ABSTRACT

WATKINS, SETH QUENTIN. Realization of $L(2\Lambda_0)$ for the Lie Algebra $A_{2n-1}^{(2)}$. (Under the direction of Kailash C. Misra.)

In affine Lie algebra representation theory, one of the primary objectives is to give an explicit construction of integrable highest weight modules. It is known that for each dominant weight there exists an integrable highest weight module. The principally specialized character of an integrable highest weight module has a product form. When the product of the principally specialized character is equal to the generating function of a specific class of partitions, then those partitions can be used to construct a minimal spanning set for the module. In order to do this, one must use Z-operators. Lepowsky and Wilson developed a theory of Z-operators to realize affine Lie algebras and their integrable highest weight modules. Harger used Z-operators to construct the integrable highest weight module $L(2\Lambda_0)$ for the affine Lie algebra $A_7^{(2)}$. In my dissertation, I will construct the integrable highest weight module $L(2\Lambda_0)$ for the general case $A_{2n-1}^{(2)}$. The module $L(2\Lambda_0)$ can be realized as a submodule of $L(\Lambda_0) \otimes L(\Lambda_0)$. Then the Z-operators for $L(2\Lambda_0)$ can be built from the Z-operators for $L(\Lambda_0) \otimes L(\Lambda_0)$. Using the generalized Euler identity, I have shown that the principally specialized character of $L(2\Lambda_0)$ for $A_{2n-1}^{(2)}$ is the generating function for partitions where each part is even and each part occurs at most n times. Therefore a basis for $L(2\Lambda_0)$ will have basis elements which correspond to those partitions.

In addition to giving a realization of $L(2\Lambda_0)$ for $A_{2n-1}^{(2)}$, I have also computed the principally specialized characters for the level 3 modules of $A_{2n-1}^{(2)}$, $n = 3, 4, 5, 6, 7$. In the last chapter I provide a conjecture which gives a formula for the principally specialized characters for the level 3 modules for the general case of $A_{2n-1}^{(2)}$.

Realization of $L(2\Lambda_0)$ for the Lie Algebra $A_{2n-1}^{(2)}$

by
Seth Quentin Watkins

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APPROVED BY:

Ernest Stitzinger

Naihuan Jing

Arnab Maity

Kailash C. Misra
Chair of Advisory Committee

DEDICATION

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BIOGRAPHY

The author was born in Raleigh, NC. He got his B.S. in Mathematics from Harding University.

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TABLE OF CONTENTS

Chapter 1 INTRODUCTION **1**

Chapter 2 Preliminaries **3**
- 2.1 Lie Algebra and the Generalized Cartan Matrix 3
- 2.2 The Root System and Root Space Decomposition 5
- 2.3 Affine Lie Algebras and the Lie Algebra of Type $A_{2n-1}^{(2)}$ 6
- 2.4 Lie Algebra Representations and Modules 7
- 2.5 The Principally Specialized Character 8

Chapter 3 The Principal Picture **12**
- 3.1 The Automorphism ν 12
- 3.2 The Principal Heisenberg Subalgebra and the Principal Realization of $A_{2n-1}^{(2)}$ 13
- 3.3 $L(2\Lambda_0)$, the Vacuum Space, and Z-operators 15
- 3.4 Results for $A_{2n-1}^{(2)}$ 16

Chapter 4 Realization of $L(2\Lambda_0)$ **21**
- 4.1 Orbit Representatives and The Generalized Commutator 21
- 4.2 Reducing The Spanning Set 23

Chapter 5 Principally Specialized Character For Level 3 Weights For $A_{2n-1}^{(2)}$ **31**

CHAPTER

1

INTRODUCTION

Lie algebras were discovered in the 19th century by the Norwegian mathematician Sophus Lie. Lie studied what he called infinitesimal transformations or transformation groups. The term Lie algebra did not come about until the 1930's.

A Lie algebra is a vector space V over a field F equipped with a bilinear map $[,]: V \times V \to V$ called the Lie bracket. The Lie bracket satisfies the Jacobi identity, and the bracket of any element with itself is 0. As a result the bracket is also anticommutative. In my work I will always take the field F to be the complex numbers $\mathbb{C}$.

Let $\mathfrak{g}$ be a Lie algebra. A subspace I of $\mathfrak{g}$ is an ideal if $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$. The derived series [3] is a series of ideals of $\mathfrak{g}$ given by $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots$. We say $\mathfrak{g}$ is solvable [3] if $\mathfrak{g}^{(n)} = 0$ for some n. From [3] we know $\mathfrak{g}$ has a unique maximal solvable ideal called $rad(\mathfrak{g})$. If $\mathfrak{g} \neq 0$ and $rad(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is semisimple [3]. The finite dimensional semisimple Lie algebras are classified by their Cartan matrices, listed in [3].

In 1968, Victor Kac and Robert moody defined a class of Lie algebras generalizing finite dimensional semisimple Lie algebras which are now known as Kac-Moody Lie algebras. these algebras are associated with an $n \times n$ matrix $A = (a_{ij})_{i,j=1}^{n}$ with $a_{ii} = 2$, $a_{ij} \leq 0$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ for $i, j = 1, \ldots, n$ and $i \neq j$ called a generalized Cartan matrix (GCM). A GCM is symmetrizable if it can be written as the product of two matrices where one is an invertible diagonal matrix and the other is a symmetric matrix [4]. A GCM is decomposable if it decomposes into a nontrivial direct sum after applying the same permutation to the rows and columns [4]. We say a GCM is indecomposable if it is not decomposable. A GCM is affine if the corank is 1 and the null space is spanned by a vector with positive entries [4]. The affine GCMs are listed in [4]. Given a symmetrizable, indecomposable, GCM A one can construct the Kac-Moody algebra $\mathfrak{g}(A)$ associated with that GCM [4]. If the GCM is affine, then we say $\mathfrak{g}(A)$ is affine.

A representation of a Lie algebra $\mathfrak{g}$ is a linear map $\pi : \mathfrak{g} \to gl(V)$, where $gl(V)$ is the set of linear operators on some vector space V, such that $\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$. In this case V is called a $\mathfrak{g}$-module. In Lie algebra representation theory one of the main objectives is to give an explicit description of a module or family of modules. From [1] and [7] we get the principal realization which provides a way to construct a Lie algrebra so as to provide a basis for a given integrable highest weight module. A Heisenberg Lie algebra of order n has a basis $e_i,\ f_i\ (i = 1, \ldots, n),\ z$, such that $[e_i, f_j] = \delta_{ij} z\ (i, j = 1, \ldots, n)$, and all other brackets are zero [4]. An affine Lie algebra in the prinipal picture contains an infinite dimensional Hiesenberg subalgebra with a triangular decomposition. The subspace containing the highest weight vectors of the Heisenberg subalgebra is called the vacuum space. In [7] and [6] the authors describe the vacuum space of a given module. Now the problem becomes describing the vacuum space instead of the entire module. In this thesis, I will provide an explicit construction for the vaccum space of the integrable highest weight module $L(2\Lambda_0)$. The vacuum space is denoted $\Omega(L(2\Lambda_0))$

The principally specialized character is the prinipal specialization of the Weyl character formula, which has an infinite product form. The principally specialized character described in [5], along with the grading of the vacuum space described in [7] and [6], provides information about the basis elements of the vacuum space. In the case that the principally specialized character can be written as the generating function for a specific type of partitions, then one can construct basis elements of the vacuum space that meet the partition criteria. The principally specialized character of the vacuum space is denoted $\chi(\Omega(L(2\Lambda_0)))$ Using the principally specialized character and the generalized Euler identity [8], I have proved the following theorem.

Theorem 1.

$$\chi(\Omega(L(2\Lambda_0))) = \prod_{j \not\equiv 0 \, mod(n+1)} (1 - q^{2j})^{-1} = \sum_{\ell \geq 0} c_\ell q^\ell$$

where $c_\ell =$ the number of partitions of ℓ with even parts such that each part appears at most n times.

In order to construct a basis for $\Omega(L(2\Lambda_0))$ one can use $Z-$operators, defined in [7] to construct a spanning set. Then the relations given by the generalized commutator, also defined in [7], allow one to reduce the original spanning set to a minimal spanning set. In [2], Harger constructed bases for $\Omega(L(2\Lambda_0))$, $\Omega(L(\Lambda_2))$, and $\Omega(L(\Lambda_4))$ in the case of $A_7^{(2)}$. My work is a generalization of Harger's result for $\Omega(L(2\Lambda_0))$ extended to the case of $A_{2n-1}^{(2)}$. In the last chapter I will also provide a list of the principally specialized characters for level 3 modules of $A_{2n-1}^{(2)}$, $3 \leq n \leq 7$, and a conjecture for a general formula for these principally specialized characters.

CHAPTER

2

PRELIMINARIES

2.1 Lie Algebra and the Generalized Cartan Matrix

Definition 2.1.1. *A Lie Algebra is a vector space* $\mathfrak{g}$ *over a field* F*, with an operation* $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$*, called the Lie bracket or commutator, which satisfies the following conditions:*

1. *The bracket is bilinear.*
2. $[x,x]=0$ *for all* $x\in\mathfrak{g}$
3. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ *for all* $x,y,z\in\mathfrak{g}$

I will always take F to be the field of complex numbers, denoted $\mathbb{C}$. Also observe that criteria 1. and 2. above, when applied to $[x+y,x+y]$, imply that $[x,y]=-[y,x]$ for all $x,y\in\mathfrak{g}$.

Definition 2.1.2. *A Lie algebra* $\mathfrak{g}$ *is abelian if* $[x,x]=0$ *for all* $x\in\mathfrak{g}$.

Definition 2.1.3. *Let* $\mathfrak{g}$ *be a Lie algebra. A subspace* I *of* $\mathfrak{g}$ *is an ideal if* $[x,y]\in I$ *for all* $x\in\mathfrak{g}$ *and* $y\in I$.

Let $\mathfrak{g}$ be a Lie algebra, then there is a sequence of ideals of $\mathfrak{g}$ defined by $\mathfrak{g}^{(0)}=\mathfrak{g}$, $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}^{(2)}=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}],\ldots$. This sequence is called the derived series [3]. We say $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)}=0$ for some n [3]. From [3] we know $\mathfrak{g}$ contains a maximal solvable ideal, call it rad($\mathfrak{g}$).

Definition 2.1.4. *A Lie algebra* $\mathfrak{g}$ *is semisimple if rad*$(\mathfrak{g})=0$ *[3].*

Definition 2.1.5. *The matrix* $A=(a_{i,j})_{i,j=1}^{n}$ *is called a generalized Cartan matrix (GCM) if it satisfies the following conditions [4]:*

1. $a_{ii} = 2$

2. a_{ij} *are nonpositive integers for* $i \neq j$

3. $a_{ij} = 0$ *implies* $a_{ji} = 0$

Definition 2.1.6. *A GCM* $A = (a_{ij})_{n\times n}$ *is symmetrizable if there exists an invertible diagonal matrix* D *and a symmetric matrix* B *such that* $A = DB$ *[4].*

Definition 2.1.7. *A GCM* $A = (a_{ij})_{n\times n}$ *is called decomposable if, after applying the same permutation to the rows and columns, A decomposes into a nontrivial direct sum. A is indecomposable if it is not decomposable [4].*

Consider the column vector

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

We say $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, n$.

Definition 2.1.8. *Let* $A = (a_{ij})_{n\times n}$ *be an indecomposible GCM. Then A is of Affine type if corankA* $= 1$*, or if there exists a vector* $u > 0$ *such that* $Au = 0$ *[4].*

In 1968, V. Kac and R. Moody independently discovered the Lie algebra associated with a given GCM. From [4], we know that each symmetrizable indecomposable Cartan matrix has a corresponding Lie algebra called the Kac-Moody Lie algebra.

Definition 2.1.9. *The Cartan Datum of a GCM A is a triple* $(\mathfrak{h}, \Pi, \Pi^\vee)$*, where* $\mathfrak{h}$ *is a complex vector space,* $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ *and* $\Pi^\vee = \{h_1, \ldots, h_n\} \subset \mathfrak{h}$*, which satisfies the following criteria [4]:*

1. Π *and* $\Pi^\vee$ *are linearly independent.*

2. $\alpha_i(h_j) = a_{ji}, (i, j = 1, \ldots, n)$

3. $n - \ell = dim\mathfrak{h} - n$*, where* ℓ *is the rank of* A

Let $\mathfrak{g}$ be a Lie algebra. For each $x \in \mathfrak{g}$ there is a linear operator on $\mathfrak{g}$ called ad_x defined by $ad_x(y) = [x, y]$ for all $y \in \mathfrak{g}$. Given a symmetrizable GCM A and its Cartan datum, $(\mathfrak{h}, \Pi, \Pi^\vee)$, we can construct the Kac-Moody Lie algebra, $\mathfrak{g}(A)$, associated with A. The Lie algebra $\mathfrak{g}(A)$ is generated by the set $\{e_i, f_i \mid i \in J\} \cup \mathfrak{h}$, where $J = \{1, \ldots, n\}$, with the following relations [4].

$$\begin{aligned}
[e_i, f_j] &= \delta_{i,j} h_i \,(i, j \in J) \\
[h, h'] &= 0 \,(h, h' \in \mathfrak{h}) \\
[h, e_i] &= \alpha_i(h) e_i \\
[h, f_i] &= -\alpha_i(h) f_i \,(i \in J;\ h \in \mathfrak{h}) \\
(ad_{e_i})^{1-a_{ij}} e_j &= 0 \,(i, j \in J;\ i \neq j) \\
(ad_{f_i})^{1-a_{ij}} f_j &= 0 \,(i, j \in J;\ i \neq j)
\end{aligned}$$

With the above relations $\mathfrak{h}$ forms an abelian Lie algebra called the Cartan subalgebra. Denote by $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the subalgebras of $\mathfrak{g}(A)$ generated by the e_i's and f_i's respectively. Then from [4] we know $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as vector spaces. We say $\mathfrak{g}(A)$ is affine of type A if A is of affine type.

2.2 The Root System and Root Space Decomposition

Definition 2.2.1. *Let $\alpha \in \mathfrak{h}^*$. Then α is a root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x,$ for all $h \in \mathfrak{h}\} \neq \{0\}$. $\mathfrak{g}_\alpha$ is called the alpha root space.*

The set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is called the *root basis* and the α_i's are called *simple roots*. We set

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

$$Q_\pm = \sum_{i=1}^{n} \mathbb{Z}_\pm \alpha_i$$

The lattice Q is called the root lattice. For $\mu, \lambda \in \mathfrak{h}^*$, we say $\mu \geq \lambda$ if $\mu - \lambda \in Q_+$. From Theorem 1.2 in [4] we have the root space decomposition of $\mathfrak{g}(A)$.

$$\mathfrak{g}(A) = \bigoplus_{\mu \in Q} \mathfrak{g}_\mu$$

Each root space is one dimensional and $\mathfrak{g}_{\alpha_i} = span\{e_i\}$ and $\mathfrak{g}_{-\alpha_i} = span\{f_i\}$ for $i \in J$. Let Φ denote the set of all roots. The sets of positive and negative roots are defined as $\Phi_\pm = \Phi \cap Q_\pm$. From [4] we get $\Phi = \Phi_+ \sqcup \Phi_-$. For $\alpha \in \Phi$, $\alpha = \sum_{i=1}^{n} a_i \alpha_i$, where all coefficients a_i are either nonnegative or nonpositive integers. The *height* of α, denoted $ht\,\alpha$, is $\sum_{i=1}^{n} a_i$.

Definition 2.2.2. *For each simple root α_i we have a simple reflection given by σ_i, $i = 1, \ldots, n$ defined by $\sigma_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$, for all $\lambda \in \mathfrak{h}^*$ [4].*

Definition 2.2.3. *The group W generated by the simple reflections is called the Weyl group.*

For $w \in W$, the expression $w = \sigma_{i_1} \ldots \sigma_{i_s}$ is reduced if s is minimal among all possible representations of $w \in W$ [4]. In this case s is called the length of w and denoted $\ell(w)$.

Definition 2.2.4. *A root $\alpha \in \Phi$ is called a real root if there exists $w \in W$ such that $w(\alpha)$ is a simple root. A root α which is not real is called an imaginary root [4].*

The set of real roots are denoted Φ^{re} and the imaginary roots are denoted Φ^{im}. Again we have $\Phi^{re} = \Phi^{re}_+ \sqcup \Phi^{re}_-$ and $\Phi^{im} = \Phi^{im}_+ \sqcup \Phi^{im}_-$ [4].

The dimension of $\mathfrak{g}_\alpha$ is called the multiplicity of α, denoted mult(α). The following results from [4] give us mult(α) for each $\alpha \in \Phi$.

Theorem 2.2.1. *([4] Proposition 5.1) Let α be a real root of a Kac-Moody algebra. Then mult(α) $= 1$.*

Theorem 2.2.2. *([4] Theorem 5.6) Let A be an indecomposable GCM. If A is of affine type, then* $\Phi^{im}_+ = \{\ell\delta \; (\ell = 1, 2, \dots)\}$

The GCM $A^{(2)}_{2n-1}$ is of affine type. The δ in the above theorem refers to the null root which will be defined in the next section.

Theorem 2.2.3. *([4] Corollary 8.3) Let $\mathfrak{g}(A)$ be an affine algebra of rank $n+1$ and let A be of type $X^{(r)}_N$. Then the multiplicity of the root $jr\delta$ is equal to n, and the multiplicity of the root $s\delta$ for $s \not\equiv 0 \mod r$ is equal to $\frac{N-n}{r-1}$.*

2.3 Affine Lie Algebras and the Lie Algebra of Type $A^{(2)}_{2n-1}$

My work is on affine Lie algebras of type $A^{(2)}_{2n-1}$ which has GCM of the following form for $n \geq 3$.

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & & & 2 & -2 \\ 0 & 0 & 0 & & & -1 & 2 \end{pmatrix}$$

The above matrix is $(n+1) \times (n+1)$. The null space of the above matrix is spanned by

$$u = \begin{pmatrix} 1 \\ 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{pmatrix}$$

We say the Lie algebra $\mathfrak{g}$ is of type $A^{(2)}_{2n-1}$ if $\mathfrak{g} = \mathfrak{g}(A)$ where A has the form above. If $\mathfrak{g}$ is type $A^{(2)}_{2n-1}$, then $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ and $\Pi^\vee = \{h_0, h_1, \dots, h_n\}$. Since the null space of the GCM $A^{(2)}_{2n-1}$ is spanned by the vector u above, this means that there is a nonzero root $\delta \in \Phi$ such that $\delta(h_i)=0$ for all $h_i \in \Pi^\vee$. The root δ is called the null root [4] and is $\delta=\alpha_0+\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$. There is another vector $v=(1,1,2,\ldots,2)$ such that $vA=0$. This means that there exists a nonzero element $c\in\mathfrak{h}$ such that $\alpha_i(c)=0$ for all $\alpha_i\in\Pi$. c is called the canonical central element [4], and $c=h_0+h_1+2h_2+\cdots+2h_n$. As a result we have $[c,\mathfrak{g}]=0$.

According to Definition 2.1.6, the dimension of $\mathfrak{h}$ is $n+2$. To complete the basis of $\mathfrak{h}$ fix an element $d\in\mathfrak{h}$ that satisfies the following conditions.

$$\alpha_0(d)=1$$
$$\alpha_i(d)=0, \text{ for } i\neq 0$$

The set $\{h_0,h_1,\ldots,h_n,d\}$ forms a basis for $\mathfrak{h}$, and d is called the scaling element [4].

2.4 Lie Algebra Representations and Modules

Definition 2.4.1. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space each over $\mathbb{C}$. Denote by $\mathfrak{gl}(V)$ the set of linear operators on V. A linear map $\pi:\mathfrak{g}\to\mathfrak{gl}(V)$ is a representation if $\pi([x,y])=\pi(x)\pi(y)-\pi(y)\pi(x)$. In this case the vector space V is referred to as a $\mathfrak{g}$-module.*

If V is a $\mathfrak{g}$-module, then π defines an action of $\mathfrak{g}$ on V where $x.v=\pi(x)(v)$ for all $x\in\mathfrak{g}$ and $v\in V$. A module V is called $\mathfrak{h}$-*diagonalizable* if $V=\bigoplus_{\lambda\in\mathfrak{h}^*}V_\lambda$, where $V_\lambda=\{v\in V|h.v=\lambda(h)v, \text{ for all } h\in\mathfrak{h}\}$. If $\dim V_\lambda\neq 0$, then λ is called a *weight*. Denote by $P(V)=\{\lambda\in\mathfrak{h}^*|\dim V_\lambda\neq 0\}$ the set of all weights of V [4].

For a set of *simple coroots* $\Pi^\vee=\{h_0,\ldots,h_n\}$, we can define the set of *fundamental weights* $\{\Lambda_0,\ldots,\Lambda_n\}\subset\mathfrak{h}^*$ by $\Lambda_i(h_j)=\delta_{i,j}$. Now we define the *weight lattice*, denoted P, as follows.

$$P=\bigoplus_{i=0}^{n}\mathbb{Z}\Lambda_i\oplus\mathbb{Z}\delta$$

For $\lambda\in\mathfrak{h}^*$ set $D(\lambda)=\{\mu\in\mathfrak{h}^*|\mu\leq\lambda\}$. From [4] we get the following definition of category $\mathcal{O}$

Definition 2.4.2. *The category $\mathcal{O}$ is defined as follows. Its objects are $\mathfrak{g}(A)$–modules V which are $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces and such that there exists a finite number of elements $\lambda_1,\ldots,\lambda_s\in\mathfrak{h}^*$ such that*

$$P(V)\subset\bigcup_{i=1}^{s}D(\lambda_i)$$

The morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}(A)$-modules.

Definition 2.4.3. *The universal enveloping algebra of $\mathfrak{g}$ is a pair $(\mathfrak{U},i)$, where $\mathfrak{U}$ is an associative algebra with 1 over $\mathbb{C}$, $i:\mathfrak{g}\to\mathfrak{U}$ is a linear map satisfying $i([x,y])=i(x)i(y)-i(y)i(x)$ for $x,y\in\mathfrak{g}$, and the following holds: for any associative $\mathbb{C}$-algebra $\mathfrak{V}$ with 1 and any linear map $j:\mathfrak{g}\to\mathfrak{V}$ satisfying $j([x,y])=j(x)j(y)-j(y)j(x)$, there exists a unique homomorphism of algebras $\phi:\mathfrak{U}\to\mathfrak{V}$ which sends 1 to 1 and $\phi\circ i=j$*

The universal enveloping algebra of $\mathfrak{g}$ is denoted $\mathfrak{U}(\mathfrak{g})$.

Definition 2.4.4. *A $\mathfrak{g}$-module V is called a highest-weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$ such that*

$$\begin{aligned} \mathfrak{n}_+(v_\lambda) &= 0 \\ h(v_\lambda) &= \lambda(h)v_\lambda \textit{ for } h \in \mathfrak{h} \\ \mathfrak{U}(\mathfrak{g})(v_\lambda) &= V \end{aligned}$$

where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. In this case the vector v_λ is called the highest-weight vector. The highest weight vector is unique up to scalar multiple.

The set $P_+ = \{\lambda \in P | \lambda(h_i) > 0,\ i = 0, \ldots, n\}$ is called the set of dominant integral weights. Let V be a highest weight $\mathfrak{g}$-module with highest weight vector v. The module V is integrable if an only if there exists an integer $N_i > 0$ such that $f_i^{N_i}.v = 0$ [4]. From sections 10.1 an 9.3 in [4] we know that for each $\Lambda \in P_+$ there exists a unique integrable highest weight module with highest weight Λ and highest weight vector v_Λ, call it $L(\Lambda)$. From section 9.2 in [4] we also get the following.

$$L(\Lambda) = \bigoplus_{\lambda \leq \Lambda} L(\Lambda)_\lambda;\ \dim L(\Lambda)_\lambda < \infty$$

2.5 The Principally Specialized Character

For all $\lambda \in \mathfrak{h}^*$ the formal exponentials, denoted $e(\lambda)$, define a linearly independent set where multiplication is defined by $e(\lambda)e(\mu) = e(\lambda + \mu)$ and extends linearly.

Definition 2.5.1. *Let $V \in \mathcal{O}$ and $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be its weight space decomposition. The formal character of V is defined by*

$$chV = \sum_{\lambda \in \mathfrak{h}^*} (dimV_\lambda)e(\lambda)$$

Recall the positive root lattice $Q_+ = \sum_{i=0}^{n} \mathbb{Z}_+ \alpha_i$ and the set of positive roots Φ_+. Now for $V \in \mathcal{O}$ set

$$D = \prod_{\alpha \in \Phi_+} (1 - e(-\alpha))^{\text{mult}(\alpha)}$$

Define ρ to be a fixed element in $\mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i = 0, \ldots, n$. Recall from section 2.2 the length $\ell(w)$ of an element in the Weyl group W.

Theorem 2.5.1. *([4] Theorem 10.4) Let $\mathfrak{g}(A)$ be a symmetrizable Kac-Moody algebra, and let $L(\Lambda)$ be an irreducible $\mathfrak{g}(A)$-module with highest weight $\Lambda \in P_+$. Then*

$$chL(\Lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w(\Lambda + \rho) - \rho)}{\prod_{\alpha \in \Phi_+} (1 - e(-\alpha))^{mult(\alpha)}}$$

Definition 2.5.2. *Let $(s_0,\dots,s_n)$ be a sequence of positive integers and q an indeterminate. The q-specialization of type $(s_0,\dots,s_n)$ is the homomorphism between the two rings of formal power series*

$$\mathbb{Z}[[e(-\alpha_0),\dots,e(-\alpha_n)]]\to\mathbb{Z}[[q]]$$

which assigns $e(-\alpha_i)$ to q^{s_i}. When $(s_0,\dots,s_n)=(1,\dots,1)$, this homomorphism is called the principal specialization.

For $\mathfrak{g}=\mathfrak{g}(A)$, the Kac-Moody algebra associated to the GCM A, we can construct the *dual Lie algebra* $\mathfrak{g}'$ as $\mathfrak{g}'=\mathfrak{g}(A^t)$, the Kac-Moody algebra associated to the transpose of A. Similarly we can define D' in the following way

$$D'=\prod_{\alpha\in\Phi'_+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}$$

where Φ' is the root system for $\mathfrak{g}'$.

Theorem 2.5.2. *([5] Corollary 4.3) Consider the highest weight module $L(\Lambda)$. The principal specialization of $e(-\Lambda)ch\,L(\Lambda)$ is called the principally specialized character of $L(\Lambda)$, denoted $\chi(L(\Lambda))$, and has the following product expansion.*

$$\chi(L(\Lambda))=\frac{D'_\Lambda}{D'_0}$$

where D'_Λ is the $((\Lambda+\rho)(h_0),\dots,(\Lambda+\rho)(h_n))$ q-specialization of D' and D'_0 is the principal q-specialization of D'.

If $\mathfrak{g}$ is type $A^{(2)}_{2n-1}$, then $\mathfrak{g}'$ is type $B^{(1)}_n$, which is itself an affine Lie algebra. The root system for $B^{(1)}_n$ is

$$\Phi'=\{\ell\delta+\alpha\,|\,\alpha\in\mathring{\Phi},\ell\in\mathbb{Z}\}\cup\{\ell\delta\,|\,\ell=1,2,3,...\}$$

Here $\delta=\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_n$ is the null root for $B^{(1)}_n$, and $\mathring{\Phi}$ is the root system of type B_n. The following is the set of positive roots for B_n.

$$\mathring{\Phi}_+=\{\alpha_i,\,\alpha_j+\alpha_{j+1}+\cdots+\alpha_k,\,\alpha_j+\alpha_{j+1}+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_n\,|\,1\le i\le n,\,1\le j<k\le n\}$$

Theorem 2.5.3. *For an algebra $\mathfrak{g}$ of type $A^{(2)}_{2n-1}$, and $F=\chi(L(\Lambda_0))=\prod_{\ell>1}(1-q^{2\ell-1})^{-1}$ we get*

$$\chi(L(2\Lambda_0))=F\prod_{\ell\not\equiv 0\,mod(n+1)}(1-q^{2\ell})^{-1}$$

Proof. In order to prove this we need to look at the root system of $\mathfrak{g}'$, which is type $B^{(1)}_n$. Let Φ' be the set of roots of $\mathfrak{g}'$. From [4] we know
$\Phi'=\{\ell\delta+\alpha\,|\,\alpha\in\mathring{\Phi},\ell\in\mathbb{Z}\}\cup\{\ell\delta\,|\,\ell=1,2,3,...\}$ where
$\delta=\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_n$,
and
$\mathring{\Phi}_+=\{\alpha_i,\,\alpha_j+\alpha_{j+1}+\cdots+\alpha_k,\,\alpha_j+\alpha_{j+1}+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_n\,|\,1\le i\le n,\,1\le j<k\le n\}$ is the

positive root system for B_n.
$\Rightarrow \Phi'_+ = \{\ell\delta + \alpha \,|\, \alpha \in \mathring{\Phi}_+, \ell \geq 0\} \cup \{\ell\delta - \alpha \,|\, \alpha \in \mathring{\Phi}_+, \ell \geq 1\}$

To compute $\chi(L(2\Lambda_0))$, I will use the following formula from Definition 2.5.3

$$\chi(L(2\Lambda_0)) = \frac{D'_{2\Lambda_0}}{D'_0}$$

where D'_0 and $D'_{2\Lambda_0}$ are the principal and $(2\Lambda_0(h_0)+1, 2\Lambda_0(h_1)+1, \ldots, 2\Lambda_0(h_n)+1)$ specializations of $D' = \prod_{\alpha\in\Phi'_+}(1-e(-\alpha))^{dim\mathfrak{g}'_\alpha}$ respectively.
Let $\alpha = \ell\delta + \mathring{\alpha} \in \Phi'_+$ where $\mathring{\alpha} \in \mathring{\Phi}$. Since $(2\Lambda_0(h_0)+1, 2\Lambda_0(h_1)+1, \ldots, 2\Lambda_0(h_n)+1) = (3, 1, \ldots, 1)$, it follows that $e(\alpha) \mapsto q^{(2n+2)\ell + ht\mathring{\alpha}}$. From this we get

$$D'_{2\Lambda_0} = \prod_{\ell\geq 0}(1-q^{(2n+2)\ell+1})^{f_1}(1-q^{(2n+2)\ell+2})^{f_2}\cdots(1-q^{(2n+2)\ell+(2n-1)})^{f_{2n-1}}$$

$$\prod_{\ell>0}(1-q^{(2n+2)\ell-1})^{f_1}(1-q^{(2n+2)\ell-2})^{f_2}\cdots(1-q^{(2n+2)\ell-(2n-1)})^{f_{2n-1}}(1-q^{(2n+2)\ell})^n$$

where $f_i =$ the number of roots $\alpha \in \mathring{\Phi}_+$ such that $ht\alpha = i$. The $1-q^{(2n+2)\ell}$ term has a multiplicity of n because imaginary roots have a multiplicity of n.

Now we need to determine the value of each f_i. Let $\mathring{\Phi}_i = \{\alpha \in \mathring{\Phi}_+ \,|\, ht\alpha = i\}$, so $f_i = |\mathring{\Phi}_i|$.
Each $\alpha = \sum_{\ell=1}^n a_\ell \alpha_\ell \in \mathring{\Phi}_i$ is uniquely defined by the lowest ℓ such that $a_\ell = 1$. Let $N_i = \{\ell \in \mathbb{Z} \,|\, \ell$ is the lowest integer such that $a_\ell = 1$ for some $\alpha \in \mathring{\Phi}_i\}$.
$max(N_i) = f_i$. Based on $\mathring{\Phi}_+$, we get the following for f_i.
$f_i = n - m$ where $i = 2m+1$ if i is odd and $i = 2m$ if i is even.
$f_1 = n$
$f_2 = n-1$
$f_3 = n-1$
$f_4 = n-2$
$f_5 = n-2$
$\vdots$
$f_{2n-2} = 1$
$f_{2n-1} = 1$
$f_{2n} = 0$
$f_{2n+1} = 0$
Using the above values for f_i and re-indexing, we get the following for $D'_{2\Lambda_0}$

$$D'_{2\Lambda_0} = \prod_{\ell>0}(1-q^{(2n+2)\ell-1})^n(1-q^{(2n+2)\ell-2})^{n-1}(1-q^{(2n+2)\ell-3})^n\cdots$$
$$(1-q^{(2n+2)\ell-(2n-1)})^n(1-q^{(2n+2)\ell-2n})^{n-1}(1-q^{(2n+2)\ell-(2n+1)})^n(1-q^{(2n+2)\ell})^n$$

Since $D_0' = \prod_{\ell>0}(1-q^{2\ell-1})(1-q^\ell)^n$, we can now find $\chi(L(2\Lambda_0))$.

$$\chi(L(2\Lambda_0)) = \frac{D'_{2\Lambda_0}}{\prod_{\ell>0}(1-q^{2\ell-1})(1-q^\ell)^n}$$

$$= F\prod_{\ell>0}(1-q^{(2n+2)\ell-2})^{-1}(1-q^{(2n+2)\ell-4})^{-1}\cdots(1-q^{(2n+2)\ell-(2n-2)})^{-1}(1-q^{(2n+2)\ell-2n})^{-1}$$

$$= F\prod_{\ell\not\equiv 0\, mod(n+1)}(1-q^{2\ell})^{-1}$$

□

The principally specialized character $\chi(L(2\Lambda_0))$ gives information about the dimension of $L(2\Lambda_0)$. We can use this in construcing a realization of $L(2\Lambda_0)$. In particular, the following identity known as the generalized Euler identity [8] plays an important role in our construction.

$$\prod_{j\not\equiv 0\, mod(n+1)}(1-q^{2j})^{-1} = \sum_{\ell\geq 0} c_\ell q^\ell$$

where c_ℓ = the number of partitions of ℓ with even parts such that each part appears at most n times.

CHAPTER

3

THE PRINCIPAL PICTURE

In order to construct a realization of $L(2\Lambda_0)$, we need a conducive realization of $A_{2n-1}^{(2)}$. We get such a construction from [1].

3.1 The Automorphism ν

Let Q be the root lattice of type A_{2n-1} with the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_{2n-1}\}$. For each simple root α_i we have a *simple reflection* given by $\sigma_i,\ i = 1, \dots, n$ defined by

$$\sigma_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \text{ for all } \lambda \in \mathfrak{h}^*$$

Let $\sigma : Q \to Q$ be the linear map defined by

$$\sigma(\alpha_i) = \alpha_{2n-i},\ i = 1, \dots, n$$

Now we define the automorphism $\nu : Q \to Q$ by

$$\nu = \sigma_1\sigma_2 \dots \sigma_n\sigma$$

Let m be the order of ν. From [1] we know that $m = 4n-2$. The automorphism ν partitions Φ into n orbits, denoted O_i, $1 \le i \le n$.

$$\begin{aligned}O_1 = \{&\alpha_1, \alpha_{2n-1}, \alpha_2, \alpha_{2n-2}, \alpha_3, \alpha_{2n-3}, \dots, \alpha_{n-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1},\\ &\alpha_n + \cdots + \alpha_{2n-1}, -\alpha_1, -\alpha_{2n-1}, \dots, -\alpha_{n-1},\\ &-(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}), -(\alpha_n + \cdots + \alpha_{2n-1})\}\end{aligned}$$

$$\begin{aligned}O_2 = \{&\alpha_1 + \alpha_2, \alpha_{2n-1} + \alpha_{2n-2}, \alpha_2 + \alpha_3, \alpha_{2n-2} + \alpha_{2n-3}, \dots, \alpha_{n-2} + \alpha_{n-1},\\ &\alpha_1 + \alpha_2 + \cdots + \alpha_{n+2}, \alpha_{n-1} + \cdots + \alpha_{2n-1}, \alpha_2 + \cdots + \alpha_{n+1},\\ &\alpha_n + \cdots + \alpha_{2n-1}, -(\alpha_1 + \alpha_2), -(\alpha_{2n-1} + \alpha_{2n-2}), \dots,\\ &-(\alpha_{n-2} + \alpha_{n-1}), -(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+2}), -(\alpha_{n-1} + \cdots + \alpha_{2n-1}),\\ &-(\alpha_2 + \cdots + \alpha_{n+1}), -(\alpha_n + \cdots + \alpha_{2n-1})\}\end{aligned}$$

$$\vdots$$

$$\begin{aligned}O_{n-1} = \{&\alpha_1 + \cdots + \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{2n-1}, \alpha_2 + \cdots + \alpha_{2n-1},\\ &\alpha_2 + \cdots + \alpha_{2n-2}, \alpha_3 + \cdots + \alpha_{2n-2}, \alpha_3 + \cdots + \alpha_{2n-3}, \dots,\\ &\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_n + \alpha_{n+1}, -(\alpha_1 + \cdots + \alpha_{n-1}),\\ &-(\alpha_1 + \cdots + \alpha_{2n-1}), \dots, -(\alpha_n + \alpha_{n+1})\}\end{aligned}$$

$$\begin{aligned}O_n = \{&\alpha_n, -(\alpha_1 + \cdots + \alpha_n), -(\alpha_{n+1} + \cdots + \alpha_{2n-1}), -(\alpha_2 + \cdots + \alpha_n),\\ &-(\alpha_{n+1} + \cdots + \alpha_{2n-2}), \dots, -(\alpha_{n-1} + \alpha_n), -\alpha_{n+1}, -\alpha_n,\\ &\alpha_1 + \cdots + \alpha_n, \alpha_{n+1} + \cdots + \alpha_{2n-1}, \dots, \alpha_{n-1} + \alpha_n, \alpha_{n+1}\}\end{aligned}$$

From each orbit O_i we select an orbit representative denoted β_i, $1 \leq i \leq n$ where $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2, \dots, \beta_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$, $\beta_n = \alpha_n$.

3.2 The Principal Heisenberg Subalgebra and the Principal Realization of $A_{2n-1}^{(2)}$

Now let $\underline{a}$ be the $\mathbb{C}$ vector space $\underline{a} = \mathbb{C} \otimes_{\mathbb{Z}} Q$, and let ω be a primitive mth root of unity. Then we have the following decomposition for $\underline{a}$.

$$\underline{a} = \coprod_{p \in \mathbb{Z}_m} \underline{a}_{(p)}$$

where $\underline{a}_{(p)} = \{x \in \underline{a} \mid \nu x = \omega^p x\}$. Now we can construct the *principal Heisenberg subalgebra* $\mathfrak{s}$.

$$\mathfrak{s} = \coprod_{i \in \mathbb{Z}} \underline{a}_{(i)} \otimes t^i \oplus \mathbb{C}c$$
$$\mathfrak{s}_{\pm} = \coprod_{i \in \mathbb{Z}_{\pm}} \underline{a}_{(i)} \otimes t^i$$

(we take $\underline{a}_{(i)} = \underline{a}_{(i \bmod m)}$). The bracket on $\mathfrak{s}$ is defined as follows.

$$[x \otimes t^i, y \otimes t^j] = \frac{1}{m} i \langle x, y\rangle \delta_{i+j,0} c, \text{ for } x \in \underline{a}_{(i)},\ y \in \underline{a}_{(j)}$$
$$[c, \mathfrak{s}] = 0$$

The bilinear form $\langle , \rangle$ above is defined on the simple roots by $\langle \alpha_i, \alpha_j \rangle = a_{ji}$ where a_{ji} is the jith entry in the GCM of type A_{2n-1}. The algebra $\mathfrak{s}$ is referred to as a subalgebra because it is a subalgebra of the affine Lie algebra of type $A^{(2)}_{2n-1}$. In [1] the construction of $\mathfrak{s}$ extends to the construction of the algebra of type $A^{(2)}_{2n-1}$. Let $\mathfrak{g}$ be the vector space over $\mathbb{C}$ with basis $\Pi \cup \{x_\alpha | \alpha \in \Phi\}$. Where Φ is the root system of type A_{2n-1} and x_α are distinct symbols. Define the map $\epsilon : \underline{a} \times \underline{a} \to \mathbb{C}^*$ in the following way.

$$\epsilon(\alpha, \beta) = \prod_{p=1}^{m-1} (1 - \omega^{-p})^{\langle \nu^p \alpha, \beta \rangle}$$

Now we define the bracket on $\mathfrak{g}$ by

$$[\alpha_i, \alpha_j] = 0$$

$$[\alpha_i, x_\alpha] = \langle \alpha_i, \alpha \rangle x_\alpha = -[x_\alpha, \alpha_i] \text{ for } 1 \le i \le 2n-1 \text{ and } \alpha \in \Phi$$

$$[x_\alpha, x_\beta] = \begin{cases} \epsilon(-\alpha, \alpha)\alpha & \langle \alpha, \beta \rangle = -2 \\ \epsilon(\alpha, \beta) x_{\alpha+\beta} & \langle \alpha, \beta \rangle = -1 \\ 0 & \langle \alpha, \beta \rangle \ge 0 \end{cases}$$

Now extend the the form $\langle , \rangle$ on $\underline{a}$ to a $\mathbb{C}$-bilinear form on $\mathfrak{g}$.

$$\langle \alpha, x_\beta \rangle = 0$$

$$\langle x_\alpha, x_\beta \rangle = \epsilon(\alpha, \beta) \delta_{\alpha+\beta,0}$$

for $\alpha, \beta \in \Phi$. We also extend ν to an automorphism of $\mathfrak{g}$ by $\nu x_\alpha = x_{\nu\alpha}$ for $\alpha \in \Phi$. Let $\mathfrak{g}_p = \{x \in$

$\mathfrak{g} | \nu x = \omega^p x\}$. We now construct the Lie algebra $\tilde{\mathfrak{g}}(\nu)$.

$$\tilde{\mathfrak{g}}(\nu) = \coprod_{i \in \mathbb{Z}} \mathfrak{g}_{(i)} \otimes t^i \oplus \mathbb{C}c \oplus \mathbb{C}d$$

The bracket on $\tilde{\mathfrak{g}}(\nu)$ is defined as follows

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^i + \frac{1}{m} i \langle x, y \rangle \delta_{i+j,0} c$$

$$[c, \tilde{\mathfrak{g}}(\nu)] = 0$$

$$[d, x \otimes t^i] = i x \otimes t^i$$

for $x \in \mathfrak{g}_{(i)}$ and $y \in \mathfrak{g}_{(j)}$. The algebra $\tilde{\mathfrak{g}}(\nu)$ is the principal realization of $A^{(2)}_{2n-1}$ [1].

3.3 $L(2\Lambda_0)$, the Vacuum Space, and Z-operators

For the GCM A of type $A^{(2)}_{2n-1}$ we have the set of *simple coroots* $\Pi^\vee = \{h_0, \ldots, h_n\}$. Recall the set of *fundamental weights* $\{\Lambda_0, \ldots, \Lambda_n\} \subset \mathfrak{h}^*$ by $\Lambda_i(h_j) = \delta_{i,j}$. The module $L(2\Lambda_0)$ is the $\mathfrak{g}(A)$-module with highest weight $2\Lambda_0$.

Definition 3.3.1. *The vacuum space of $L(2\Lambda_0)$ is denoted $\Omega(L(2\Lambda_0))$ and defined as follows*

$$\Omega(L(2\Lambda_0)) = \{v \in L(2\Lambda_0) \mid \mathfrak{s}_+ . v = 0\}$$

Theorem 3.3.1. *([6] Prop.3.1) $L(2\Lambda_0) \cong \mathfrak{U}(\mathfrak{s}) \otimes \Omega(L(2\Lambda_0))$ as $\mathfrak{s}$-modules.*

From the above theorem it follows that $L(2\Lambda_0)$ is determined by $\Omega(L(2\Lambda_0))$. Therefore $L(2\Lambda_0)$ can be realized by finding a basis for $\Omega(L(2\Lambda_0))$.

Denote by $\pi_{(p)} : \underline{a} \to \underline{a}_{(p)}$ the pth projection and let $x_{(p)} = \pi_{(p)} x$ for $x \in \underline{a}$. From [1] and [7] we get the following definitions for $E^\pm(\beta, \zeta, 2)$, $X(\beta, \zeta)$, and $Z(\beta, \zeta, 2)$ for $\beta \in \Phi$.

$$E^\pm(\beta, \zeta, 2) = exp(m \sum_{\pm j > 0} (\beta_{(j)} \otimes t^j) \frac{\zeta^j}{2j})$$

$$X(\beta, \zeta) = \frac{1}{m} E^-(-\beta, \zeta, 1) E^+(-\beta, \zeta, 1)$$

$$Z(\beta, \zeta, 2) = E^-(\beta, \zeta, 2) X(\beta, \zeta) E^+(\beta, \zeta, 2) = \sum_{i \in \mathbb{Z}} Z(\beta, i) \zeta^i$$

Since $L(2\Lambda_0)$ is an $\mathfrak{s}$-module, coefficient $Z(\beta,i)$ of ζ^i acts on $L(2\Lambda_0)$ and is called a *homogeneous operator of degree i*.

Theorem 3.3.2. *([7] Prop.4.7) Let $v_{2\Lambda_0}$ be a highest-weight vector for $L(2\Lambda_0)$.*

$$\Omega(L(2\Lambda_0)) = span\{Z(\gamma_1,i_i)Z(\gamma_2,i_2)\dots Z(\gamma_s,i_s)v_{2\Lambda_0} \mid \gamma_k \in \Phi,\ i_k \in \mathbb{Z},\ k=1,\dots,s\}$$

From [7] and [2] we can define the $\mathfrak{s}$-filtration on $\Omega(L(2\Lambda_0))$ by

$$0 = \Omega_{[-1]} \subset \Omega_{[0]} \subset \Omega_{[1]} \subset \cdots \subset \Omega(L(2\Lambda_0))$$

where $\Omega_{[i]} = span\{Z_{j_1}(\beta_{j_1})Z_{j_2}(\beta_{j_2})\cdots Z_{j_s}(\beta_{j_s})v_{2\Lambda_0}|0 \le s \le i\}$.
We know that $L(2\Lambda_0) \cong \mathfrak{U}(\mathfrak{s}) \otimes \Omega(L(2\Lambda_0))$ as $\mathfrak{s}$-modules. Therefore the principally specialized character $\chi(L(2\Lambda_0))$ has the following factorization.

$$\chi(L(2\Lambda_0)) = \chi(\mathfrak{U}(\mathfrak{s}))\chi(\Omega(L(2\Lambda_0)))$$

We get the following expression of $\chi(\Omega(L(2\Lambda_0)))$ from [7] section 7.

$$\chi(\Omega(L(2\Lambda_0))) = \sum_{n\ge 0} \chi(\Omega_{[n]}/\Omega_{[n-1]})$$

By computing $\chi(L(2\Lambda_0))$ and $\chi(\mathfrak{U}(\mathfrak{s}))$ in conjunction with the generalized Euler identity [8], I have proved the following result.

Theorem 3.3.3.

$$\chi(\Omega(L(2\Lambda_0))) = \prod_{j \not\equiv 0 \, mod(n+1)} (1-q^{2j})^{-1} = \sum_{\ell \ge 0} c_\ell q^\ell$$

where c_ℓ = the number of partitions of ℓ with even parts such that each part appears at most n times.

3.4 Results for $A^{(2)}_{2n-1}$

Proposition 3.4.1. $\underline{a}_{(p)} = 0$ *for p even.*

Proof. Recall that $\underline{a} = \coprod_{p\in\mathbb{Z}_m} \underline{a}_{(p)}$ and $\underline{a}_{(p)} = \{x \in \underline{a} \mid \nu x = \omega^p x\}$. ν acts on the simple roots in the following way:

$$
\begin{aligned}
&\nu\alpha_1 = \alpha_{2n-1}\\
&\nu\alpha_2 = \alpha_{2n-2}\\
&\nu\alpha_3 = \alpha_{2n-3}\\
&\vdots\\
&\nu\alpha_{n-2} = \alpha_{n+2}\\
&\nu\alpha_{n-1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}\\
&\nu\alpha_n = -\alpha_1 - \alpha_2 - \cdots - \alpha_n\\
&\nu\alpha_{n+1} = \alpha_n\\
&\nu\alpha_{n+2} = \alpha_{n-1}\\
&\nu\alpha_{n+3} = \alpha_{n-2}\\
&\vdots\\
&\nu\alpha_{2n-2} = \alpha_3\\
&\nu\alpha_{2n-1} = \alpha_2
\end{aligned}
$$

Now let $h = \sum_{\alpha_i \in \Pi} a_i \alpha_i \in \underline{a}_{(p)}$, then $\nu h = \omega^p h$. Using the above values for ν, we get the following relations:

$$
\begin{aligned}
&\omega^p a_1 = a_{n-1} - a_n\\
&\omega^p a_2 = a_{n-1} - a_n + a_{2n-1}\\
&\omega^p a_3 = a_{n-1} - a_n + a_{2n-2}\\
&\omega^p a_4 = a_{n-1} - a_n + a_{2n-3}\\
&\vdots\\
&\omega^p a_{n-1} = a_{n-1} - a_n + a_{n+2}\\
&\omega^p a_n = a_{n-1} - a_n + a_{n+1}\\
&\omega^p a_{n+1} = a_{n-1}\\
&\omega^p a_{n+2} = a_{n-2}\\
&\vdots\\
&\omega^p a_{2n-2} = a_2\\
&\omega^p a_{2n-1} = a_1
\end{aligned}
$$

Solving for each a_i in terms of a_1, gives the following:

$$
\begin{aligned}
&a_2 = a_1 + \omega^{-2p} a_1 = a_1(1+\omega^{-2p}) \\
&a_3 = a_1 + \omega^{-2p} a_2 = a_1(1+\omega^{-2p}+\omega^{-4p}) \\
&a_4 = a_1 + \omega^{-2p} a_3 = a_1(1+\omega^{-2p}+\omega^{-4p}+\omega^{-6p}) \\
&\vdots \\
&a_{n-1} = a_1 + \omega^{-2p} a_{n-2} = a_1(1+\omega^{-2p}+\omega^{-4p}+\cdots+\omega^{-2(n-3)p}+\omega^{-2(n-2)p}) \\
&a_n = a_1 + \omega^{-2p} a_{n-1} = a_1(1+\omega^{-2p}+\omega^{-4p}+\cdots+\omega^{-2(n-3)p}+\omega^{-2(n-2)p}+\omega^{-2(n-1)p}) \\
&a_{n+1} = \omega^{-p} a_{n-1} = a_1(\omega^{-p}+\omega^{-3p}+\omega^{-5p}+\cdots+\omega^{-(2(n-3)+1)p}+\omega^{-(2(n-2)+1)p}) \\
&\vdots \\
&a_{2n-3} = \omega^{-p} a_3 = a_1(\omega^{-p}+\omega^{-3p}+\omega^{-5p}) \\
&a_{2n-2} = \omega^{-p} a_2 = a_1(\omega^{-p}+\omega^{-3p}) \\
&a_{2n-1} = \omega^{-p} a_1
\end{aligned}
$$

Notice that $\omega^p a_1 = a_{n-1} - a_n = -\omega^{-2(n-1)p} a_1$. Therefore either $a_1 = 0$ or $\omega^p + \omega^{-2(n-1)p} = 0$. Now suppose $a_1 \neq 0$, then $\omega^p + \omega^{-2(n-1)p} = 0$. Since ω is a primitive $4n-2$ root of unity, we have $\omega^{2n-1} = -1$. Therefore $\omega^p + \omega^{-2(n-1)p} = \omega^p + \omega^{2np} = \omega^p + (\omega^{2n-1}\omega)^p = \omega^p + (-1)^p \omega^p = 0$. Since $\omega^p \neq 0$, we know $1 + (-1)^p = 0$, which is only true if p is odd. Hence if p is even, then $\underline{a}_{(p)} = 0$.

$\square$

Proposition 3.4.2. *$L(2\Lambda_0)$ is a submodule of $L(\Lambda_0) \otimes L(\Lambda_0)$ and $v_{\Lambda_0} \otimes v_{\Lambda_0}$ is the highest weight vector of $L(2\Lambda_0)$*

Proof. Let $\mathfrak{g}$ be a Lie algebra of type $A^{(2)}_{2n-1}$ and let $h \in \mathfrak{h}$ and $e \in \mathfrak{n}_+$.

$$
\begin{aligned}
e.(v_{\Lambda_0} \otimes v_{\Lambda_0}) &= e.v_{\Lambda_0} \otimes v_{\Lambda_0} + v_{\Lambda_0} \otimes e.v_{\Lambda_0} \\
&= 0 \otimes v_{\Lambda_0} + v_{\Lambda_0} \otimes 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
h.(v_{\Lambda_0} \otimes v_{\Lambda_0}) &= h.v_{\Lambda_0} \otimes v_{\Lambda_0} + v_{\Lambda_0} \otimes h.v_{\Lambda_0} \\
&= \Lambda_0(h) v_{\Lambda_0} \otimes v_{\Lambda_0} + v_{\Lambda_0} \otimes \Lambda_0(h) v_{\Lambda_0} \\
&= \Lambda_0(h)(v_{\Lambda_0} \otimes v_{\Lambda_0}) + \Lambda_0(h)(v_{\Lambda_0} \otimes v_{\Lambda_0}) \\
&= 2\Lambda_0(h)(v_{\Lambda_0} \otimes v_{\Lambda_0})
\end{aligned}
$$

$\square$

Proposition 3.4.3. *$Z(\beta,\zeta,2)$ acting on $L(\Lambda_0)\otimes L(\Lambda_0)$ is given by*

$$Z(\beta,\zeta,2)v\otimes w = \frac{1}{14}E^-(-\beta,\zeta,2)E^+(-\beta,\zeta,2)v\otimes E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)w$$

$$+E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)v\otimes\frac{1}{14}E^-(-\beta,\zeta,2)E^+(-\beta,\zeta,2)w$$

Proof.

$$\begin{aligned}Z(\beta,\zeta,2)v\otimes w =& E^-(\beta,\zeta,2)X(\beta,\zeta)[E^+(\beta,\zeta,2)v\otimes E^+(\beta,\zeta,2)w]\\ =&E^-(\beta,\zeta,2)[X(\beta,\zeta)E^+(\beta,\zeta,2)v\otimes E^+(\beta,\zeta,2)w\\ &+E^+(\beta,\zeta,2)v\otimes X(\beta,\zeta)E^+(\beta,\zeta,2)w]\\ =&E^-(\beta,\zeta,2)X(\beta,\zeta)E^+(\beta,\zeta,2)v\otimes E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)w\\ &+E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)v\otimes E^-(\beta,\zeta,2)X(\beta,\zeta)E^+(\beta,\zeta,2)w\end{aligned}$$

Since

$$E^+(-\beta,\zeta,1)E^+(\beta,\zeta,2)=E^+(-\beta,\zeta,2)$$

and

$$E^-(\beta,\zeta,2)E^-(-\beta,\zeta,1)=E^-(-\beta,\zeta,2)$$

we get

$$Z(\beta,\zeta,2)v\otimes w = \frac{1}{14}E^-(-\beta,\zeta,2)E^+(-\beta,\zeta,2)v\otimes E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)w$$

$$+E^-(\beta,\zeta,2)E^+(\beta,\zeta,2)v\otimes\frac{1}{14}E^-(-\beta,\zeta,2)E^+(-\beta,\zeta,2)w$$

$\square$

Proposition 3.4.4. *For $\mathfrak{g}=A^{(2)}_{2n-1}$, $Z(\beta,i)=0$ for i odd.*

Proof. From the previous proposition we get $Z(\beta,\zeta)=Z(-\beta,\zeta)$. This along with the fact that $E^{\pm}(-\beta,\zeta,2)=E^{\pm}(\beta,-\zeta,2)$ gives us that $Z(\beta,\zeta,2)=Z(\beta,-\zeta,2)$.

Therefore $\sum_{i\in\mathbb{Z}} Z(\beta,i)\zeta^i = \sum_{i\in\mathbb{Z}} Z(\beta,i)(-\zeta)^i$, which implies $Z(\beta,i) = Z(\beta,i)(-1)^i$. Therefore $Z(\beta,i)$ can only be nonzero if i is even.

$\square$

CHAPTER

4

REALIZATION OF $L(2\Lambda_0)$

By Theorem 3.3.1 we know $L(2\Lambda_0) \cong \mathfrak{U}(\mathfrak{s}) \otimes \Omega(L(2\Lambda_0))$ so it suffices to give a realization of $\Omega(L(2\Lambda_0))$ for constructing the module $L(2\Lambda_0)$. In order to construct a basis for $\Omega(L(2\Lambda_0))$, the spanning set given in Theorem 3.3.2 needs to be reduced in such a way that the monomials of Z-operators count the partitions described in Theorem 3.3.3. To do this I establish an ordering on the Z-operator monomials and use the generalized commutator to reorder the homogonous components within each monomial.

4.1 Orbit Representatives and The Generalized Commutator

We take ν and Φ as in Section 3.1. From [1] we know that ν partitions Φ into n orbits each of size $4n-2$. From each orbit we choose an *orbit representative*. The set of orbit representatives is denoted by $O = \{\beta_1, \dots, \beta_n\}$. For A_{2n-1} the orbit representatives are

$$\begin{aligned}
\beta_1 &= \alpha_1 \\
\beta_2 &= \alpha_1 + \alpha_2 \\
&\vdots \\
\beta_{n-1} &= \alpha_1 + \cdots + \alpha_{n-1} \\
\beta_n &= \alpha_n
\end{aligned}$$

Where the α_i, $i = 1, \dots, n$ are simple roots of A_{2n-1}. We also have from [1] and [7] that $Z(\nu^p \beta, \zeta, 2) =$

$Z(\beta,\omega^p\zeta,2)$ for all $\beta\in\Phi$. From this it follows that

$$\Omega(L(2\Lambda_0))=\text{span}\{Z(\beta_{i_1},j_i)Z(\beta_{i_2},j_2)\ldots Z(\beta_{i_s},j_s)v_{2\Lambda_0}\mid\beta_{i_k}\in O,\ j_k\in\mathbb{Z},\ k=1,\ldots,s\}$$

Theorem 4.1.1. *([1] Theorem 7.3) The generalized commutator* $[[Z(\alpha,\zeta_1,2),Z(\beta,\zeta_2,2)]]$ *for* $\alpha,\beta\in\Phi$ *has the following identity*

$$[[Z(\alpha,\zeta_1,2),Z(\beta,\zeta_2,2)]]$$

$$=\prod_{p\in\mathbb{Z}_m}(1-\omega^{-p}\frac{\zeta_1}{\zeta_2})^{\frac{<\nu^p\alpha,\beta>}{2}}Z(\alpha,\zeta_1,2)Z(\beta,\zeta_2,2)-\prod_{p\in\mathbb{Z}_m}(1-\omega^{-p}\frac{\zeta_2}{\zeta_1})^{\frac{<\nu^p\beta,\alpha>}{2}}Z(\beta,\zeta_2,2)Z(\alpha,\zeta_1,2)$$

$$=\frac{1}{m}\sum_{p\in C_{-1}}\epsilon(\nu^p\alpha,\beta)Z(\nu^p\alpha+\beta,\zeta_2,2)\delta(\frac{\omega^{-p}\zeta_1}{\zeta_2})$$

$$+\frac{2}{m^2}<x_\alpha,x_{-\alpha}>\sum_{p\in C_{-2}}(D\delta)(\frac{\omega^{-p}\zeta_1}{\zeta_2})$$

(where $C_i=\{p\in\mathbb{Z}_m\mid<\nu^p\alpha,\beta>=i\}$, $i=-1,-2$*).*

In the theorem above $\delta(\zeta)=\sum_{i\in\mathbb{Z}}\zeta^i$ and $D\delta=\sum_{i\in\mathbb{Z}}i\zeta^i$. Now let $\prod_{p\in\mathbb{Z}_{14}}(1-\omega^{-p}\frac{\zeta_1}{\zeta_2})^{\frac{<\nu^p\alpha,\beta>}{2}}=\sum_{i\geq 0}k(\alpha,\beta)_i\left(\frac{\zeta_1}{\zeta_2}\right)^i$ and $\prod_{p\in\mathbb{Z}_{14}}(1-\omega^{-p}\frac{\zeta_2}{\zeta_1})^{\frac{<\nu^p\beta,\alpha>}{2}}=\sum_{i\geq 0}k(\beta,\alpha)_i\left(\frac{\zeta_2}{\zeta_1}\right)^i$ Then isolating the coefficient of $\zeta_1^r\zeta_2^s$ gives us

$$\sum_{i\geq 0}k(\alpha,\beta)_iZ(\alpha,r-i)Z(\beta,s+i)-\sum_{i\geq 0}k(\beta,\alpha)_iZ(\beta,s-i)Z(\alpha,r+i)$$

$$=\frac{1}{m}\sum_{p\in C_{-1}}\epsilon(\nu^p\alpha,\beta)Z(\nu^p\alpha+\beta,s+r)\omega^{-pr}+\frac{2}{m^2}<x_\alpha,x_{-\alpha}>\sum_{p\in C_{-2}}r(\omega^{-p})^r\delta_{s,-r}$$

The generalized commutator is primarily useful because it allows me to reorder Z-operators and write Z-operators of one orbit representative in terms of another. Using the generalized commutator, I can reduce the current spanning set for $\Omega(L(2\Lambda_0))$ down to a smaller spanning set which counts the appropriate partitions. My goal is to construct a spanning set consisting of Z-operator monomials $Z(\beta_{i_1},j_i)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)$ acting on $v_{2\Lambda_0}$ which satisfy the following criteria:

1. j_k is even for all $1\leq k\leq s$

2. $j_1 \le j_2 \le \cdots \le j_s \le 0$

3. $j_k = j_{k+1} \Rightarrow i_{k+1} = i_k + 1$

4. $j_k \ne j_{k+1} \Rightarrow i_k = n$

These criteria ensure that the elements of the spanning set count the partitions with even parts such that each part appears at most n times.

4.2 Reducing The Spanning Set

The first criterion from Section 4.1 is satisfied as a result of Proposition 3.4.4.

Proposition 4.2.1. *For* $\mathfrak{g} = A_{2n-1}^{(2)}$

$$\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1}, j_i)Z(\beta_{i_2}, j_2)\ldots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} \mid \beta_{i_k} \in O,\ j_k \in 2\mathbb{Z},\ k = 1,\ldots,s\}$$

Proof. This is an immediate result of Proposition 3.4.4. □

In order to satisfy criteria 2., 3., and 4. we need to first establish an ordering on the Z-operator monomials. Let $(n_1,\ldots,n_k)$ and $(m_1,\ldots,m_k)$ be in $\mathbb{Z}^k$. We say $(n_1,\ldots,n_k) \le_T (m_1,\ldots,m_k)$ if

$$\begin{gathered} n_k \le m_k \\ n_{k-1} + n_k \le m_{k-1} + m_k \\ \vdots \\ \sum_{i=2}^{k} n_i \le \sum_{i=2}^{k} m_i \\ \sum_{i=1}^{k} n_i \le \sum_{i=1}^{k} m_i \end{gathered}$$

Set $\beta_1 < \beta_2 < \cdots < \beta_n$ and let $\mu = (\mu_1,\ldots,\mu_k)$ and $\gamma = (\gamma_1,\ldots,\gamma_k)$ where $\mu_i, \gamma_j \in \{\beta_1,\ldots,\beta_n\}$. Order μ and γ by the reverse lexicographic ordering. Now we say $(\mu; n_1,\ldots,n_s) <_M (\gamma; m_1,\ldots,m_s)$
If $(n_1,\ldots,n_s) <_T (m_1,\ldots,m_s)$
or
If $(n_1,\ldots,n_s) = (m_1,\ldots,m_s)$ and $\mu < \gamma$

Proposition 4.2.2. *Let $r > s$, then $Z(\beta_i, r)Z(\beta_j, s) \equiv Z(\beta_j, s)Z(\beta_i, r)$ mod shorter terms or terms higher in the $<_M$ ordering.*

Proof. Consider $Z(\beta_i, r)Z(\beta_j, s)$ such that $r > s$. Then by the generalized commutator we have

$$k(\beta_i,\beta_j)_0 Z(\beta_i,r)Z(\beta_j,s)$$

$$= k(\beta_j,\beta_i)_0 Z(\beta_j,s)Z(\beta_i,r)$$

$$-[\sum_{\ell\geq 1} k(\beta_i,\beta_j)_\ell Z(\beta_i,r-\ell)Z(\beta_j,s+\ell) - \sum_{\ell\geq 1} k(\beta_j,\beta_i)_\ell Z(\beta_j,s-\ell)Z(\beta_i,r+\ell)]$$

$$+\frac{1}{m}\sum_{p\in C_{-1}} \epsilon(\nu^p\beta_i,\beta_j)Z(\nu^p\beta_i+\beta_j,s+r)\omega^{-pr} + \frac{2}{m^2} < x_{\beta_i}, x_{-\beta_i} > \sum_{p\in C_{-2}} r(\omega^{-p})^r\delta_{s,-r}$$

$$= k(\beta_j,\beta_i)_0 Z(\beta_j,s)Z(\beta_i,r)$$

$$-[\sum_{\ell\geq 1} k(\beta_i,\beta_j)_\ell Z(\beta_i,r-\ell)Z(\beta_j,s+\ell) - \sum_{\ell\geq 1} k(\beta_j,\beta_i)_\ell Z(\beta_j,s-\ell)Z(\beta_i,r+\ell)]$$

$$+ \text{shorter terms}$$

Since
$Z(\beta_i,r)Z(\beta_j,s) <_T Z(\beta_i,r-\ell)Z(\beta_j,s+\ell)$
and
$Z(\beta_i,r)Z(\beta_j,s) <_T Z(\beta_j,s-\ell)Z(\beta_i,r+\ell)$ for $\ell\geq 1$
it follows that $Z(\beta_i,r)Z(\beta_j,s) \equiv Z(\beta_j,s)Z(\beta_i,r)$ mod shorter terms or terms higher in the $<_M$ ordering. □

Proposition 4.2.3. $\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0} | \beta_{i_k}\in O,\ j_k\in 2\mathbb{Z},\ j_1\leq j_2\leq \ldots j_s,\ 1\leq k\leq s\}$

Proof. Consider the two sets

$$T = \{Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0} | \beta_{i_k}\in O,\ j_k\in 2\mathbb{Z}\}$$

and

$$T' = \{Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0} | \beta_{i_k}\in O,\ j_k\in 2\mathbb{Z},\ j_1\leq j_2\leq \ldots j_s,\ 1\leq k\leq s\}$$

Let $Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}$ be an arbitrary element in T. Our induction hypothesis is that any element shorter or higher in the $<_M$ ordering is in the span of T'. From Proposition 4.2.2 we get

$$Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s) \equiv Z(\beta_{i'_1}, j'_1)Z(\beta_{i'_2}, j'_2)\cdots Z(\beta_{i'_s}, j'_s)$$

mod terms that are shorter or higher in the $<_M$ ordering, where $j'_1 \leq j'_2 \leq \ldots \leq j'_s$. Now by our induction hypothesis we have $\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} \mid \beta_{i_k} \in O,\ j_k \in 2\mathbb{Z},\ j_1 \leq j_2 \leq \ldots j_s,\ 1 \leq k \leq s\}$. □

Proposition 4.2.4. $\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} | \beta_{i_k} \in O,\ j_k \in 2\mathbb{Z},\ j_1 \leq j_2 \leq \ldots j_s \leq 0,\ 1 \leq k \leq s\}$

Proof. Since $L(2\Lambda_0)$ is a highest weight module of weight $2\Lambda_0$, we have that $L(2\Lambda_0)$ is generated by the f_i's in $A^{(2)}_{2n-1}$. We know that $d(f_i) = -1$ for all i.

Therefore $L(2\Lambda_0)$ decomposes into the eigenspaces of d in the following way.

$$L(2\Lambda_0) = \bigoplus_{n\leq 0} L_n$$

Where $L_n = \{v \in L(2\Lambda_0) \mid dv = nv\}$. By assuming $2\Lambda_0(d) = 0$, we get $dv_{2\Lambda_0} = 2\Lambda_0(d)v_{2\Lambda_0} = 0$. This means $L_0 = span\{v_{2\Lambda_0}\}$. This along with the fact that $Z(\beta, i)L_j \subset L_{j+i}$ for all $\beta \in O$ gives us that $Z(\beta, i)$ annihilates $L(2\Lambda_0)$ for all $\beta \in O$ and $i > 0$. □

Proposition 4.2.5. $Z(\beta_i, 2m)Z(\beta_j, 2m) \equiv 0$ *mod shorter terms or terms higher in the* $<_M$ *ordering.*

Proof. Consider $Z(\beta_i, 2m)Z(\beta_j, 2m)$ for $m \in \mathbb{Z}$. Then setting $r = 2m+1$ and $s = 2m-1$ in the generalized commutator gives the following:

$$
\begin{aligned}
&k(\beta_i,\beta_j)_1 Z(\beta_i,2m)Z(\beta_j,2m)\\
=&-k(\beta_i,\beta_j)_0 Z(\beta_i,2m+1)Z(\beta_j,2m-1)+k(\beta_j,\beta_i)_0 Z(\beta_j,2m-1)Z(\beta_i,2m+1)\\
&+k(\beta_j,\beta_i)_1 Z(\beta_j,2m-2)Z(\beta_i,2m+2)\\
&-[\sum_{\ell\geq 2} k(\beta_i,\beta_j)_\ell Z(\beta_i,2m+1-\ell)Z(\beta_j,2m-1+\ell)\\
&-\sum_{\ell\geq 2} k(\beta_j,\beta_i)_\ell Z(\beta_j,2m-1-\ell)Z(\beta_i,2m+1+\ell)]\\
&+\text{shorter terms}\\
=&k(\beta_j,\beta_i)_1 Z(\beta_j,2m-2)Z(\beta_i,2m+2)\\
&-[\sum_{\ell\geq 2} k(\beta_i,\beta_j)_\ell Z(\beta_i,2m+1-\ell)Z(\beta_j,2m-1+\ell)\\
&-\sum_{\ell\geq 2} k(\beta_j,\beta_i)_\ell Z(\beta_j,2m-1-\ell)Z(\beta_i,2m+1+\ell)]\\
&+\text{shorter terms}
\end{aligned}
$$

Since $Z(\beta_k,\ell)=0$ for all k and for ℓ odd.

Since
$Z(\beta_i,2m)Z(\beta_j,2m) <_T Z(\beta_j,2m-2)Z(\beta_i,2m+2)$,
$Z(\beta_i,2m)Z(\beta_j,2m) <_T Z(\beta_i,2m+1-\ell)Z(\beta_j,2m-1+\ell)$
and
$Z(\beta_i,2m)Z(\beta_j,2m) <_T Z(\beta_j,2m-1-\ell)Z(\beta_i,2m+1+\ell)$ for $\ell\geq 2$
it follows that $Z(\beta_i,2m)Z(\beta_j,2m)\equiv 0$ mod shorter terms or terms higher in the $<_M$ ordering.

$\square$

Proposition 4.2.6. $\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} \mid \beta_{i_k}\in O,\ j_k\in 2\mathbb{Z},\ j_1\leq j_2\leq \ldots j_s\leq 0,\ j_k=j_{k+1}\Rightarrow i_{k+1}=i_k+1,\ 1\leq k\leq s\}$

Proof. Consider the two sets

$$T=\{Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}\,|\,\beta_{i_k}\in O,\ j_k\in 2\mathbb{Z},\ j_1\le j_2\le\ldots j_s\le 0,\ 1\le k\le s\}$$

and

$$\begin{aligned}T'=\{&Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}\,|\,\beta_{i_k}\in O,\\ &j_k\in 2\mathbb{Z},\ j_1\le j_2\le\ldots j_s\le 0,\\ &j_k=j_{k+1}\Rightarrow i_{k+1}=i_k+1,\ 1\le k\le s\}\end{aligned}$$

Let $Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}$ be an arbitrary element in T. Our induction hypothesis is that any element shorter or higher in the $<_M$ ordering is in the span of T'.

Now suppose there exists a pair of Z-operators $Z(\beta_{i_k},j_k)Z(\beta_{i_{k+1}},j_{k+1})$ such that $j_k=j_{k+1}$ and $i_{k+1}\neq i_k+1$. Then Proposition 4.2.5, tells us that $Z(\beta_{i_k},j_k)Z(\beta_{i_{k+1}},j_{k+1})$ can be written as a linear combination of terms that are shorter or higher in the $<_M$ ordering. From this it follows that $Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}$ is also a linear combination of terms that are shorter or higher in the $<_M$ ordering.

Therefore, by the induction hypothesis, $Z(\beta_{i_1},j_1)Z(\beta_{i_2},j_2)\cdots Z(\beta_{i_s},j_s)v_{2\Lambda_0}$ is in the span of T'. □

Proposition 4.2.7. *For $s\in 2\mathbb{Z}$, the homogenous component $Z(\beta_i,s)$, $i\neq n$, is a linear combination of monomials in $Z(\beta_n,r)$ of length two, and the value of r is not fixed.*

Proof. By computing the generalized commutator $[[Z(\beta_n,\zeta_1),Z(\beta_n,\zeta_2)]]$ and isolating the coefficients of $\zeta_1^r\zeta_2^{s-r}$ we get the following identity.

$$\sum_{i\ge 0}k(\beta_n,\beta_n)[Z(\beta_n,r-i)Z(\beta_n,s-r+i)-Z(\beta_n,s-r-i)Z(\beta_4,r+i)]$$

$$=m^{-1}\sum_{j=1}^{n-1}\frac{(1-\omega^{2j-1})(\omega^{-(2j-1)r+(2n-2+2j-1)s}-\omega^{(2j-1)r+(2n-2)s})}{1+\omega^{2j-1}}Z(\beta_{n-j},s)+2m^{-2}(-1)^r\delta_{s,0}$$

The coefficient matrix $A=(a_{ij})_{i,j=1}^{n-1}$ is obtained from the equation. The goal is to show that for each $s\in 2\mathbb{Z}$ there exist values $r_1,\ldots,r_{n-1}\in\mathbb{Z}$ which make A nonsingular. I have found that the values of a_{ij} are given by

$$a_{ij}=\frac{(1-\omega^{2j-1})(\omega^{-(2j-1)r_i+(2n-2+2j-1)s}-\omega^{(2j-1)r_i+(2n-2)s})}{1+\omega^{2j-1}}$$

A is diagonalizable as $A=A'D$. Where $A'=(a'_{ij})_{i,j=1}^{n-1}$

$$a'_{ij}=\omega^{(2j-1)(s-r_i)}-\omega^{(2j-1)r_i}$$

and D is a diagonal matrix with the values $d_1,\ldots,d_{n-1}$ on the diagonal.

$$d_j = \frac{\omega^{(2n-2)s}(1-\omega^{2j-1})}{1+\omega^{2j-1}}$$

Since $det(D) \neq 0$, we have $det(A) = 0$ if and only if $det(A') = 0$.

$$\begin{aligned} det(A') = &(-1)^k \omega^{-(2(n-1)-1)(r_1+\cdots+r_{n-1})} \\ &\prod_{1\leq i<j\leq n-1} (\omega^{2r_i} - \omega^{2r_j}) \\ &\prod_{1\leq i\leq n-1} (\omega^{2r_i} - \omega^{s}) \\ &\prod_{1\leq i<j\leq n-1} (\omega^{2(r_i+r_j)} - \omega^{2s}) \end{aligned}$$

Recall that ω is an mth root of unity, where $m = 4n-2$. If any of the following hold, then $det(A') = 0$.

$$\begin{aligned} &r_i \equiv r_j \ mod\ \frac{1}{2}m \text{ for } 1 \leq i < j \leq n-1 \\ &r_i \equiv \frac{1}{2}s \ mod\ \frac{1}{2}m \text{ for } 1 \leq i \leq n-1 \\ &r_i + r_j \equiv s \ mod\ \frac{1}{2}m \text{ for } 1 \leq i < j \leq n-1 \end{aligned}$$

For each $s \in 2\mathbb{Z}$ we want to show that there exist values $r_1,\ldots,r_{n-1} \in \mathbb{Z}$ such that $det(A) \neq 0$. If s is even, then s is even mod m because m is even. This means we only need to show the existence of the r values for values of s that are even mod m. Observe that $\frac{1}{2}m = 2n-1$ and let $s' \in \mathbb{Z}_{2n-1}$ such that $s \equiv s'\ mod\ (2n-1)$. We want to find the number of distinct pairs $(a,b) \in \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1}$ such that $a \leq b$ and $a+b \equiv s'\ mod\ (2n-1)$.

Now let $a \in \mathbb{Z}_{2n-1}$. Then $a + (s'-a) = s'$.

If $a = s'-a$, then $2a \equiv s'\ mod\ (2n-1)$ and either $a = \frac{1}{2}s'$ or $a = \frac{1}{2}s$.

If $s < 2n-1$, then $s = s'$ and $\frac{1}{2}s' + \frac{1}{2}s' = s'$.

If $s > 2n-1$, then $\frac{1}{2}s \in \mathbb{Z}_{2n-1}$ and $\frac{1}{2}s + \frac{1}{2}s = s \equiv s'\ mod\ (2n-1)$.

Let $a,b,c \in \mathbb{Z}_{2n-1}$. $a+c = b+c \Rightarrow a = b$.

The above statements imply that there are $\frac{1}{2}(2n-1-1) = n-1$ pairs of distinct numbers in

$\mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1}$ that sum to s' and one pair, namely $(\frac{1}{2}s, \frac{1}{2}s)$, with the same number that adds to s'.

This gives a total of n different pairs that add to s'. By taking each r_i value from a different pair such that no r_i is equal to $\frac{1}{2}s$ we can guarantee that none of the relations mentioned above hold, thus ensuring that $det(A) \neq 0$. □

Proposition 4.2.8. $\Omega(L(2\Lambda_0)) = span\{Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} | \beta_{i_k} \in O,\ j_k \in 2\mathbb{Z},\ j_1 \leq j_2 \leq \ldots j_s \leq 0,\ j_k = j_{k+1} \Rightarrow i_{k+1} = i_k + 1, j_k \neq j_{k+1} \Rightarrow i_k = n,\ 1 \leq k \leq s\}$

Proof. Consider the two sets

$$\begin{aligned} T = \{&Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} | \beta_{i_k} \in O, \\ &j_k \in 2\mathbb{Z}, \\ &j_1 \leq j_2 \leq \ldots j_s \leq 0, \\ &j_k = j_{k+1} \Rightarrow i_{k+1} = i_k + 1, \\ &1 \leq k \leq s\} \end{aligned}$$

and

$$\begin{aligned} T' = \{&Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} | \beta_{i_k} \in O, \\ &j_k \in 2\mathbb{Z}, \\ &j_1 \leq j_2 \leq \ldots j_s \leq 0, \\ &j_k = j_{k+1} \Rightarrow i_{k+1} = i_k + 1, \\ &j_k \neq j_{k+1} \Rightarrow i_k = n,\ 1 \leq k \leq s\} \end{aligned}$$

Let $Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0}$ be an arbitrary element in T such that $Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} \notin T'$. Then there exists a j_k such that the highest Z-operator of degree j_k in the $<_M$ ordering is $Z(\beta_i, j_k)$ where $i \neq n$. Let $j_k = 2m, m \in \mathbb{Z}$, since j_k cannot be odd.

Proposition 4.2.7 allows us to write $Z(\beta_i, 2m)$ in the following way:

$$Z(\beta_i,2m) = k_1\left[\sum_{\ell\geq 0} a_\ell(Z(\beta_n, r_1-\ell)Z(\beta_n, s_1+\ell) - Z(\beta_n, s_1-\ell)Z(\beta_n, r_1+\ell))\right]$$

$$+ k_2\left[\sum_{\ell\geq 0} a_\ell(Z(\beta_n, r_2-\ell)Z(\beta_n, s_2+\ell) - Z(\beta_n, s_2-\ell)Z(\beta_n, r_2+\ell))\right]$$

$$\vdots$$

$$+ k_{n-1}\left[\sum_{\ell\geq 0} a_\ell(Z(\beta_n, r_{n-1}-\ell)Z(\beta_n, s_{n-1}+\ell) - Z(\beta_n, s_{n-1}-\ell)Z(\beta_n, r_{n-1}+\ell))\right]$$

Where k_i, r_i, s_i are scalars and $r_i+s_i = 2m$, $1 \leq i \leq n-1$. Now looking at the degrees of the Z-operators of the right hand side of the above equation, notice that $(r_i \pm \ell)+(s_i \mp \ell) = 2m$ for $i = 1,\ldots,n-1$ and $\ell \geq 0$. This, along with the fact that there must be a lowest element in the set $\{r_i, s_i | i = 1,\ldots,n-1\}$, gives us that there exists a pair of Z-operators, call them $Z(\beta_n,\ell_1)Z(\beta_n,\ell_2)$, which is lowest in the $>_M$ ordering on the right hand side. As a result we get the following equation:

$$Z(\beta_i,2m) = Z(\beta_n,\ell_1)Z(\beta_n,\ell_2) + \text{terms higher in the } >_M \text{ ordering}$$

Our induction hypothesis is that terms higher in the $>_M$ ordering are in the span of T'. Since each monomial in T has finite length, we can only replace Z-operators of orbit representatives that are not β_n a finite number of times. Therefore any element in T can be written as a linear combination of elements in T'. □

From Proposition 4.2.8 we have a spanning set that satisfies the four partition criteria given in Section 4.1. This leads to the following result.

Theorem 4.2.1. $\{Z(\beta_{i_1}, j_1)Z(\beta_{i_2}, j_2)\cdots Z(\beta_{i_s}, j_s)v_{2\Lambda_0} |\ \beta_{i_k} \in O,\ j_k \in 2\mathbb{Z},\ j_1 \leq j_2 \leq \ldots j_s \leq 0,\ j_k = j_{k+1} \Rightarrow i_{k+1} = i_k+1, j_k \neq j_{k+1} \Rightarrow i_k = n,\ 1 \leq k \leq s\}$ *is a basis for* $\Omega(L(2\Lambda_0))$.

Proof. The grading described in Section 3.3 indicates that the basis elements of $\Omega(L(2\Lambda_0))$ need to count the partitions described in Theorem 3.3.3. The monomials described in our spanning set count these partitions. Therefore our spanning set is a basis. □

CHAPTER

5

PRINCIPALLY SPECIALIZED CHARACTER FOR LEVEL 3 WEIGHTS FOR $A_{2N-1}^{(2)}$

In this chapter I will provide a list of the principally specialized characters of level 3 weights that I have computed. I will then give a conjecture for a general formula of the specialized characters of level 3 weights.

Recall the canonical central element c of the Lie algebra $A_{2n-1}^{(2)}$. From [4] we know that $c = h_0 + h_1 + 2h_2 + 2h_3 + \ldots + 2h_n$.

Definition 5.0.1. *Given a weight λ. The level of λ (also the level of $L(\lambda)$) is given by $\lambda(c)$.*

For $A_{2n-1}^{(2)}$, the level 3 weights are $2\Lambda_0 + \Lambda_1$, $\Lambda_0 + 2\Lambda_1$, $3\Lambda_0$, $3\Lambda_1$ and $\Lambda_i + \Lambda_j$ where $i \in \{0,1\}$ and $j \in \{2, \ldots, n\}$.

$n = 3$

$$\chi(L(2\Lambda_0 + \Lambda_1)) = \chi(L(\Lambda_0 + 2\Lambda_1)) = F \prod_{i>0} (1 - q^{3i-1})^{-1}(1 - q^{3i-2})^{-1}$$

$$\chi(L(3\Lambda_0)) = \chi(L(3\Lambda_1)) = F \prod_{\substack{i>0 \\ i \not\equiv 0, \pm 1 \; mod \; 9}} (1 - q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_2)) = \chi(L(\Lambda_1+\Lambda_2)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 4 \; mod \; 9}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_3)) = \chi(L(\Lambda_1+\Lambda_3)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 2 \; mod \; 9}} (1-q^i)^{-1}$$

$n = 4$

$$\chi(L(2\Lambda_0+\Lambda_1)) = \chi(L(\Lambda_0+2\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 3 \; mod \; 11}} (1-q^i)^{-1}$$

$$\chi(L(3\Lambda_0)) = \chi(L(3\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 1 \; mod \; 11}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_2)) = \chi(L(\Lambda_1+\Lambda_2)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 5 \; mod \; 11}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_3)) = \chi(L(\Lambda_1+\Lambda_3)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 4 \; mod \; 11}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_4)) = \chi(L(\Lambda_1+\Lambda_4)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 2 \; mod \; 11}} (1-q^i)^{-1}$$

$n = 5$

$$\chi(L(2\Lambda_0+\Lambda_1)) = \chi(L(\Lambda_0+2\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 3 \; mod \; 13}} (1-q^i)^{-1}$$

$$\chi(L(3\Lambda_0)) = \chi(L(3\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 1 \; mod \; 13}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_2)) = \chi(L(\Lambda_1+\Lambda_2)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 5 \; mod \; 13}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_3)) = \chi(L(\Lambda_1+\Lambda_3)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 6 \; mod \; 13}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_4)) = \chi(L(\Lambda_1+\Lambda_4)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 4 \; mod \; 13}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_5))=\chi(L(\Lambda_1+\Lambda_5))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 2\ mod\ 13}}(1-q^i)^{-1}$$

$n=6$

$$\chi(L(2\Lambda_0+\Lambda_1))=\chi(L(\Lambda_0+2\Lambda_1))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 3\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(3\Lambda_0))=\chi(L(3\Lambda_1))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 1\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_2))=\chi(L(\Lambda_1+\Lambda_2))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 5\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_3))=\chi(L(\Lambda_1+\Lambda_3))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 7\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_4))=\chi(L(\Lambda_1+\Lambda_4))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 6\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_5))=\chi(L(\Lambda_1+\Lambda_5))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 4\ mod\ 15}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_6))=\chi(L(\Lambda_1+\Lambda_6))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 2\ mod\ 15}}(1-q^i)^{-1}$$

$n=7$

$$\chi(L(2\Lambda_0+\Lambda_1))=\chi(L(\Lambda_0+2\Lambda_1))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 3\ mod\ 17}}(1-q^i)^{-1}$$

$$\chi(L(3\Lambda_0))=\chi(L(3\Lambda_1))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 1\ mod\ 17}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_2))=\chi(L(\Lambda_1+\Lambda_2))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 5\ mod\ 17}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_3))=\chi(L(\Lambda_1+\Lambda_3))=F\prod_{\substack{i>0\\ i\not\equiv 0,\pm 7\ mod\ 17}}(1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_4)) = \chi(L(\Lambda_1+\Lambda_4)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 8 \; mod \; 17}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_5)) = \chi(L(\Lambda_1+\Lambda_5)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 6 \; mod \; 17}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_6)) = \chi(L(\Lambda_1+\Lambda_6)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 4 \; mod \; 17}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_7)) = \chi(L(\Lambda_1+\Lambda_7)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 2 \; mod \; 17}} (1-q^i)^{-1}$$

Conjecture 5.0.1. *For $A^{(2)}_{2n-1}$ the level 3 principally specialized characters have the following formulas.* $2 \leq j \leq n$

$$\chi(L(3\Lambda_0)) = \chi(L(3\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 1 \; mod \; (2n+3)}} (1-q^i)^{-1}$$

$$\chi(L(2\Lambda_0+\Lambda_1)) = \chi(L(\Lambda_0+2\Lambda_1)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm 3 \; mod \; (2n+3)}} (1-q^i)^{-1}$$

$$\chi(L(\Lambda_0+\Lambda_j)) = \chi(L(\Lambda_1+\Lambda_j)) = F \prod_{\substack{i>0 \\ i\not\equiv 0,\, \pm(2j+1) \; mod \; (2n+3)}} (1-q^i)^{-1}$$

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