned by examining algebraic structure and the concept of function. In the next section, each term and its implication for learning algebra is discussed in detail.

**Algebra**

Learning algebra can be described as learning about algebraic structure and the concept of function (Sfard & Linchevski, 1994). What follows is a discussion on what it means to learn algebra through learning about algebraic structure and function.

**Algebraic Structure.** The term structure is widely used in research but is rarely consistently defined. In different contexts, the term structure can mean different things to different people (e.g., Dreyfus & Eisenberg, 1996; Hoch & Dreyfus, 2004; Stehlíková, 2004). The term algebraic structure when used in abstract algebra is understood to consist of a set closed under one or more operations, satisfying some axioms (Linchevski & Livneh, 1999).

Linchevski and Vinner (1990) suggested that one of the components of success in school mathematics is the ability to identify hidden structures in algebraic terms. The characterization of a student’s ability to use structure is known as structure-sense (Linchevski and Livneh, 1999). Structure-sense is an extension of the notion of symbolic-sense which is also an extension of number sense (Novtona and Hoch, 2008).
To back up for a moment, number sense is described as a student’s intuition for numbers such as knowing simply by looking when a computation produced an obvious wrong answer and an instinct for choosing the correct computation for a given problem (Greeno, 1991). Similarly, symbol sense is one having a complex feel for symbols (Arcavi 1994). It is the ability to manipulate and to interpret symbolic expressions, and a sense of the different roles symbols can play in different contexts.

Finally, Kirshner and Awtry (2004) added that students using structure have an understanding of the visual salience of algebraic rules: “Visually salient rules have a visual coherence that makes the left- and right-hand sides of the equation appear naturally related to one another” (p. 229). For example, the rule $\frac{x \cdot r}{y \cdot c} = \frac{xr}{yc}$ is visually salient because a student can easily see how the numerator of the product $xr$ is related to the factors $x$ and $r$. On the other hand, the rule $\sin(x) = \sqrt{\sec^2(x)-1}$ is not because it is not visually obvious how $\sec(x)$ is related to $\sin(x)$. Therefore, structure sense requires a sense of anticipation which, as noted by Boero (2001), supports one’s understanding of an algebraic problem and how best to approach an algebraic situation. Tall and Thomas (1991) indicated that one needs to have versatility in thought when analyzing an algebraic form from a structural context. This versatility allows one to freely shift their focus from specific computational situations to global structural thinking and back as situations may require and would allow one to use their understanding of structure as they formally resolve algebraic situations.

**Function.** The concept of function is central to students’ ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs. Not surprisingly, Principles and Standards for School Mathematics (NCTM 2000, p. 296) advocates
instructional programs from prekindergarten through grade 12 that “enable all students to understand patterns, relations, and functions.” Research has shown that an understanding of function develops over an extended period of time. Although the function concept is a central one in mathematics, many research studies of high school and college students have shown that it is also one of the most difficult for students to understand (Tall 1996; Sierpinska 1992; Markovits, Eylon, and Bruckheimer 1988; Dreyfus and Eisenberg 1982).

Conceptualizing function. The concept of function in mathematics dates back to at least the seventeenth century and has evolved considerably over that time. Early in the 18th century Bernoulli used ‘function’ to describe an expression with at least one variable and constant: “One calls here function of a variable a quantity composed in any manner whatever of this variable and of constants” (cited in Kleiner, 1989, p. 284). Kleiner notes that it was Euler who brought the concept to prominence by treating calculus as a formal theory of functions. Eisenberg (1992) remarks that the mathematics education literature on functions can be divided broadly into two groups: (1) encouraging the teaching of functions and (2) best practices for teaching functions and misconceptions related to learning functions.

Vinner and Dreyfus (1989) note that “the examples used to illustrate and work with the concept are usually, sometimes exclusively, functions whose rule of correspondence is given by a formula.” Consequently, students are able to give a definition, but their work on identification or construction tasks are solely based on a formula conception. Vinner and Dreyfus (1989) also suggest that students do not necessarily use the definition when deciding whether a given mathematical object is a function, deciding instead on the basis of a ‘concept image’; that is, the set of all the mental pictures associated in the student’s mind as a result of his or her experience with examples and non-examples of functions.
Tall and Bakar (1992) report on a study with twenty-eight students (aged 16-17) who had studied the notion of a function during the previous year and had used functions in a college course, but with little emphasis on aspects such as domain and range. The students were tasked: “Explain in a sentence or so what you think a function is. If you can give a definition of a function, then do so” (p. 105). Tall and Bakar (1992) note that none of the students gave satisfactory definitions but were able to provide explanations, such as the following:

- a function is like an equation which has variable inputs, processes the inputted number, and gives an output.
- a process that numbers go through, treating them all the same to get an answer.
- an order which plots a curve or straight line on a graph.
- a term which will produce a sequence of numbers, when a random set of numbers is fed into the term.
- a series of calculations to determine a final answer, to which you have submitted a digit.
- a set of instructions that you can put numbers through. (p. 105)

Tall and Bakar (1992) note that “most of the students expressed some idea of the process aspect of function – taking some kind of input and carrying out some procedure to produce an output – but no one mentioned that this only applies to a certain domain of inputs, or that it takes a range of values (p. 1050).” This lack of precision, as noted by Tall and Bakar (1992) could lead to potential difficulties for students learning more advanced mathematical concepts.

**Learning about functions.** Tall and Bakar (1992) claim that even though curriculum documents include the definition of the function concept, the definition “is not stressed and proves to be inoperative, with student understanding of the concept reliant on properties of familiar prototype examples” (p. 212) with the result that students have many misconceptions.
Tall and Bakar (1992) added by example, that 44% of a sample of 109 students starting a university mathematics course considered a constant function as not a function in at least one of its graphical or algebraic forms, usually because $y$ is independent of the value of $x$. Furthermore, 62% of the students thought that a circle is a function.

Carlson, in 1998, also studied student understanding of function. They undertook a study which included high achieving students who had either just completed college algebra or, at least, some calculus subjects. The college algebra course included an introduction to functions. Carlson (1998) found that many of the students did not understand function notation, had difficulty understanding the role of the independent and dependent variables in a given functional relationship, could not explain what is meant by expressing one quantity as a function of another, and were unable to speak the language of functions. Students reported that due to the amount of information learned over the short period of time, forced them to replace understanding with memorization.

In addition to lacking understanding of the definition of function and over memorization of facts discussed in the previous two paragraphs, Oehrtman, Carlson and Thompson (2008) also found that students confuse visual attributes of a real-world situation with similar attributes of a graph of a function modeling the situation. School mathematics tends to focus on special features of graphs, for example, turning points, points of inflection and gradient. Function models of real-world situations sometimes exhibit similar features, for example, a road going up a hill, a curve in a road, or a vehicle slowing down. Oehrtman, Carlson and Thompson (2008) note that the superficial similarity of these features of graphs and the real world setting often leads to confusion, even for students with a strong understanding of functions. They assert that “students are thinking of the graph of a function as a picture of a physical situation rather than as a
mapping from a set of input values to a set of output values. Developing an understanding of function in such real-world situations that model dynamic change is an important bridge for success in advanced mathematics” (p. 154).

Though developmental mathematics programs improve one’s chances of success in their first college algebra course, it is clear from the high failure rate mentioned earlier that more needs to be done. As stated in chapter one, one means to improve student success with learning algebraic concepts is to address an individual’s algebraic habits of mind (Papadopoulos, 2019; Eroğlu & Tanışlı, 2017; Goldenberg et al, 2010). Therefore, the next section will explore and discuss the literature as it relates to habits of mind.

**Habits of Mind**

Habits of Mind are a collection of thinking behaviors that when used appropriately can lead to improved sense-making (Costa, 2008). In figure 1, below, one can see that there are general habits of mind as well as subject specific habits of mind for mathematics.

![Figure 1: Relationships among habits of mind constructs](image.png)
In this section, we will discuss and define terms within general habits of mind, mathematical habits of mind, and algebraic habits of mind, as well as examine current research on algebraic habits of mind.

**General habits of mind.** Cuoco et al. (1996) delineates between general and subject-specific mathematical habits of mind. General habits of mind, such as experimenting, patterning, and describing are habits of mind that will help students make sense of problems regardless of the subject or context. For example, productive thinkers, would search out patterns when in a function table much the same way they would seek out patterns in the structure of a sentence to gleam both of their meanings. If they are presented a new situation, they study their environment and look for relationships that will assist them to infer what actions they should take. This type of habit leads to behaviors that will encourage one to be successful in a multitude of situations.

**Mathematical habits of mind.** Cuoco et al. (1996) described mathematical habits of mind as means to "close the gap between what mathematicians do and what they say" (p. 376). According to Levasseur and Cuoco (2009), mathematical habits of mind, or mathematical modes of thought, enable us to reason about the world from a quantitative and spatial perspective and to reason about math content to empower us to use our mathematical knowledge and skills to make sense of and solve problems. Further, Oesterle et al. (2016) states that Mathematical Habits of Mind are drawn upon "when we habitually choose actions and strategies, pose questions, and display attitudes that are productive in a mathematical context." Even more specifically, Goldenberg (1996) defined habits of mind as thinking that "one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become mental habits – one not only can draw upon them easily, but one is likely to do so” (p. 13).
Though Cuoco et al. (1996) first coined the term Mathematical Habits of Mind, other researchers have used the term differently. Where Cuoco et al. (1996) define it to mean productive thinking that leads towards resolving a problem, Lim (2009) argues that we should use the term more broadly to simply describe the habitual thinking that occurs when one is solving a mathematical problem. Lim (2009) claims, that focusing solely on productive behaviors of successful school students, researchers may miss an opportunity to examine habitual thinking that occurs in individuals who think differently. Currently, by the definition of habits of mind, only those habits that are demonstrated by mathematicians (though the term mathematician is not carefully defined in the literature) are prized as the essential mathematical habits students should develop.

**Differing views on unproductive habits.** The literature is not in agreement on unproductive thinking behaviors. Lim (2009) and Oesterle et al (2016) argued that unproductive thinking behaviors are a set of thinking behaviors distinct from the habits of mind. They find that students who lack command of mathematical concepts tend to demonstrate behaviors that can be characterized as impulsive, overreliance on visual clues, lacking persistence, and dependent on procedural and memorization. These results were consistent with the findings by Carlson (1998) and Stigler et al. (2010) but neither actually classified such behaviors as unproductive habits.

On the other hand, researchers like Goldenberg (1996) and Driscoll and Moyer (2001) argue that the habits described above are not unique behaviors but are underdeveloped habits of mind, implying that these habits exist as a continuum in regards to sophistication. Goldenberg (2016, personal communication, August 28, 2016) argues that experiences one has in schools likely discouraged the growth and development of a particular habit leaving the student to use it in other settings but not when making sense of mathematics:
Sorry! But when you mentioned the self-identified poor math students, I couldn’t help myself. These are—nearly all, I believe—kids who got a bad break over and over. Anyway, that’s what I see both as teacher and as researcher. They did use child-versions of the necessary habits of mind when they were little, and they got taught not to develop them (not always just in school). In particular, they (like most of us, even those who wind up with some success in mathematics and even many who become mathematics teachers) learn that “mathematics is about learning the facts, calculating fast and accurately, and following rules,” not about tinkering, looking for structure, making conjectures, testing them.

**Subject-specific habits of mind.** Subject-specific habits of mind are habits of mind that are consistently drawn upon by students as they work with problems in a particular subject. Researchers like Driscoll and Moyer (2001) have worked to classify and study subject-specific habits of mind. Currently only algebraic habits of mind and geometric habits of mind have been thoroughly explored in this regard; however, Lee and Tran (2015) have begun examining statistical habits of mind.

**Geometric habits of mind.** Geometric habits of mind contain a distinct set of habits that mathematicians use when considering geometric tasks. These habits, like algebraic habits, can be seen in other mathematical fields but they dominate in use in geometry. For example, when presented with a problem asking one to relate two different shapes, given nothing more than an image, one is more likely to draw upon the geometric habit of mind of invariance and reasoning with relationships as opposed to the mentioned algebraic habits (Driscoll et al, 2007). In this problem, since the student is not provided with numbers or much in the way of a formula to act upon, naturally the habits of mind most closely associated with shape will be drawn upon.
However, if the shapes given, are shapes that one knows formulas for, then there is potential for one to use their algebraic habits of mind to interact with the formulas as a means of supporting their geometric habits of mind (Driscoll et al, 2007).

**Algebraic habits of mind.** According to (Driscoll, 1999), algebraic habits of mind consist of three major habits: (1) Doing and Undoing, (2) Building rules to represent functions, and (3) Abstracting from Computation, see figure 2 below. Driscoll (1999) argues that these habits of mind enables us to learn algebra. By using the algebraic habits of mind, the learner is able to access deeper meaning during algebraic tasks and consider concepts at a higher level that would not be available without the algebraic habits (Driscoll, 1999). The algebraic habits of mind allow for one to consider the nature of functions, how they work, and the impact that a system’s structure has on calculations (Driscoll & Moyer, 2001).

![Diagram](image.png)

Figure 2: Driscoll (1999) algebraic habits of mind (p. 1)
**Algebraic habits of mind: building rules to represent functions.** Building rules to represent functions consists of pattern seeking, pattern recognition and generalization components, which occurs in the analysis of the problem solving process (Driscoll, 1999). Building rules to represent functions is the ability to recognize patterns, organize data to represent situations relating to well-defined functions (Driscoll, 1999). According to Driscoll and Moyer (2001), there are seven defined habits that fit into this category of behavior. The habits are: organizing information, predicting patterns, chunking the information, describing a rule, different representations, describing change, justifying a rule.

Students regularly draw on algebraic habits of mind when they are seeking patterns in algebra; that is, when they study which aspects change and which stay the same. For example, if students were tasked to examine the following tables:

![Tables](image)

Figure 3: Three distinct tables with the same slope.

The students could see that the slope, \( \frac{\Delta y}{\Delta x} \), is constant and the same for all of the tables meaning that each function is representative of the family of functions \( y = 2x + b \). Understanding this relationship, students could then easily extend each table and make inferences.
The category building rules to represent functions is mostly associated with the study of concepts that fall within the domain of function as described in the CCSS-M (Magiera, van den Kieboom, & Moyer, 2013). When one is using this habit of mind they are seeking out patterns, rules, and are attempting to find alternative representations of a situation, these are all techniques that enhance one’s understanding of function (Driscoll 1999). However, this algebraic habit of mind which Driscoll (1999) calls the natural complement to Doing-Undoing, also assists learners with their understanding of how algebraic function families and situations work, even if only as an interplay with abstracting from computation, and therefore does contribute to one’s understanding of structure (Magiera, van den Kieboom, & Moyer, 2013). Building rules to represent functions enable students to reason about the input and output behavior over a domain of a function as well as understand the concept of what situations can and cannot be a function. In figure 4, Driscoll breaks down the specific behavior’s one sees in connection to building rules to represent functions.

<table>
<thead>
<tr>
<th>Habit of Mind</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Organizing information</td>
<td>Organizing information in ways useful for uncovering patterns and the rules that define the patterns</td>
</tr>
<tr>
<td>Predicting Patterns</td>
<td>Noticing a rule at work and trying to predict how it works</td>
</tr>
<tr>
<td>Chunking the information</td>
<td>Looking for repeating chunks of information that reveal how a pattern works</td>
</tr>
<tr>
<td>Describing a rule</td>
<td>Describing the steps of a rule without using specific inputs</td>
</tr>
<tr>
<td>Different representations</td>
<td>Wondering what different information about a situation or problem may be given by different representations</td>
</tr>
<tr>
<td>Describing change</td>
<td>Describing change in a process or relationships</td>
</tr>
<tr>
<td>Justifying a rule</td>
<td>Justifying why a rule works for &quot;any number&quot;</td>
</tr>
</tbody>
</table>

Figure 4: Description of habits related to building rules to represent functions
Algebraic habits of mind: abstracting from computation. The category of habits abstracting from computation require students to use structures and formulate generalizations about computation (Driscoll, 1999). Abstracting from computation requires the capacity to consider computations independent from the particular numbers used, “...thinking algebraically involves being able to think about computations freed from the particular numbers to which they are tied in arithmetic” (Driscoll, 1999, p. 15).

For example, in the stacking cans problem by Steins and Smith (1998), figure 5,

![Figure 5: Stacking Cans](image)

the students explore figurate numbers with the goal of having students use functions to model relationships between quantities. The students in the study attempted to determine how many cans are needed to form a stack if the base had thirty cans. Throughout the activity, many unique approaches to solving this task were used culminating in a group of students generalizing a function for n-cases. The students made the generalization by examining several smaller stack sums. Using these sums, the participants made a function that described the nature of change they were observing to generalize beyond their examples and create a function that models the
number of cans in any stack when given the row amount, (as this description indicates, the participants used habits of mind from multiple categories).

Student understanding of structure and function as domains of algebraic understanding is assisted by the algebraic habit of mind of abstracting from computation. It allows them to generalize a mathematical structure, adding to their structure-sense, allowing them to understand how broad concepts interrelate while also allowing the student to understand the input/output nature of functions. However, this algebraic habit of mind is most often elicited from mathematicians when one needs to make sense of structure (Driscoll 1999). In figure 6, Driscoll breaks down the specific behavior’s one sees in connection to abstracting from computation.

<table>
<thead>
<tr>
<th>Habit of Mind</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational shortcuts</td>
<td>Looking for shortcuts in computation based on an understanding of how operations work</td>
</tr>
<tr>
<td>Calculating without computing</td>
<td>Thinking about calculations independently of the particular numbers used</td>
</tr>
<tr>
<td>Generalizing beyond examples</td>
<td>Going beyond a few examples to create generalized expressions, describe sets of numbers, state or conjecture the conditions under which particular mathematical statements are valid</td>
</tr>
<tr>
<td>Equivalent expressions</td>
<td>Recognizing equivalence between expressions</td>
</tr>
<tr>
<td>Symbolic expressions</td>
<td>Expressing generalizations about operations to justify computational shortcuts</td>
</tr>
<tr>
<td>Justifying shortcuts</td>
<td>Using generalizations about operations to justify computational shortcuts</td>
</tr>
</tbody>
</table>

Figure 6: Description of habits related to abstracting from computation.

Algebraic habits of mind: doing and undoing. The category of doing and undoing is a collection of algebraic habits that are considered the middle ground for the other two algebraic
habits. The ability to not only use a process to achieve a goal, but also have the understanding to reverse the process from an answer to its starting point (Driscoll, 1999, p. 15). Doing and undoing is essentially about the user’s understanding of reversibility (Driscoll, 1999). As an exemplar, student examination of the features of the quadratic function can require the utilization of doing and undoing. Say a student is considering the function $f(x) = 20(x - \frac{3}{20})^2 - \frac{49}{20}$, the student can use the function in its transformation form as given to determine the vertex, but they can also undo this form and rewrite it into standard form to find the y-intercept and go even further to rewrite the equation in factored form to determine the zeros of the function. Though the finding of multiple representations for the same function is part of the habit of equivalent expressions (within the abstracting from computation category), the act of changing the function by calculation and manipulation of expression to go between those equivalent expressions is what is referenced by the habits of doing and undoing.

According to Magiera, van den Kieboom, and Moyer (2013), doing and undoing is the most balanced algebraic habits of mind in that students draw on it when working on problems related to function and those that involve algebraic structure. For example, as noted by Boero (2001), when working backward from a solution, students are undoing a pattern or rule they notice about a function or process but when they finish and assess the reasonableness of their findings, they often check their work from a structural perspective (e.g. does my number look right?)

In figure 7, Driscoll (1999) defines the specific behavior’s one sees in connection to Doing and Undoing:
Doing and Undoing (Driscoll, 1999)

<table>
<thead>
<tr>
<th>Habit of Mind</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input from output</td>
<td>Finding input from output or initial conditions from a solution</td>
</tr>
<tr>
<td>Working backward</td>
<td>Working the steps of a rule or procedure backward</td>
</tr>
</tbody>
</table>

Figure 7: Description of habits related to doing and undoing.

Review of research on algebraic habits of mind. Of the seventeen studies found by this researcher that specifically focus on algebraic habits of mind, thirteen were case studies and four were a phenomenology. Current research on Algebraic Habits of Mind fall into two categories with one exception. What follows is a discussion of those two categories and the one study that does not fit within these groups.

Category 1: How are students using algebraic habits of mind to successfully make sense of problems? Students in these studies participate in task-based interviews where they make sense of the algebraic task. The researchers question the participants to learn about their thinking and then their algebraic habits of mind are discussed. Studies that focused on the actual utilization of algebraic habits of mind use participants who are identified as highly proficient in mathematics in terms of classroom performance.

From these studies, they found that students with sufficient command of algebraic habits of mind generally draw upon the same algebraic habits of mind as similar peers when doing the same tasks regardless of their personal background (Driscoll & Moyer, 2001; Eroğlu & Tanışlı, 2017); that is, if high achieving students used equivalent expressions on a particular task other high achieving students also used equivalent expressions on the same task. Furthermore, researchers have shown that students who already demonstrate algebraic habits of mind can sharpen their use of associated skills (e.g. improved pattern recognition) through practice and
modeling of sophisticated thinking behavior (Driscoll & Moyer, 2001; Magiera, van den Kieboom, and Moyer, 2013). Lastly, researchers have found that learners at any stage of their mathematical development (elementary, secondary, post-secondary) can improve their algebraic habits of mind (Driscoll, 1999; Eroğlu & Tanışlı, 2017; Matsuura 2013). Therefore, even though it is not stated in the literature, one could logically conclude that individuals who have not demonstrated success in algebraic courses could improve their algebraic habits of mind which as stated before is strongly linked with successful performance in algebra courses and thus improve their academic proficiency in the subject matter.

**Category 2: How do we teach to identify/improve algebraic habits of mind?** These studies generally had two components: (1) Assisting teachers in recognizing algebraic habits of mind and (2) examining the qualitative nature of the observed habit of mind. In assisting teachers to recognize algebraic habits of mind, researchers provide educators with artifacts of student thinking, such as videos of students interacting with a problem and student work. The participants would then discuss in small or whole groups, and in some instances one-on-one the habits they were observing relative to their understanding of the algebraic habits of mind as defined by Driscoll (1999). When examining the qualitative nature of an individual’s algebraic habits of mind, the researcher would encourage the participants to rank samples of student work as strong indications of command of algebraic habits of mind to weak and/or limited evidence of algebraic habits of mind.

The findings from the studies can be summarized into four major points. Researchers found that educators were better at identifying algebraic habits of mind in their students and implement algebraic habits of mind lessons if they themselves had sufficient command of their own algebraic habits of mind (Matsuura, 2013). Also, Participants were more willing to
showcase their algebraic habits of mind in one-on-one sessions and in small groups (Driscoll & Moyer, 2001). If one’s algebraic habit of mind improves, so does their understanding of algebraic concepts (Papadopoulos, 2019; Eroğlu & Tanışlı, 2017), Lastly, researchers found that participants are most able to improve their understanding of identifying and teaching others algebraic habits of mind if they first improve their own algebraic habits of mind (Matsuura, 2013). The exception, “K-8 pre-service teachers’ algebraic thinking: Exploring the habit of mind ‘building rules to represent functions’” (Magiera, van den Kieboom, & Moyer, 2017, p. 50). Magiera et al (2017), had similar elements as its predecessors. It was a case study that used the building rules to represent functions as described by Driscoll (1999) as a framework to examine whether and how eighteen pre-service teachers use the algebraic habits of mind while making sense of an algebraic tasks.

The participants were enrolled in a course designed to engage them with tasks that cause them to regularly draw upon the algebraic habits of mind as opposed to explicitly teaching them. The results of the tasks were rated in terms of the qualitative nature of the habits used to make sense of algebraic concepts. Using a one-on-one task-based interview format, the participants engaged with tasks that were chosen to elicit the algebraic habits of mind. The observation and usage of habits were reported in the study. This portion of the study is consistent with much of the other studies that examined teacher understanding, usage, and pedagogical implications of algebraic habits of mind. What makes this study unique from the rest, was that it is the first of its kind, that this researcher found, that also examined the associations between habits.

Magiera et al (2017) examined the relationship between the occurrences of specific habits as well as to the qualitative ratings assigned to each habit to determine if there were any consistent associations. That is, this study not only examined whether two specific habits
occurred in conjunction (if someone uses the habit of describing a rule they also used the habit of describing change) they also examined if they consistently had a direct relationship (if someone uses the habit describing a rule well, they also used the habit of describing change well). From their work, as indicated in the table below, Magiera et al (2017) found some significant associations (p.39):

Table 1: PST’s Mean AT-Feature Scores

<table>
<thead>
<tr>
<th>AT Feature</th>
<th>Feature 1</th>
<th>Feature 2</th>
<th>Feature 3</th>
<th>Feature 4</th>
<th>Feature 5</th>
<th>Feature 6</th>
<th>Feature 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Organizing Information</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Predicting Patterns</td>
<td>0.717**</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Chunking Information</td>
<td>0.535*</td>
<td>0.914**</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Different Representations</td>
<td>0.387</td>
<td>0.466</td>
<td>0.399</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Describing a Rule</td>
<td>0.508*</td>
<td>0.771**</td>
<td>0.727**</td>
<td>0.277</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Describing Change</td>
<td>0.462</td>
<td>0.337</td>
<td>0.321</td>
<td>0.122</td>
<td>0.173</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>7. Justifying a Rule</td>
<td>0.444</td>
<td>0.537*</td>
<td>0.484*</td>
<td>0.324</td>
<td>0.360</td>
<td>0.376</td>
<td>-</td>
</tr>
</tbody>
</table>

*p < 0.05, ** p < 0.01

As this table shows, within the domain of building rules to represent functions, organizing information and predicting patterns, organizing information and chunking information, organizing information and describing a rule, predicting patterns and chunking information, predicting patterns and describing a rule, predicting patterns and justifying a rule, chunking information and describing a rule, and chunking information and justifying a rule all have positive associations. That is, they found that each of those aforementioned pairings have a direct relationship in that if one habit in a pairing was used well, they would expect its pair to also be used well. In their conclusion they added that since these habits were strongly associated with each other than if one was able to improve one of these habits we would expect its pair to also improve. The study did not examine the other categories of habits so it is unclear if the associations only exist within the domain of building rules to represent functions.
In regards to algebraic habits of mind, different researchers have hypothesized about students enrolled in developmental mathematics programs with contrasting claims. For example, Oesterle et al (2016) argues that students who lack grade-level proficiency (e.g. students who are enrolled in developmental mathematics courses) have most likely developed 'bad habits' of mind; habits that are unique and different from the idealized mathematical habits of mind that work to hinder their sense-making of algebraic tasks.

Oesterle et al (2016) claim comes from their study comparing the effects of assigning leaders to small groups with differing qualitative levels of algebraic habits of mind as determined by the researchers. Oesterle et al (2016) noted that the groups that struggled tended to possess general 'bad habits of mind': (1) a tendency to look for a quick answer, (2) had a lack of persistence when the answer is not obvious, (3) had a preference for memorization over understanding, (4) and tried to guess or recall what is expected rather than engage in the problem. Oesterle et al. (2016) discusses multiple instances and observations of students demonstrating the above behaviors when the group struggled to make sense of the task but did not observe these behaviors in groups that had leaders who were adept at using their algebraic habits of mind. In one particular example, they discuss a task where students, having worked on polygonal area for a week, had to find the area of two scalene triangles (p. 67).

In this study, students worked in heterogeneous academic groups. Some groups performed better than others on the task despite each group having students assigned a leader at or above grade-level. The groups that performed well were led by students who consistently utilized their algebraic habits of mind. While, on the other hand, the groups that underperformed were led by students who demonstrated the bad habits mentioned before. The researchers noted that once a leader elicited a certain type of habit (algebraic habit of mind or unproductive habit
of mind) it was rare for the participant or the group to switch to the other type of habits. They did not set out to study ‘bad habits of mind’ of students so there was little exploration beyond anecdotal observations made by the researchers.

Contrarily, Goldenberg (1996) does not believe that habits are dichotomously good or bad. As stated earlier, Goldenberg (2016, personal communication with researcher) argues that these two characteristics may not be mutually exclusive. He argues that 'bad habits' are simply underdeveloped habits of mind, meaning that mathematical habits of mind, specifically algebraic habits of mind, exists on a continuum as opposed to being from two different sets of drawn upon thinking habits. He adds that all individuals have the potential to refine and develop algebraic habits of mind but either due to social, cultural, or identity reasons may have had some aspect of their habit development stunted. Goldenberg also never studied this claim explicitly, it has always been a personal belief he has held about algebraic habits of mind (2016, personal communication with researcher) and has been an underpinning for his research.

Neither researcher has formally studied the population of students who are classified as needing developmental support and thus have never studied community college students enrolled in developmental mathematics courses. They used only anecdotal observation and conjecture from their understanding of the meaning of mathematical habits of mind to speculate the thinking behavior of students who need more support to learn algebraic concepts. It is only with formal study of students enrolled in developmental algebra courses can one determine if this population can elicit the same algebraic habits of mind or if they utilize a different set of habits as they make sense of algebraic topics.

With a high percentage of students in developmental mathematics failing to achieve a credential within six years of starting at a community college, one must act now to understand
how students enrolled in developmental mathematics make sense of algebraic tasks. This understanding will have implications across many aspects of mathematics education including pedagogical, curriculum design, and educational research. An important step in this direction is the examination of the elicited habits when completing tasks designed to encourage the usage of algebraic habits of mind by this population.
Chapter 3: Methods

Introduction

Using a case study approach, the researcher examined the productive habits elicited from six algebraic tasks. The tasks were adapted from prior studies with the explicit intention to encourage the use of the algebraic habits of mind. Six participants enrolled in developmental mathematics courses, after completing their algebraic portion of the program, completed these six tasks in a task-based interview format. Using six participants provided this researcher thirty-six opportunities to observe the habits of mind consistently used while making sense of the algebraic tasks. The observed habits were coded according to Driscoll’s (1999) framework for algebraic habits of mind. Any productive habits that could not be described using Driscoll’s (1999) framework were characterized, triangulated, and themed with other similar observations. These themes were then checked against Driscoll’s (1999) four criteria for inclusion as a new habit into the algebraic habits of mind. The goal of this study, again, is to answer the following research questions:

RQ1: What algebraic habits of mind are drawn upon by community college students enrolled in a developmental mathematics course when solving algebraic problems?

RQ2: What other productive mathematical habits do community college students enrolled in a developmental mathematics course use while solving algebraic problems?

Study Design

The underlying goal of this study is to understand the habits of mind of students enrolled in developmental mathematics courses at community colleges. This study focused on six purposefully selected tasks designed to elicit the algebraic habits of mind. The participants that engaged the tasks were selected from a pool of students meeting specific criteria related to the
goal of this study. Each participant was enrolled in developmental mathematics, completed their algebraic module, communicated their thoughts clearly, and were community college students. The researcher chose to only target developmental students who completed the algebraic module because they felt that this cohort had the prerequisite skills and vocabulary needed to understand the algebraic questions posed in the tasks.

This study was bounded in space (all participants attended the same large southeastern community college) and time (one academic semester) and examined the algebraic habits of mind elicited while solving six algebraic tasks. The goal of the study aligns with the qualitative methods used in a case study (Yin, 2013). Since there is limited scholarship on algebraic habits of mind of students enrolled in developmental mathematics courses, rich-detailed descriptions of the observed habits are necessary to build understanding of algebraic habits of mind as it relates to the population. Specifically, this study is an exploratory multiple case study as the researcher is trying to understand the algebraic habits of mind observed within each case (Creswell et al, 2011).

The six algebraic tasks were the cases that the researcher used to observe the algebraic habits of mind. The researcher conducted task-based interviews in order to observe the algebraic habits of mind of community college students enrolled in developmental mathematics programs. A task-based interview provides an opportunity for a participant to not only interact with the interviewee but also with a task (Goldin, 2000). Task-based interviews “can serve as research instruments for making systematic observations in the psychology of learning mathematics and solving mathematical problems (p. 520). Task-based interviews allow for researchers to access more than just the artifacts created by the participants as they can observe and ask questions regarding the participant thinking dynamically while the participant engages with a task (Goldin,
1997). As such, this researcher, while conducting the task-based interviews, engaged the participants as they worked through the six cases by asking questions about their comprehension of the tasks and encouraged the participants to actively think aloud their thoughts.

The participants chosen to represent a population grossly underrepresented in research, the community college student enrolled in developmental mathematics. Data came from observations made during six task-based interviews with six participants. It consisted of artifact collection, researcher notes, video and audio recordings.

**Context**

This study is situated in the context of a large southeastern community college. This community college is situated on the cusp between a major city and several rural counties. Therefore, the population drawn to this campus is quite diverse. Figure 8 shows the breakdown of the school by in-state vs out-of-state students, gender, race/ethnicity, as well as the general graduation rates of major groups compared to the national average (Overgrad.com, 2018). The light blue bars in the graduation rates bar graph represents the national average, while the dark blue bars represent this college’s graduation rates. As it can be observed, this college struggles with graduating students in all compared categories.

![Figure 8: School Demographics and Graduation Rates](Image)
The researcher’s role outside of the study. This researcher works as a coordinator for a math tutoring facility at a community college campus. The researcher actively encourages developmental mathematics students to receive free additional academic support outside of their class at their center through email, classroom appearances, and face-to-face conversations. In the fall of 2017, 95 students came a total of 335 times and in the spring of 2018, 100 students came a total of 443 times for assistance with a developmental mathematics course. Though there are multiple facilities like this at the college to allow for convenient access for these students, the researcher’s center serves a disproportionately high number of these students.

Participants

Hycner (1999, p. 156) notes “the phenomenon dictates the method (not vice-versa) including even the type of participants.” In this case, the phenomenon was the algebraic habits of mind evoked by tasks presented to students enrolled in a developmental math course at a community college. Therefore, the participants in this study were students currently enrolled in developmental math courses at a community college after having completed their algebraic module. This group was chosen from the broader population of any developmental math student because they received the necessary training, skills, and vocabulary to understand the algebraic tasks used in this study.

The researcher asked students who regularly used the tutoring center for academic support to volunteer their participation in the study. The researcher targeted students enrolled in developmental mathematics who could clearly express their ideas, completed their algebraic module, and who had rapport with the researcher. The researcher sought six participants to ensure that enough opportunities to observe habits were available (six tasks, six participants, provided for thirty-six opportunities). The researcher asked students who came to the center if
they were willing to come to the center after-hours to participate in the study with the understanding that there would be no gain or harm from directly participating in the study beyond furthering the study of algebraic habits of mind. The researcher continued to ask students if they were willing to participate until the researcher was able to find six participants. Ten total students were asked to participate, the six task-based interviews each occurred in the evening after the center closed.

The choice to observe only six participants complete six tasks is justified in the literature. In the view of Gonzalez (2009), when undertaking research that is reliant on a qualitative approach, the sample size is usually driven by the need to uncover all the main variants within the approach, hence the choice to provide thirty-six opportunities to demonstrate the algebraic habits of mind. Furthermore, Guests et al. (2006) argues that data saturation can occur within the first five interviews and after that very few phenomena are likely to emerge thus the selection of six task-based interviews provided enough information and enough redundancy to allow for data triangulation.

The six participants. The six participants were chosen because they provided the researcher with the best opportunity to access the phenomenon to be studied; that is, they were selected because they were enrolled in developmental mathematics curriculum at a community college and appear to be able to clearly communicate their ideas. Each of the participants completed through at least the introductory algebraic module of the developmental curriculum. The college has eight total modules but the number of modules needed for each student is directly tied to their intended concentration. All students, regardless of their intended concentration must complete the first three modules with the third labeled as “an introduction to algebra”. Since having completed module three was the only curricular requirement for
participating in this study, and each student having different concentrations, some of the students were completing advanced algebraic concepts as opposed to those who only needed the introductory course. Figure 9 shows a brief comparison of module three (introductory algebraic concepts) and four (advanced algebraic concepts). What follows the figure is a brief description of the background of each of the participants: Dupe, Alex, Nikki, Liam, Roger, and Bob (all pseudonyms).

<table>
<thead>
<tr>
<th>Modules Completed</th>
<th>Content</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Module 3 Completed</td>
<td>Solving equations with one variable, Identifying graphical features of first, second, and third degree polynomials, translating text to algebraic symbols, rewriting expressions and functions in terms of other variables, and factoring.</td>
<td>Dupe, Nikki, Liam</td>
</tr>
<tr>
<td>Module 4 Completed</td>
<td>Solving systems of equations, using the quadratic formula, solving inequalities, identifying, using, and graphing features of rational and radical functions, and domain and range.</td>
<td>Alex, Roger, Bob</td>
</tr>
</tbody>
</table>

Figure 9: Module characteristics that the participants completed prior to the study.

**Dupe.** Dupe is a twenty-six-years-old South African national. She moved to the United States three years before the timing of this study. Dupe is a former au pair who decided to change industry to health sciences and as a result is pursuing an associate with a health sciences focus. She is a regular at the tutoring center, she is friendly, and provides a genuinely positive disposition towards mathematics. Because of her concentration, Dupe only needed to complete through module three of the developmental mathematics program. She participated in the study
three days after completing module three. Dupe took one semester to complete the three modules.

*Alex.* Alex is a nineteen-year-old Indian national. He is a non-native speaker but has lived in America for the past five years. Alex intends to become an Engineer and as a result needs to complete all eight modules before beginning his college curriculum. Alex completed the first four modules within the first six weeks of school. Alex is a regular to the center, he is inquisitive, and often asks why the skills he is taught works and often requests more challenging problems than what his modules provide.

*Nikki.* Nikki is a proud mother of two adult children. Growing up in the community she attended a high school less than ten miles from the community college. She owns and operates a successful construction company and is seeking to sharpen her bookkeeping skills as she feels it will best support her company. As a result, Nikki’s concentration is on accounting. Students with an accounting focus only need to complete through module three. Nikki completed the three modules over three semesters. Nikki’s busy schedule does not allow her to regularly visit the tutoring center but when she comes she stays for great lengths of time. She often asks about how the concepts she is learning relate to her company.

*Liam.* Liam is a twenty-three-year-old student who has expressed interest in majoring in information technology and engineering. His parents emigrated to the United States when he was six from India. He was taught both Hindi and English at the same time in school and can speak a total of three languages. Liam is eager to please and works hard. He has accumulated the most visits and hours logged at the tutoring center of the six participants. Liam completed high school calculus and despite achieving a satisfactory grade in the course, he did not pass the advanced placement exam to receive college credit for the course. As a result, Liam had to take the
community college’s placement test which he did not meet proficiency standards. Liam took four years to complete the first three modules and began module four at the time of this study.

**Roger.** Roger is a former marine who was honorably discharged after ten years of service. Roger intends to enter engineering at a four-year college after completing his associates. He completed the first four modules after one semester of work. Roger is known to ask the staff variations of problems he is studying to understand the limits of the concepts he is learning. For example, he would often ask how changing a value or power effect his method for solving or the solution in general. Roger is from the North East of the United States only relocating to the area being discharged from the military. Roger blames his placement into the developmental math modules on his length from being in school and overreliance on the calculator. His family obligations, wife, six children, and work keeps his time at the tutoring center limited but he visits the center frequently.

**Bob.** Bob is a fifty-year-old student seeking a promotion within his current company. Bob writes briefs and edits technical reports for engineers at his firm. Bob already has two associate degrees one with a concentration in information and technology and the other in business analytics. Bob is seeking his third associates but now with a concentration in engineering. His company has offered him a position as an engineer if he can complete the program. Bob completed the four modules over three semesters. Bob’s work requires him to travel frequently, so it is not uncommon for Bob to miss weeks of school at a time, which partly explains the time it took to complete the modules.

**Unit of Analysis**

The unit of analysis were the habits of mind elicited during the six algebraic tasks by the participants. The participants engaged with these cases by reading and attempting to solve each
task. They were encouraged to think aloud and write down their thoughts as much as possible. Their verbal utterances, and artifacts were observed and collected.

**Task-based interview.** The tasks used in this study were purposefully chosen from previous studies that focused on eliciting participants’ algebraic habits of mind. The six algebraic tasks chosen came from four studies (Stein and Smith, 1998, Driscoll, 1999; Driscoll, 2003, Matsuura et al, 2013; & Magiera, van den Kieboom, Moyer, 2017). Figure 10 shows a brief description of each task.

While engaged with the tasks, the participants had at their disposal, whiteboards, markers, paper, pens, calculator, and a math textbook to look up terms or vocabulary words they did not understand. These are the typical resources provided to the students in the tutoring center and thus they will have some familiarity with using them while engaged with the tasks. Also, while interacting with the tasks, the participants were encouraged to talk aloud their thoughts and were prompted to speak out their ideas when writing or while contemplating during the tasks.
## The six tasks

<table>
<thead>
<tr>
<th>Tasks Title</th>
<th>Task Description</th>
<th>Study associated with task</th>
<th>Algebraic habits of mind category targeted by tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Stamp Task</td>
<td>Using 5 cent and 7 cent stamps, what amounts cannot be made.</td>
<td>Driscoll (2003)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>Create a function rule</td>
<td>Write polynomial function rules that have zeros of 4/3 and -4/3</td>
<td>Driscoll (1999)</td>
<td>Input from output</td>
</tr>
<tr>
<td>The carnival of bears task</td>
<td>There are two sets of bears on each side, each attempting to move to the other side. Determine the number of slides and jumps needed to make this happen.</td>
<td>Driscoll (2003)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>The stacking cans task</td>
<td>There is a stack of cans with each row having one more can than the next, how many cans would make a stack with thirty cans on the bottom row?</td>
<td>Steins and Smith (1998)</td>
<td>Abstracting from computation</td>
</tr>
<tr>
<td>The patterns at work task</td>
<td>Identify the patterns that would enable you to recreate the given information if told the initial conditions. Use your pattern rules to create a new relationship with new values. Adjust your rules as necessary to allow you to make the new table and still describe the first.</td>
<td>Matsuura et al (2013)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>The flowerbeds task</td>
<td>Each flowerbed needs to be surrounded by six paving slabs. The flowerbeds are arranged in an array such that four flowerbeds use eighteen slabs. How many slabs are needed for n flowerbeds?</td>
<td>Magiera et al (2017)</td>
<td>Abstracting from computation</td>
</tr>
</tbody>
</table>

Figure 10: Description of tasks

**The tasks**

Six tasks were chosen because they are currently used in algebraic habits of mind research and had demonstrable success in eliciting algebraic habits of mind in their participants.
Since the point of my study is to see what habits are used when completing algebraic tasks, the researcher felt it was important to select tasks that have already been verified in this field. Each task will be described, next, and the semi-structured interview protocol explained.

**The stamp task.** The first task referred to in this study was from the postage-stamps problem by Driscoll, (2003, p. 3). The task asks the reader to determine which amounts are impossible to make using five cent and seven cent stamps. Specifically, the problem states:

The post office has only five cent and seven cent stamps. By combining different number of stamps of five cent and seven cent stamps, customers can usually get the amount of postage they need. For example, one five cent stamp and two seven cent stamps make 19 cents in postage; two five cent stamps make 10 cents in postage. Which amounts of postage is it impossible to make using only five cents and seven cents stamps? Explain how you know that each of these amounts is impossible to make. Explain how you know you have found all the amounts that are impossible to make (that is, tell how you know that all the other amounts of postage are possible).

Driscoll (2003) completed the task with students identified as advanced eight graders in the latter part of their school year. In this task, he found that the students had multiple avenues to make sense of this problem but mostly used skills associated with the category of building rules to represent functions habits. Driscoll noted that the students, working in groups, consistently created tables and lists showing impossible values and then used various strategies to try to find patterns to describe in the lists and tables.

**Create a function rule task.** The second task in this study, known as create a function rule, was adapted from Driscoll (1999). This task expects the participants to work backwards to
determine a function rule that would have zeros at \( \frac{4}{3} \) and \(-\frac{4}{3}\). It was adapted by the researcher by adding a second question. This task reads:

Find algebraic functions where \( \frac{4}{3} \) and \(-\frac{4}{3}\) would be zeroes using only positive and negative whole numbers. How many unique polynomials, by degree, can you find like this?

In this task, Driscoll (1999) outlined its use as a task in his book *Fostering Algebraic Thinking* (p. 19). Driscoll (1999) argues that as students make sense of the task they will primarily elicit habits associated with the algebraic habit of mind category of doing and undoing. He states that individuals using this habit of mind will be able to reverse the solution into a function demonstrating they understand the process enough to work backwards. Once students arrive at a function rule that satisfies the first question, the participants should be able to extend their solution to represent functions of any degree with the same zeros.

**The carnival of bears task.** The third task in this study will be referred to as the carnival bears task (Driscoll, 2003, p. 9-10). Driscoll (2003) found that this task most often elicited algebraic habits linked to building rules to represent functions. In his study, using advanced middle school participants in a small group, he observed them discussing the solution and the mathematics to solve the task. He noted that the participants were able to derive the solution but struggled with determining a solution for a general case. The participants created charts and solved the problem by first examining a simpler case. Using the simpler case, the participants generalized the linear relationship correctly but could not determine the pattern and thus generalization of the non-linear relationship. This task reads:

Connie, Jeff, and Kareem went to the circus. They saw bears do tricks. Three brown bears and three black bears did one of the tricks. At the beginning of the trick, the three
black bears were on the left side of a long mat divided into seven squares, and the three
brown bears were on the right side. Each bear had its own square with an empty square in
the middle. The bears could only do two different types of moves:

1. They could slide onto the next square if it was empty, or
2. If the next square was not empty, they could jump over one bear to an empty square.

The black bears moved only from left to right and the brown bears only moved from right
to left. When the trick was over, the bears had switched places. All the black bears were
on the right side, and all of the brown bears were on the left side.

1. The bears needed 15 moves to switch places. Explain how they did it. How many
   moves were slides? How many moves were jumps?

2. What is the smallest number of moves that five black bears and five brown bears need
to switch places? Explain how to do it. How many moves would be slides? How many
   moves would be jumps?

3. If 20 black bears and 20 brown bears were involved, how many slides would be
   needed? How many jumps would be needed? How many moves would be made
   altogether? Explain how you got your answer.

4. Pretend that someone told you how many black bears and brown bears did the trick.
   Explain how you could figure out how many slides, jumps, and moves altogether the
   bears needed to switch places.

**The stacking cans task.** The fourth task in this study is called the stacking cans task
(Steins and Smith, 1998). An additional question was added below the image to discourage
students from using the calculator exclusively as they engage in this task. The task, shown in the
figure below, states:
Figure 11: Stacking cans task adapted from Stein and Smith (1998)

This task is like a problem Driscoll (1999) describes in his book Fostering Algebraic Thinking. In this book, he explains that participants could recognize that the first and last term combine to the same sum as the next terms, and so on. He described this composition and decomposition as a form of doing and undoing (p. 19). Stein and Smith (1998) also observed the act of composing and decomposing numbers to determine the sum but they also found the participants extending and simplifying the number of elements in the row to pattern sought to identify relationships and used symbolic expressions to represent sums for the general case, which would mean that they would expect to see habits from both abstracting from computation and building rules to represent functions as well.

The patterns at work task. The fifth task in this study is called the patterns at work task it was adapted from Matsuura et al, (2013). Their task states:

Look at the figure below:
Figure 12: The patterns at work task.

How many patterns can you find? What are the patterns that you notice? If you were to continue down, would $A$ or $B$ ever list the number 1005? If there was a new table that followed the same patterns as noted above, but the third value under $B$ was 10, what would be the value directly across from $B$ under $A$? What would be the value under 10? What would be the first values at the start of the table? Which pattern rules adequately allow the construction of both tables if given values for each table?

This task was used in the Matsuura study to encourage the algebraic thinking related to the domain of building rules to represent functions and doing and undoing. The students in Matsuura’s study were encouraged to describe patterns observed within the data then determine which of these patterns would still hold and assist in creating a new table. The researchers found that the participants were able to create function rules to relate the values within the two columns and recursive rules to relate values within their own set. Some participants used the recursive rules to infer the slope of the function rule that related the two columns. Matsuura et al (2013) allowed pre-service attempt this problem and then after a discussion on algebraic habits of mind,
encouraged the participants to identify habits of mind in student work and reflect on their own habits.

**The flowerbeds task.** The flower beds task was created by the Shell Center for Mathematical Education (1984, p. 64). It was used by Magiera, van den Kieboom, & Moyer in 2016 and again in 2017. The researchers chose this task to study pre-service elementary and middle school teachers building rules to represent functions algebraic habits of mind. The figure below provides the description of the task:

![Flower Beds](image)

The city council wishes to create 100 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds.)

1. How many slabs will the council need?
2. Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

*Shell Centre for Mathematical Education, 1984, p. 64.

Figure 13: The flowerbeds task.

While completing the task, Magiera et al (2017) noted use of many algebraic habits including describing change, describing a rule, different representations, and predicting patterns from the building rules to represent functions category but they also noted the use of the abstracting from computation habits such as symbolic expression and equivalent expressions. With the latter especially prevalent throughout the study as the participants attempted to model the task for a general case. Interestingly this study also noted the usage of a proportional thinking habit that did not appear to be defined within Driscoll’s (1999) framework. The researchers noted that the individuals who used this method in the task were unsuccessful in determining the
correct number of slabs from the number of garden beds and did not declare this behavior as a
new algebraic habit of mind.

Data Collection

Data for this study was drawn from six task-based interviews using a think aloud protocol
(the protocol for each task is supplied in the appendix). This includes the participants' written
work, video recordings of their task-based interviews, and researcher field notes. The following
sections will describe in detail each data source collected.

Participant’s written artifacts. During each task, two video recorders captured the
dialogue and written artifacts of the participants. Five out of six of the participants chose to use
whiteboards to express their ideas in written form. The other participant completed the first three
tasks on paper then transitioned to the whiteboards. The artifacts were collected or recorded
when appropriate, written works by the participants were stored in a locked filing cabinet
belonging to the researcher.

Researcher artifacts. The researcher also created artifacts. During the interview, the
researcher took notes on the verbal utterings made by the participants and observations of their
work that best assisted the researcher in recalling the participants’ habits during each task as well
as ensuring the researcher remains unbiased. The notetaking by the researcher added to their
reflective process ensuring they stay aligned with the algebraic habits of mind framework
(Driscoll, 1999).

Video recording. The researcher used two cameras to record the overall engagement of
the participant with the tasks. One camera was angled such that it recorded written work and
hand gestures. Observing gestures will allow the researcher to capture the experiences of the
participants when articulation may be limited (Cook et al, 2012). Some gestures could include a
person demonstrating the movement of discrete objects by motioning with their fingers or the
swipe of an arm to indicate the direction or slope a line is taking. Since the gestures could occur
in front of the person or down on their paper, the researcher utilized two cameras to capture both
potential locations.

The second camera had an enhanced microphone and was pointed directly at both the
participant and researcher in order to clearly capture the dialogue exchanged during each task. In
order to save space on both devices, the cameras were stopped after each task in order to upload
the recordings to a secure server and clear memory on the devices. The next section describes
each task and protocol for the task in detail.

**Data Storage**

All electronic data was stored on a secured server that was password protected and only
accessible by this researcher. Physical artifacts were kept in a filing cabinet that only the
researcher could access with a key. The videos, artifacts, and transcription were added to a
digital program to assist with analysis and transcription. This encrypted file was kept on the
researcher’s personal computer which is password protected.

**Data Analysis**

As explained by DeCuir-Gunby et al (2011), the codes, from Driscoll’s (1999)
framework, were applied to raw data to see how well the theory and research literature is
supported (p. 138-139). “Analyzing interview data is a multistep “sense-making’ endeavor
(DeCuir-Gunby, Marshall, McCulloch, 2011). The transcribed and artifact data was coded.

Driscoll’s (1999) framework is derived from both theoretical and empirical research, this
framework was the source of the theory driven codes as it is commonly used in research on
algebraic habits of mind (e.g. Driscoll & Moyer (2001); Matsuura et al (2013); Eroğlu, D., &
Tanışlı, D. (2017); Magiera et al (2017); Magiera et al (2017)). With the unit of analysis being the observed acts and statements of the participants as they made sense of the task, some of the acts modeled multiple habits and thus sometimes a single act was coded as two habits. The a priori codes used in this study with description and examples is provided in the figure below.
Figure 14: Codes used in the study
<table>
<thead>
<tr>
<th>Algebraic habit of mind category</th>
<th>Algebraic habit of mind</th>
<th>Description</th>
<th>Codes</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building rules to represent functions</td>
<td>Building rules to represent functions</td>
<td>Organizing information in ways useful for uncovering patterns and the rules that define the patterns</td>
<td>Table, chart, list</td>
<td>I'll make a table. The table will allow me to see what’s going on.</td>
</tr>
<tr>
<td>Predicting patterns</td>
<td>Predicting patterns</td>
<td>Noticing a rule at work and trying to predict how it works</td>
<td>Stated pattern, described pattern</td>
<td>The first flowerbed has six slabs, the second has ten, the third has fourteen, and fourth has eighteen. So, I predict the nth flowerbed will have $4n + 2$ slabs.</td>
</tr>
<tr>
<td>Chunking the information</td>
<td>Chunking the information</td>
<td>Looking for repeating chunks of information that reveal how a pattern works</td>
<td>Described grouped information, attributes of categories</td>
<td>Numbers that are multiples of five and seven can clearly be eliminated as potential impossible numbers since they are always divisible by five and seven.</td>
</tr>
<tr>
<td>Describing a rule</td>
<td>Describing a rule</td>
<td>Describing the steps of a rule without using specific inputs</td>
<td>Explains steps, wrote rule</td>
<td>Each new row has one additional can.</td>
</tr>
<tr>
<td>Different representations</td>
<td>Different representations</td>
<td>Wondering what different information about a situation or problem may be given by different representations</td>
<td>Changed to table, change to graph, change to list, change to expression</td>
<td>I am going to make a graph from my table so that I can see what kind of function we are talking about.</td>
</tr>
<tr>
<td>Describing change</td>
<td>Describing change</td>
<td>Describing change in a process or relationship</td>
<td>States change between numbers, states change in function, indicates slope, discusses variance</td>
<td>With each additional flowerbed, I noticed that there are four additional slabs.</td>
</tr>
<tr>
<td>Doing and undoing</td>
<td>Abstracting from Computation</td>
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<tr>
<td><strong>Justifying a</strong></td>
<td><strong>Computation shortcuts</strong></td>
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<tr>
<td><strong>rule</strong></td>
<td>looking for shortcuts in</td>
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<td></td>
<td>computation based on an</td>
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<td></td>
<td>understanding of how</td>
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<td></td>
<td>operations work</td>
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<tr>
<td><strong>Justifying why</strong></td>
<td><strong>skipped steps, skipped</strong></td>
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<tr>
<td><strong>a rule</strong></td>
<td><strong>computation, simplified</strong></td>
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<tr>
<td><strong>works for</strong></td>
<td><strong>the calculation</strong></td>
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<tr>
<td><strong>&quot;any number&quot;</strong></td>
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<td><strong>Explains why</strong></td>
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<td><strong>rule</strong></td>
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<td><strong>works</strong></td>
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<td><strong>I can have</strong></td>
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<td><strong>infinite</strong></td>
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<td><strong>functions</strong></td>
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<td><strong>with these</strong></td>
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<td><strong>two zeros</strong></td>
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<td><strong>because I</strong></td>
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<td><strong>can always add</strong></td>
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<td><strong>another zero</strong></td>
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<td><strong>and still</strong></td>
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<td><strong>Uses specific</strong></td>
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<td><strong>b</strong> is ten,****</td>
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<td><strong>my function</strong></td>
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<td><strong>b = 5a + 3,</strong></td>
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<td><strong>I am setting</strong></td>
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<td><strong>b to</strong></td>
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<td><strong>ten,</strong></td>
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<td><strong>and I find</strong></td>
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<td><strong>a = 7/5</strong></td>
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<td><strong>x = 4/3</strong></td>
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<td><strong>then by multiplying each side</strong></td>
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<td><strong>by three and subtracting four,</strong></td>
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<td><strong>I get the</strong></td>
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<td><strong>expression of</strong></td>
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<td><strong>(4x - 3)</strong></td>
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<td><strong>Abstracting</strong></td>
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<td><strong>Computation</strong></td>
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<td><strong>work</strong></td>
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<td><strong>Skipped steps,</strong></td>
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<td><strong>skipped</strong></td>
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<td><strong>computation,</strong></td>
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<td><strong>simplified the</strong></td>
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<tr>
<td><strong>calculation</strong></td>
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<td><strong>I could add</strong></td>
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<tr>
<td><strong>1 + 2 + 3 + 4 ... + 30,</strong></td>
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<td><strong>but I see that</strong></td>
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<tr>
<td><strong>1 + 30 = 2 + 29, 3 + 28,</strong></td>
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<tr>
<td><strong>and so on,</strong></td>
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<tr>
<td><strong>giving me</strong></td>
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<tr>
<td><strong>fifteen pairs</strong></td>
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<tr>
<td><strong>of 31,</strong></td>
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<td><strong>so the answer is</strong></td>
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<td><strong>15 x 31.</strong></td>
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<tr>
<td><strong>calculating</strong></td>
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<td><strong>without</strong></td>
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<tr>
<td><strong>computing</strong></td>
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<tr>
<td><strong>Thinking about</strong></td>
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<td><strong>calculations</strong></td>
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<td><strong>independently of</strong></td>
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<td><strong>the particular</strong></td>
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<td><strong>numbers used</strong></td>
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<td><strong>found solution to</strong></td>
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<td><strong>calculation without</strong></td>
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<td><strong>calculating,</strong></td>
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<td><strong>found solution to</strong></td>
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<td><strong>calculation</strong></td>
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<td><strong>inferring from</strong></td>
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<tr>
<td><strong>another calculation</strong></td>
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<tr>
<td><strong>I know that five rows of cans would be</strong></td>
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<tr>
<td><strong>fifteen total cans,</strong></td>
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<td><strong>I also know that five</strong></td>
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<tr>
<td><strong>more rows of cans would be</strong></td>
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<td></td>
<td></td>
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<tr>
<td><strong>5 + 1,5 + 2,5 + 3,5 + 4,5 + 5 which is 5x5 +</strong></td>
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<td></td>
<td></td>
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<tr>
<td><strong>(1 + 2 + 3 + 4 + 5)</strong></td>
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<tr>
<td><strong>meaning that the new</strong></td>
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<tr>
<td><strong>amount would be</strong></td>
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<tr>
<td><strong>2 * 15 + 5 * 5 = 55.</strong></td>
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<tr>
<td><strong>Generalizing</strong></td>
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<tr>
<td><strong>beyond examples</strong></td>
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<tr>
<td><strong>Going beyond a few</strong></td>
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<tr>
<td><strong>examples to create</strong></td>
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<tr>
<td><strong>generalized</strong></td>
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<tr>
<td><strong>expressions,</strong></td>
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<tr>
<td><strong>describe sets of numbers,</strong></td>
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<td><strong>state or conjecture the</strong></td>
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<tr>
<td><strong>conditions under</strong></td>
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<tr>
<td><strong>which particular</strong></td>
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<tr>
<td><strong>Uses examples to</strong></td>
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<tr>
<td><strong>write an expression for</strong></td>
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<tr>
<td><strong>operation,</strong></td>
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<tr>
<td><strong>uses examples to make</strong></td>
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<tr>
<td><strong>claims about</strong></td>
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<tr>
<td><strong>structure</strong></td>
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<tr>
<td><strong>I noticed that the numbers of bears always</strong></td>
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<tr>
<td><strong>match the number of jumps,</strong></td>
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<tr>
<td><strong>So for six bears there will be six jumps</strong></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Equivalent expressions</td>
<td>Recognizing equivalence between expressions</td>
<td>State that two expressions are equivalent</td>
<td>The expression 3(2x+5) is the same as 6x+15</td>
<td></td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------------------------------------</td>
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<td>-----------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Symbolic expressions</td>
<td>Expressing the generalizations about operations symbolically</td>
<td>Representing operations symbolically, representing (in)variance symbolically</td>
<td>To move from the left column (a) to the right column (b) I would multiply five and then add three, so b=5a+3</td>
<td></td>
</tr>
<tr>
<td>Justifying shortcuts</td>
<td>Using generalizations about operations to justify computational shortcuts</td>
<td>Visual saliency, defense of computational shortcut</td>
<td>See, if I list out the pairs, we can see that I am always making 31 until I get to my final pair of 15, which is why 15 x 31 works.</td>
<td></td>
</tr>
</tbody>
</table>
Data-driven codes were created to assist with the coding of the raw data following the five steps as outlined by Boyatzis (1998): reduce raw information, identify subsample themes, compare themes, create codes, and determine the reliability of codes. The data-driven codes were used to describe major themes that had not been captured by the established Driscoll (1999) framework. For example, when solving the stamp task, the researcher noted the participants consistently demonstrated a habit of finding the complementary solution to the task (i.e. they were finding the values that were possible) and used this information to infer claims about impossible numbers, this act was coded as ‘finding the complement’ as it did not appear to be represented in Driscoll’s (1999) framework. The researcher then compared the theme to other themes identified to see if it would be better situated into a broader theme. Next the researcher sought out literature on the observed coded act in both the context of the task and in mathematics in general, if it was supported by the literature in either case, the researcher presented the code to peers for review. If the peers agreed with the researcher’s conclusion, the code was kept.

The researcher’s journal assisted with the theming of the data-driven codes and also helped to ensure that the researcher was consistent in their coding process. The constant comparative method also aided reliability as it encouraged the constant checking of data to existing codes as highlighted below in figure 15.
Reliability and Trustworthiness

In this section, issues regarding reliability and trustworthiness will be identified and how the researcher attempted to limit the effects of such concerns will be stated. The issues will include concerns regarding the limited amount of research on this topic, the researcher's personal bias, and access to participant thinking.

Limited research on the algebraic habits of mind of community college students enrolled in developmental mathematics programs. The researcher acknowledges that there is no study that currently examines this population’s algebraic habits of mind. As a result, this work’s claims must be taken with caution. The researcher made every attempt to validate their observations through peer review of work, triangulation of artifacts and observations, and comparison of findings to the established literature.

Researcher personal bias. This researcher holds the unavering belief that all students are capable of being good at math. They do not accept the notion that mathematical ability is genetically-linked or otherwise only beholden to a select type of individual. Therefore, the
researcher will need to take caution and have their worked peer checked to ensure they are not overreaching their interpretations of artifacts and observations staying grounded in the theory and literature.

**Access to participant thinking.** The researcher acknowledges that they cannot immediately access the thinking processes of the participants. Inferences will be made about the participants’ verbal utterings, gestures, written work, and body language. These inferences are knowingly, to the researcher, culturally dependent and limited to the researcher’s understanding of the participants’ actions (Cochran & Chambers, 1965). Therefore, the researcher worked with peer reviewers and kept a journal to ensure accurate, adequate, and consistent interpretation of observations reported and analyzed in subsequent chapters.
Chapter 4: Results

Introduction

This chapter presents the qualitative findings for each of the six cases of the study. The researcher defines a case as an algebraic task. The focus in each case is, while using six algebraic tasks, observe the habits and algebraic habits of mind of college students in developmental algebra courses use. In this chapter we will describe the habits of mind of the participants as revealed through their work on each of the tasks.

For a habit to be characterized as an algebraic habit of mind it must meet four criteria (adapted from Driscoll’s Geometric Habits of Mind Criteria, 2007): (1) the algebraic habit of mind should reflect mathematically important thinking relevant to students’ solution to a task, (2) the algebraic habit of mind should connect to literature on the learning of algebra and the development of algebraic thinking, (3) Evidence of an algebraic habit should appear across multiple tasks or multiple times and should represent a student’s thinking, and (4) The algebraic habit of mind should be teachable. There are other habits that students might use while solving these problems. The other habits discussed in this chapter have similar criteria: (1) The habit should reflect mathematically important thinking and it should be relevant to students’ solution to a task, (2) the other habit should connect to the research literature on the learning of mathematics and the development of mathematical thinking, (3) there should be evidence that the habit appears across multiple tasks or multiple times within a task and it should reflect a participant’s thinking, and (4) the habit should be teachable. The current algebraic habits of mind framework are shown in figure 16 below:
Algebraic Habits of Mind

<table>
<thead>
<tr>
<th>Category: Building rules to represent functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Organizing information</td>
</tr>
<tr>
<td>Predicting Patterns</td>
</tr>
<tr>
<td>Chunking the information</td>
</tr>
<tr>
<td>Describing a rule</td>
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<tr>
<td>Different Representations</td>
</tr>
<tr>
<td>Describing Change</td>
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<tr>
<td>Justifying a rule</td>
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<table>
<thead>
<tr>
<th>Category: Doing and undoing</th>
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<tbody>
<tr>
<td>Input from output</td>
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<tr>
<td>Working backwards</td>
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<tr>
<th>Category: Abstracting from computation</th>
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<tbody>
<tr>
<td>Computational shortcuts</td>
</tr>
<tr>
<td>Calculating without computing</td>
</tr>
<tr>
<td>Generalizing beyond examples</td>
</tr>
<tr>
<td>Equivalent expressions</td>
</tr>
<tr>
<td>Symbolic expressions</td>
</tr>
<tr>
<td>Justifying shortcuts</td>
</tr>
</tbody>
</table>

Figure 16: Algebraic Habits of Mind (Driscoll, 1999)
Each section in this chapter, will introduce the task, discuss the algebraic habits of mind observed, and descriptions of other productive habits not in Driscoll’s (1999) framework observed in each task. We begin with the stamp problem.

**Description of Task one: The Stamp Task**

The post office has only five cent and seven cent stamps. By combining different number of five cents and seven cents stamps, customers can usually get the amount of postage they need. For example, one five cent stamp and two seven cent stamps make 19 cents in postage; two five cent stamps make 10 cents in postage. Which amounts of postage is it impossible to make using only five cents and seven cents stamps? Explain how you know that each of these amounts is impossible to make. Explain how you know you have found all the amounts that are impossible to make (that is, tell how you know that all the other amounts of postage are possible).

**Introduction to Task one**

The postage stamp task required the participants to determine the values, using multiples of five and seven in combination, that are impossible to make. These numbers are referred to in this paper as unattainable or impossible. In previous research, Driscoll (2003) used this task to examine algebraic habits of mind closely linked to *Building Rules to Represent Functions*. Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. In particular, the next section will highlight the algebraic habits that were used by the participants as they solved the task. In addition, there were two other habits identified: Brute Force and Complement.

**Algebraic Habits of Mind Observed in Task One**

**Building rules to represent functions - Organizing information.** Organizing information is about arranging information in a structured way. Three of the six participants,
Dupe, Nikki, and Liam, used a table to organize information. Dupe and Nikki created a table at the beginning of their work on the task as they sought attainable values. On the other hand, Liam created a table midway through his work as he attempted to generate a list of values from expressions he created (i.e. $5x, 7x$). Dupe and Nikki created a table that listed the values that could be obtained by adding combinations of five and seven in order to find values that were not obtainable. Dupe created her table in three stages. Initially, Dupe listed out multiples of fives from five to 50, then she listed out multiples of sevens that also did not exceed 50, lastly, she began to use combinations of multiples of fives and sevens added together. She first added $5 + 7$ and then repeatedly added fives to this calculation and recorded this value. Then she started over and found the solution to $5 + 7 + 7$ and added repeated fives to this calculation. She continued in this manner to complete her table.

Nikki’s tables were created through listing out combinations of five and seven. Initially, Nikki considered multiple fives and then began to incorporate more sevens into her sums to discover more attainable numbers. She found her organization in this table lacking and requesting permission to start over. In her new table, Nikki created rows that add an additional digit as you progress down and increased in the number of sevens as you move to the right (figure 17).

Figure 17: Dupe and Nikki organizing information during the stamp task
Liam’s organized list was generated by his expressions \(0.5x\) and \(0.7x\). Liam struggled with abstracting math from the context. Because the task description stated five cent and seven cent stamps, he attempted to model his expressions in terms of money. Unfortunately, he made an error in representing cents as he represented five cents as \(0.5\) and seven cents as \(0.7\) instead of \(0.05\) and \(0.07\), respectively. He constructed a table that had three columns and five rows. The first column was for his x-inputs, the second for the expression \(0.5x\), and the last for \(0.7x\). Under the first column’s title, Liam wrote \(0.5\) and counted by \(0.5\) until he reached the value of \(2.0\), then in each of the other columns, Liam used his calculator to evaluate these inputs to determine the output values of the other columns (figure 18).

![Figure 18: Liam representing \(f(x) = 0.5x\) and \(f(x) = 0.7x\) symbolically, in his table, and graphically.](image)

The participants used a variety of methods to organize information that was provided to them. These three individuals demonstrated the algebraic habit of mind, organizing information by arranging their information from the task in the form of a table and in Liam’s case a graph as well. Liam’s table was derived from graphing his expressions. His use of an expression is similar to the habit related to describing rules, which is presented next.
**Building rules to represent functions - Describing a rule.** To describe a rule, the participants would need to be able to describe the steps of an observed rule without the emphasis being on specific inputs. Three participants described rules as a means of making sense of the task. Alex, an ESL student, wrote down the directions in the task, but changed the wording so that the directions became rules that reflected what the task asking. For example, he converted the sentence “By combining different number of stamps of five cents and seven cents stamps, customers can usually get the amount of postage they need.” into the rule: seven or five every time any amount can make.” This rewrite, he explained, meant that he could use an unlimited number of five and seven, only, to make numbers (figure 19). Bob came to the same conclusion from reading the prompt and stated, “given any combinations of five, given that I have an unlimited amount, I can always make multiples of five, this multiple [rule] then would also have to apply for seven too.”

![Figure 19: Alex describing the rules for finding impossible values.](image-url)
Liam wrote expressions to model the multiples of five cents and seven cents. Liam determined that five cents meant .5 and seven cents meant .7 and since each stamp was either five or seven cents, then .5x and .7x would model the situation. Using these expressions, he concluded that at a certain point these two functions would intersect and would bound the values of impossible numbers, “So we know that you have five cents and you have seven cents and we know that there has to be a certain combination of five cents um plus seven cents can’t make and it is to a limit and there is one point where that limit kind of fails.” It was not uncommon for Liam to use the word “limit” in his description of when impossible values would no longer exist. As seen by Liam and Alex, the participants described a rule by translating the problem description into constraints set by the situation. These constraints were verbalized and expressed using mathematical notation. The different types of representations participants used is described in the next section.

Building rules to represent functions - Different representations. Participant actions observed to model the task with different representations for the same concept or process were identified as exhibiting the algebraic habit of mind, different representations. In this task, only one participant attempted to employ this habit. Liam created two functions to model his interpretation of the task, he then translated his expressions into table values, and finally proceeded to graph the data from the table. Liam attempted to model the multiple property of five cents and seven cents in the task by writing the expressions .5x and .7x respectively. He did not consider the fact that one can have combinations of multiple five cents and seven cents at the same time, so he created the expressions as independent functions of each other.

After graphing, Liam determined that his graph should have bounds bringing back up an earlier argument, "You know I have infinite numbers that I can input into these [expressions] this
means I can make infinite five cent and seven cent numbers so eventually, as I have more five and seven cents, I will have less and less impossible numbers, meaning the possible numbers limit." As a result, Liam chose to bound his expressions between $0$ and $1$. He argued that five cents can evenly divide $1$ and seven cents could get “close or equal to a dollar” and therefore the two functions should cross around $1$, which to him would represent where this ‘limiting’ should occur. With this information, Liam created a table with input values for the number of stamps, and the resulting evaluations at $0.5x$ and $0.7x$.

Graphing the information from the table (figure 20) by making a double line graph for the expressions along the x-axis (number of stamps) and cost along the y-axis, he realized the curves did not cross beyond $x = 0$. He double checked his table calculations to confirm the accuracy of his graph. Liam contemplated, and determined that the solution instead must be where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. When asked to explain his bounds, all Liam could say was because “the solution had to be below $1$.”

Figure 20: Liam’s bounds on his graphed functions.

Liam’s different representation allowed him to change perspectives as he made sense of the problem. In the next section, the participants will describe change without changing perspective.

**Building rules to represent functions - Describing change.** To describe change, participants must relate how a change in a value or condition would relate to the change to a
value or condition of something else. Liam was the only participant observed describing change during task one. Liam made the argument that since he can use an infinite number of fives and sevens to make numbers; that as he used more multiples of fives and sevens, the number of impossible numbers would decrease and eventually stop. He used this underlying theory as his justification for his entire approach. He believed that eventually the multiples of fives and sevens would be able to sum to any natural number after a certain point. It was this point he was searching for using his \(5x\) and \(7x\) expressions as he believed this point was at their intersection, “initially there will be lots of numbers we cannot find but as we look at larger numbers there will be less and less of these numbers until we can find those numbers.” Here Liam conjectured that the occurrence of impossible numbers would have to lessen as he increased the number of five cent and seven cent stamps and eventually the amount of impossible numbers between possible numbers would have to cease.

The act of comparing the effect of increasing the amount of fives and sevens and relating it to the decrease in unattainable numbers is the demonstration of the algebraic habit of mind describing change. Describing change is similar to predicting patterns in that both are looking at relationships, but as you will see below, predicting patterns relies on a bigger picture point of view.

**Building rules to represent functions - Predicting Patterns.** Only one participant attempted to use this habit. Alex made two observations to build an argument for finding impossible numbers: “so five and seven are our lowest values and so all the numbers smaller than seven but five is impossible and if I add any of these low numbers to five or seven I also get impossible numbers. So I think this would apply to multiples of five and seven too.” As described above, first he recognized that values below five, and the number six, cannot be made
by adding multiples of five and seven in combination while also observing that adding multiples of five and seven will produce numbers that were possible. As a result of these two observations, he predicted that impossible numbers greater than six would only be found by adding an impossible number to a possible number, “since we are adding five and seven each time, if you simply add a number that five and seven cannot make to a value that can be made then you will always produce a value that is impossible to make.”

His prediction allowed him to build a list of values that he feels were impossible to make (figure 21). Alex demonstrated the algebraic habit of mind of predicting patterns by trying to argue the relationship that a possible number + an impossible number = an impossible number. Though this logic is faulty (e.g. 22 (possible) + 4 (impossible) = 26 = 5 + 7 + 7 + 7), it still demonstrates the habit as Alex still tried to predict the form that impossible numbers have. Justifying a rule, is similar to predicting patterns in that they are explaining relationships, except at this point one is attempting to prove that their understanding or conjecture is consistent for any number.

Figure 21: Alex’s list of impossible numbers.
Building rules to represent functions - Justifying a rule. To demonstrate the justifying a rule algebraic habit of mind, individuals would have had to argue why a rule or conjecture would hold for any number. Alex justified his rule that for any combination of multiples of five and seven, if you simply add a number that you know five and seven cannot make then you will create a number that is impossible to make. He justified this rule by showing cases where his rule worked he went beyond simply checking his work and instead attempted to prove to the researcher that his claim was correct by demonstrating that his algorithm holds for several cases. For example, in figure 22, he showed the researcher that if you add four to five plus seven you would make the number 16 which is impossible to make using fives and sevens. Alex may not have explicitly proven that his algorithm works for any ‘n’ but his attempts to show multiple examples to the researcher demonstrated that to him he justified his rule works for any value. Justifying rules require the participant to make claims about how rules remain consistent within a task. Chunking information is similar but it is more about how a collection of rules or concepts describes a pattern.

Figure 22: Alex shows two calculations where he is adding possible to impossible values which both result in impossible values.

Building rules to represent functions - Chunking Information. Chunking information was characterized by students who broke concepts down into simpler chunks in order to look for
patterns in the chunks with the aim to apply this information to the whole. Both Dupe and Bob looked for repeated chunks of information in hopes of finding a pattern in the task. Dupe examined the possible numbers to look for a pattern while Bob looked at the impossible numbers.

Dupe created her chart in layers (figure 7). Initially she only listed the multiples of fives, when this did not give form to the impossible numbers, she added in multiples of sevens to her chart. This layer, also did not provide the information she needed as missing numbers were both possible and impossible, so she added a third layer. The third layer consisted of combinations of multiples of fives and sevens. Examining her three-layer chart, Dupe attempted but failed to see a pattern for the impossible numbers. Notice that Dupe started off with a chunk of numbers, namely multiples of fives. She examined them and did not discern a pattern of impossible numbers. She then added multiples of sevens and reexamined her chart. With no significant pattern gleaned she added combinations of fives and sevens to the chart and once again searched for a pattern but to no avail. Each new iteration to the chart, fives only, fives and sevens only, and five, seven in combinations, represented a new chunk that Dupe created in order to seek out a pattern for the impossible numbers.

Bob’s chunked his information into two categories. After examining a few cases of possible and impossible numbers, Bob concluded that there appears to be a pattern with prime numbers, “The reason I am doing this is I am looking at possibilities, seeing where this is leading me…. 17, and 19 are prime numbers, they are not possible to make using five and seven, the answer is probably related to those facts but I am not exactly sure what that would entail.” He appears to have observed that each of the impossible numbers he discovered are a member of the
prime number set. After further examination, however, he realized that there exist numbers that are not prime but impossible and are prime but possible (figure 23).

Figure 23: Bob attempting to chunk his list of observations of impossible numbers into either prime or composite categories as an attempt to find a pattern.

Bob demonstrated chunking when he attempted to classify outcomes as either prime or composite. As he stated before, he was sure that possible and impossible numbers were “related to those facts”. He had hoped by identifying that the adding of fives and sevens were either prime or composite only, in order to discover a pattern within the task. Chunking information is an algebraic habit of mind associated with examining patterns by grouping information into chunks, this is much different from the habit of mind input from output. As shown below, this habit is much more about direct computation.

**Doing and Undoing - Input from Output.** For one to use the algebraic habit of mind input from output, they would have to demonstrate computationally the calculation of specific inputs when given a solution and conditions. Liam was the only participant who demonstrated this habit. Liam found input from output by solving for \( x \) for each expression equated to one (i.e. \( .5x = 1, .7x = 1 \)) (figure 24). Input from output is similar to working backwards in that both are trying to find the start position. Working backwards however, concerns itself not with specific values but rather the description of the steps or procedure itself that is done to get to the starting position.
Doing and Undoing - Working Backwards. For a participant to have used the algebraic habit of mind of working backwards they would have had to described the reversal of a process, algorithm, or steps to a rule from the final state to an earlier state. Nikki was the only example of a participant using the habit of mind working backwards. Nikki used it when she attempted to prove that she found all of the impossible numbers. She picked relatively large numbers randomly and sought to find combinations of multiples of five and seven that could make this number; in other words, she chose numbers and then sought to work backwards to derive the combinations of fives and sevens she used to make this number.

She checked these numbers by first examining numbers that she knew combinations of multiples of five and seven could make that were as close to the randomly chosen number. Using subtraction, she determined the difference between the random number and the known number to determine if the difference was reachable by five and seven (figure 25). This is working backwards, as opposed to input from output, as Nikki worked backwards without specifically acting on an algebraic expression. Instead, she decomposed a value into multiples of either five
or seven and a remainder. Then broke the remainder portion down using combinations of five and seven.

![Image of calculation steps]

Figure 25: Nikki using the habit working backwards from 102 to determine if it is a possible value or not.

This example closely resembles input from output but the main distinction here is that Nikki is not attempting to derive back to a specific initial condition. In this case, Nikki is using logic in order to reverse how a large number is constructed. As she explained, “I think these large numbers are made by multiples of five and seven, so I want the largest multiple of five or seven I can have as it relates to these numbers and when I subtract it, I think the result will also be a multiple if not, I’ll use the next largest.” Notice in her argument, that she intends to deconstruct numbers that she perceives has consisting of multiples of five and seven into a large multiple and small multiple, the example discussed above is her doing exactly that. Working backwards requires the participant to recognize or conjecture a process that can be reversed, computational shortcut is similar in terms of recognizing a situation but it places its emphasis on operations themselves.

**Abstracting from Computation - Computational shortcuts.** The algebraic habit of mind computational shortcuts is the act of identifying a pattern in computation and skipping the
steps of actually computing while also being able to determine the solution to a computation. Liam demonstrated this by creating a list of values associated with his expression, $0.5x$, starting at $x = 1$ and incremented by one, to create a list of values that increased by 0.05. He continued making this list calculating each subsequent iteration until he realized the values were increasing by 0.05. Once he made this observation, he completed his list by listing multiples of 0.05 as a shortcut (figure 26). Liam, realizing the pattern in his calculation, the increasing by 0.05 allowed him to use the algebraic habit of mind computational shortcut as he was able to extend the pattern as opposed to having to continue calculating each input. Computational shortcut allows an individual to determine a solution without the added burden of computing multiple calculations independently this is similar to symbolic expression in that the goal is to compactly represent a situation using symbols.

Figure 26: Picture shows that Liam stopped computing each expression and began counting by 0.05 to complete his list to 1.00.

Abstracting from Computation - Symbolic expression. Demonstration of the habit of mind of symbolic expression required the use of symbols to represent operations. In Figure 27, Liam and Nikki used this habit by creating symbolic expressions. Liam represented the concept of multiples of $0.05$ and $0.07$ through the symbolic expressions $0.5x$ and $0.7x$ respectively. Liam’s
use of symbolic expression allowed him to generalize a list and later make a table by iterating different values of $x$. 

Figure 27: [Left] Liam shows his symbolic expressions $5x$ and $7x$ and [right] Nikki shows her use of superscript notation to represent multiple sums.

On the other hand, Nikki use a nonstandard superscript notation to represent multiple sums of a number, (e.g. $5 + 5 + 5 + 7 + 7 = 5^37^2$). The notation allowed her to attempt to compactly organize her list, giving her the opportunity to list high multiples of five and seven in a confined space.

Both participants demonstrated the algebraic habit of mind, using symbolic expressions, by using symbols to represent a bigger idea. In Liam’s case he used $5x$ and $7x$ to represent the relationship between price and stamp count, while for Nikki, the use of superscript notation allowed her to write multiple sums of five compactly. One of the goals of using symbolic expressions is to provide the participant a sense of structure and form of the task in order to be able to generalize beyond their work and make bigger connections.

**Abstracting from Computation - Generalizing beyond examples.** The algebraic habit of mind of generalizing beyond examples required the participant to make a conjecture, rule, or statement about a mathematical situation after working examples. Two people demonstrated this
habit. Liam generalized early in his task exploration while Roger generalized near the end. Liam argued that “I know that five cents can divide into a dollar [he writes $20 \times 0.5 = 1.00$] and seven cents can either get to or close to a dollar too, …as I said earlier there will eventually be an upper limit where the numbers that cannot happen, or stop, well if they both go to a dollar then somehow that dollar like somehow represents part of that limit.”

Liam used his understanding that $0.05$ and $0.07$ can equal/approximate a dollar as indication that the curves $f(x) = 0.5x$ and $g(x) = 0.7x$ should intercept around 1. The argument that the curves should intersect around one strictly comes from his personal experience and knowledge of how these monies, five cents and seven cents, could combine to values close to $1$. Liam’s real-life understanding and experience with money led him to generalize the relationship between his two expressions, conjecturing that they must limit around a number they both can relatively divide into.

Roger used his understanding of the number system to make conjectures about the nature of impossible numbers. Roger was checking each number sequentially to find ones that were impossible to make. While listing out his impossible numbers, Roger made the statement that “there should not be an end all be all impossible number." When asked to elaborate, Roger argued that “all numbers are infinite, all types, so even this type is infinite." Thus Roger is using his knowledge and experience with infinite number sets to support his conjecture that since number sets, according to Roger are infinite, then the solution set, given that it is a number set, must also be infinite. In addition to algebraic habits of mind, two other habits were also observed.
Other habits observed in task one

Habits are described as ‘other habits’ because they describe observed productive thinking behaviors that relate to solving the task but do not match the definitions of established algebraic habits of mind as defined in current literature. In this task, two habits that fit this description were observed, they are labeled as: Brute Force and Complement. Brute force is defined as attempting to exhaustively check relationships in order to arrive at the solution, the participant need not completely check all relationships, as they may find the solution before then, but the participant must have made the declaration to do so if the need arises. Complement is the exploration of a problem by trying to determine or prove an inverse situation in order to conclude the original statement; in other words, people using this habit attempt to prove what cannot be the solution to arrive at what is the solution.

**Brute force.** Most of the participants used a brute force strategy to find the list of impossible numbers after being unable to generalize a form for the impossible numbers (the exceptions were Alex and Liam who both were able to create conjectures). Roger attempted to check every number starting at one, sequentially, identifying impossible numbers. Nikki and Dupe created organized charts to list out every number that was possible in hopes of identifying the numbers not listed as impossible. Bob switched to brute force when his theory on the relationship between prime numbers and impossible values proved to be false. He mentally checked each number from one to 20 and labeled values he felt were impossible to make from combinations of multiples of five and seven. In each case, the above strategies showed the participants checking whole numbers in sequential order as they attempted to uncover impossible values. All three participants described in the next section use brute force in conjunction with the complement habit.
**Finding the Complement.** Three participants used a complement strategy. The complement strategy in this task was demonstrated by participants determining what was not the solution to in-turn find the correct solution. Specifically, some of the participants argued that it was simpler to find values that could be made using multiples of five and seven. They recorded these values and compared them to the set of natural numbers. They argued that numbers found in the natural number list but not in their list of attainable numbers must be numbers that five and seven could not make. Nikki and Dupe used this approach to determine most of their impossible numbers; as mentioned before, they created tables with attainable values with the understanding that any number not listed would be unattainable. Liam abandoned this approach after suggesting it was a possible technique; but instead, chose to model this situation with the expressions of \(5x\) and \(7x\) arguing that using algebra rather than this technique was better as it would allow him to find the answer faster.

**Conclusion for Task one**

As stated earlier, when Driscoll chose to use the stamp task in prior research (2003), it was intended to elicit habits that belong to the category of building rules to represent functions; similar results occurred in this study as well. The nature of this task, the requirement of a list of impossible numbers, encouraged the participants to check numbers in a systematic way. Therefore, the algebraic habit *organized list* made sense as a chosen habit. Similarly, many students chose to use brute force as a means to ensure they found an exhaustive list. Of the participants who used brute force, two disparate behaviors were observed. Some of the participants used brute force to manually check each number and create a list of impossible values. Others used multiples and combinations of five and seven to create a list of possible
values and then identified natural numbers that were a complement to this list, creating their impossible values.

The most common algebraic habits of mind used were describing rules and organizing information. The algebraic habits that were not observed were calculating without computing, equivalent expressions, and justifying shortcuts. Other habits that were observed in this task were brute force and finding the complement. A chart showing the observed habits in this task follows (Figure 28).

![Chart showing the count of observed habits](image)

**Figure: 28** Chart showing the frequency of observed habits in task one.

**Description of Task Two: Create a function rule**

Find algebraic functions where $\frac{4}{3}$ and $-\frac{4}{3}$ would be zeroes using only positive and negative whole numbers. How many polynomials by degree can you find like this?
Introduction to Task Two

In this task, students are expected to work backwards from the solutions of $\frac{4}{3}$ and $-\frac{4}{3}$ to create polynomials where each solution represents an x-intercept. Driscoll (1999) found that this task most often elicited algebraic habits linked to Doing and Undoing, specifically input from output and working backwards. Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. This includes, describing a rule, different representations, justifying a rule, working backwards, computational shortcuts, calculating without computing, generalizing beyond examples, symbolic expression, justifying shortcuts. In addition, there were two additional habits identified: output from input and focusing on features and behavior.

Algebraic Habits of Mind Observed in Task Two

Building rules to represent functions - Describing a rule. In making sense of the task, both Nikki and Alex described rules. However, the purposes of their rules appear to be different. Nikki’s description of a rule, appears to be a global rule, it explains the general requirements of what a zero is, while Alex’s rule specifically explains how to determine an even degree polynomial function that would contain the given solutions. Nikki explains that a zero is, “for your given x, the graph will cross the x-axis at that point. In the function, if you know the x-value, the equation will come out to zero.” Here Nikki is describing a rule for zeros. That if a function contains a zero, the function itself must contain an x-input that will evaluate to zero and that function must graphically cross the x-axis.

Alex used rules to describe the process of his algorithm. His rules allow for calculating infinite even degree functions, “first, since you know what the inputs are, and their signs are different we must use an even degree exponent, if we switch it up [the exponent] then we will
have to switch the other numbers as well. So like if we use the fourth power, we would first need
to find what $\frac{4}{3}$ to the fourth power is then we will know what to times and what to subtract. So in
this case, the function would be $81x^4 - 256$ (Figure 29).” Alex’s algorithm, shows that you can
raise either solution to an even exponent (since the even exponent will result in a positive
solution) and use the result to determine the coefficient (the denominator) and y-intercept
(negative value of the numerator) given a polynomial in the form of $f(x) = mx^n + b$, where n is
an even number.

Figure 29: Alex showing how to determine his coefficients and constant for his algorithm.

**Building rules to represent functions - Different representations.** Liam, Nikki, and
Roger drew graphs of functions on a coordinate plane to represent the types of polynomial
functions that could cross the x-axis at the given points. Liam and Nikki began with attempting
to find the x-intercepts algebraically; unsuccessful they drew coordinate planes to attempt to see
the types of functions that would cross at these points. Liam determined that the graph of a
quadratic function could open up or downward with the zeroes $\frac{4}{3}$ and $-\frac{4}{3}$. Similarly, Nikki drew
the graph of a quadratic function and a cubic function through the zeroes; however, she only
considered an upward facing parabola (Figure 30).
Figure 30: The various graphical representations of functions that contain the zeros.

On the other hand, Roger was successful in determining a quadratic function opening upward that contains the zeros $\frac{4}{3}$ and $-\frac{4}{3}$ using the coordinate plane, like Nikki and Liam. Afterwards, he later conjectured that he could construct an infinite number of function rules of even and odd degree. He showed this on the graph by first adding an additional x-intercept at $x = -3$ and then drawing a cubic function that contained all three x-intercepts making the argument that he could simply add more x-intercepts to a get function to be of any degree that he wanted.

Each participant represented functions graphically and all three attempted to transform their graphical representation to an equivalent algebraic one, though only Roger was successful. These three participants demonstrated an understanding that the graph functions had an equivalent algebraic representation thus demonstrating the habit of mind of different representations. Though Liam could not determine the specific function rules that would represent his graphs, he did try to justify a form that the graphical representation should take.

Building rules to represent functions - Justifying a rule. Liam was the only participant who demonstrated which habit of mind during task two. Liam first attempted to find a particular function with the given zeros and a specific y-intercept but was unsuccessful. He claimed there was not enough information given. After he failed to write a specific function, Liam decided to write a general case for a function that would cross through the zeros regardless of a specific y-
intercept, “Okay, so I can’t seem to find this equation because I don’t seem to have enough information but I can make a general one so that when you give me more, I can find the equation next time”. As his quote showed, he tried to represent a generalized form for a quadratic function $ax^2 + bx + c$ that would cross through the zeros.

While exploring, he only searched for the ‘c’ value in his equation, which he labeled ‘b’. Liam argued that since the x-intercepts cannot be changed, the only value in the function that he needs to be concerned with was the y-intercept, “the x-intercept cannot change, so the only thing that can really change is how high or low the y-intercept is relative to the x-intercept you see. So by keeping everything, you know, at $\frac{4}{3}$ then I can adjust the function by some value to create all of the functions.” In this example, Liam is arguing that there are infinite functions that contain the zeros $\frac{4}{3}$ and $-\frac{4}{3}$; these zeros, he argues are invariant while the y-intercept can vary resulting in infinite polynomials.

It this concept that led him to search for the ‘c’ value in $ax^2 + bx + c$. Figure 31 shows the relationship Liam found. He claimed it would give infinite functions for any given y-intercept (b value). Liam used this generic form, arguing that since b can be any number, then this must mean that there are infinite quadratic function rules of this form (though he did not consider $b = 0$) that would cross the x-axis at $\frac{4}{3}$ and $-\frac{4}{3}$. 
Doing and undoing - Input from output. Liam, Roger, and Nikki demonstrated the input from output habit of mind. Initially, Liam interpreted the task as one that required him to solve a linear equation of the form $ax = 0$ for $x$ with values of $a$ equal to $\frac{4}{3}$ and $-\frac{4}{3}$.

Roger found the expression $3x - 4$ by setting $x = \frac{4}{3}$ and multiplied both sides of the equation by three and subtracting four. When explaining how he knew that, Roger explained that to get the solution $x = \frac{4}{3}$, “I would have to isolate $x$, so I would have added four and then divided by three giving me that answer.” Lastly, Nikki also attempted to use the habit input from output as she made sense of the problem; however, she did not seem to demonstrate an understanding of the concept of input/output. Instead, she focused exclusively on the fact that a zero means $y = 0$, “Since $\frac{4}{3}$ is a zero, we know $y = 0$, which means if I add 0 to both sides I get $y + 0 = 0$. This means that I can write $P(\frac{4}{3}) = y + 0$” finding $y = -4/3 - 0$ and $y = 4/3 - 0$ as her equations. Liam, Roger, and Nikki used the algebraic habit of mind, input from output, as they made sense of the task, see figure 32. Each wrote an equation that required the finding of a solution (Liam and Nikki’s case) or an initial expression that would provide the solution given (Roger).
Doing and undoing - Working backwards. Four out of the six participants demonstrated the working backwards habit. Dupe, Roger, and Bob used this habit to find an expression that contained the zero $\frac{4}{3}$. Dupe identified that for a linear function where $x = \frac{4}{3}$ to have a zero the slope of three and a constant of $-4$ was necessary, “I need the three as my coefficient so that the threes cancel out giving me four and then I minus four and I get zero.” Roger wrote the equation $x = \frac{4}{3}$ and proceeded to reverse the operations until he had $3x - 4$, specifically he did this by multiplying both sides of the equation by three and subtracted four. Bob also found the same expression, he reasoned allowed his approach, “I am going to try a backward construction. If I have $\frac{4}{3}$ then I would need to times by three that gives me four then minus four gives me zero. Okay.” Dupe, Roger, and Bob demonstrated working backwards in this example, as opposed to input from output (with the exception of Roger who did both), by focusing on reversing the operations as opposed to specifically finding an input or initial condition, figure 33.
Figure 33: Participants working backwards to find an expression.

Alex used this habit to search for a single polynomial that would contain both zeros. Alex realized that squaring $\frac{4}{3}$ and squaring $-\frac{4}{3}$, produced the same result. Since both equated to $\frac{16}{9}$, Alex wrote the function $f(x) = 9x^2 - 16$, and stated, “I square the value to make sure it is positive, then it has to start with nine times to get rid of the fraction and now that would leave 16, so I would subtract that and now I know it makes zero no matter what.” Like the others, Alex emphasis was not on finding a specific input but rather working backwards from a current state to an original state. In this case, after determining that squaring the inputs provided the same rational result $\frac{-b}{m}$, Alex was able to work backwards from this solution to arrive at a function that models the expectation of the task.

**Abstracting from computation - Computational shortcuts.** Three participants demonstrated the algebraic habit of mind of computational shortcuts. Alex’s even degree
polynomial algorithm relies on the shortcut that all real values to an even degree will become positive, “They would all work for even degrees because when you square it, whether it is a positive or negative number it gives you the same value.” As his explanation shows, Alex is using the algebraic habit of mind, computational shortcuts by showing that there is no need to compute both inputs \( x = \frac{4}{3}, x = \frac{-4}{3} \) to check if they work in his function rules. The act of making function rules with an even degree not only makes them both positive but also the same value and therefore if one works, so does the other.

Roger also used this habit. As shown in figure 34, After determining his first expression, Roger concluded that since the sign was different in solution \( x = \frac{4}{3}, \frac{-4}{3} \) it would only change the constant from −4 to +4, giving him two expressions without having to compute the other, \((3x + 4)(3x - 4)\). Roger demonstrated this habit by skipping the steps of working backwards from \( x = \frac{-4}{3} \) to \( 3x + 4 \). He recognized that since only the sign changed, the numbers used in the problem would be identical just with a change in operation from plus to minus.

![Figure 34: Roger explaining why the constant value switched in his expressions.](image)

Bob demonstrated computational shortcuts by deriving three trivial functions. He used three identities in crafting his expressions, finding functions \( f(x) = 0, f(x) = \frac{0}{x} \) (though this is not a polynomial), and \( f(x) = x^0 - 1 \) produced zeros at \( x = \frac{4}{3} \) and \( x = \frac{-4}{3} \) (figure 35);
though he identified these solutions as *cheats*, “I know it is a bit of a cheat but these seem to work.” Bob used the habit computational shortcuts in the above example by writing functions that use identities which by their nature reduced any input down to a constant value. As a result of these identities, any input, including the ones given in the task will always reduce to the same value. Computational shortcuts allowed these participants to skip computational steps, calculating without computing is similar but requires the participant to consider computations independently of numbers.

Abstracting from computation - Calculating without computing. The algebraic habit, calculating without computing, requires the participant to consider calculations independent of specific numbers described in the task. Roger and Bob were the only participants who demonstrated this habit. After Roger determined his first expression \((4x - 3)\) he found that he needed a second expression to multiply his first in order for his function to work, he explained, “for these to be zeros, we need to write them as a product, that way when either answer goes in, the result is always zero.” Roger is demonstrated calculating without computing by showing that he need not evaluate the entire expression with the zero inputs. If either expression, since they are a product, evaluates to a zero then the whole expression will evaluate to zero.

On the other hand, Bob used calculating without computing to incorrectly determine why he could not find a quadratic function, “I’m stuck here because when I square \(\frac{4}{3}\), the fraction only
gets bigger (larger numerator and denominator values) and to get rid of this fraction I would need to times it by the reciprocal, but the reciprocal is not a whole number, so I guess this can’t be done.” Bob demonstrated calculating without computing in this example by walking through his process for how he would determine the coefficient for his function rule. In his explanation, he recognized that squaring a fraction resulted in another fraction and to further simplify this expression, by multiplying a single number, would also require a fraction which is in violation of the task; therefore, Bob used calculating without computing to conclude the task was impossible.

**Abstracting from computation - Generalizing beyond examples.** Two participants, Roger and Alex, showed the habit of mind of generalizing beyond examples. Roger, after determining the cubic function by adding an additional x-intercept, realized that he could keep adding x-intercepts to his expression creating any degree of polynomial greater or equal to two. Roger used the graphical representation of the zeros and the resulting functions he drew as examples to allow him to generalize his solution to the task that infinite functions both even and odd degree, with positive and negative coefficients could meet the requirements of the task.

Alex also generalized a form that would enable him to generate infinite polynomials. His algorithm, unlike Roger’s method, only allowed for creating even degree function rules that only used the two x-intercepts given in the problem. Alex found that even degree polynomials removed the negative value from the inputs allowing him to find one set of coefficients and constants for his function rule; however, Alex also attempted to use the same logic for odd degree polynomials, "but odd degrees will not work, when they are negative they will stay negative and when they are positive they will stay positive so they cannot make both values become zero." Here Alex is arguing that because the cubing of the x-intercepts would produce values of different signs, he cannot find a function with the same set of coefficients and constant
in the form of \( f(x) = mx^n + b \). This form was the only function family he considered and therefore concluded that odd degree polynomials were impossible. Even though Alex did not generalize his result to represent all solutions, he still demonstrated the algebraic habit of mind, generalizing beyond examples by conjecturing from his examples (working with degree of two and four) a general form for even degree functions that meet the requirements of the task. Those who generalized beyond examples represented their solutions with symbolic expressions.

**Abstracting from computation - Symbolic expression.** Every participant used symbols to model \( x = \frac{4}{3} \) and \( x = -\frac{4}{3} \). The nature of the task required participants to use and reason from the expressions \( x = \frac{4}{3} \) and \( x = -\frac{4}{3} \), as a result, each participant worked with symbols. The purpose of symbolic expression in this task is to encourage the participants to find functions that contains the given zeros \( x = \frac{4}{3} \) and \( x = -\frac{4}{3} \). One habit that supported finding such a solution was justifying shortcuts.

**Abstracting from computation - justifying shortcuts.** Alex justified why polynomials in his algorithm had to be of an even degree, “When I square it there, any number you put in there will make it a positive”; that is, by applying an even degree he is guaranteeing that any input would evaluate to an even number. Alex demonstrated justifying shortcuts in the example above by arguing that by using a polynomial of even degree, he would know the result would always be positive. Two other habits were observed in this task and are discussed below.

**Other Habits Observed in Task Two**

**Output from input.** Output from input was a habit used by three of the participants. This habit allows the participant to determine an output given a selected input. The input they derived was then checked against the expectations of the task to determine if further modification to their function was needed. It is similar to the problem solving strategy guess and check with the
exception that guess and check implies that the inputs themselves are guessed and then verified for correctness, while in input from output the inputs are known in advance and the outputs that result are used to make decisions about modifying a process, rule, or understanding.

Dupe and Roger, for example, showed that though their linear expression $4x - 3$ would work for $\frac{4}{3}$ it would not work for $-\frac{4}{3}$. While for Alex, this habit aided him in realizing that he needed to square his values in order to remove the negative sign. Like with Dupe and Roger, when he used his inputs with his linear function, he kept having issues with the negative solution not combining with the constant. He then realized that if he squared his input, the solution would work regardless if the input is positive or negative (figure 36).

Figure 36: Examples of Liam and Dupe using an x-input to verify the y-output.

**Focusing on features of a function.** Focusing on features of a function is defined by participants making use of their sense of structure and features of an algebraic function. Two participants demonstrated this algebraic habit of mind. Dupe and Roger both make the argument that since the $\frac{4}{3}$ and $-\frac{4}{3}$ are zeros the function must have a minimum of degree two. Dupe elaborated, “a zero is nothing more than a way of saying x-intercept, and since this has two, the function must be at last a quadratic in order to cross the axis twice.”

Roger made a similar argument gesturing after drawing a coordinate plane to show that a possible solution could cross the x-axis at both locations making either an upward or downward
facing parabola. Later, Roger also gestured and stated that higher degree functions could also have the same x-intercepts by ‘bouncing’ at the zeros instead of crossing. Notice how both participants used the features of the function, relating the x-intercepts to the minimum degree of the function and Roger’s notion of the graph ‘bouncing’ at the x-intercepts creating even higher degree functions both of these demonstrate a structural understanding of the concept of function.

**Conclusion for Task Two**

The create a function rule task’s design is to elicit habits of mind associated with Doing and undoing. This task provided the participants with two zeros and prompted the participants to find polynomial functions that possess these zeros while only using integer values in the function rule. Given this set-up, it is not surprising to see participants use the algebraic habits working backwards, symbolic expression, different representations, and input from output. Given that the students had the solution, it was only logical for many of them to work backwards to find a function, similarly, in order to make sense of the solution or the known polynomial forms (e.g. standard quadratic form: \( ax^2 + bx + c \)) each participant chose to interact with symbolic representations of the zeros and functions that could contain the zeros, and additionally, some also drew graphical representations to demonstrate curves that would contain the zeros.

Interestingly, there were two habits observed that are not identified in Driscoll’s algebraic habit framework. The first habit, referred to as output from input was a habit where individuals evaluated expressions by substituting known values for x and evaluating. As the name suggests, it is the counterpart to the algebraic habit of mind - input from output. This habit allowed individuals to not only check their work but also encouraged participants to make sense of why their solutions were not working. The other habit was the habit of focusing on features of a
function. This habit was observed when participants used their sense of algebraic structure as they made sense of the task. A chart of the observed habits in task two follows (figure 37).

![Count of Observed habits](image)

Figure 37: Chart showing the frequency of observed habits of task two.

**Description of Task Three: The Carnival of Bears Task**

Connie, Jeff, and Kareem went to the circus. They saw bears do tricks. Three brown bears and three black bears did one of the tricks. At the beginning of the trick, the three black bears were on the left side of a long mat divided into seven squares, and the three brown bears were on the right side. Each bear had its own square with an empty square in the middle. The bears could only do two different types of moves:

1. They could slide onto the next square if it was empty, or
2. If the next square was not empty, they could jump over one bear to an empty square.
The black bears moved only from left to right and the brown bears only moved from right to left. When the trick was over, the bears had switched places. All the black bears were on the right side, and all of the brown bears were on the left side.

1. The bears needed 15 moves to switch places. Explain how they did it. How many moves were slides? How many moves were jumps?

2. What is the smallest number of moves that five black bears and five brown bears need to switch places? Explain how to do it. How many moves would be slides? How many moves would be jumps?

3. If 20 black bears and 20 brown bears were involved, how many slides would be needed? How many jumps would be needed? How many moves would be made altogether? Explain how you got your answer.

4. Pretend that someone told you how many black bears and brown bears did the trick. Explain how you could figure out how many slides, jumps, and moves altogether the bears needed to switch places.

**Introduction to task three**

In this task, students are expected to identify the patterns of and make rules for the relationship between the number of bears and number of jumps and slides. Driscoll (2003) found that this task most often elicited algebraic habits of mind predicting patterns and generalizing beyond examples. Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. In particular, the next section will highlight the algebraic habits, organizing information, predicting patterns, describing a rule, different representations, describing change, justifying a rule, working backwards, calculating without computation, and generalizing beyond examples that were used by the participants as they solved
the task. In addition, there were two additional habits that were identified: brute force and output from input.

**Algebraic habits of mind observed in task three**

**Building rules to represent functions - Organizing information.** The bears, in the task, would change position as each one moved to the opposite side of their starting position. To represent the bears, every participant created a drawing that modeled their starting position; however, only five participants modeled and tracked the movement of the bears (Alex, Dupe, Nikki, Bob, and Roger). There were distinct methods for modelling and tracking the movement, figure 38 below shows the four methods for tracking position.

![Figure 38: The four ways the participants modeled the movement of the bears.](image)

Alex and Dupe both chose to model the movement of the bears by erasing the current location of the bears and drawing the new location of the bears; however, Dupe did not track the
moves while they happened like Alex. Instead she determined how the bears moved first then attempted to work backwards to find the moves that were used. When this did not work she completed the first part of the task again by using arrows to track the movement of the bears. Nikki also used arrows to indicate movement. An arrow to the left represented a bear on right moving to the left, an arrow with a large arc symbolized a jump, and an arrow with a small arc symbolized a slide. Nikki switched to a more organized list to track her bear movements using type of bear and beginning and ending locations to represent her moves as a separate list from her initial drawing. Bob created a new diagram underneath his previous diagram each time a bear made a move. Lastly, Roger used markers to complete the first part of the task. Three black markers represented three black bears and three blue markers represented three brown bears. To show movement, he picked up a marker and moved it to the empty space.

As stated before, all participants initially showed the basic setup of the first scenario in the task (three black bears, empty center square, and three brown bears). From here, we saw four different methods for modelling how the bears moved. Though four methods for showing movement were observed there were only three methods for tracking the bears’ movements (tallies, listing, and symbols). Though Dupe modeled her movements like Alex, she did not keep a running record like Alex and Roger. As a result, she was unable to determine the amount of each type of moves. As a result, she completed this scenario a second time using arrows like Nikki’s arrow method to track the movements of the bears. Lastly, Bob tracked movement by indicating the direction with an arrow and a shorthand description (e.g. SL for Slide, and J for jump) of the type of move between each diagram.

The four methods for representing the bear movements and the three ways to record the results demonstrated the algebraic habit organizing information by showing various ways the
participants visually organized the task’s information. By constructing these figures, and developing a tracking system the participants were able to make sense of this part of the task and determine the solution to the first part of the task. Unfortunately, success in the first part of the task did not translate to success in the later parts of the task as they were unsuccessful to predict the patterns hidden within the task.

**Building rules to represent functions - Predicting patterns.** Three participants described their approach to the problem by attempting to make sense of a pattern they were observing when initially moving the different types of bears between squares. Each of these individuals predicted a pattern in how the bears needed to move in order to have the bears switch sides, (Bob) “I think there is probably a sequence I’m looking for, one side, then one side, back forth back forth. There should be a \(1 - 1\) trade if I do it on one side things would box up”, (Dupe) “So I think when a black bear moves, I think a brown bear needs to move too”, (Alex) “One bear should move, either side, then the other side should move next." This pattern, as described by the participants, assumes that each side must take a turn. After a bear on the left moves, a bear on the right moves, and so on.

Even though this approach cannot lead to a solution to the task directly, as such an approach ignores spacing and only focuses on equality of movement, it did help them to make sense to the underpinnings of the task. These descriptions above fall under predicting patterns because the participants are recognizing that the bears need to switch sides and recognize that the bears are starting in a balanced position (three brown bears on one side and three black bears on the other, separated by a single square), and using these facts they attempted to predict that the bears would move in an alternating pattern in order to maintain this sense of balance. The failure of these predictions allowed for most of them to find the solution to how the bears will move and
encouraged some to use this failure to more accurately create rules for how the bears should move.

**Building rules to represent functions - Describing a rule.** Most of the rules observed in this task involved descriptions on how to start, (Nikki) “If they are sliding it doesn’t matter which one you start with”, (Alex) “I moved the rights, if you move the left ones it will still work” or how to avoid getting stuck, (Alex) “You have to move one, then you have to move one from each side to allow the whole thing to keep moving, if you allow two from the same group to move you will get stuck”, (Roger) “I know I can’t block them in cause if I have another brown bear here then they are stuck and if one of the black bears move then these are stuck.”

However, two individuals described rules for how to score whether a bear jumps or slides, and the other attempted to describe the procedure for having the bears switch sides. Dupe, in her second attempt on the three brown and three black bears problem used arrows to indicate how her bears moved from one space to the next. She explained the benefits of this by saying “[I am] essentially moving one bear each time to an empty box. By either jumping or moving. Then we are moving the dot to the empty box with this [an arrow] then there always has to be bears moving. Um Then eventually then we count, then we count the loops and moves and wonder if we have more moves or less moves.” Here Dupe is describing a rule for tracking and counting the types of moves needed to solve the puzzle. Without this method, Dupe was unable to complete the task the first time she attempted to work backward.

Alex used his failure of his pattern prediction to recognize that he was not focusing on spacing, “oh, I see, back and forth won’t work, in order to get this to work we need each side to like, um spread out. When spread we can then move the bears between other and um yeah, then the bears can go around each other.” By recognizing that the bears needed to spread out to allow
the bears to pass by one another, Alex was able to complete the first task relatively quickly; however, he did not attempt to use this ‘spread’ rule on the larger sets of bears. Instead he attempted to use his solution from the first part to generalize solutions for the bigger sets which was a common practice of the other participants.

After Alex found his solution of six slides and nine jumps, he examined the numbers to determine how these numbers could be derived from the starting amount of bears. He assumed that he could find the slides by doubling the number of brown bears and find the number of jumps by adding the number slides to one half the number of slides. With this understanding he was able to describe how to find the number of slides and jumps in the subsequent parts of the task, for example when discussing how to find the slides and jumps in the five bears problem he said, “A \( \frac{1}{2} \) was increased to that number [slides to find jumps], cause here I added three (6 + 3 = 9 jumps) and here I added five (10 + 5 = 15 jumps).” Another method the participants used to make sense of the first part was to use different representations.

**Building rules to represent functions - Different representations.** Dupe and Roger used different representations to understand the three brown bear and three black bear part of the task. Roger, figure 39, initially drew ‘stickmen’ to represent his bears and used arrow symbols to show the movement of the bears. He then determined that drawing the stickmen was too labor intensive for the task and requested manipulatives to complete the task, “I need props. Can I have extra markers? ...um three black and three blue.”
Figure 39: Roger representing the bears with symbols then with markers and tallying his moves.

Changing representation of the bears from drawings to manipulatives gave Roger more immediate access to representing the bears’ movement. Now instead of erasing and redrawing each bear and then analyzing the consequences of the move, Roger was able to make a quick movement and invest more time into analyzing the consequences of each move. Using this alternative method, Roger was able to understand how the movements led to changes on the board, much like the descriptions of change in the next section.

**Building rules to represent functions - Describing change.** Two participants described change in this activity. Roger described change by assuming a multiplicative relationship between the number of brown bears and black bears to the results. He stated that it did not matter which color he chose but, “The way I imagined it is if I have six slides for three brown bears and three jumps for three black bears, then adding in two more brown bears would give me four more slides which is ten, then the same concept would apply here. If my three black bear gets me nine, then two more will give me six more which makes 15.” In this description, Roger is assuming that where \( n \) is the change in brown bears and \( m \) is the change in black bears then \( 2n = \text{additional slides} \), and \( 3m = \text{additional jumps} \), and thus he can account for change by adding his current count of six slides and nine jumps to the relationship he found for slides and
jumps as it relates to the increased number of brown bears and black bears. Roger showed changed by describing the direct relationship between bears count to slides and jumps.

On the other hand, Liam used a probabilistic approach to determine the expected number of moves or slides for each bear. He stated that because a bear could slide or jump it had a $\frac{1}{2}$ chance of committing either action, and also stated that the bears take up $(n)$ out of $(n + 1)$ squares, and lastly argued that since the ‘trick’ takes 15 moves, the product of all three numbers would result in the amount of slides and moves. He used this understanding and the change in the context of each problem to attempt to find the number of slides and jumps the bear used, “Okay in the next problem, there are five black bears and five brown bears, so there are more bears meaning we will have some quick number changes. I have $\left(\frac{10}{11}\right)$ for the bears, $\left(\frac{1}{2}\right)$ chance, and 15 moves, giving 6.8 moves, either jump or slides.” In this sense, Liam is describing how the change in the quantity of bears affects different parts of his algorithm for calculating jumps and slides. Though Liam’s approach may seem unorthodox, he did provide a justification for his probabilistic model for the number of jumps and slides.

**Building rules to represent functions - Justifying a rule.** Liam and Dupe justified their rule. Liam looked at the problem through a probability lens. He determined that each bear had a random choice to either slide or jump and did not consider the dependent nature of such moves in the context of the question. Since each bear had two options and could only pick one move then there was $\frac{1}{2}$ chance that bear would jump or slide, “My half means that I have one bear that can do two things so one out of two.” He also argued that the sample space that the bears were operating on was “always one more than the number of bears” (figure 40). He drew the first two examples (three brown/three black & five brown/five black with the empty center square) to illustrate this. Next, he took the meaning of the first questions description of the bears taking 15
moves to mean that even when the number of bears increased there were still a total of fifteen moves. Lastly, he claimed that since this is a probability of multiple events, then “like in my stats class, all I have to do is times them all together.”

Figure 40: Liam listing out his justifications for his procedure to finding any number of bears.

Even though Liam is bringing in atypical techniques to approach this task, i.e. probabilistic thinking, he did have a justification for each of his choices in his algorithm. Working backwards, like justifying a rule, required the participant to have a deep understanding of how an observed rule works for they cannot justify or traverse the rule in the opposite direction without knowing how the rule works.

**Doing and Undoing - Working backwards.** Two participants demonstrated this algebraic habit, Dupe and Alex, but the manner in which they showed this is unique. Dupe worked through the first part of the task, by moving the dots (bears) by drawing their new location and erasing their old across her board until she successfully moved all of the dots. She then attempted to work backwards by moving the dots back to the start to count the number of jumps and slides the bears used but she was unable to do this. As a result, she reset the board and kept track of the movement using arrows that she later counted.
Alex worked backwards by using his conclusion from the first part in reverse. After working backwards, he developed an algorithm which he subsequently used to answer the rest of the task. For example, in the second part, Alex said, “Okay, so the last one took six slides and nine jumps and I see I have six bears, so…. Oh I see... Since this one would require a six and nine jumps, this one would require a 10 and 15. So like when we did the six bears, one would have to move all the way across the board by sliding so that was six but for nine bears they all had to jump plus three, So this one, if we move from here to the end that would be 10, and if we jump the bears five would have to jump five bears each and then we would have 10 more.” As can be seen, Alex’s description of the bear movement is an example of working backwards. In this case, he followed the rules for the bear movements to the ending state, then reversed his movements to describe the numbers of slides and jumps needed and then described how he could reverse the solution for an even bigger case, 10 black and brown bears.

**Abstracting from computation - Calculating without computing.** Liam demonstrated this habit by recognizing that since both sliding and jumping had an equal probability then the expected value of slides which he computed would be the same for jumping, “Since it is also $\frac{1}{2}$ chance of jumping we would have the same amount of approximate jumps.” With the understanding that the probability of each event was the same, Liam was able to rationalize that jump amount without computing it.

**Abstracting from computation - Generalizing beyond examples.** Roger and Alex used their work on the first task to make a generalization about how to find the number of slides and jumps for the general $n$ bears case. Alex and Roger had similar methods for calculating the number of slides and jumps. The distinction in their generalizations is Alex described his relationship between bears, jumps, and slides as an explicit formula while Roger made a
recursive connection. This slight distinction led to the creation of two similar but different
generalizations. Alex found that slides were always double the bear count of one type and jump
count was 1.5 times the slide count.

Roger, on the other hand, found that given there were initially three bears of each type
and this amount led to six slides and nine jumps then adding additional bears $n$ of each type
would result in $2n$ more slides and $3n$ more jumps. Both Alex and Roger derived independently
equivalent formulas that would allow one to derive the same number of slides and jumps as the
other. Unfortunately, they made an incorrect assumption on the relationship between jumps and
bears. This task was difficult for all of the participants. There were no other observations of
algebraic habits of mind beyond the ones already mentioned; however, two other habits were
observed.

Other habits observed in task three

**Brute Force.** Bob claimed that the only way he would figure out this task was to list out
all of the possible combinations after realizing that his ‘balanced pattern’ described earlier was
not going to work, “If I move this it blocks it and if I move that one it blocks it. Apparently that
pattern isn’t gonna work. If I had infinite time I would mix-and-match combinations but at this
point, I am not seeing it. Here Bob is suggesting that the only way he could successfully navigate
this task was to list out every combination of movement by the bears that is possible. Though he
suggested that that would provide the solution, Bob did not pursue this thinking.

**Output from Input.** Liam demonstrates his calculation in figure 4 adding “Since there
are 15 moves and the probability of sliding is $\frac{1}{2}$ and since there are six bears out of seven squares
we get 15 times $\frac{1}{2}$ times $\frac{6}{7}$ which is $\frac{45}{7}$ or about 6.4 this would mean that 6.4 approximate the
amount of slides.” It is clear in this section that Liam is using his various inputs in order to derive an output.

Figure 41: Liam deriving the number of slides and Jumps using given inputs.

**Conclusion for task three**

As stated earlier, this task’s design is to elicit habits of mind associated with predicting patterns and generalizing beyond examples. This task required the participants to determine how six ‘bears’ can switch sides when in starting position of three on each side and an empty square in the middle only by moving to an empty square through either sliding one square or jumping one bear. The goal of this task was to have students complete the first question (and second if necessary) to determine the algebraic relationship of bears to sliding and jumping. Giving this set-up it is not surprising to see the participants use the algebraic habits of mind organizing information, predicting patterns, and generalizing beyond examples.

Unfortunately, only three of the participants were able to complete the first portion of the task and derive values for jumping and sliding (Dupe, Alex, and Roger) and of those three, only
two derived the correct values (Alex and Roger). As a result, these two were able to use their answers to conjecture about the amount of slides and jumps needed for the other questions, with one exception, Liam. Liam did not attempt to switch the bears at all, instead he used probabilistic thinking. Liam created an algorithm for determining the slides and jumps of the bears relying on the assumption that a bear has an equal opportunity to slide and jump \( P(\text{jump}) \times (\text{bears/Total squares}) \times 15 \text{ total moves} \). Additionally, there were two habits observed that were not listed in Driscoll’s algebraic habit framework: Brute force and Output from Input. A chart showing the observed habits in this task follows (Figure 42).

![Chart showing the frequency of observed habits of task three.](image)

Figure 42: Chart showing the frequency of observed habits of task three.
Description of task four: Stacking Cans Task

Figure 43: Stacking cans, task four, adapted from Stein and Smith (1998)

Introduction to task four

In this task, students are expected to determine a function relating the amount of cans in the bottom row to the total number of cans. In order to accomplish this, Stein and Smith (1998) argued that participants should be able to reason about change as rows are added, generalize the effect of adding rows, and examine the input/output relationship of row to total cans. Therefore, the algebraic habits of mind that one would expect to observe while participants complete this task are predicting patterns, describing change, generalizing beyond examples, equivalent expressions, and symbolic expressions. Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. In particular, the next section
will highlight the algebraic habits, organizing information, different representations, describing change, input from output, computational shortcuts, generalizing beyond examples, doing and undoing, and symbolic expression, that were used by the participants as they solved the task. In addition, there were one additional habit that was identified: brute force.

**Algebraic habits of mind observed in task four**

**Building rules to represent function - Organizing information.** Two participants demonstrated this algebraic habit of mind. One participant drew representations of the stacking cans in addition to an organized list of calculations, the other made a table relating row number to number of cans in that row. In the figure 44 below, the stacking representation is shown.

![Figure 44: Visual representation of stacking cans (Dupe).](image)

Unlike other people who drew the stacking tower from top to bottom, Dupe decided to start at the base and attempted to work her way to the top of the tower. After drawing thirty circles at the base, she claimed that it would be easier to break the base up into smaller ‘groups’
so that she could keep track of the amounts better, “I think it would be too hard um, to like do them all at once, so I am going to break them up.” She then grouped the thirty bottom cans into six groups of five and treated each stack as if it was its own self-contained stack. Since each stack was a group of five, she put four cans on the row above, then three, then two, and finally one. Recognizing that she has six groups of five, four, three, two, and one, Dupe multiplied each value by six and summed to find her total and listed these calculations in an organized list as shown in the above figure.

Liam did not invest time in drawing the stack. Instead he examined the picture given in the task and began making an organized table relating the row number to the amount in the row (figure 45). He recognized that the row amount always equaled the row number declaring that “$y = x$ in this case”.

Figure 45: Liam’s table

Both participants demonstrated the algebraic habit of mind of organizing information. Dupe did this by methodically drawing out her understanding of how the cans stack and by organizing her calculations into a list while Liam made an organized table.

**Building Rules to represent functions - Different representation.** Four participants (Dupe, Alex, Liam, and Bob) used the different representations habit to assist in making sense of
the task (see figure 46). The figure below shows their representations. An explanation of how these were used to assist the participant in this task follows.

Figure 46: Different representations of the can task (Dupe top left, Alex top right, Bob bottom left, Liam Bottom right).

Dupe initially drew the first three rows of the can-stack stating that the overall shape looks like a pyramid with an extra can in each row. She then decided to leave space and draw the bottom row of thirty cans. Stating that she needed to fill in the center, Dupe contemplated the best way to fill in the gap between the first three rows and the last row. Ultimately, Dupe settled on dividing the bottom row into multiples of five, making six stacks, “Okay so essentially I just broke it up into pieces. So, there is thirty cans in the bottom, thirty is easily divisible by five, so
that is why I stacked them into six different stacks, I guess. And If you know you are going to use six at the bottom, or I mean five at the bottom, then the next row would have four, then multiplying four by six, three by six, two by six, they would all come, they would all make, if you put all of this together, they would make the triangle, they would fill it up.” Dupe used her reasoning that ‘each subsequent row would add a can’ in reverse by subtracting a can off of each of her groups of five, giving six groups of four, and subtracted a can off each of these groups making three groups of six, and so on. Thus the multiplication on the lower left of her work represents the numeric representation of her understanding of the can behavior (i.e. there are five cans in six stacks, four cans in six stacks, three cans in six stacks…).

Bob and Alex also commented on the pyramidal shape the stacked cans made. However, they considered the cans from the top-down rather than starting at the base like Dupe. Bob drew the first three rows of the cans and recognized that the cans were adding by one. As a result, he listed the integers from one to thirty and used a calculator to sum their value.

Alex also mentioned that he could just sum the numbers one to thirty but claimed “there has to be a formula for this, or why you give me the question with 500 cans”. Using this logic, Alex drew a modified stack of cans of five rows to see if he could find a relationship in a similar problem, “I am trying to do it for a small amount so I can do it for the total cans.” From his smaller problem, he was able to determine that for odd number rows, “the middle row [count] times the bottom row [count] will work”, but could not see how to make this technique work for even rows of cans. In each participant’s case, they were attempting to use a visual representation to reason about the change in the quantity as more rows were added.

**Building rules to represent functions - Describing change.** Dupe, Alex, Nikki, Roger, and Bob used the algebraic habit of mind describing change to make sense of this task. Dupe,
Alex, Nikki, and Roger recognized that each subsequent row had one additional row than the previous row. Nikki and Roger used this information to add integers from one to thirty on a calculator. Similarly, Dupe attempted to model this behavior starting at thirty cans and counted down as described in the different representations section, “Cause every time you stack it up, you take one away.”

Bob, unlike the other participants attempted to create a recursive function to model the change he noticed, “30 in the bottom row, so you must have 30 rows, it looks almost like you have an exponential function here, no, it isn’t exponential, you are going $x + 1$ in this case. In each case you are adding one each time, so it is an iterative function.” He listed the sequence $(1,2,3,4,5,\ldots,30)$ on the white board table, underneath he wrote ordinal numbers representing the position of the first five numbers. Above the sequence, starting at number two, he wrote the sum of the sequence to that point, in other words, above three, he wrote six because $1 + 2 + 3 = 6$ (figure 47).

![Figure 47: Bob considering the change in the sums.](image)

Next he considered the relationship between the running total and the ordinal number, this row was labeled “Add” in figure 36, “The question to me is then, what factor will it be when we reach 30. I know the answer is staring me in the face. If I want an $x$ value to be a $y$-value.
Then I need to know how to make the row value into the number of cans.” Bob divided the sum by the row number for the first five numbers and recognized that each division is increasing by a half for each row he increases. He tested this approach by multiplying six by 3.5 and seven by four and noticed that this worked. Excited, he wondered “what would be this value if I went to 30 times?” However, Bob did not determine the factor he needed to multiply by thirty, instead he made a list of factors to determine the factor associated with ten (5.5) and multiplied 5.5 by 30 to determine the number of cans he thought would be in the stack (figure 37). Further explanation of his choice to use 5.5 as the factor for thirty will be discussed in more detail in the computational shortcut section.

![Bob’s list of factors](image)

**Figure 48: Bob’s list of factors**

**Doing and Undoing - Input from Output.** Roger was the only participant to use input from output. Roger used brute force to calculate the total for the first part of the task (how many cans will there be in a stacked tower, when the bottom row has thirty). After finding this solution, Roger determined the ratio of total cans to the number of cans in the bottom row. He used this information to answer the next part of the task, “The answer for this one (total number of cans when there are 500 cans in the bottom row) is 125,250 cans. I took the amount that I summed up the first time and divided it by 30. It gave me $\frac{1}{2}$ of 30 and .5 and um I am thinking this .5 is from the cans balancing on top of another can. So um, I took 500 and divided that by
two added that . 5 which gave me 250.5 and then multiplied it by 500 cans to give me my total.”

Roger started with his first solution and sought a relationship between last row number and total cans. When he divided by the number of rows he noticed that the solution was precisely . 5 more than half of the divisor (row $n$). Using this relationship, Roger determined a method to correctly calculate the second part of the task. This is an example of input from output as it shows the participant starting with a solution and manipulating it back to discover a relationship between the solution and row number.

**Abstracting from Computation - Computational shortcuts.** Computational shortcut is the exploitation of calculation situations based on one’s understanding of operations. Bob attempted to use this algebraic habit of mind. “Well if 10 has a factor of 5.5 then times three will give me a factor for thirty which is 16.5, so 16.5 times by 30 is 495. I would want to verify this but I think I’ll stop there. For the 500 one I would do 50x5.5x500 = 137,500.” Bob discovered a relationship between the ‘sum of row n’: ‘row n’ and found that as n increased, the ratio also increased by .5. This ratio is described in more detail in the describing change section. This quote shows Bob demonstrating the algebraic habit of mind computational shortcuts by discussing how he determined the factors for thirty and five hundred. Instead of listing out the factors for each row n, progressively adding .5 to each total and eventually determining the appropriate factor, Bob used a shortcut by determining the factor for ten and recognized that $10 \times 3 = 30$ and $10 \times 50 = 500$. Using this knowledge, Bob thought he could scale the factor at ten (5.5) by multiplying by three for thirty and 50 for five hundred to determine each part of the task ‘s respective total number of cans.
Abstracting from Computation - Generalizing beyond examples. Two participants, Alex and Roger, used the algebraic habit generalizing beyond examples. Both participants found equivalent and accurate functions for determining the total sum from row number.

Alex used simpler examples, examining the sum of three, five, ten, and thirty rows. As stated earlier, Alex noticed that for odd rows, the middle row number times the last row number always produced the correct total. However, this method did not work for the even row examples, ten and thirty. As a result, Alex determined the sum of the ten row problem and the thirty row problem. He then looked at the ratio of the sum of cans to the row number and came to the realization, “So we are off by 15, oh wait here we were off by 5, oh, I got it.” Alex discovered that by halving the can total and multiplying it by the can total, he will always be off by half the can total. He elaborated, “so to find any total, I half it, times it by itself, and then add the half”. As shown in figure 49, Alex used this method to determine both solutions for the task. In effect, Alex used examples in order to generalize this information into a formula that he could use to find any total given the amount of rows which could be represented as \( y = \frac{x}{2} \ast x + \frac{x}{2} \), where \( y \) is total cans and \( x \) is the row number.

Figure 49: Alex’s total can’s calculations
Roger found the same function rule but described it differently. After Roger used brute force to calculate the sum from one to 30, he divided the sum by the row number to see if he can discover the relationship between the row number and the sum. When he divided 465 by 30, he saw the solution was 15.5 which he described as half of thirty and .5. He explained away the .5 by stating that this $\frac{1}{2}$ just represented how the cans are stacked, placing one can on the half of two other cans. As a result of this, he generalized that the total can be found by adding .5 to the row number divided by two and then multiplying that result by the row number (i.e. $(\frac{n}{2} + .5)n$). Roger did not consider more examples before making this conjecture.

As shown above, both Alex and Roger demonstrated the algebraic habit of mind generalizing beyond examples. They both examined examples to describe generalized function rules that show the relationship between row number and total cans.

**Abstracting from computation - symbolic representation.** Liam and Bob were the only participants who demonstrated the use of symbolic representation in this task. Liam, a student who took but did not pass AP calculus in High School, identified the question as a ‘volume problem’, he made a table showing that the row number ($x$) and number of cans in that row ($y$) always matched, concluding that “This is a volume question. What we have to do is set up an integral from one to 30, and we know our function, $y = x$, and my reasoning is that if I do this, right, and go from $1 - 30$ right, then I should be able to figure how many cans are there from top to bottom.” Liam then set up an integral as shown in figure 50 below and used it to compute the amount of cans. Liam’s use of an integral symbolically represented to him the summing of the stacking of thirty cans. Liam recognized the need to sum the thirty cans, “I can just add all thirty up but that isn’t algebra” but instead considered an approach that he thought was symbolically equivalent.
As seen in figure 51, Bob also attempted to use symbolic representation to make sense of this task. In Bob’s case, he attempted to use symbols to represent the iterative nature of the rows (i.e. each row is one more than the last). The picture on the left was his attempt to write a series of functions that would tell the number of cans in each row, “it works by plugging in one for x, but this is silly as we already know that each row is the number of cans. Bob erased this work, then attempted to model the sum behavior. He wrote \( x + (x + n) \) in an attempt to show the previous total \( x \) plus the new rows total \( x + n \) but decided “this does not make a lot of sense either as I don’t know how to show this for thirty rows. It works for two though (laughs)”.

The attempt to model this task recursively benefited Bob later when he correctly determined the change between row sets explained in the describing change section later.
The use of symbols by both Liam and Bob allowed them to express generalizations of the task algebraically and allow them to connect or consider different aspects of the problem. In Liam’s case, the use of symbols allowed him to connect the idea of volume of a solid object to counting cans, and in Bob’s case it allowed him to consider the change he was witnessing to the sum from one row to the next.

**Other habits observed in task four**

**Brute force.** Five of the six participants (Liam, Bob, Alex, Nikki, and Roger) discuss a brute force approach as an option for finding the solution to the first part of the task; however, Liam and Bob decided to abandon the approach for as Liam put it “that way does not use enough algebra.” Nikki and Alex used a calculator while Roger manually added to sum the integers from one to thirty, figure 52, all chose not to attempt to sum the integers from one to five hundred by brute force.
Conclusion for task four

As stated earlier, this researcher anticipated this task to elicit the algebraic habits of mind predicting patterns, describing change, generalizing beyond examples, equivalent expressions, and symbolic expressions. This task required the participants to determine how the can sum increased each time a row was added with the goal that they could determine a function rule that explains how to find the total when given the row number, both Alex and Roger were successful in describing function rules that model this relationship. Surprisingly, most of the participants (four out of six) immediately used brute force as their technique in determining the total cans without much consideration for other methods. The most popular habits observed were describing change and brute force. A chart showing the observed habits in this task follows (Figure 53).
Figure 53: Chart showing the habit observed in task four.
Description of task five: The patterns at work task

Look at the two-column figure:

<table>
<thead>
<tr>
<th>A</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
</tr>
<tr>
<td>6</td>
<td>33</td>
</tr>
</tbody>
</table>

Figure 54: Columns of values used in patterns at work task.

1. What are the patterns that you notice?

2. If you were to continue the patterns vertically, would a, b, or both reach the number 1005?

3. If a new pair of columns were created that followed the same patterns as noted in #1, but the third value under b was 10, using your rules: What would be the value directly across under A? What value would be under 10? What values would be at the beginning of the columns?
4. Which pattern rules adequately allow the construction of both tables given the initial values for each table?

**Introduction to task five**

This task was adapted from Matsuura et al (2013). In their study, they designed the task to elicit the algebraic habits of mind predicting patterns, symbolic expressions, and input from output. The task has three major components. Initially, they are asked to describe any patterns they can find encouraging them to look for patterns within each column as well as between the columns. Then the participants were asked if the table continued, would either column reach the value 1005. Lastly, the participants are asked to make a new table where the third row under column must have the value ten using the rules they considered in the first part. If their rules prohibited them from creating the second table, they were then tasked to adjust their rules so that they would apply to both tables.

Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. In particular, the next section will highlight the algebraic habits, organizing Information, predicting patterns, different representations, input from output, working backwards, computational shortcuts, symbolic expressions, justifying shortcuts, that were used by the participants as they solved the task. In addition, there were two additional habits that was identified: brute force, refinement.

**Algebraic habits of mind observed in task five**

**Building rules to represent functions - Organizing Information.** Four of the six participants used the algebraic habit of organizing information. Bob created a list of his data, while Dupe, Nikki, and Roger each created a table. The list enabled Bob to examine the various relationships found by combining the columns in various ways. For example, Bob recognized
that if he added the columns A and B it would produce a list of 3, 9, 15, 21, 27, 33, and 39, a pattern that is increasing by six each time (figure 55).

Figure 55: Bob analyzing his lists.

On the other hand, Nikki, Dupe, and Roger each used a table to connect their rules from the first table to the new table. Each of their tables is shown in figure 56. For example, as Dupe’s table shows below, she identified the rules “+1” under A and “+5” under B from the first table and transferred these rules by literally writing the rules at the top of the new table and then constructing it.

Figure 56: The tables created by Dupe (left), Roger (center), Nikki (right).

**Building rules to represent functions - predicting patterns.** The nature of the task encouraged predicting patterns to be the most popular algebraic habit in this task with all six
participants demonstrating this habit. The task itself encouraged participants to look for patterns within the table in an attempt to predict the rules that best describe the table and then use the predicted rules to create a new table. The intention of this was to encourage students to develop explicit function rules that describe $A$ and $B$ (e.g. $b = 5a + 3$), only one student found such a relationship. The rest found rules that either made the connection between the two tables impossible or described the relationship by recursion only.

Those that produced recursive rules examined the initial table as if the columns changed independently of each other. Alex for example determined that ‘column $A$’ represented ordinal numbers while ‘column $B$’ represented numbers that counted by five. So when tasked to find the values around 10 in the new table, he claimed, “If we replace this by 10 then underneath will be 15 of course because it is 10 and I am adding five. The left side is going to stay the same cause the left side is just counting the numbers and if this is 10, then this is a five and this is a zero. Again the left side does not change.” His interpretation is not wrong, of course, these are patterns that are identifiable in the original table, but he was not attending to how the columns co-vary. Dupe also treated the columns independently and came to the same conclusion as Alex. Her rules are shown in the figure below:

Figure 57: Dupe’s illustration of the rules for column $A$ and $B$. 
In addition to identifying the function rules of each column, Bob, Nikki, and Roger attempted to relate Column $A$ and $B$ together; however, instead of looking for a function rule relationship, they examined the relationship from adding and/or subtracting the two columns, “Well if I add $A$ and $B$ as I go down I get 3, 9, 15, 21, 27, 33, and 39 which is a difference of six (Bob); “Column $A$ is sequential, column $B$ goes 3, 13, 23, 33 and 8, 18, 28, they are like multiples of ten but switching. Also I have 3, 7, 11, 15, If I take 0 from three and if I do one from eight, two from 13, and three from 18 it gives a pattern there, just add four. So my rules are sequential, every other ten and the difference is four. (Nikki); “The difference between these are 3, 7, 11, 15, 19 so we are adding four for the differences. Another pattern is 3, 8, 3, 8, and 1, 1, 2, 2, 3, 3, in column $B$ and it appears that $A$ is going in descending to ascending order. (Roger).”

By combining the rules of “column $A$ is sequential” and “$B - A = 4$”, the participants created rules that would not allow them to create the new table where ten was in the third item in the row, as Bob states “Hmm what I would expect to happen is a 15 below, five and zero above. But that contradicts my first and second rules (see figure 58). These rules appear to be on top of each other. These rules are not portable. Let me rethink my rules.” Each of these three participants had the opportunity to either adjust their current rules to allow for the creation of the new table or abandon these rules to determine new rules that would allow for the creation of the new table. Nikki could not complete this portion of the task.
Bob and Roger revised their rules in order to construct the new table. In Bob’s case, he removed the restrictions that $A - B$ and $B - A$ had to equal specific values, instead having $A - B$ equal a negative value, and $B - A$ equal a positive value. However, even with the broader rules, Bob could not resolve his rules with the fact that his table began with 0 in both columns. Roger on the other hand, simply removed his $B - A$ rule and used the same rules as Dupe and Alex to create the recursive table.

Liam was the only participant who examined the relationships between columns $A$ and $B$ through functions. First Liam identified the same column relationships as the others, representing column $B$ using a recursive function rule: $f(x) = y + 5$, this notation meaning that each additional $B$ value was found by adding five to the previous $B$ value. He then created a function that showed the relationship between column $A$ and $B$, $y = 5x + 3$ (figure 59). How he derived this function will be explored in more detail in the Doing and Undoing - Input from Output section later. These two functions allowed Liam, to create the required table.
Building rules to represent functions - different representations. Liam was the only participant who used different representations to make sense of the task. After copying down the table, Liam represented the change in column $B$ recursively with the function $f(x) = y + 5$. He also graphed the table, designating values in $A$ as $x$ values, and values in $B$ as $y$ values concluding, “Graphically if you look at the same relationship and graph a few points it looks linear, and for each $x$ value there is one specific $y$ value, so I am going to find the slope to get my function (figure 60).” These different representations for the values given in the table, the recursive function, graph, and function relating $A$ and $B$ allowed Liam to gain a deeper understanding as to how the columns relate. An understanding that the other participants did not gain as a result.
Doing and undoing - input from output. Liam was the only participant who used this algebraic habit of mind. After deriving his recursive function of \( f(x) = y + 5 \) by determining the constant difference in column B and graphing the table, he found that the function must be linear. He then used two points to find the slope and wrote \( y = 5x + b \) (he used y to represent B and x to represent A), “So \( y = 5x + b \) and I’ll choose one point from here, I’ll choose (1,8). So our other slope [he meant y-intercept] we have here is 3. Giving us \( y = 5x + 3 \), now let’s see if that works…” Liam used input from output in this case when he used the point (1,8) in his function to determine the value of B. Liam also used input from output to determine the second and third parts of the activity as well. In the second part, he was tasked to determine if either column would reach 1005 if the columns continued on forever. In this case he set 1005 equal to both his recursive function and his explicit function, “Best thing to do is take our functions and
set it equal to 1005. So, \(1005 = y + 5\), so \(y = 1000\), so it works in this relation after \(y\) is 1000 and now \(1005 = 5x + 3\), so \(x = 200.4\), so yes they will meet. At the \(x\) value of 200.4 you will get the \(y\) value of 1005.”

Now, while his calculations are accurate, Liam neglected to notice that column \(A\) contained only whole numbers and as a result column \(B\) would never equal 1000 nor 1005. Though Liam’s analysis is incomplete in this part, it is clear that his use of input from output allowed him to make sense of the conditions in which 1005 would need to exist (i.e. decimal values for \(A\) would have to exist or the previous entry in the \(B\) column would have to be 1000).

The last use of this habit, demonstrated by Liam, allowed him to construct the new table. The third part of the task required Liam to reuse the same rules he derived from the first table to create a new table where the third entry under column \(B\) is 10. Using his recursive function of \(f(x) = y + 5\), Liam derived the values in column \(B\) \((0, 5, 10, 15)\). Next Liam, set each value in turn equal to his explicit function of \(5x + 3\) to determine his corresponding \(A\) values. The figure 61, shows a sample of his calculations. Liam repeatedly demonstrated this habit by deriving the column \(A\) values using his function and various outputs.

Figure 61: Liam’s use of input from output to calculate values on the table.
**Doing and undoing - working backwards.** Only one participant demonstrated this habit. Liam’s recursive function modeled how to find the next term when given the current term, in effect \( B_{\text{next}} = B_{\text{current}} + 5 \). After careful consideration, Liam determined how to use this information to find the values in reverse, “Wait a minute! We can use our \( y + 5 \) function, so if I want to go up, I would go backwards and minus five, giving me five and zero.” Liam derived the values that came before ten by successfully reversing his \( f(x) = y + 5 \) rule, recognizing that the values that came before had to be five less.

**Abstracting from computation - computational shortcuts.** Two participants used computational shortcuts. Alex, when determining if column \( B \) ever reached 1005, initially started to add five repeatedly to three but realized that he could jump to 33 since the computation to that point was already completed, “Hmmm, well we could do \( 3 + 5 + 5 + \ldots \), oh wait, I can just start at 33.” Recognizing that column \( B \) repeatedly added five as the table increased allowed for Alex to skip some of the computation as he checked to see if 1005 would indeed appear in that column.

Bob showed this habit when he quickly determined the result of subtracting column \( A \) from \( B \) after already subtracting column \( B \) from \( A \), “I could also do \( B - A \). I would get the same numbers as \( A - B \) just positive.” Here Bob is demonstrated his understanding of computational shortcuts by recognizing that inverting a subtraction will derive the same absolute values but with an inverted sign without having to compute.

**Abstracting from computation - symbolic expression.** As stated earlier, Roger, Bob, Nikki, and Liam all used symbolic representations in this task. Roger, Bob, and Nikki used such notation to represent the addition \( (A + B) \) and subtraction \( (A - B) \) of the columns. Liam used symbolic expressions to represent the explicit relationship between columns \( A \) and \( B \) \( (y = 5x + \)
3) and recursive behavior of column $B$ ($f(x) = y + 5$). They used these equations to represent generalizations of the operations they were either doing themselves (Roger, Bob, and Nikki) or to represent function behavior (Liam).

**Abstracting from computation - justifying shortcuts.** Justifying a shortcut refers to generalizations one makes about operations in order to avoid computing. Alex, Dupe, and Roger demonstrated this algebraic habit of mind. Each participant demonstrated this habit during the second part of the activity, where they had to determine if column A or column B if extended would reach 1005. All three used the argument that column A is the set of whole numbers and would naturally contain the number 1005, “A would reach 1005 of course. Since it starts at zero and counts by one it will reach every number. (Alex)”; “1005 is a number that you can count up from A since you can count by one and you have whole numbers (Dupe)”; and “A is in order counting by one to infinity so it can (Roger).”

Additionally, both Dupe and Roger reason that column $B$ cannot reach 1005 because the one’s digit will always end in three or eight, as Dupe argues, “But $B$, even though you are counting by five it only ends in three and eight so it can’t be 1005. If it were 1003, or 1008 then yes (Dupe)”. As this section shows, Alex, Dupe, and Roger justified that 1005 could be reached in column $A$ and not in column $B$ by using logic and their understanding of computing with one and 5.

**Other habits observed in task five**

**Brute force.** Alex demonstrated this habit, shown in figure 62, when he was considering if 1005 would be reached by column $B$. Starting at three, Alex repeatedly added five, building towards 1005 but stopped short when he recognized a pattern, the one’s digit repeating three and
eight. By brute forcing, Alex was able to extend the list and pattern of values enough to recognize that the ones digit will never end in a five.

Figure 62: Alex using brute force.

**Refinement.** Refinement was a phenomenon observed while some individuals were trying to extend their rules from the first table to create another table with specific values, it was specifically demonstrated by Bob, Nikki, and Roger. Refinement, in each case, was observed

Figure 63: Nikki struggling to apply her rules to both tables.
when the participants recognized that their rules did not work and fluidly switched back and forth between their original table, their rules, new table, and back to the original table, modifying, checking, and reevaluating constantly. Bob for example demonstrates this constant shifting,

Okay, let’s start with the first rule, If $A - B$ has a negative difference of four, then I need a six here [across from 10]. $A$ increases by one, so this has to be seven, and here is five, and here is four, no way around it based on that rule. So that’s the easy part. Hmm, okay, so for column $B$, each one increases by five. Hmm well that (fifteen under the ten), is not happening. Okay so let me adjust rule three, that will still fit what’s happening on this table [the original table], now let’s see if I can go back to a difference of four. So in $B$, I will have 0, 5, 10, 15 and then in $A$, I would have 0, 1 ... oh that still won’t apply… [removes the restrictions that $A - B$ and $B - A$ must be constants], Let’s check it this way, column $B$ is still increasing by five, so let me rewrite $B$, and $A$ is 0, 1, 2, 3, 4. ... Ahh, finally my rules hold for both.

Notice in his description that Bob continually cycled between the old table, rules, and the new table, adjusting both his understanding of his original table, rules, and the feedback he was receiving from employing his understanding to the new table. Both Nikki and Roger experience similar refinement as they attempted to create the second table using their rules. The following figure demonstrates the habit:
As the model illustrates, those demonstrating refinement go through this constant state of change, moving between examining their understanding of the original information (in this case the table), their expression of the original information (the rules), and their desired result (the new table).

**Conclusion for task five**

Matsuura et al (2013) found that the most common algebraic habits of mind observed while completing this task were predicting patterns, symbolic expression, and input from output. Two out of three of these habits were also prominent with this sample as well. Like in their study predicting patterns and symbolic expression were the most common. However, organizing information and justifying shortcuts were also popular algebraic habits observed in this study. It was surprising that not more than one participant attempted to find a function rule using input from output given the design of the study and the encouragement to “find a relationship between $A$ and $B$.” A chart showing the observed habits in this task follows (Figure 65).
Figure 65: Habits observed during task five.
Description of Task six: Flowerbeds Task

![Flower Beds](image)

The city council wishes to create 100 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds.)

1. How many slabs will the council need?
2. Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

*Shell Centre for Mathematical Education, 1984, p. 64.

Figure 66: Description of the flowerbeds task.

**Introduction to task six**

This task was developed by the Shell Center for Mathematical Education in 1984. As a result, the task was developed before Driscoll described the Algebraic Habits of Mind in 1999; however, Magiera et al, (2017) highlight the use of this task to elicit the algebraic habits of mind. In their study, they examined in-service teacher’s habits of minds and developed strategies for the teachers to improve their students’ algebraic habits of mind. In their study, they found the algebraic habits of mind observed were predicting patterns, describing a rule, organizing information, describing change, generalizing beyond examples, and symbolic expression; they also observed participants using proportional reasoning behavior in the task as well.

Below describes the activity of each participant (Dupe, Alex, Liam, Nikki, Roger, and Bob) as they engaged with this task. In particular, the next section will highlight the algebraic habits, organizing information, predicting patterns, describing a rule, different representations, describing change, input from output, generalizing beyond examples, and symbolic expression,
that were used by the participants as they solved the task. In addition, there were three additional habits that was identified: Proportional thinking, output from input, and brute force.

**Algebraic habits of mind observed in task six**

**Building rules to represent functions - organizing information.** Liam was the only participant who demonstrated the habit of organizing information while completing this task. As seen in the figure below, Liam listed out the details of the task that appeared relevant to him, including the goal of the task (CC wants 100 flowerbeds), the relations observed in the image provided (one flowerbed to six slabs; four flowerbeds to 18 slabs), and an image of how two flowerbeds would be arranged with slabs (figure 67). By organizing his information in this manner, Liam was able to first examine a ratio relationship then when that failed, he used his image in conjunction with his relation to uncover a linear relationship. This is an example of organizing information because by listing his information in the manner that he did directly led him to consider the pattern between flowerbeds and slabs.

![Figure 67: Liam organizing the task four’s information.](image)
Building Rules to represent functions - predicting patterns. Four participants used the predicting patterns habit as they attempted to make sense of the task. Each focused on an aspect of visual invariance; that is, each examined the visual with the intention of finding aspects that were not changing as additional flowerbeds were added. Alex and Liam described the pattern as adding four each time and demonstrated this by covering two slabs of the left side of the image for each flowerbed to show that each flowerbed had exactly four slabs on the right, (see figure 68). This conclusion led naturally to the discovery that the flowerbeds to slabs had a relationship of \( y = 4x + 2 \), where x is the number of flowerbeds and y is the number of slabs. Alex and Liam demonstrated predicting patterns by visually inspecting the image for consistency (i.e. each flowerbed had four slabs) by covering the slabs on the left side of the flowerbed. Doing this allowed them to predict the rule that describes the number of new slabs added with new flowerbeds and ultimately led to the creating of a function rule to describe this behavior.

Figure 68: Liam (left), Alex (right) covering the left end of the flowerbeds to show that each flowerbed is associated with four slabs.

Dupe also found a visual consistent representation of the figure given in the task; however, her representation examined the task from a proportional thinking perspective. Dupe deduced that since four flowerbeds have 18 slabs then two flowerbeds should have nine slabs as indicated in figure 69. This interpretation led her to conclude that one flowerbed must have 4.5
slabs (the unit ratio) and, “Okay so for every ten flowerbeds, you will have 45 slabs. Since one has 4.5, then for 100 flowerbeds you would need 450 slabs.” Here Dupe demonstrated the habit of predicting patterns by identifying a pattern that she noticed and attempted to extend this pattern.

Figure 69: Dupe showing the ratio of two flowerbeds to 9 slabs and the subsequent ratios.

Like the others, Roger also searched for visual invariance. He noted that this task was unusually challenging because some of the slabs were “shared,” “By looking at it, it is hard to simply count it because some of them share.” By shared, he is referring to the idea that the slabs between flowerbeds are used to fill in the spaces of both. After further examination, he discussed the invariance that he noticed, “Hmmm, well only the top and bottom ones of each flowerbed are the ones not being shared and the outsides won't be shared either. That’s four. The shared ones are taking the place of four so that’s half the amount we need in those spaces.” The figure below shows Roger’s understanding of the pattern.
As a result of this prediction of the rule, Roger determined that, “for every 100 you are going to need a top and a bottom so that is going to be 200 and for every two you are going to need two (every two flowerbeds share two slabs), so that’s 100 (50 pairs need two each) and the ends have four total, so it will be 304” (figure 71). Roger demonstrated the algebraic habit of mind, predicting patterns, like the others, by identifying what aspects of the image would remain the same and what aspects of the image would change as new flowerbeds were added. By searching for invariance and variance the participants were able to create/describe relations between the flowerbeds and the slabs.
Building rules to represent functions - Describing a rule. Two participants, Alex and Bob demonstrated the algebraic habit, describing a rule. Predicting a rule and describing a rule are similar habits, but have two key differences. One is that describing a rule moves beyond identifying patterns in a problem and instead has the participants describing the actual steps one takes to carry out an observed rule. The second is that in describing a rule, the participant moves beyond the given information and describes the observed process or algorithm divorce of any specific input.

Alex, using this habit in combination with the algebraic habit describing change, recognized that each new flowerbed added four to the old total to find the current total, “You see each gets four to the right, the first has four, then the next needs four, and four, and four, but there is also the two at the start that we need. So If I know how many I have and I want the next one, of course the easiest thing would be to just add four; however, since I need to skip to 100 flowerbeds, I can also use my thinking that each flowerbed has four plus two more from the beginning, so the math for the four, we can see that it would be four times four then plus two for the beginning and for the end it will be four times 100 then plus two.” As shown above, Alex
demonstrated this algebraic habit of mind (describing a rule) by showing how to find the number of slabs needed, multiplying the number of flowerbeds by four then adding two.

Bob, also demonstrated this algebraic habit of mind by attempting to show how to calculate the number of slabs needed, “yeah to find this, you need the inner portion first, since each center flowerbed has six slabs and the outer ones have four unique slabs, to find the slabs, you would multiply the inner portion by six and add the outside which will be eight. For an illustration of Bob’s description see figure 72.

![Figure 72: Bob’s basis for his algorithm.](image)

“I in this case, 98 times six will be 588 plus the outer four on each side will give me 596 slabs for 100 flowerbeds” (see figure 73). Though this is an incorrect solution, given that Bob did not account for the sharing of slabs in the figure, this example still highlights his algebraic habit of mind of describing a rule as he describes the steps to the algorithm that he observed.
Figure 73: Roger’s calculation of slabs using his algorithm, \( slabs = (6)('inner flowerbeds') + 8. \)

**Building rules to represent functions - different representations.** Liam was the only participant who used different representations while making sense of this task. As shown in figure 74, Liam initially drew the figure in his workspace, then he translated this figure (incorporating information from the one shown in the task) into a table, and finally represented this information as a linear function. Liam used multiple representations to uncover different patterns and information from the task. Changing the figure from a visual representation to a table allowed Liam to uncover that the slope was consistently four meaning that it is linear. Then using his understanding of the representation of the general form of \( y = mx + b \), Liam was able to further represent this information with the function \( y = 4x + 2 \), showing that with each different representation, Liam was able to uncover further meaning in the relation between flowerbeds and slabs.
Building rules to represent functions - Describing change. Five out of six participants (Alex, Dupe, Liam, Nikki, and Roger) used this habit to make sense of the task. Each of these participants described how adding an additional flowerbed changed the total number of slabs. Three of the five participants described the change as a linear growth discussing how the next slab would be precisely four more, “Each next one is four more than the last (Alex)”, “Okay so there is 18 for four but for just one in particular that next one only gets four. So you are really only adding four more (Liam)”, and “one flowerbed equals six slabs. However, they play into each other, so for the next flower I only need four slabs more. I start off with six but for every number, I have to add four.”

In each of these three cases, the three participants described changed as linear, with a constant rate of change of four. This is the algebraic habit of mind, describing change, as the
participants were able to explain how the relationship between the change in the flower beds and slabs, that is for every additional flower bed there were four additional slabs.

Two other participants also described change in the relationship between flowerbeds and slabs; however, they appeared to examine the task through a ratio lens. For example, Dupe initially described her rate of change for each additional flowerbed as adding four slabs but then adjusts it to match her unit ratio of one flowerbed to \( \frac{4.5}{1} \) slabs, “So for every four flowers there are 18 slabs. If I had a fifth one, I would plus four, yeah no, that’s wrong. Each slab has four and a half because between two flowerbeds they have nine slabs, so one has four and one has five. So they share the extra one. The fifth bed will have 4.5 slabs.”

Similarly, Roger also examined the ratio of the number of flowerbeds to the sum of slabs for that amount. In figure 75, Roger is representing the various ratios of total slabs needed to flowerbeds, \(\frac{6 \text{ slabs}}{1 \text{ flowerbeds}}, \frac{10 \text{ slabs}}{2 \text{ flowerbeds}}, \frac{14 \text{ slabs}}{3 \text{ flowerbeds}}, \frac{18 \text{ slabs}}{4 \text{ flowerbeds}}, \frac{22 \text{ slabs}}{5 \text{ flowerbeds}}, \frac{26 \text{ slabs}}{6 \text{ flowerbeds}}\). He then used his calculator to examine the decimal approximation of each of the ratios, noting, “the ratio appears to be approaching four but I don’t know what this means. I don’t know how to write a function since the ratio seems to be changing every time I add a flowerbed. I know the pattern, you are adding four, but since my result changes with each new flower, I can’t think how to represent it.” In Roger’s case, he was attempting to find the constant of proportionality, even though this relation is not directly proportional; however, as he takes the ratio of slabs needed to the number of flowerbeds towards infinity (specifically the number of flowerbeds) the ratio will become four. In both Dupe and Roger’s case, they attempted to describe change through the use of ratios. Dupe determined the unit ratio of one flowerbed to 4.5 slabs and argued that each additional flowerbed would thus contain 4.5 more slabs while Roger looked at how the ratio
changed as he added four slabs and an additional flowerbed. Both used their understanding of how the relations were changing to attempt a description of the change they were witnessing.

Figure 75: Roger examining the ratios between slabs and flowerbeds as he increased both quantities (four additional slabs per one additional flowerbed).

**Doing and undoing - Input from output.** Liam was the only participant to use this algebraic habit of mind. Liam demonstrated this behavior when he used points from his table to determine slope as well as determining the constant b in his equation \( y = mx + b \). Figure 76, shows his calculations. Both cases are input from output, as Liam is using given quantities or outputs (i.e. points) to determine the initial conditions, slope and constant.
Two participants demonstrated this habit of mind (Dupe and Nikki). Dupe used the examples of four flowerbeds to 18 slabs and two flowerbeds to nine slabs to determine that the flowerbed to slab ratio is $1:4.5$. Using this information, she generalized the function rule to be $y = 4.5x$.

Nikki also used examples to describe her observation between flowerbeds and slabs but had difficulty writing the generalization down, “Can I just say it to you? My first flower equals six slabs, if I add an additional flower then I am adding an additional flower to my original flower. The same thing happens to slabs but instead I add four. So like if I have one flower I have six slabs but if I go to two I am adding four, so then, for three flowers would be 13, no 14. Four flowers would be 18. So the flowerbeds give me four times as many slabs so for 100 flowerbeds, I’ll need 400 slabs, in other words, all I have to do is multiple by four.” In Nikki’s case, she generalizing the change of adding four by describing it as a multiplier ($slabs = four * flowerbeds$) however, she forgot to consider the fact that the initial flowerbed started with six and not four. In both Nikki and Dupe’s case, they attempted to generalize the examples given in the task by identifying a relationship they observed in the given figure and extending it beyond this given case.
Abstracting from computation - symbolic expression. Five of the six participants (Alex, Bob, Dupe, Liam, and Nikki) used symbolic expressions to generalize the task’s phenomena that they observed. Of the five, Nikki could not represent her generalization of the task to her satisfaction. Below is an excerpt of her thinking process as she attempted to write a function rule:

This is plus six since we are starting off with the first flower. No it is $x + 4$ because it is increasing by four and $6(x + 4)$ because we are starting with six. No, I am writing this wrong. 18 slabs go with four flowerbeds and I want 100 flowerbeds...Hmm wait. (Lists out the slab count per flowerbed, $6 + 4 + 4 + 4$). These fours are the new slabs. Then all of this right here is all slabs but how do I show these as a function?

As shown in this explanation and in the figure below, Nikki was building towards a symbolic expression but struggled with modeling her numbers with a function. As shown in the figure below, specifically the last image, Nikki almost found an accurate symbolic expression for the task but could not determine what to put in the parentheses that would allow her to determine the correct number of slabs.

Figure 77: Nikki’s process for building a symbolic expression.
Alex, Liam, Dupe and Bob demonstrated less difficulty in modeling the task as they observed it. As stated earlier, Liam used two data entries on his table to determine slope and then determined the constant to write the function rule \( y = 4x + 2 \). Similarly, Alex found an equivalent expression but he did this by examining the image itself. After covering the first two slabs, he saw that around each new flowerbed there would be four slabs (see figure 78) meaning that given the number of flowerbeds, there would be \( 4 \times \text{flowerbeds} + \text{two slabs} \).

![Figure 78: Alex’s description of how the slabs relate to the flowerbeds.](image)

Dupe and Bob did not write symbolic expressions that correctly modeled the task but their expressions or equations did accurately represent their thinking (figure 79). In Dupe’s case, as stated earlier, she deduced a unit ratio for the number of flowerbeds to slabs and determined that this must be the multiplier to arrive at the number of slabs given the number of flowerbeds \((f(x) = 4.5x)\).
Bob saw the flowerbeds as either being on the ends or between the ends (inner flowerbeds). The ones that he considered to be the inner flowerbeds he argued contained six slabs and the ends contain four each leading him to write \((N \times 6) + 8\) to represent slabs given that \(N\) is the number of flowerbeds, he forgot to adjust his \(N\) to account for the first and last flowerbed being represented by the \(+8\). These examples represent symbolic expression as each of the five participants used symbols to represent generalizations of their understanding of the operations symbolically.

**Other habits observed in task six**

**Proportional thinking.** Proportional thinking describes the phenomena of the participants using ratio thinking as they attempt to make sense of the task. A ratio is defined as the comparison of two objects showing the number of times one object is contained inside another. Three participants used their Proportional thinking on this task. As stated before, Dupe determined the unit ratio between slabs and flowerbeds by converting 18 slabs to four flowerbeds to 4.5 slabs per one flowerbed. Similarly, Roger examined multiple ratios of slabs to flowerbeds by increasing the number of slabs by four and the flowerbeds by one. He determined that the ratio looked like it was becoming one but was not sure what this meant. Lastly, Liam also
initially considered whether the relation between flowerbeds and slabs were directly proportional, “the committee wants 100 flowerbeds. For every one there appears to be six slabs, and we know for four flowerbeds there are 18 slabs. Okay so if one is six then we have two 12, three is 24, and four is 30, hmm okay something seems off”. Notice in Liam’s case he attempted to verify the ratio he first observed of one flowerbed to six slabs by counting multiples of six until he got to four flowerbeds. When this result did not match the given information (i.e. that four flowerbeds corresponds with 18 slabs), he recognized that this was not indeed a directly proportional relationship which led him to consider a linear relation.

**Output from Input.** Roger used output from input when he used inputs from the given example to check and see if his algorithm worked, “Hmm let me check this. So for four flowerbeds, I would have two times four for the tops and bottoms, four for the outside, and eight being shared making it 20, but this shows only 18, so something is wrong. Hmm, I’m off by two, so then for 100 flowerbeds then I am off by 50. Right? I am off by two for four and four goes into 100, 25 times, so yeah that makes me off by 50. (Roger corrects his original guess from 304 to 254).

Notice in this example, Roger used his known inputs and outputs to check the accuracy of his algorithm, modifying it to make it work when it does not; in other words, he is using the input to output relationship to verify that his process holds, tweaking his model as needed to achieve the calculation he needs. As a result, translating his explanation, his final function rule becomes \( y = 2x + 4 + 2x - \frac{2x}{4} \), where \( x \) is flowerbeds and \( y \) is slabs. The first \( 2x \) represents his top and bottom for each flowerbed, the four represents the two outer slabs on each end, the next \( 2x \) represents the sharing of two slabs between the flowerbeds, and the \(-\frac{2x}{4}\) represents his adjustment, given that his formula over estimated by two for every four flowerbeds. It should
be noted that had Roger correctly considered the relationship between the number of times sharing to the number of flowerbeds in his original algorithm, that is there is one less share per flower bed, making the relation actually $y = 2x + 4 + 2(x - 1)$, which is equivalent to the $y = 4x + 2$ function rule found by others.

**Brute force.** At the end of the task, Roger frustrated with being unable to correctly write a function relating flowerbeds to slabs and being unable to make sense of the ratios he was exploring stated that he should just extend the relationship of +4 slabs per +1 flowerbed until he reaches 100 flowerbeds, “This pattern doesn’t work either. I struggle with these. I don’t know how to write the formula for the pattern. If I had to, I would run this all the way to 100, by increasing the denominator by one to 100 and the numerator by adding four,” figure 80. Roger ultimately decided not to extend his pattern out.

Figure 80: Roger’s ratios.

**Conclusion for task six**

Magiera et al, (2017) observed the algebraic habits of mind observed were predicting patterns, describing a rule, organizing information, describing change, generalizing beyond examples, and symbolic expression as well as another habit of proportional reasoning in this task. All of these habits were likewise observed in this study. It was surprising though, that organizing information and generalizing beyond examples were not as prevalent in this study as
it was in Magiera et al (2017). The most popular habits observed in this study were predicting patterns, describing change, symbolic expression, and proportional reasoning. A chart showing the observed habits in this task follows (Figure 81).

![Count of Observed habits](image)

**Figure 81:** A chart showing the observed habits in task six.
Chapter 5: Cross-Case Analysis

Introduction

The preceding chapter presented, by case, findings from this research study. This chapter will present a cross-case analysis in order to synthesize knowledge from the individual case studies. As stated in previous chapters, the researcher defines a case as a task and, therefore, this study examined six cases. This cross case analysis will examine themes that emerged from the cases as well as describe similarities and differences between these cases as it relates to the research questions.

Theme one: Relationships among the features of algebraic habits of mind

There is limited research on the association among the features of algebraic habits of mind. However, this researcher did find one study that examined such a concept. Magiera, van den Kieboom, and Moyer (2017) studied pre-service teachers’ algebraic habits of mind. They sought to answer the research question: Which features of the habit of mind Building Rules to Represent Functions appear to support and strengthen one another in our PSTs’ [pre-service teacher’s] written solutions to algebra-based problems? To be clear, Magiera, van den Kieboom, and Moyer (2017) refer to building rules to represent functions, doing and undoing, and abstracting from computation as the algebraic habits of mind and subcategories such as organizing information, predicting patterns, chunking information, etc. as features of those algebraic habits of mind; however, this study refers to these features as the algebraic habits of mind.

The Magiera et al study had eighteen pre-service teachers complete activities that were designed to elicit algebraic habits of mind related to building rules to represent functions. The researchers examined how well the participants used these habits and then looked at how the
habits supported each other. They examined relationships among the different habits within the building rules to represent functions category by conducting Pearson’s correlation tests among all seven algebraic habits. Table 2 below (p. 39), shows the results of their test. Their research shows that there is significant association between organizing information and predicting patterns, organizing information and chunking information, organizing information and describing a rule, predicting patterns and chunking information, predicting patterns and describing a rule, predicting patterns and justifying a rule, chunking information and describing a rule, and chunking information and justifying a rule. Magiera, van ken Kieboom, and Moyer (2017) explain that since these habits are significantly associated with one another, one would not only expect these habits to occur during the same algebraic activity but strengthening one of these habits would likely strengthen the associated habit as well.

The results shown in chapter four of this study found similar clustering of habits. The following chart (figure 82) shows the frequency of combination of related habits. The chart was made by summing all of the pairwise events observed by a participant within each task. For example, if Dupe demonstrated the habits of organizing information [oi], predicting patterns...
[pp], and chunking information [ci] in task one, the pairs oi-pp, oi-ci, and pp-ci would each receive a tally. These tallies were reported on the chart showing the results of all 36 events (six participants completing six tasks each). Therefore, each possible pairwise combination had 36 opportunities to occur. As the chart shows, eight associations were more common than others.

These associations are:

<table>
<thead>
<tr>
<th>Count of Associated Habits</th>
<th>Habits</th>
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<tbody>
<tr>
<td>1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21</td>
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<td>1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21</td>
</tr>
</tbody>
</table>

*The green represents the 95th percentile*

Figure 82: Count of associated habits among the six tasks by the six participants, the upper 95% are colored green.

organizing information and predicting patterns, organizing information and different representations, organizing information and symbolic expression, predicting patterns and symbolic expression, different representations and describing change, different representations and input from output, different representations and symbolic expressions, and describing change and symbolic expressions.

On the diagonal in figure 82, notice that in yellow we have the actual count of occurrence of each observed productive habit. The top 6 most occurant habits being, in order, symbolic expressions, organizing information, predicting patterns, describing change, describing change,
and brute force. These habits represent the habits most often elicited by the participants throughout all the tasks.

These findings are like those mentioned above by Magiera et al (2017). That is, the habits associated with building rules to represent functions demonstrate a positive mutual relationship between each other (e.g. when participants used the habit of organizing information they were also likely to use predicting patterns). However, the chart also shows potential relationships between the building rules to represent function habits (Organizing information, predicting patterns, different representations, describing change) and two habits (symbolic expressions and input from output) in Driscoll’s (1999) other categories. What follows are examples that demonstrate how these specific pairs of habits were used in conjunction to support the participants’ productive mathematical thinking as they navigated the algebraic tasks starting with organizing information and predicting patterns.

**Organizing information and predicting patterns.** Organizing information and predicting patterns were habits that supported each other. This relationship is identified in column 1 and row 1 of figure 82. There were eight occurrences where a participant used both habits in the same task.

Participants who structured their information in a clear organized manner often were able to examine relationships given and make predictions of rules they observe. For example, Bob identified rules that could be used to describe a given table and can also support the creation of another table with specific values is an example of a participant using both organizing information and predicting patterns habits of mind. As Figure 83 shows, Bob used the given table to find patterns within the table. He organized his information by grouping operations together. He made columns showing the result of adding and subtracting the columns and
identified the change between these calculations. Organizing this information allowed him to identify computational patterns between $A$ and $B$, noticing this relationship led him to study the recursive relationships within column $A$ and column $B$. As a result, Bob was able to predict patterns by writing out rules that described these behaviors.

Figure 83: Bob organizing his observations and predicting rules that could transfer between two tables.

**Organizing information and different representations.** Organizing information and different representations were habits that supported each other to allow the participant to make sense of the task. Participants were able to organize the details given in the task in order to gleam key information and then change this representation into others to uncover more information. The number of co-occurrences can be found in figure 82 in column 1 and row 5. They occurred together five times.
Liam, while engaging with the first task, demonstrated the use of organizing information and different representations. Figure 84 shows a table, algebraic functions, and graph. Task one, required the students to determine which numbers cannot be computed through the addition of combinations of five and seven cent stamps. Liam attempted to model combinations of five and seven cents by the functions $y = .7x$ and $y = .5x$ (he did not consider that five cents and seven cents could combine together). Using this information, he created a table of values. The table of values represent his use of the habit of organizing information. His goal in creating the table was to see if a value repeated itself in either of the output columns. Liam claimed that eventually graphs of the two functions would cross resulting in a “limit” in the sense that every $x$ beyond that point of intersection would represent values that five and seven can always combine to make. His use of the table led Liam to graph the function, as the table did not show where the functions crossed. After graphing, Liam found that the functions never crossed. The use of using different representations (function, table, and graph) and organization information allowed Liam to think about the assumptions he was making about the task, namely that eventually the functions $y = .5x$ and $y = .7x$ would cross.

Figure 84: Liam’s use of organizing information and different representations.
Organizing information and symbolic expression. Organizing information and symbolic information often supported each other in the tasks. The number of co-occurrences can be referenced in figure 82 in column 1 and row 14, there were eight. Part of the reason may have been that four out of six tasks specifically requested the participant to determine an algebraic expression that models the observed rule for any ‘n’ case and given that organizing information is a natural step in the problem-solving process (Schoenfeld, 1987), it should be of little surprise that participants would use both habits while completing many of the tasks. Though most of the occurrences of these habits were at distinct periods of time within the tasks (e.g. individuals would create a table of input/output values and then seek to model the behavior symbolically), some participants used the symbolic expression habit while organizing information, such as Nikki.

Nikki created a table during the first task in an attempt to list every possible combination of 5 and 7. She found it difficult to list out all of the multiples of 5 and 7 and so created her own non-standard symbolic expressions to represent these multiples. Using superscript, Nikki represented 5 and 7 to represent a sum of multiple of a number. This superscript notation allowed Nikki to create an organized table, compactly.

Predicting patterns and symbolic expressions. Logically, one can see how predicting patterns and symbolic expression are tightly linked. Predicting patterns requires the participants to notice and predict how a rule works and symbolic expressions is used in algebra as a means to express the predicted rule’s operations in general terms. Figure 82, column 2 and row 14 shows seven co-occurrences of these habits.

Alex for example, noticed that adding a flowerbed resulted in the addition of four additional slabs with an additional two at the beginning (figure 85). As a result of noticing this
rule, Alex wrote the equation $y = 4x + 2$ using $x$ to represent the number of flowerbeds and $y$ equal to the slabs. He then used his equation to check his work by calculating for 1, 2, and 4 slabs before finding the solution for 100 flowerbeds. Each task encouraged the participant to describe the given relationships to a general case. Therefore, it was reasonable to expect the participants to seek to identify the rule at work and then try to give it a symbolic form.

![Image of Alex writing an equation](image.png)

Figure 85: Alex noticing a rule at work as a precursor to writing his symbolic expression

**Different representations and describing change.** Different representations allowed the participants to learn different information as they changed the representation of the problem. Figure 82, column 5, row 6, shows six co-occurrences of these habits.

In some cases, different representations would lead to describing change. For instance, changing the representation of the bear movements into a table showing bear count, jumps, and slides, allowed some participants to determine the change that occurs by increasing the number of bears. However, in other cases, describing change was the necessary precedent. For example, Liam first noticed that each additional flowerbed added four additional slabs. He then decided to create a table to model the count of flowerbeds to slabs using his understanding of change. After
creating his table, Liam chose to graph the relationship to determine the relationship between flowerbeds and slabs. Because of the graph, he discovered the relationship was linear. With that information, he created a function rule to model the relationship and successfully determined the amount of slabs for one hundred flowerbeds.

Different representations and input from output. Figure 82, column 5, row 8 shows 6 co-occurrences of these habits. These habits were paired the most in task two: create a function rule. Participants attempted to use the solutions to derive the original expressions. Once they had one of the one factor, some individuals would graphically show a quadratic function with both zeros in an attempt to determine the other factor.

As an example, Roger started with $x = 4/3$ and working with the knowledge that to arrive at that answer he would have isolated $x$, and described how he would have arrived at that input, “I would have to isolate $x$, so I would have added four and then divided by three giving me that answer.” As a result of that understanding Roger (also using the habit working backwards) arrived at the expression $3x - 4$. He then wrote the expression $(3x - 4)(3x + 4)$ and graphed an upward facing parabola. Then when considering more possible function rules that share the same two zeros, Roger erased part of the left side of the parabola and added a new x-intercept and added $(x + 3)$ to his symbolic representation of the function rule for the zeros (figure 86). In this case, Roger moved between symbolic and graphical representations of the function as he examined the task using input from output to derive the first portion of his function.
Figure 86: Using input from output to derive the function rule and then different representations to extend the method for the general case

**Different representations and symbolic expressions.** Figure 82, column 5 and row 14, shows that this pairing had the highest frequency of co-occurrences. Different representations allowed the participants to learn new information by changing the representation of the task’s information. Often symbolic expression, as mentioned in the previous section, was used to assist the habit of different representations as they translated information from different forms to symbols (i.e. converting a chart into an explicit function or changing an arithmetic pattern to a function rule).

As an example, Dupe used both habits in support of each other during task six. Dupe started with the task by redrawing the four flowerbeds and 18 slabs given to her in the task. She then made two groups, each with two flowerbeds and nine slabs. After making the groups, Dupe attempted to represent this information graphically by creating ratios from this information. She represented the groups using the ratio \( \text{flowerbeds} : \text{slabs} \). By changing the representation to ratios, Dupe determined the unit ratio \( 1 \text{ flowerbed} : 4.5 \text{ slabs} \). Using this information, Dupe wrote the proportion (figure 87). This example demonstrated how Dupe, converting the geometric representation into ratios, to a unit ratio, and subsequently setting up the proportion using symbol expressions, made sense of this particular task.
Describing change and symbolic expressions. Often participants used the algebraic habit of mind of symbolic expressions as an antecedent to describing change; that is, those who were able to correctly determine how a relationship or process was changing, for the most part, were able to use symbols to represent this change. There were six co-occurrences as shown in figure 82, column 6, row 14.

During the patterns at work task, Liam determined three different types of change and represented each one symbolically. In column b for example, Liam determined that the b column is increasing by five and wrote the recursive formula $y = f(x) + 5$, and subsequently determined the relationship between column a and b by writing the function $y = 5x + 3$. In order to show how columns $A$ and $B$ co-vary, Liam not only used the recursive information he found (column $A$ increases by one while column $B$ increases by five) he also connected this information by calculating the rate of change between the columns and represented an explicit function rule symbolically to represent how column $A$ can be changed, or mapped, to column $B$. 

Figure 87: Dupe’s use of different representations and symbolic expression
Summary for Theme One. As this section demonstrated, many of the algebraic habits used by students enrolled in developmental mathematics programs at community colleges occurred in clusters. As a result, one could assume that after the observation of one habit, one could reliably predict the occurrences of other subsequent habits. Additionally, the algebraic habits of mind different representations and symbolic expressions were the most commonly linked habits. They were part of every described connection noted as a significant co-occurrence (outlined in green in figure 82) with the exception of the organizing information and predicting patterns relation. Clearly, the algebraic habits of mind of different representations and symbolic expressions were essential habits that not only aided in sense-making for the participants but also connected with the usage of all other algebraic habits of mind.

Theme two: Preference towards certain habits over others

Across the cases, some habits were observed in greater frequency than others. It was not uncommon to see participants use arithmetic thinking as opposed to algebraic thinking when engaging the tasks. As stated in chapter one, arithmetic thinking is defined as using only numeric constants in computations to make sense of a mathematical situation. An example of an arithmetic thinking habit can be observed during the can task. When the participants were determining the number of cans in a stack where thirty cans were on the bottom row, the overwhelming approach was to add consecutive numbers from one to thirty as opposed to trying to generalize an algebraic expression or computational shortcut for the calculation, approaches that are more algebraic in nature.

The researcher believes that there may be three reasons why the participants relied so heavily on arithmetic habits. (1) The participants are still developing an understanding of algebraic concepts and may not be comfortable with or have fully developed algebraic habits, (2)
the tasks did not contain the type of questions to make a significant barrier to using an arithmetic habit, and/or (3) the provision of the calculator made the use of arithmetic approaches too feasible.

Even though the participants demonstrated a tendency to prefer arithmetic approaches, this is not to say that they did not use algebraic ones as well. The three most popular habits in order of usage were symbolic expressions, organizing information, and predicting patterns. As shown above, all three habits show a strong frequency of occurring together in tasks, this would imply that there was significant interplay between these three habits. Logically, one could see how they would be used together. Possibly, a participant would read the task and organize details they gleaned from the task into a table (organizing information), then they could feasible examine the table, looking to explain any rules they observe in the table (predicting patterns), and finally they could describe the rules they observed using symbols to describe the rule in general terms (Symbolic expressions). Though this explanation does seem to logically explain how the three habits worked together, there is also evidence as to why these three habits would appear in isolation as well.

**Theme three: Observed habits not in Driscoll’s Algebraic Habits of Mind Framework**

When Driscoll published his framework for algebraic habits of mind (1999) he allowed for the possibility of inclusion of additional algebraic habits. They concluded that any new habit must meet four specific criteria: (1) The algebraic habit of mind should reflect mathematically important thinking relevant to students’ solution to a task; (2) The algebraic habit of mind should connect to literature on the learning of algebra and the development of algebraic thinking; (3) Evidence of an algebraic habit should appear across multiple tasks or multiple times and should represent a student’s thinking; and (4) the algebraic habit of mind should be teachable.
When developing algebraic habits of mind, Driscoll used findings from research on advanced middle school and early high school math students (1999). Subsequent research confirmed his findings when examining pre-service teachers and middle school students (Papadopoulos, 2019; Eroğlu & Tanışlı, 2017; Goldenberg et al, 2010). However, this is the first study of its kind, that this researcher can find, that examines the algebraic habits of mind of students enrolled in developmental math courses at a community college. This population is relatively distinct from both advanced middle school students and pre-service teachers and as a result some of the findings reported in this paper show productive thinking habits inconsistent with the literature. As a result, productive habits not outlined in Driscoll’s (1999) framework occurred. Furthermore, some of these acts appear to be habits and moreover, algebraic in nature and thus will be considered according to Driscoll’s (1999) guidelines mentioned above.

What follows is an analysis of the productive thinking habits, observed across the six task, not described by Driscoll’s (1999) framework. First, the researcher will discuss the observation of arithmetic and general habits, and then there will be an examination of potentially uncovered algebraic habits of mind.

**Arithmetic and general habits of mind.**

**Brute force.** Brute force is a productive arithmetic habit where the participant reasons about a task by exhaustively computing. Brute force was the fifth most popular productive habit (out of twenty-one) demonstrated by the participants behind only symbolic expressions, organizing information, predicting patterns, and describing change. Task one (stamp task) and task four (stacking cans task) showed the highest incidence of this habit. Interestingly the participants demonstrated this habit differently on those two tasks. In task one, the participants needed to determine the numbers that combinations of five and seven cannot make. In this case,
the participants used brute force by checking values in a list consecutively. This usage of brute force is subtly different from how it was used in task four. In task four, the participants needed to determine the sum of consecutive integers from one to thirty. The hope was that the participants would be able to determine a function rule that describes the relationship between row number (or number in a row) and the total (Gauss’ equation for sum natural numbers). However, five out of six participants chose to sum each of the integers up consecutively to determine the total. So in the first case, brute force was used by checking every possible solution; while in task four, brute force was used for executing long computations.

*Refinement.* Refinement, appeared only in task five. As the chart below shows, it involves the fluid cycle of thought between examining the original information, making assumptions, planning an approach, implementing the approach, interpreting the results, reevaluating the original information, and making new or adjusted assumptions, and so on. Until the assumptions and plan finally produce the results desired in the context of their understanding of the given information, as outlined in figure 88.

![Figure 88: The refinement model](image-url)
The reason this habit was most likely observed in this task as opposed to the others was because this was the only task that forced the participants to verify the assumptions they were making about the task. This task required the participants to consider a table and determine rules that can be specific enough to create the given table but flexible enough to allow the participant to complete a new table with different given values. Some participants created rules for the table that were too specific to the first table and as a result found frustration when trying to use them to complete the new table. As a result, the participants used the productive habit of refinement as they carefully considered what rule needed to be relaxed or removed in order to complete the task.

Carefully checking one’s understanding and assumptions about a mathematical situation appears to be a general concept that transcends algebra since understanding and assumptions are necessary to understand mathematics regardless of subject specificity. Therefore, this researcher argues for this habit to be considered a general mathematical habit.

**Potential algebraic habits of mind.** When productive acts were observed, the researcher attempted to code. In most cases, the a priori codes were sufficient in coding. However, there were a few instances that a productive act was not describable by Driscoll’s (1999) framework. As a result, the act was initially coded as other with a note in the researcher’s journal describing the act, the participant, and a timestamp. After coding all of the acts observed in the tasks, the ‘other’ acts were reexamined. First the researcher checked to see if the act occurred more than once to determine if the act was habitual. If the act was habitual, the researcher created a code to represent these collections of acts. Lastly, the researcher shared the codes and the examples that derived the codes with their peers for peer review, critique, and validation. What follows below are the potential algebraic habits of mind that came from the data-driven codes.
**Output from input.** Output from input is similar to refinement and the process of guess and check. The main distinction between output from input and the other two habits is the intention behind the act. Refinement and guess and check have the intention of adjusting thinking or guessing systematically until the correct solution is found. Output from input is not done with the purpose of finding a specific solution. The intention with this habit is not the solution but rather to uncover a hidden pattern or relationship by examining a given set of outputs and/or how the change in inputs relate to the change in outputs.

**Mathematically important thinking.** As an example of how output from input can support mathematically important thinking, Alex, while completing task two, sought a function rule that would equate to zero with the inputs $\frac{4}{3}$ and $-\frac{4}{3}$. Alex first experimented with a linear function with a coefficient of three in order to simplify the inputs to whole numbers. When he used both inputs in the expression $(3x)$ he recognized that both calculations had the same absolute value but had different signs. As a result, Alex determined that squaring this result would produce the same output making it possible to create a function rule that would produce his desired effect, namely $y = (3x)^2 - 16$. As was shown, Alex used inputs to find outputs even though outputs were not his end goal. He experimented with outputs by acting on inputs in order to create a function rule which ultimately led him to uncovering a family of functions that would always satisfy the task, namely $y = (3x)^n - 4^n$, where $n$ is even.

**Connected to literature.** Output from input is regarded as an important development stage in learning the procept related to function (DeMarois, 1996). Gray and Tall (1994) regard the concept of procept as an essential component for improving one's ability to think critically about algebraic situations without relying on a collection of disconnected procedures. DeMarois (1996) argues that as students develop the function procept, it would be natural for them to progress
through key stages. Initially one would demonstrate an understanding that the function notation \( f(x) \) relates rules for calculating output from input, then they would develop a process for reversing the calculation rules, then ultimately, the concept could mature in one’s mind as an object, a function named ‘\( f \)’ in this case. Output from input, as described in this paper, demonstrates the participants’ development towards the concept of function procept. The participants are exploiting their understanding of input to output as it relates to the concept of function.

**Reliability.** Output from input occurred five times over three of the six tasks representing the twelfth most popular habit (out of 21 habits). The repeated use of this habit as well as its use across multiple tasks seems to suggest that this habit is reliable and thus representative of student thinking.

**Teachable.** According to common core standards (CCSS.MATH.CONTENT.HSA.CED.A, CCSS.MATH.CONTENT.HSA.REI.B.3), students are expected to be able to write functions using variables. Also, the common core standards (CCSS.MATH.CONTENT.HSF.BF.A.1) expect participants to be able to write expressions and functions that model situations. Given that the standards are learning goals, one can conclude that it is possible for one to learn to use the habit of output from input as it requires the participant to abstract operations from a context, use inputs to generate outputs, and then use the outputs to modify and describe operations symbolically (e.g. create a function).

**Conclusion.** Output from input appears to be a logical bridge between computing related values and determining a function rule. That is, it seems to be the logical leap that is necessary in participants to transition from doing and undoing to abstracting from computation. Given its
computational nature though, this researcher argues that this habit belongs within the category of doing and undoing.

**Finding the complement.** The mathematical complement is the finding of an amount needed to make a value or set whole; therefore, the habit of finding the complement requires the recognition that the set of incorrect answers and correct answers belong to a larger superset. Those using this habit determine one set of answers (whether correct or incorrect groups) in comparison to the superset to find the other set.

**Mathematically important thinking.** In the first task, many participants considered determining values that were impossible to make using five and seven to be too challenging. When checking if a number was impossible, they struggled with determining if they considered all possible combinations of five and seven. As a result, some students focused on listing out combinations of five and seven to evaluate values that were possible. They then went through this list of possible numbers and checked it against the set of natural numbers identifying numbers not on both list as impossible values. This approach proved to be the most consistent approach in finding the task’s solution.

**Connected to literature.** Though it did not appear in Driscoll’s (1999) study, it is listed as a possible approach by the Educational Development Center [EDC](http://mathpractices.edc.org/pdf/Integer_Combinations-Postage_Stamps_Problem(HS_Version).pdf). The EDC refer to this type of habit as providing the participants with an entry point as they examine values the can and cannot be made. In the example they provide, the participants constructed a table that listed values from 0 to 84 and eliminated numbers that five and seven can make. The participants then completed an additional
step to describe a pattern they noticed in the impossible values. No further discussion on this approach is noted by the EDC.

*Reliability.* Three students made sense of task one using this approach. Though this habit did not transcend multiple tasks, it was used by half of the participants. Furthermore, this task was also the only task that required the participant to determine a list of values. It is unclear if this approach would appear again in other similar task but given that fifty percent of the participants naturally thought to approach this problem in this manner, and given the success of the participants that used it, it does appear that the participants would consider using this approach again in a similar setting.

*Teachable.* Complementary thinking is an essential part of problem solving and is used in many mathematical fields (Ito, 1993). Evidence of needing to understand the concept of complement is found in abstract algebra, set theory, computer science, logic, and geometry. As a result of being such a critical component to many mathematical fields, one can find many sources of lessons on the internet that teach this concept and therefore one can reasonably conclude that this habit is teachable.

*Conclusion.* Given the importance of complementary thinking within the field of algebra, this researcher argues that finding the complement be considered an algebraic habit of mind. Since the habit emphasizes ‘ruling out incorrect solutions’ over computation, one can see that this habit does not fit into either doing and undoing or abstracting from computation and therefore it is recommended that finding the complement be considered a habit of mind within the domain of building rules to represent functions.

*Focusing on features of a function.* Focusing on features of a function uses structure and features of functions to describe a relation. For some individuals, a function is more than a
relationship defining a computational rule for a set of inputs to a set of outputs. For those individuals, a function describes graphical features that one would observe if the function rule was graphed.

*Mathematically important thinking.* In task two, participants demonstrated this habit by considering the definition of the feature of zeros. Both Dupe and Roger argued that a function with two zeros must have, at a minimum, a degree of zero because the graph must come into contact with the x-axis at least twice. Using this logic, the participants knew to search for a parabolic function that described such behavior as this was a function that would pass through both points. Roger even used this habit to a greater degree, realizing that a function could also ‘bounce’ at the given zeros given an infinite amount of possible function rules. These participants demonstrated the habit of focusing on features of a function by relating the concept of zeros to x-intercepts and describing the behavior they would expect at the points and using this to reason about function rules.

*Connected to literature.* Moving beyond thinking of functions as a command for ‘action’ (finding an output from an input) and understanding that it is actually a description of a ‘process’ is an essential development in one’s thinking about functions (Godino et al, 2015; Carlson & Oehrtman, 2005; Cooney & Wilson, 1996; Tall & Bakar, 1992; Kleiner, 1989). Godino et al, (2015) describes this development as algebraization levels of mathematics. There are seven levels in this model with level zero showing no understanding of algebraic concepts through level six which characterize advanced understanding and skill mastery of algebraic concepts. Godino et al (2015) describes the habit of focusing on features within levels four and six. At level four, an individual is able to work with identifying families of functions and at level six, an individual can use algebraic structure and features to make sense of a function. These levels,
Godino et al (2015) argues, demonstrate the natural progression that one should encounter the concepts related to functions. Students learning to think about functions at levels four through six should contribute to how they think and habitually act when encountering functions. With this habit being an integral part of the natural development of the concept of function as defined by Godino et al (2015), it should be evident that research supports the existence of this habit.

Reliability. Only two participants demonstrated this habit during one task, so it was not heavily observed during this study. However, this approach of using structure to model polynomial functions based on features given in a graph is a concept covered in their mathematics course. So this researcher believes that this habit may become more common with these participants as they continue to develop their mathematical skills and confidence. Recall, the nature of their developmental course is self-pace, so it is possible that the participants have not begun learning this topic or mastered this particular type of thinking yet.

Teachable. Modeling functions by focusing on features and structure is already a class topic at this particular institution. The software company this course uses to deliver their instruction, including this lesson, is a national company that provides similar services nationally. Therefore, it is reasonable to assume that this habit is already being taught at multiple sites and thus is a teachable concept.

Conclusion. Though this habit did not occur frequently throughout the tasks or by many participants, given that it is an integral part of learning functions and part of the assigned curriculum, this researcher believes that there is enough evidence to support focusing on features of a function as a potential habit of mind. Using graphical features and structure to uncover information about a function appears most related to building rules to represent functions. This category describes the participant searching information to uncover hidden relationships,
patterns, and rules in a given problem. Likewise, focusing on the features of functions is a method that an individual is using features and structure to uncover information that otherwise may not be obvious.

*Proportional reasoning.* Proportional reasoning is the act of comparing ratios to make an inference.

*Mathematically important thinking.* Roger demonstrated this habit during task six. Roger was attempting to determine how the slabs changed in relation to increasing flowerbeds. As a result, he wrote ratios of \( \frac{\text{slabs}}{\text{flowerbeds}} \) for multiple flowerbeds. As he wrote more and more ratios he saw that the ratio was approaching four which was the rate of change between the quantities. Unfortunately, he was unable to make the connection precisely between the two concepts (the ratio as the number of flowerbeds approached infinity and the slope). Nevertheless, this type of reasoning is a common method to describe end behavior of rational functions with equivalent degree in the numerator and denominator and using the divergence test in calculus (Dawkins 2019).

*Connected to literature.* Proportional thinking is an essential skill that naturally leads to the development of advanced algebraic concepts and problem solving strategies (Jitendra et al, 2011). McNeil and Fyfe (2012) describe proportional thinking as a bridge skill that connects the concrete to the abstract as it is often the first opportunity many young mathematicians have at abstracting or projecting. Proportional thinking allows the participant to take two known and directly related quantities to determine how many there would be of one of the quantities given the scale of another. Post, Behr, and Lesh (1988) add that incorporating the variable to the proportion as a representation of the unknown quantity is often the first experience for many students with a variable and the resulting act of cross multiplication allows for many an
opportunity to solve their first algebraic equation. As shown, proportional thinking is a desirable habit is supported in the literature.

Reliability. Proportional thinking was clearly observed twice in one task and potentially observed twice in another. In task six, Dupe and Roger use proportional thinking to try to determine the amount of slabs needed for one hundred flowerbeds. In task four, both Roger and Bob examined relationships through division in hopes of determining the number of cans in a stack. However, neither event was coded as proportional thinking because neither participant explicitly showed symbolic representations of ratios or proportions while they were making sense of the problem. Even though this habit was only observed twice in this study, one can assume, given its relevance in literature and its exposure in the curriculum, that others would also use this habit in similar situations. Case in point, Magiera et al (2017), who also used task six in their qualitative study, also observed students using proportional reasoning to calculate the number of slabs needed.

Teachable. Ratios and proportions is a concept first outlined in the common core standards at grade six (CCSS.MATH.CONTENT.6.RP). Educators teach the concept during grades six and seven introducing the concept of variable as students seek to determine a missing value given a ratio and a scaled value (CCSS.MATH.CONTENT.7.RP.A.2.C). If this productive thinking behavior is included in the common core standards, then one could conclude that this habit is teachable.

Conclusion. Given proportional thinking’s essentialness as the bridge between concrete understanding and advanced algebraic thinking, and its relevant usage in understanding advanced mathematical concepts, this researcher recommends the identification of proportional thinking as an algebraic habit of mind. Since this habit encourages abstracting from concrete and processing
for finding unknown quantities in a proportion (such as cross multiplication and balancing) strongly relates to finding equivalent expressions, it is recommended that this habit be added to abstracting from computation.
Chapter 6: Discussion

The majority of community college students, 59%, enroll in a developmental course prior to taking their first college-level course (Hodara, 2013). Despite the support these students receive in the developmental programs only 22% successfully complete their first college algebra course within two years of first enrolling in college (Complete College America, 2012). Even worse, once students finish their first college algebra class 70% cease their education altogether. Therefore, it should be clear that something more than the current practices is needed.

Algebraic habits of mind, like other mathematical habits of mind, have been shown to successfully support learning of mathematical concepts (Papadopoulos, 2019; Eroğlu & Tanışlı, 2017; Goldenberg et al, 2010) and it is theorized by this researcher that improving one’s understanding of mathematical concepts would translate to improved academic purpose. Therefore, it is argued that improving the development of algebraic habits of mind of students at a community college enrolled in developmental courses, would also improve the success rates mentioned above. Yet the algebraic habits of mind of community college students enrolled in developmental mathematics programs research is virtually nonexistent. The purpose of this study was to add to the literature on algebraic habits of mind by addressing the aforementioned void in research.

This chapter discusses the findings outlined in chapter four and five. First, this chapter will provide explicit answers to this study’s two research questions based on the findings followed by connections of the research answers to the literature. Then there will be a discussion of the implications from the findings as it relates to math educators, curricular designers, and math education researchers; then the limitations of this study will be acknowledged; finally, ending the chapter with a discussion of future research.
RQ1: What algebraic habits of mind are drawn upon by community college students enrolled in a developmental mathematics course when solving algebraic problems?

The six tasks were chosen because they were designed to elicit algebraic habits of mind, based on prior research. Figure 89, serves as a reminder of the six algebraic tasks:

<table>
<thead>
<tr>
<th>Tasks Title</th>
<th>Task Description</th>
<th>Study associated with task</th>
<th>Algebraic habits of mind category targeted by tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Stamp Task</td>
<td>Using 5 cent and 7 cent stamps, what amounts cannot be made.</td>
<td>Driscoll (2003)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>Create a function rule</td>
<td>Write polynomial function rules that have zeros of 4/3 and -4/3</td>
<td>Driscoll (1999)</td>
<td>Input from output</td>
</tr>
<tr>
<td>The carnival of bears task</td>
<td>There are two sets of bears on each side determine the number of slides and jumps needed to allow them to switch.</td>
<td>Driscoll (2003)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>The stacking cans task</td>
<td>There is a stack of cans with each row having one more can than the next, how many cans would make a stack with thirty cans at the bottom?</td>
<td>Steins and Smith (1998)</td>
<td>Abstracting from computation Building rules to represent functions</td>
</tr>
<tr>
<td>The patterns at work task</td>
<td>Identify the patterns that would enable you to recreate the given table if told the initial conditions, use your pattern rules to create a new table with the given entry. Adjust your rules as necessary but make sure they still describe the original table sufficiently</td>
<td>Matsuura et al (2013)</td>
<td>Building rules to represent functions</td>
</tr>
<tr>
<td>The flowerbeds task</td>
<td>Each flowerbed needs to be surrounded by six slabs. The flowerbeds are arranged in an array such that four flowerbeds use eighteen slabs. How many slabs are needed for n flowerbeds?</td>
<td>Magiera et al (2017)</td>
<td>Building rules to represent functions Abstracting from computation</td>
</tr>
</tbody>
</table>

Figure 89: The tasks revisited
In short, all the algebraic habits of mind were observed with the exception equivalent expressions. For this study, six individuals enrolled in a developmental mathematics program at a community college engaged in six tasks designed to elicit their algebraic habits of mind from each of Driscoll’s (1999) broad categories: Building rules to represent functions, Doing and undoing, and Abstracting from computation.

The majority of prior research on algebraic habits of mind examined the Algebraic habit of mind of building rules to represent functions and doing and undoing (Eroğlu, D., & Tanışlı, 2017; Magiera, van den Kieboom, & Moyer, 2017; Magiera et al, 2013; Matsuura et al, 2013; Driscoll, 2003; Van den Kieboom & Magiera, 2012; Driscoll, 2001; Driscoll, 1999). The observation of habits in these studies were consistent with the findings observed in this study.

Magiera et al (2017) was the only study to also examine algebraic habits of mind from the abstracting from computation category. Their findings in this category were also similar to the findings from this study with the exception of equivalent expressions. They found equivalent expressions to be a rather commonly used algebraic habit of mind while this study did not observe the habit throughout all six tasks.

Magiera et al (2017) was also the only study to examine the relationships between habits but only sought to examine associations of habits within the building rules to represent functions category. This research found the same relationships between habits as the Magiera et al (2017) study and additional ones in the other categories they did not study, which was discussed in chapter five. What follows in this section will be further discussion as to the relationships observed between the algebraic habits of mind and then a discussion as to possible explanations as to why there were inconsistent findings in terms of the observation of the algebraic habit of mind of equivalent expressions.
Connections between habits. This researcher found that some habits occurred together frequently, these findings were like the findings by Magiera et al (2017). Organizing information, predicting patterns, and symbolic expressions were three habits that individuals consistently drew upon together. Organizing information is similar to Polya’s (1945) “understand the problem” step when problem solving, in this step, the problem solvers write facts, draw diagrams, and mark in the text of the task itself. Polya (1945) and later Schoenfeld (1985) argue that this behavior of making sense of the question being asked is a natural first step for problem solvers. As a result, it should be of no surprise that this behavior was observed frequently.

Similar to organizing information, predicting patterns is also a behavior described in Polya and Schoenfeld’s work on problem solving. Polya’s “devising a plan” or Schoenfeld’s “analysis and exploration” stages both discuss the participant examining their understanding of the task in an attempt to anticipate the rule being observed and, they argue, is a natural follow up step to organizing information. As a result, it should also be no surprise that predicting patterns was observed.

Lastly, symbolic expressions, a habit from the abstracting from computation category, is the habit of expressing computations with symbols. As each task had a question that wanted the participants to generalize their information to a general case it should be of no surprise that this habit was the most common.

Given that predicting patterns and organizing information are natural steps in the problem solving process, as described by Polya and Schoenfeld and given that these individuals, who are now in college, would have most likely been exposed in some form or another to a formal problem solving process (Stephan, 2014) it was not surprising to this researcher, to see these habits consistently occur together. Moreover, since each task required the participant to
generalize the observed relationship for a general or large ordinal case while completing the problem solving task, it should also be of no surprise that the participants also used the algebraic habit of symbolic expressions in conjunction.

**Equivalent Expressions.** The algebraic habit Equivalent expressions was the only algebraic habit of mind that was not observed in this study. This is contradictory to the findings by Magiera et al (2017) who found this habit had a high frequency of use in task six, the garden task. It is possible that their population of pre-service math teachers and this study’s population of community college students enrolled in developmental mathematics have different ways of making sense of mathematics and thus could approach the tasks differently. Their population consisted of pre-service math teachers within their last two years of college, while this study’s population consisted of students who were enrolled in pre-curricular courses and thus have not even begun their college math curricular journey yet. However, this conclusion seems unlikely given that both groups demonstrated the same habits of mind, with the exception of one habit, and also shown similar co-occurring of specific habits. Therefore, another possible explanation for the lack of observation of equivalent expressions may exist that better explains the differing results. This difference could be the result of a semantical understanding of the term equivalent expressions by the researchers.

Magiera et al (2017) never explicitly define their understanding of the algebraic habit of mind, equivalent expressions. They simply state Driscoll’s (1999) framework and include the definition that Driscoll provided, “Equivalent expression: recognizing equivalence between expressions”. To this researcher, it was not clear precisely what was meant by *expression* or the word *recognize*. As a result, before conducting the study, the researcher turned to other sources for a better understanding of these concepts.
The researcher chose to use Redden’s (2001) definition of *mathematical expression*. They define a mathematical expression as a finite set of symbols that are well-formed defining a rule and depends on context. Similarly, the researcher also chose to use the definition provided by Oxford English Dictionary (1989) to provide a meaning for the word *recognize*. According to the dictionary, *recognize* means to acknowledge the existence, validity, or legality of. Thus, to this researcher, since they do not have direct access to their thinking, the only way an observer could tell if a participant *recognized* an equivalent expression would be if the interlocutor made an explicit (verbal or written) reference between the equivalence of two mathematical expressions. As a result, the researcher only looked for incidences where the participants described two or more mathematical expressions and stated that these two expressions were equivalent as they were productively thinking about the mathematical tasks. No participant in this study was observed expressing an equivalence between two or more explicitly stated mathematical expressions.

Though no participant made such a connection between two expressions explicit, there were multiple occurrences where the participants appeared to use equivalent expressions. For example, some participants were observed writing algebraic expressions that appeared to model their arithmetic work but they never made explicitly stated they considered the two expressions equivalent. It is also unclear in the literature if the act of graphing one’s function or expression counts as a form of expression. For example, Roger, and others, drew curves containing the zeros $\frac{4}{3}$ and $-\frac{4}{3}$ then sought function rules that modeled these curves. Afterwards they provided an explanation that they considered the two descriptions (the function and the graph) to be equivalent. However, going by the definition of equivalent expression as the researcher understood it this act was coded as a ‘different representation’ and not an equivalent expression.
As it can be seen, to remove any ambiguity, the researcher chose to use a strict definition of the algebraic habit of mind of equivalent expression and it is possible that this strict understanding led to the under identification of the habit.

**RQ2: What other productive mathematical habits do community college students enrolled in a developmental mathematics course use while solving algebraic problems?**

Eight productive mathematical habits not described in Driscoll’s (1999) framework were observed: Brute force, Refinement, Finding the complement, Output from input, Focusing on the features of functions, Proportional thinking, persisting, and switching. With the exception of finding the complement and proportional reasoning, other researchers did not find these habits in their studies or at least did not explicitly discuss them. It is possible, that these habits were beyond the scope of their studies as they focused on examining the algebraic habits in terms of teaching individuals to recognize and improve the established algebraic habits of mind.

It should also be noted that the previous studies also only focused on students identified as advanced in mathematics or pre-service teachers. Moreover, pre-service teachers have historically been shown to have been advanced students themselves when they were in k-12 education, by completing school with relatively high grade point averages (US Department of Education, 2016), and so, it can be argued that prior research only examined a population who were classified as advanced students. Given that students in this study were enrolled in a developmental math course, one can arguably classify their mathematical understandings as not consistently advance. Therefore, it is logical to expect one to see different thinking habits, hence the observation of productive thinking habits not described by Driscoll (1999) in this study.

Just because these new habits were not elicited or identified by the group of advanced students does not mean that these are not important mathematical/algebraic habits. In fact, this
researcher argues that all of these habits are important in helping individuals think about mathematics productively. Figure 90 shows the suggested algebraic habits of mind taking in consideration the findings from this study.
### Algebraic Habits of Mind (Strickland 2019), Adapted from Driscoll, 1999

<table>
<thead>
<tr>
<th>Category: Building rules to represent functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Organizing information</td>
</tr>
<tr>
<td>Predicting Patterns</td>
</tr>
<tr>
<td>Chunking the information</td>
</tr>
<tr>
<td>Describing a rule</td>
</tr>
<tr>
<td>Different Representations</td>
</tr>
<tr>
<td>Describing Change</td>
</tr>
<tr>
<td>Justifying a rule</td>
</tr>
<tr>
<td>Finding the complement</td>
</tr>
<tr>
<td>Focusing on features of a function</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category: Doing and undoing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input from output</td>
</tr>
<tr>
<td>Working backwards</td>
</tr>
<tr>
<td>Output from input</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category: Abstracting from computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational shortcuts</td>
</tr>
<tr>
<td>Calculating without computing</td>
</tr>
<tr>
<td>Generalizing beyond examples</td>
</tr>
<tr>
<td>Equivalent expressions</td>
</tr>
<tr>
<td>Symbolic expressions</td>
</tr>
<tr>
<td>Justifying shortcuts</td>
</tr>
<tr>
<td>Proportional reasoning</td>
</tr>
</tbody>
</table>

Figure 90: The adjusted algebraic habits of mind.
What follows is an explanation as to how these habits can potentially help individuals make sense of algebraic concepts.

**Brute Force.** Brute force is a productive arithmetic habit where the participant reasons about a task by exhaustively computing. It was the fifth most observed habit in this study. Brute force was also a habit that worked in concert with finding the complement and output from input. Brute force is also a natural precursor to the guess and check strategy (Schoenfeld, 1987). The main difference being that brute force has a more iterative focus (checking if one works, then two, then three, and so on…) as opposed to the oscillating approach that guess and check provides (first guess was too high, second guess was too low, try a number between guess one and two, and so on…). With guess and check being a productive strategy in thinking about algebraic problems Capraro, Capraro, & Rupley, 2012) and brute force potentially being a precursor for it, one can see that encouraging the use of brute force could logically lead to students developing more advanced thinking.

**Refinement.** Refinement is the fluid cycle of thought between examining the original information, making assumptions, planning an approach, implementing the approach, interpreting the results, reevaluating the original information, and making new or adjusted assumptions, and so on until sense is made of a mathematical situation. This habit clearly represents metacognitive thinking and is not only relevant to algebra. A common error students make is when they incorrectly infer or make assumptions about mathematical situations (Clement, 1981). This habit allows one to step back and examine their assumptions and possibly correct any mistakes they are thinking in how they are thinking about the problem. Without having an understanding of the mathematical concept at hand, it will be difficult for most individuals to effectively think about it.
Finding the complement. Finding the complement essentially combines the brute force habit with ‘solving a simpler problem’ strategy. Essentially, participants determine that it would be simpler to identify what is not a solution and determine those values using brute force to then switch this incorrect solution for the correct one. This habit is one that is used often in algebra. For example, when determining the domain of a rational function it is common for an individual to determine where the discontinuity exists first, by finding where the denominator equals zero, and then describe the domain by removing this value from the set of all real numbers.

Output from input. Output from input is the examination of outputs from inputs in order to find a relationship between the outputs and inputs. For example, one could examine the various slopes of a function at different values of \( x \) along a curve, then they could seek the relationship between each \( x \) input and the slope output to discover the function rule that describes the derivative of the curve at any given \( x \) allowing the participant to develop an intuitive understanding as to the meaning of the derivative beyond the limit definition.

Focusing on features of functions. The habit of focusing on features of functions, encourages individuals to use structure and features of functions to describe a relation. This type of thinking is a natural inverse to using a function to draw a graph. Individuals who can use this habit can recognize the general shape of families of functions by looking at an image, and then use known information about the graph and their understanding of the ‘nature’ of the function that appears to correspond with the graph, they can model the observed behavior using function rules.

Proportional reasoning. Proportional thinking is extremely important algebraic habit. It is used throughout most individuals mathematical learning. For example, understanding that a right circular cone has a constant ratio of \( r/h \) is a fundamental property allows an individual to
solve related rates problems. Another example is the proportional thinking algebra students use when they are searching for the end behavior of rational expressions, namely when they compare the degree of a function’s numerator to its denominator to describe the end behavior.

Proportional thinking allows an individual to search for relationships between quantities and units by examining the values of known to unknown quantities providing the learner opportunities to simplify computation. For example, an individual using the proportional thinking habit can effectively recognize and flexibly solve the following proportions:

\[
\frac{3 \text{ books}}{5 \text{ people}} = \frac{9 \text{ books}}{x \text{ people}}, \quad \frac{3 \text{ books}}{12 \text{ people}} = \frac{7 \text{ books}}{x \text{ people}}, \quad \frac{4 \text{ books}}{x \text{ people}}
\]

In the first case, the proportional thinker will recognize that horizontally there is a relationship, one can see a factor of three. The second case, shows a vertical relationship with a factor of four, and in the last case, we can use the cross product identity to determine the missing values.

**Persisting.** The mathematical habit of mind of persisting was also observed in this study. On average, the participants spent approximately two hours completing the six tasks and demonstrated significant patience while making sense of the algebraic concepts. The amount of time these participants dedicated to each task contradicts the habit of settling on the quick answer as described by Oesterle et al (2016).

**Switching.** While the participants demonstrated significant persistence in working on a task, it was observed that they did not persist with a particular habit within a task for long periods of time. Throughout the study, the participants demonstrated the habit of changing habits to make sense of the tasks. For example, a participant would start a task by creating a table in order to seek out a pattern, if the pattern was not clear from the table, the participant would switch habits from organizing information to different representation, to see if changing how the information is presented provides further insight. It was not uncommon in other studies
to observe participants switching between habits (Magiera et al, 2017) however the frequency that the participants in this study switched between habits was a lot more than those reported in other studies (Magiera et al, 2017) and the amount of time invested on using a habit much less (Matsuura et al, 2013).

**Implications**

The results of this study show that college students enrolled in developmental mathematics education courses have the ability to use algebraic habits of mind to solve difficult and unfamiliar problems. The results also show that they drew upon a set of habits not currently identified in Driscoll’s Algebraic Habits of Mind Framework (1999). Lastly, this research shows that the participants, though they possess and use algebraic habits of mind tend to switch to arithmetic techniques when they do not show immediate success with the algebraic habits. These results have implications that reach stakeholders of mathematics education across several levels including mathematics instructors, curriculum designers, and researchers.

**Mathematics instructors.** Even though the students would, at times, abandon an algebraic approach to the tasks, they did not give up on the problem. On average the students persisted on each tasks for twenty minutes. This contrasts with Oesterle’s (2016) claim, from their anecdotal observations, that students struggling to learn mathematical concepts struggle to persist. This seems to imply that there may need to be a distinction made between an individual ‘who struggles at learning concepts’ and an individual who is enrolled in developmental mathematics courses.

Though the students spent a lot of time on each task, much of the time was observed spent on exploring, testing out ideas using different algebraic habits, rather than planning, analyzing, implementing, and verifying. This is evident of what Schoenfeld (1992) refers to as a
metacognitive divide. According to Schoenfeld (1992), people who struggle with problem solving have less metacognitive control and get stuck in stages like exploring and as a result will test out many ideas until they find something that works which can be a highly inefficient and frustrating process. Mathematics instructors as a result need to encourage students to effectively use the algebraic habits of mind. In order to achieve this, the researcher encourages instructors to explicitly model the algebraic habits of mind, have students practice identifying the various habits of minds at use by the instructor, as research has shown this to be a highly effective method of learning to use the habits (Magiera, van den Kieboom, and Moyer, 2017), provide activities that encourage the thinking necessary to elicit a variety of algebraic habits, and allow students to productively struggle.

**Modeling.** The modeling by the instructor should demonstrate to the learner the algebraic habit of mind. To effectively model this, the instructor needs to talk aloud their thoughts, demonstrate how they effectively problem solve and think about problems. Driscoll (2003) argues that teacher can monologue aloud a few examples and later dialogue through directed questioning with students while problem solving to help develop algebraic habits of mind effectively.

**Productive struggle.** Productive struggling is important for anyone’s development in learning mathematics. Individual's need time to think about, try to use, and understand the material they are learning. If one only is told when to use specific algebraic habits of mind, then such practice becomes strategy or trained approach losing the natural feel of exploration and pattern sniffing that we want to instill in our learners. Students benefit greatly from struggling productively (Warshaur, 2015), let them. Having time to make sense of a problem on their own will encourage them to become independent thinkers and consumers of information.
Curriculum designers. The standards for mathematical practices (CCSM) are linked to the mathematical habits of mind including the algebraic habits of mind (Blanton & Kaput, 2005; Hirsch, Lappan, & Reys, 2012; Matsuura et al, 2013; Suh & Seshaiyer, 2014). With the identification of additional habits of mind, this researcher encourages the inclusion of these new habits within the mathematical practices to continue encouraging the development of these productive behaviors. Additionally, it is evident that the participants within this study inconsistently persisted with choosing and finishing with an algebraic habit of mind, many of the participants would abandon approaches even if it appeared to the researcher that the participant was making significant progress. Frequent abandoning of algebraic habits, this researcher believes, made it hard for the participants to complete the tasks in the study.

One potential reason for abandoning a productive thinking behavior is a lack of familiarity with open-ended tasks. The modules, the participants completed for the developmental mathematics program emphasized skill and concept development but did not offer many open-ended tasks. Having a lack of experience with problems that require the participants to experience the problem solving process greatly hinders one’s thinking (Pelfrey, 2000); therefore, curriculum designers should ensure the inclusion of problems that not only align to the mathematical practices but also allow the participant multiple avenues of entry.

Researchers. For researchers, the results of this research indicate the potentially new algebraic habits of mind within the three categories as defined by Driscoll (1999). As shown in figure 90, this study suggests two new habits for building rules to represent functions, and one additional habit for doing and undoing and one for abstracting from computation.

Adding these habits, as new algebraic habits of mind, gives researchers additional behaviors to study and assists the framework in describing productive algebraic thinking.
behavior for all students. The additional habits of mind, being newly discovered in this research, not only need verification in future studies, but also further exploration as to the pervasiveness of these behaviors. The other algebraic habits that were already established in Driscoll’s framework have been established in the literature as recurring habits in varied situations (Van den Kieboom & Magiera, 2012; Magiera et al, 2017; Matsuura et al, 2017); however, the habits observed in this study, though seemingly consistent with the literature, were noticed in a limited capacity within the tasks. That is, these observed behaviors did not repeatedly occur in many multiple cases. Therefore, the argument could be made, that some of these habits, like finding the complement which only occurred in the first task, may not be truly an algebraic habit if it does occur in multiple algebraic situations. However, this inconsistency in observation could be more due to a limitation of task selection which will be discussed in the next section.

Limitations

Retrospectively, this researcher identified six limitations of this study that could potentially influence the results and therefore conclusions drawn from this research, which are: selection of tasks, access to student thinking, interaction between interviewer and interviewee, timeliness of the study, broader use of the framework, and limited repetitive tasks.

Selection of tasks. The researcher intentionally chose tasks used in prior research to elicit specific algebraic habits of mind. The vast majority of the studies that used the tasks chosen for this study desired to study the productive habits associated with building rules to represent functions and therefore chose tasks that would elicit these behaviors. By using tasks that would specifically target these algebraic habits of mind, this researcher is potentially encouraging the elicitation of habits within this category over others that would occur instead if different tasks
were chosen. Therefore, one should cautiously conclude the absence of a habit as an indication that such a habit is not used by the population sampled.

**Access to participant thinking.** In a task-based interview, the researcher observes and engages with the participant in an attempt to understand the thinking of that individual. There are three significant problems with this approach (Assad, 2015), one it assumes that the observer/researcher understands what it is they are observing, two it assumes that the researcher possesses the capacity to describe this observation accurately and adequately, and it assumes that observing a participant’s actions and descriptions are enough to adequately understand their thinking. In order to mitigate this problem, the researcher used the constant comparative method in addition to peer review of findings.

**Interaction between interviewer and interviewee.** The analysis that was used in this study did not account for the interventions and contributions made by the interviewer. At times the researcher would deviate from a task’s script to ask a participant to explain or expound on an action or statement. The potential cuing by the researcher, ‘that something may be worth considering’ could have potentially influenced the focus and thinking of some of the participants that may not have been observed had the researcher stayed on script.

**Timeliness of the study.** Each of the participants chosen for this study were beginning the last of eight modules of developmental mathematics curricula. Completing the first seven modules, the researcher believed, provided the participants enough of a foundational understanding in both skills and vocabulary in order to be able to engage with the tasks. However, this researcher did not study participants with only completed other amounts of modules in order to verify this claim. This study, by including participants not normally studied with the algebraic habits of mind framework, has demonstrated the potential inclusion of new
habits; however, by not including more diverse groups of developmentally-enrolled students the exploration of habits that would fully represent the population of community college developmental mathematics students would be more encompassing.

**Broader use of the framework.** These findings were inconsistent with the established literature in that it included many observations of algebraic habits of mind from the doing and undoing and abstracting from computation categories. However, the established literature that used these tasks had a much narrower focus. They typically were examining the elicitation of algebraic habits of mind associated with building rules to represent functions category and evaluated the quality of use of those habits. This study had a much different focus. This researcher examined and described the occurrence of any mathematically productive habit observed. The broader focus could account for the observations of differing habits of mind within the same task. Nonetheless, having a broader focus also means the researcher had to manage and organize a larger code list which means there is more opportunity to incorrectly code a behavior.

**Limited repetitive tasks.** One of Driscoll’s (1999) requirements for determining if an observed act is an algebraic habit of mind. He argues that for the act to be an algebraic habit of mind it needed to be observable across multiple tasks and repeatedly within a task. However, this study only used six tasks that were specifically chosen because they covered distinct content. So, in some cases, the participants did not have the opportunity to demonstrate the same habits across all of the tasks; therefore, this researcher chose to make the assumption that an observed habit would manifest in other situations if a similar task was presented to the participants but it is unclear if this is the case.
Future Research

In reflecting on this study, the researcher recommends four areas for future research as it relates and/or builds off this study:

- What context clues lead developmental algebra students to choose specific algebraic habits of mind over others?
- What causes students enrolled in developmental algebra to decide to switch between habits?
- Why are college students enrolled in developmental algebra relying more on arithmetic than algebraic approaches?
- How well will a college student enrolled in developmental math perform at recognizing and using the algebraic habit of mind of equivalent expressions in a task designed to elicit such a response?

What context clues lead students enrolled developmental algebra courses to choose specific algebraic habits of mind over others? The six participants completing the six tasks chose to use a wide-variety of algebraic habits. At times, the participants would remark that it was obvious to them which approach they should take to work towards the solution; however, in most cases, the participants were unable to adequately describe what exactly about the task made their approach so obvious.

The researcher suspects that context clues within the tasks, as perceived by the participants, inspired the habit of mind that was used by the participants and if this habit was one that ultimately led to unproductive results, the participants would experience more frustration than those who chose a more productive habit initially. Given that frustration can affect persistence (Schoenfeld, 1999), further understanding into what exactly, students enrolled in developmental
algebra programs, perceive as context clues as they relate to choosing algebraic habits should be explored.

**What causes students enrolled in developmental algebra to decide to switch between habits?** At times, it appeared to the researcher that the participant was making significant progress towards a solution using a particular habit, for only the participant to suddenly stop their approach, and drastically change habits to begin the analysis and exploration anew. It was not clear to the researcher exactly why the participants would abandon one approach to begin a different one but at times, such actions not only halted the progress the participants were making but in some cases it sabotaged their work altogether.

**Why are college students enrolled in developmental algebra relying more on arithmetic than algebraic approaches?** This study has shown that developmental students would utilize arithmetic approaches as opposed to algebraic habits of mind at times. Gray & Tall (1994) refer to the using one approach over another as the proceptual divide and is, they argue, an important divide that separates those who do well at mathematics and those who “make mathematics intolerably hard”. There are several factors that could have led the participants to use arithmetic approaches over algebraic, the participants had a calculator, the values that they had to calculate may not have been complex enough to necessitate the need for algebra, and/or the students may not have a strong understanding of proceptual thinking yet.

This researcher theorizes that practicing the algebraic habits of mind is how one will become more adept at using them and by building algebraic habits of mind, one will also build their proceptual understanding as they can make more connections between mathematical ideas. Therefore, understanding the reason that students learning developmental algebra use arithmetic
approaches over using algebraic habits of mind can assist those desiring to promote and study the use of the algebraic habits of mind.

How well will a college student enrolled in developmental math perform at recognizing and using the algebraic habit of mind of equivalent expressions in a task designed to elicit such a response? The only algebraic habit of mind not observed in this study is the algebraic habit of mind of equivalent expressions. This habit belongs to the category of abstracting from computation. The habit requires the participant to make the declaration that two expressions are equivalent (e.g. \(4 + 4 + 4 = 4 \times 3\)). The tasks chosen, though equivalent expressions were an observed habit in other studies, with these tasks they were not explicitly designed to elicit such thinking. Therefore, it is unclear as to whether students enrolled in developmental math courses understand and use this habit. In order to assess this, this researcher recommends that a task that explicitly requires the use of the algebraic habit of mind of equivalent expressions be developed and used to examine this group’s perception of the habit.

Conclusion

As this study shows, community college students enrolled in developmental mathematical programs use algebraic habits of mind. Moreover, through the study of this population, a population not normally examined using the algebraic habits of mind framework, the researcher has uncovered potentially new algebraic habits of mind not previously identified which could have implications throughout the field of algebraic habits of mind. Through better understanding of the algebraic habits of mind of community college students enrolled in developmental mathematics one can better understand how to address these habits both pedagogically and through curricular development.
As stated in chapter one, improving how one uses their algebraic habits of mind links to success in algebra classes and given the low success rate of community college students enrolled in developmental mathematics programs in persisting beyond their first math course, it is imperative that understanding of their algebraic habits and how to improve these habits continues to build.
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APPENDICES
Appendix A

Interview protocol for stamp task.

Below is the protocol for the semi-structured interview for the stamp task.

• Prepare the table, have a box across from the student with materials, including markers, paper, rulers, calculator, scissors.

• Ensure all audio and video equipment is ready.

• Say: You are about to begin your first tasks. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.

• Are you ready to begin?

• This task is called the Stamp task. Would you like to read this task yourself, or should I read it to you?

• Allow for the reading of the task.

• Allow for a few minutes for the interviewee to understand the task at hand.

• Say: What is the task asking you to do?

• Allow for response, follow up as needed.

• Say: How are you going to solve this task?

• Allow for response, follow up as needed.
• Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.
• If they feel they have a solution,
• Say: why do you think that is a solution?
• Say: how do you know this is the best solution?
• Say: how have you proven your answer? or Say: can you prove your answer is correct?
• When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.
Appendix B

The interview protocol for create a function rule.

Below is the protocol used for this task.

- Prepare the table, have a box across from the student with materials, including markers, paper, rulers, calculator, scissors.
- Ensure all audio and video equipment is ready.
- Say: You are about to begin your second task. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.
- Are you ready to begin?
- This task is called the create a function rule. Would you like to read this task yourself, or should I read it to you?
- Allow for the reading of the task.
- Allow for a few minutes for the interviewee to understand the task at hand.
- Say: What is the task asking you to do?
- Allow for response, follow up as needed.
- Say: How are you going to solve this task?
- Allow for response, follow up as needed.
• Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.
• If they feel they have a solution,
• Say: why do you think that is a solution?
• Say: how do you know this is the best solution?
• Say: how have you proven your answer? or Say: can you prove your answer is correct?
• When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.
Appendix C

Interview protocol for the carnival bears task.

- Below is the protocol used for this task.
- Prepare the table, refill any materials used in the previous task.
- Ensure all audio and video equipment is ready.
- Say: You are about to begin your third task. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.
- Are you ready to begin?
- This task is called the carnival bears task. Would you like to read this task yourself, or should I read it to you?
- Allow for the reading of the task.
- Allow for a few minutes for the interviewee to understand the task at hand.
- Say: What is the task asking you to do?
- Allow for response, follow up as needed.
- Say: Are any of the questions related?
- Allow for response, follow up as needed.
- Say: What different activities will you have to do answer these questions?
- Allow for response, follow up as needed.
• Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.

• If they feel they have a solution,

• Say: why do you think that is a solution?

• Say: how have you proven your answer? or Say: can you prove your answer is correct?

• When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.
Appendix D

The task protocol for stacking cans.

- Below is the protocol used for this task.
- Prepare the table, refill any materials used in the previous task
- Ensure all audio and video equipment is ready.
- Say: You are about to begin your third task. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.
- Are you ready to begin?
- This task is called the stacking cans. Would you like to read this task yourself, or should I read it to you?
- Allow for the reading of the task.
- Allow for a few minutes for the interviewee to understand the task at hand.
- Say: What is the task asking you to do?
- Allow for response, follow up as needed.
- Say: Are any of the questions related?
- Allow for response, follow up as needed.
- Say: What different activities will you have to do answer these questions?
- Allow for response, follow up as needed.
- Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.
- If they feel they have a solution,
- Say: why do you think that is a solution?
- Say: how have you proven your answer? or Say: can you prove your answer is correct?
- When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.
Appendix E

Interview protocol for describe the pattern of the table.

- Below is the protocol used for this task.
- Prepare the table, refill any materials used in the previous task
- Ensure all audio and video equipment is ready.
- Say: You are about to begin your third task. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.
- Are you ready to begin?
- This task is called describe the pattern of the table. Would you like to read this task yourself, or should I read it to you?
- Allow for the reading of the task.
- Allow for a few minutes for the interviewee to understand the task at hand.
- Say: What is the task asking you to do?
- Allow for response, follow up as needed.
- Say: Are any of the questions related?
- Allow for response, follow up as needed.
- Say: What different activities will you have to do answer these questions?
- Allow for response, follow up as needed.
• Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.

• If they feel they have a solution,

• Say: why do you think that is a solution?

• Say: how have you proven your answer? or Say: can you prove your answer is correct?

• When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.
Appendix F

The flowerbeds task protocol.

- Below is the protocol used for this task.
- Prepare the table, refill any materials used in the previous task
- Ensure all audio and video equipment is ready.
- Say: You are about to begin your third task. As you complete this task, I am going to encourage you to speak your thoughts out loud. My goal in this study is to understand exactly what you are thinking as you are completing this task. I also may ask you questions about your thought process and ask you to elaborate on things that you express or write to ensure I understand what you mean. Nothing I question or write down should be taken in consideration of your correctness while working on the task. If at any time you want to stop working on this task, you merely must inform me.
- Are you ready to begin?
- This task is called the Flowerbeds Task. Would you like to read this task yourself, or should I read it to you?
- Allow for the reading of the task.
- Allow for a few minutes for the interviewee to understand the task at hand.
- Say: What is the task asking you to do?
- Allow for response, follow up as needed.
- Say: Are any of the questions related?
- Allow for response, follow up as needed.
- Say: What different activities will you have to do answer these questions?
- Allow for response, follow up as needed.
• Allow the student to proceed through the task asking them to explain any drawings and encourage them to consistently speak their thoughts.
• If they feel they have a solution,
• Say: why do you think that is a solution?
• Say: how have you proven your answer? or Say: can you prove your answer is correct?
• When students decide that they have answered the question or can no longer continue, end the task by removing the scenario, any created documents, and marking the time at the end of this task for later reference.