ABSTRACT

MARTIN, KRISTI. Pre-Service Secondary Mathematics Teachers' Problem-Solving Across Trigonometric Domains (Under the direction of Dr. Karen Hollebrands).

This study examined pre-service secondary mathematics teachers' knowledge of trigonometry as revealed while solving problems across four different trigonometric domains (triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry). Trigonometry is known to be difficult for both students and pre-service teachers (Bressoud, 2010; Demir & Heck, 2013; Gür, 2009; Thompson, 2008). The study employed a multiple case study design (Creswell, 2013). Participants engaged in a series of task-based interviews (Goldin, 2000). The task-based interviews were analyzed for problem-solving phases (e.g., orienting, planning, executing, checking) and problem-solving attributes (e.g., resources, heuristics, affect, monitoring) using the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005). The pre-service secondary mathematics teachers who participated in this study generally began each task with a problem-solving cycle, but there were no other discernable patterns in the problem-solving cycles across domains. The participants were able to use knowledge from the greatest number of domains when solving triangle trigonometry tasks. Participants used a total of four different heuristics: algebra, diagrams, simpler cases, and substituting values. Further analysis included the Mathematical Understanding for Secondary Teaching (MUST) Framework (Heid et al., 2015) for the mathematical understandings shown by the pre-service secondary mathematics teachers as they solved the trigonometry tasks. The participants in this study were all able to demonstrate procedural fluency with trigonometry. However, only some of them were able to demonstrate conceptual understanding of trigonometry. This study contributes to the scarcity of research literature on pre-service teachers' knowledge of trigonometry (Akkoç, 2008; Chigonga, 2016; Topçu et al., 2006; Weber, 2008).

Pre-Service Secondary Mathematics Teachers' Problem-Solving Across Trigonometric Domains

by
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BIOGRAPHY

Kristi Martin was born and raised in Holland, Michigan. She graduated from West Ottawa High School. After high school, she attended Western Michigan University in Kalamazoo, Michigan, where she earned a Bachelor of Arts in Mathematics Education. A year later, she relocated to Raleigh, North Carolina, where she spent a decade teaching mathematics to students in the Wake County Public School System and at Wake Technological Community College.

Kristi earned a Master's of Education in Mathematics Education from North Carolina State University. She then continued at North Carolina State University to earn her Doctorate of Philosophy in Mathematics Education. During her time as a graduate student, she worked as a teaching assistant for the course Methods of Teaching Mathematics in Secondary Schools and supervised secondary mathematics student teachers. She also worked as a research assistant on the Fraction Activities and Assessments for Conceptual Teaching (FAACT), Wolfpack WORKS, and Pathways to Practice North Carolina projects. She also worked in the Individualized Learning Center at Wake Technological Community College.

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Chapter 1: Introduction

Problem

The domains of trigonometry (triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry) appear in the algebra, geometry, and function standards of the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics (NCTM), 1989), *Principles and Standards for School Mathematics* (NCTM, 2000), and the Common Core State Standards Initiative for Mathematics (CCSSI) (CCSSI, 2010). Previous research has included the three trigonometric domains of: triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs (Demir & Heck, 2013). Some research also includes a fourth domain: analytic trigonometry. This fourth domain of analytic trigonometry has been identified by other researchers to be an essential context for students to reason trigonometrically (Delice, 2002; Delice et al., 2009). Triangle trigonometry is typically the first introduction students have to trigonometry and includes problems that require students to find the length of missing sides and measures of unknown angles in triangles. Usually, this will consist of the use of trigonometric ratios, inverse trigonometric functions, and the Laws of Sines and Cosines. Unit circle trigonometry includes the unit circle and related concepts, such as angle measure and radians. Trigonometric functions and graphs include the graphs of all six trigonometric functions, the idea of periodicity, and the effects of changing the values of parameters in the equations on the graphs. Analytic trigonometry includes solving trigonometric equations and verifying trigonometric identities, as well as applications such as infinite series in calculus.

Because trigonometry is taught across several courses, it can help students to make connections between algebra and geometry (Dejarnette, 2018). However, the domains of trigonometry are frequently presented in isolation from one another across the curriculum. Thus, students and pre-service teachers may have different ways of reasoning about trigonometry across these various domains.

Trigonometry has many applications in science and engineering that make it useful for students to learn and can be used in many STEM careers (Galle & Meredith, 2014; Gathing, 2011; Sultan, 2011). For example, students use trigonometry in designing bridges (Gathing, 2011) or determining how robots move (Sultan, 2011). Other applications include calculating the distance between planets (Rosenkrantz, 2004), uncovering an unknown location (Goetz, 2016), and solving an engineering question about a rotating arm (Rule, 2006). All of these applications move beyond the traditional Ferris wheel (Kaplan, 2008) and circular motion (Wilcock, 2013) problems frequently used in mathematics courses to introduce trigonometry and provide stronger motivation for why students should learn about trigonometry.

Trigonometry is a problematic topic in mathematics for many students, including pre-service teachers (Akkoç & Akbaş Gül, 2010; Calzada & Scariano, 2006; Ross et al., 2011; Thompson, 2008; Tuna, 2013). For example, despite having learned some trigonometry in a previous course, only four high school students out of 178 in a study by Kendal & Stacey (1998) were able to correctly answer even a single question on a pretest of their knowledge. Pre-service teachers showed misconceptions about radians and tended to rely on experience with degrees when evaluating basic trigonometric functions (Tuna, 2013). Even mathematics graduate students in a study by Yiğit Koyunkaya (2016) showed a primarily procedural understanding of right triangle trigonometry. Hypotheses about why these difficulties occur in trigonometry include students not being taught the connection between right triangle trigonometry and the unit circle (Markel, 1982), missing the relationship between the unit circle and trigonometry graphs

(May & Courtney, 2016; Peterson et al., 1998), students experiencing difficulties connecting abstract and concrete representations (Ross et al., 2011), and students not comprehending the connections between the domains of trigonometry (Thompson, 2008).

Previous research has shown that students at all levels have a predominantly procedural understanding of trigonometry across all domains. The purpose of the proposed research study is to look beyond procedural fluency to examine how pre-service secondary mathematics teachers draw upon their conceptual understanding and use mathematical reasoning when solving high cognitive demand trigonometry problems.

Other researchers have used several theories to examine students' knowledge of trigonometry. Martínez-Planell and Delgado (2016) used APOS to examine what students who had completed a college trigonometry course knew about inverse trigonometric functions. DeJarnette (2018) assessed high school students' work on trigonometric functions in the context of a Ferris wheel problem using the cK¢ model. Weber (2005) analyzed students' mental and physical actions when estimating values on the unit circle using the notion of procept (Gray & Tall, 1994). Moore (2009, 2012, 2013, 2014) used a lens of radical constructivism to analyze both student's and pre-service teacher's understanding of trigonometric concepts. This wide range of theories and frameworks used by researchers shows that no one framework is dominant in the research about trigonometry.

This study will utilize the Multidimensional Problem-Solving Framework by Carlson and Bloom (2005) and the Mathematical Understanding for Secondary Teaching (MUST) Framework (Heid et al., 2015). The first framework is the result of analyzing the problem-solving behaviors of expert mathematicians. The framework includes problem solving phases (e.g., orienting, planning, executing, checking) and problem solving attributes (e.g., resources, heuristics, affect, monitoring). This framework has been used previously to analyze the problem-solving of pre-service teachers in other mathematical domains (Andrews & Xenofontos, 2015; Chamberlin, 2018; Koichu & Kontorovich, 2013).

The Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005) was used to examine the problem-solving process of the participants. This framework was chosen because it allows analysis of both the progression that the pre-service secondary mathematics teacher uses to solve the tasks and the knowledge and strategies they draw upon to solve the tasks. The Mathematical Understanding for Secondary Teaching (MUST) Framework (Heid et al., 2015) was used for further analysis of the mathematical understandings displayed by the pre-service secondary mathematics teacher as they solved the high cognitive demand trigonometry tasks.

Even with the acknowledged difficulties for students, trigonometry is an under-researched branch of mathematics content (Akkoç, 2008; Topçu, Kertil, Akkoç, Yilmaz, & Önder, 2006; Weber, 2008), with most of the research occurring after 2000. Thus, this research will add to the body of literature concerning trigonometry. The majority of the studies about trigonometry are on the subject of the understandings of high school and college students (Bagni, 1997; Dejarnette, 2018; Martínez-Planell & Delgado, 2016; Weber, 2005). Because trigonometry is spread across the secondary curriculum, all pre-service secondary mathematics teachers need to have a strong understanding of the various domains that they will be expected to teach (Ball et al., 2001; Eisenhart et al., 1993; Ferrini-Mundy & Findell, 2001; Murray & Star, 2013). Despite the importance of this content domain across the high school mathematics curriculum, most pre-service secondary mathematics teachers do not take a course that is focused on trigonometry beyond what they studied in high school.

Research Questions

This study will address the following research questions:

1. How do pre-service secondary mathematics teachers engage in problem solving phases (e.g., orienting, planning, executing, checking) when solving high cognitive demand trigonometric tasks?
    a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?
2. How do pre-service secondary mathematics teachers draw upon problem solving attributes (e.g., resources, heuristics, affect, monitoring) as they solve high cognitive demand trigonometric tasks?
    a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?
3. How do pre-service secondary mathematics teachers engage in mathematical proficiency (e.g., conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (e.g., mathematical noticing, mathematical reasoning, and mathematical creating) as they solve high cognitive demand trigonometric tasks?

Conceptual Framework

This research will use the conceptual framework of the Mathematical Understanding for Secondary Teaching (MUST) framework (Heid et al., 2015). This framework views the knowledge needed by secondary mathematics teachers from the three perspectives of mathematical proficiency, mathematical activity, and mathematical context of teaching. Pre-service secondary mathematics teacher knowledge of trigonometry will lie within the strands of strategic competence in mathematical proficiency and mathematical reasoning in mathematical activity in this research.

The MUST framework views the teaching of secondary mathematics from the three perspectives of mathematical proficiency, mathematical activity, and mathematical context for teaching. Within these perspectives, this research focused on the mathematical proficiency and mathematical activity strands that the pre-service teachers who participated in the study demonstrated as they solve the high cognitive demand trigonometry tasks. Mathematical context for teaching was not included as a strand to be analyzed since the focus of the research was on the participants' content knowledge, rather than their pedagogical knowledge. Though two of the tasks do address some pedagogical knowledge, the focus was on content knowledge.

The strands of mathematical proficiency include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Because the tasks were chosen to address the four trigonometric domains, historical and cultural knowledge did not end up being addressed by any of the tasks. Thus, this research was only analyzed for the pre-service secondary mathematics teachers' use of the first five strands of mathematical proficiency. The strands of mathematical activity include mathematical noticing, mathematical reasoning, and mathematical creating. Each of the strands of mathematical activity is broken down into further strands. Mathematical noticing includes the strands of structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Mathematical reasoning includes the strands of justifying/proving, reasoning when conjecturing and generalizing, and constraining and extending. Mathematical creating includes the strands of representing, defining, and modifying/transforming/manipulating. The strand of structure of mathematical systems strand was not present in any of the tasks that were chosen to represent the four trigonometric domains. All of the remaining strands of mathematical activity are addressed by the tasks and will be analyzed.

The MUST framework and the Multi-Dimensional Problem-Solving Framework complement each other in the analysis of the pre-service secondary mathematics teaches' problem-solving. The Multi-Dimensional Problem-Solving Framework was first used to analyze the problem-solving process each participant engaged in. This included analyzing the phases that each participant progressed through and the knowledge from each trigonometric domain and problem-solving strategies that were used to solve each task. The MUST framework was then used to analyze the mathematical understandings that were showcased by each pre-service secondary mathematics teacher as they solved the high cognitive demand trigonometry tasks in the interviews.

Research Methods

The study used the qualitative design of a case study (Creswell, 2013). The participants, who were pre-service secondary mathematics teachers from a university in the southeastern United States. The research consisted of task-based interviews for each participant to investigate the problem solving phases engaged in (e.g., orienting, planning, executing, checking) and the problem solving attributes drawn upon (e.g., resources, heuristics, affect, monitoring) by pre-service secondary mathematics teachers when solving high cognitive demand trigonometry tasks. The analysis of the problem-solving process of the pre-service secondary mathematics teachers when solving trigonometric tasks utilized the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005). This framework has previously been used to analyze the problem-solving of elementary to undergraduate students (Aljaberi & Gheith, 2015; Elia et al., 2009;

Furinghetti & Morselli, 2009; Hannula, 2015), teachers (Andrews & Xenofontos, 2015; Chamberlin, 2018; Kuzle, 2011; Yimer & Ellerton, 2010), and mathematicians (Harlim & Belski, 2013). Further analysis of their work used the MUST Framework to analyze the mathematical understandings each pre-service secondary mathematics teacher showed in their work on the tasks.

Researcher Positionality

I completed my undergraduate studies in mathematics education and spent more than a decade teaching mathematics to high school and community college students, so my background influenced how I interpreted the knowledge of the pre-service teachers in the study. Additionally, I spent two years working with pre-service secondary mathematics teachers in a methods course and supervising student teachers. Though I now believe I have a strong understanding of the domains of trigonometry and their connections, I did not feel well prepared by my own pre-service secondary mathematics education. I attempted to view the knowledge of my participants through a conceptual frame, rather than through a lens of my own experiences. I also needed to be careful not to let my experiences working with students in the classroom influence what I believe that the participants need to know to teach trigonometry successfully. For example, I made sure not to read into what knowledge my participants were drawing upon based on what I have seen students do in my classroom (Bourke, 2014).

Overall, I needed to be mindful of the fact that I have my own experiences with mathematics content on my path from being a pre-service to a novice to an experienced mathematics teacher and did not to let them influence my interpretation of the knowledge of the pre-service secondary mathematics teachers that participated in the study. As I completed and analyzed my research, I was reflective of how my background and experience affected my interest in collecting and interpreting my data. I also bracketed off my own experiences as I gathered and analyzed the data (Creswell, 2013).

Rationale and Significance

With little research available on what pre-service teachers know about trigonometry, this study will begin to fill a gap. Though some research exists on what pre-service secondary mathematics teachers' knowledge of trigonometry, it tends to be narrow in its focus. For example, Moore, LaForest, and Kim (2016) examined the knowledge of pre-service teachers about the unit circle, Akkoç (2008) examined pre-service teachers' knowledge of the concept of radians, and Kahan, Cooper, and Bethea (2003) examined the relationship between content knowledge and planning to teach periodicity. While these studies and others provide insight into a single domain of trigonometry, this study will examine knowledge of pre-service secondary mathematics teachers' knowledge across all four domains of trigonometry.

Limitations

There are several limitations to this study. Because the study used a convenience sample of pre-service secondary mathematics teachers from a single university, the results of the study may not generalize to other pre-service secondary mathematics teachers. This research assumed that there are four domains of trigonometry, but there are significant connections between the domains. Thus, the researcher needed to be aware of how pre-service secondary mathematics teachers' knowledge within and across domains influences their work. The pre-service secondary mathematics teachers in the study had not taken a course in trigonometry since they were high school students, which may mean that their difficulties are a result of time lapsed from one to three years, rather than lack of knowledge. Because of differences in interviewers and participants, task-based interviews do not lead to generalizable results (Goldin, 2000). The interventions and questions asked by the interviewer can also influence the direction of the participants' thinking and work on the task (Koichu & Harel, 2007), and this was important to consider during analysis.

Chapter 2: Literature Review

Trigonometry is addressed across the high school mathematics curriculum. Within *Curriculum and Evaluation Standards* and *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics (NCTM), 1989, 2000), trigonometry appears in both the algebra and geometry standards. Within the *Common Core Standards Initiative for Mathematics* (CCSSI), trigonometry appears in the strands for algebra, geometry, and functions. Because trigonometry appears across the mathematics standards, it can provide a bridge for students to connect their knowledge of algebra and functions to their understanding of geometry (Calzada & Scariano, 2006). Students need to connect their knowledge of the four trigonometric domains: triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. For this to happen, "… [students] must understand angle measure, the unit circle, and right triangles in ways that let them see trigonometric functions as relations between two quantities" (Moore & LaForest, 2014, p. 617). Thus, teachers need to have a strong understanding of the connections themselves, so that they can help their students to make these connections as well.

Trigonometry is useful for students because it has many real-life applications, including the traditional Ferris wheel problem (Dejarnette, 2018; Kaplan, 2008), within the physical sciences (Kramer, 1948), and robotics (Sultan, 2011). Students can see applications of trigonometry when designing bridges (Gathing, 2011) or solving physics problems (Galle & Meredith, 2014). Students can also use trigonometry to understand how it resolves challenges in their everyday lives, such as noise-canceling headphones (McCulloch et al., 2017) or finding the best solution in a video game (Lamb, 2013). Other applications of trigonometry include determining where the light from a lighthouse can be seen (Wilcock & Haynes, 2012), calculating orbits of planets (Rosenkrantz, 2004), and solving an engineering problem involving a robotic arm (Rule, 2006). The wide variety of applications of each domain of trigonometry means that many students will use trigonometry in their future careers.

Trigonometry is a difficult topic for many students and pre-service secondary mathematics teachers to master (Bressoud, 2010; Gür, 2009; Kendal & Stacey, 1996; Moore et al., 2012; Nejad, 2016; Ross et al., 2011; Thompson, 2008). Though many pre-service secondary mathematics teachers have procedural knowledge of trigonometry, they lack conceptual knowledge of the content (Bryan, 1999). In a study by Tuna (2013), 60% of the pre-service mathematics teachers were able to define the concept of a degree correctly, but only 8% of the same pre-service teachers were able to define the concept of a radian accurately, and only 17% were successfully able to identify that there are 2π radians in a full circle. If pre-service teachers have these misunderstandings of concepts underlying trigonometry, they will likely pass them along to their future students (Moore et al., 2012).

Despite the recognition that trigonometry is an important topic in secondary mathematics and that many students and pre-service teachers struggle to understand trigonometry, "… relatively little research has focused on this subject" (Weber, 2008, p. 144). Though some research has been conducted, most of it has taken place since 2000 and has concerned the learning of high school students (Akkoç, 2008; Topçu et al., 2006; Weber, 2008). This scarcity of research leaves many unanswered questions about how to best teach specific trigonometric concepts (Martínez-Planell & Delgado, 2016).

Conceptual Framework

The Mathematical Understanding for Secondary Teachers (MUST) framework details the mathematical understandings that secondary teachers need to have to do their work with students

(Heid et al., 2015). This framework focuses specifically on secondary mathematics teachers (and by extension pre-service secondary mathematics teachers) because the knowledge needed by secondary mathematics teachers is distinct from the knowledge required by elementary mathematics teachers. This difference is because secondary mathematics involves a broader range of mathematical content, includes a greater emphasis on formal mathematics and proof, consists of a focus on abstraction, and secondary students can reason about mathematics differently from elementary students.

The MUST framework incudes the three perspectives of mathematical proficiency, mathematical activity, and mathematical context for teaching (Heid et al., 2015). Mathematical proficiency "… includes aspects of mathematical knowledge and ability … that teachers need themselves and that they seek to foster in their students" (p. 11). Mathematical activity "… can be thought of as 'doing mathematics'" (p. 11) and includes "… specific mathematical actions in which the teacher should be able to engage" (p. 11). Mathematical context of teaching is what differentiates teachers from other professions that use mathematics and includes knowledge of teaching, students, and curriculum. Secondary mathematics teachers require knowledge from all three perspectives. All three perspectives are intertwined as secondary mathematics teachers use their mathematical proficiency to participate in mathematical activity while engaged in the mathematical context of teaching.

Within mathematical proficiency are the strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Mathematical activity includes the strands of mathematical noticing, mathematical reasoning, and mathematical creating. Mathematical context of teaching consists of the strands of probing mathematical ideas, accessing and understanding the mathematical thinking of learners, knowing and using the curriculum, assessing the mathematical knowledge of learners, and reflecting on the mathematics of practice. These can be seen in Figure 1.

1. **Mathematical Proficiency**
    - Conceptual Understanding
    - Procedural Fluency
    - Strategic Competence
    - Adaptive Reasoning
    - Productive Disposition
    - Historical and Cultural Knowledge

2. **Mathematical Activity**
    - Mathematical noticing
        - Structure of mathematical systems
        - Symbolic form
        - Form of an argument
        - Connect within and outside mathematics
    - Mathematical reasoning
        - Justifying/proving
        - Reasoning when conjecturing and generalizing
        - Constraining and extending
    - Mathematical creating
        - Representing
        - Defining
        - Modifying/transforming/manipulating

3. **Mathematical Context of Teaching**
    - Probe mathematical ideas
    - Access and understand the mathematical thinking of learners
    - Know and use the curriculum
    - Assess the mathematical knowledge of learners
    - Reflect on the mathematics of practice

Figure 1

Mathematical Understanding for Secondary Teaching Framework (p. 14)

Specifically, this research will focus on the strands of mathematical proficiency and mathematical activity (Heid et al., 2015). The framework defines mathematical proficiency as

> … aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students. The mathematical proficiency that teachers need goes well beyond what one might find in secondary students or even in the average educated adult (p. 11).

Some examples of mathematical proficiency in trigonometric contexts might include:

1. Seeing the connections between trigonometric domains

2. Recognizing when to use the law of sine or law of cosines and using them to solve for missing information in a given triangle
3. Figuring out how to break apart a trigonometric expression into known identities to simplify the expression
4. Working with the right triangle and unit circle definitions for trigonometric functions
5. Persevering when multiple attempts are needed to solve a problem
6. Knowing the history of the development of angle measurement in radians and degrees

As seen in Figure 1, mathematical proficiency includes six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. The first five strands were potentially addressed in solving the nine interview tasks used in the current study.

The MUST framework defines mathematical activity as “doing mathematics” (p. 11). The three strands of mathematical activity are mathematical noticing, mathematical reasoning, and mathematical creating. Mathematical noticing “involves recognizing similarities and differences in structure, form, and argumentation both in mathematical settings and in real-world settings” (p. 18) and it “… requires identifying mathematical characteristics that are particularly salient for the purpose at hand…” (p. 18). Some examples of mathematical noticing in trigonometric contexts include:

1. Awareness that $f(\alpha) + f(\beta) \neq f(\alpha + \beta)$ for trigonometric functions
2. Recognizing missing or redundant portions of a trigonometric proof
3. Noticing the trigonometric models present in real-world applications

Mathematical reasoning "… includes activities such as justifying and proving as well as reasoning in the context of conjecturing and generalizing" (p. 21) and "… results in the production of a mathematical argument or mathematical rationale that supports the plausibility of a conjecture or generalization" (p. 21). Some examples of mathematical reasoning in trigonometric contexts include:

1. Justifying why "doing the same thing to both sides of an equation" does not apply when verifying an identity
2. Analyzing properties of a set of trigonometric values to determine properties that are shared
3. Extending the concepts of right triangle trigonometry to angles more than 90° to create the unit circle

Mathematical creating "entails generating new ways to convey a mathematical object … and transformations of existing representations of a mathematical entity" (p. 23) and "[t]he essence of mathematical creating is the production of new mathematical entities" (p. 23). Some examples of mathematical creating in trigonometric contexts include:

1. Creating a set of examples and non-examples of a trigonometric concept
2. Defining the sine of an angle in multiple trigonometric domains and connecting these definitions for students
3. Performing symbolic manipulations on a trigonometric identity to obtain a desired trigonometric identity

The framework describes the difference between mathematical proficiency and mathematical activity as:

> Whereas the mathematical proficiency perspective describes the general types of mathematical understandings that a teacher should have and use, the mathematical activity perspective describes specific mathematical actions in which the teacher should be able to engage (p. 11).

Both of these perspectives and their strands will be analyzed in conjunction with the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005). Since this research is focused on the knowledge of the teacher and not their enactment of the knowledge when teaching, it will not focus on any strands from the mathematical context of teaching perspective.

Research on Secondary Mathematics Teacher Knowledge

There are many types of knowledge that secondary mathematics teachers need. High school mathematics teachers need a strong understanding of the mathematics content they are expected to teach (Ball et al., 2001; Even, 2011; Harel, 2008; Usiskin, 2001). The achievement of students is related to the content knowledge of their teachers (Ball et al., 2008; Begle, 1972; Hill et al., 2005; Monk, 1994). There is not enough time during their training for secondary mathematics teachers to gain all of the content knowledge they need (CBMS, 2012; Heid et al., 2015). In a study of pre-service secondary mathematics teachers and secondary mathematics teachers who would be or were licensed to teach seventh grade, van der Sandt and Nieuwoudt (2003) found that only 38% of the secondary mathematics teachers and 24% of the pre-service secondary mathematics teachers scored at a level 3 or above on a test of van Hiele levels. Secondary mathematics teachers also exhibit misconceptions that are similar to students about the concept of functions, such as a function can only be represented by an equation (Even, 1993). If secondary mathematics teachers have gaps in their knowledge of mathematics content, they will pass those gaps along to their students.

PCK, MKT, and Other Forms of Knowledge

Beyond content knowledge, teachers need other forms of knowledge, such as pedagogical content knowledge (PCK) and mathematical knowledge for teaching (MKT) (Ball et al., 2008; Shulman, 1986). Both PCK and MKT are part of the "complex bod[y] of knowledge and skills" (Shulman, 1987, p. 4) required by teachers. Research has shown that teachers' MKT is a statistically significant predictor of student achievement gains, with the most considerable difference noted for teachers with MKT in the lowest deciles (Hill et al., 2005), which means that it is an essential form of knowledge for mathematics teachers to acquire.

According to Shulman and Shulman (2004), accomplished teachers possess the characteristics of being ready, willing, able, reflective, and communal. Thus teachers need to have both the content and pedagogical knowledge to implement learning in their classrooms, as well as the beliefs that students can learn the material they are teaching. The beliefs that teachers hold about students and the nature of mathematics are related to the instruction teachers implement in their classrooms (Thompson, 1984). Neither beliefs nor content knowledge is enough on its own, though. Teachers need both to be able to implement mathematics instruction that aims for conceptual understanding (Charalambous, 2015). Studying advanced mathematics can impact teachers' beliefs and practices in the classroom, resulting in teachers focusing more on conceptual understanding (Wasserman, 2014). Helping teachers to address any gaps in the characteristics named by Shulman and Shulman (2004) can help them to become better teachers. The seeds of these characteristics can be planted when secondary mathematics teachers are being prepared to teach as pre-service secondary mathematics teachers and reinforced with professional development once they are in the classroom.

Content Knowledge of Pre-Service Secondary Mathematics Teachers

Mathematics content knowledge is an essential component of secondary mathematics teachers' knowledge, which means that the preparation of pre-service secondary mathematics teachers needs to focus on helping them to gain mathematics content knowledge. Pre-service secondary mathematics teachers have knowledge of the mathematics content from their own time as students, but it tends to be primarily procedural fluency, which is insufficient for the work of teaching (Ball, 1990; Heid et al., 2015). Research has shown that pre-service secondary mathematics teachers struggle to explain the content they will be expected to teach, even if it is elementary content, such as fractions (Depaepe et al., 2015; van der Sandt & Nieuwoudt, 2003).

Kahan et al. (2003) developed a framework to analyze teacher content knowledge that includes elements of teaching, such as goals, selection of tasks, and discourse, and processes of teaching, such as preparation, instruction, and reflection. This framework was applied to a cohort of student teachers, and the authors found that those who ranked highest in content knowledge tended to also rank highest in lesson planning. This study suggests that pre-service secondary mathematics teachers who have weaker content knowledge are less able to plan high-quality lessons for their future students and were more likely to emphasize procedural fluency in their lessons. However, even the student teacher with the highest-ranked content knowledge struggled to take advantage of teachable moments during the implementation of her lesson, so content knowledge alone is not sufficient to ensure the enactment of high-quality lesson plans.

Even when pre-service secondary mathematics teachers want to teach for conceptual understanding, they may be unable to do so because they lack conceptual understanding of the content themselves (Eisenhart et al., 1993). Both the development of PCK (Shulman, 1986) and the ability to teach for conceptual understanding depend on pre-service secondary mathematics teachers' content knowledge (Depaepe et al., 2015; Eisenhart et al., 1993). Morris et al. (2009) suggest that pre-service secondary mathematics teachers should learn to analyze learning goals for lessons, because this requires knowledge of the content and PCK, MKT. Pre-service secondary mathematics teachers need to have opportunities during their education to learn mathematics connected strongly with pedagogy to develop conceptual understanding and pedagogical content knowledge (CBMS, 2012; Eisenhart et al., 1993; Ferrini-Mundy & Findell, 2001; Murray & Star, 2013).

Learning Trigonometry Across Domains

Though most research on learning trigonometry has been published in the years since 2000, there is a long history of recommendations on how to teach trigonometry. The limited body of research on trigonometry shows that students have difficulty with all domains of trigonometry. In right triangle trigonometry, a study found that high school students relied on diagrams and memorized mnemonics like SOH CAH TOA (Pritchard & Simpson, 1999). In unit circle trigonometry, undergraduate students, pre-service teachers, and in-service teachers have difficulties with radians (Akkoç, 2008; Moore, 2010; Topçu et al., 2006). Trigonometric functions and graphs present many issues, including the concept of periodicity (Nejad, 2016) and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, Carlson, & Silverman, 2007) for undergraduate students and in-service teachers. In analytic trigonometry, high school students struggle to solve equations and verify identities because they rely on memorized formulas that they try to use without understanding (Bagni, 1997; Gür, 2009).

History of Teaching Trigonometry

Though research on trigonometry is fairly recent, recommendations on how to teach trigonometry have appeared in *The Mathematics Teacher* for over a century, with the earliest found in 1914. The advice from 1914 to the 1930's focused primarily on preparing students to study further mathematics in a college setting and related issues, such as the use of textbooks for formulas on examinations (Bailey, 1932; Glover, 1934; Tripp, 1919; Van Ingen, 1914). In the 1940's, recommendations were mixed between a focus on formal mathematics in trigonometry (Gottschalk, 1940; Newson & Randolph, 1946) and the practical uses of mathematics in trigonometry (Beaverman, 1946; Newson & Randolph, 1946). Specific proposals from the 1940's included combining Advanced Algebra and Geometry to allow students to see the connections between them and also allowing students to understand the usefulness of mathematics, including trigonometry, in their future careers.

The post-World War II 1950's brought a variety of suggestions about the teaching of trigonometry, including debate of formal (Schumaker, 1959; Tierney, 1957) vs. practical applications of trigonometry (Andree, 1955; Rosenberg, 1958). There was also a discussion about which grade trigonometry should be introduced to students (Bruns & Frazier, 1957; Schumaker, 1959; Steinen, 1957; Wolfe, 1956). A unique suggestion was that trigonometry should be taught in 8th grade as an extension of learning ratios (Steinen, 1957). A survey of large school districts found that most have students take algebra in ninth grade and geometry in tenth grade followed by a variety of courses, including trigonometry, for eleventh and twelfth grades (Bruns & Frazier, 1957). A comparison of college freshman found that those who had only a gap of summer before taking college trigonometry scored better on a pre-test, but students with a longer gap ended the course with higher final grades in the trigonometry course (Wolfe, 1956).

The early years of the space race in the 1960's and 1970's renewed a focus on formal uses of trigonometry (Rodgers, 1975; Satty, 1976; Troyer, 1968). Several suggestions for how to best present trigonometry were given, including historical development by Ptolemy (Brendan, 1965), the use of geometric functions and linear algebra (Troyer, 1968), formal definitions from the complex number system (Hamming, 1971), and combined right triangle trigonometry and unit circle trigonometry (Satty, 1976). Though these authors present multiple paths to trigonometry, they all suggest a more formal use of trigonometry.

The late 1970's and 1980's encouraged the teaching of trigonometry in a way that was relevant to students' lives and future careers (Lando & Lando, 1977; Saunders, 1980; Sconyers, 1986). Several suggestions from this time focused on graphing. Lando and Lando (1977) suggest using the real-life application of local yearly temperature data to determine a trigonometric function to model the data. Sconyers (1986) suggests that students should learn about other periodic functions before trigonometric functions so that students are not overwhelmed by so many new concepts at one time. Walton and Walton (1987) suggest that students create computer programs to graph polar functions.

Throughout the 1990's, there was a strong focus on teaching trigonometry in a manner that was consistent with the *Curriculum and Evaluation Standards* (Hirsch et al., 1991; Moody, 1992), which included plenty of applications and student explorations (Daniels, 1993; Stephens, 1997; Vonder Embse & Enbretsen, 1996; Zerger, 1993). Some of the applications suggested include archeology (Hersberger & Farlow, 1990), modeling the motion of a clock hand (Shultz & Bonsangue, 1995), locating a hidden treasure (Lewis, 1994), and creative writing (Barnes, 1999). There were also suggestions to make connections between geometry and trigonometry more explicit (Bonsangue, 1993; Seydel, 1994). Further recommendations included the use of dynamic geometry (Vonder Embse & Enbretsen, 1996) and connecting the unit circle to trigonometric graphs with spaghetti (Peterson et al., 1998). All of these suggestions place students at the center of their own learning about trigonometry and align with the standards presented by NCTM in 1989.

The early 2000’s to the present has offered a varied focus. Some have suggested a focus on the uses of trigonometric formulas, such as the law of sines and law of cosines (Bannon, 2018; Dobbs, 2001; Harrison, 2002; McMullin, 2003). Others have suggested building student knowledge through activities, such as determining how trigonometry is used in careers, using algorithms that calculators use, and walking a radian (Dorner, 2013; Keleher, 2006; Touval, 2009). A variety of activities (Kaplan, 2008; Lancaster & Bentele, 2011; Landers, 2013) and applications (Keiser, 2003; Madden et al., 2005; McCulloch et al., 2017; Özgün-Koca et al., 2013; Wilcock, 2013) were suggested for helping students to learn about trigonometric functions and graphs. Other suggestions include using history and mathematics (Bressoud, 2010; Caglayan, 2013; Rosenkrantz, 2004; Santucci, 2011). The focus on teaching trigonometry has shifted from a very formal teacher-centered form of instruction with the goal of preparing students to study higher-level mathematics in college during the early to mid-1900s to a more useful student-centered approach with a goal of letting students explore and see the usefulness of trigonometry during the late 1900s and early 2000s.

Triangle Trigonometry

Triangle trigonometry is frequently the first introduction students have to trigonometry. Matos (1990, 1991) suggests that students learn triangle trigonometry similar to the way that trigonometry was developed so that students can build up an understanding of the concepts of angles and the uses of trigonometric ratios. The six trigonometric ratios used today were first developed by Arabic cultures in the 1200’s (Massa et al., 2006), and the earliest uses of trigonometry were to calculate distances in astronomy in ancient times (Massa et al., 2006;

Matos, 1990). Though the historical development of these concepts may not mirror modern student learning of the concepts, they can provide an insight as to why the concepts were developed and used.

Pritchard & Simpson (1999) performed a series of task-based interviews after high school students learned right triangle trigonometry and determined that the students relied on diagrams and visual representations to solve the triangle trigonometry problems. Students created sketches to identify whether the given information was the opposite side, adjacent side, or hypotenuse and used that information to write an equation with a trigonometric ratio of sides and then solve the equation. Pritchard & Simpson (1999) argue that the reliance on diagrams is because students do not have a significant concept image of what trigonometric ratios mean because students are given definitions of the trigonometric ratios before they become familiar with their meanings. Rather than just memorizing the common mnemonic SOH CAH TOA to remember $\sin\theta = \frac{opposite}{hypotenuse}$, $\cos\theta = \frac{adjacent}{hypotenuse}$, and $\tan\theta = \frac{opposite}{adjacent}$ for right triangles, students need to have an understanding of trigonometric ratios and how to solve problems involving them. Students should understand that the trigonometric ratios result from similar triangles and will remain constant.

Inverse Trigonometry

Martínez-Planell and Cruz Delgado (2013, 2016) interviewed college students about their knowledge of inverse trigonometric concepts. These students had completed a trigonometry course that had been taught in a traditional manner, though this was not described. The students who had the highest grades in the class were found to only be in transition from an action to a process conception when they were asked to solve problems relating inverse trigonometry to the unit circle. The majority of students were found to be in a pre-action conception, despite passing the course. All of the students had difficulty relating angle measure and arc length, as well as difficulty connecting the domains of triangle trigonometry and unit circle trigonometry. These difficulties show that even students who are successful in coursework on trigonometry do not have a deep understanding of the content they are learning and rely on memorized facts, which is supported by other research on trigonometric learning (Bagni, 1997; Gür, 2009; Moore, 2012, 2009; Thompson et al., 2007; Weber, 2005).

Unit Circle Trigonometry

There is limited research available on how students understand the unit circle. Much of the available research concerns students' understanding of the concepts of radians and angles, which are foundational to unit circle trigonometry (Moore, 2010).

Radians and Angles

Radians are a key to making sense of trigonometry. Still, many students, pre-service teachers, and in-service teachers do not have a strong understanding of the relationship between a radian and the radius of a circle (Akkoç, 2008; Maknun et al., 2018; Moore, 2010; Topçu et al., 2006). Students learn about measuring angles in degrees as early as elementary school, but do not learn about measuring in radians until high school (CCSSI, 2010), which leaves many students confused about the necessity of radians. Historically, the development of measuring angles with radians preceded the development of degrees (Matos, 1990, 1991). Akkoç (2008) found that pre-service teachers' concept of radians was dominated by their concept of degrees. For instance, over 90% of the pre-service teachers in Akkoç's study considered $\sin(30)$ to be $\sin 30°$ despite the lack of the degree symbol and follow up interviews showed that many believed an angle was only in terms of radians if it was in terms of π. High school calculus students in a study by Orhun (2010) relied on converting from radians to degrees when solving problems. This overreliance on concepts of degrees persists for in-service teachers (Topçu et al., 2006). If teachers have difficulties with the concept of radians, they are likely to pass along these difficulties to their students. Without a strong understanding of foundational concepts such as radians, students are likely to have obstacles with the rest of trigonometry.

However, research has shown that students can develop a strong understanding of radians. In an experiment designed to address common difficulties with radians, Akkoç and Akbaş Gül (2010) found that when common misconceptions about radians were addressed during lessons, high schools students overcame three common challenges, including seeing real numbers as radians, defining radians and seeing them as a length, and using the number π. Similarly, after a teaching experiment that presented students with tasks designed to develop their understanding of the radius as a unit of measure, Moore (2010) found that undergraduate pre-calculus students were able to "… [develop] an image of the radius such that they can imagine measuring quantities relative to the radius … [and are] prepared to coherently use the radius and trigonometric functions in multiple contexts" (p. 821). Pre-service teachers need to engage in these types of experiences with the trigonometric content they will be expected to teach so that they can confront any misconceptions and gain a strong conceptual understanding before they are in the classroom.

In one teaching experiment, Moore (2009) designed a series of lessons for a small group of undergraduate trigonometry students to explore the relationship between angle measure, radian, and the unit circle. The use of dynamic applets in Geometer's Sketchpad helped the students participating in the experiment to overcome a weak understanding of the connection between triangle trigonometry and unit circle trigonometry by building a more robust comprehension of the relationships between angle measure, radian, and the unit circle. In order to examine how students develop an understanding of angles, Moore (2010, 2013) designed a second teaching experiment that used Geometer's Sketchpad for a small group of undergraduate trigonometry students to develop the concept of the radius as a unit of measure. The student with the most substantial concept of angle was the student who was most able to solve problems and make further connections during the teaching experiment (2013). Still, all students were able to build a more robust understanding of degrees and radians by the end of the teaching experiment. While both of these were small qualitative studies, they do provide an insight into how students come to understand angle measure and its connections to other topics in trigonometry. Experiences such as these can deepen the knowledge of pre-service secondary mathematics teachers and the activities they complete may be modified for their future students.

Trigonometric Functions and Graphs

Students typically first encounter periodic functions when they learn about graphing sinusoidal and other trigonometric functions, but other periodic functions, such as music notes and lunar cycles, could be introduced to students before learning about trigonometry (Kramer, 1948; Minerva Martinez et al., 2017; Sconyers, 1986). The many real-life applications of trigonometric graphs, such as the sound waves created by snoring, average temperature over a year, and the amount of light visible during a solar eclipse, can help students make sense of the effects of the parameters for amplitude, period, phase shift, and vertical shift (Barnes, 1999; Kaplan, 2008; Landers, 2013; Lando & Lando, 1977; Madden et al., 2005; McCulloch et al., 2017; Shultz & Bonsangue, 1995).

Periodicity and Parameters on Graphs

Connecting triangle trigonometry with the unit circle can help students see why the shape of the graph of $y = \sin x$ is curved and not straight (Peterson et al., 1998). High school students who learned about graphing of sinusoidal functions beginning with the unit square and increasing the number of sides to approach the unit circle were able to make deep connections between the unit circle and the properties of the graphs of sinusoids (Demir & Heck, 2013). Students should also need to develop an understanding of periodicity to be able to prove that $f(x) = \sin(3x + 5)$ will repeat when *x* varies by a multiple of $\frac{2\pi}{3}$ (Thompson et al., 2007) and show that functions such as $g(x) = \cos(x^2)$ are not periodic (Dreyfus & Eisenberg, 1980).

Graphing calculators and other graphing software can help students to explore the effects of the parameters in sinusoidal functions on their graphs (Choi-Koh, 2003; Dejarnette, 2018; Demir & Heck, 2013; Kepceoğlu & Yavuz, 2016; Ng & Hu, 2006). Because graphing calculators and other graphing programs can allow students to see many graphs, students can quickly explore relationships, rather than just viewing a few static images (Choi-Koh, 2003). Technology can also help relieve the tedium of creating multiple graphs by hand since dynamic graphs can show the changes as students vary each parameter (Ross et al., 2011). By allowing the technology to do the work of creating the graphs, students can explore features of the graphs and make stronger connections between the graphs and equations. Graphing technology allows high school students to receive immediate feedback about their work as they translate from word problems to equations to graphs, which can increase their understanding of the relationships between the forms of the problem (Dejarnette, 2018). The use of graphing technology results in high school students who have higher achievement on researcher-created tests of their knowledge of graphing trigonometric functions than their peers who do not learn with graphing technology (Kepceoğlu & Yavuz, 2016; Ng & Hu, 2006).

Function Concept

Functions are a foundational topic through mathematics from algebra to calculus (CCSSI, 2010; NCTM, 2000). The mathematical definition of a function has changed over time (Gök et al., 2019; Malik, 1980), as has the emphasis on the concept of function in the mathematics curriculum (CCSSI, 2010; Gök et al., 2019; NCTM, 1989, 2000). Even though this is a foundational concept throughout the mathematics curriculum, students demonstrate a weak understanding of function (Carlson, 1998; Carlson et al., 2007; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). These difficulties persist in pre-service secondary mathematics teachers (Even, 1993; Sánchez & Llinares, 2003) and in-service mathematics teachers (Even, 1990, 1993). Students need a deep understanding of functions to build covariational reasoning and quantitative reasoning (Moore & Carlson, 2012).

The concept of inverse functions also presents difficulties for students and pre-service teachers. Many pre-service secondary mathematics teachers in a teaching experiment relied on the technique of switching *x* and *y* coordinates to find the inverse algebraically, but were unable to explain the meaning of their result consistently or implement a switching method graphically (Paoletti et al., 2018). Though an undergraduate student in a teaching experiment by Paoletti (2018) was able to reorganize parts of her meaning of the concept of functions, she was unable to shift her understanding of inverse functions. Weak understandings of the inverse function concept will lead to difficulties understanding inverse trigonometric functions and the relationships between the inputs and outputs of trigonometric functions and inverse trigonometric functions.

With trigonometric functions, students show similar misconceptions to functions in general. For example, when creating a graph of the height of a person riding a Ferris wheel over time, high school students initially created a circle like in the context, rather than a sinusoidal graph showing the height over time (Dejarnette, 2018). When reasoning about the graph of $y = \sin^{-1}\theta$, the undergraduate students in a teaching experiment were able to successfully create the graph by switching known coordinates from the graph of $y = \sin\theta$, but unable to reason about whether the new graph showed the same relationship as the original graph (Paoletti, 2018). Without a strong understanding of the concepts of function and inverse function, using these in trigonometric contexts becomes difficult for students.

Covariational Reasoning and Quantitative Reasoning

Another difficulty with understanding the graphs of trigonometric functions is the difficulty with covariational reasoning and quantitative reasoning (Moore, 2012, 2013, 2014). Carlson et al. (2002) define *covariational reasoning* as "… the cognitive activities involved in coordinating varying quantities while attending to the ways in which they change in relation to each other" (p. 54). Moore (2013) defines *quantitative reasoning* as "… an individual's mental actions when conceiving of a situation, constructing measurable attributes of the situation (which are called quantities) and constructing about relationships between conceived quantities" (p. 228). These two types of reasoning are vital to understanding the connections across the domains of trigonometry (Moore, 2013, 2014; Moore et al., 2012), because students need to be able to reason about how the angle and the value of the trigonometric function vary together. Even students who had completed Calculus 2 with an A struggle with covariational reasoning tasks (Carlson et al., 2002). Given that these types of reasoning are challenging for students who have completed Calculus 2, it shouldn't be surprising that covariational reasoning and quantitative reasoning are also problematic for high school and undergraduate trigonometry students.

When students are given tasks specifically designed to address quantitative reasoning, their skills can improve (Moore, 2013, 2014). As the undergraduate students in a series of teaching experiments developed better covariational reasoning and quantitative reasoning skills, they were better able to make connections across tasks, which shows that students are capable of improving their covariational reasoning and quantitative reasoning skills if they are specifically focused on when teaching trigonometry. Though covariational reasoning and quantitative reasoning can be difficult, when they are addressed explicitly through teaching, they are accessible to high school and undergraduate trigonometry students.

Analytic Trigonometry

Trigonometric equations and identities present several difficulties for students. Among the misconceptions that high school students show are logically invalid inferences, incorrect definitions, and misused data (Gür, 2009). Over half of the high school students in a study believed there was a solution to $\sin x = \frac{\pi}{3}$ or $\sin x = \sqrt{3}$, despite the range of the sine function being $[-1, 1]$ (Bagni, 1997). Another study found that nearly half of high school students were unable to use the property of the cosine function is even to justify why $\cos(-\theta) = \cos(\theta)$ (Gür, 2009). In follow-up interviews, Gür (2009) found that the majority of the high school students were unable to justify trigonometric identities because they had relied on memorizing the formulas without a conceptual understanding of why they were true.

Students must first recognize and recall trigonometric properties before they are able to complete the simplification to simplify a trigonometric expression or prove a trigonometric identity (Delice, 2002). Another concern with proving trigonometric identities is dealing with indeterminate forms, since students do not have to deal with that issue with many other types of equations (Tripp, 1919). These research studies suggest that students may be able to complete a procedure of solving trigonometric equations or proving trigonometric identities. Still, if they rely on memorizing trigonometric properties without a conceptual understanding of why those properties are correct, they will have only procedural fluency in these skills.

Disconnect Between Trigonometric Domains

Another common difficulty in learning trigonometry is that students do not understand the connections between the various domains of trigonometry, including triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry (Moore, 2009; Moore et al., 2012; Yiğit Koyunkaya, 2016). This difficulty persists even in preservice teachers (Moore et al., 2012) and mathematics graduate students (Yiğit Koyunkaya, 2016), so it is not surprising that it is a challenge for high school and undergraduate students first learning trigonometry to understand these connections. Mathematics graduate students, who we would expect to have a strong understanding of mathematics content, relied on knowledge of the SOH CAH TOA mnemonic when asked to justify trigonometric identities involving right triangles and sine and cosine (Yiğit Koyunkaya, 2016), but were unable to explain the connections between the mnemonic and the relationships in identities, such as the cofunction identity $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.

Similarly, in two teaching experiments conducted by Moore (2009) and Moore et al. (2012), undergraduate students were initially unable to see the connections between different branches of trigonometry and relied on memorized procedures to solve problems. However, the students in both experiments were able to develop an understanding of the relationships relevant to the concepts of the study. These studies show that students can develop a strong conceptual understanding of the connections if they are taught explicitly in a way that makes the relationships apparent. However, Moore et al. (2012) acknowledge that many teachers typically lack sufficient knowledge of these relationships for themselves, which makes it unlikely that they will be able to teach their students to recognize the relationships between the various domains of trigonometry.

Learning Trigonometry with Technology

There are many suggestions for learning trigonometric concepts with the use of technology. Dynamic geometry software, such as Geometer's Sketchpad and GeoGebra, can be used for learning about triangle trigonometry and graphs of trigonometric functions, as well as the connections between branches of trigonometry (Kepceoğlu & Yavuz, 2016; Moore, 2009; Peterson et al., 1998; Strayer & Tessema, 2017). Graphing calculators can be used for learning about graphing trigonometric functions (Choi-Koh, 2003; Cruthirds, 2007; Szetela, 1979). Researchers have also used other existing technologies (Dejarnette, 2018; Guttenberger, 1992) or created their own technology programs (Ng & Hu, 2006; Sander & Heiß, 2014) to address learning a variety of trigonometry concepts, such as graphing trigonometric functions (Ross et al., 2011) or simplifying trigonometric expressions with a computer algebra system (Stacey & Ball, 2001; Weigand & Weller, 2001).

Technology has also been used to help students with physical disabilities (Figueira et al., 2015) and to assist students who are low achieving (Blackett & Tall, 1991; Puteh & Rahman, 2015; Rahman & Puteh, 2016). Further, the use of technology in teaching trigonometry can increase motivation (Rahman & Puteh, 2016; Zengin et al., 2012) and decrease the achievement gap between male and female students (Blackett & Tall, 1991).

Graphing Calculators

Suggestions for the use of graphing calculators in the learning of trigonometry date back to the late 1970's (Szetela, 1979). The TI-89 calculators can help students to learn trigonometric concepts, such as graphing, by allowing them to examine the output of the calculator and determining why a graph is not being accurately portrayed (Cruthirds, 2007). With the use of a calculator, Choi-Koh (2003) helped a high school student to explore the effects of the parameters in $y = a\sin(bx + c) + d$. Though the calculator did create some difficulties with scaling making graphs appear inaccurate, overall it was a very useful tool for the student to explore and develop an understanding of the effects of each parameter, because the teacher followed up with why and how questions to help the student make sense of what he was seeing.

Calculators can also be used to learn about other trigonometric domains, such as right triangle trigonometry. In a study of two groups of low-achieving high school students learning right triangle trigonometry, the group that learned the concepts with the use of calculators scored higher on researcher-created achievement tests of procedural skills of solving for missing sides of right triangles at the end of the study (Szetela, 1979). Szetela theorizes that "[c]alculators may allow teachers to direct their energies more productively by eliminating the tedium of computation in concept learning and problem-solving" (p. 118). The use of graphing calculators can also remove some barriers for students that struggle and allow them to learn more advanced concepts in trigonometry (Wulandari et al., 2018). Despite these benefits, Naidoo and Naidoo (2009) that introducing the use of the calculator may cause students to become overly reliant on calculators and that their use may inhibit conceptual understanding of trigonometry.

Dynamic Geometry

Another recommendation for learning trigonometry with technology is the use of dynamic geometry programs, such as Geometer's Sketchpad (Brakoniecki et al., 2018; Moore, 2009; Vonder Embse & Enbretsen, 1996) or GeoGebra (Kepceoğlu & Yavuz, 2016; McCulloch et al., 2017; Özgün-Koca et al., 2013; Puteh & Rahman, 2015; Rahman & Puteh, 2016; Strayer

& Tessema, 2017; Zengin et al., 2012). The dynamic capabilities of these programs allow students to see connections more easily than with static diagrams, such as on a graphing calculator or in a textbook. Learning about trigonometric relationships within right triangles with dynamic geometry software is a natural outlet for the software (Vonder Embse & Enbretsen, 1996). Dynamic geometry can also allow students to make conjectures about graphs of trigonometric functions and receive immediate feedback (McCulloch et al., 2017; Özgün-Koca et al., 2013). Using dynamic geometry software can allow students to make connections between trigonometric domains. For example, in a teaching experiment with the use of Geometer's Sketchpad, Moore (2009) found that undergraduate students were able to develop a strong understanding of the relationships between angle measure, radians, and the unit circle because of the dynamic nature of the tasks presented to them. High school students can make connections between triangle trigonometry and the unit circle that enable them to understand inverse trigonometry when using GeoGebra (Strayer & Tessema, 2017). Dynamic geometry software provides benefits within individual domains of trigonometry and allows students to make connections across domains because of the connected dynamic nature of the software.

The use of dynamic geometry software has benefits for student performance on assessments of their knowledge. High school students who learned trigonometry with the use of dynamic geometry performed better on researcher-created tests of their knowledge of periodicity and graphing trigonometric function than their peers who had learned without dynamic geometry (Kepceoğlu & Yavuz, 2016; Puteh & Rahman, 2015; Zengin et al., 2012). Low achieving high school students also showed increased motivation for learning mathematics on a survey when they learned with the use of dynamic geometry (Rahman & Puteh, 2016). The use of dynamic geometry can allow pre-service mathematics teachers to construct more accurate representations than can be created by hand and will enable them to explore relationships in ways that are not possible with static diagrams (Brakoniecki et al., 2018). Despite these benefits in achievement and motivation, teachers need to be aware of the limitations of the software, such as inaccurate representations that might result in student misconceptions (Antohe, 2009). Students and teachers also need to make sure they create sketches that maintain their desired properties when they are manipulated so that they are able to accurately examine the relationships (Brakoniecki et al., 2018).

Other Software

Many other software programs have been used to help students learn trigonometric concepts. Some of these are preexisting with a range of functionality (Dejarnette, 2018; Guttenberger, 1992; Ross et al., 2011; Stacey & Ball, 2001; Weigand & Weller, 2001) and others have been designed explicitly by researchers to help high school students learn trigonometric concepts (Antoro & Samosir, 2015; Ng & Hu, 2006; Prabowo et al., 2018; Richtarikova, 2018; Sander & Heiß, 2014). There are a variety of existing web tools and Excel spreadsheets that can be used to help students visualize trigonometric concepts and receive feedback on their work with right triangle trigonometry and trigonometric graphs (Wilson, 2008). DeJarnette (2018) used Etoys notebooks to allow high school students to explore graphs of trigonometric graphs for the Ferris wheel problem after students had learned about graphing in class and found that students displayed major misconceptions about the sine function.

Computer algebra systems (CAS) can also be used to assist high school students with learning trigonometry (Stacey & Ball, 2001). Still, it is not a panacea that will solve all of the issues students face when learning trigonometry (Weigand & Weller, 2001). The use of CAS can help high school students to find exact trigonometric values without the use of identities.

However, they need to be able to recognize the forms that are used by the software may not appear identical to the forms that are calculated by hand in class. Despite knowing how to find the values of the parameters in a sinusoidal equation, high school students in Weigand and Weller's (2001) study relied on approximating the curve when using CAS, even when they zoomed in and could visibly see that their function was not the correct one. Working with CAS will change the nature of the work that high school students do with trigonometry because they will be free from computing and solving equations so that they can work with more complicated situations and systems of trigonometric equations (Stacey & Ball, 2001). However, students need to be able to recognize when answers are in different forms or to determine when solving a problem by hand is more beneficial.

Software designed by researchers can also benefit students who are learning about trigonometry. Researchers in Brazil developed a dynamic environment that interacted with a computer to enable visually impaired high school students to learn about triangle trigonometry and found that the students were able to retain and apply abstract concepts after using the program (Figueira et al., 2015). Other software, Math Learn, was explicitly designed as an app so that it would be accessible to high school students outside of school (Prabowo et al., 2018). COTACSI, another researcher designed software, was designed to help high school students learn about trigonometric equations and found that students who used the software performed better than their peers who did not learn from the software on a researcher-created test of trigonometric content (Lotfi & Mafi, 2012). Antoro and Samosir (2015) believed that the interactive program they designed to learn trigonometry would be more engaging to students. However, they did not explicitly test this hypothesis on their software with students. The software can be designed to help students overcome barriers to learning trigonometry.

The use of technology to learning trigonometry can have benefits beyond a single unit of study. Sander and Heiß (2014) designed an interactive program for right triangle trigonometry that provided varying amounts of feedback and hints to high school students. They found that students who did not receive hints before beginning the problems performed better in long term retention of concepts, though they found no difference in the short-term performance. With the use of computer software, high school students are more actively involved in their own learning, so they can gain a more robust understanding of the material and retain their knowledge in the long term, as shown as researcher-created tests of trigonometry content (Guttenberger, 1992). Blackett and Tall (1991) tested the use of a computer program for triangle trigonometry. They found that high school students who had used the program achieved higher on researcher-created tests of trigonometry knowledge than those who had not used the program, with even greater benefits for those who had been the lowest-achieving on the pre-tests and female students.

Though most of the research on teaching trigonometry with technology is focused on increasing student knowledge and achievement, Ross et al. (2011) were interested in determining whether the sequencing of traditional instruction and technology-based instruction impacted high school student achievement. They used a program, which had been designed by the Ontario Ministry of Education, named CLIPS: Trig, which included seven interactive activities with quizzes to provide formative feedback to students. The study was inconclusive about the impact of the order because whether the traditional instruction or the instruction with CLIPS: Trig happened second was enhanced by having already experienced the other first. Ross et al. (2011) concluded that it was better to mix the use of CLIPS: Trig with traditional instruction, rather than using one in isolation from the other.

Other Benefits

The use of technology has been shown to have a variety of benefits for students outside of greater content knowledge. The use of GeoGebra was found to increase both content learning on researcher-created tests of knowledge and motivation on surveys for under-achieving high school students (Puteh & Rahman, 2015; Rahman & Puteh, 2016). Technology has been shown to decrease the amount of time it takes for high school students to learn trigonometric concepts (Figueira et al., 2015). Though all the high school students in Blackett and Tall's (1991) study showed a greater increase in content knowledge when learning with technology, the increase was greater for the girls in the study. This means that the use of technology may help to eliminate the gender gap in STEM. For groups, such as females and under-achieving students, that are traditionally left out of learning trigonometry and other STEM topics, the use of technology can benefit their learning.

Teachers' Content Knowledge of Trigonometry

Pre-service secondary mathematics teachers need a strong understanding of the mathematics content they will be expected to teach to their future students (Ball et al., 2001; Even, 2011; Harel, 2008; Usiskin, 2001). This is because research has shown that student achievement is related to the content knowledge of their teachers (Ball et al., 2008; Begle, 1972; Hill et al., 2005; Monk, 1994). Research has shown that pre-service teachers do not have a strong conceptual understanding of the mathematics content they will be expected to teach (Ball, 1990; Charalambous, 2015; Eisenhart et al., 1993), and this extends to trigonometric concepts. Pre-service secondary mathematics teachers show procedural fluency with trigonometry problems, but generally lack conceptual understanding and mathematical reasoning for these same problems (Brakoniecki et al., 2018; Bryan, 1999; Kahan et al., 2003; Tuna, 2013). Even pre-service secondary mathematics teachers who desire to teach in ways that allow their students to gain both procedural fluency and conceptual understanding may be unable to do so if they lack the conceptual understanding themselves (Eisenhart et al., 1993). Without support to build this understanding, this lack of conceptual understanding will likely continue to be an issue as pre-service teachers transition to their careers as in-service teachers. For content such as trigonometry that is already difficult for students to make connections and build conceptual understanding, this is even more of a difficulty for both pre-service secondary mathematics teachers and in-service secondary mathematics teachers.

In-Service Teachers' Knowledge of Trigonometry

In a study of in-service secondary mathematics teachers' knowledge of trigonometry, the teachers' concept of angle was dominated by their understanding of degree and lacked a robust conception of a radian (Topçu et al., 2006). In follow-up interviews, the participants who had a more robust concept of radian were able to describe how they would make more productive connections when working with students (Topçu et al., 2006). In a study of teachers from underachieving schools in South Africa, Brijlall and Maharaj (2014) found that teachers were generally weak in their pedagogical content knowledge of trigonometry. Another study from South Africa acknowledges the difficulties of teaching trigonometry in culturally and linguistically diverse classrooms (Adler, 1999), though the teacher was able to rely on her strong mathematical content knowledge to navigate the difficulties present in her classroom. However, teachers with weak pedagogical content knowledge as in the study by Brijlall and Maharaj may lack the ability to navigate these challenges. Despite the need for professional development in teaching trigonometry, there are many complexities to overcome to make professional development successful (Aaron et al., 2015).

In-service teachers need to be able to strengthen their own knowledge of trigonometry to teach it so that their students gain conceptual understanding. Thompson et al. (2007) worked with a group of in-service teachers to create coherent meanings of trigonometry, through tasks, such as explaining why the graph of the function $h(x) = \cos(sin(x))$ has the appearance it does. These tasks required the teachers to coordinate the meanings of the input of trigonometric functions, the unit circle, key features of graphs of trigonometric functions, and covariational reasoning. Though some of the teachers were able to advance their thinking about trigonometric functions, others were unable to see the issues in their own conceptions and what they were teaching students, because they did not see the long-term picture of what they were preparing students for.

Pre-Service Teachers' Knowledge of Trigonometry

Pre-service secondary mathematics teachers also demonstrate issues with their knowledge of trigonometry. Of the pre-service secondary mathematics teachers surveyed by Tuna (2013), only 8% were able to correctly able to define radian, and only 17% were able to identify that a circle contains 2π radians. Knowledge of radians is foundational to trigonometry and pre-service teachers who do not know basic facts will not be able to teach this content. Though all of the pre-service secondary mathematics teachers in Bryan's (1999) study were able to identify $\sin^2\theta + \cos^2\theta$ as equivalent to 1, only 2 of the 9 were able to offer a sound conceptual explanation for why this was true. If pre-service teachers do not know these basic ideas about trigonometric concepts, they cannot be expected to be able to teach more complicated trigonometric concepts in a coherent way to students.

In a group of pre-service secondary mathematics teachers, those who ranked highest on a test of content knowledge were able to create lesson plans about trigonometric graphs that focused on both conceptual understanding and procedural fluency compared with those who ranked lower whose lessons focused primarily on procedural fluency (Kahan et al., 2003). However, even the student teacher who ranked highest in content knowledge and planning to teach for conceptual understanding struggled to take advantage of teachable moments when enacting her lesson with students. Creating lesson plans on a topic can allow pre-service secondary mathematics teachers to examine their own procedural fluency and conceptual understanding of a mathematical concept. However, planning alone is not enough to ensure the enactment of high-quality lesson plans as intended. By participating in a lesson study about right triangle trigonometry as part of her coursework, one pre-service teacher was able to grow in her knowledge of the content and consider how to teach for conceptual understanding, rather than just procedural fluency (Cavey & Berenson, 2005). This research shows that pre-service teachers can gain a greater understanding of trigonometric concepts and that it is beneficial to their planning of lessons on trigonometry to have a greater understanding.

Pre-service teachers can also learn how to teach trigonometric concepts with technology, just as students can. For example, they can use dynamic geometry software to build the connection between triangle trigonometry, the unit circle, and graphs of trigonometric functions (Garofalo et al., 2000), which can be followed with applications of trigonometric functions. Pre-service teachers can also explore apparent patterns in trigonometric ratios with the use of dynamic geometry (Brakoniecki et al., 2018). Lessons such as these can enable the pre-service teachers to refresh their own knowledge of trigonometry and consider the prelogical implications of using technology for such a lesson with future students.

Multidimensional Problem-Solving Framework

The Multidimensional Problem-Solving Framework was developed by studying the behavior of expert mathematicians while they were solving non-routine tasks (Carlson & Bloom, 2005). They found that the experts engaged in four phases of problem-solving (orienting, planning, executing, and checking). The orienting phase consists of "… predominant behaviors of sensemaking, organizing, and constructing" (p. 62). Behaviors in the planning phase include constructing a conjecture, imagining the solution approach, and evaluating the feasibility of the conjectured solution approach. The execution phase consists of "… making constructions and carrying out computations" (p. 63). Finally, the checking phase includes the behaviors of verifying the solution, determining the reasonableness of the solution, and deciding whether to accept the solution or cycle through the problem-solving phases again. These cycles may not occur linearly, because the participant may need to return to the planning phase if they find a solution that they determine to not be reasonable in the checking phase.

The experts also drew upon four attributes when engaged in solving non-routine problems (resources, heuristics, affect, and monitoring). Resources include "… knowledge, facts, and procedures" (p. 64). Heuristics include "… constructing a diagram, attempting a parallel problem, etc." (p. 64). Affect includes showing feelings of "… enjoyment, pride, frustration, and mathematical integrity" (p. 64). Monitoring includes "… reflections on the effectiveness and efficiency of the solution process" (p. 64). The attributes may be used across phases, and more than one attribute may be used in each problem-solving phase.

In the MUST Framework, Heid et al. (2015) define the strand of strategic competence within mathematical proficiency as "… the ability to generate, evaluate, and implement problem-solving strategies" (p. 16) and the strand of mathematical reasoning within mathematical activity as "… activities such as justifying and proving as well as reasoning in the context of conjecturing and generalizing … [that] results in the production of a mathematical argument or mathematical rationale that supports the plausibility of a conjecture or generalization" (p. 21). Analyzing how pre-service secondary mathematics teachers engage in the problem-solving phases and draw upon the problem-solving attributes will help to determine how they show mathematical proficiency and mathematical activity when solving high cognitive demand problems. For example, while solving one of the tasks, a participant might consider solving a triangle using the Pythagorean Theorem, the law of sines or law of cosines, and the unit circle, before deciding that using the Pythagorean Theorem will be the easiest and solving the triangle with it. In this example, the participant is drawing upon knowledge of geometry and two trigonometric domains, triangle trigonometry and unit circle trigonometry. She is also engaged in a phase of planning when they are considering how to solve the triangle and then a phase of executing when they solve the triangle. This example also shows the strand of strategic competence since they considered several possible solutions and used the ones they determined to be best.

Prior Use of the Multidimensional Problem-Solving Framework

The Multidimensional Problem-Solving Framework has been used to analyze the work of K-12 students (Albarracín & Gorgorió, 2014; Elia et al., 2009; Hannula, 2015; Salado et al., 2018), undergraduate students (Aljaberi & Gheith, 2015; Antonini, 2011; Bowling, 2014; Burtch, 2005; Chong & Shahrill, 2016; LaRue, 2016; Lockwood & Gibson, 2016; Moore & Carlson, 2012; Savic, 2012; Thornburg, 2010), elementary pre-service teachers (Bjuland, 2007; Chamberlin, 2018; Hancock, 2018) and secondary pre-service teachers (Bloom, 2008; Furinghetti & Morselli, 2009; Koichu & Kontorovich, 2013; Kuzle, 2011; Ryve, 2007; Yimer & Ellerton, 2010), in-service teachers (Andrews & Xenofontos, 2015; Koichu & Kontorovich, 2013), and mathematicians or experts (Harlim & Belski, 2013; Lockwood et al., 2016; Savic, 2012; Troudt, 2015). Though the framework was initially developed based upon mathematician's work on non-routine tasks, it can be used to analyze a wide variety of individuals' problem-solving work. Some research has also extended the Multidimensional Problem-Solving Framework to create frameworks such as the Optimization Problem-Solving Framework (LaRue, 2016) and the Modified Multidimensional Problem-Solving Framework (Thornburg, 2010).

Bloom (2008) used the Multidimensional Problem-Solving Framework to analyze the work of two pre-service secondary mathematics teachers, Amy and Ben, in a course aimed at teaching problem-solving. The framework was used to analyze how the two participants engaged in problem-solving cycles and drew upon problem-solving attributes over the length of the course. Early in the class, Amy was generally unable to orient herself to the problem and monitor her progress as she began working, which led to difficulties completing many of the given problems. However, later in the course, as her ability to orient herself to the problems and monitor her work improved, Amy was able to increase her ability to engage in all phases of the problem-solving cycle fully. She was also able to utilize a greater variety of heuristics to help herself solve the problems. Because Ben began the course with more robust background knowledge of mathematics and better skill in engaging in the phases of problem-solving, he did not show as much growth. However, he was able to more consistently reflect upon his work when he encountered a problem and displayed better attitudes towards problem-solving at the end of the course.

Elia et al. (2009) analyzed the work of 152 high-achieving Dutch fourth-grade students on three non-routine tasks. The authors concentrated on the strategies students used and whether the students were successful in finding the answer. Because the tasks were presented in paper and pencil form, rather than an interview, the authors were only able to analyze the written work and not how students engaged in the problem-solving phases. However, they were able to draw some conclusions about the problem-solving attributes that students drew upon to solve the problems. The most common successful strategy identified was trial-and-error. Though this study relied upon written work, rather than videos of problem-solving used in many other studies utilizing the Multidimensional Problem-Solving Framework, the authors were still able to analyze how the students were solving the problems.

Conclusion

Trigonometry is an important topic in high school mathematics that bridges between standards for algebra, geometry, and functions (CCSSI, 2010; NCTM, 1989, 2000). Trigonometry is also essential because it has many applications in the world outside of the mathematics classroom (Barnes, 1999; Galle & Meredith, 2014; Gould & Schmidt, 2010; Hirsch et al., 1991; Kramer, 1948; McCulloch et al., 2017). Trigonometry is known to be difficult for students to learn (Bressoud, 2010; Chigonga, 2016; Demir & Heck, 2013; Gür, 2009; Thompson, 2008; Wulandari et al., 2018; Yi et al., 2013), yet the body of research on trigonometry is inadequate (Akkoç, 2008; Topçu et al., 2006; Weber, 2008).

Since trigonometry is an under-researched topic in mathematics education compared with other mathematics topics such as algebra, geometry, or functions (Ross et al., 2011; Weber, 2005), there is a wide range of future research to be done. There have been many promising small-scale studies about learning from different teaching strategies (Inan, 2013; Moore, 2009; Naidoo & Naidoo, 2009; Puteh & Rahman, 2015; Rahman & Puteh, 2016; Weber, 2005), so larger follow-up studies would be a critical area of future research, to determine whether results observed with small numbers of students are consistent when implemented with higher numbers of students. Future research should also consider, both quantitatively and qualitatively, what concepts students struggle within and across the different domains of trigonometry and why students struggle with them.

More research is needed about how teachers understand trigonometry themselves and how their understanding relates to their teaching of trigonometry. There is limited research on preservice teachers and their knowledge of trigonometry (Bryan, 1999; Moore et al., 2016; Topçu et al., 2006; Tuna, 2013) or thoughts about teaching trigonometry (Cavey & Berenson, 2005; Garofalo et al., 2000), but there is even less research to be found about in-service teachers and trigonometry (Thompson et al., 2007; Topçu et al., 2006). Knowledge of how future and current teachers understand trigonometry concepts for themselves and consider teaching them to students will provide more information about how to best go about teaching students and how to help students to understand trigonometry better.

With the limited amount of existing research specifically on the teaching and learning of trigonometry, it is difficult to determine the best methods for increasing students' conceptual understanding and achievement. Research on trigonometry agrees with the larger body of research in mathematics education that students learn better when student-centered teaching methods are used (Gür, 2009; Inan, 2013; Naidoo & Naidoo, 2009; Orhun, 2004; Weber, 2005), and that technology can enhance student learning (Kepceoğlu & Yavuz, 2016; Moore, 2009; Rahman & Puteh, 2016; Zengin et al., 2012). Given the limited existing research, it is difficult to conclude how to best help students learning trigonometry with a conceptual understanding of the connections between triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. More research is needed to paint a clearer picture of how students and teachers overcome the difficulties they face in understanding trigonometry.

Chapter 3: Methodology

Research Design

This study utilized a qualitative research design. Qualitative research allows a complex and detailed picture of the issue to emerge. In this research study, the goal was to describe what pre-service secondary mathematics teachers know about trigonometry and what strategies they used to solve high cognitive demand trigonometry problems. According to Creswell (2013)

> "Qualitative research begins with assumptions and the use of interpretive/theoretical frameworks that inform the study of research problems addressing the meaning individuals or groups ascribe to a social or human problem. To study this problem, qualitative researchers use an emerging qualitative approach to inquiry, the collection of data in a natural setting sensitive to people and places under study, and data analysis that is both inductive and deductive and establishes patterns or themes. The final written report or presentation includes the voices of participants, the reflexivity of the researcher, a complex description and interpretation of the problem and its contributions to the literature or a call for change" (p. 44).

This research used the qualitative method of a case study. According to Creswell (2013), "[c]ase study research is a qualitative approach in which the investigator explores a real-life, contemporary bounded system … over time, through detailed, in-depth data collection involving multiple sources of information …. And reports a case description and case themes" (p. 97). This research used four cases of individual pre-service secondary mathematics teachers to answer all three research questions. Specifically, this study used a multiple case study approach. In this approach, "one issue … is selected, but … multiple case studies [are selected] to illustrate the issue" (Creswell, 2013, p. 99). This study examined four representative cases to allow for generalization (Creswell, 2013). Descriptions of how these four cases were selected will follow.

Each case is a pre-service secondary mathematics teacher who demonstrated different ways of solving trigonometric tasks. Each pre-service secondary mathematics teacher who was selected as a case had a diverse background in terms of factors such as major, year in an undergraduate program, and background procedural knowledge of trigonometry. All cases are bounded by their participation in the test of procedural trigonometric knowledge and the series of three task-based interviews. Since the test and each of the interviews were designed to take about one hour each, this amounted to about four hours of time, with the research primarily focused on the three one-hour task-based interviews. Based on the schedule of each participant and the interviewer, these interactions were spread across three to six weeks.

Participants

The participants in this study were pre-service secondary mathematics teachers who were currently enrolled in a mathematics education program at a large southeastern university in the spring semester of 2019. The mathematics education program at this university consists of five mathematics education courses that are spread out from their sophomore year to senior year. The participants were all undergraduate students who were preparing to teach middle grades or high school mathematics. Because the study requires knowledge of trigonometry, participants must have completed a mathematics course in pre-calculus or equivalent before the study. This course may have been taken in high school or college. However, since the study does not require knowledge of teaching, participants may have been enrolled in any level of mathematics education course.

This study is interested in pre-service secondary mathematics teachers because they will be expected to teach trigonometry topics across the secondary curriculum. Previous research has primarily assessed what procedural knowledge pre-service secondary mathematics teachers have about trigonometry (Akkoç, 2008; Çetin, 2015; Kahan et al., 2003; Naidoo & Naidoo, 2009; Topçu et al., 2006; Tuna, 2013), with only a small amount assessing conceptual understanding or problem-solving skills about trigonometry (Bryan, 1999; Cavey & Berenson, 2005; Moore, Silverman, et al., 2014; Saleh et al., 2018). This research will add to the literature on both the conceptual understanding and problem-solving on trigonometry of pre-service secondary mathematics teachers.

Seven participants were recruited for the study. Initially, all students taking mathematics education courses were contacted, but none agreed to participate. After this, 21 students in the mathematics education program who were suggested by a mathematics education faculty were contacted individually through email, and seven agreed to participate in the study. The number of participants in the study was limited by the small number of pre-service secondary mathematics teachers who are currently enrolled in the mathematics education courses at the university. However, since the primary focus of the study is qualitative, the small sample size is acceptable. All seven of the participants completed both the test of trigonometry knowledge and the three task-based interviews. The characteristics of the participants are presented in Table 1.

Table 1

Participant Characteristics

Name	Year	Major	Most Recent Math Course
Bonnie	Junior	High School Mathematics Education	Calculus 3 Linear Algebra
Daniel	Junior	High School Mathematics Education & Mathematics	Analysis Combinatorics Abstract Algebra
Emma	Sophomore	High School Mathematics Education	Calculus 3
Evanna	Freshman	High School Mathematics Education	Calculus 2
Jessie	Senior	Middle School Mathematics Education	Intro to Euclidian Geometry
Katie	Senior	High School Mathematics Education	Intro to Euclidian Geometry
Rupert	Sophomore	High School Mathematics Education & Statistics	Calculus 3 Foundations of Advanced Mathematics

Only four cases were analyzed to answer the research questions. These cases were Daniel, Emma, Evanna, and Jessie. These cases were chosen because they were representative of the range of trigonometric thinking and solutions presented by all seven participants. They were also chosen to represent a range of background factors. For example, Jessie was the only participant who was a middle school mathematics education major, and Daniel was one of two participants with a double major and the only one who had taken graduate-level mathematics coursework. When two participants were similar in characteristics, the one who had been more clearly able to describe their thinking during the interviews was selected as a case.

Data Collection and Analysis by Research Question

Initially, all participants took a paper test of their knowledge of trigonometry. They were allowed to use a graphing calculator and formula sheet as they completed the test with unlimited time. Because there is no existing test of trigonometric knowledge, one was developed based on existing standardized exams, such as the North Carolina End-of-Course tests, NAEP, the SAT, and the ACT. The test had 20 questions that were equally distributed between triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, analytic trigonometry, and connections between the domains. The items were not be grouped by domain on the test but were presented in a randomized order. This was to make sure that the order of the questions did not influence how the participants solve them. The test was examined for face validity by experts to ensure that it measured knowledge of trigonometry appropriate for pre-service secondary mathematics teachers.

The test was also used to help determine which participants would be analyzed in the cases. As seen in Table 2, the participants' scores varied across the domains. Daniel and Rupert had the highest scores and were the most consistent across the domains. Bonnie, Emma, and Jessie scored in the middle overall, and all three scored the lowest on the domain of trigonometric functions and graphs. Evanna and Katie scored the lowest total scores but varied in their scores across domains.

Table 2

Participants Scores on Test of Trigonometry

Name	Total Score	Triangle Trigonometry	Unit Circle Trigonometry	Trigonometric Functions and Graphs	Analytic Trigonometry
Bonnie	70%	71%	83%	50%	79%
Daniel	98%	93%	100%	100%	93%
Emma	78%	100%	83%	38%	79%
Evanna	63%	71%	75%	75%	43%
Jessie	75%	71%	75%	63%	93%
Katie	63%	64%	50%	38%	79%
Rupert	95%	100%	92%	88%	100%

Research Question #1: How do pre-service secondary mathematics teachers engage in problem solving phases (e.g., orienting, planning, executing, checking) when solving high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #2: How do pre-service secondary mathematics teachers draw upon problem solving attributes (e.g., resources, heuristics, affect, monitoring) as they solve high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #3: How do pre-service secondary mathematics teachers engage in mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (mathematical noticing, mathematical reasoning, and mathematical creating,) as they solve high cognitive demand trigonometric tasks?

To answer all three research questions, each participant engaged in a series of three task-based interviews with the researcher (Goldin, 2000). Each interview involved a series of three tasks across the trigonometric domains (triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry). Since each interview involved three tasks across four domains, each interview had at least one task that involved relationships between two domains. There was a series of three interviews so that the domains could be presented in different orders in each interview. For example, the triangle trigonometry problem was presented as the second task in interview one, the first task in interview two, and the last task in interview three. This was to ensure that the order of the domains did not influence how the participants solve the tasks. A complete table of the tasks by domain is in Table 3 and the complete set of tasks is available in Appendix A.

Table 3

Interview Tasks by Domain

Interview	Task	Triangle Trigonometry	Unit Circle Trigonometry	Trigonometric Functions and Graphs	Analytic Trigonometric	Connections Between Trigonometric Domains
1	1			X	X	X
1	2	X				
1	3		X			
2	1	X	X			X
2	2			X		
2	3				X	
3	1		X	X		X
3	2				X	
3	3	X				

The tasks were selected from an initial list of 60 tasks. This list was cultivated from a variety of resources, including Illustrative Math resources by content and practice standards (*Illustrative Mathematics*, n.d.), the classroom-based situations included in with MUST framework (Heid et al., 2015), NCTM Problems of the Week (NCTM, n.d.), and NRICH secondary curriculum (*NRICH*, n.d.). These initial tasks were examined to determine what domain or domains were involved in solving the task. Tasks were selected from the list to ensure that content from each of the four domains of trigonometry was present in the set of three tasks for each interview. During each interview, participants were provided with a TI-84 graphing calculator and a formula sheet, available in Appendix C, to use as they completed the tasks.

Descriptions of Tasks

Below is a brief description of each of the nine tasks that were presented to participants. The tasks are described in the order that they were presented to participants. Interview 1 consisted of Task 1, Task 2, and Task 3. Interview 2 consisted of Task 4, Task 5, and Task 6. Interview 3 consisted of Task 7, Task 8, and Task 9. Tasks in each interview addressed all four of the trigonometric domains of triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. The complete tasks, as presented to participants, are available in Appendix B.

Task 1: Identities from a Graph

Task 1 consists of four parts that address the domains of trigonometric functions and graphs and analytic trigonometry. Participants are first asked to sketch graphs of $f(x) = \cos x$ and $g(x) = \sin x$. These graphs can then be referenced throughout the rest of the task to aid in problem-solving. In the second part, participants are asked to find a translation of the graph of $f(x) = \cos x$ that maps onto itself and a reflection that also maps the graph onto itself. Participants are then asked to identify a trigonometric identity associated with each transformation. The third part of the task is identical to the second part, but it is for the graph $g(x) = \sin x$. For the fourth and final part of the task, participants are asked to find a combination of translations and reflections that map $f(x) = \cos x$ onto $g(x) = \sin x$ and that map $g(x) = \sin x$ onto $f(x) = \cos x$. This is followed by asking participants to identify a trigonometric identity associated with each transformation.

Task 2: Pythagorean Theorem and Pythagorean Identity

Task 2 consists of three parts and addresses the domain of triangle trigonometry. Participants are presented with a diagram of a right triangle ABC, with legs AB and BC and hypotenuse AC. Participants are then asked to verify the equation $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$ for the triangle pictured. Next participants are asked to deduce that $\sin^2\theta + \cos^2\theta = 1$ is true for any acute angle θ. Finally, participants are asked to say any information they know about $\cos\theta$ if they know that θ is in the second quadrant and $\sin\theta = \frac{8}{17}$. This part also asks participants to provide a picture and explain their answers.

Task 3: Which One Doesn't Belong?

Task 3 consists of a single part and addresses the domain of circle trigonometry. The task presents participants with a diagram showing three squares filled in with $\sin 150°$, $\sin 225°$, and $\cos 120°$. The fourth square was filled with a question mark. Participants were asked to add a fourth member to the group so that each group of three members would exclude the fourth member. Participants were also asked to explain why each member was excluded from the other three members.

Task 4: Define Sine of an Obtuse Angle

Task 4 presents participants with a diagram of a triangle and contains two parts that address right triangle trigonometry and unit circle trigonometry. Participants were presented with a diagram of a right triangle with legs AB and BC and hypotenuse AC. On the diagram $\angle BAC$ was marked as $\angle\alpha$. Participants were told that a student knew the definition of the sine of α is the length of the side opposite A divided by length of the hypotenuse or $\sin\alpha = \frac{|BC|}{|AC|}$. The student said "the sine of an obtuse angle does not make any sense because I can't make a right triangle with an obtuse angle." The first part of the task asked participants to help the student understand how to define the sine of an obtuse angle and to include a picture or diagram. The second part of the task asked participants to find $\sin\frac{3\pi}{4}$ and $\sin\pi$ and to explain their answers.

Task 5: Graphs of Foxes and Rabbits

Task 5 consists of four parts that address the domain of trigonometric functions and graphs. The task begins by presenting the two graphs of the populations of rabbits and foxes in a national park over 24 months. The first part of the task asks participants to explain why a trigonometric model is an appropriate model for the data. The second and third parts of the task each ask participants to find the trigonometric functions that model the number of rabbits and foxes as a function of time. The fourth part of the task asks participants to graph both functions together and provide an explanation for why one of the functions seems to "chase" the other function.

Task 6: Sum and Difference Identities

Task 6 consists of five parts that all address the domain of analytic trigonometry. Participants are presented with the sum angle identity for sine as $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$. In the first part of the task, participants are told to use the even and odd identities along with the sum angle identity for sine to conclude the difference angle identity of $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$. In the second part of the task, participants are told to derive the sum angle identity for cosine, $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, from the sum and difference angle identities for sine and other previously known identities. The third part of the task asks participants to derive the difference angle identity for cosine of $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ from the previously verified identities. The fourth part of the task tells participants to derive the sum angle identity for tangent of $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ from any previously shown identities. The final part of the task asks participants to derive the difference angle identity for tangent of $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ by using any previously derived identities.

Task 7: Horizontal Shrink vs. Horizontal Stretch

Task 7 is a single part task that addresses the domains of unit circle trigonometry and trigonometric functions and graphs. The task asks the participants to explain why the graph of $y = \sin(2x)$ is a horizontal shrink of the parent graph $y = \sin(x)$ and not a horizontal stretch of the parent graph.

Task 8: Proving an Identity

Task 8 consists of a single part and addresses the domain of analytic trigonometry. The task presents participants with the work from a hypothetical student for a proof of $\sin x * \cos x * \tan x = \frac{1}{\csc^2 x}$. Participants are told that this is the work of a student, and the student has explained their reasoning as "I just treated it the equation like any algebra equation. You know, what you do to one side, you have to do to the other, and then I showed it was the same as $1 = 1$. I know $1 = 1$ is true, so the identity must be true." Based on the work and explanation, participants are asked whether the student is correct and to explain why or why not.

Task 9: Angles in a Rectangle

Task 9 consists of two parts and addresses the domain of triangle trigonometry. Participants are given a diagram with the information that *ABCD* is a rectangle, $BC = 3AB$, and P and Q are points on *BC* such that $BP = PQ = QC$. In the first part of the task, participants are asked to show that $\angle DBC + \angle DPC = \angle DQC$. For the second part of the task, participants are asked to generalize their results from the first part of the task. That is, generalize that the result of the first part holds true for any rectangle $ABCD$ without any of the other given information.

Task-Based Interviews

Each task-based interview is a situation between the interviewer, the participant, and the task (Goldin, 2000; Koichu & Harel, 2007). The role of the interviewer during a task-based interview is to understand the participant's thinking and solution to the task, and the role of the participant is to explain their reasoning at the moment as they solve or attempt to solve the task. To do this, the interviewer should question the participant, which opens up nearly infinite paths for the interview to continue (Hunting, 1997). Good questions should

- "be open-ended so that students are allowed some freedom to choose their own preferred ways of responding
- maximize opportunity for discussion or dialogue so that thought processes can be revealed, and
- allow both student and interviewer to reflect on their respective thought processes" (Hunting, 1997, p. 153).

According to Goldin (2000), the exploration of each task should include four stages of posing the question, minimal suggestions from the interviewer, guided heuristic suggestions as needed, and exploratory metacognitive questions.

The interviews between the researcher and each participant were video recorded and transcribed (Davidson, 2009). Transcripts were also annotated to include actions relevant to solving the given tasks. Copies of any work that the participants did while completing the tasks were also kept to triangulate results (Creswell, 2013). Each interview was first coded using a priori codes based on the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005). This framework, shown in Figure 2, includes codes for problem-solving phases (e.g., orienting, planning, executing, checking) and problem-solving attributes (e.g., resources, heuristics, affect, monitoring). Open coding was also used for any instances that did not fit within the a priori codes (Creswell, 2013). A codebook with examples of each problem-solving phase and problem-solving attribute was created to ensure consistent coding (DeCuir-Gunby et al., 2011).

Phase • Behavior	**Resources**	**Heuristics**	**Affect**	**Monitoring**
Orienting • Sense making • Organizing • Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does that look like?"
Planning Conjecturing Imagining Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me where I want to go?", "How efficient will Approach X be?"
Executing • Computing • Constructing	Conceptual knowledge, facts and algorithms were accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking • Verifying • Decision making	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver

Figure 2

Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005)

Each task was coded independently for problem-solving phases and problem-solving attributes. Coding was completed only on what each participant verbalized in the transcript or what the participant wrote on paper while solving the task. The coding of the transcripts was completed in Atlas.ti. The result method was similar to Carlson and Bloom (2005) and Kuzle (2013), who included a column for the transcript excerpt, a column for the phase, and a column for the attribute.

For problem-solving phases, each transcript was coded with non-overlapping codes for the entirety of the transcript. The participants' solving was coded as being in an orienting phase if they were describing their thinking as "constructing either a picture or a mental image of the problem situation as they attempted to make sense of the question" (Carlson & Bloom, 2005, p. 62). During an orienting phase, the predominant behaviors are "sense making, organizing, and constructing" (Carlson & Bloom, 2005, p. 62) and specific behaviors included "defining unknowns, sketching a graph, constructing a table" (Carlson & Bloom, 2005, p. 62). To be coded as a planning phase, participants displayed "initially devised conjectures about a viable solution approach" (Carlson & Bloom, 2005, p. 62) or "contemplate[d] various solution approaches by imagining the playing-out of each approach, while considering the use of various strategies and tools" (Carlson & Bloom, 2005, p.63). In a planning phase, participants displayed behaviors of "The sequence of behaviors that were exhibited included (a) the construction of a conjecture; (b) either the verbalization of a solution approach or silence, with the subject appearing to imagine how the solution approach would play out; and (c) evaluation of the viability of the conjectured approach" (Carlson & Bloom, 2005, p. 63). In an executing phase, the predominant behaviors "involved making constructions and carrying out computations" (Carlson & Bloom, 2005, p. 63). Some specific behaviors that were coded as an executing phase "included writing logically connected mathematical statements, accessing resources (including conceptual and factual knowledge), executing strategies and procedures, and carrying out computations" (Carlson & Bloom, 2005, p. 63). For a checking phase, the participants' changed from active solving to "verification behaviors as they spontaneously assessed the correctness of their computations and results" (Carlson & Bloom, 2005, p. 63). Some specific behaviors displayed during a checking phase might include "spoken reflections about the reasonableness of the solution and written computations" (Carlson & Bloom, 2005, p. 63).

For problem-solving attributes, only the relevant sections of the transcript were coded. The framework defines resources as "conceptual understandings, knowledge, facts, and procedures used during problem solving” (Carlson & Bloom, 2005, p. 50). This was operationalized as knowledge from each of the four trigonometric domains. For example, a participant’s work would be coded as resources of triangle trigonometry when using SOH CAH TOA to solve a triangle. The use of the provided physical resources of the formula sheet and graphing calculator was also coded as resources. From the framework, heuristics are defined as “…specific procedures and approaches” (Carlson & Bloom, 2005, p. 50), such as “… constructing a diagram, attempting a parallel problem, etc.” (Carlson & Bloom, 2005, p. 64). Thus, a section of the transcript was coded with exactly one phase and as many attributes as were applicable. The attributes of resources and heuristics were broken down into specific resources and heuristics that were used for more detailed analysis. A complete list of resources and heuristics coded is provided in Table 4. Though affect and monitoring were included in the framework as problem-solving attributes, they were not analyzed because they appeared in very limited numbers across the participants and tasks. A sample coded transcript of Task 3 from Interview 1 is in Table 5. In this sample transcript, the interviewer supported Evanna to orient herself to the task. While the interviewer did support the problem-solving at times, it was not consistent across participants or tasks.

Table 4

Resources and Heuristics

Resources	Heuristics
Formula Sheet	Draw a Diagram
Graphing Calculator	Arithmetic or Algebraic Manipulation
Geometric Properties	Substitute Values
Pythagorean Theorem	Solve a Simpler Problem
Coordinate Transformations	
Triangle	
SOH CAH TOA	
Special Right Triangles	
Law of Sine or Law of Cosines	
Unit Circle	
Graphs	
Trigonometric Identities	

Table 5

Sample Transcript Coding

Transcript	**Phase**	**Attribute**
Interview 1 Task 3: Add a fourth member to the set $\{\sin(150°), \sin(225°), \cos(120°)\}$ so that each group of three excludes the fourth. Explain why each group of three excludes the fourth.		
Interviewer: So your job is to add a fourth member to the set so that each group of three excludes the fourth one. Evanna: Mm-hm. Interviewer: Does it make sense what that's asking? Evanna: I'm not... Interviewer: So for example, you're going to, you're going to pick something here, but you want to pick something so that if you pick this group of three, this one's excluded, but you could also pick, say this group of three and this one's excluded. Evanna: Okay. Interviewer: So each group of three that you pick excludes the other one. Evanna: Right. Interviewer: Okay, does that makes sense? Evanna: I think so.	Orienting	NA

Table 5

Sample Transcript Coding, Continued

Transcript	**Phase**	**Attribute**
Evanna: Okay, so I'm going to use the unit circle to see what these mean. I don't want exactly remember one. So $\sin 150°$ is the y value. So $\frac{1}{2}$. So this is like $\frac{1}{2}$. $225°$ sine, that's the y value, $-\frac{\sqrt{2}}{2}$and then $\cos 120°$ is the x value, which is $-\frac{1}{2}$. So... Interviewer: So you're kind of staring at it and looking at your unit circle. What? Evanna: Yeah. Interviewer: What are you thinking about or what are your...	Planning	Resource – Formula Sheet Resource – Unit Circle
Evanna: Well, um, if these two are $\frac{1}{2}$, then if I made this one equal to$\frac{1}{2}$, that would exclude this. Interviewer: Okay. Evanna: But if I made this a $\frac{1}{2}$, but if it was $\frac{1}{2}$, I'm not sure what these would do to exclude this	Executing	NA
Evanna: … Um. I'm making sure I did these right. Huh? And if I made this negative something that would exclude this one. Interviewer: Okay. [pause] So if I'm hearing correctly, your idea so far, maybe make it equal to $\frac{1}{2}$ and maybe equal negative something? Evanna: Right. But if it's $-\frac{1}{2}$, I don't know what these, if these two, these three would not exclude this one. Uh.	Checking	NA
Evanna: Okay. Well $150°$ is, oh I don't think that has anything to do with it. $150°$ is $\frac{5\pi}{6}$, $225°$ is $\frac{5\pi}{4}$, and $120°$ is…	Executing	Resource – Unit Circle
Evanna: … no that doesn't. Interviewer: So what were you trying to look at with those? Evanna: To see what their radians were, to see if that would make a difference.	Checking	NA

Table 5

Sample Transcript Coding, Continued

Transcript	Phase	Attribute
Evanna: So just I know it doesn't have anything to do with sine or cosine, but... Interviewer: That's all right. Evanna: Um, $150°$ is $\frac{5\pi}{6}$, $120°$ is $\frac{2\pi}{3}$, and then sine or $225°$ is $\frac{5\pi}{4}$. That's not gonna do anything. [pause] I have no idea.	Executing	Resource – Unit Circle
Interviewer: So pick something that you think will fit the first couple things you noticed and see if you can then make it fit the, so you wanted something that was negative, so you could exclude this one and something that was one half so you could exclude that one. Evanna: Uh, okay. So, this, if it were a $-\frac{1}{2}$ in your mind, cross this out. If you were $-\frac{1}{2}$, it could either be $\sin 120°$ or, not $\cos 120°$. Evanna: It can be $\sin 210°$, $\cos 240°$. Okay. Oh well that was just right there. …	Planning	NA
Evanna: … Um, if it were $\sin 210°$, that would exclude this because it's cosine. Interviewer: Okay. So then we've excluded...	Executing	NA
Evanna: Did I exclude all of them? Interviewer: I've think we've got three of them. Right? Cause this one's positive. Evanna: So those two, so that's excluded. Interviewer: This one's not $\frac{1}{2}$. And then this one is cosine. Evanna: Mm-hm. Interviewer: So, I think you need to exclude that one. Evanna: Oh, what makes that one different? Hmm. So that's not right. Interviewer: It's not that it's not right. There's lots of possibilities here. So, you just have to figure out a way that that one's different than the other three.	Checking	NA
Evanna: Just a $\sin 210°$. Does it has to do with their opposite?	Planning	NA
Evanna: … Like this is y. So $\sin 150°$ is $\frac{1}{2}$. But the first value is -3, $-\sqrt{3}$. Um, $\sin 225°$. The other value is $-\frac{\sqrt{2}}{2}$. $\cos 120°$ is $-\frac{1}{2}$. I mean, yeah.	Executing	Resource – Unit Circle

Table 5

Sample Transcript Coding, Continued

Transcript	Phase	Attribute
Evanna: … And then $\sin 210°$, no that's not right. Interviewer: So what were you kind of looking at there? Evanna: They're other, so the corresponding x value and then the corresponding y value to see if that excluded it. Interviewer: See if there was something different about those? Evanna: And unless I looked over it, that did not do anything. Um, [pause] is, no, it's not that it's in a different quadrant.	Checking	Resource – Unit Circle
Interviewer: So you picked $\sin 210°$. Are there other...? Evanna: $\cos 240°$. Interviewer: $\cos 240°$. Would that one? Cause that's also $-\frac{1}{2}$, right? Evanna: Mm-hm. So if I did $\cos 240°$... Interviewer: And I'm not saying which one is... I have no idea which one is better. Just trying the other option.	Planning	Resource – Unit Circle
Evanna: $\cos 240°$ is $-\frac{1}{2}$. Interviewer: Okay. Evanna: So then the corresponding value is $-3, -\frac{\sqrt{3}}{2}$. $\cos 120$ the other one is $-\frac{\sqrt{3}}{2}$. Yes. We're trying to exclude this one. Interviewer: Yep. Evanna: It's $\sin 150°$, $\sin 225°$...uh. Interviewer: Were these all the options that were $-\frac{1}{2}$? I feel like there are more.	Executing	Resource – Unit Circle
Evanna: Anything $-\frac{1}{2}$. So $\cos 120°$, $\sin 210°$, $\cos 240°$, $\sin 330°$. Interviewer: Maybe that's a better option. Evanna: Mm-hm. So.. different quadrant? Interviewer: Okay. Evanna: Cause the other ones are in that quadrant. Interviewer: Okay. So that's the only one in the fourth quadrant. Makes sense to me. So we're going to change it to? Evanna: Yeah.	Executing	Resource – Unit Circle

Table 5

Sample Transcript Coding, Continued

Transcript	**Phase**	**Attribute**
Interviewer: That's our winner? Evanna: Yeah. Interviewer: And then can you do me a favor and just write down why we excluded each one to help me remember? So we excluded... Evanna: I'm gonna name these, for me. One, two, three. So excluded. So we excluded one by all of these being negative. Does this make sense how I'm doing it? Interviewer: Yup. Evanna: Okay. We excluded two because the others were $\frac{1}{2}$. We excluded three because it was cosine, and then we excluded four because of the quadrant.	Checking	Resource – Unit Circle

After the transcripts were individually coded for problem-solving phases, they were visually represented as in Lee and Hollebrands (2006), as seen in Figure 3. The figures show the instances of each of the problem-solving phases in order as the task was solved, but do not necessarily indicate the amount of time spent in the phase. The codes for phases and attributes were analyzed by the participant as a whole and by trigonometric domain and across all participants for common themes.

	Part 1															
Orienting	x															
Planning		x						x					x			
Executing			x		x		x		x		x			x	x	
Checking				x		x				x		x				x

Figure 3

Sample Coding of Phases

All interviews were coded multiple times to make sure nothing was missed and to ensure validity in the coding. Further, a subset of the interviews were coded by one additional coder to reach intercoder agreement (Creswell, 2013). IRR was 86% for phases and 92% for attributes.

Research Question #3: How do pre-service secondary mathematics teachers engage in mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (mathematical noticing, mathematical reasoning, and mathematical creating,) as they solve high cognitive demand trigonometric tasks?

There was no new data collected to answer the third research question, but further analysis of the task-based interviews was performed. The interviews were further analyzed holistically using the Mathematical Understanding for Secondary Teaching (MUST) Framework (Heid et al., 2015). This framework states that the knowledge needed for teaching secondary mathematics should be viewed from the perspectives of mathematical proficiency, mathematical activity, and mathematical context for teaching. This research is focused on the strands of mathematical proficiency and mathematical activity since it is focused on the knowledge of the pre-service secondary mathematics teachers and not on their teaching. Each case was analyzed for examples or non-examples of each of the strands of mathematical proficiency and mathematical activity.

Chapter 4: Results by Case

Four cases were analyzed for this study. All participants were pre-service secondary mathematics teachers at a large research university in the southeastern United States. Participants each engaged in a series of nine high cognitive demand trigonometry tasks across three task-based interviews. These four cases were chosen as representative exemplars of all participants.

Case 1: Daniel

At the time of the study, Daniel was a traditional undergraduate student. He was a junior with a double major in mathematics education and mathematics. He had nearly completed his mathematics coursework, including several graduate-level mathematics courses. He anticipated completing his student teaching in the spring of the following school year. On a test of trigonometric content knowledge, Daniel showed a strong procedural understanding across all four domains. He received an almost perfect score of 98%, missing only part of a question about triangle trigonometry.

Daniel worked diligently on the tasks during the interviews. He was generally able to find the correct solution quickly and persisted on the few that gave him more difficulty. Daniel rarely relied upon the provided graphing calculator or formula sheet while completing the tasks, so it will be noted explicitly in the description when he does utilize these resources. He would often relate the tasks to the ones he had seen done with students in the classroom during his practicum that semester, though none of these examples specifically utilized trigonometry.

Daniel's Problem-Solving by Task

Task 1: Identities from a Graph

Daniel progressed through the problem-solving phases during the parts of Task 1, as seen in Figure 4.

	Part 1				Part 2								Part 3		
Orienting	x				x		x						x		
Planning		x						x							
Executing			x	x		x					x			x	
Checking									x	x		x			x

	Part 3									Part 4				
Orienting	x													
Planning		x		x			x							
Executing			x		x			x		x		x		
Checking						x			x		x		x	x

Figure 4

Daniel's Phases on Task 1: Identities from a Graph

Each time he began a new cycle of orienting, this indicated that he was on a new part of the question. For the three of the four parts of the task, Daniel was successful in completing the task with a single cycle of the problem-solving phases, but he required multiple cycles for the fourth part. This part asked him to find a reflection from sine to itself using the graphs he had created and then relate those to identities. Initially, he conjectured that it would be similar to his solution for cosine since the translations for the two functions had been identical, but when he realized that he would have to translate sine to use a reflection over the $y - axis$. His next conjecture was to reflect the graph of the sine function over the line $y = x$ by switching the x and y coordinates, but he reasoned that would result in the inverse of the sine function, arcsine, rather than the sine function. Finally, he successfully conjectured that he should reflect across the origin. He reasoned through this reflection by substituting ordered pairs to test whether his conjectures resulted in the desired result, before recalling that sine was an odd function as part of his explanation of why this solution worked. Across the parts of the problem, Daniel was generally successful with his first conjectured solution. He only required one problem-solving cycle, but he was able to cycle through the phases planning, executing, and checking multiple times when his first conjecture was not the solution.

Throughout his work on Task 1, Daniel utilized the problem-solving attributes of resources and heuristics. Specifically, he used the resources of trigonometric functions and graphs and analytic trigonometry and the heuristic of substituting values.

For his work on the first part of Task 1, part 1 prompted Daniel to begin by making a graph of a single period of sine and cosine, as shown in Figure 5. His graph was accurate and assisted him in drawing upon the resource of his knowledge of graphs of trigonometric functions as he worked through the remaining parts of the task. For example, he explained the reason that a translation of 2π would translate cosine onto itself by saying “Because cosine has a period of 2π and so this is called the phase shift [points to $\cos(x + 2\pi)$]. Which would be, if you add 2π to x you're just shifting it to the left [motions to left] by 2π and since it repeats every 2π [makes several cosine graphs with hand to show repeating] and you are just shifting everything over to the left.” This statement shows that he connected his knowledge of the period of cosine on a graph with the translation he was being asked to find. He was able to use a similar understanding of the graphs of sine and cosine to reason about his solutions across Task 1.

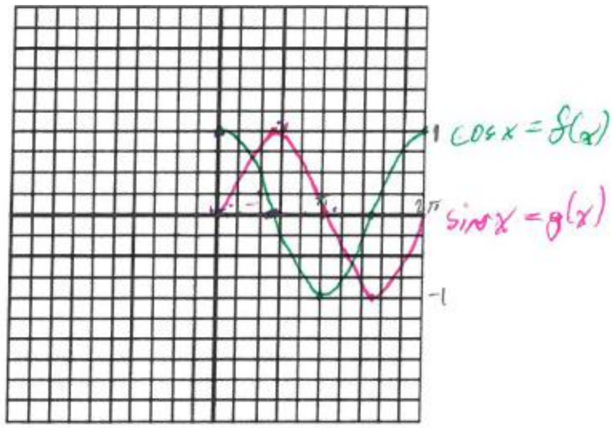

Figure 5

Daniel's Graphs of Sine and Cosine

Daniel also reasoned about his transformations using the heuristic of substituting values. For example, when he was trying to determine the reflection for sine, he reasoned through his solution with specific values.

> Yeah, reflected across the origin. Which I don't [pause] don't remember what to do. … flip it, and you'd switch x and y. And then you'd make it… $(1,1)$ go to $(-1,-1)$ … $(1,2)$ I would want to go to $(-2,-1)$, so then I would just make everything negative, so I think it'll be like negative y equals … so okay, okay, I got it, $-sin(-x)$.

Using a specific value from the coordinate plane to test his reasoning confirmed that he was correct to negate both the x coordinate and y coordinate.

Task 2: Pythagorean Theorem and Pythagorean Identity

While solving Task 2, Daniel engaged in the problem-solving phases, as seen in Figure 6.

	Part 2		Part 1				Part 3				
Orienting	x						x				
Planning			x					x			x
Executing		x		x					x	x	
Checking					x	x					

Figure 6

Daniel's Phases on Task 2: Pythagorean Theorem and Pythagorean Identity

When asked to show $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$ for right triangle ABC, Daniel immediately recognized the equation as the Pythagorean Identity. This enabled him to jump from orienting to executing without verbalizing any planning. When asked to explain it without trigonometry, Daniel was able to use his previous insights of the Pythagorean Identity to explain how it was a modified form of the Pythagorean Theorem resulting from dividing each term by AC. In the checking phase, Daniel acknowledged that if he were to write up his explanation formally, he would start with the Pythagorean Theorem and not the given equation, so what he had written as work was backward from the formal proof.

For the final part of Task 2, Daniel initially grappled with reasoning about the location of θ in relation to the triangle, because he was unsure whether to place theta inside of his second quadrant triangle or as the rotation from the positive $x - axis$ to the hypotenuse of the second quadrant triangle. Despite this initial uncertainty, he was able to use his knowledge of SOH CAH TOA to identify side lengths. As a result of his initial questioning of the location of theta, he required a planning phase before he was able to enter an executing phase for this part of the task. Finally, when Daniel was asked if there was another way to find the solution, he again cycled through planning and executing. However, his alternative solution was nearly identical to his first solution.

While working on Task 2, Daniel utilized several resources and attributes. These included the resources of the Pythagorean Theorem, trigonometric identities, SOH CAH TOA, and the unit circle. He also used the heuristics of drawing a diagram, seen in Figure 7, and written arithmetic computations.

When reasoning about the given equation, $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$, Daniel used resources of both the Pythagorean Identity and Pythagorean Theorem. He first identified the equation as a form of the Pythagorean Identity, because he recognized that $\frac{|AB|}{|AC|}$ and $\frac{|BC|}{|AC|}$ were equivalent to $\sin\theta$ and $\cos\theta$. When asked to reason about the equation without using trigonometry, Daniel identified it as a form of the Pythagorean Theorem, since multiplying through by AC would give the equation $AB^2 + BC^2 = AC^2$ with AC as the hypotenuse.

Working on the final part of the question, Daniel immediately drew the right triangle in the second quadrant, seen in Figure 7, and labeled the sides, explaining that he knew where to label the sides "[b]ecause sine is opposite over hypotenuse."

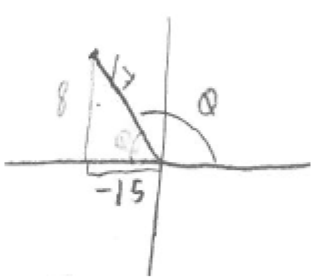

Figure 7

Daniel's Diagram of Triangle in Second Quadrant

This showed that he was using his knowledge of the trigonometric relationships between the sides and angles in a right triangle. He also solved for the missing side of the triangle using both $17^2 - 8^2$ and $1^2 - \frac{8}{17}^2$, since he reasoned that he could use the unit circle to make the hypotenuse 1 unit long or scale the unit circle by 17 to make the hypotenuse 17 units long. He then explained that his calculation of 15 or $\frac{15}{17}$ had to be negative, because cosine is negative in the second quadrant. Though Daniel was able to draw upon his knowledge of the Pythagorean Identity when reasoning about the initially given equation, he did not draw upon this knowledge when solving for a specific triangle in the final part of Task 2, even when asked if there was another solution.

As part of his initial solution for finding cosine, given $\sin\theta = \frac{8}{17}$ in the second quadrant, Daniel drew the diagram in Figure 7. This diagram helped him to reason about where theta was located in relation to the second quadrant triangle with sides 8 and 17 because he was able to draw the location of theta. As part of his later explanation for why the 15 had to be negative, he labeled the quadrants with ASTC for the mnemonic "All Seniors Take Calculus" that he had been taught in high school. This mnemonic helps him to know which trigonometric functions were positive in each quadrant (<u>a</u>ll in the first quadrant, <u>s</u>ine in the second quadrant, <u>t</u>angent in the third quadrant, and <u>c</u>osine in the fourth quadrant). He knew from the mnemonic that the sine value was positive in the second quadrant, and thus cosine must be negative in the second quadrant. Also, as part of his solution, he wrote out the arithmetic to calculate the missing side of the triangle.

Task 3: Which One Doesn't Belong?

For his two solutions to Task 3, Daniel engaged in problem-solving phases across two solutions, as seen in Figure 8.

	Solution 1						Solution 2									
Orienting	x															
Planning		x					x		x		x		x			
Executing			x		x			x		x		x		x		
Checking				x		x									x	x

Figure 8

Daniel's Phases on Task 3: Which One Doesn't Belong?

For his first solution, Daniel quickly oriented himself to the task and then planned by noticing, "Um, one thing that I see here is that we got degrees that end in 0. So that could be something." After this, he completed two cycles of executing and checking. During these phases, he identified commonalities between two or three of the given expressions and the expression he was trying to come up with, then determining whether it would exclude his potential fourth expression. For example, during his first checking phase, he stated that his exclusions would be, "So then there would be these three, I think the group would be, this one's all the zeroes. This one's all got twos and then one that excludes this one." Daniel was able to determine his first solution of $\sin 20°$ in under four minutes, so he was challenged to find a second solution.

For his second solution, Daniel engaged in four alternating phases of planning and executing with a concluding phase of checking. During these planning and executing phases, Daniel considered various properties of the expressions, including which quadrant the angle was located in, their evaluated value, and the value of their angle. Each pair of planning and executing phases generally consisted of identifying a characteristic and determining what about that characteristic was shared by several of the expressions before crafting his expression to include or exclude that common characteristic. For example, one phase of planning consisted of Daniel saying, "So now I just gotta worry about this one. Uh, [pause] $\cos 120°$, $\cos 150°$, $\sin 225°$? This one is something special. Wait a minute. This one's supposed to be negative."

This was followed by an executing phase where Daniel said, “So all these ones are negative. Is that supposed to be negative? It's the cosine in the second quadrant, and so it should be negative. So, these three are all negative and that one's not.” This back and forth between planning for a characteristic of a solution followed by executing that characteristic in the solution allowed Daniel to work towards his solution efficiently. He determined a solution of $\cos 150°$ and concluded by checking that his solution did exclude each of the four expressions.

Throughout his second solution to Task 3, Daniel consistently used his knowledge of the unit circle. He began by finding the values of several of the given expressions, “…this one is in the second quadrant. It's $30°$ here. So, what's the opposite of $30°$? That's the $\frac{1}{2}$, so this one is $\frac{1}{2}$, opposite of $30°$ is $\frac{1}{2}$. So, you had $\frac{1}{2}$. So, I'm going to go with this one is $225°$ is in the uh, third quadrant. So, and how much above $180°$? $45°$, so, this one is going to be $\frac{\sqrt{2}}{2}$, which is $-\frac{\sqrt{2}}{2}$, yeah.” By evaluating the expressions, Daniel was able to notice that none of the given expressions contained a three and used that to begin formulating his solution. His initial conjecture for an answer was $\cos 30°$ since it equals $\frac{\sqrt{3}}{2}$. Next, Daniel again used his knowledge of the unit circle by noticing which quadrant the angles were located in and said, “Um, all these ones are in the second quadrant, or they're not. … How about, how about now I can use that though. … Yeah, let's do $\cos 150°$.” Noticing that two of the three given expressions were located in the second quadrant caused him to modify his solution to also be in the second quadrant. Daniel concluded by checking that he had found a characteristic to exclude each of the expressions.

Task 4: Define Sine of an Obtuse Angle

Daniel’s work on the multiple parts of Task 4 proceeded through the problem-solving phases, as seen in Figure 9.

	Part 1					Part 2		
Orienting	x							
Planning		x				x		
Executing			x		x		x	x
Checking				x				

Figure 9

Daniel's Phases on Task 4: Define Sine of an Obtuse Angle

In the first part of the task, which required helping a student define the sine of an obtuse angle, Daniel proceeded through a single problem-solving cycle of orienting, planning, executing, and checking. After reading the task prompt, Daniel repeated: “Draw a picture to explain how Joyce might find the sine of an obtuse angle.” This showed that he was orienting himself to the problem and understanding its goal. During the planning and executing phases, Daniel drew the diagram seen in ?. He explained this as “Draw, um, and then draw the angle. Whatever it is, you're the 90° ends, and then it'd be like that. And then if I just go down straight down here, then I have a right triangle right here. … Um, and it turns out that if you take the sine of this angle here, then that is the same as the sine of this angle.” Daniel was able to explain further that the triangle in the second quadrant would have the same sine value as the triangle in the first quadrant, because it would have the same opposite and hypotenuse, thus allowing the student to use their SOH CAH TOA definition of sine.

Figure 10

Daniel's Diagram of Sine of an Obtuse Angle

For the second part of the task, Daniel planned for both expressions and then executed once for each expression. In the executing phase for $\sin\frac{3\pi}{4}$, Daniel explained using “bow-tie” angles, as seen in Figure 11, that he had been taught in high school.

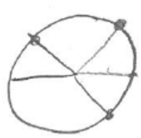

Figure 11

Daniel's "Bow-Tie" Angles

For him, this meant that the triangle for $\frac{3\pi}{4}$ was created by using the angle $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$ and indicated that he could use his knowledge of the triangle formed in the first quadrant by the acute angle of $\frac{\pi}{4}$. Then he simply needed to use his knowledge of the unit circle or coordinate plane to determine the correct signs. For the executing phase for $\sin\pi$, Daniel used the unit circle definition, which he explained as “Um, so that would be like on the unit circle, that would be like directly on the other side. So, if we start at y of 0, and then we go all the way halfway around the circle, then we're still gonna end up back at y is $0\ldots$ So $\sin\pi = 0$.”

Throughout his work on Task 4, Daniel used the resource of the unit circle and the heuristic of diagrams. As he completed the first part of the task, Daniel explained that the definition of sine could be extended from SOH CAH TOA to a unit circle definition as

> But if we want to like extend our definition of sine if we want sine to mean more things like … in an obtuse angle or like a negative angle, then we need to like extend its definition. … so, if we like draw like a unit circle and then we like draw an angle somewhere in the unit circle sine, sine is going to be the y value of that point when the angle intersects to the unit circle. So that means that we can do it with an obtuse angle. Which we would do it like this, or we can do it of a negative angle cause you could go down here, and it will be here. … negative angles are also the same thing as just a very big angle. … you can plug in any number now, any angle into sine now, and you can get a number out, and sometimes that number will be negative or positive. …So that's like extending the definition of what sine is past just what it looks like in a right triangle. I guess that's how I think about it.

This showed his understanding of the unit circle definition of sine and how it extends the right triangle trigonometric definition of sine. Later he used his knowledge of the unit circle to find the value of $\sin\frac{\pi}{4}$ as $\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$. He explained that $\sin\frac{3\pi}{4}$ would be positive since "… sine is the y value, and then we know it's going to be positive in the second quadrant here." On Task 4, Daniel was able to use his knowledge of the unit circle to both find values for trigonometric expressions and explain how the unit circle definition of sine extends for any angle, unlike the right triangle trigonometry definition of sine. Though he described the All Seniors Take Calculus (ASTC) trick to remember the signs of the trigonometric functions in each quadrant, he was able to relate that trick to his knowledge about how each trigonometric function is defined.

Daniel created three diagrams in the process of solving Task 4. First, he created a diagram showing an obtuse angle, as seen in Figure 10, and explained how a student could draw a right triangle from it to use the SOH CAH TOA definition of sine with an obtuse angle. Next, he created a diagram to show $\sin\frac{3\pi}{4}$, seen in Figure 12, which was followed by a diagram showing "bow-tie" angles, seen in Figure 11. Daniel used these two diagrams to explain that the $\sin\frac{3\pi}{4}$ was equal to $\sin\frac{\pi}{4}$, since they would have the same opposite and hypotenuse. Throughout this task, Daniel used diagrams to demonstrate how to visualize the relationships between right triangles and their trigonometric values in ways that extended to the unit circle.

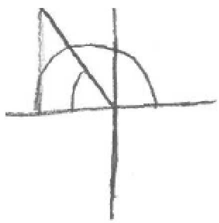

Figure 12

Daniel's Diagram of $sin\frac{3\pi}{4}$

Task 5: Graphs of Foxes and Rabbits

On Task 5, Daniel's work progressed through the problem-solving phases, as seen in Figure 13.

	Part 1			Part 2							Part 3		Part 4	
Orienting	x			x										
Planning					x						x		x	
Executing		x				x		x		x		x		x
Checking			x				x		x					

Figure 13

Daniel's Phases on Task 5: Graphs of Foxes and Rabbits

The first part of the task asked Daniel to explain why it was appropriate to model the two graphs with a trigonometric function, and Daniel was able to immediately move from orienting himself to the problem to executing his solution, without having to plan in between them. During the executing phase, Daniel described why a trigonometric function was an appropriate choice as

> It looks like to me that these are functions that repeat. They go up and down like on a set schedule. … we only have 24 months here, but it looks like there's like a place where it just starts over, and it goes up and down again.

In the checking phase of his solution, Daniel questioned whether trigonometric functions were the only functions that would fit his explanation of having a repeated period and concluded that they were the ones we know best if they were not the only family of functions that repeated.

In the second part of the task, when asked to find a trigonometric function for the fox data and the rabbit data, Daniel again oriented himself to the task. He then jumped to executing without planning in between the two phases. Daniel then alternated between several extended executing phases and checking phases. During the executing phases, Daniel determined the values of the parameters for the amplitude, period, phase shift, and vertical shift for both the function for the foxes and the function for the rabbits. For example, when working on the function for the foxes, Daniel explained his calculations for the amplitude as "… so the amplitude, it goes from 110 all the way up to 170. So that would be 60. And then also just to check $110 - 60 = 50$. So that would mean that we're going to multiply by 60, the amplitude is 60." During the executing phase, Daniel was able to connect the values on the graph with the values of the parameters he was calculating. During one of the checking phases, Daniel was unsure about his calculation of the parameter for the period, so he checked his calculation by substituting values and determined that he had been correct.

During the solving of Task 5, Daniel used the resource of knowledge of trigonometric graphs and heuristic of substituting values. Daniel first used his knowledge of trigonometric graphs to explain why they would be an appropriate model for the data, since "…it's appropriate to model them as trigonometric functions because they have a repeated period." This shows that the data given matched what he visualized as the graph of a trigonometric function. Daniel used his knowledge of trigonometric graphs throughout finding the equations for the foxes and rabbits. For example, when finding the vertical shift and parent function for the rabbits, Daniel described his process as,

> So, we can make 1000, since that seems like it's in the middle, it would be like close to the average. We would say we want to start at 1000. And if we're starting at the middle, then we want to use sine cause sine starts at zero. Whereas cosine starts at its maximum. So, sine of something plus $1000\ldots$ when sine is 0, we want it to be 1000.

This shows that he knows how to locate the midline of sine as the average and to place it in the equation by adding it to the end as the vertical shift. When he continued on to determine the amplitude for the equation for the number of rabbits, Daniel explained

> We want it to fluctuate above and below. Um, so then the number that we want to multiply by would be the amplitude, and it looks like it varies by like 500 … So, when the sine is one, which is its' maximum, it will be multiplied by 500 [and] plus 1000. So that would be 1500, which is what it looks like the maximum is. The minimum is similar because it looks like it goes down to 500, so we have sine's minimum is -1. So, it would be $-500 + 1000$ would give you 500, so that would be the minimum.

Daniel is again using his knowledge of trigonometric graphs since he is estimating the amplitude by how much the data vary above and below the midline. He is further using his knowledge of the graph to identify the minimum and maximum of the parent function and identify those as corresponding to his function for the rabbits. Next, Daniel identified that he didn't need to find a phase shift because he had chosen to use the sine function.

> I don't have to add anything to the t because that is the phase shift, and that would just shift it over, and if I wanted to shift it over, I would have just used cosine. Because cosine is the sine shifted over. If it didn't start at its maximum, minimum, or the average, then I would have to use some kind of phase shift. But I'm just going to assume in both of these that it starts at the maximum or the average because I don't want to deal with the phase shift.

Daniel was able to use his knowledge of trigonometric graphs to identify that he didn't need a phase shift because he had chosen the sine function. Because the sine function starts at the average, as the rabbit data starts at the average, Daniel did not need to have a phase shift to move the function to the correct starting point. Finally, Daniel described finding the parameter, b, for the period as

> So, then what I multiply by t is some is like, I think they call it b… and then the period would be $\frac{2\pi}{b}$. … And so, the period here looks like it starts repeating after like 12 months, which is a year. … So, to make $\frac{2\pi}{b} = 12$. … $b = \frac{\pi}{6}$.

Daniel was able to identify the period of the data as 12 months and use that to solve for the parameter b using the equation $\frac{2\pi}{b}$. The only written work Daniel had in finding the two equations was the algebra to verify the calculations for b related to the period. Daniel also noticed that the data started by decreasing from the average, rather than increasing from the average like the sine function, so reasoned that there needed to be a negative in front of the amplitude to flip the function to match the data. With this final calculation, Daniel was able to find the equation for the rabbits of $r(t) = -500\sin(\frac{\pi}{6}t) + 1000$. When working on finding the equation for the foxes, Daniel provided similar reasoning using his knowledge of trigonometric graphs for each parameter to find the equation of $f(t) = 60\cos(\frac{\pi}{6}t) + 110$.

When Daniel questioned whether the parameter he had calculated for the period was correct, he checked his answer by substituting values.

> I feel like I have the $\frac{2\pi}{b}$ formula wrong. I'm going to test this. Well, I can just plugin. So, t is 12. This will be π, no 2π. And that's what I want it to be. Never mind, that works. And then $1t$ is six, which is half the period, it would be π, which would be half of 2π.

Substituting in 12 for a full period and 6 for a half period allowed Daniel to verify that his parameter of $\frac{\pi}{6}$ was correct.

Task 6: Sum and Difference Identities

As Daniel worked on verifying the five sum and difference identities for Task 6, his problem-solving phases progressed as seen in Figure 14.

	Part 1			Part 2								Part 4				Part 5		
Orienting	x			x								x				x		
Planning					x					x			x				x	
Executing		x				x		x			x			x				x
Checking			x				x		x						x			

Figure 14

Daniel's Phases on Task 6: Sum and Difference Identities

For the sine difference angle identity, Daniel first oriented himself to the task by reading the prompt and then immediately began an executing phase. Daniel was able to recognize that $\sin(\alpha - \beta) = \sin(\alpha + -\beta)$ and use this fact to apply the given sum angle identity for the difference. He was able to work through the algebra to simplify, as seen in Figure 15.

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Figure 15

Daniel's Work on Task 6 Sine Difference Angle Identity

During the checking phase, Daniel explained his reasoning as

> "So, cosine is even. So that means that if you put a negative number into cosine, it's the same as the positive number. … and then sine is different in that if you take the sine of a negative number, it's the negative sine of that number positive. So, I just got for the $\cos -\beta$, that meant I could just get rid of the negative. And then for the $\sin -\beta$, I just made that $-\sin\beta\ldots$"

Daniel was able to use this reasoning about the even and odd identities to explain later how he could also transition from the sine and tangent angle sum identities to the sine and tangent difference angle identities. For the first identity of the task, Daniel was quickly able to identify his solution and execute that solution.

For his work on the cosine sum angle identity, Daniel oriented himself to the next part of the task and formulated a plan to use the identity $\cos\theta = \sin(\theta + \frac{\pi}{2})$ that had been used on a previous task. During the executing phase, Daniel applied the sine sum angle identity that had been given and began to simplify, as seen in Figure 16.

$$\cos(\alpha+\beta) = \sin\left((\alpha+\beta)+\frac{\pi}{2}\right) = \cancel{\sin(\alpha+\beta)\cos\left(\frac{\pi}{2}\right)} + \cos(\alpha+\beta)\sin\left(\frac{\pi}{2}\right) =$$
$$\sin\left(\frac{\pi}{2}-(\alpha+\beta)\right) = \sin\left(\frac{\pi}{2}\right)\cos(\alpha+\beta)$$

Figure 16

Daniel's Work on Task 6 Cosine Sum Angle Identity

Daniel recognized that $\cos\frac{\pi}{2} = 0$, so that term containing that would equal 0, and he would be left with just the other term. Since $\sin\frac{\pi}{2} = 1$, simplifying gave him a redundant conclusion of $\cos(\alpha+\beta) = \cos(\alpha+\beta)$. In another cycle of planning, executing, and checking, Daniel also tried the version of the identity from the formula sheet, $\cos\theta = \sin(\frac{\pi}{2} - \theta)$, but he recognized that he would end up in the same redundant conclusion, so he did not complete the work, as seen in Figure 16.

Once Daniel realized that he had reached a dead-end with his first strategy, he entered a planning phase of trying to find a new strategy. He was able to identify a variation of the Pythagorean Identity, $\cos(\alpha+\beta) = \sqrt{1-\sin^2(\alpha+\beta)}$ as another identity relating sine and cosine. His initial work on this can be seen in Figure 17.

$$\cos(\alpha+\beta) = \sqrt{1-\sin^2(\alpha+\beta)} = \sqrt{1-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2} = \sqrt{1-(\sin^2\alpha\cos^2\beta + 2\sin\alpha\cos\alpha\sin\beta\cos\beta + \cos^2\alpha\sin^2\beta)}$$

Figure 17

Daniel's Continued Work on Task 6 Cosine Sum Angle Identity

Daniel continued to work through the algebra required by this solution in an executing phase. As Daniel worked, he described some of his process as

> "What a mess. ... So, I want to get this so I can factor out so that I can have my other stuff. It would be really nice if I could get like a, [pause] so I've got $\sin\alpha\sin\beta$ here. But

it's also got $\cos\alpha\cos\beta$ and there's also this 2. So, there's just a lot of stuff right here, right there. And then these ones, I'm not sure what to do with them, but I like, cause I could like take out like a... Um, if we were to factor out like a $\sin\alpha$, we'd get this. Maybe I should have split this middle one up because $\sin\alpha\cos^2\beta + \sin\beta\cos^2\alpha$... I can just kind of factoring back into what it was."

Daniel spent a total of nearly 18 minutes working in this executing phase in his attempt to get the algebra to simplify to what he was looking for. Eventually, he identified another possible solution path for the algebra within this form of the Pythagorean Identity that he believed would work. For the sake of time, he was asked to move on to the tangent sum angle identity.

Daniel began by orienting himself to the next part of the task and then planning how to approach the problem. He said, "So usually when you do stuff with tangent, you just make it sine and cosine." He was able to proceed with this plan, as seen in Figure 18.

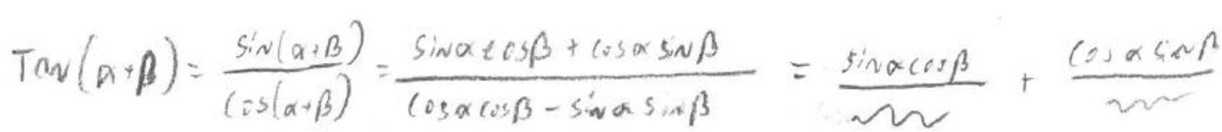

Figure 18

Daniel's Work on Task 6 Tangent Sum Angle Identity

During the executing phase, Daniel rewrote the initial expression for tangent in terms of sine and cosine. He then used the sum angle identities for sine and cosine to rewrite the expression again. He then said, "Can multiply by the conjugate, that we give me this square minus this square and then a bigger mess on the top." At this point, he was unsure how to transition from the expression that he had found to the expression that he wanted to end up with. He decided to start with the form he was hoping to end up with and work backward, as seen in Figure 19.

$$\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\frac{\sin\beta}{\cos\beta}\right)} = \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\frac{\sin(\alpha+\beta)}{\cancel{\cos\alpha\cos\beta}}}{\frac{\cos(\alpha+\beta)}{\cancel{\cos\alpha\cos\beta}}} = \tan(\alpha+\beta)$$

Figure 19

Daniel's Continued Work on Task 6 Tangent Sum Angle Identity

Daniel explained that the work he did in Figure 19 was backward from what should be his final answer as,

> So, if we did all that backward where we turn this into this and then un-cross those out and then went and then kind of dissected that out. Yeah, we like wrote out the old thing for these formulas and then went backwards further and split it up and then multiple, and then like all this stuff went backwards. If you just read it backwards…

Daniel was able to cycle through the problem-solving phases by recognizing that he could rewrite tangent in terms of sine and cosines or sines and cosines in terms of tangents. This allowed him to derive the tangent sum angle identity successfully.

Throughout the task, Daniel used the resources of knowledge of trigonometric graphs and trigonometric identities as well as the heuristic of algebra. When Daniel worked on the sine difference angle identity, he used his knowledge of trigonometric graphs when he identified that cosine is an even function and sine is an odd function. He used his knowledge of the graphs to reason about $\cos -\beta$ and $\sin -\beta$ as he simplified the identity. He used his knowledge of identities several times as he worked on the different identities he was verifying. For example, he started the sine sum angle identity by saying "So we just want to start with $\sin(\alpha - \beta)$ is the same thing as $\sin(\alpha + -\beta)$, which is using this sum-angle formula." He recognized the relationship between the identity he was asked to verify and the one he was given. When working on the cosine sum angle identity, Daniel identified that he could use both the cofunction identity of $\cos(\alpha + \beta) = \sin((\alpha + \beta) + \frac{\pi}{2})$ and the form of the Pythagorean Identity of $\cos(\alpha + \beta) = \sqrt{1 - \sin^2(\alpha + \beta)}$. Though he was not able to complete verifying the cosine sum angle identity during the interview, his knowledge of trigonometric identities allowed him to pursue two paths. Daniel did not write the work down for the tangent difference angle identity, but he identified his potential strategy as

> I guess I could also do it using the even/odd thing and turn these into sines and cosines, and then they would just be, all the cosines would get rid of the negative on the β and then … the sine in that $\tan \beta$ would take the negative out, and that's why it would be a plus here instead of a one minus.

He is again using his knowledge of identities, by recognizing that he could rewrite tangent in terms of sine and cosine and that he could use the even/odd identities to explain why the signs changed from the tangent sum angle identity to the tangent difference angle identity. As Daniel worked on the various parts of the task, he wrote out the algebraic steps to demonstrate how he was verifying the identities, as seen in Figure 15, Figure 16, Figure 17, Figure 18, and Figure 19. Having the algebra written down helped him both when he was struggling to figure out how to proceed on the cosine sum angle identity and when he was working both forward and backward on the tangent sum angle identity. He even asked himself, "Do I have anything resembling what I want?" as he looked at his work, which shows that he was using his algebraic work to help with his reasoning.

Task 7: Horizontal Shrink vs. Horizontal Stretch

For Task 7, Daniel's work proceeded in phases, as seen in Figure 20.

	Part 1				
Orienting	x				
Planning		x		x	
Executing			x		x
Checking					

Figure 20

Daniel's Phases on Task 7: Horizontal Shrink vs. Horizontal Stretch

Daniel's initial orienting and planning phases were very brief and followed by a longer executing phase. During the first executing phase, Daniel described that $\sin(2x)$ is a shrink of $\sin x$ by

> When you plug in a number into $\sin x$, um, you get out, you get out something, but when you plug in that same number into $\sin(2x)$, you're twice as far along as you were before. … So, you're, so you're like twice, you're going twice as far with the x. You're like, you're going, you're starting with one, and then you're going, you've might as, it's the same thing as if you plugged in two into f, so that shrinks it because you're getting there faster because you're going places, you're going everywhere twice as far.

When asked how he would explain this to a student, Daniel expanded his explanation as

> So, everything you plug into $\sin x$, you get there twice as fast when you have $\sin(2x)$. … If you think of like x as like, I guess I'm thinking of x as like time. So, you're in your truck, and like how far you go is how far you're traveling along the line. So, as you walk with, walk along sin(x), um, you get to, you get back to zero in π seconds, I guess. But when you walk along the $\sin(2x)$, you get to zero, you get to zero in $\frac{\pi}{2}$ seconds because

> you're walking twice as fast. So, if you want to go through a whole period, you, it takes you 2π seconds inside of x and when you want to go a whole period. Or what you could do is since you do everything twice as fast and $\sin(2x)$ and it only takes you π to go through. I don't know if that would make sense to some students I feel like, but not every student.

These descriptions show that Daniel is considering the input of the two functions as the reason for $\sin(2x)$ being a horizontal shrink of $\sin x$. He describes the reason as plugging the same x into both functions resulting in an input that is twice as far along the x-axis for $\sin(2x)$. Thus, a smaller x-value in completing a full period of the function, so the result is a horizontal shrink.

Daniel's explanation also used the heuristics of a simpler example and substituting values. Daniel described the shrink for the function $f(x) = x^2$, because he considered that to be a more straightforward example to understand.

> Okay, well the way I like to think about it is like $f(x) = x^2$. That one makes more sense to me. When you plug in 1 into $f(x) = x^2$, you get out 1. So, like $f(1) = 1$. And then $g(x) = (2x)^2$, $g(1)$ you double it and then before you do anything else. … You're starting with 1, and then it's the same thing as if you plugged in 2 into f, so that shrinks it because you're getting there faster because you're going everywhere twice as far. So, like when you plug in 1 into $f(x)$, you'd get the one, but when you plug 1 into $g(x)$, you double it and then square it. … if you go over 1, you're going to go up 1 in $f(x)$. But if you go over 1 in $g(x)$, you're going to go up twice as far … and then same for $\sin x$. If you go over like π in $\sin x$, you're going to go up and down. But if you go over π in $\sin(2x)$, it's like going 2π, so you're going to go up, down and back up. I don't think a student would understand what I just said.

Daniel explained that he had recently taught some students in his practicum about the concept of a horizontal shrink using the example of $f(x) = x^2$. Because he thought it was easier for students to calculate. He had shown them by plugging in numbers to both functions to demonstrate that the function shrinks horizontally and said this is a pattern that students could notice and extrapolate to $\sin x$.

Task 8: Proving an Identity

Daniel's work on Task 8 proceeded through the problem-solving phases, as seen in Figure 21.

	Part 1				
Orienting	x		x		
Planning					
Executing		x		x	
Checking					x

Figure 21

Daniel's Phases on Task 8: Proving an Identity

After orienting himself to the task, Daniel immediately entered an executing phase. During this phase, Daniel immediately recognized that the students' proof was not valid as it was written. He explained,

> Technically if the student wrote it backwards, this would be a good proof because … you start with something that's true. And so basically what she did was say 1 = 1 she knows that's true. And then she turned one into a different format. … And then she just converted everything, switched everything, did the fractions or whatever. She just used identities and then got here. So, what she did, but this going the other way is not, is not a proof because writing it like this, starting with this assumes what you're trying to prove. So, if she had started the other direction and started with 1 = 1 and then went from there,

> then this would be a valid proof. But since she started with what she wanted to prove, then it's not.

Daniel recognized that the students had performed mathematically correct manipulations, but that the order the proof was written in started with what she wanted to prove. This meant that she had assumed the identity was true with her proof, rather than starting with something that was known to be true.

Following this, Daniel was asked if it was possible for a student to start with the given equation of $\sin x * \cos x * \tan x = \frac{1}{\csc^2 x}$ and complete a valid proof. Again, he oriented himself to the follow-up question and moved directly to an executing phase. He explained that it was possible to achieve a correct proof if she hadn't multiplied both sides of the equation by $\cos^2 x$, since that assumed the equation was true.

> She could have, she could have started with them not equal … And then manipulated from here and then didn't multiply on both sides by $\cos^2 x$. … She would have to keep both sides separate. She can't multiply both sides by $\cos^2 x$, cause that assumes what she's trying to prove.

Though Daniel stated that it was possible to start with $\sin x * \cos x * \tan x = \frac{1}{\csc^2 x}$ and end up with a correct proof by only manipulating the expression on one side of the equal sign, he stated that this would be more confusing for students. During his checking phase, he concluded

> Because if you do it backwards, it is a valid proof. So, there's no reason to tell her not to do that. Cause if I was like trying to prove something … this would be like on scratch paper and then on the original paper I would just write exactly this backwards. Once I've proved it, I mean it's better to try to make it more streamlined mind, but it's not really that big a deal.

Daniel clearly showed an understanding of the conventions of writing proofs as well as strategies for solving the proof before writing it up following the conventions. He was also able to consider how to support students in writing proofs and addressing their misconceptions.

Task 9: Angles in a Rectangle

For Task 9, Daniel’s problem-solving phases proceeded, as seen in Figure 22.

	Part 1									
Orienting	x		x							
Planning				x			x			x
Executing		x			x			x		
Checking						x			x	

Figure 22

Daniel's Phases on Task 9: Angles in a Rectangle

Note. The video cut off after 31 minutes, so approximately 5 minutes of work was not included.

Daniel began orienting himself to the task by reading the prompt and marking information on the provided diagram. He was able to conclude that $BP = PQ = QC = 1$ since it was given that $BP = PQ = QC$ and $BC = 3$, so the segment was split into thirds of length 1 or $1AB$. He moved from a planning phase of writing down information that was given and that he had concluded to an executing phase. During the executing phase, Daniel identified, “ΔDQC is a 45-45-90 triangle. Cause it's one on both sides, on both the legs. Um, so that means that $DQ = \sqrt{2}$.” Daniel continued to make a few more conclusions and notes based on the initial information and continued to mark the diagram seen in Figure 23. Because he had identified that $m\angle DQC = 45°$, he knew that his goal was to show $m\angle DBC + m\angle DPC = 45°$.

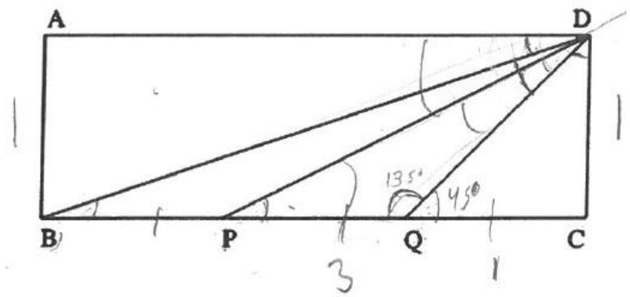

Figure 23

Daniel's Task 9 Marked Diagram

As Daniel worked through his next cycle of planning-executing-checking, he continued to keep a record of information that he had deduced, as seen in Figure 24. He was focused on his knowledge of angles and their properties.

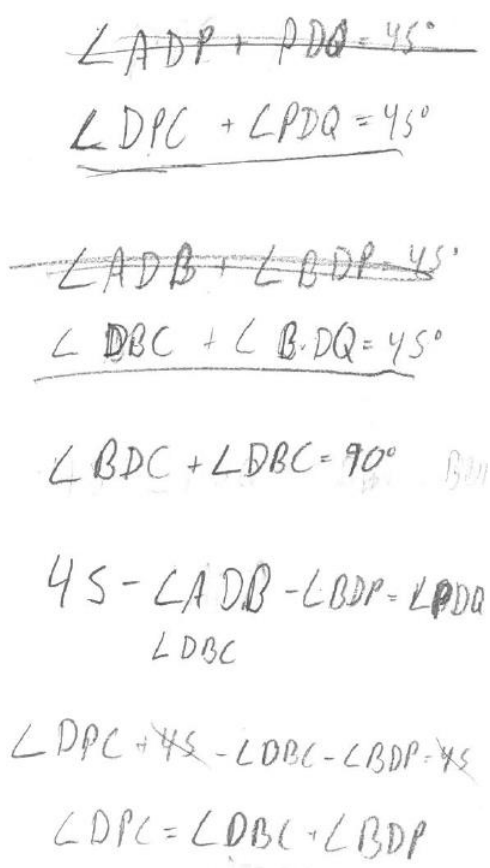

Figure 24

Daniel's Task 9 List of Information

> How do we get PDQ? We get PDQ to equal. That's equal to DPC, DBC. That's weird. Um, okay, so we got DPC, and now we just want some angle here. This angle here, which is the same as this angle here, to be the same as this angle, PDQ. Um, why would this angle be the same as that one? So, this is PDQ, BDP. Well, I mean, I can write $\angle ADB + \angle BDP$, it was 45°. And, $\angle ADP$ is the same as $\angle DBC$. … so, then if I wrote these two, these two are equal to each other, but we knew that. We can't really get rid of anything with that either. This PDQ is part of this triangle, BDP is part of this triangle. Well, writing down more stuff like this, same as they are here, but $45° - \angle PDQ - \angle BDP = \angle ADB$, which is the same as $\angle DPC$. Uh oh. But I don't think I did it correctly. If we wanted to like to move one of these over and then like add all these together, it would give us 90°. Well, let's do that. Let's put $\angle TBC$ here, then $\angle BDP$ here. And then if we add all these, so we add this to this. We get, oh, if we add, if we put this in for this, then we just get... We get rid of this? Which would mean that this is your, that's not true, so that this must be, something's wrong.

Daniel was using both angles within triangles and alternate interior angles to work towards the conclusion that $m\angle DBC + m\angle DPC = 45°$. He was able to identify several other pairs of angles that summed to 45° but hadn't found the relationship he was trying to show yet. After discovering that he had made a mistake somewhere in his calculations, Daniel started a new cycle with a new strategy.

In his next problem-solving cycle, Daniel identified that he had three right triangles in the diagram and tried to work within the triangles.

> I've got two triangles here, and then we know these two things are true, and we want to do something with this. We also have another bigger triangle, a 90° triangle, right

> triangle is here. The right triangles... So $\angle DPC + \angle BDC = 90°$. The side is not… This triangle is the same as the big triangle here. I don't know if that helps. Also QDC … I just want to turn PDQ into DBC. So that's BDQ is this angle here in this triangle, so this triangle, that's the same thing. … Which would mean this angle and that angle, which means these are similar triangles, but how can I show that?

Daniel continued to work with the angles in the right triangles by adding and subtracting to show that $m\angle DBC + m\angle DPC = 45°$. Throughout the task, Daniel was able to keep in mind his goal of trying to show that $m\angle DBC + m\angle DPC = m\angle DQC = 45°$ and work with relevant properties of angles and triangles to try to reach that goal. Unfortunately, despite spending nearly 40 minutes working on the task, he was not able to find a solution to the task.

Throughout the task, Daniel used his knowledge of geometry to approach the task. Initially, he used properties of rectangles in “Since as a rectangle, then this side is also one” and “$AB = DC = BP = PQ = QC$. Check to make sure that's right. So that's three. And then one-third of this is AB.” He then began using properties of angles, such as “Well, if this is a 45° angle, then this is a $135°$ angle. These are alternate interior angles, right? AD and BC, that means that's got to be 45°.” He recognized that two supplementary angles have to sum to $180°$ and that alternate interior angles are congruent. Daniel also used his knowledge of special right triangles. He was able to recognize that ΔDQC with legs of length 1 and hypotenuse length $\sqrt{2}$, though he was not sure how to use this information in his problem-solving and disregarded it.

Analysis by Domain

Daniel used a variety of resources and heuristics as he solved the tasks across the four trigonometric domains, as seen in Figure 25. This diagram shows what resources and heuristics Daniel used as he solved tasks in each domain. For example, when working on tasks in the domain of triangle trigonometry, Daniel used resources from geometry, analytic trigonometry, triangle trigonometry, and unit circle trigonometry, as seen in the top left corner of the diagram. He also used heuristics of written algebra and created diagrams when solving triangle trigonometry tasks. Similarly, each of the other quadrants shows what resources and heuristics Daniel used to solve tasks in the domains of unit circle trigonometry (top right), trigonometric functions and graphs (bottom left), and analytic trigonometry (bottom right).

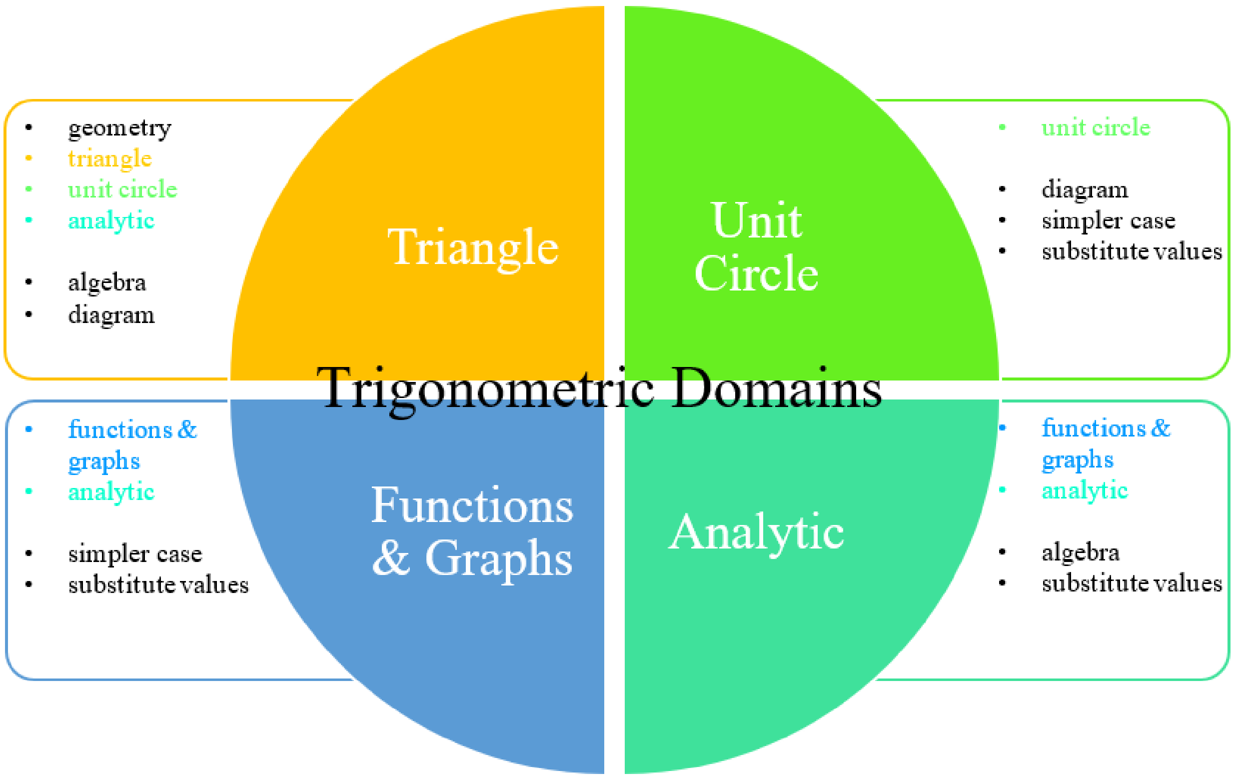

Figure 25

Daniel's Use of Resources and Heuristics Across Domains

Daniel used knowledge of some domains across multiple domains, but others in only that domain. He used knowledge of geometry and triangle trigonometry only on tasks in the domain of triangle trigonometry. He used knowledge of circle trigonometry in the domains of triangle trigonometry and circle trigonometry and knowledge of trigonometric functions and graphs on tasks in the domains of trigonometric functions and graphs and analytic trigonometry. Daniel used knowledge of analytic trigonometry when solving tasks in the domains of triangle trigonometry, trigonometric functions and graphs, and analytic trigonometry.

Daniel’s use of heuristics also varied across the domains. He used diagrams on tasks in the domains of triangle trigonometry and unit circle trigonometry. He used the strategy of substituting values in the domains of unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. He used written algebra in the domains of triangle trigonometry and analytic trigonometry and used a simpler case in the domains of unit circle trigonometry and trigonometric functions and graphs.

Triangle Trigonometry

The three tasks that addressed the domain of triangle trigonometry were Tasks 2, 4, and 9. For Tasks 2 and 4, Daniel was generally able to successfully solve each part of the task within a single problem-solving cycle, as seen in Figure 26. Within these single cycles, he generally proceeded through the phases in the order of orienting, planning, executing, and checking. For Task 9, Daniel required multiple problem-solving cycles, as seen in Figure 26. Though his initial orienting phase was interrupted by a short executing phase, his work on this task otherwise follows the cycle theorized by the Multi-Dimensional Problem-Solving Framework (Carlson & Bloom, 2005). Daniel initially orients himself to the problem and then proceeds through a planning, executing, and checking phase and then a second cycle of planning, executing, and checking.

Task 2											
	Part 2		Part 1				Part 3				
Orienting	x						x				
Planning			x					x			x
Executing		x		x					x	x	
Checking					x	x					

Task 4								
	Part 1					Part 2		
Orienting	x							
Planning		x				x		
Executing			x		x		x	x
Checking				x				

Task 9										
	Part 1									
Orienting	x		x							
Planning				x			x			x
Executing		x			x			x		
Checking						x			x	

Figure 26

Daniel's Phases on Triangle Trigonometry Tasks

When solving tasks in this domain, Daniel used the greatest variety of resources. This included knowledge of geometry, as well as knowledge of three of the four trigonometric domains. Daniel frequently used his knowledge of geometry when solving Task 9. One example of this is when he was initially trying to determine measures of angles, “Well, if this is a $45°$ angle, then this is a $135°$ angle, yes.” As Daniel worked on the task, he relied on the knowledge of angle properties to try to deduce the relationship he was trying to find. An example of when Daniel used knowledge of triangle trigonometry was in Task 2 when he used the given information of $\sin\theta = \frac{8}{17}$ to place the lengths 8 and 17 on the sides of a triangle, “Because sine is opposite over hypotenuse.” Using the knowledge of right triangles allowed Daniel to solve for the missing side needed to calculate $\cos\theta$. Daniel used knowledge of unit circle trigonometry when he was solving for $\sin\frac{3\pi}{4}$ in Task 4. He explained "So this is a $\frac{\pi}{4}$ angle and um, $\sin\frac{\pi}{4}$ is going to be $\frac{\sqrt{2}}{2}$ or um, $\frac{1}{\sqrt{2}}$." His knowledge of the unit circle allowed him to evaluate the necessary angle. Finally, an example of using analytic trigonometry occurred while Daniel was solving Task 2 and recognized the given equation as equivalent to the Pythagorean Identity. "Well, that's just the Pythagorean Identity. So this is, if this is our angle theta, then this is the same thing as $\cos^2\theta$ plus BC is the opposite side. I see that's $\sin^2\theta$." Recognizing this identity allowed Daniel to see how the sides in the given relationship were related to the trigonometric functions in the identity.

Unit Circle Trigonometry

The domain of unit circle trigonometry is addressed by Tasks 3, 4, and 7. Daniel began each of these tasks by engaging in the problem-solving phases of orienting, planning, executing, and checking. After this initial cycle, his phases varied by the task. For Task 3, he followed the initial phase with a phase of executing and checking where he concluded a solution. For his second solution on Task 3, he alternated between planning and executing phases multiple times before two concluding checking phases. On both Task 4 and Task 7, Daniel followed the initial cycle with a series of planning and executing phases. On all three of these tasks, Daniel's phases were primarily planning and executing.

	Task 3															
	Solution 1						Solution 2									
Orienting	x															
Planning		x					x		x		x		x			
Executing			x		x			x		x		x		x		
Checking				x		x									x	x

	Task 4							
	Part 1					Part 2		
Orienting	x							
Planning		x				x		
Executing			x		x		x	x
Checking				x				

	Task 7				
	Part 1				
Orienting	x				
Planning		x		x	
Executing			x		x
Checking					

Figure 27

Daniel's Phases on Unit Circle Trigonometry Tasks

For this domain, Daniel only used knowledge of unit circle trigonometry and no other trigonometric domains. For example, when solving Task 3, Daniel used knowledge of the unit circle to identify a possible solution.

> So across from or next to the $30°$ is $\sqrt{3}$. So, it's $\frac{\sqrt{3}}{2}$. Um, all these ones are in the second quadrant, or they're not. How about now I can use that, though. Yeah, let's do $\cos 150°$. Since I know that's going to be the same thing, just a negative. So $-\frac{\sqrt{3}}{2}$.

His knowledge of the unit circle allowed him to identify possible characteristics of his solution that would exclude some of the other given expressions. He was also able to reject one of his possible solutions because it did not have the characteristics he was looking for.

Daniel used three heuristics of diagrams, simpler cases, and substituting values to solve the three unit circle trigonometry tasks. Daniel used a diagram to explain why two angles had the same sine value when solving Task 4.

> Well, I would draw like the coordinate grid and then draw the angle. Whatever it is, where the 90° ends and then it'd be like that. And then if I just go down straight down here, then I have a right triangle right here. Um, and it turns out that if you take the sine of this angle here, then that is the same as the sine of this angle.

Using the diagram in Figure 11, Daniel was able to show that the two angles formed congruent triangles that had been reflected over the y-axis, and thus the two triangles would have the same since value. Daniel used both a simpler case and substituting values when solving Task 7. He explained that you would progress along the function $\sin(2x)$ more quickly than along the function $\sin x$ because "So, as you walk with walk along $\sin x$, um, you get to, you get back to 0 at in π seconds. But when you walk along the $\sin(2x)$, you get to 0 in $\frac{\pi}{2}$ seconds because you're walking twice as fast." He explained that when he had taught students about function transformations, he had used the function $f(x) = x^2$ rather than the function $g(x) = \sin x$, "… I used like the x^2 as an example cause I knew they would understand x^2 because it's like easier to calculate x^2." Using a combination of a simpler case and substituting values resulted in an explanation that Daniel felt would be more accessible to students.

Trigonometric Functions and Graphs

Tasks 1, 5, and 7 addressed the domain of trigonometric functions and graphs. Daniel's work on these three tasks generally followed the orienting, planning, executing, and checking cycle of problem-solving. For Task 1, Daniel completed a single cycle of orienting, planning, executing, and executing to complete the task of graphing sine and cosine in part 1. For part 2 of

Task 1, Daniel did not follow the problem-solving cycle as he completed the task of finding identities related to cosine. In part 3, which gave Daniel the most difficulty in Task 1, he completed four cycles of planning, executing, and checking before completing the task. Finally, in part 4 of Task 1, Daniel used knowledge from the first three parts of the task and only alternated between executing and checking. For Task 5, Daniel completed part 1 with a cycle or orienting, executing, and checking. For part 2 of Task 5, he started with a cycle of orienting, planning, executing, and checking, followed by executing, checking, and executing. For both parts 3 and 4, he only required a phase of planning and executing for each part. For Task 7, Daniel completed the task with phases of orienting, planning, executing, planning, and executing.

	Task 1														
	Part 1				Part 2								Part 3		
Orienting	x				x		x						x		
Planning		x						x							
Executing			x	x		x					x			x	
Checking									x	x		x			x

	Part 3									Part 4				
Orienting	x													
Planning		x		x			x							
Executing			x		x			x		x		x		
Checking						x			x		x		x	x

	Task 5													
	Part 1			Part 2							Part 3		Part 4	
Orienting	x			x										
Planning					x						x		x	
Executing		x				x		x		x		x		x
Checking			x				x		x					

	Task 7				
	Part 1				
Orienting	x				
Planning		x		x	
Executing			x		x
Checking					

Figure 28

Daniel's Phases on Trigonometric Functions and Graphs Tasks

Daniel used knowledge of both trigonometric functions and graphs and analytic trigonometry to solve tasks in the domain of trigonometric functions and graphs. When solving Task 1, Daniel explained that his transformation of $\cos x = \cos(x + 2\pi)$ worked “[b]ecause cosine has a period of 2π, and so this is called the phase shift. Which would be, if you add 2π to x, you're just shifting it to the left by 2π and since it repeats every 2π and you're just shifting everything over to the left.” This knowledge of the graph allowed him to connect the translation he had identified with the identity he had found for cosine. Also during Task 1, Daniel used his knowledge of analytic trigonometry when he explained the connection between the identities he found for cosine as $\cos x = \cos(x + 2\pi n)$ is the identity because translating the graph by 2π or any multiple of 2π will result in the same graph as $\cos x$ and $\cos x = \cos(-x)$ because the graph of cosine is even.

Daniel used the heuristics of simpler cases and substituting values to help solve tasks in the domain of trigonometric functions and graphs. When explaining why $\sin(2x)$ was a horizontal shrink of $\sin x$ in Task 7, Daniel explained with the simpler case of "I tried to teach something similar to this, and I used the x^2 as an example cause I knew they would understand x^2 because it's like easier to calculate x^2." Using this example that Daniel believed would be easier, he was able to explain how multiplying by 2 inside the function moved along the $x-axis$ twice as fast, so resulted in shrinking the function horizontally. Daniel substituted values to determine whether his calculation for b in the equation for the foxes was correct in Task 5. "I feel like I have the $\frac{2\pi}{b}$ formula wrong. I'm going to test this. I can just plugin. So t is 12. This will be π, no, 2π. And that's what I want it to be. Never mind that works. And then $1t$ is 6, which is half the period, it would be π, which would be half of 2π." By substituting values, he was able to verify that his calculation had been correct.

Analytic Trigonometry

Analytic trigonometry was addressed by Tasks 1, 6, and 8. While solving these three tasks, Daniel generally began each part with a problem-solving cycle of orienting, planning, executing, and checking. For Task 1, Daniel completed part 1 with an orienting, planning, and two executing phases. For part 2 of Task 1, Daniel jumped around a bit in his phases and completed the task using a combination of all four phases, but none in the order of the problem-solving cycle. For part 3 of Task 1, Daniel completed four cycles of orienting, planning, executing, and checking to solve the task. For part 4, Daniel alternated between executing and checking phases. On Task 6, Daniel began each part he completed with a cycle of orienting, planning, executing, and checking. Part 2 of this task was the only part that required more than a single cycle, and Daniel alternated between two phases of executing and a single phase each of planning and checking. To solve Task 8, Daniel initially completed just an orienting and executing phase, but solved a follow-up question with an orienting, executing, and checking phase.

Task 1															
	Part 1				Part 2								Part 3		
Orienting	x				x		x						x		
Planning		x						x							
Executing			x	x		x					x			x	
Checking									x	x		x			x

	Part 3									Part 4				
Orienting	x													
Planning		x		x			x							
Executing			x		x			x		x		x		
Checking						x			x		x		x	x

Task 6																		
	Part 1			Part 2								Part 4				Part 5		
Orienting	x			x								x				x		
Planning					x					x			x				x	
Executing		x				x		x			x			x				x
Checking			x				x		x						x			

Task 8					
	Part 1				
Orienting	x		x		
Planning					
Executing		x		x	
Checking					x

Figure 29

Daniel's Phases on Analytic Trigonometry Tasks

To solve the three analytic trigonometry tasks, Daniel used knowledge of trigonometric functions and graphs and analytic trigonometry. While working on Task 1, Daniel explained that the transformation for sine would be the same as the transformation for cosine, because "… [sine] has the same period as cosine, so I can just add 2π." His knowledge of the graphs having the same period allowed him to quickly find the transformation for sine since he had already found the transformation for cosine. On Task 6, Daniel explained his use of identities to simplify as "So I just got for the $\cos(-\beta)$That meant I could just get rid of the negative. And then for the

$\sin(-\beta)$, I just made that $-\sin(\beta)$ and then I just turned the negative sine into the minus, and that's how you get that." He was able to use his knowledge of the even and odd identities to simplify the two functions with negatives inside.

Daniel used the heuristics of written algebra and substituting values to solve the analytic trigonometry tasks. Daniel kept track of his work with written algebra on Task 6, as seen in Figure 15, Figure 16, Figure 17, Figure 18, and Figure 19. Having a record of his work allowed him to keep track of his progress towards his goal. On Task 1, when Daniel was finding the transformation from sine to cosine, he knew that he needed to shift sine by $\frac{\pi}{2}$ to the left, but questioned whether that meant addition or subtraction in his equation. He checked which direction by substituting values into the expression and found that his transformation should be $\sin(x+\frac{\pi}{2}) = \cos x$.

Mathematical Understanding for Secondary Teachers

The tasks that were completed address two of the three perspectives of the Mathematical Understanding for Secondary Teachers framework. These include the perspectives of mathematical proficiency and mathematical activity. "*Mathematical Proficiency* includes aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students" (Heid et al., 2015, p. 11). For these tasks, mathematical proficiency includes the knowledge of the trigonometric domains that are accessed to solve each task. "Engaging in *Mathematical Activity* can be thought of as 'doing mathematics'" (Heid et al., 2015, p. 11). Mathematical activity for these tasks includes the heuristics and strategies that are used to solve each of the tasks. "Whereas the Mathematical Proficiency perspective describes the general types of mathematical understandings that a teacher should have and use, the Mathematical Activity perspective describes specific mathematical actions in which the teacher should be able to engage" (Heid et al., 2015, p. 11). Both mathematical proficiency and mathematical activity work together as the participants solve each of the trigonometric tasks.

Mathematical Proficiency

Mathematical proficiency consists of six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Daniel demonstrated five of the six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

Daniel demonstrated conceptual understanding and procedural fluency across the tasks and domains. For Task 2, Daniel showed conceptual understanding when he was able to explain why the cosine value he had found in the second quadrant was negative.

> And it's got to be, so it's got to be negative in this direction. So, the cosine is going to be the, we're going back to the unit circle. So, the cosine is going to be $-\frac{15}{17}$, because it's the... I mean, I guess, if you just drew a triangle and then you could just do like this side is 8, and this side is 17, and then you did the Pythagorean Theorem and then you remembered like, my teachers said all seniors take calculus. So, you remember that cosine is a negative in the second quadrant, whatever number you got here, you would just say, is the negative.

Though he referenced the mnemonic ASTC that he had drawn on his diagram, he was able to conceptually explain why the cosine had to be negative since the x value would be negative in the second quadrant. On Task 5, Daniel was able to demonstrate procedural fluency in his work with the given data for the rabbits and foxes to find the values of the parameters in the equations, such as, "So we can make [the vertical shift] a thousand since that seems like it's in the middle, it would be like close to the average." On Task 6, he was able to square fluently the expression $(\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2$ when trying to verify the cosine sum angle identity.

Daniel's evidence of the other three stands was less evident because he tended not to verbalize it. On Task 6, Daniel considered several different identities to verify the cosine sum angle identity. He showed strategic competence in first using an identity that he considered would be easier and less messy, the cofunction identity. However, he realized that this would be redundant and then used the Pythagorean Identity, though he suspected it would be messier. As he started with the Pythagorean Identity, he stated, "I do feel like that might work though. So, I'm trying to get the cosine terms. Yeah, that may actually. … Be really messy." On Task 2, Daniel showed adaptive reasoning when he was able to change how he viewed the given equation of $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$. Initially, he stated that it was the Pythagorean Identity, since $\frac{|AB|}{|AC|} = \sin\theta$ and $\frac{|BC|}{|AC|} = \cos\theta$. However, when asked to consider it in another way, he was able to view it in terms of the Pythagorean Theorem. Daniel showed a productive disposition throughout the tasks with his persistence, working on part 2 of Task 6 for 18 minutes. He also stated, "I'm glad I thought that through." after determining the reflection that would map sine back onto itself for Task 1.

Mathematical Activity

Daniel demonstrated all three strands of mathematical activity: mathematical noticing, mathematical reasoning, and mathematical creating across the tasks and domains. Mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside of mathematics. Daniel's work on the tasks used three of the four strands: symbolic form, form of an argument, and connect within and outside of mathematics. Though Daniel did not show an understanding of the structure of mathematical symbols, it was likely because the tasks did not require him to do so.

Daniel was able to demonstrate an understanding of the symbolic form of trigonometric functions on Task 6. On this task, he was able to use the sum and difference identities in several ways. First, he was able to recognize that $\sin(\alpha - \beta) = \sin(\alpha + -\beta)$, which allowed him to use the sine sum angle identity to verify the sine difference angle identity. He was also able to rewrite $\sin^2(\alpha + \beta)$ by squaring the sine sum angle identity to get $\sin^2 \alpha \cos^2 \beta + 2 \sin \alpha \sin \beta \cos \alpha \cos \beta + \cos^2 \alpha \sin^2 \beta$. This showed that he understood that the result of the sine is squared, and not the input is squared.

Daniel demonstrated an understanding of the form of an argument on both Tasks 6 and 8. For Task 6, he ended up working backward to verify both the tangent sum and difference angle identities. He recognized that his written work was backward from a formal proof and stated,

> So if we did all that backward where we turn this into this and then un-cross those out and then went and then kind of dissected that out. Yeah, we like wrote out the old thing for these formulas and then went backwards further and split it up and then multiple and then like all this stuff went backwards. If you just read it backwards…

He recognized that by working backward from the solution, he would need to reverse his written work to have a formal proof of the identities. Daniel also recognized the flaw in the hypothetical student's proof in Task 8. He explained that the student could have reversed her steps or not multiplied both sides, because either of those assume what she was trying to prove and make the proof invalid. On both of these tasks, Daniel showed that he had an understating of what constitutes a valid proof.

Daniel was able to connect outside of mathematics on Task 5 when he recognized the real-world relationship between the populations of the rabbits and foxes. He described the relationship as, "… as the foxes, there's more foxes that eat the rabbits, so the rabbit population goes down, but then as the rabbits go away, then the foxes have less to eat, so rabbits go away." Daniel also connected within mathematics on several occasions when he considered simpler cases to explain a task. For example, on Task 7, Daniel considered $y = x^2$ and $y = (2x)^2$ as a simpler example of the relationship between $y = \sin x$ and $y = \sin(2x)$. He was able to describe how the 2 influences the behavior of both parent functions in the same manner and how considering the relationship in a parabola might be easier for students to understand since it is more familiar to them.

Mathematical reasoning consists of three strands: justifying/proving, reasoning when conjecturing and generalizing, and constraining and extending. Daniel showed evidence of two of the strands: justifying/proving and constraining and extending. On Task 8, Daniel considered a proof written by a hypothetical student. He stated that the proof she had written assumed what she was trying to prove, but if she had written in backward from its' current form that it would be valid. He then explained what he would say to the student to help them understand how to turn their work into a valid proof.

> I mean, I would, if she did this when it was originally written here, I would just tell her that she was on the right track, but her thinking was backwards. And I would also tell her that what she did here was kind of assumed what she was trying to prove. Because this is, I mean, if you do it backwards, it is a valid proof as far as I can tell. Um, so there's no reason to tell her not to do that. Cause if I was like trying to prove something and then I got this, I would just, when I would just go back to my original piece of paper, this would

> be like on scratch paper and then on the original paper I would just write exactly this backwards. Yeah. Once I've proved it, I'm not trying to like, I mean it's better to try to make it more streamlined mind, but it's not really that big a deal.

He considered her thinking valid and would help her to formalize her answer, but stated that he would likely do some of the same work on scratch paper and then formalize his own solution, just as he would support her to do. Daniel considered constraining and extending the inverse of the sine function when he was working on finding a reflection from sine onto itself for Task 1. One option he considered for the reflection was the inverse.

> So, I just did like a $\sin y = x$ that would not look the same [as $y = \sin x$] … Because that was just that would be just the arcsine or the inverse sine, which is just like… Yeah, and it wouldn't, and it wouldn't look like, I mean like I guess if you were like to extend it, so it wasn't a function it would just keep, but it would just keep going up.

When considering this option, he first recognized that visually it would not look the same, but then expanded his explanation to state that the domain of arcsine was restricted to make it a function. However, if it were to be extended, then it would look like the sine graph, but going up the $y - axis$, rather than across the $x - axis$.

Mathematical creating consists of three strands: representing, defining, and modifying/transforming/manipulating. Daniel showed evidence of two of the three strands: representing and modifying/transforming/manipulating. Daniel created multiple representations of his thinking across the tasks and domains, primarily through creating diagrams to help him explain his thinking. For example, on Task 4, Daniel created a diagram showing his “bow-tie” angles. He used this representation to describe why all of the angles had the same sine and cosine values, with just the signs changed depending on the quadrant. Daniel showed modifying/transforming/manipulating on Task 1 by transforming his graphical transformations to trigonometric identities and on Task 6 by algebraic manipulations of the sum and difference angle identities. On Task 1, for example, Daniel was able to identify from the graph that translating cosine 2π to the left resulted in the equation $\cos(x + 2\pi) = \cos x$ and this was the periodic identity.

> Because cosine has a period of 2π and so this is called the phase shift. Which would be, if you add 2π to x you're just shifting it to the left by 2π and since it repeats every 2π and you're just shifting everything over to the left.

He was able to transform from the graphical representation to an equation to a trigonometric identity and explain the connections between all three forms. On Task 6, Daniel was able to perform algebraic manipulations beginning with given identities to end up with the form of the identity he was asked to verify. For example, he was able to begin with $\cos^2(\alpha + \beta)$ and use the Pythagorean Identity to rewrite it as $\sqrt{1 - \sin^2(\alpha + \beta)}$. He then worked with the algebraic manipulations, as seen in Figure 16. Though he was unable to verify this particular identity, he was able to verify the other four in the task using algebraic manipulations.

Case 2: Emma

At the time of her participation in the study, Emma was a traditional undergraduate student completing her sophomore year. She was majoring in mathematics education and currently enrolled in Calculus 3. The only mathematics education course she had taken was a one-credit introduction to mathematics education course. On the test of trigonometric content knowledge, Emma scored a 78% overall. She scored 100% on triangle trigonometry, 83% on unit circle trigonometry, 79% on analytic trigonometry, and 38% on trigonometric functions and graphs.

Emma was enthusiastic about solving the tasks. She talked about liking math puzzles and how much fun it was to solve tricky math problems. When she encountered interview tasks that gave her difficulty, she would tell herself, “think Emma.” On one difficult task, she said, “It worked! An idea. I got insight. Oh, precalculus Emma would've been so proud of that one just now” after she solved the task. She frequently made use of the formula sheet, but rarely used the graphing calculator. On the few tasks that she did use a calculator, she generally used her TI-89 instead of the provided TI-84.

Emma’s Problem-Solving by Task

Task 1: Identities from a Graph

For Task 1, Emma’s problem-solving phases are shown in Figure 30.

	Part 1					Part 2							Part 3		
Orienting	x					x		x		x			x		
Planning		x									x				
Executing			x		x		x		x			x		x	x
Checking				x											

	Part 4							
Orienting	x		x		x			
Planning						x		
Executing		x		x			x	
Checking								x

Figure 30

Emma's Phases on Task 1: Identities from a Graph

Emma began Task 1 by orienting herself to the first part of the task. She then had a planning phase of “Okay, so I'm just kind of plotting out points that I know I'm gonna have to hit.” She began to mark on the provided grid during an executing phase but realized that she had made a mistake written in pen during a checking phase and drew herself a new set of coordinate axes. During a new executing phase, she narrated some of her thinking as “So sine starts at 0 and then $\sin\frac{\pi}{2}$ is…", but primarily worked silently to create the graphs of $y = \sin x$ and $y = \cos x$ in Figure 31.

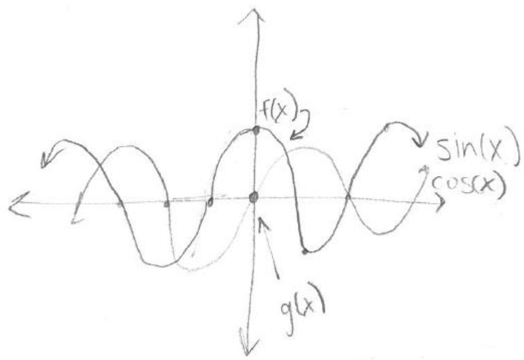

Figure 31

Emma's Graphs of Sine and Cosine for Task 1

Next, Emma oriented herself to part 2 of the task. Immediately, Emma entered an executing phase and stated, "So just shift it by its period, shift either $+2\pi$ or -2π." While saying this, she also gestured shifting the graph to the right with her hands. Emma then oriented herself to the goal of finding a reflection of cosine back to itself. During an executing phase, she looked at the graph and said, "Just like reflect across the $y - axis$." For the final portion of part 2, Emma oriented herself to finding identities that matched the translation and reflection that she had just found. During a planning phase, she grabbed the formula sheet and looked at it. She explained, "I'm just like looking for identities that might resemble what I have." During an executing phase, she explained her solution as,

> That [periodic identity] looks like the period of sine and cosine that's very similar to my $f(x)$, x being θ, plus 2π of however many you're trying to shift it. So $\cos(\theta + 2\pi)$… But $\cos(\theta + 2\pi * n) = \cos(\theta)$. However many 2π you shift it, it's gonna be the same. And the reflection, oh the negative is equal to the positive $\cos(-\theta) = \cos(\theta)$.

Using both her graphs and the formula sheet, Emma was able to connect the translation and reflection that map cosine to itself with the identities associated with those two transformations.

Emma oriented herself to part 3 of the task, which asked her to repeat part 2, but for cosine. She stated, “So same idea.” In an immediate executing phase, she gave her solution for the translation as, “So it's gonna be the same deal, just $g(x + 2\pi)$.” For the reflection, she was also able to transition to an immediate executing phase.

> And then again just reflect across the $y - axis$. And then the same idea applies for the second question any multiple of it, but it's just the same thing. $\sin(-\theta) = \sin\theta$. Wait no $-\sin\theta$, so that's the identity that's listed.

In a checking phase, Emma considered why the identity was not the same for the reflection for both sine and cosine, even though the identity was the same for the translation.

> Sine is odd, so you'd really just reflect across the x… and y? x would not make it the same. So you would have to reflect across the y and the x. So really, that's the line $y = x$. No. Yes. No. No. … [We have to both x and y for sine, but only had to do one for cosine], because cosine is even and sine is odd. [Cosine] has symmetry across the $y - axis$ but sine does not have symmetry across the $y - axis$, but if you flip it once, and then you flip it again it lays over itself. I'm sure there might be another logical math way to do it, but that's how my brain processes it. And you when you flip it once if you just flip across the $x - axis$ or just the y, it's the opposite, so that's why not identity makes senses so if you have $-\theta$ it's gonna be $-\sin\theta$. They’re opposites.

Even though Emma’s first solution was to use the same reflection for sine as she had for cosine, once she found the identity on the formula sheet, she was able to recognize her mistake. She was then able to reason about why the reflection was different using the graph and thinking about reflecting the graph of sine just over the $x - axis$ or $y - axis$ resulting in the opposite of the graph of sine to explain the presence of the extra negative. She also recalled that cosine is even, and sine is odd to help explain the identities.

For part 4, Emma began by orienting herself to finding transformations from sine and cosine to each other. She again immediately began executing her solution.

> If you want cosine to equal sine, then you want a shift. So $f(x + \frac{\pi}{2}) = g(x)$. And that really could be plus or minus, cause you could shift it either way. No, you would it would either be $+\frac{\pi}{2}$ or $f(x - \frac{3\pi}{2})$. Is it $\frac{\pi}{2}$? Now I'm thinking, no… Yeah, $\frac{3\pi}{2}$. Gotten to the little point where it just I haven't had to think about the really simple ones in so long.

Emma recognized that she needed to shift cosine to equal sine. She further extended her reasoning by explaining that it could be shifted either left or right, but by different amounts depending on the direction. Emma again oriented herself to finding the transformation, this time from sine to cosine. She was able to use her previous solution to move to an executing phase immediately and said, "So we would do generally the same thing just in the other direction so you'd have to do $+\frac{3\pi}{2}$ or $-\frac{\pi}{2}$." After orienting herself to the goal of finding identities matching her two translations, Emma began planning her solution by looking at the formula sheet. She named the cofunctions as her solution in an executing phase and explained,

> That kind of look like looks like cofunctions. Yeah, so relating the two based on part of their period. x being θ that $\frac{\pi}{2}$ follows. Look at that! Okay so if f was cosine, $\cos(\frac{\pi}{2} - \theta) = \sin\theta$. That might not have been completely correct on my part, but it was like the related identity, and I'll trust the formula sheet over my own brain. That's why we like those. And then the other one would have been $\sin(\frac{\pi}{2} - \theta) = \cos\theta$.

During a checking phase, Emma considered why the identity on the formula sheet was different from the translation she had identified from the graph.

> I wonder why… I just didn't think about that quite the same way, which may not be correct but… So I have, saying that x and θ are equivalent, in $\theta - \frac{\pi}{2}$ rather than where here it says $\frac{\pi}{2} - \theta$. Which I was, oh, it's because when you're shifting things you have to do it backwards. That's just silly, and I never really understood why you had to do it backwards, but I know that the math works out. When you shift things to the right, you subtract, and when you shift to the left, you add, and that's why I had all of those backwards. So we add, subtract, subtract, add. Switch… No, I did that backwards. I just… no… maybe, maybe it was right. Who knows? We'll just leave it.

Emma had recognized that the cofunction was the relationship that she was looking for, but noticed that it was different from the relationship she had identified. She recalled that the left and right translations are "backward" from the equation, but said that she just knew the math worked out even if she didn't fully understand why it worked.

While solving Task 1, Emma made frequent use of the formula sheet when she was trying to identify the identities that matched each of her transformations. She also used knowledge of unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. She used knowledge of the graphs of sine and cosine throughout the task. For example, she was able to sketch graphs and reflecting them over the $x - axis$ and $y - axis$ to determine how to map them back onto themselves. Emma was also able to work with trigonometric identities, such as describing that $\cos(\theta + 2\pi * n) = \cos(\theta)$ results in shifting the graphs by the period of 2π or any whole number multiple to the left or right and would map the function back to itself.

Task 2: Pythagorean Theorem and Pythagorean Identity

On Task 2, the problem-solving phases that Emma used are shown in Figure 32.

	Part 2					Part 1				Part 3				
Orienting	x					x				x				
Planning		x		x	x		x				x		x	
Executing			x					x				x		
Checking									x					x

Figure 32

Emma's Phases on Task 2: Pythagorean Theorem and Pythagorean Identity

Emma stated Task 2 by orienting herself to parts 1 and 2 of the task. She then began a planning phase and recognized parts of the given equation in terms of their sides being opposite, adjacent, and hypotenuse. She wrote the work in Figure 33 and explained her thinking during an executing phase.

$$\left(\frac{a}{h}\right)^2 + \left(\frac{o}{h}\right)^2$$
$$\cos^2\theta + \sin^2\theta = 1$$

Figure 33

Emma's Pythagorean Identity for Task 2

> And we know that [adjacent over hypotenuse] is $\cos^2\theta$ plus and then opposite over hypotenuse is $\sin^2\theta$, and we know that that identity equals 1 because it is there. We deduce that I bet there's a secret here that could probably help me cause it's been shown to me. But like, doesn't mean, I remember how it actually happened. Hmm. Recently, this lovely identity has made all of my calculus really fun. Lots of things cancel out. Yes, spherical coordinates and stuff. They're so pretty. Okay. Can we deduce that that equals 1? Hmm. It's just the truth.

Though Emma could see the relationship between the sides given in the equation, the sides in the triangle, and the Pythagorean Identity, she was unsure why it was equal to 1. She considered this further in planning phases and was prompted to reconsider her initial thoughts about using the Pythagorean Theorem.

> Maybe I should write this down here. Just wishful, can we see something is true? So knowing that this is, that this equals one or it's just kind of like circular logic though? Just what, why, what I was thinking?

Emma had actually solved part 2 by identifying the Pythagorean Identity, so she returned to the original equation and considered further the use of the Pythagorean Theorem to understand where the 1 in the equation came from.

Emma reoriented herself to part 1 of the task. She started a planning phase by relabeling the sides of the triangle with a, b, and c. She then wrote the version of the Pythagorean Theorem in Figure 34 and said to herself, “Well, this is a, and this is b and this is c^2. Is there any way I can rewrite that?”

$$\left(\frac{\overset{a}{|AB|}}{|AC|}\right)^2 + \left(\frac{\overset{b}{|BC|}}{|AC|}\right)^2 = \overset{c^2}{1}.$$

Figure 34

Emma's Pythagorean Theorem for Task 2

Emma continued working in an executing phase and began to work from the Pythagorean Theorem. She was asked to explain her thinking about why she wanted to rewrite the equation.

> It was just a theory of can math happen. Um, just why can I manipulate to maybe make something more obvious, but I don't really see it. And I know that there is a beautiful way to show it because I just have seen it before, but it's not in my brain. Hmm.

So when you have the $\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2$, that's the same as $\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$. And you can add that. So $\frac{AB^2+BC^2}{AC^2}$. I don't think that helped me. It showed something. Um, and that's $a^2 + b^2 = c^2$. That makes sense cause you would multiply both sides by c^2. This is just roundabout math but...

Her written work for this phase is shown in Figure 35. She proceeded through a few steps on paper that she did not verbalize.

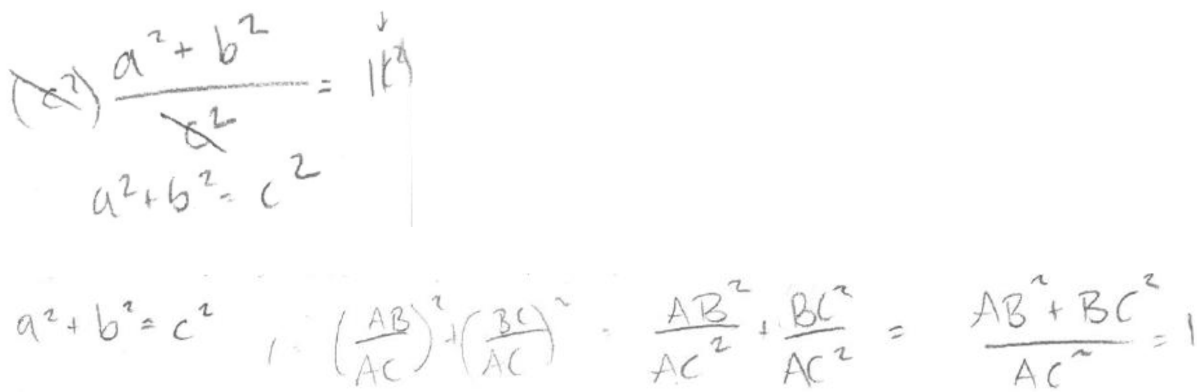

Figure 35

Emma's Expansion of the Pythagorean Theorem for Task 2

During a checking phase, Emma explained her final thoughts on where the 1 came from.

So I don't know, but I assumed that I knew that that was one. I guess that was there that was given. I'm not sure I know another way to prove that. Other than just the roundabout math that existed right there.

Since she was stumped about the 1, Emma moved onto part 3 of the Task.

Emma began part 3 with an orienting phase and followed that with a planning phase. Her initial thoughts were, “What can you say about typically when I'm thinking about θ? Opposite over hypotenuse. So cosine of the same θ would just use Pythagorean Theorem to find your other side." During an executing phase, Emma solved the triangle using the Pythagorean Theorem, as shown in Figure 36. She explained as she worked,

> So $b = 17^2 - 8^2$. So cosine would be $\frac{15}{17}$. So I guess the θ doesn't really matter, but if it were, if it were in the second quadrant, that would make it an obtuse angle, which means the right triangle trig doesn't necessarily work. Because right triangle trig is assuming that those are acute angles and that … hmm. Thinking, thinking, thinking. Actually, well I know, it would just be the negative of that because if you have the same, no, no, no, yes cause well cause a θ, any angle in the first quadrant the like $\pi - \theta$ is gonna equal the same sine. In the second quadrant. And it's going to have the same cosine but negative. So I guess you could say that $\cos(\pi - \theta)$ would be equal to $-\frac{15}{17}$.

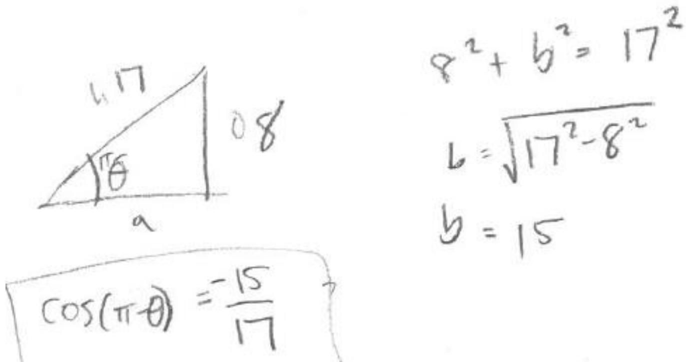

Figure 36

Emma's Solved Triangle for Task 2

Emma continued to think about the relationship between a θ in the first quadrant and in the second quadrant. She drew the diagram in Figure 37 to show the two θ she was thinking about.

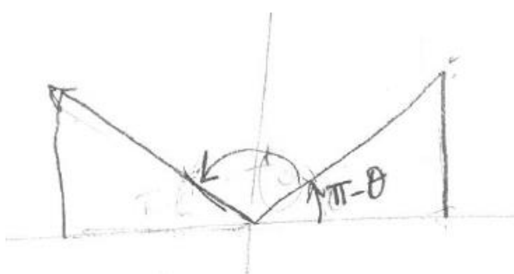

Figure 37

Emma's Two θ for Task 2

> Cause if I did it with like a sine, of the original θ. I don't know if that is sound math, but that's what the way my brain wants to do it. If you have a triangle in the first quadrant and a triangle in the second and this is θ and this is $\pi - \theta$, then your sine, oh wait, this might be backwards in some way because this is the triangle we're working with to find, well I guess you want, no cause that is, that is the θ you're actually working with. Actually deducing from trigonometry is not my strong suit is what I'm discovering here. Like doing a problem that just has a like set way to do it. It makes a lot of sense for me. But like coming up with the math without like, I don't feel like I have a strong enough

> foundation in it. Trigonometry is one of those things that, like every class just sprinkles like a little bit more of, and it doesn't really get taught, at least in our state or at the school system that I went through, solid trigonometry. So like a lot of the original test, a lot of what I was doing, I was like, I know how to do this. And I learned all of these things at different times and actually I don't know what that is. So question mark.

Though Emma stated the phase thinking about θ and the relationship between angles in the first and second quadrants, she ended the phase thinking about her own trigonometric knowledge. While working on the task, she started to realize that her trigonometric was primarily procedural or memorized from school. During a checking phase, she further her reasoning about the two θ. She started by explaining why she was confident that the solution is negative.

> So I got this answer from talking about like a right triangle in the first quadrant. Um, and the sine of these two triangles would be the same because they're both positive. Um, but I am debating about the angle cause I don't think I used the right angle in either of the cases because neither of these angles is the one we're working with. But I guess the sine of the angle [in the first quadrant] would be the $\pi - \theta$ technically. I'm thinking I had it backwards because this is, I think it actually, it represents both angles. Just this one from [the positive $x - axis$] and this one from [the negative $x - axis$]. Cause this is π and this is $\theta,$ and this angle is the same because of other geometric properties. But like in the other triangle, so sine would be the same. So sine would still do the same thing. It would just, this would be $\pi - \theta$. And then so the cosine would be the positive $\frac{15}{17}$ but if you're working with this angle, you're cosine here. That like horizontal component is the negative of this one. I worked that out a little visually, but I think that's right.

Emma was able to make use of the relationship between reference angles in the first and second quadrants. Still, she was unable to explain why that relationship worked to her own satisfaction.

While working on Task 2, Emma accessed her own knowledge of geometry, triangle trigonometry, and unit circle trigonometry. She also used heuristics of diagrams and written algebra. Emma was able to apply knowledge of the Pythagorean Theorem to label sides on the given triangle and to solve for the missing side of the triangle in part 3. She was able to use knowledge of SOH CAH TOA to label sides on the given triangle in relation to the opposite, adjacent, and hypotenuse and translate those sides to sine and cosine. Though she did not directly reference the unit circle, she used knowledge of it to find that the sine value would be the same in the first and second quadrants, and the cosine value would negate in the second quadrant. Emma used written algebra to work through her thinking for parts 1 and 2 of the task. She used a diagram in part 3 to think about the relationship between θ in the first and second quadrants.

Task 3: Which One Doesn't Belong?

While solving Task 3, Emma progressed through the problem-solving phases, as seen in Figure 38.

	Part 1									
Orienting	x									
Planning		x			x			x		
Executing			x	x		x			x	
Checking							x			x

Figure 38

Emma's Phases on Task 3: Which One Doesn't Belong?

Emma initially oriented herself to the goal of the task and then said, "These are all obtuse angles" during a planning phase. She followed this with an executing phase, saying, "So I want to make this one acute." She had identified the first characteristic that she would use to exclude her eventual solution of $-\sin 30°$. In her next executing phase, Emma stated, "I also want to make it sine because then these three are sine and that's cosine."

Emma pulled out her calculator as she began a planning phase. "So if it's acute and sine, how can I isolate this dude? Cause I need these to have something in common. How about I find out if any of these are equal? So we have some ideas in that respect." In the evaluating phase that followed, Emma found the value of each of the given expressions in her calculator.

> Hmm, so yes, these three could all equal some multiple of 0.5. Let's see. Have a puzzle to solve. So if I made it... equal to some multiple of 0.5 or I could just say ± 0.5, is where my brain is going.

Emma had now identified three characteristics she could use to exclude expressions, only one acute angle, only one cosine, and only one not equal to ± 0.5. She then entered a checking phase of determining her progress so far.

> That gives these three something in common. So does that check all my boxes? It's acute. Sine excludes that one and ± 0.5 would exclude that one. So is there a value of sine that is an acute angle, that would be $\sin \frac{\pi}{6}$, I think in radians. So in degrees, that would be $30°$. So $\sin 30°$.

She realized that she had only excluded three of the four expressions, so she returned to work.

Next Emma entered a planning phase of trying to determine how to exclude $\sin 150°$, since it had not yet been excluded by any of her characteristics. "Now we're going to puzzle again. Hmm, that is frustrating. … So, what do these two have in common? They're both negative. Can we make it negative?" Once she had identified the characteristic of being positive, she moved on to an executing phase and said "How about we make it $-\sin 30°$?" During a final checking phase, she checked whether her solution worked.

> So here it's plus, cause [$\sin 150°$] doesn't fit, that doesn't negate that point. So for [$\sin 150°$] to be excluded, all three of these are negative. For [$\sin 225°$]to be excluded, all three of these are ± 0.5. For [$-\sin 30°$] to be excluded, all of these are obtuse angles. And for [$\cos 120°$] to be excluded, all are sine. Negative sign fixed it.

Thus, Emma determined her solution to be $-\sin 30°$.

To solve Task 3, Emma used the graphing calculator to evaluate each given expression. She also used knowledge of unit circle trigonometry to determine an expression that was sine with an acute angle that equaled ± 0.5. She did not use any heuristics to solve Task 3.

Task 4: Define Sine of an Obtuse Angle

Emma's work on Task 4 proceeded through the problem-solving phases seen in Figure 39.

	Part 1				Part 2				
Orienting	x				x				
Planning		x							
Executing			x			x	x		x
Checking				x				x	

Figure 39

Emma's Phases on Task 4: Define Sine of an Obtuse Angle

Emma began by orienting herself to part 1 of the task and started a planning phase by drawing the diagram of the sine of an obtuse angle seen in Figure 40.

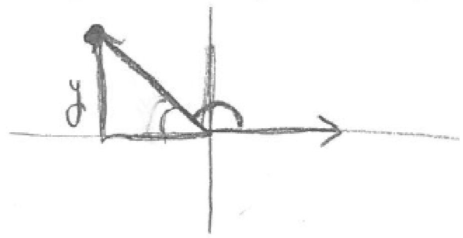

Figure 40

Emma's Diagram of Sine of an Obtuse Angle for Task 4

Her initial explanation of how to find the sine of an obtuse angle was,

> So I guess if you were dealing with that obtuse angle, you could find the sine of this angle and subtract from π to find, no, that's not, that's not the same thing. I guess the sine of this angle would be the same as the angle as it's like supplementary with. … So the sine would be the same there cause you could just make the angle. You could make the right triangle with that other angle. And it just would only matter if the obtuse angle went into one of the negative quadrants because you could do a right triangle, but then your sine would be equal and opposite potentially.

She expanded her explanation during an executing phase.

> So like if you had an angle here and you did that angle, and you wanted the sine of that angle, you could make the right triangle right there and find that one. Cause you'd have opposite and hypotenuse, and your opposite is in the negative. Actually, yeah, the sine was negative. So, oh my brain's kind of off today. Your opposite angle is negative there, 'cause it's in the negative x direction. And then your hypotenuse is technically also in the negative direction, but I don't know what I was thinking there. It probably works in some way, but not in the way I was thinking of it. I've never thought of like sine and cosine as like trig identities as like I knew that right triangle trig was just like a way to use them, but they weren't like exclusive to right triangles. I guess I've never like seen them related

> to each other in a way that like made sense always. And the math always worked out, and that's where my brain goes for like relating the two. But I'm not sure that always applies like for every obtuse angle, I guess.

During this phase, Emma was grappling with how to define sine of an obtuse angle, since she hasn't considered how to relate it to right triangle trigonometry previously. During a checking phase, Emma was asked to expand upon why she knew that the sine value for her triangle in

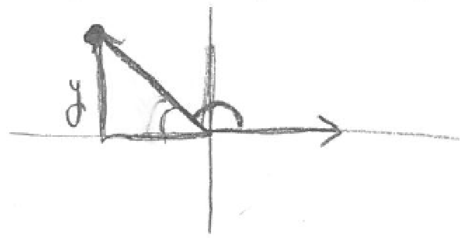

Figure 40

Emma's Diagram of Sine of an Obtuse Angle for Task 4

Figure 40 had the same sine value as a triangle in quadrant 1 and she explained,

> Because the sine value is like I always remember that x is cosine and y sine and I had on my unit circle because it's really $r * cosine$ and $r * sine$. But on a unit circle, you have radius is just 1. So any coordinate on the circle is just $(cosine, sine)$. So the y value here is right there. So that would be your sine. The y value would be the sine up there, and the x coordinate would be the cosine, and right triangle trig relates that they probably can like write out a bunch of stuff to make that make more sense. But I just learned it that way.

During this explanation, Emma wrote out Figure 41.

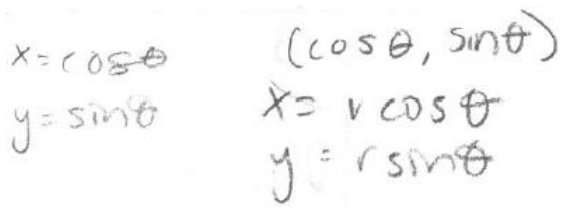

Figure 41

Emma's Coordinate Definition of Sine of an Obtuse Angle for Task 4

Since Emma didn’t have any more thoughts about part 1, she moved onto part 2, which she began with an orienting phase. She then began an executing phase in which she drew and labeled the unit circle in Figure 42.

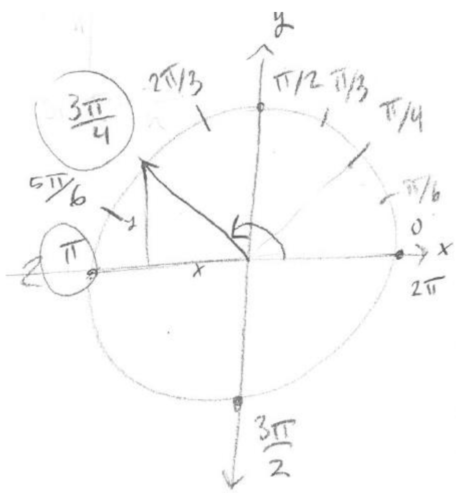

Figure 42

Emma's Unit Circle for Task 4

She explained her reasoning about finding $\sin\frac{3\pi}{4}$ and $\sin\pi$ as,

> Unit circle! And if we're doing why, I'm just going to draw a unit circle for the fun of it. So only the top half of it is really important right now because we're doing these two and the [$\sin\frac{3\pi}{4}$] will be $\frac{\sqrt{2}}{2}$. Cause I just have these angles memorized, and then I changed their positive or negative values based on which quadrant we're in. Um, so then $\sin\pi$. It's 0.

In an executing phase, Emma tried to expand upon her explanation and connect it to right triangle trigonometry to help the hypothetical student in the task understand.

> [$\sin \pi$] doesn't help you a lot with a triangle cause it's just straight. Like you don't really can't really draw a triangle when it's touching the axis. But that, I guess you could explain that in the way that if you can't draw the triangle, it's not going to have a value of sine because it's zero. It's touching itself.

She continued her explanation in a checking phase.

> That explanation only works for me knowing after the fact, had somebody looked at me and tried to explain it that way, I'd be like, uh no. So another way you could look at it is just like teach that perspective of the way you parametrically represent a circle is $x = r\cos\theta$ and $y = r\sin\theta$. And just demonstrate that somehow so that they have more of an understanding of the right triangle is an easy way to represent. Often classes teach right triangle trigonometry before they get to unit circle stuff. But for me, it makes a whole lot more sense if you do unit circle first and understand why the unit circle is the way it is. So, um, I would explain it showing the y component. Like there is no y component of that. I want to say vector, but at this point, they also probably wouldn't be looking at vectors, but there is no y component of that angle. So it's not going to have any value of sine.

Emma recognized that she and many students would not have understood her explanation for $\sin \pi$ based upon the unit circle, so she advocated teaching the unit circle first to help students understand how to find trigonometric ratios of angles outside of right triangles. She completed her explanation of $\sin \frac{3\pi}{4}$ during an executing phase.

I was just always taught that for, for the $\frac{3\pi}{4}$ case that if you have an angle, you just drop a perpendicular to the $x - axis$. And just for my math brain, drop a perpendicular to the $x - axis$. And I guess this is my physics experience talking, but find the components. And so I was just always taught that like your y and x components were important for finding your sine and cosine.

Emma was able to relate evaluating the sine of an obtuse angle to both the unit circle and to vectors. Though she was able to make these connections, she felt like she relied on memorization and procedures to find them.

Emma used resources of knowledge of right triangle trigonometry and unit circle trigonometry to solve Task 4. She also used the heuristic of diagrams. When Emma was explaining why the triangle in the second quadrant would have the same sine value as the triangle in the first quadrant, she relied upon knowledge of SOH CAH TOA. She explained that the two triangles would have the same opposite and hypotenuse, so their sine values would be the same ratio. She further explained that this would be true in any quadrant; just the signs would change depending on the quadrant. Emma relied heavily on knowledge of the unit circle to solve the task. When first defining the sine of an obtuse angle, she immediately thought of defining it in terms of the unit circle. She continued to rely upon and expand her definition in terms of the unit circle. She also used memorized knowledge of the unit circle to evaluate $\sin\frac{3\pi}{4}$ and $\sin\pi$. Emma created two diagrams to help her solve Task 4. She drew a triangle in the second quadrant to show how the sine of an obtuse angle could create a right triangle. She also drew a unit circle to reference when evaluating $\sin\frac{3\pi}{4}$ and $\sin\pi$ at the end of the task.

Task 5: Graphs of Foxes and Rabbits

For Task 5, Emma's problem-solving phases as she worked are seen in Figure 43.

	Part 1			Part 2			Part 3		Part 4		
Orienting	x			x			x		x		
Planning		x			x					x	
Executing			x			x		x			x
Checking											

Figure 43

Emma's Phases on Task 5: Graphs of Foxes and Rabbits

After orienting herself to the task, during a planning phase, Emma described why a trigonometric function was appropriate as, “They just do, they're beautiful. They're nice, wobbly sinusoidal curves.” During an executing phase, she expanded her explanation as,

> I'm just seeing the values oscillate, like at a relatively like consistent pace. I guess you could say. I'd probably call [the rabbits] sine time with some sort of different... All the different components of sine. Which I've been terrible at, but just because it, it starts in the middle of it's… I guess that doesn't necessarily reflect that, but [the foxes] looks a little more like a cosine. I guess that I'm kind of working ahead of myself, but I explain why. So sinusoidal, because the oscillation is consistent cause that could go, oscillation technically could be like [crazy]. Like smaller into an asymptote. It's crazy that like improper like infinite integrals can have like finite, what... brain tangent.

Emma was able to recognize the graphs as sinusoidal because of their consistent oscillations.

Next, Emma oriented herself to the task of finding a trigonometric model for the rabbits. In a planning phase, Emma said “So I was thinking $r(t)$ is $\sin t$, but remembering isn't it like you multiply by the amplitude or the amplitude in the thing? I don't really remember.” She proceeded to an executing phase to find the parameters for the equation.

> That amplitude looks like 400 rabbits, so I want to say like $400 \sin t$ and it's shifted up by a thousand. For some reason, I want to say that's not right that that much. No, because

> he's shifted right and left within the parentheses. See if I... these are all just like not math tricks but just like how a teacher taught it to me, and I don't remember their tricks. Maybe that's not right, but that's... 1000 is what I'm believing to be where I'm supposed to put the amplitude and amplitude is just the different distance from the crest to the equilibrium. Or the trough to the equilibrium. And in both cases we went from 600 to 1000 or 1000 to 1400, so I called the amplitude 400, and then a $y - intercept$ being 1000.

Emma wrote down her solution as $r(t) = 400 \sin t + 1000$. She identified the graph as sine because it started at the midline. She then described an amplitude of 400 and a vertical shift of 1000.

Emma oriented herself to finding an equation for the foxes next. During her planning phase, she stated, "I'd call this probably cosine maybe, but also maybe a negative cosine cause it doesn't… thinking about cosine, that's one." In an executing phase, Emma further described her solution for finding the equation for the foxes as

> No, that's, that's the same as cosine. So it'd be on the amplitude here, it's about 50. So $50 \cos t$ plus that starts at ... it counts by 10, so 165. So 50, again, was a relative amplitude of where the data is falling. And then 165, I saw that these tick marks were counting by 10. So this highest number, like highest data point that serves as the $y - intercept$ was between one and two ticks away from 150. So I call that kind of 165 as a rough estimate.

Emma's solution for the foxes was the equation $f(t) = 50 \cos t + 165$. She chose cosine because the data did not start at the midline. She determined the amplitude to be 50 and the vertical shift to the top of the data to be 165.

Emma then oriented herself to the final part of the task. During this, she stated, “That's a biological application than I actually did in BIO 181 last semester.” She started by making the graph in Figure 44 and narrated her process during a planning phase.

> So the harder part of this is that axes actually is followed different. You need like, it's just really difficult though. Reconciling axes is annoying for me. Cause foxes are just such a smaller population. Um, there are obviously lots and lots of rabbits.

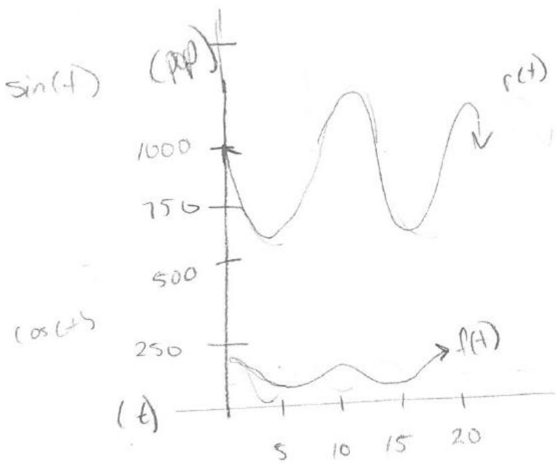

Figure 44

Emma's Graph of Rabbits and Foxes for Task 5

She continued to work on her graph during an executing phase and then explained why the two functions seem to chase one another.

> Just for the sake of spreading out-ish. I'm just too OCD for this. So this is our $r(t)$. This is not nearly as long as it was, though. Sorry, it's a little closer. This one's a little wide. We'll stick with that just for my graphing skills. So they seem to chase each other because there's a predator-prey relationship and so the more foxes there are, the less rabbits there's going to be, cause a lot of foxes are eating rabbits and when there's lots of rabbits then um, there's a lot of food for the foxes, and they thrive. But as the foxes thrive and they eat

> lots of rabbits, the rabbit population declines and then a few foxes starve. And so you then have a dip in the fox population. So they chase one another. … Predator-prey relationship, more foxes equals less rabbits, which in turn means less foxes equaling more rabbits. I guess that's an attempt of a mathematical way to explain it.

During the last part of the task, Emma was able to address the real-world application of the task. She was able to describe the relationship between the two functions and relate them to her original comment about learning about it in her biology class.

To solve Task 5, Emma used knowledge of trigonometric functions and graphs and the heuristic of a diagram. Emma used knowledge of the parent functions of $y = \sin\theta$ and $y = \cos\theta$ to determine which would be easier to use for the rabbit and fox data based on their shapes. She also used knowledge of graphs to find the amplitude and vertical shift, though she did not find the period or phase shift for either function. She also used a diagram when she was asked to relate the real-world reason why the two graphs seem to chase one another.

Task 6: Sum and Difference Identities

On Task 6, Emma worked through the problem-solving phases, as shown in Figure 45.

	Part 1					Part 2						Part 3		
Orienting	x					x						x		
Planning		x					x			x				
Executing			x		x			x					x	
Checking				x					x		x			x

	Part 4						Part 5		
Orienting	x						x		
Planning		x		x					
Executing			x		x			x	
Checking						x			x

Figure 45

Emma's Phases on Task 6: Sum and Difference Identities

After orienting herself to the task, Emma moved into a planning phase. During the planning phase, she considered how to assign variables to proceed.

> So $\sin -\theta = -\sin\theta$, which means that sine, if we're calling $\alpha + \beta = \theta$, the $-\theta$ equals a negative. Both of them are negative in that case. Hmm. Where does go math go? $\alpha - \beta$. We want to prove $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. Using that. Hmm. What if we use $\cos -\theta = \cos\theta$. Well if we do that because we have $\cos\beta$ and $\cos\alpha$ and $\cos -\beta$, need $\cos -\beta = \cos\beta$ is with that says. So this could be β or positive, and this could be cosine... Is this going to help me at all? Maybe not. Maybe, maybe not.

Emma moved forward into an executing phase of trying her thought.

> But that means we could also do the $\sin -\alpha$. Oh wait, no. Why? Negative also, $-\beta$ because of this identity equals $sin * cos + cos * sin$. But that doesn't necessarily work. Those would have to be negative to make this true because we made β negative, and then that would…

Emma continued to consider how to assign the variables to use the even and odd identities during a checking phase.

> Trying to make it true for all of it is the difficult part. It's kind of like a puzzle. My problem is making one of these positive and one of them negative, making β negative but to still have sines and cosines. Back up a little. I'm gonna get a bigger eraser.

Emma returned to a planning phase since she was still unsure how to use the even and odd identities. “So what if you're talking about having $-\beta$, so $\cos -\beta$, cause this is really like plus $-\beta$.” In her next executing phase, she seemed to realize that she had found the way to work with the even and odd identities within the sine difference angle identity.

So $\cos -\beta$ because of cosine, this identity. So that's equivalent, and then we talk about this $\sin\beta$. $\sin -\beta = -\sin\beta$. Oh, I might just have gotten there. Because then we're going to have the original. This, this one's confusing me. So if we have $\sin(\alpha + -\beta)$, we'll have $\sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$, which is the same as $\sin\alpha$. And this is going to be the same as $\cos(+\beta)$ because of the cosine identity. And then you just switch it around a little. Move that negative to the front. Minus $\cos\alpha\sin(+\beta)$. There we go.

During this phase, Emma also completed the work shown in Figure 46. The work matches what she described in the executing phase. After completing part 1 she said, “Okay, that makes me feel better about my ability to find things.”

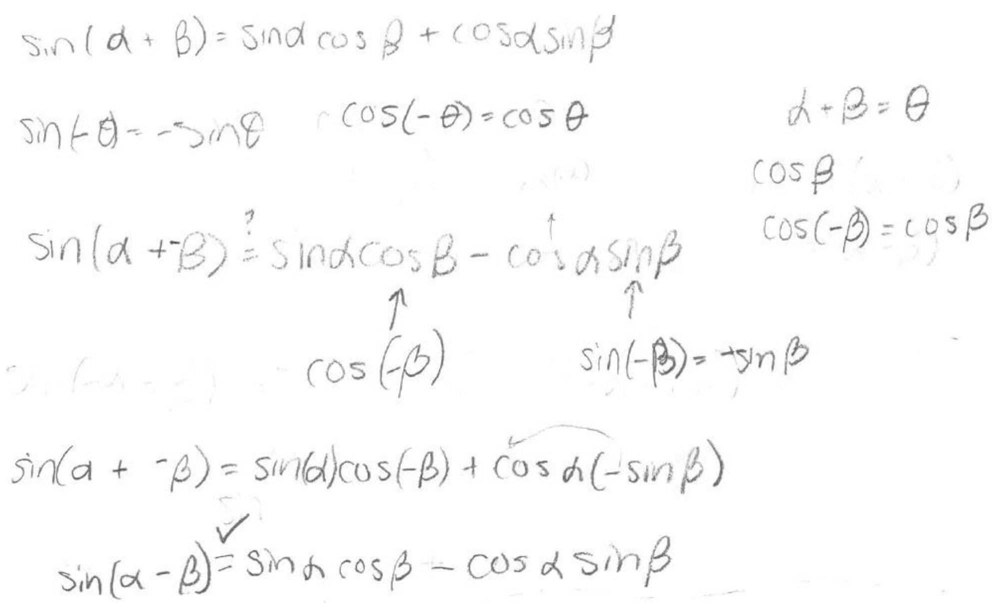

Figure 46

Emma's Work for the Sine Difference Angle Identity for Task 6

Emma then oriented herself to part 2 of the task. She began with a long planning phase of trying to identify which identities would be useful.

The question is which [identities] will be helpful. I wanna say… this is a moment like um, one of my high school calculus textbooks when teaching infinite sums and series. It literally an explanation for a problem was by staring we see, and I was like that helps me absolutely zero, and it's, I feel like that textbook would do this. It'd be like by staring we see that this is equal to that just because… Obviously, you staring works a little different than mine cause staring to me is just staring, not even thinking.

First, I'm going to write down what I'm trying to show. Hmm, using… because I need to relate… See I could, now I know how to show you that the negative would work, but I'm not exactly sure how I could start to relate like to get this one from this one. I want to say Pythagorean Identity was my initial reaction. I also thought it might get complicated, but we could give it a shot.

During the executing phase, Emma wrote the work seen in Figure 47. She described it as,

So basically the question I have here, $\sin^2\theta + \cos^2\theta = 1$. So does $\sin^2(\alpha+\beta) + \cos^2(\alpha+\beta) = 1$? So I went to square. It's going to be messy. … We're going to get a lot of sines, and cosines squared that hopefully can cancel. … So we have two of those. So how about I just simplify that? … Does that all equal 1? Who knows? Right now, okay. We have... This is where I need colors.

So, I want to identify some, some like squares that are like each other. But these don't really want to factor out. So these terms are like, but it just makes it $4\sin\alpha\sin\beta\cos\alpha\cos\beta$ and we have $\sin^2\alpha\cos^2\beta$, $\cos^2\alpha\sin^2\beta$, $\cos^2\alpha\cos^2\beta$. I can factor $\cos\beta$ out of those two. Oh, wait. So let's cross that out and call this one four... Just only working with five terms. Um, wait, can that one, that one, that one I can factor out a cosine but I don't want to factor out a $\cos\beta$. I want, I want a $\cos\beta$ added to a $\sin\beta$ there. This is just one because you could factor the $\cos\beta$ and the $\sin\beta$ out and that would be one. And it would leave you with one multiplied by $\sin\alpha\cos\alpha$. So that's just 1. Does the same thing happen over here? Yes, it does. So that's just 1.

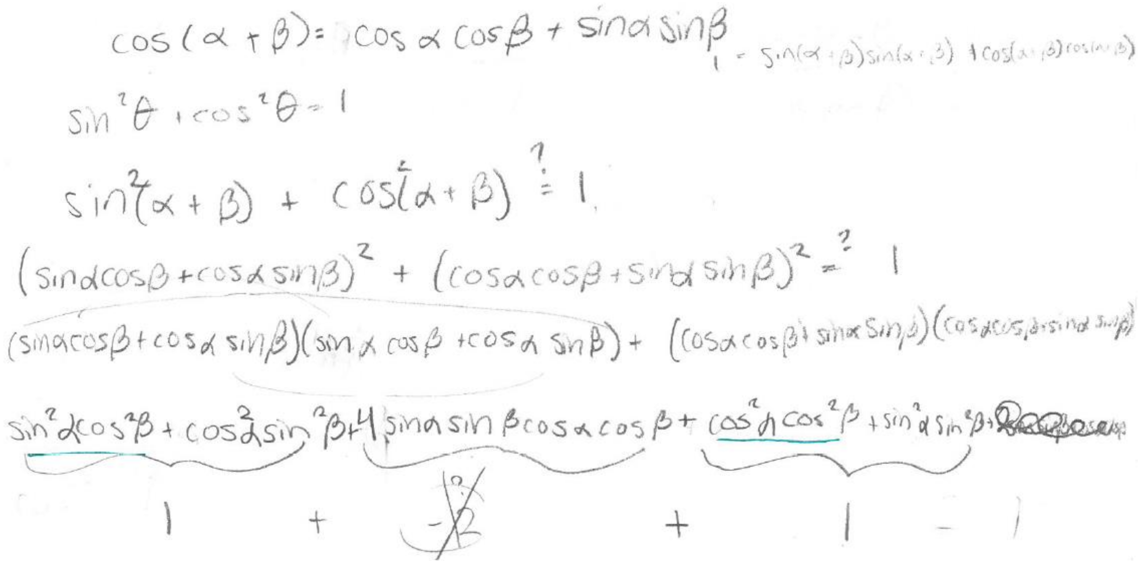

Figure 47

Emma's Work for the Cosine Sum Angle Identity for Task 6

Emma continued to a checking phase and looked at her algebra to try to find a way to simplify it.

> Right here is the conundrum. If these are being added together, I need this to be -2 for it to equal 1. But I don't know how on planet earth it would equal -2, and I believe that's the place I get stuck on that route.

Emma was unsure how to proceed and returned to a planning phase.

> Could there have been a simpler way? The one... Is there an identity that lets me split you up? Hmm. The only identity would be that one that I'm trying to show. I mean... Wait... The one to show.

Emma was still unsure of how to proceed and went on to a checking phase.

> So, okay. I think this is the place that I'm stuck. My brain is returning to some circular logic, trying to back up and find another way to go about it. Um, it doesn't really

necessarily do anything other than just like we want to know if this is true so that we could... Using this identity for $\cos(\alpha + \beta)$ because if it does equal 1, then that shows that that's the correct one. When you FOIL it all out you get this monster. And I don't know what each, what this term is equal to cause I don't really have identities that allow me to simplify it any further. And I want it to equal -2 so that this actually shows that it's 1, but it doesn't necessarily do that cause I don't know what that term is, but I know that both of these terms would go to 1 because you could factor out part of it. They would just call it 1 multiplied by another two terms added and those two terms also equal 1 and the $1 * 1$ you end up with 1 and that happens in both of these places, which is why I want that to be -2, but I don't know that I could get it to be that exclusively and that is where I'm stuck.

At this point, Emma was stuck in the algebra for the cosine sum angle identity, so she moved on to the next part of the task.

After orienting herself to part 3 of the task, to show the cosine difference angle identity, Emma moved onto an executing phase. She talked to herself through the algebra and wrote the work in Figure 48.

So it's definitely going to plunk out to be fine cause $\cos \alpha$, there it is. And $\cos(-\beta)$ is equal to $\cos \beta$ from your identities. And then $\sin(-\beta)$ is equal to $-\sin(+\beta)$. So that negative can go out front and be our subtraction there. We call it $\sin(+\beta)$.

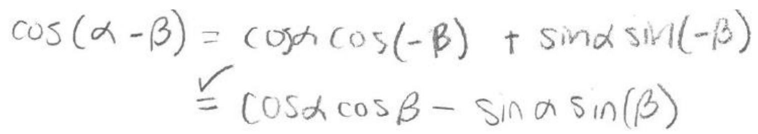

Figure 48

Emma's Work for the Cosine Difference Angle Identity for Task 6

During a checking phase for this part of the task, she simply stated, “That matches up.”

Next, Emma oriented herself to part 4. She began a planning phase.

Okay, so I'm going to bet that the $\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$. Which is going to equal ...

Where are you at? So, my bet is if we split... This is not going to be helpful to me. Can you factor it? I don't want to. I wanted to end up to be $\tan \alpha$ and $\tan \beta$. I going to rewrite what this one is, so I know what I'm going for.

During this phase, she began the work in the middle of Figure 49. She rewrote tangent in terms of sine and cosine, then used the sum angle identities to rewrite sine and cosine. She was unsure how to turn that back into what she was looking for, so she started a different approach.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\,\cancel{\cos\beta}}{\cos\alpha\,\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\,\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \checkmark\,\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Figure 49

Emma's Work for the Tangent Sum Angle Identity for Task 6

Emma followed this with an executing phase in which she rewrote the given tangent sum angle identity in terms of sine and cosine, seen at the top of Figure 49. She then entered a planning phase of trying to figure out how to work between the two forms she currently had.

> That 1 could be a $\sin^2 + \cos^2$. … What if we divide everyone by $\cos\alpha\cos\beta$, so that it goes to 1? Ooh, actually, that might be really helpful.

In the executing phase, Emma talked to herself as she worked through the algebra at the bottom of Figure 49. This was followed by a checking phase during which Emma confirmed that she had matched the given identity and said, “It worked! An idea. I got insight. Oh, precalculus Emma would've been so proud of that one just now.”

Finally, Emma oriented herself to part 5 of the task. She said, “So we'll do it again” during a planning phase and then moved onto an executing phase. She began writing the work in Figure 50 and described her process as, So I'm going to bet that it, this is the same as… which I expect to look like … So does this equal whatever sine... So sine minus was equal to $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ all over the cosine. And now it will be positive. Plus $\sin\alpha\sin\beta$. And once again, I think dividing everyone by $\cos\alpha\cos\beta$ will plunk out a lot.

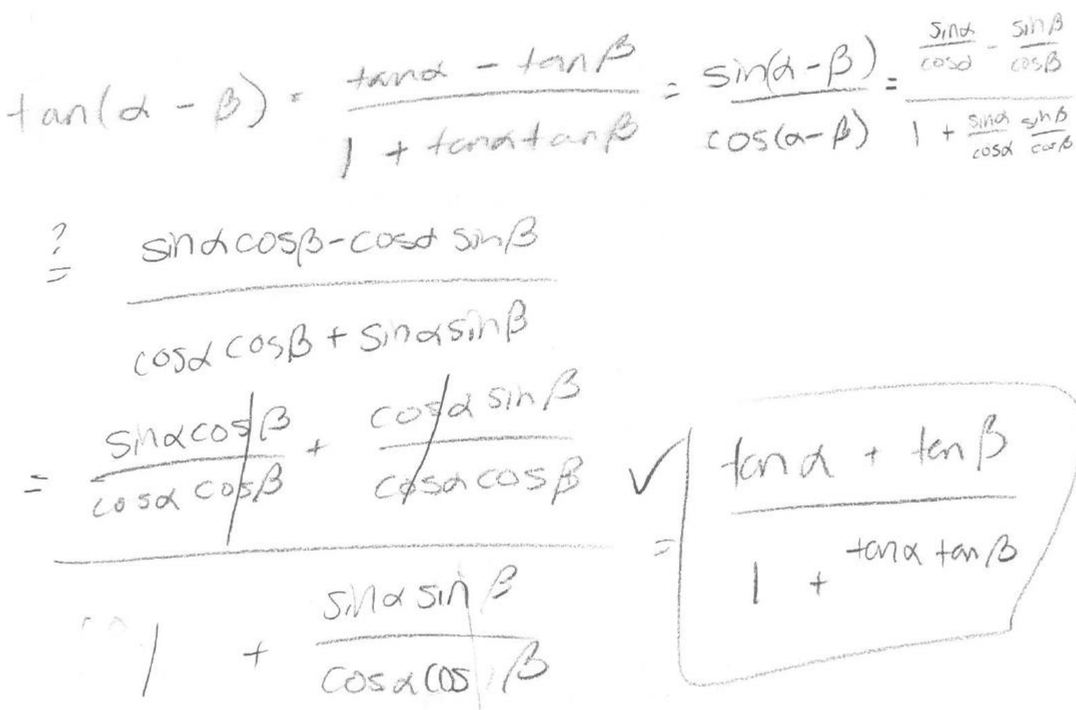

Figure 50

Emma's Work for the Tangent Difference Angle Identity for Task 6

After this, she entered a checking phase of working out the rest of the algebra to get the given identity.

That cancels, that cancels, and that cancels, and you get $\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}$ and hmm, I missed something. Oh, that's why. Hello? You don't cancel cause I should just divided by $\cos\beta$. So we have $\tan\alpha\tan\beta$. Check. Lovely.

Emma was able to use what she had done for the tangent sum angle identity to make the tangent angle difference identity easier.

To solve Task 6, Emma used knowledge of analytic trigonometry and used the heuristic of algebra. Emma was able to use the even and odd identities when solving the difference angle identities for sine and cosine. She recognized how those two identities would affect the negative and apply them to simplify. Emma also used written algebra throughout the task to keep track of her reasoning. The written algebra was crucial to her solving several of the tasks.

Task 7: Horizontal Shrink vs. Horizontal Stretch

To solve Task 7, Emma proceeded through the problem-solving phases in Figure 51.

	Part 1			
Orienting	x			
Planning		x		
Executing			x	
Checking				x

Figure 51

Emma's Phases on Task 7: Horizontal Shrink vs. Horizontal Stretch

Emma began Task 7 by orienting herself to the task. She had a planning phase in which she thought about how to explain the reason for the shrink.

> So if you're multiplying what you're plugging into the function, you're going to have like a larger, I guess is that the, I should have paid attention to this before I came back and did this again. But if we're looking at the graph of...

At the end of this, Emma began sketching $y = sin\, x$ and $y = sin(2x)$ as seen in Figure 52.

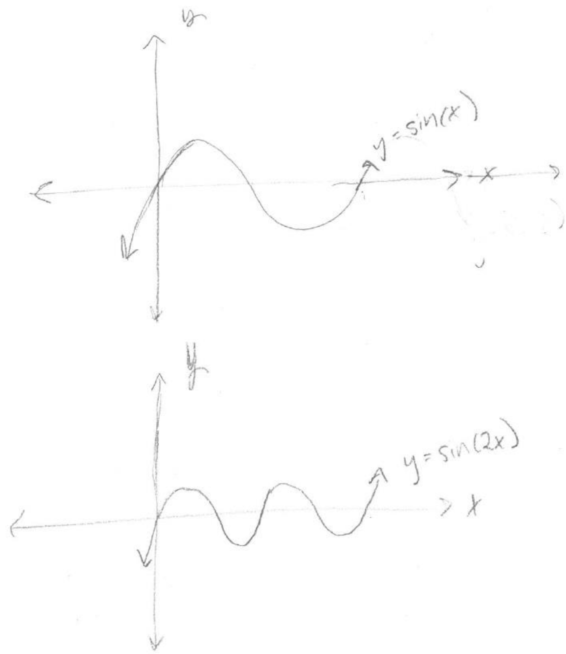

Figure 52

Emma's Sketch of $y = sin\,x$ *and* $y = sin(2x)$ *for Task 7*

After Emma completed her graphs, she began an executing phase and continued her explanation of the shrink.

> Sine, that's not perfect. But $y = sin\,x$ horizontal shrink. These like terms were always really confusing for me because two of them looked the same, and I could never like tell them apart visually, which is like how I learned. So I never truly learned them. But if you increase like what you're plugging into a function and the way that sine works being an oscillating curve... I think at some point we talked about this ... potentially helpful for my brain and mind on this one. So it was really the number in front of [sine] that changed the amplitude, which is the vertical portion too, that helps me. Um, when you multiply the whole thing, it gets bigger. But when you alter, when you alter what you're plugging into the function, it just changes like the actual value and where it lands like horizontally. So I'm not sure that that's like a sound way to mathematically describe it, but if it's a

> horizontal shrink, you're going to get a period that like happens more often. So if this was, um, I'm going to say $2x$. So you just, like if you had a period and a half here, now you have like, well, you'd really want just for like the sake of scale, you want just like one period in the same amount of space. So if you really just have like that in the same amount of space, it's gonna happen twice rather than once.

Emma considered that multiplying by an a-value in $y = a * sin(b * x)$ would stretch the graph vertically, but multiplying by a b-value would shrink the graph horizontally. She rationalizes that this was because the period would change, and thus more periods would fit into the same horizontal distance. Emma concluded with a checking phase in which she wrapped up her explanation.

> If you shrink it, and I don't know if that, that answers the question of why it's a horizontal shrink, but the idea of horizontal shrink is that you take the graph and you smush it so that the same thing happens in half the distance when you're dealing with a factor of two. And that has to do with you like changing what you're plugging in by a factor of two. So that's my thought process there.

Emma’s explanation of why 2 was a shrink rather than a stretch was that it would squish two periods into the same horizontal length as one period of the parent function.

To solve Task 7, Emma used knowledge of trigonometric functions and graphs and the heuristic of a diagram. Emma struggled to verbalize her reasoning until she had sketched graphs of $y = sin\, x$ and $y = sin(2x)$. Once she had the visual to reference, she was able to reason about the effect of multiplying by 2 on the period. She realized that the 2 would make twice as many periods fit in the same horizontal distance, which is the reason the graph shrinks horizontally.

Task 8: Proving an Identity

Emma worked through the problem-solving phases in Figure 53 to complete Task 8.

	Part 1					
Orienting	x					
Planning		x				
Executing			x		x	
Checking				x		x

Figure 53

Emma's Phases on Task 8: Proving an Identity

Emma first oriented herself to the task, and during a planning phase, tried to determine what the hypothetical student had done to solve the proof.

> So let's see what she did. $\sin x * \cos x * \tan x$, $\frac{1}{\csc^2 x}$. But is this true? And she's trying to do it by [what she did there]. Yeah, but this is it equal to $\frac{1}{\csc^2 x}$. I don't think it is. That's my instinct because these cancel and that's just $\sin^2 x$.

She continued to analyze the student's work in an executing phase.

> But actually, it might be, because just for the sake of thinking about it, $\cos x$ equals one over $\csc x$, hypotenuse over opposite. So it's, yeah, it's inverse sine or not inverse sine, but $\frac{1}{\sin x}$. So $\sin x$ equals $\frac{1}{\csc x}$. So that makes that true because if you square both sides, you can make that a one? Yeah. So that is true.

During this executing phase, Emma wrote the proof out herself, as seen in Figure 54.

$$\csc x = \frac{1}{\sin x}$$

$$\sin x * \cancel{\cos x} * \frac{\sin x}{\cancel{\cos x}} \stackrel{?}{=} \frac{1}{\csc^2 x} \qquad (\sin x)^2 = \left(\frac{1}{\csc x}\right)^2$$

$$\sin^2 x \stackrel{\checkmark}{=} \frac{1}{\csc^2 x} \qquad \sin^2 x = \frac{1}{\csc^2 x}$$

Figure 54

Emma's Proof for Task 8

Once Emma had completed the proof for herself, she returned to thinking about the student's version of the proof in a checking phase. She asked herself, "Is her logic sound? I don't think this logic necessarily proves it, maybe."

In another checking phase, Emma considered again the work that was provided by the hypothetical student.

> Well, if you have a question mark [over the equal sign] and you do what she did, you multiply cosecant by both sides… I think her logic does work. Um, the only problem I have with like how she started is she said, it was equal, and I always do the like question mark thing of like, I'm checking this. But I think her logic does work because she does end up, she never has an equation that isn't a sound math concept rather, except for like right here where she's like, is this true? But I definitely, this is sound, so cause she has a sine and a sine, they can cancel with that and then the cosines cancel. It does work.

During this time, she wrote on the provided student's work, as seen in Figure 55. She was primarily concerned with adding a question mark above the initial equal sign to indicate that the student didn't know the first statement was true.

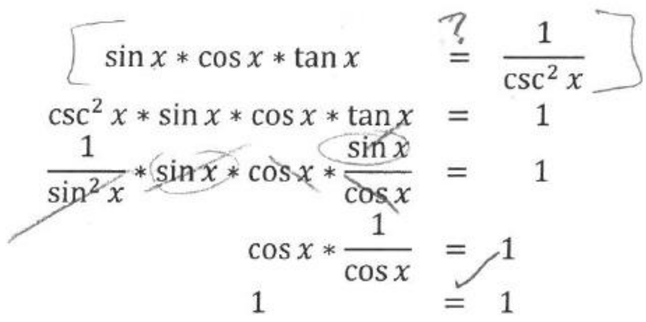

Figure 55

Emma's Work on Student's Proof for Task 8

When asked to think about how the student's work related to the work she had created in

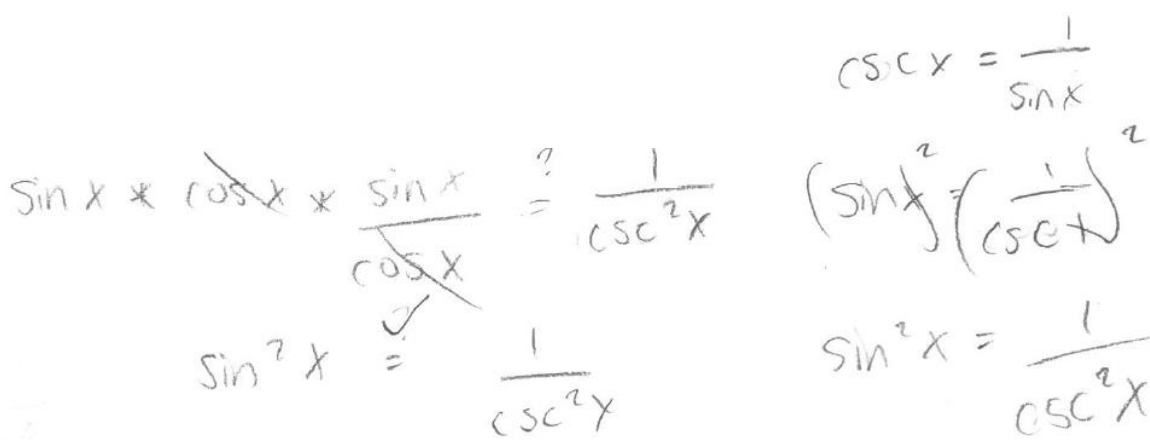

Figure 54

Emma's Proof for Task 8

Figure 54, Emma entered a checking phase to explain the relationship.

So basically she, I changed the $\tan x$ into $\frac{\sin x}{\cos x}$ immediately. Before moving it over. Um, and that just made it a little simpler. So I, I didn't end up with a $1 = 1$. I ended up with an identity that I could show was true, which does the same thing essentially. As long as you're doing sound algebraic manipulation and you come up with something that equals it's the other thing. And that's true then both are solid ways to check. Mine just happened to be two steps rather than her five. Both are true.

Emma considered her solution and the student's solution to both be valid solutions. She wanted the student to modify her solution by including a question mark above the first equal sign. Otherwise, She accepted the fact that the student had manipulated both sides of the equal sign.

While solving Task 8, Emma used the heuristic of written algebra. As Emma began to reason about the solution that the student had written, she chose to write her version of the proof to determine whether or not the given statement was true. Then she returned to thinking about the student's reasoning presented in the proof.

Task 9: Angles in a Rectangle

Emma's problem-solving phases for Task 9 are shown in Figure 56.

	Part 1													
Orienting	x													
Planning				x		x	x		x		x	x		
Executing		x			x			x		x			x	
Checking			x											x

Figure 56

Emma's Phases on Task 9: Angles in a Rectangle

Emma oriented herself to Task 9 and then began an executing phase during which she began to mark the provided rectangle, as shown in Figure 57.

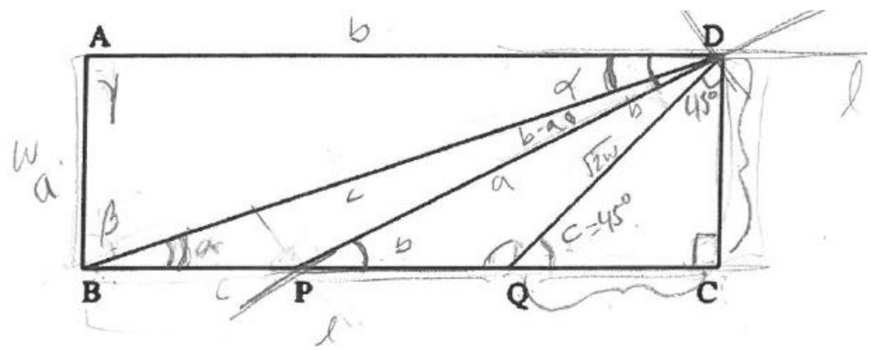

Figure 57

Emma's Marked Diagram for Task 9

She explained that initial markings that she made as,

> So if we know, hmm interesting show something that my brain just assumes is true. How do you show this? So the length? Yeah. Alright, so these two angles, this is a right angle. Can I get B? BD is cutting the whole rectangle by a diagonal. So this angle has to be $45°$. So that angle has to be $45°$.

She continued with her explanation in a checking phase, where she analyzed her reasoning for identifying $m\angle DQC$ as $45°$.

> Any quadrilateral… No, that's a lie. Any rectangle or square, each angle is $90°$. And if you … hmm, no, that is wrong. That is wrong. So $45°$ would make that a right triangle—the triangle, like ΔABD or whatever. Whichever one you do that would make it a right triangle, a 45-45-90, because the other one would also have to be $45°$, and that's obviously not okay. And that's just where my brain wanted to make it easier.

Emma believed that $m\angle DQC = 45°$, but she was unable to mathematically justify that fact during her initial executing and checking phases.

Emma continued to reason about the figure and its angles. During a planning phase, she considered whether another triangle in the figure might be a 30-60-90 triangle.

> Okay. Um, but actually could be a special triangle. Just a different one. And it makes me want it to be a 30-60-90. Because that could be a strategy. I don't know how to prove that it would be that, other than the fact that with special triangles, you know that AD is going to equal BC cause it's a rectangle.

In an executing phase, Emma continued to reason about whether any of the triangles in the figure were special right triangles.

> So you know that $3AB$ is also equal to AD just the way it is BC. And 30-60-90 rules. Have, it's like square root something, 1, 2, 3 or something. That seems right because 45-45-90, 1, 1, 2, maybe? I think that's what it was right, though. No, but it was one-two short leg. So that would mean it's not a 30-60-90 because it's three times rather than two times.

During this executing phase, Emma was able to recall the ratio between the sides of 30-60-90 triangles and determine that ΔABD was not a 30-60-90 triangle because the legs did not fit the correct proportion. She also drew the diagrams of special right triangles shown in Figure 58.

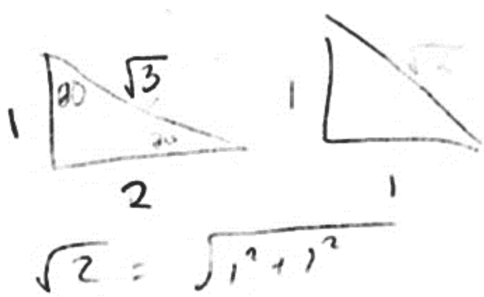

Figure 58

Emma's Special Right Triangles for Task 9

Emma continued to think about the special right triangles in another planning phase.

> So how else could I approach this? The addition of this angle plus this angle has to equal this angle. Well this, this triangle is right. Okay, so in... well both triangles are right. I

want to make this a 45-45-90 also, but it's not. There's no way to prove that with this scale actually. Maybe?

Emma continued to consider whether ΔDQC was a 45-45-90 triangle in another planning phase.

Cause this, these are equal actually. It is. This length and this length are equal. So 45-45-90 rules can't let do, let me call that a $45°$. Each of these little sections [from BC] are equal to [AD] cause it says that [BP, PQ, and QC] are all equal and that [BC] is three times one of these. So if these are all equal parts and it's split into 3, and this is 3 times that, then each little one is going to be equal to the side. Throwing it back to eighth grade when I did geometry. Um, this also isn't a lovely sound proof but, so that means that [$m\angle QDC$] has to be 45° to add up to that 90°.

Which means if we look at... the opposite, ideas, Emma, insight. I'm not sure I would have told you that [$m\angle DBC + m\angle DPC = m\angle DQC$], which means I feel like I don't have the insight, just been like searching for it in the things that I know.

Yeah. I was wondering if I could do something [with the right angles]. That would be helpful for you. I really don't think they will. But actually, that would make those transversals. Does that do anything? I was thinking about the parallel lines and the fact that this cuts, would I have any sort of help from that angle? Cause it would be equal to that angle, right? But I wasn't looking in the right places, and I don't think it's at a place that would be helpful for me.

While Emma considered the usefulness of the transversal during this phase, she drew the diagram in Figure 59 to help her reason about the angles that would be equivalent.

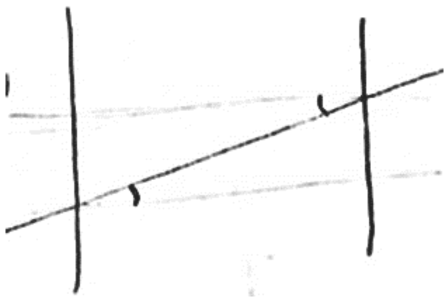

Figure 59

Emma's Diagram of a Transversal for Task 9

During an executing phase, Emma continued to list what she knew about the figure in hopes of stumbling upon a fact that would be useful.

> So we know that $m\angle DBC$ and $m\angle DPC$ are going to add up to be $45°$. These two angles are quite obtuse. Hmm. These three angles have to equal $45°$. If you look at the big triangle, no, you have to find $90°$ degrees in every corner somehow. Yeah. I don't know where the answer is going to come from here. … I know that these are two like scalene obtuse triangles. I know that this length is going to be, I don't remember if it's $\sqrt{2}$ or not, but I guess it is because if you did Pythagorean Theorem, that would be the 2, so it's really just like w, w, $\sqrt{2}w$ or something if we did it algebraically.

This listing of known facts about the diagram and considering what might be useful continued in a planning phase.

> I'm not sure if [the side lengths] would be helpful. Hmm, we know what this length... Ideas, ideas, ideas. … Wait. Okay. This big triangle. Little triangle, I just don't know how much each like adds to, I need to know these top angles to find the bottom ones. I don't know how to try to find the top angles. I feel like I'm dancing around the answer, but I have no idea what I'm dancing around. Well, if I can find this angle, I could definitely find $m\angle DBC$. But I'm not sure how to find that angle without knowing at least one of

> these or I'd actually probably need two of them. I know that the sum of these is $45°$. But I don't know how much each gets. Could I use [the formula sheet]? I'm wondering if law of sines or law of cosines would be helpful. Would you have to solve all three triangles?

Emma began an executing phase of using the law of sines to solve for missing angles in the triangles, as seen in Figure 60.

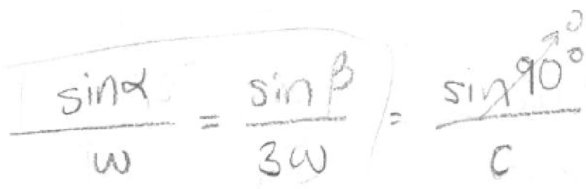

Figure 60

Emma's Law of Sines for Task 9

She explained her work as,

> So I don't want any of those lines. So should I do that with, cause that would find DB, not that it matters it's equal? So $\sin 45°$, same as $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$, which just proves that $QC = DC$. But that means it also equals 0, which just is not helpful.

When the law of sine did not lead to success, Emma returned to a planning phase.

> Hm. Hm. Hm. I mean for ΔDBA might be more helpful cause I have at least one angle in it. I tried doing the ΔDQC to see if I get, could get like the length of QC, but I'm not sure that would have helped me with this. But $\sin 90° = 0$, so that sends the other values to 0 again without helping me find... I'm missing too many variables for anything that I'm thinking about to work. I really don't know.

After this, Emma looked at the diagram and thought for about 2 minutes before explaining what she was thinking.

> I'm just like, my brain wants to go to law of sines or, or which is what I've been doing, or law of cosines but those really only help. Like law of sines, I liked that it had more equal signs, so I had the necessity for less variables, but none of my like triangles that I could work with have enough variables that don't go to 0. Um, because it's defined. So that does have some sort of value on the bottom. But because the numerator goes to zero, it doesn't help me find that value. Then you figure out what you want, and then law of cosines has too many variables that are missing. Um, I was hoping the sine of some of the angles might like, well I have this, but that's so symbolic that I'm not sure I can get much from it. Just is lovely. Um, I do not know. I'm stuck.

Emma stated that she was stuck, but continued to stare at the diagram for about two more minutes. She then started a new planning phase and identified equivalent angles based upon the transversal idea that she had briefly mentioned earlier in the task.

> Hang on—one more thought. Let me draw something to see if my thought works. Helps if I draw what I'm trying to draw correctly. [$m\angle DBC$ and $m\angle BDA$] are equal. Does that tell me anything is the question, does that help me? I don't know.

In an executing phase, she began to write down the information she knew about many of the angles in the diagram, seen in Figure 61.

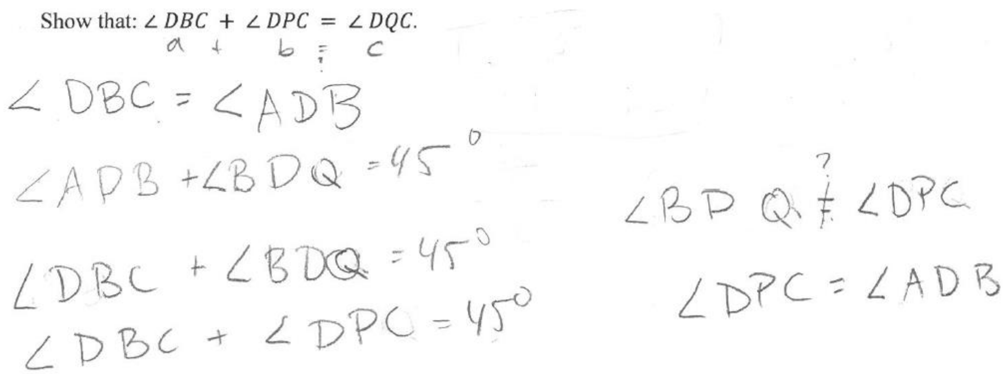

Figure 61

Emma's Known Information About Angles in Task 9

She explained this information as,

> Which means those two angles, if this is true, if $m\angle DBC + m\angle DPC = 45°$, these two angles are equal to $\angle DPC$. Because this is just a line cut by two transfers. So these two angles, great. $\angle DBC = \angle ADB$ and $\angle ADB + \angle BDQ = 45°$. So this is $\angle ADB$, this angle, $\angle PDQ$. This angle equals 45°. Which means I can sub in because these are equal. So $\angle DBC + \angle BDQ = 45°$. And this angle, 180 minus this angle and this angle? No, B is the sum of $\angle BDP$ and $\angle PDQ$. I don't know how to get from this being equal to this. Is $\angle BDQ$ equal to $\angle DPQ$? $\angle DBC$, so that's that. $\angle DPQ$, that's that. So that is the statement.

Emma made progress in identifying several relationships between various angles in the figure but hadn't yet found a path to the angles that she needed to show. She progressed to a checking phase of considering how the information she knew might help her reach a solution.

> I guess it's $\angle DPC$, just, I don't know if that necessarily proves it, but knowing that this angle is equal to that angle. And knowing that this angle is 45° and that this angle is equal, then I don't know if this is the true statement. It needs to be to make that true, but I

> don't know how to prove that statement. BP, $\angle BDC$, no I wanted $\angle BDQ$ that was right there. $\angle DPC$ is the one I need to change. Okay. I don't think my transversal is going to happen there. Cause there's not two parallel lines. I chose that these two are equal to $\angle DPC,$ and we showed that those two are equal. So this angle is equal to $\angle DPC - \angle DBC$. So $\angle ADP$ is equal to $\angle P$, $\angle DPQ,$ or $\angle DPC$. Which actually shows that this isn't true. BDQ, cause now B is equal to the top two angles, but that means I'm getting lost in all the things that I'm labeling, and I'm not sure I'm connecting dots that are actually being helpful. I'm not sure I have it. I'm trying so hard. I was angry that I was stopping and like trying to do it again and staring at it longer, and still, just like the transversal was like more of a development. But I don't think I have any more insight.

Though Emma attempted to persist in solving Task 9, she was unable to find a path that led her to a solution.

Emma used a variety of knowledge from geometry and the domain of triangle trigonometry to try to solve Task 9. She also used heuristics of written algebra and diagrams. Emma drew upon knowledge of multiple angle relationships within the diagram, including supplementary angles and alternate interior angles. She used these relationships to try to determine information about the angles she was looking for. Emma also used the Pythagorean Theorem to solve for missing sides in some of the triangles. Though she did not ultimately use this information to help her solve the task, she did solve for it and consider whether it might be useful. In triangle trigonometry, Emma used special right triangles and the law of sines to help her attempt the task. Emma initially considered whether she was working with 45-45-90 and 30-60-90 triangles. Though her instinct to include these was visual, she was able to justify that ΔDQC was a 45-45-90, but there were no other special right triangles in the diagram. Emma also attempted to use the law of sines but discovered that substituting in $\sin 90°$ resulted in an equation that was not useful. Near the end of her solution, Emma started keeping a written record of the information she had found. This record helped her to try to determine whether there were any relationships she had found, but not used. Throughout her solution, Emma drew several diagrams and annotated the given diagram. These helped her to visualize her thought process and clarify what information she had found.

Analysis by Domain

The resources and heuristics that Emma used to solve the tasks varied across the four trigonometric domains, as shown in Figure 62. When working on triangle trigonometry tasks, Emma used knowledge of geometry, triangle trigonometry, and unit circle trigonometry. For unit circle trigonometry tasks, Emma used knowledge of triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs. For the domains of trigonometric functions and graphs and analytic trigonometry, Emma used knowledge from both domains on the tasks. Thus, Emma used knowledge of trigonometric functions and graphs across three domains, and the knowledge of each of the other three domains when solving tasks in a total of two domains.

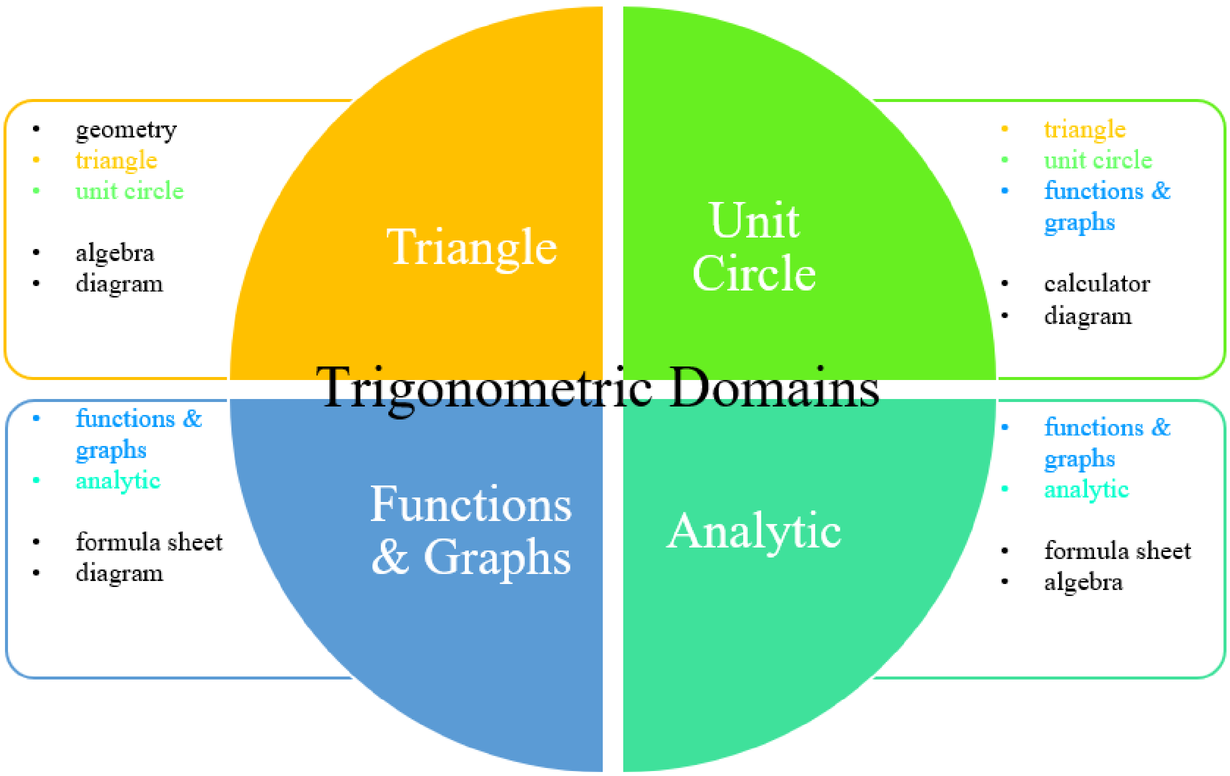

Figure 62

Emma's Use of Resources and Heuristics Across Domains

Emma’s used only two heuristics across the domains. She used diagrams in the three domains of triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs. She used written algebraic manipulations in just two domains, triangle trigonometry, and analytic trigonometry.

Triangle Trigonometry

Tasks 2, 4, and 9 addressed the domain of triangle trigonometry, as seen in Figure 63. For these three tasks, Emma generally began the task with a cycle of orienting, planning, executing, and checking, as described by Carlson and Bloom (2005). For Task 1, Emma completed the cycle without a checking phase for part 2 and part 3 but did include the checking phase for part 1. On Task 4, Emma completed the cycle for part 1, but skipped the planning phase and instead used two executing phases for part 2. Emma also skipped an initial planning phase for Task 9, but alternated between planning and executing multiple times after the initial problem-solving cycle.

Task 2														
	Part 2					Part 1				Part 3				
Orienting	x					x				x				
Planning		x		x	x		x				x		x	
Executing			x					x				x		
Checking									x					x

Task 4									
	Part 1				Part 2				
Orienting	x				x				
Planning		x							
Executing			x			x	x		x
Checking				x				x	

Task 9														
	Part 1													
Orienting	x													
Planning				x		x	x		x		x	x		
Executing		x			x			x		x			x	
Checking			x											x

Figure 63

Emma's Phases on Triangle Trigonometry Tasks

Emma used knowledge of geometry, as well as from the domains of triangle trigonometry and unit circle trigonometry to solve tasks in the domain of triangle trigonometry. Emma used geometry knowledge of the Pythagorean Theorem on Tasks 2 and 9 to find side lengths of right triangles and knowledge of angle relationships, such as supplementary angles and alternate interior angles, on Task 9 to find measures of missing angles. Emma used a variety of knowledge from triangle trigonometry across the tasks. She used knowledge of SOH CAH TOA on Tasks 2 and 4 to describe the relationship between the side opposite an angle and the hypotenuse. She used special right triangles to identify a triangle as a 45-45-90 triangle and the law of sines to try to solve for missing angles on Task 9. Emma also used knowledge of the unit circle on Tasks 2 and 4 to determine the values of sine and cosine, including their signs.

Emma used heuristics of written algebra and diagrams. Emma used written algebra on Task 2 to reason about how to show the given equation $\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$ was true and on Task 9 to reason about the angle relationships she had identified and how those might work to help her shown the angle relationship she was looking for. Emma created several diagrams for each of the triangle trigonometry tasks. In Task 2, she created two diagrams for part 3 of the task. These included a diagram of the triangle that she was working with and a diagram showing the relationship between the triangle in the first quadrant and the second quadrant. For Task 4, she created diagrams showing a triangle in the second quadrant and how to use that to define the sine of an obtuse angle and also a diagram of the unit circle. On Task 9, Emma marked extensively on the given diagram. She also created diagrams of the two special right triangles and two parallel lines with a transversal as she considered whether those applied to the diagram.

Unit Circle Trigonometry

Unit circle trigonometry is addressed by Tasks 3, 4, and 7. For all three of these tasks, Emma began with a problem-solving cycle of orienting, planning, and executing. In two of the three tasks, this was followed by a checking phase. While working on Task 3, Emma engaged in an initial cycle of orienting, planning, and two executing phases. This was followed by a cycle of planning, executing, and checking twice. For Task 4, on part 1, Emma engaged in a single cycle of orienting, planning, executing, and checking. For part 2, she primarily engaged in executing phases. Task 7 was also a single cycle of orienting, planning, executing, and checking.

Task 3										
	Part 1									
Orienting	x									
Planning		x			x			x		
Executing			x	x		x			x	
Checking							x			x

Task 4									
	Part 1				Part 2				
Orienting	x				x				
Planning		x							
Executing			x			x	x		x
Checking				x				x	

Task 7				
	Part 1			
Orienting	x			
Planning		x		
Executing			x	
Checking				x

Figure 64

Emma's Phases on Unit Circle Trigonometry Tasks

Emma used knowledge from three trigonometric domains, triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs, to solve tasks in the domain of unit circle trigonometry. She used knowledge of triangle trigonometry only on Task 4 and knowledge of trigonometric functions and graphs only on Task 7, but she used knowledge of unit circle trigonometry on both Tasks 3 and 4. On Task 4, Emma explained that a triangle in the second quadrant would have the same sine value as the reference triangle in the first quadrant because they would have the same length opposite and hypotenuse and thus the same sine value. On Task 7, Emma explained that $y = \sin(2x)$ is a horizontal shrink of the graph of $y = \sin x$ since you would be able to fit two periods of $y = \sin(2x)$ into a single period of $y = \sin x$. For Task 3, Emma determined that she wanted an expression that had an acute angle with the sine value equal to $\pm.5$ and used her knowledge of the unit circle to determine that this should be $\sin 30°$. For Task 4, Emma used memorized knowledge of the unit circle initially to evaluate $\sin\frac{3\pi}{4}$ and $\sin\pi$. She was also able to explain the relationships between sine values in the first and second quadrant of the unit circle. Emma did use the graphing calculator as a resource to evaluate the given expressions in Task 3.

Emma created diagrams on two of the three unit circle trigonometry tasks. For Task 4, she created two diagrams. One diagram showed how to create a right triangle in the second quadrant for an obtuse angle, and the other showed the unit circle. For Task 7, she created a pair of graphs showing the relationship between $y = \sin(2x)$ and $y = \sin x$. This visualization helped her to explain why $y = \sin(2x)$ is a horizontal shrink.

Trigonometric Functions and Graphs

The domain of trigonometric functions and graphs is addressed in Tasks 1, 5, and 7. For part 1 of Task 1, all parts of Task 5, and Task 7, Emma engaged in an initial orienting, planning, executing, and checking problem-solving cycle. Emma used a problem-solving cycle of orienting, planning, executing, and checking for part 1 of Task 1, but alternated between orienting and executing for the remaining parts of Task 1 when she was finding each transformation and matching identity. For Task 5, Emma completed a cycle of orienting, planning, and executing for three of the four parts of the task with just an orienting and executing for the remaining part of the task. Emma completed Task 7 with a single problem-solving cycle of orienting, planning, executing, and checking.

	Task 1														
	Part 1					Part 2							Part 3		
Orienting	x					x		x		x			x		
Planning		x									x				
Executing			x		x		x		x			x		x	x
Checking				x											

	Part 4							
Orienting	x		x		x			
Planning						x		
Executing		x		x			x	
Checking								x

	Task 5										
	Part 1			Part 2			Part 3		Part 4		
Orienting	x			x			x		x		
Planning		x			x					x	
Executing			x			x		x			x
Checking											

	Task 7			
	Part 1			
Orienting	x			
Planning		x		
Executing			x	
Checking				x

Figure 65

Emma's Phases on Trigonometric Functions and Graphs Tasks

To solve the tasks in the trigonometric functions and graphs domain, Emma used knowledge from trigonometric functions and graphs and analytic trigonometry. Emma used knowledge of graphs for all three of the tasks. On Task 1, she used a variety of knowledge about the graphs of sine and cosine. This included knowing that translating either graph by their period would translate the graph onto itself and reflecting cosine over the $x - axis$ or sine over both the $x - axis$ and $y - axis$ will reflect the graphs back onto themselves. On Task 5, Emma was able to describe the general shape of sinusoidal curves as oscillating consistently and then find the values of the amplitude and vertical shift for the given fox and rabbit data. For Task 7, Emma reasoned about the period of the graph and used the period to explain why $y = \sin(2x)$ is a horizontal shrink of $y = \sin x$. Emma only used knowledge of identities on Task 1. During this task, she was able to relate the translations she had found to identities for sine and cosine. Emma also used the formula sheet as a resource during Task 1.

Emma created a diagram to assist her in thinking about Task 7. During this task, she drew graphs of $y = \sin(2x)$ and $y = \sin x$. This allowed her to reason about why the relationship between the two is a horizontal shrink, since twice as many periods of $y = \sin(2x)$ fit into the same horizontal distance as $y = \sin x$.

Analytic Trigonometry

The analytic trigonometry tasks were Tasks 1, 6, and 8. Emma generally began each part of these tasks with a problem-solving cycle of orienting, planning, executing, and checking though she frequently skipped either the planning or checking phase. For Task 1, Emma completed all four phases on part 1 and completed just an orienting and two executing phases on part 3. For parts 2 and 4, Emma completed a phase of orienting and executing twice before completing the cycle of orienting, planning, executing, and checking. She did follow the initial cycle with an executing phase on part 1. Emma completed Task 6 with all four phases on parts 1 and 2, but skipped planning on parts 3 and 5 and skipped checking on part 4. In part 4 she also followed with a cycle of planning, executing, and checking. Emma completed Task 8 with a cycle of orienting, planning, executing, and checking, followed by another executing and checking phase.

Task 1															
	Part 1					Part 2							Part 3		
Orienting	x					x		x		x			x		
Planning		x									x				
Executing			x		x		x		x			x		x	x
Checking				x											

	Part 4							
Orienting	x		x		x			
Planning						x		
Executing		x		x			x	
Checking								x

Task 6															
	Part 1						Part 2						Part 3		
Orienting	x						x						x		
Planning		x	x					x			x				
Executing				x		x			x					x	
Checking					x					x		x			x

	Part 4						Part 5		
Orienting	x						x		
Planning		x		x					
Executing			x		x			x	
Checking						x			x

Task 8						
	Part 1					
Orienting	x					
Planning		x				
Executing			x		x	
Checking				x		x

Figure 66

Emma's Phases on Analytic Trigonometry Tasks

Emma used knowledge from trigonometric functions and graphs and analytic trigonometry to solve the analytic trigonometry tasks. The only task on which Emma used knowledge of graphs was Task 1. For this task, Emma accurately graphed both sine and cosine and used her graphs to reason about transformations that would map them back onto themselves and each other. On Tasks 1 and 6, Emma used knowledge of identities. For Task 1, Emma was able to identify which identities matched the transformations that she had found from the graphs. On Task 6, Emma used the even and odd identities to verify the sine and cosine difference angle identities from the sine and cosine sum angle identities. Emma also used the formula sheet as a reference while completing Tasks 1 and 6.

Emma used the heuristic of written algebra on Tasks 6 and 8. For Task 6, Emma wrote out the steps to verify each identity. For some of the identities, this was only a few steps, but for most, she wrote out several steps. This written record of her work helped her reason through verifying each identity. Emma also recorded some algebraic work on Task 8. Though a proof was provided, Emma wrote out the proof using her own reasoning. She then compared her reasoning to the provided poof when determining whether the provided proof was valid.

Mathematical Understanding for Secondary Teachers

Mathematical Proficiency

Mathematical proficiency consists of six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Emma demonstrated four of the six strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, and productive disposition. Emma demonstrated conceptual understanding when she made connections across the domains of triangle trigonometry and unit circle trigonometry on Task 4. She explained that a triangle in the second quadrant would have the same sine value as its’ reference triangle in the first quadrant.

> I always remember that x is cosine and y sine and I had on my unit circle because it's really $r * cosine$ and $r * sine$. But on a unit circle, you have the radius is just 1. So, any coordinate on the circle is just $(cosine, sine)$. So, the y value here is right there.

By connecting knowledge from these different domains, Emma showed that she had a conceptual understanding of how they fit together. Emma demonstrated procedural fluency on Task 6. On this task, she was able to work through the algebra for all of the different sum and difference angle identities fluently. She was able to substitute the value of $-\beta$ into the sum angle identities to verify the difference angle identities. She was also able to execute the fraction operations required to simplify the two tangent angle identities.

Emma showed strategic competence with the variety of heuristics she was able to use across the tasks. On most tasks, her first strategy was successful, but she tried multiple approaches on Task 9. She first tried using knowledge of angles, then special right triangles, then the law of sines, before finally returning to angle properties. Even though she was unable to solve the task with any of the strategies, she was able to consider all of them. Emma showed a productive disposition throughout the tasks. She was also eager to work and persisted in solving the tasks that gave her difficulty. When she gave up on Task 9 after working on it for more than 30 minutes, she stated, “I'm not sure I have it. I'm trying so hard. I was angry that I [was] stopping and like trying to do it again and staring at it longer…”

Mathematical Activity

Emma demonstrated all three strands of mathematical activity: mathematical noticing, mathematical reasoning, and mathematical creating across the tasks and domains. The strand of mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Emma demonstrated an understanding of two strands: symbolic form and connect within and outside mathematics, but demonstrated a lack of understanding of one strand: form of an argument. Emma was able to work with the symbolic form on Task 6. She was able to apply the sum and difference angle identities and used the symbolic form correctly without overapplying the distributive property. She was able to connect outside of mathematics on Task 5. As she was working, she stated, “Ooh, that's a biological application than I actually did in BIO 181 last semester.” She was further able to describe the predator-prey relationship between the two populations. Emma demonstrated a lack of understanding of the form of an argument on Task 8. On this task, Emma stated that the hypothetical student’s proof was correct, “as long as you're doing sound algebraic manipulation, and you come up with something that equals the other thing.” Emma had noticed that the student multiplied both sides of the equation by $\cos^2 x$ as the first step, but did not consider this to be a problem.

Mathematical reasoning consists of three strands: justifying/proving, reasoning when conjecturing and generalizing, and constraining and excluding. Emma demonstrated one strand: justifying/proving. Emma demonstrated this strand on Task 6. For example, on Task 6, she was able to write a sequence of logically connected algebraic statements to verify the tangent sum angle identity. She began with $\tan(\alpha + \beta)$ and initially rewrote it as $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$. She then used the sine and cosine sum angle identities to rewrite it as $\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$. She considered how to transform this into the form she was looking for and said, “What if we divide everyone by $\cos\alpha\cos\beta$, so that it goes to 1? Ooh, actually, that might be really helpful.” After dividing through by $\cos\alpha\cos\beta$, the result simplified to the form she was looking for, $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

Mathematical creating consists of three strands: representing, defining, and modifying/transforming/manipulating. Emma demonstrated two of the three strands: representing and modifying/transforming/manipulating. Emma created representations of her thinking through diagrams on several tasks. For example, on Task 2, Emma created a diagram showing a triangle in the second quadrant and its' reference triangle in the first quadrant, seen in Figure 37. She then described that these would have the same sine value, but the cosine would be negative in the second quadrant because the horizontal part of the triangle is in the negative direction. Emma showed an understanding of modifying/transforming/manipulating on Tasks 1 and 6. On Task 1, she was able to identify transformations mapping sine and cosine to themselves and each other from the graph. For example, she initially assumed that both the translation and reflection would be the same for sine as they had been for cosine. However, when she looked at the identity for the reflection, she noticed that it was $\sin(-\theta) = -\sin\theta$. This caused her to reconsider the reflection graphically and determine that sine needed to be reflected over both the $x - axis$ and $y - axis$ to map onto itself. She was able to algebraically manipulate the identities on Task 6. For example, on the sine difference angle identity, she first rewrote $\sin(\alpha - \beta)$ as $\sin(\alpha + -\beta)$ and then applied the sine sum angle identity. She then used the even and odd identities to simplify $\sin -\beta$ and $\cos -\beta$ to verify the identity.

Case 3: Evanna

Evanna was a traditional undergraduate at the time of her participation in the study. She was a freshman majoring in mathematics education. She had yet to take any mathematics education courses and was currently enrolled in Calculus 2 as a mathematics class. On a test of her knowledge of trigonometry, Evanna scored a 63% overall. She left a total of four questions blank, including one triangle trigonometry, one unit circle trigonometry, and two analytic trigonometry. She scored a 71% on triangle trigonometry, a 75% on unit circle trigonometry, a 75% on trigonometric functions and graphs, and a 43% on analytic trigonometry.

Evanna was not confident about solving the tasks. She would get stuck easily and give up. She also frequently would seek confirmation about the correctness of her work or answers from the interviewer. Evanna frequently grabbed the formula sheet and graphing calculator, but she was unsure how to use them to assist her in solving the tasks.

Description by Task

Task 1: Identities from a Graph

Evanna's phases for Task 1 can be seen in Figure 67.

	Part 1				Part 2										
Orienting	x				x		x		x			x			
Planning														x	
Executing		x		x		x		x		x			x		x
Checking			x								x				

	Part 3							Part 4							
Orienting	x		x					x			x			x	
Planning						x									
Executing		x		x					x			x			x
Checking					x		x			x			x		

Figure 67

Evanna's Phases on Task 1: Identities from a Graph

Evanna began by orienting herself to the first task. She then began an executing phase of creating the graphs of $f(x) = \cos x$ and $g(x) = \sin x$.

> All right, so sine starts at the origin. Does it matter where I put one [on the $y - axis$]? Well, I'll make this 1. I want to make it clear. π [on the $x - axis$]. π. 2π. Want me to draw it like over [on the negative x-axis] too?

Evanna first labeled her scale on the $x - axis$ and $y - axis$, then plotted the points $(0,0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right), (2\pi, 0)$ for sine and $(0,1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$ for cosine. She also extended her graphs back into the negative x-values and plotted the appropriate points. She worked quietly but did talk to herself as she completed her graphs. Next, she entered a checking phase and asked, “Did I do the period right? Is it 2π?” After receiving confirmation that the period was 2π, Evanna returned to working in an executing phase. “And then cosine starts it one. I think that's right. I haven’t done this in so long.” At the end of part 1 of the task, she had created the graphs in Figure 68.

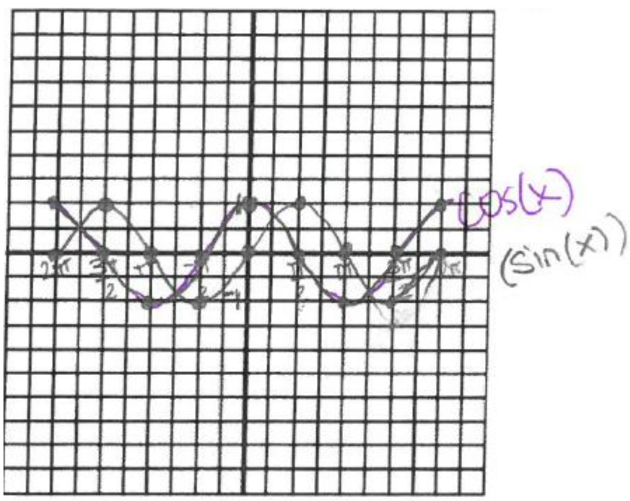

Figure 68

Evanna's Graphs of Sine and Cosine for Task 1

Evanna then oriented herself to part 2 of the task and finding a translation that would map cosine on top of itself. Evanna then began with an executing phase of finding the translation.

> So to move it left and right that would be cosine … and you do it within the parentheses to move it left or right? 2π moves over a half as a whole π… so a period … 2π. To shift it, so the negative sign means to the … left? I can't remember what exactly the negative and plus within the parentheses means. Either way, if you subtract 2π that's the entire

> period, so it'll move it … [This point at 2π] will end up here [at 0]. [It will] make the cosine again.

Evanna immediately stated that cosine would have to be translated a whole period to end up back on top of itself, but she was unsure how this would show up in the equation. She eventually wrote $\cos(x - 2\pi)$ as her solution.

For part 2 of the task, Evanna oriented herself and then proceeded to an executing phase during which she stated, “Reflection you do the negative on the outside. So, flip it and then move it.” During this explanation, she motioned reflecting over the $x - axis$ with her hand. She then clarified during another orienting phase that the question was asking her only to use a reflection, not a reflection and translation, as she had initially suggested. This was followed by another executing phase.

> Oh, across the $y - axis$. I’m not sure where to put that. Well, my immediate thought when I hear reflection, I think of up-and-down [over the $x - axis$]. So I saw that it won't work if you flip it across the $x - axis$, but if you flip it across the $y - axis$ it's the same exact thing. I just don't remember where you put that negative sign… with the x or is it out here [in front of cosine]?

Once Evanna considered using only reflections, she realized that she could do a single reflection over the $y - axis$ to map cosine onto itself. However, she was unsure of how to place the negative in the equation, so she decided to try graphing both options on the calculator. Using the calculator, she determined that $\cos(-x)$ results in the same graph as $\cos x$ and wrote that as her solution. Evanna then oriented herself to the goal of finding an identity that matched each of her translations. She used the formula sheet and noted that there were many identities with which she was unfamiliar. During an executing phase, she looked at the formula sheet and identified “The periodic formulas. Because it shows that if you move it this over by the period, it'll be the same. Yeah, this one $\cos(\theta + 2\pi * n) = \cos\theta$." For the reflection, Evanna next used a planning phase. She again reviewed the identities on the formula sheet and stated, "And then the reflection … Um, it's not an inverse. I don't really know what I'm looking for. I'm not sure." She continued to look at the formula sheet and, during an executing phase, found a solution. "Oh, oh. Even and odd formulas. $\cos(-\theta) = \cos\theta$."

Next, Evanna oriented herself to part 3, which was the same as part 2, but for sine instead of cosine. She again began with an executing phase for the translation. "So that's the same thing. You want to subtract it or move it by the period. So this one would be $\sin x$ … and I guess it could be either plus or minus 2π. It doesn't matter if either way. Um, so yeah, 2π." For the reflection, she oriented herself and then entered an executing phase. "So sine, as you flip it across the $x - axis$ is going to be negative sine, which is backwards. And same thing if you flip it across the $y - axis$. Because I don't … this point will end up over here. Umm." When it was suggested that she could do more than one reflection if necessary, Evanna said, "Oh, do both. So, if you did negative, so if you flipped it across the $x - axis$ that would put it here and then flipped it across $[y - axis]$ … so flipping it across both. So that would be a $-\sin(-x)$." During a checking phase, Evanna typed her solution in the graphing calculator to double-check that she was correct. She then began to think about identifying the identities in a planning phase that led to an executing phase. During the executing phase, Evanna stated the two identities that she believed matched her transformations. "So even/odd, well periodic first $\sin(\theta + 2\pi * n) = \sin\theta$. Odd, well this one shows that $\sin(-\theta) = -\sin\theta$. So it'd be two formulas." The even/odd identity for sine did not identically match what she had written down, so Evanna used a checking phase to consider whether it was her solution or whether she needed to combine it with another identity. “Could you still use that same formula and say $-\sin\theta$ instead of $-\theta$? Like do it backwards or is there another one? So if it's just $-\sin(-x) = \sin x,$ which we saw because that was it was backwards. It was flipped, and so putting a negative on the outside will flip this to a positive sine.” She was able to conclude that the even/odd identity for sine was equivalent to the one she had found.

Then Evanna oriented herself to the goal of finding a translation from cosine onto sine. During an executing phase, she looked at her graph and stated, “Okay. So cosine move it over $\frac{\pi}{2}$. And that would make it sine. So for making cosine, okay $\cos x$I guess it … $\frac{\pi}{2}$.” She decided to check her solution using the graphing calculator during a checking phase. “I'm gonna check that. That's not right. … Oh, well, that does make it sine. Subtracting $\frac{\pi}{2}$. I think. I'm gonna change the sign. Yeah, so subtracting $\frac{\pi}{2}$. I think you can add it too. $\frac{\pi}{2}$ will shift it.” She then oriented herself to finding the translation from sine to cosine. She looked at the graph in an executing phase and stated, “Well, you can just shift that one … as well. It's a $\sin x$. I don't know what I did. No, so I have to reflect it as well. So I have $\sin(x - \frac{\pi}{2})$, but then I also have to reflect it across the $x - axis$.” During a checking phase, Evanna used the graphing calculator to verify her solution. “Which I believe you put on the outside, but I'll check.” Thus, her solution was $-\sin(x - \frac{\pi}{2})$. For the final portion of part 4, Evanna oriented herself to finding the identities that were associated with her solutions. During an executing phase, she found the co-function identities on the formula sheet. “Okay, so I guess there are these co-function formulas. So $\cos(\frac{\pi}{2} - \theta) = \sin\theta$. I guess it's a form of this kind of.”

While solving Task 1, Evanna made frequent use of both the formula sheet and the graphing calculator. She used the formula sheet each time that she was asked to find an identity matching her transformations. She used the graphing calculator to verify the majority of her solutions did indeed map back onto sine or cosine as needed.

Evanna used knowledge of both trigonometric functions and graphs and analytic trigonometry to solve this task. She used knowledge of graphs on several occasions. She began by creating accurate graphs, including a scale, of both sine and cosine from memory. She also was able to explain that translating by a full period would map both sine and cosine back onto themselves. She used knowledge of analytic trigonometry to find identities that matched her transformations. She was able to recognize similar identities on the formula sheet and reason about whether they showed the same relationships that she had found.

Task 2: Pythagorean Theorem and Pythagorean Identity

The phases that Evanna progressed through to solve Task 2 can be seen in Figure 69.

	Part 2	Part 1	Part 3
Orienting	x		x
Planning	x	x	x
Executing	x	x	x
Checking		x	x

Figure 69

Evanna's Phases on Task 2: Pythagorean Theorem and Pythagorean Identity

After an orienting phase, Evanna began a planning phase and stated, "Oh, that's like the $\sin^2\theta + \cos^2\theta = 1$. So $\sin^2\theta + \cos^2\theta = 1$." She recognized the given equation as a form of the Pythagorean Identity and explained further in an executing phase. "So AB, well, I'll label this. So this is hypotenuse, adjacent, and opposite. So AB is adjacent over AC which is hypotenuse. So that's cosine. And then BC is opposite. And then AC is hypotenuse, which is $\sin^2\theta$. Equals 1." She also labeled the sides of the given triangle as opposite, adjacent, and hypotenuse, as seen in Figure 70.

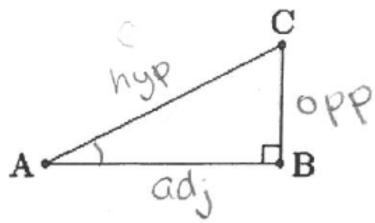

Figure 70

Evanna's Labeled Triangle for Task 2

Evanna had solved part 2 of the task, so she was asked to take another look at part 1 and see if she could explain the relationship in the original equation.

Evanna returned to looking at part 1 with a planning phase. She identified it as Pythagorean Theorem with, “Um, the $a^2 + b^2 = c^2$.” She expanded upon this in an executing phase, “I guess. c^2, but that would be if this equals to 1. Because that's our c. So like what? So this would be the length $AB^2 + BC^2$ has to equal the hypotenuse squared. And so in order for that to be 1, this would have to be 1.” During a checking phase, she asked, “Did I answer that question?” She was able to relate the given equation to the Pythagorean Theorem, but only if the hypotenuse was 1.

Next, Evanna oriented herself to part 3 of the task. She started a planning phase by creating and labeling the sides of the triangle on the right in Figure 71.

> Okay, so sine, I have to write all this out. Is opposite over hypotenuse, so if, I guess we go the other way but doesn't matter. Um, theta. So sine equals opposite. So 8 and then the hypotenuse is 17. So we need this to be able to figure out cosine.

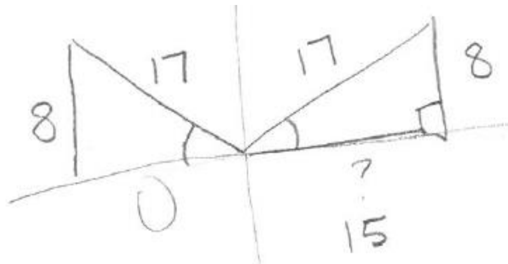

Figure 71

Evanna's Triangles in the First and Second Quadrant for Task 2

Evanna knew that sine was the ratio of the opposite and hypotenuse on a triangle and that cosine was the ratio of the adjacent and hypotenuse. Thus, she was able to label the opposite and hypotenuse from the given information. Then, during an executing phase, she used the Pythagorean Theorem, as seen in Figure 72, to find the length of the adjacent side.

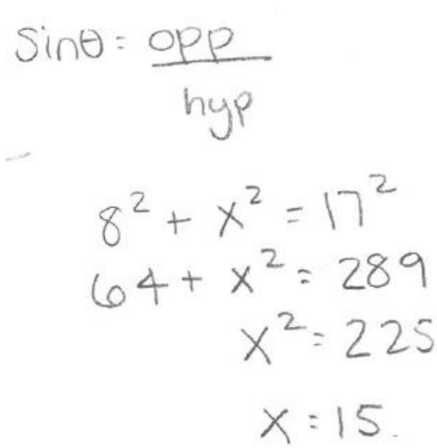

Figure 72

Evanna's Solving for Cosine in Task 2

> So that's where 8^2 plus whatever this is, x^2 equals 17^2. So that's $64 + x^2 = 17^2$, 289, and just making sure I do this right. 15, so that is 15 which means cosine is, which is adjacent over hypotenuse is $\frac{15}{17}$.

During a checking phase, she was asked about her earlier statement that the triangle she drew should go the other way. As she started her explanation, she drew the triangle on the left in Figure 71.

> So if this were the $x - axis$ and $y - axis$, I can draw this the same way because you said the second quadrant. And that means this would still be 17. Oh, but that would be negative. Yeah, that does make [a difference]. Um, no, it doesn't. It would just be a -15, yeah. Because you still, this stuff doesn't change because that's going up. So that's still positive. So you still find x, but because it's going this way it made this, whatever goes here has to be negative. So whatever this answer is, you make it negative. I don't know. I haven't done this stuff in so long. Um, well, if sine is $\frac{8}{17}$, opposite over hypotenuse, uh, cosine, no. I think I'll, I'm going to go with that.

She concluded by modifying her answer from $\cos\theta = \frac{15}{17}$ to $\cos\theta = -\frac{15}{17}$.

Evanna used knowledge of geometry, triangle trigonometry, and unit circle trigonometry to solve Task 2. She used the Pythagorean Theorem to explain the given equation and to solve for a missing side of a triangle. She used SOH CAH TOA to relate the given equation to the Pythagorean Identity and to assign values to sides in her triangle in Figure 71. She used knowledge of the signs of sine and cosine in the first and second quadrant to modify her final answer to be negative.

She also used heuristics of algebra and a diagram. Evanna wrote down her work to solve for the missing side of the triangle needed to find cosine in Figure 72. She used a diagram to show the triangle she was working with and labeled the sides in Figure 71. She added to this diagram when she was asked about the triangle being in the second quadrant, and it helped her to recognize that her solution should be $\cos\theta = -\frac{15}{17}$ instead of $\cos\theta = \frac{15}{17}$.

Task 3: Which One Doesn't Belong?

Evanna solved Task 3 by proceeding through the problem-solving phases as seen in Figure 73.

	Part 1															
Orienting	x															
Planning		x						x					x			
Executing			x		x		x		x		x			x	x	
Checking				x		x				x		x				x

Figure 73

Evanna's Phases on Task 3: Which One Doesn't Belong?

Evanna clarified the task during an orienting phase and then began a planning phase. She referenced the provided unit circle on the formula sheet and stated,

> Okay, so I'm going to use the unit circle to see what these mean. I don't exactly remember one. So $\sin 150°$ is the y value. So $\frac{1}{2}$. So, this is like $\frac{1}{2}$. $\sin 225°$, that's the y value, $-\frac{\sqrt{2}}{2}$ and then $\cos 120°$ is the x value, which is $-\frac{1}{2}$. So...

Evanna then considered what she had found in an executing phase.

> Well if these two are $\frac{1}{2}$, then if I made [my solution] equal to $\frac{1}{2}$That would exclude this. But if I made this a $\frac{1}{2}$ but if it was $\frac{1}{2}$, I'm not sure what these would do to exclude this.

During a checking phase, she continued to consider her initial thought of making her solution equal to $\frac{1}{2}$.

> I'm making sure I did these right. Huh? And if I made this negative something that would exclude this one. But if it's $-\frac{1}{2}$, I don't know what these, if these two, these three would not exclude this one. Uh.

Evanna had decided that her potential solution should be equal to $-\frac{1}{2}$.

Evanna then returned to thinking about her solution in an executing phase. “Okay. Well, $150°$ is, oh, I don't think that has anything to do with it. $150°$ is $\frac{5\pi}{6}$, $225°$ is $\frac{5\pi}{4}$, and $120°$ is…” In a checking phase, she explained that she was considering the angles in radians. “No, that doesn't… To see what their radians were, to see if that would make a difference.” She continued to consider the radian values in an executing phase. “So just I know it doesn't have anything to do with sine or cosine, but... Um, $150°$ is $\frac{5\pi}{6}$, $120°$ is $\frac{2\pi}{3}$, and then sine or $225°$ is $\frac{5\pi}{4}$. That's not gonna do anything. I have no idea.”

It was suggested to Evanna that she pick something that fit the first couple characteristics she had identified and then see if she could find other characteristics. She used this advice and began a planning phase. “Okay. So, this, if it were a $-\frac{1}{2}$, in your mind, cross this out. If you were $-\frac{1}{2}$., it could either be $\sin 120°$ or not $\cos 120°$. It can be $\sin 210°$, $\cos 240°$. Okay. Oh well, that was just right there.” During an executing phase, she stated, “Um, if it were $\sin 210°$, that would exclude this because it's cosine.” Next, she entered a checking phase to determine if she had excluded all four or which she had excluded.

> Did I exclude all of them? So those two, so that's excluded. [So I need to exclude that one.] Oh, what makes that one different? Hmm. So that's not right.

Evanna thought she was not correct but was encouraged to think about other possibilities that fit her constraints or other characteristics that might exclude the remaining expression.

Evanna then moved onto a brief planning phase of “Just a $\sin 210°$. Does it has to do with their opposite?” before moving onto an executing phase of “Like this is y. So $\sin 150°$, is $\frac{1}{2}$. But the first value is -3, $-\sqrt{3}$. Um, $\sin 225°$. The other value is $-\frac{\sqrt{2}}{2}$. $\cos 120°$ is $-\frac{1}{2}$. I mean, yeah." This was followed by a checking phase, during which she described more about what she was considering.

> And then $\sin 210°$, no that's not right. The other, so the corresponding x value and then the corresponding y value to see if that excluded it. And unless I looked over it, that did not do anything. Um, no, it's not that it's in a different quadrant.

She began this cycle by considering the corresponding x or y coordinates that went with the values that she had, but couldn't find any common characteristics. At the end of the phase, she started considering which quadrant the angles were located in.

She began a planning phase of looking for other possible expressions that might fit her constraints, but be in a different quadrant. "$\cos 240°$. So if I did $\cos 240°$..." This was followed by an executing phase during which she considered whether $\cos 240°$ was the solution she wanted. "$\cos 240°$ is $-\frac{1}{2}$. So then the corresponding value is -3, $-\frac{\sqrt{3}}{2}$. $\cos 120°$ the other one is $-\frac{\sqrt{3}}{2}$. Yes. We're trying to exclude this one. It's $\sin 150°$, $\sin 225°$...uh." Since she could not identify another characteristic, she entered another executing phase of considering other values on the unit circle that were equal to $-\frac{1}{2}$. "Anything negative one half. So $\cos 120°$, $\sin 210°$, $\cos 240°$, $\sin 330°$. So… different quadrant? Cause the other ones are in that quadrant." She decided upon a solution of $\sin 330°$. She then checked her solution in a final checking phase.

> So we excluded [$\sin 150°$] by all of these being negative. Does this make sense how I'm doing it? Okay. We excluded [$\sin 225°$] because the others were $\frac{1}{2}$. We excluded [$\cos 120°$] because it was cosine, and then we excluded [$\sin 330°$] because of the quadrant.

Evanna concluded the task with a solution of $\sin 330°$.

Evanna used knowledge of the unit circle to complete this task. She used the provided unit circle on the formula sheet to evaluate the given expressions and to find the possibilities for a solution once she had determined some characteristics to look for. She did not use any heuristics while solving this task.

Task 4: Define Sine of an Obtuse Angle

For Task 4, Emma worked through the problem-solving phases, as seen in Figure 74.

	Part 1			Part 2		
Orienting	x					
Planning		x		x		
Executing					x	
Checking			x			x

Figure 74

Evanna's Phases on Task 4: Define Sine of an Obtuse Angle

Evanna began part 1 of the task with an orienting phase and then moved into a planning phase. Her initial thought was to use the y-value on the unit circle.

> Well, on the unit circle, if you did sine of an obtuse angle like over here, like of $210°$, sine is just the y value. Um, so don't really know how to explain it. The only thing I can

> come up with was like with any other angle that's not a right or with any angle. When you find sine on the unit circle, it's just the y value.

During this phase, she created the diagram in Figure 75.

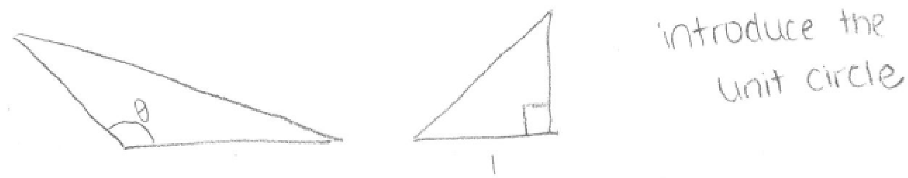

Figure 75

Evanna's Diagram of a Triangle with an Obtuse Angle for Task 4

When asked to explain how she would help a student who only knows the right triangle definition of sine to understand the connection to the unit circle definition of sine, Evanna responded in a checking phase. “Well, in a regular right triangle. Um, I'm fried. Been doing calculus all day. Um, sine of an obtuse angle. I really don't know. Like I'm blanking. [I would introduce the unit circle.]” Evanna knew that students should learn about the unit circle, but was unsure how to help them connect that to their previous knowledge.

For part 2 of the task, Evanna immediately jumped to a planning phase of finding the value of $\sin\frac{3\pi}{4}$ and $\sin\pi$.

> Well, I know with the unit circle there's rule of like 1. Cause it's $\sqrt{2}$? I think that was… The triangles might be... maybe not. I can't remember, but it's like 1 and like the hypotenuse and is the $\sqrt{2}$.

Evanna’s initial thoughts were around the ratios of sides of 45-45-90 triangles. She then transitioned to an executing phase, during which she used the unit circle to evaluate the two expressions.

> Well just looking at a unit circle $\sin\frac{3\pi}{4}$ is the $\frac{\sqrt{2}}{2}$ and the $\sin\pi$ is 0. Because sine is the y value, and then $\sin\pi$ is the same thing as $180°$. So it's just along the x axis. So vertically, it's just 0. And then the y value of $\frac{3\pi}{4}$, if you drew a triangle that's where that rule [for the 45-45-90 triangle] comes in that, I can't exactly remember, but it's something like over here.

She finished with a checking phase by asking, “Is that good?”

Evanna used knowledge from the domains of triangle trigonometry and unit circle trigonometry to complete Task 4. When she was trying to evaluate $\sin\frac{3\pi}{4}$, she tried to recall the ratios of sides of a 45-45-90 triangle as a way to explain why $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$. She also used knowledge of the unit circle, such as the fact that sine is the y-value of each coordinate. Evanna also used the heuristic of a diagram as she began to consider how to define the sine of an obtuse angle.

Task 5: Graphs of Foxes and Rabbits

On Task 5, Evanna worked through the problem-solving phases, as seen in Figure 76.

	Part 1		Part 2			Part 3			Part 4	
Orienting	x		x			x			x	
Planning				x			x			
Executing		x			x			x		x
Checking										

Figure 76

Evanna's Phases on Task 5: Graphs of Foxes and Rabbits

Evanna began by orienting herself to Task 5. She then began an executing phase of describing why a trigonometric model was appropriate for the data.

> I guess it would be important because of the, what am I trying to think of, prey. Predator-prey model. Um, so the more foxes there are, the less rabbits you're going to have because there's more of them to consume the rabbits. So I guess like naturally it's gonna do that, which is like a sine. Yeah. I can't think of the word, but, um, so I guess that's why it would be appropriate to model it. I don't know. … Naturally predator-prey model is going to fluctuate, similar to trig graphs.

She was able to recognize the real-world application and the predator-prey relationship between the population of the foxes and rabbits.

Next, Evanna oriented herself to the second part of the task. During a planning phase, she began to consider how to find the model for the rabbits and stated, "So it starts at 1000, and then it goes down, which is like sine." In an executing phase, Evanna continued to work on finding the model.

> Okay, so negative sine. Because from the axis, it's going down first. So it's flipped. Oh, is this where like you have to find the period and everything? Oh gosh. So the period is $\frac{2\pi}{\omega}$. Um, so I'm gonna just put this as sine. Negative sine starts at 1000, so I guess plus 1000 will be on the outside. I would never, and then the period over 24 months. So 2π over, is that just where the 24 is? Or no, that's four. No. ω is a fixed number for the period. This t, oh, so 24 is the period? Is that what the period is? Yeah. 24 was the period. And then we're trying to find this number. Okay. Times x, $24x = 2\pi$ and then so $\frac{2\pi}{24} = \frac{\pi}{12}$ putting that back. No, you don't put that back again. Yeah, you do. And so the period is $\frac{2\pi}{\frac{\pi}{12}}$ flip

> that. That's not right, is it? So I found x. Yeah. Just the α that goes on the bottom. It's a fixed number, so we just put this number in here. So $-\sin t + 1000$. I can't think of anything else.

Evanna was able to find values for the amplitude, period, and vertical shift. She stated that the negative in front of sine was to account for the fact that the graph was upside down from a parent sine curve. She found the period to be 24 months, rather than 12 months, but then correctly found the value of ω. Her written solution was $-\sin\left(\frac{\pi}{12}t\right) + 1000$, which deviated slightly from her spoken solution.

Evanna then oriented herself to find an equation for the foxes and said, "Oh, same thing." She started with a planning phase of, "So this is like cosine because it starts above and then goes down." In her executing phase, she used the solution she had found for the rabbits to assist her with finding the solution for the foxes. "So cosine. So same thing in the period. Um 24, oh is the period, the periods gonna be the same. Plus, it looks like 200. No, not even like 160." Thus, Evanna had determined the equation for the foxes to be $\cos(\frac{\pi}{12}t) + 160$. As with the equation for the rabbits, Evanna was able to find the values of the period and vertical shift, as well as identifying the parent function she would use.

Evanna started part 4 by orienting herself to the question and then began an executing phase of describing the real-world reasoning for the two graphs seeming to chase each other. Rather than creating a new graph, she chose to sketch the curve on the provided graphs, as seen in Figure 77. She stated,

> So I guess what I mentioned earlier? So when it's around 12 here, foxes and around 12, here it's, there's more foxes and less rabbits, which makes sense because the more

predators, the less prey. So, and I guess the vice versa is true as well. So just it, the more foxes, less rabbits, and when there's less foxes.

Evanna reiterated her original thinking about the relationship between the foxes and rabbits and the real-world implications of that relationship.

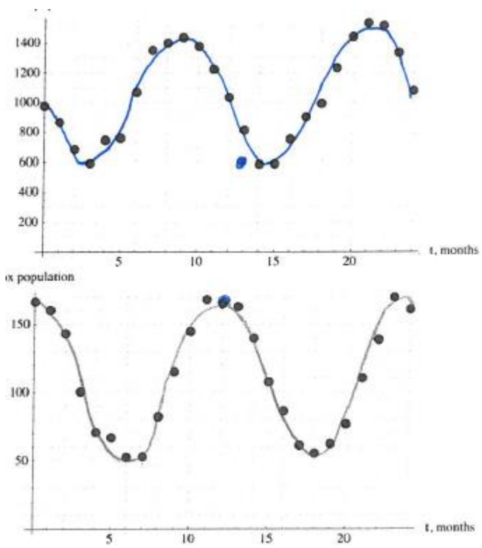

Figure 77

Evanna's Sketch on the Fox and Rabbit Data for Task 5

Evanna used knowledge of trigonometric functions and graphs to solve Task 5. She was able to determine which parent function to use for both the rabbits and foxes and explain why those parent functions made sense based on the shape of those graphs. She was also able to identify values for some of the parameters within the equations. She did not use any heuristics to help her solve this task.

Task 6: Sum and Difference Identities

For Task 6, Evanna’s problem-solving phases can be seen in Figure 78.

	Part 1				Part 2				Part 3				Part 4						Part 5			
Orienting	x				x				x				x						x			
Planning		x				x	x			x										x		x
Executing			x					x			x			x		x					x	
Checking				x								x			x		x	x				

Figure 78

Evanna's Phases on Task 6: Sum and Difference Identities

After orienting herself to the task, during a planning phase, Evanna stated, "well I'm not sure if this has anything to do with it, but $\sin(-\theta) = -\sin\theta$." During an executing phase, she used this identity to assist in simplifying.

> So using that, sine it would be like negative of this, is negative sine which may, which is like right here. I don't, I'm not really sure how to put that in words, but it's like sine of a positive angle. So, it produces a positive sine of that angle. And it's also asking you to find the sine of a negative, which gives you according to this a negative sine and there's your negative sine.

As she explained, she wrote the work in Figure 79.

Figure 79

Evanna's Work on the Sine Difference Angle Identity

In a checking phase, she further explained her thinking.

> So I guess the same is that sine of an angle is gonna produce like sine times cosine, which stays the same in both. What changes is that here you have both positive angles. Which is gonna produce positive. And here like according to this rule sine of a negative angle, produces a negative sine. I don't know how to explain it. Um, so there's your negative sine. I guess what this is saying the same, but producing sine and cosine together. I don't

really know how to put that into words. Yeah, I guess that goes back to the cosine formula because cosine of a negative degree is still positive. So here what it's asking for a negative, like a $-\beta$, like that doesn't change the sign of cosine. I don't know. That's the only thing I can do on that one.

Though she was able to use the even and odd identities to simplify the sine difference angle identity from the sine sum angle identity, she was not confident in her explanation of why this worked.

Evanna then oriented herself to part 2 of the task, which asked her to verify the cosine sum angle identity from the sine sum and difference angle identities. She started with a planning phase.

So $\cos(-\theta)$ is just positive, so... hm. I guess... So, I'm assuming that if $\cos(-\theta)$ is positive then $\cos(+\theta)$ is negative. No, that wouldn't make sense. Um, well with the, I don't know, maybe just but the double angle, no. Because it almost looks like it because cosine of a double angle is equal to $\cos^2\theta - \sin^2\theta$. So it's almost like, cause I mean if you multiplied like let's say this was x and this was x that would be $\cos^2\theta$. [But] I don't know if that makes sense because that's not necessarily a double angle.

Evanna began this planning phase trying to continue to use the even and odd identities as she had in part 1 but did not find success with that route. She then looked at the formula sheet and noticed similarities between the cosine sum angle identity and the cosine double angle identity. Since this also did not seem to lead anywhere, she began a new planning phase.

Well tangent relates sine and cosine, like just in general but, or the $\sin^2\theta$ and $\cos^2\theta$. I'm not sure how that would… I mean, I can kind of see it [with the Pythagorean Identity], but at the same time, I can't see it.

Since she had identified the Pythagorean Identity as a possible avenue, she was encouraged to write out what she was thinking and see if that gave her any insight. During the executing phase that followed, she stated, “I mean like these two are these basically if the, if α and β were the same. That would be cosine, oh, but that's minus... Hm. So if cosine squared, 1 minus sine squared. I really don't see it.” Since she was not able to determine how to proceed, she moved onto part 3 with the cosine difference angle identity.

She began part 3 with an orienting phase and then a planning phase. At the beginning of the planning phase, she noticed a difference between the sine sum angle identity and the cosine sum angle identity.

> Okay, well I don't know if this has anything to do with it, but $\cos(\alpha + \beta)$ gives you a negative sine, but $\sin(\alpha + \beta)$ gives you a plus, like they're flip-flopped. Um, I don't know. I just caught that. So cosine of a negative, I guess going back to this and kinda how I got those, cosine of a negative angle gives you a positive cosine. So it's like asking for a $-\beta$.

She again recognized the even and odd identities within the sum and difference angle identities. In an executing phase, she tried to use the even and odd identities to work with the cosine difference angle identity.

> So that's where that negative comes in, and it produces a positive cosine. Oh, but that also does… You're multiplying two cosines together like positive $\cos\alpha \cos\beta$. It's just a matter of either subtracting the sine term or adding a sine term. They're opposite of what's actually there [in the sum angle identity].

Evanna did not record any work for part 3 but did explain further in a checking phase.

> I don't know if [the sine one changing signs] has anything to do with going back to this cosine… Well, I know, I don't think it's that because it's sine and we're looking at cosine, and I don't know if there's any connection there. Because sine of a negative is a negative sine. So cosine of a positive gives you a negative sine and cosine of a negative gives you a positive sine. Yeah, I'm sorry.

Though she was again able to recognize the use of the even and odd identities to simplify, she was not confident in her explanation of what was happening.

Evanna next oriented herself to part 4 of the Task. She had a brief planning phase of "Yeah, so tangent relates sine and cosine. That's like saying, I guess that's like saying $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$." and then transitioned to an executing phase.

> Um, and if those are true, if these are true, that would be… And then, hmm. I guess the top, I don't know if that'll work, like taking out the factor of one of them. But you can't take out a factor from the bottom, unless, I mean, I didn't think you could do this, but like this turns into $\tan\alpha$, that turns into 1, that no that doesn't. It's negative tangent. But you can't do that. You can't factor anything out on the bottom. Hmm.

During this phase, Evanna wrote the work in Figure 80. She started by rewriting tangent in terms of sine and cosine, then used the previous sum and difference angle identities to rewrite. From there, she was considering factoring something to change it into what she was looking for.

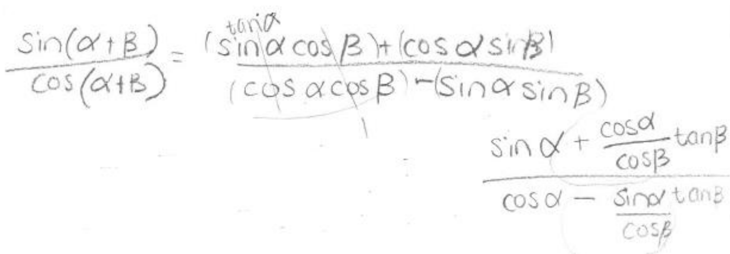

Figure 80

Evanna's Work on the Tangent Sum Angle Identity

She had considered dividing everything by $\cos\beta$, but explained why she did not think that would work in a checking phase.

> Unless you divide everything by cosine? Because if you did that, this term will leave you with... I don't know, never mind. Cause if you divide, like for instance, if you divide this one by cosine, they would just simply cancel out, and you're left with sine. That's not what we want to do. We want this to be one.

In an executing phase, Evanna considered what would happen if she divided by $\cos\beta$. She wrote the second step shown in Figure 80. She continued to evaluate her solution so far in a checking phase.

> But then you can't cancel out $\frac{\cos\alpha}{\cos\beta}$, like it'd be left together. But if you divided by $\cos\beta$, that would give you the $\tan\beta$ and the $\tan\alpha$, which is what we want. Here you're left with $\frac{\cos\alpha}{\cos\beta}$. Hmm. It fixes some stuff.

In another checking phase, Evanna continued to think about the solution she found when she divided by $\cos\beta$.

> So you divided by that you're just left with $\sin\alpha$, no, yeah. Yeah. Um, plus you're left with $\frac{\cos\alpha}{\cos\beta} * \tan\beta$and then you're left with $\cos\alpha$, cause that turns into 1. Minus

> $\tan\alpha\tan\beta$ or no, this one. No, just $\frac{\sin\alpha}{\tan\beta}$ and then $\frac{\sin\alpha}{\cos\beta}$. I'm not sure what to do with the α's.

Evanna continued to look at part 4 for a moment, but could not figure out how to proceed.

She then oriented herself to part 5 of the task. During a planning phase, she stated, "I guess you go about it the same way." During an executing phase, she started working through the algebra similar to what she had tried for part 4, as seen in Figure 81.

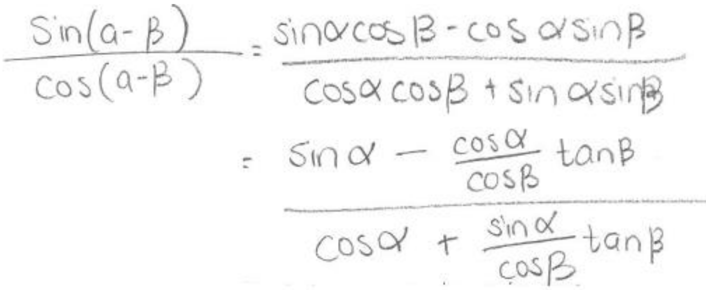

Figure 81

Evanna's Work on the Tangent Difference Angle Identity

While working, she explained, "Um, so if we divided this by $\cos\beta$, we'll be left with $\frac{\sin\alpha}{\cos\beta}$, you're left with $\frac{\cos\alpha}{\cos\beta} * \tan\beta$, that cancels so left with $\frac{\cos\alpha+\sin\alpha}{\cos\beta}$." She entered a planning phase as she considered how to proceed.

> I just don't know how to... with the α and the β. I'm kind of looking over here [on the formula sheet]. Because we want this to be negative, well to turn into negative somehow. We want that to would be 1. We want this to be positive.

Evanna was unable to proceed with the task beyond this.

Evanna used knowledge of analytic trigonometry to solve task 6. She was able to use the even and odd identities to simplify the sine and cosine sum angle identities into the difference angle identities. However, she did not name them as the even and odd identities. She also was able to rewrite tangent in terms of sine and cosine for parts 4 and 5. She also used the heuristic of written algebra throughout the task. She wrote out the identities she was using to help herself simplify on part 1. She also wrote a few steps of algebra on parts 4 and 5. This included rewriting tangent in terms of sine and cosine and rewriting $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of the given identities. She also made use of the formula sheet when trying to determine what identities she could use to help her simplify each of the sum and difference angle identities.

Task 7: Horizontal Shrink vs. Horizontal Stretch

To solve Task 7, Evanna worked through the problem-solving phases, as seen in Figure 82.

	Part 1			
Orienting	x			
Planning		x		
Executing			x	x
Checking				

Figure 82

Evanna's Phases on Task 7: Horizontal Shrink vs. Horizontal Stretch

Evanna first oriented herself to the task. Her initial thoughts were in the planning phase. “I know a fraction would be a stretch. I just don't know why. Um, I guess sine of that's the, that's not the period. $\sin\theta$, no that's not going to do anything. I don't know why.” Evanna knew that a fraction smaller than one would create a stretch and a number larger than one would create a stretch, but she was initially unsure why that occurred. She then entered an executing phase and looked at the formula sheet as a reference.

> I guess the periodic formulas. I don't think that's gonna help though. Oh, because it's applying to the x and x is the horizontal axis. So, like in an $x-y$ coordinate, here's y,

here's x. So if you're changing the x term, um, then that'll affect its horizontal behavior. I think. That's all I got.

During this phase, Evanna realized that the reason it was a horizontal shrink was because it was affecting the horizontal behavior based on its location with x. She then expanded upon her explanation of why it was a shrink for $y = \sin(2x)$ in a checking phase.

Yeah, what I think verticals the exact opposite. But I forgot like where you put it in the equation to do it. But I know that if it was 2, it would make it stretch like we would think. And if it was $\frac{1}{2}$, it would shrink just like we would think. I really don't know. No, I don't remember it, how these equations get set up.

Evanna knew that a horizontal stretch or shirk was the opposite of what you would think, but that a vertical stretch or shrink was the same as what you would think. However, she was unsure where these showed up in the equation or why they were a shrink or stretch in each direction.

While solving this task, Evanna referenced the formula sheet but did not ultimately use anything specific from it to help her solve the task. She did not use any other resources or heuristics.

Task 8: Proving an Identity

Evanna worked through the problem-solving phases, as seen in Figure 83 for Task 8.

	Part 1				
Orienting	x				
Planning					
Executing		x		x	
Checking			x		x

Figure 83

Evanna's Phases on Task 8: Proving an Identity

To start Task 8, Evanna used an orienting phase. She then proceeded to an executing phase. During the executing phase, she worked out the proof herself, as seen in Figure 84.

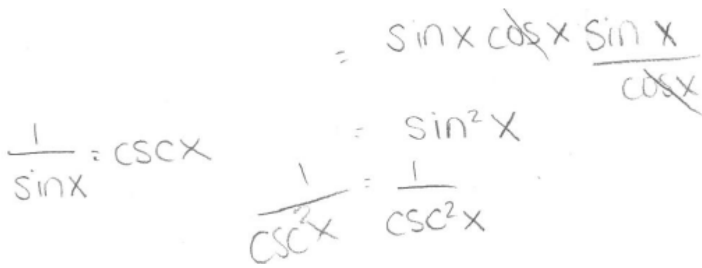

Figure 84

Evanna's Proof on Task 8

She explained her thinking as she worked,

> Tangent cosine sine, well $\sin x \cos x \tan x$, well $\tan x$ is the same thing as $\frac{\sin x}{\cos x}$. Um, so cosines would cancel, and this would just be $\sin^2 x$. And then, which is the same thing? I think as $\frac{1}{\csc^2 x}$. Because $\frac{1}{\sin x}$ is $\csc x$.

After verifying the identity was true, she entered a checking phase.

> So I don't think that was the right way to go about that, I don't think. I don't know. I've never solved trig. I don't think given algebra like that. Well maybe, well maybe not because I mean what she, ah! what she did is right.

During this checking phase, Evanna was unsure whether the work the student has done was correct or incorrect because it did not feel like how she had previously solved similar proofs. Evanna then proceeded to an executing phase of considering whether the student’s proof was correct and why it was correct or incorrect.

> Well, bringing over cosecant, flipping it, just like we did here. Canceling the terms out. I don't know. I mean I guess that is a valid way to do it. I don't know. I've just never, I don't think I've ever solved it. I don't know. I just, I mean, I guess the way I went about it

> was just solving this side to get to here. When she tried to make them equal. I really don't know [if it's right or wrong]. Cause I mean, she did prove, because I guess when you do it this way, you're making them equal each other.

Evanna believed that the students' proof was mathematically correct but still was unsure of whether it was a valid proof. She entered another checking phase to consider whether it was a valid proof and how it compared to her proof.

> Well, I guess that is right because if you did mine, is right, cause I think because the only difference really is that she multiplied this on both sides and then did exactly what I did. And then they both equal one. Okay. I'll go with, yes it is right. And then just multiplying cosecant would give you $1 = 1$ [in mine]. I'll go with that.

Evanna decided that the only difference between her proof and the student's proof was that the student had multiplied both sides of the equation by cosecant. Thus, she determined that the students' proof was indeed a valid proof.

Evanna used knowledge of analytic trigonometry to solve Task 8. She was able to identify that $\frac{1}{\sin x}$ was equivalent to $\csc x$ and was able to simplify $\sin x \cos x \tan x$ as she worked to verify the identity. She also used the heuristic of written algebra as she solved the tasks for herself.

Task 9: Angles in a Rectangle

For Task 9, Evanna's problem-solving phases can be seen in Figure 85.

	Part 1					
Orienting	x					
Planning		x		x	x	
Executing			x			x
Checking						

Figure 85

Evanna's Phases on Task 9: Angles in a Rectangle

Evanna first oriented herself to the task and then entered a planning phase.

> Okay. Um, well, this is, I'm trying to remember this. They're all 90° as rectangles. So, and then this whole angle right here, would have is 180°, 90°, don't know if this is gonna help any. Um, they're all the same length, how much is important? Oh, these added together equal that. I don't know if you can work it with like actual numbers. I don't know if you're allowed to do that.

As she was planning, she noted that all of the corners would be 90° and that the opposite sides were the same length. She proposed that she should substitute a value and see if she could use that to help her proceed. She then entered an executing phase of assuming $\angle DQC = 45°$.

> Like if [$\angle DQC$] were 45° and [$\angle QDC$] were 45°. Oh no, that's not gonna... that would mean all of [$\angle ADQ$] was 45° split up. Yeah, this was 45°, $180° - 45° = 135°$. Yeah. And then if that was 135°. No, I don't think that's going to do any good. Um, I was just going to see if that would be… [I'm just picking a number to kind of make it easier to work with.]

During this phase, Evanna began marking the given diagram with what she knew and what she had assumed, as seen in Figure 86.

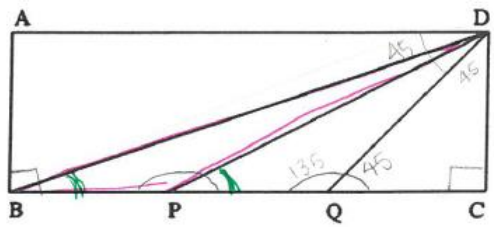

Figure 86

Evanna's Marked Diagram for Task 9

Evanna then continued with a planning phase. She started by drawing ΔBCD separately from the rectangle, as seen in Figure 87.

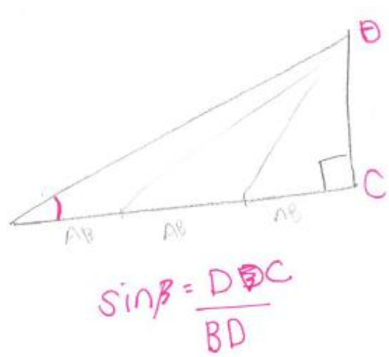

Figure 87

Evanna's Triangles for Task 9

> So we want this angle plus this one to get to that one. Ah. Well, maybe if I drew [the triangles] separately. This is 90°. Ah. … I just [know] the length of [the big one]. Or this length. Um, so it's the length of AB. I know, I really don't, I don't remember this stuff. I mean, that's also AB, but… I'm just starting at it. I'm sorry, I don't, I really don't know. No, I think any of these [formulas] are going to do anything. Let's see. I don't even know what I'm looking for. It's really not clicking at all.

During this phase, Evanna also spent about 2 minutes looking at the formula sheet, but she did not find anything that she considered useful. Since she was stuck, the interviewer helped her recap what she had determined so far, and then she entered another planning phase.

> I mean, are you looking like the sum to product formula? … Or sum to difference. So I guess for... sine is equal to, well we don't have AB, I don't know if this is going to do anything. Over or is this just for right triangles? I mean this one is, like just the whole thing or the smaller one.

Evanna had found the sum to product identity on the formula sheet and tried to use it to help her show that $\angle DBC + \angle DPC = \angle DQC$ in an executing phase.

> This is DC, which is the same thing as AB. Um, sin $\angle DBC$ is equal to opposite, DC, over hypotenuse, which here is BD. That's, we're not adding anything to it unless we subtract. No, no. I really don't see anything.

Evanna attempted to proceed with the task but was unable to find a way, so she stopped.

Evanna used knowledge of geometry, right triangle trigonometry, and analytic trigonometry to attempt to solve Task 9. She started the task by considering the angles. She assumed that $m\angle DQC$ was $45°$ and found several other angles by using geometric properties. She considered using triangle trigonometry and started to set up an equation using SOH CAH TOA near the end of her solution. In the middle of her solution attempt, Evanna considered using the sum to product identity. Evanna also used the formula sheet as a reference when she was unsure how to proceed with the task.

Evanna used the heuristics of a diagram and substituting values in her attempt to solve Task 9. Evanna wrote on the given diagram and then later pulled out a part of the diagram that she drew separately. She also assumed that $m\angle DQC$ was $45°$ to try to simplify her work.

Analysis by Domain

As Evanna solved the nine tasks, she used a variety of resources and heuristics across the four trigonometric domains, as seen in Figure 88. The diagram shows what knowledge from domains and heuristics she used to solve tasks in each domain. For example, as shown in the upper right quadrant, to solve triangle trigonometry tasks, she used knowledge of geometry, triangle trigonometry, unit circle trigonometry, and analytic trigonometry. She also used heuristics of algebra, diagrams, and substituting values to solve triangle trigonometry tasks. Each of the other quadrants similarly shows what knowledge and heuristics she used to solve tasks in the domains of unit circle trigonometry (upper right), trigonometric functions and graphs (bottom left), and analytic trigonometry (bottom right).

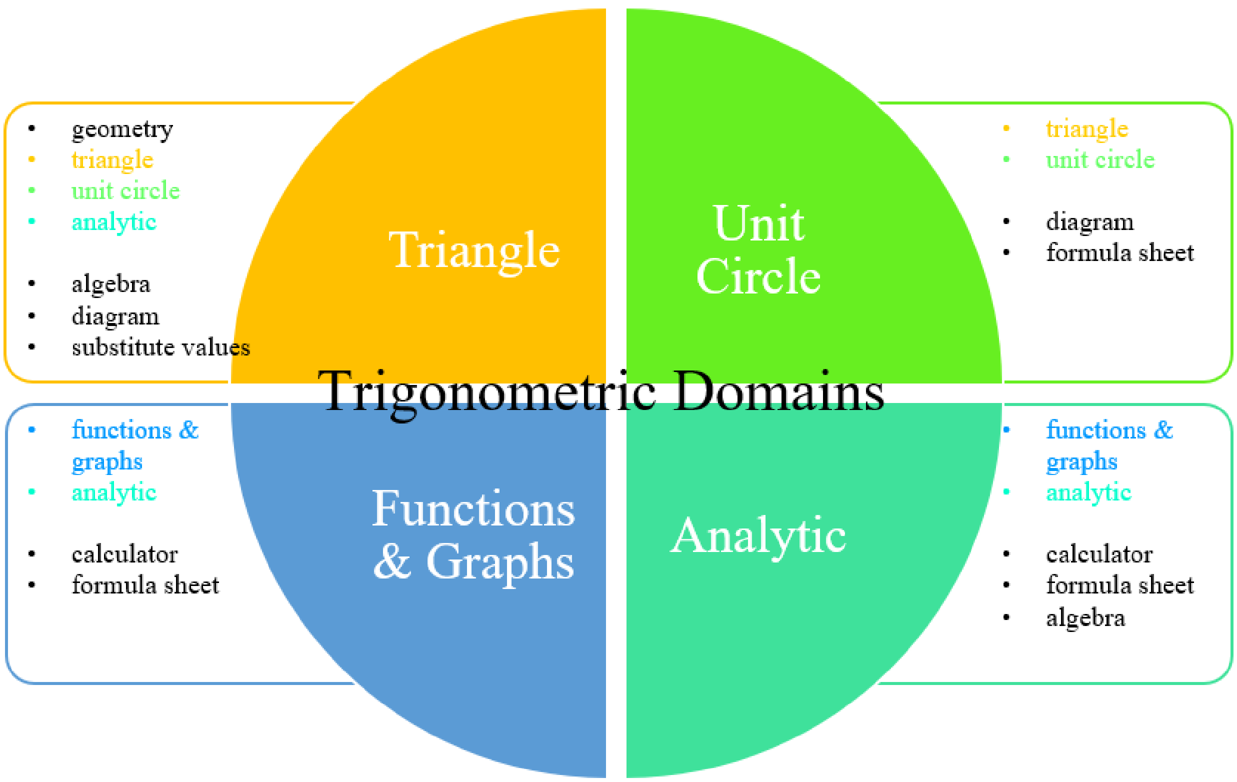

Figure 88

Evanna's Use of Resources and Heuristics Across Domains

Evanna’s use of heuristics was limited across the tasks and domains. She used written algebra on three tasks, one triangle trigonometry and two analytic trigonometry. She used diagrams on all three triangle trigonometry tasks, including one task that addressed both triangle trigonometry and unit circle trigonometry. She used the heuristics of substituting values on Task 9, a triangle trigonometry task, but she did not use any other heuristics as she completed the tasks.

Evanna did use the calculator and formula sheet across several domains. She used the calculator in the domains of trigonometric functions and graphs and analytic trigonometry and the formula sheet in the domains of unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry.

Triangle Trigonometry

The domain of triangle trigonometry was addressed by Tasks 2, 4, and 9. On these three tasks, Evanna generally used a single cycle of orienting, planning, executing, and checking for each part of each task, as seen in Figure 89. For Task 1, Evanna completed an orienting, planning, and executing phase for part 2, then completed a planning, executing, and checking phase for part 1. Since the two were related, she did not need to have another orienting phase. For part 3, she completed the full cycle of orienting, planning, executing, and checking. Task 4 was similar to Task 1, but she did not have an executing phase for part 1 or an orienting phase for part 2. For Task 9, she began with a cycle of orienting, planning, and executing. This was followed by two planning phases and an executing phase.

	Task 2									
	Part 2			Part 1			Part 3			
Orienting	x						x			
Planning		x		x				x		
Executing			x		x				x	
Checking						x				x

	Task 4					
	Part 1			Part 2		
Orienting	x					
Planning		x		x		
Executing					x	
Checking			x			x

	Task 9					
	Part 1					
Orienting	x					
Planning		x		x	x	
Executing			x			x
Checking						

Figure 89

Evanna's Phases on Triangle Trigonometry Tasks

Evanna used knowledge from the highest number of domains to solve triangle trigonometry tasks. This included knowledge of geometry and three of the four domains, triangle trigonometry, unit circle trigonometry, and analytic trigonometry. She used knowledge of geometry on Task 2 to find missing sides of a right triangle using the Pythagorean Theorem and on Task 9 to reason about relationships within the rectangle, such as the sum of the angles of a triangle adding to 180°. She primarily used knowledge of triangle trigonometry with SOH CAH TOA, in both Task 2 and 9. In Task 2, she used SOH CAH TOA with to set up a triangle given $\sin\theta = \frac{8}{17}$ and solve for $\cos\theta$. For Task 9, she tried to use SOH CAH TOA to find values for sine and cosine to use in the product to sum identity. For Tasks 2 and 4, she used knowledge of the unit circle by stating that the sine value is the y coordinate on the unit circle and finding values for sine on the unit circle. Evanna recognized the Pythagorean Identity in Task 2 and attempted to use the sum to product identity on Task 9.

Evanna used three heuristics to solve the triangle trigonometry tasks, algebra, diagrams, and substituting values. She used algebra on Task 2 to solve the Pythagorean Theorem, to "just mak[e] sure I do this right." Evanna used diagrams on all three of the triangle trigonometry tasks. She created a diagram of a triangle in the first and second quadrants for Task 2, a triangle with an obtuse angle to show why it could not be a right triangle for Task 4, and marked the provided diagram and pulled out a triangle for Task 9. She started Task 9 by substituting values for some of the angles, though she quickly abandoned this approach. This was also the only domain that she did not use either the graphing calculator or formula sheet as a resource.

Unit Circle Trigonometry

The Tasks 3, 4, and 7 addressed the domain of unit circle trigonometry. Across the three tasks, Evanna generally began each part with a cycle of orienting, planning, executing, and checking. On Task 3, she began with the cycle of all 4 phases, then spent the rest of the task alternating between executing and either planning or checking phases. For Task 4, she used three of the four phases for each part, skipping executing on part 1 and orienting on part 2. For Task 7, she also used three of the four phases, skipping checking and including an additional executing phase.

	Task 3															
	Part 1															
Orienting	x															
Planning		x					x					x				
Executing			x		x		x		x		x			x	x	
Checking				x		x				x		x				x

	Task 4					
	Part 1			Part 2		
Orienting	x					
Planning		x		x		
Executing					x	
Checking			x			x

	Task 7			
	Part 1			
Orienting	x			
Planning		x		
Executing			x	x
Checking				

Figure 90

Evanna's Phases on Unit Circle Trigonometry Tasks

Evanna only used knowledge of triangle trigonometry and unit circle trigonometry to solve tasks in the domain of unit circle trigonometry. She used knowledge of triangle trigonometry across all three tasks. On Tasks 2 and 9, she used knowledge of SOH CAH TOA to write relationships between given sides and trigonometric ratios. On Task 4, she attempted to remember the ratios of sides of 45-45-90 triangles to evaluate $sin\,\frac{3\pi}{4}$, but ended up using knowledge of the unit circle when she could not recall the ratios. She used knowledge of unit circle trigonometry on Tasks 3 and 4. For Task 3, she evaluated the given expressions using the unit circle. She also considered other characteristics of the expressions based on the unit circle, such as the quadrant the angle was located in and the value of the angle in radians. On Task 4, she evaluated $\sin\frac{3\pi}{4}$ and $\sin\pi$ using the unit circle. She also used the formula sheet at a resource for the unit circle on these tasks.

The only heuristic that Evanna used on the triangle trigonometry tasks was to create diagrams. She drew a diagram of an obtuse triangle on Task 4 to show why it could not be a right triangle and use SOH CAH TOA to solve for missing sides or angles. She marked the given diagram on Task 9 and drew the largest triangle separately from the rectangle to simplify her drawing.

Trigonometric Functions and Graphs

Trigonometric functions and graphs were addressed in Tasks 1, 5, and 7. Evanna completed multiple cycles of orienting, planning, and executing on Tasks 5 and 7 and multiple cycles of orienting, executing, and checking on Task 1. For part 1 of Task 1, Evanna completed an orienting, executing, and checking with another executing phase. For parts 2, 3, and 4, she primarily alternated between orienting and executing phases. For Task 5, she completed orienting and executing phases for parts 1 and 4, but orienting, planning, and executing phases for parts 2 and 3. For Task 7, she completed a cycle of orienting, planning, and executing with another executing phase.

Task 1															
	Part 1				Part 2										
Orienting	x				x		x		x			x			
Planning														x	
Executing		x		x		x		x		x			x		x
Checking			x								x				

	Part 3							Part 4							
Orienting	x		x					x			x			x	
Planning						x									
Executing		x		x					x			x			x
Checking					x		x			x			x		

Task 5										
	Part 1		Part 2			Part 3			Part 4	
Orienting	x		x			x			x	
Planning				x			x			
Executing		x			x			x		x
Checking										

Task 7				
	Part 1			
Orienting	x			
Planning		x		
Executing			x	x
Checking				

Figure 91

Evanna's Phases on Trigonometric Functions and Graphs Tasks

For the trigonometric functions and graphs tasks, Evanna used knowledge from the domains of trigonometric functions and graphs and analytic trigonometry. Evanna used knowledge of trigonometric graphs on Task 1 and 5. For Task 1, she started by creating a graph of sine and cosine. She then found translations and reflections that mapped cosine and sine back to themselves, as well as translations from sine to cosine and vice versa. For Task 5, Evanna was able to identify which parent function to use for the rabbit and fox data. She was also able to calculate several parameters to find an equation to model each set of data. She used knowledge of trigonometric identities on Task 1. On this task, she was able to find identities that matched the transformations she had found to map sine and cosine either to themselves or each other.

Evanna used the graphing calculator and formula sheet as resources to help her solve the tasks in this domain. She did not use any heuristics on these three tasks.

Analytic Trigonometry

The tasks that addressed the domain of analytic trigonometry were Tasks 1, 6, and 8. Evanna generally began each part of these three tasks with a full or partial cycle of orienting, planning, executing, and checking. For Task 1, she began parts 1 and 4 with phases of orienting, executing, and checking. On parts 2 and 3, she began with a pair of orienting and executing phases for each part. Except for part 2 of Task 6, Evanna completed at least 3 of the phases of orienting, planning, executing, and checking at the beginning of each part. She completed all four phases for parts 1 and 3, skipped planning on part 4, and skipped checking on part 5. For part 2, she had a second planning phase and skipped the checking phase. For Task 8, Evanna had a planning phase before a pair of executing and checking phases.

	Task 1														
	Part 1				Part 2										
Orienting	x				x		x		x			x			
Planning														x	
Executing		x		x		x		x		x			x		x
Checking			x								x				

	Part 3							Part 4							
Orienting	x		x					x			x			x	
Planning						x									
Executing		x		x					x			x			x
Checking					x		x			x			x		

	Task 6																					
	Part 1				Part 2				Part 3				Part 4						Part 5			
Orienting	x				x				x				x						x			
Planning		x				x	x			x										x		x
Executing			x					x			x			x		x					x	
Checking				x								x			x		x	x				

	Task 8				
	Part 1				
Orienting	x				
Planning					
Executing		x		x	
Checking			x		x

Figure 92

Evanna's Phases on Analytic Trigonometry Tasks

Evanna used knowledge from the domains of trigonometric functions and graphs and analytic trigonometry to solve the three tasks in the domain of analytic trigonometry. She used knowledge of graphs only on Task 1. On this task, she was able to find transformations of sine and cosine onto themselves and each other using knowledge of their periods and reflections. Evanna used knowledge of trigonometric identities for all three tasks. For Task 1, she was able to match identities with the transformations of sine and cosine that she had found from the graph. On Task 6, she used multiple identities to verify the sum and difference angle identities, including the even and odd identities and Pythagorean Identity. For Task 8, she verified the proof for herself using several identities. This included rewriting $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$ and knowing $\frac{1}{\sin x} = \csc x$.

Evanna did use the heuristic of written algebra on Tasks 6 and 8. On Task 6, Evanna showed the sum and difference identities by writing out the steps to verify each algebraically. For some of the identities, this was just a single step, but for others, it required several steps for her to show. On Task 8, she verified the identity for herself by writing it out algebraically. She also used the formula sheet and graphing calculator as resources on the analytic trigonometry tasks.

Mathematical Understanding for Secondary Teachers

Mathematical Proficiency

Mathematical proficiency consists of six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Evanna demonstrated one of the strands: procedural fluency. She demonstrated procedural fluency across several of the tasks. For example, on Tasks 3 and 4, she was able to use the provided unit circle to identify values of sine and cosine expressions, since she knew that “sine, that's the y value.” She also demonstrated procedural fluency in determining the values of parameters on Task 5. She determined the length of the period to be 24 months, instead of the actual 12 month period but then used the formula $\frac{2\pi}{x} = 24$ to find the value of x as $\frac{\pi}{12}$. She then correctly placed the value $\frac{\pi}{12}$ into her equation to get $-\sin\left(\frac{\pi}{12}t\right) + 1000$ for the rabbits.

Mathematical Activity

Evanna demonstrated all three strands of mathematical activity: mathematical noticing, mathematical reasoning, and mathematical creating across the tasks and domains. The strand of mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Evanna demonstrated two strands: symbolic form and connect within and outside mathematics.

Evanna demonstrated an understanding of the symbolic form of a trigonometric function on Task 6. On this task, she was able to work with the given sum and difference identities for sine and cosine to verify the sum and difference identities for tangent without overapplying the distributive property. She started the tangent sum angle identity by rewriting $\tan(\alpha + \beta)$ as $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$. She then used the given sum and difference identities to rewrite this as $\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$. Though she initially debated about simplifying the terms before the addition and subtraction in the fraction to be $\tan\alpha\tan\beta$, she decided this was not valid. Instead, she divided through by $\cos\beta$, but was not able to progress beyond this step.

On Task 5, Evanna was able to demonstrate connecting outside mathematics. When asked to describe why a trigonometric function was appropriate, she stated, “Naturally predator-prey model is going to fluctuate, similar to trig graphs.” She later wrote “the more foxes, less rabbits” and “less foxes, more rabbits” as an explanation of why the peaks and valleys in the graph seem to chase one another. She was able to consider the real-world relationship between the populations of the rabbits and foxes outside of the mathematics of the task.

Mathematical reasoning consists of three strands: justifying/proving, reasoning when conjecturing and generalizing, and constraining and excluding. Evanna demonstrated one of the strands: justifying/proving. She demonstrated this strand on Task 6 when she used informal reasoning about the even and odd identities for the sine and cosine difference angle identities.

> Okay, well I'm not sure if this has anything to do with it, but $\sin(-\theta) = -\sin\theta$. So, using that, sine it would be like negative of this, is negative sine which may, which is like right here. … that goes back to the cosine formula because cosine of a negative degree is still positive.

Though she didn't record a formal proof of this, she was able to reason through why the even and odd identities resulted in a sign change from the sum angle identities to the difference angle identities. She provided a more formal proof for the tangent sum and difference angle identities. For the tangent identities, she began by rewriting tangent in terms of sine and cosine. She then rewrote the expression using the sum and difference identities for sine and cosine. Though she was not able to complete the proof, she was formally able to write down her steps for the part she did complete and show her reasoning.

Mathematical creating consists of three strands: representing, defining, and modifying/transforming/manipulating. Evanna demonstrated two of the strands: representing and modifying/transforming/manipulating. She represented her thinking in a diagram on Task 2. For this task, she initially drew a triangle and labeled the sides from the given information. When she realized that the triangle should be in the second quadrant, she added a coordinate axes and the triangle in the second quadrant. With the diagram of the triangle in the second quadrant, she was able to reason that the cosine value should be negative, "[b]ecause you still, this [y-value] doesn't change because that's going up. So that's still positive. So, you still find x, but because it's going this way, whatever [the x-value is] has to be negative."

Evanna was able to demonstrate modifying/transforming/manipulating on Tasks 1 and 6. On Task 1, Evanna was able to determine from the graph the translations and reflections that would map sine and cosine back onto themselves and onto each other. She was then able to identify identities from the formula sheet that matched each transformation. She then transformed the representation of the graph to an expression of the transformation to an identity. On Task 6, she was able to perform algebraic manipulations on the tangent sum and difference angle identities. These algebraic manipulations included rewriting tangent in terms of sine and cosine, rewriting sine and cosine in terms of the sum and difference angle identities, and performing algebraic manipulations.

Case 4: Jessie

Jessie was a traditional undergraduate student in her senior year at the time of her participation in the study. She was majoring in middle grades mathematics and planning to student teach the following semester. She had completed her mathematics coursework, including a mathematics content for middle school teachers' course and a capstone course titled School Mathematics from an Advanced Perspective. On the test of trigonometric content knowledge, Jessie scored a 75% overall. She scored a 93% on analytic trigonometry, 75% on unit circle trigonometry, 71% on triangle trigonometry, and 63% on trigonometric functions and graphs.

Jessie worked consistently on the tasks during the interviews. She was able to find correct solutions for most of the tasks and frequently used the provided formula sheet and unit circle on the tasks. She relied less on the provided graphing calculator because she stated she was unfamiliar with it but did use its calculation capabilities and graphed with some assistance. She did relate two of the tasks to the mathematics education courses she was enrolled in that semester and related one of her solutions to work with students in her practicum from that semester.

Description by Task

Task 1: Identities from a Graph

Jessie’s work on Task 1 proceeded through the problem-solving phases, as seen in Figure 93.

	Part 1		Part 2														
Orienting	x		x					x				x					
Planning		x		x		x			x				x		x	x	
Executing					x		x			x				x			
Checking											x						x

	Part 3								Part 4									
Orienting	x								x									
Planning						x				x							x	
Executing		x	x		x		x					x	x		x			
Checking				x				x			x			x		x		x

Figure 93

Jessie's Phases on Task 1: Identities from a Graph

For Part 1, Jessie began by creating a graph of sine and cosine, as seen in Figure 94. She mostly worked silently but said, “This is kind of like gonna come out of memory” as she started to work.

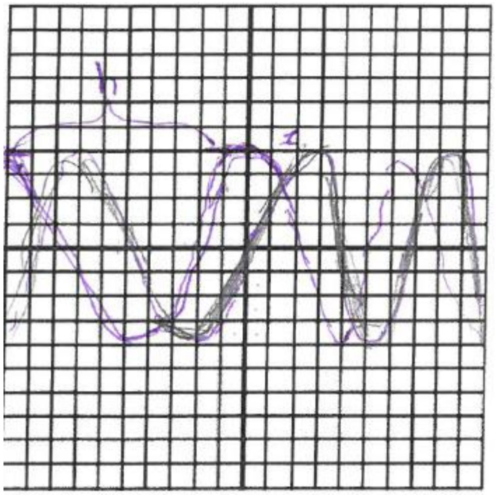

Figure 94

Jessie's Graphs of Sine and Cosine

To find the transformations of cosine for part 2, Jessie oriented herself to the task and began a planning phase. For the translation of cosine onto itself, she said, “So I guess I could shift it to the right so that one of them would end up there [points to one peak and then next one to the right], but I guess I need the distance between the two.” In the executing phase, Jessie marked two peaks of her cosine graph and explained

> So that's kind of the cutoff here, so they don't go above 1. And so that's, I think it would be the amplitude or like the highest point. So what I'm trying to do is kind of take this one and bring it here. But I'm not sure what the distance between these two would be. But what I would do is like say cosine of whatever x is and then subtract would shift to the right h units to kind of get this. I remember that.

Next, Jessie entered a planning phase and stated that it would be useful if she knew how to use the graphing capabilities of the calculator to determine the value of h for the shift. After some assistance with the calculator, she produced a graph of $y = \cos x$. In the next executing phase, while looking at the graph and counting the tick marks on the x-axis Jessie said, “So it seems… Yeah, I’d say around 6, although it doesn’t look like exactly.” After finding the translation, Jessie oriented herself to finding a reflection of cosine onto itself and began a planning phase in which she looked at the graph she had created. During her executing phase, she explained, “I guess reflecting it maybe over the $y - axis$. We kind … the same thing because they're like identical on either side. So, reflect about the $y - axis$.” When encouraged to write her answer as she had done for the translation, Jessie said, “So it would go from like minus x or negative x and y. [points to a point on the graph, then writes coordinates in terms of $(\pm x, \pm y)$ as she talks through them]. Ahh, so it would be $f(-x)$ maybe, I'm thinking $f(-x)$ Kind of… flip the x’s.” Finally, Jessie oriented herself to finding identities that matched the two transformations she has found. Her planning phase for the reflection consisted of looking at the formula sheet. Then during the executing phase, Jessie identified the even/odd formula. Similarly, for the translation, she looked at the formula sheet and identified the periodic formula.

For part 3, Jessie began by orienting herself to the task and immediately began an executing phase. During the executing phase, Jessie looked at the graph of $y = \sin x$ on the graphing calculator and said “six” before writing $\sin(x - 6) = f(x)$. For the reflection, Jessie began with an executing phase. She again looked at the graph of $y = \sin x$ on the graphing calculator and turned the calculator 90°. She explained her reasoning as

> I'm trying to see… I don't think that [reflecting over the $y - axis$] because like reflect you move about the $y - axis$ would kind of switch the direction. So it won't be cutting it that way [over $y = x$] or that way [over $y = -x$]. Um trying to think if maybe reflected about the $x - axis$ and then shifted it so I can get back to the $y - axis$.

During this phase, she considered a total of three single reflections and a combination of a reflection and a translation. After being instructed to use just reflections, Jessie entered a checking phase. She identified, “So that would be reflecting about the $x - axis$ and then maybe reflecting about the $y - axis$. That worked. Reflected about the $x - axis$ and then reflect about the $y - axis$." From this, she immediately began an executing phase for finding the identities for her transformations. "So I think I would choose the same identities for these ones too. So for the periodic formula, it's the exact same thing because it just moves 6 units and so that one would make sense." In the following planning phase, she stated that the reflections would not be the same as the identity for the reflection of cosine. She then entered an executing phase of determining the identity that matched her reflections.

> Because it's like picking a point that would be, maybe $(5,1)$ to $(-5,1)$. That would then be after flipping it over x, $(-5,-1)$. Change the x. Visually I'm convinced, but then I don't think I'm making some sort of mistake here. … Um, $(-x,-y)$ and then if I can get about the $y - axis$, the $x - axis$ for $(-x,y)$. And then flipping that change the x. Yeah, so it's taking the original points and just multiplying them, all of them of by a negative. So they go from a $(-x,-y)$ to (x,y).

Her written work from the executing phase is in Figure 95.

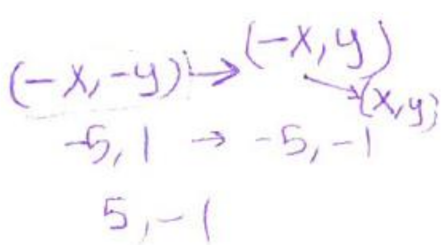

Figure 95

Jessie's Coordinate Transformation for Task 1

Finally, she entered a checking phase where she identified the even/odd formula as the correct identity for the reflections of sine.

For part 4 of the task, Jessie began by orienting herself to the goal of using transformations to map sine to cosine and cosine to sine. During her planning phase, she began by resketching her graphs and then regraphing both on the graphing calculator. She repeated her earlier strategy of counting tick marks and said, “So, I think it would just be switching it… over one. So just like moving it over one unit. That would be translations or translating.” She then clarified, “Cosine is moving one unit to the right.” In her next executing phase, she tried to identify an identity from the formula sheet for the relationship that she had just found.

> I'm looking for somehow a way of manipulating cosines to get sine and manipulating sine to get cosine. Um, because it calls for a trig identity that could be associated with it. I think I use [the Pythagorean Identity] because I'm not sure, and this was also during the first test that I took was I'm not sure if like the θ the radians applied to x's as well. But if squared [$\cos(x-1)$] and then squared [$\sin(x-1)$] and then you add them together, you would end up with $\cos^2 x + \sin^2 x$. And then expanding these two [$(x-1)$s], so like squaring these two, you'd end up x^2 and then -1. Then adding them would maybe give you this [Pythagorean] identity.

Throughout this phase, she referred to the formula sheet multiple and expressed, “but I'm not sure that would be correct” at the conclusion of her explanation. She further explained her reasoning in another executing phase as

> So, I was thinking about squaring sine and then squaring cosine. Since I said that [$\cos(x-1)$ and $\sin(x-1)$], so $\sin x$ would be [$\sin(x-1)$], they’re like equivalent. So squaring sine squared and then squaring that one and adding them together would be like squaring these two and adding them together. And I was thinking like, oh this would give you the middle value would be $-2x$ and then the middle value here would be positive $2x$, so like everything would cancel out except the…

As she explained, she wrote the work in Figure 96 showing that she was squaring the $(x - 1)$ rather than the $\cos(x - 1)$ and similarly for sine.

$$\sin^2 x + \cos^2(x) = [\cos(x-1)]^2 + [\sin(x+1)]^2$$

$$x^2 - 2x + 1 \quad + \quad x^2 + 2x + 1$$

Figure 96

Jessie's Squaring on Task 1

However, in her checking phase, Jessie seemed to realize that there was a problem with her solution.

> Yeah, no, I don't think that would work. Um, I just kind of saw these two and was like okay, these will cancel out and then end up with my one, and then the sign is weird because I couldn't prove it. So it may work, but I don't know how to get from [my answer] to [the Pythagorean identity], but it looks like you can, but I couldn't.

Although Jessie again looked at the formula sheet, she was unable to come up with another identity that might represent her translation.

Throughout the task, Jessie used the resources of the graphing calculator and formula sheet, as well as knowledge of trigonometric graphs, trigonometry identities, and coordinate transformations. She also used the heuristic of written algebra, as seen in Figure 96. Jessie relied on the formula sheet when identifying the identities that matched her transformations in parts, 2, 3, and 4 as described above. With the use of the formula sheet, she was able to correctly identify both the even/odd identities for her reflections and the periodic identities for her translations. She also used the graphing calculator to create more accurate graphs of sine and cosine to assist her in identifying the transformations she was looking to find. Her knowledge of graphs included the basic shape of the sine and cosine graph, as well as how to modify the equation to shift the graph to the left or right. Her knowledge of identities was shown when she was able to generalize her solution from cosine to her solution for sine since she knew that the periodic identity would be the translation that applied to both. She was also able to use her knowledge of coordinate transformations to reason about the reflection over both the x-axis and y-axis using an ordered pair, as described above.

Task 2: Pythagorean Theorem and Pythagorean Identity

The problem-solving phases that Jessie progressed through as she completed Task 2 are shown in Figure 97.

	Part 1	Part 2	Part 3
Orienting	x	x	x
Planning	x	x	x
Executing	x	x	x
Checking		x	

Figure 97

Jessie's Phases on Task 2: Pythagorean Theorem and Pythagorean Identity

Jessie began by orienting herself to part 1 of the task. In her planning phase, she stated, "I think it may be the 30-60-90. Um, and then there's something kind of associated with that." She then proceeded to label the diagram with the sides, as shown Figure 98.

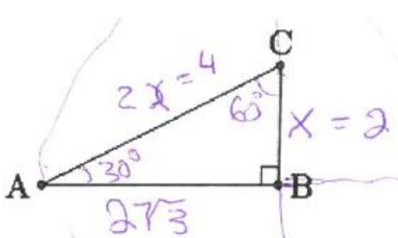

Figure 98

Jessie's Labeled Diagram for Task 2

In the executing phase, she decided to substitute values for x and the sides in the ratio of a 30-60-90 triangle. "x and then $2x$ and x times it's made of something, $x * \sqrt{3}$, I think. So I'm just going to plug in numbers. Um, so this would be 2, this would be 4, and this would be $2\sqrt{3}$ and then I'm just gonna substitute them." Once she had labeled the sides in the diagram, she substituted the values into the formula she was given to verify, as shown in Figure 99.

$$\left(\frac{2\sqrt{3}}{4}\right)^2 + \left(\frac{2}{4}\right)^2 = 1$$
$$\frac{12}{16} + \frac{4}{16} = \frac{16}{16} = 1$$

Figure 99

Jessie's Substituted Values for Task 2

For part 2 of the task, Jessie began by orienting herself to the task. In her planning phase, she related the given formula to the sides of the triangle using her knowledge of right triangle trigonometry. "Let's see on a unit circle and then $\frac{AB}{AC}$, that would be the adjacent over the hypotenuse, which is cosine. So $\cos^2\theta$ plus $\frac{BC}{AC}$. That's opposite over hypotenuse, and that's $\sin^2\theta$ that would equal." In her executing and checking phases, she clarified that this meant the formula given in part 1 would work for any right triangle.

For part 3, Jessie again oriented herself to the task. Then, in the planning phase, "I'm just going to draw the unit circle, second quadrant. $\cos\theta$, um $\frac{8}{17}$. And so cosine would be whatever x is over 17." In her executing phase, she solved for this relationship using the Pythagorean Theorem as seen in Figure 100 and drew the picture $\sin\theta = \frac{8}{17}$ seen in Figure 101.

$\cos\theta = \frac{x}{17} = \frac{15}{17}$

$17^2 - 64 = x^2$

$289 - 64 = x^2$

$x^2 = 225$

$x = 15$

Figure 100

Jessie's Solving for Task 2

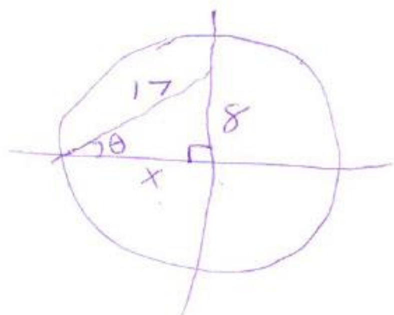

Figure 101

Jessie's Diagram of $\sin\theta = \frac{8}{17}$

To solve Task 2, Jessie used resources including the Pythagorean Theorem, triangle trigonometry (SOH CAH TOA and special right triangles), and the unit circle. She also used the heuristic of a diagram, seen in Figure 101, and substituting values. Jessie initially assumed that the triangle was a special right triangle and substituted the values into the given formula to verify it. Next, she used knowledge of SOH CAH TOA to identify the sine and cosine present in the first formula to verify the second formula. Finally, she used knowledge of SOH CAH TOA again to identify the relationships between sides and solved for a missing side using the Pythagorean Theorem.

Task 3: Which One Doesn't Belong?

For Task 3, Jessie proceeded through the problem-solving phases as shown in Figure 102.

	Part 1								
Orienting	x								
Planning		x		x		x		x	
Executing			x				x		x
Checking					x				

Figure 102

Jessie's Phases on Task 3: Which One Doesn't Belong?

Jessie began with an orienting phase of determining what the task was asking her to do. She then began a planning phase by drawing a unit circle, as seen in Figure 103.

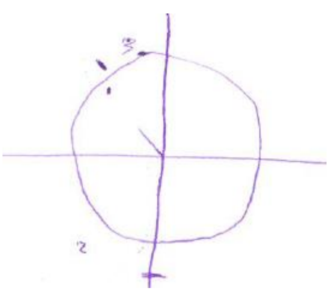

Figure 103

Jessie's Unit Circle on Task 3

After drawing the unit circle, she explained, "So which I'm just looking at which quadrant the angles would be in. So 150 and 120 would be up here. So that's [$\sin 150°$ and $\cos 120°$] in the second quadrant. Then 225 would be down here in the third quadrant." She then had an executing phase in which she considered a characteristic that she could use to exclude $\sin 225°$. "So far, probably these three and exclude that one. Probably put an angle that's less than 180° but greater than 90°. Um, maybe 140. $\cos 140°$. That's just because the angles would all be here, and that one would be in the third." She had considered her first possible solution and the first reason for excluding one of the expressions.

Next, Jessie entered a planning phase of trying to find a reason to exclude one of the other expressions. She entered each expression into the calculator and stated "So all of them would be negative, except that one's all... would give negative answers, but $\sin 150°$ would be a positive." After this, Jessie had a checking phase during which she recorded the values that she had already excluded and the reasons why she had excluded them.

During a planning phase, Jessie next considered how to exclude $\cos 120°$. In the executing phase, she stated "Well $\cos 140°$ and $\sin 225°$ give us decimals. And then $\sin 150°$ gives us a fraction and then $\cos 120°$ is the only one that gives a negative fraction." In her next planning phase, Jessie considered how to exclude her expression of $\cos 140°$ from the original three expressions. She explained her thinking as,

> Um, where $\cos 140°$ would be to kind of see if it would have like anything special about it or a different... So $\sin 225°$ is perfect. Um, 150, and then 120. So they all seem to have like places on the unit circle where they're whole multiples of π whereas the one I chose was just like a random angle degree between or angle measure between $\frac{3\pi}{4}$ and $\frac{5\pi}{6}$. How would I say that?

In this planning phase, she had found a characteristic that could exclude the last expression but was unsure how to describe the characteristic. She thought to herself for a moment and then entered an executing phase in which she stated, "No radian mark or measure on the unit circle. Or, it's not an obvious multiple of π." As Jessie worked through the task, she kept a record of her reasons for excluding each expression, as seen in Figure 104.

o Exclude sin(225°) because it's the only angle
Measure outside of Quadrant II

o Sin(150°) only 1 that'd be positive.

o cos(140) & sin(225) decimals. sin 150 → positive fraction.
cos(120°) only 1 with negative fraction.

o cos(140°) only one with no Radian marked measure on
unit circle. Obvious multiple of π.

Figure 104

Jessie's List of Reasons for Excluding Expressions on Task 3

While solving Task 3, Jessie used the provided resources of the formula sheet and graphing calculator. She also used the resource of her knowledge of the unit circle. She also used a heuristic of a diagram, when she drew her own unit circle initially. At the beginning of the task, Jessie began to draw a unit circle, as seen in Figure 104. She then relied on the unit circle that was provided on the formula sheet for the rest of the task. For example, when she was trying to exclude her expression of $\cos 140°$ she looked at the provided unit circle and noticed that her value was not marked. Jessie also used the provided calculator to evaluate the given expressions and the expression that was her solution.

Task 4: Define Sine of an Obtuse Angle

While working on Task 4, Jessie proceeded through the problem-solving phases, as seen in Figure 105.

	Part 1				Part 2						
Orienting	x				x		x				
Planning		x	x					x		x	
Executing				x		x			x		
Checking											x

Figure 105

Jessie's Phases on Task 4: Define Sine of an Obtuse Angle

After orienting herself to the task, Jessie began a planning phase. Her initial thought was that the student would believe that with an obtuse angle, it would not be possible to create a right triangle.

> She may be thinking that an obtuse angle would be like a huge one where you couldn't have a 90° angle anymore. Um, so something like this could be θ and then sin θ and then she probably thinks that the obtuse angle would like take over and not be able to have right, a 90° in one of the sides. So this would be less than 90°.

She showed this by creating the diagram in Figure 106, which shows θ as an obtuse angle and the other two angles must be less than 90°.

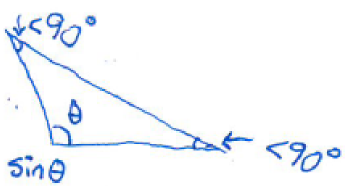

Figure 106

Jessie's Diagram of a Triangle with an Obtuse Angle for Task 4

Jessie's planning phase continued as she considered how sin θ might be defined for an obtuse triangle. She stated, "So I think you could use like the, the law of sines for like the triangles that don't have a right angle. So like, yeah, so have like sides *b* and [angle] *B* and then [angle] *A* and

[side] a, what's over here basically." While describing this solution, Jessie referenced the sides and angles of the triangle that she had drawn in Figure 106.

When asked a follow-up question about how the right triangle definition given in the task related to her solution of the law of sines, Jessie stated, "So for [the original definition] you need two sides and then an angle... You can have like an angle and then a side and then find out what the other side is. Or you can have the two sides and then find out what the angle is. So technically you only, for [the original definition and the law of sines], you only need an angle and a side, and then you can figure out like third." Thus Jessie considered using the right triangle trig definition of sine and the law of sines to be similar because you could solve for a missing side by knowing a side and an angle.

For Part 2 of the task, Jessie oriented herself to the task and suggested looking at the unit circle to find the values of $\sin\frac{3\pi}{4}$ and $\sin\pi$. She then began an executing phase of looking up the values on the unit circle. "So it's cosine, sine. So the y value would be so $\frac{\sqrt{2}}{2}$ for this one. And then it would be zero for this [one]." When again asked how these solutions were related to the right triangle definition of sine give in the task, Jessie began a planning phase. She drew the diagrams in Figure 107, because "I think it would be related cause like you could create like a right triangle with the unit circle."

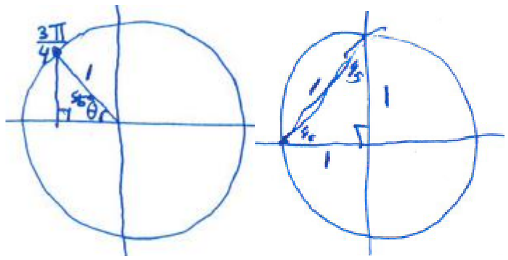

Figure 107

Jessie's Diagrams of $sin\frac{3\pi}{4}$ *and* $sin\,\pi$

In her executing phase, she then explained further,

> And then your radius would be one. And then whatever your side length over here is... Um. Yeah. And then you would have your degree angle, which is 135°. And so you have the degree angle, and then you have, and then you could find out those side measures. The angle, the right angle, and this would be one and then 135°. Then this would be 180° minus 135°, like 45°. And then you have like the side and an angle.

When considering $\sin\pi$, Jessie was unsure about how to relate it to a triangle.

> Not sure how the triangle is gonna fit in with the $\sin\pi$. Um, maybe the triangle would be better, like the top way and... yeah. The only thing I would think of is like having, don't know how this would work, but kind of having triangle be facing this way. But then yeah, this would be one, this would be, maybe create an equilateral triangle, but I'm not sure. Kinda just memorized these four points [on the axes]. Never really thought of them in terms of triangles.

She continued to look at the diagram and, in an executing phase, stated, “Yeah. I think if we were to create this one, it would be like a 45-45-90, but then these three sides would be equal, and I don't think that makes sense.” She concluded with a checking phase and stated that she was unsure about her solution because “I've never really thought about it that way, so never really made the connection between triangles and then these four.”

While working on Task 4, Jessi drew upon her knowledge of geometry, triangle trigonometry, and unit circle trigonometry. She also used the resource of the unit circle on the formula sheet and drew several diagrams. Initially, Jessie drew upon knowledge of geometry to know that if she had an obtuse angle in a triangle, then there could not also be a right angle in the same triangle since that would sum to more than 180°. Jessie then identified the law of sines as an alternative way to solve a triangle that did not have a right angle. She drew upon knowledge of right triangle trigonometry when she explained that you could solve for a missing side in a right triangle if you knew another side and angle since that would leave only one unknown in the equation $\sin\theta = \frac{opposite}{hypotenuse}$. She drew upon knowledge of the unit circle when she used it to evaluate $\sin\frac{3\pi}{4}$ and $\sin\pi$. She was able to identify that the cosine value of an angle is the x-coordinate on the unit circle, and the sine value of an angle is the y-value on the unit circle. She also considered her diagram for $\sin\pi$ to show a special right triangle of a 45-45-90 triangle. Jessie also created several diagrams to assist in her thinking, as seen in Figure 106 and Figure 107.

Task 5: Graphs of Foxes and Rabbits

The problem-solving phases that Jessie progressed through while working on Task 5 are seen in Figure 108.

	Part 1	Part 2	Part 3	Part 4
Orienting	x	x	x	x
Planning	x	x	x	
Executing	x	x	x	x
Checking				

Figure 108

Jessie's Phases on Task 5: Graphs of Foxes and Rabbits

Jessie began with an orienting phase to familiarize herself with the task. She then had a planning phase in which she described the reason that a trigonometric function would be appropriate as, “I think they don't grow like linearly. They grow exponentially and then decrease exponentially.” She then expanded this rationale and related it to the context in an executing phase.

> I do know that rabbits grow exponentially, and I don't think that they need like they carry their children very long. And then I know that the more foxes there are, then the rabbits will decrease rapidly. And so, when the rabbits decrease rapidly than the fox is decreased rapidly and because they don't have food, but then they, the rabbits increase and then the foxes increase, but then they really depend on each other.

She further explained that this would be trigonometric instead of exponential because,

> So, trig would have like an amplitude cutoff. So, like a, has to do with the unit circle, I think. But the radius, so there's like a cutoff, and for the animals, there's like a carrying capacity where they, so if there are a lot of foxes or a lot of rabbits, then they won't have enough food, and it'll be like competition. So, they would need to stay below that carrying capacity or below that amplitude measure.

Jessie’s explanation of why a trigonometric function is appropriate for the data relies upon understanding the context of the real-life situation, and the idea of the amplitude is a cutoff for the maximum number of foxes or rabbits.

For part 2 of the task, Jessie oriented herself to the goal of finding a trigonometric equation for the number of rabbits. In the planning phase, she initially stated, “So this kind of looks like the cosine one and shifted upwards...” Next, Jessie entered an executing phase and found the values of the parameters for the equation.

> Not sure what these marks stand for, but it's like shifted like right below 60 [for the foxes] and right below 600 [for the rabbits] so that's going to be shifted upwards. So, but it's kinda funny cause like they don't have the same amplitude cause this is like above that ones. But it would be like something around $\cos x$ and then shifted upwards would have... Less around like right under 600. But then the amplitudes are funny. So this is definitely not exactly cause they're higher than each other.

While finding the equation, Jessie marked the minimum and maximum on the graph, as seen in Figure 109. She then determined the equation for the rabbits to be $\cos x + 560$, since the graph had been moved up by 560.

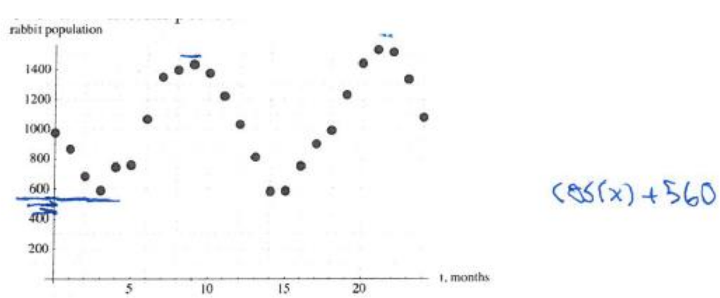

Figure 109

Jessie's Equation for the Rabbits on Task 5

For part 3 of the task, Jessie oriented herself to find an equation for the foxes and then had a planning phase during which she stated, “So this one would be like um, so cosine as well.” Then during her executing phase, she wrote the equation $\cos x + 50$. She agreed that this was for the same reason as the equation for the rabbits. She did clarify that they were both cosines, because

“Cosine cuts straight through with the y-axis, whereas sine would not. It would like start out from the bottom and then cut through that way, but then it goes like [downward].” She drew the diagram in Figure 110, showing the graph of a cosine function on the left and a sine function on the right to explain why she found both equations using a cosine function.

Figure 110

Jessie's Drawings of Cosine and Sine for Task 5

For part 4 of the task, Jessie oriented herself to the question and then entered an executing phase in which she expanded upon her explanation from part 1.

> In terms of a real-world situation, … there would be few rabbits, so few foxes could feed, and few foxes could survive. And then as the rabbits increase, they have more food. And so more foxes can survive. But then you have a ton of foxes, so they're finishing off the rabbits, so they'd die off, and then the foxes die off and so on. So, kind of in a chase, like the rabbits want to grow, but then the foxes grow. That makes the rabbits decrease.

Jessie used her knowledge of the real-life situation and the relationship between the rabbits as food for the foxes to explain why the two graphs seem to chase one another.

To solve Task 5, Jessie drew upon the knowledge of graphs of functions in general, as well as graphs of trigonometric functions. She also used a diagram to help justify her choice of parent function to use. Jessie initially described the graphs as growing and decreasing exponentially. But she later described a trigonometric model as appropriate, because she could identify the amplitude. Jessie justified the choice of cosine as the trigonometric model with the diagram in Figure 110 because cosine is at its' peak on the $y - axis$ and sine is not at its' peak on the $y - axis$.

Task 6: Sum and Difference Identities

For Task 6, Jessie proceeded through the problem-solving phases, as seen in Figure 111.

	Part 1			Part 2			Part 3				
Orienting	x			x			x				
Planning		x							x		
Executing			x		x			x		x	
Checking						x					x

	Part 4						Part 5			
Orienting	x		x				x			
Planning					x					
Executing		x		x		x		x		
Checking									x	x

Figure 111

Jessie's Phases on Task 6: Sum and Difference Identities

Jessie began the task by orienting herself to the goal of the task. During this phase, she pulled out the formula sheet and referenced it throughout solving the task. Jessie began a planning phase by saying, "So I'm not sure. I don't think I can expand this, but this would be the same as like I would attempt to do like $\sin\alpha - \sin\beta$." This expansion is shown in Figure 112. She started to explain her expansion in an executing phase, but paused and didn't continue.

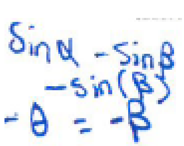

Figure 112

Jessie's Expansion of $sin(\alpha + \beta)$

She reoriented herself to the task with the assistance of the interviewer and was reminded of the given identity. She then entered an executing phase in which she tried substituting $\alpha = 30°$ and

$\beta = 60°$. This gave her $\sin(30° + 60°) = \sin 90°$ and $\sin(30° - 60°) = \sin -30°$ as seen in Figure 113. She verbalized her thinking as,

> This would be $\sin -30°$ so $\sin 90°$ verses $\sin -30°$. And that would be, but then one example wouldn't prove that. ... you're not multiplying both by a negative. It's just one of them that's negative. So the outcome of the one that you negated would be a negative whatever you get out of the first one. That's probably why just the second part is negative and not the whole thing.

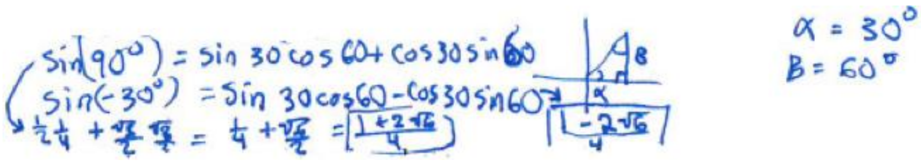

Figure 113

Jessie's Substitution for $sin(\alpha - \beta)$

Jessie then entered a checking phase. She looked at the formula sheet and stated

> So, it was $\sin -\theta = -\sin \theta$. So, the negative, they wouldn't, θ it would be the $-\beta$ here. And that alone would end up a negative of the answer. So, for the positive one, it was $\cos 30° \sin 60°$, but then because you negated β in the second point, then you would negate the answer. You would just turn the second part and not the α part.

She continued to explain her reasoning as

> So, it's kind of like in the top part. So, this would be, so the $-\theta$ would be the $-\beta$ here. And so since we were just assuming that this is right then the... since the $-\theta$ is the $-\beta$, then that would just be $-\sin \beta$ and then that would be this part cause that's the second part of it. So I guess that would be the second negated part cause you're not negating the first part.

Jessie recognized that the change in sign from the sine angle sum identity to the sine angle difference identity came from the odd identity for sine.

Next, Jessie oriented herself to part 2 of the task. She jumped to an executing phase and identified the even/odd identities from the formula sheet that she wrote down in Figure 114.

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

Figure 114

Jessie's Even/Odd Identities for Task 6

> Well for the even/odd formula for sine is that $-\theta$ becomes $-\sin\theta$ whereas $\cos -\theta$ doesn't change the sign. And so, um, I guess maybe that would mean that if you don't have that negative there, then that would make it make the answer negated. So then $\cos\theta$, but I don't know how I would put that.

When asked to explain how this fit with what she knew from the sine formulas provided and in part 1, Jessie initially identified the cosine angle sum identity from the formula sheet and then, after a long pause, stated, "I think the double angle formula looks similar." She recorded the double angle identity for cosine in Double angle Formula $\cos^2(\alpha+\beta) - \sin^2(\alpha+\beta) = \cos(2\theta)$

> Figure 115Figure 115 and explained,
>
> So that would be um, they're not squared, but it's like, cause it's cosine times cosine that would maybe be cosine squared and then the θ would be the $\alpha + \beta$, and then you subtract that from the sine. So maybe it relates to the doubling because their cosine, maybe the cosine squared would be $\alpha + \beta$. And then this one is $\sqrt{\beta}$. Not sure if that's an identity, though. And that would be, $\cos 2\theta$. So maybe you can use like the double angle formula to get that.

Double angle Formula $\cos^2(\alpha+\beta) - \sin^2(\alpha+\beta) = \cos(2\theta)$

Figure 115

Jessie's Double Angle Formula for Task 6

During a checking phase, she expanded how she saw the double angle as the solution.

> Yeah, not sure how you would because on the two there and that these two are equal and I don't think that they're always going to be equal. I'm not sure of that. Yeah, because the double angle one for sine doesn't match. That's why I'm not sure if that's the right one to use.

Thus, for part 2, she identified the double angle identity for cosine as a possible solution but was unsure how to verify that it was a solution.

Next, Jessie oriented herself to part 3 of the task, verifying the cosine angle difference identity. She immediately progressed to an executing phase. She wrote the work seen in Figure 116 and explained,

> So that would relate to the fact that cos $-\theta$ would just give the positive part of it. And similar to what I did like above, we kind of think of the θ as being the $-\beta$. And so because this one is negated, that would kind of be like the θ here, and so it would change it into a positive for the, because of like the even/odd one.

She continued her explanation and related it to the double angle identity she had identified for the cosine angle sum identity.

> So the first part is unchanged. So that's giving the constant. And the second part would be the negative theta, which becomes positive theta because of the double angle thing.

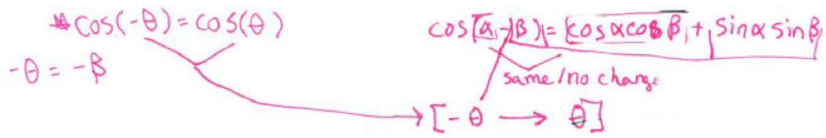

Figure 116

Jessie's Work on the Cosine Angle Difference Identity for Task 6

For part 4 of the task, Jessie began by orienting herself to the task. She then again jumped to an executing phase. During the initial executing phase, she stated,

> The tangent would be like sine over cosine. So that would be like sine of ... [writes], and then I would attempt to show that... Is equal to uh, by using like the other identities. I think you would use um, one of the product to sum formulas up here. Although I don't see where tan would come in.

During this statement, Jessie wrote the two leftmost expressions in Figure 117. She was able to rewrite $\tan(\alpha + \beta)$ as $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ and then rewrite that using the angle sum identities from earlier in the task.

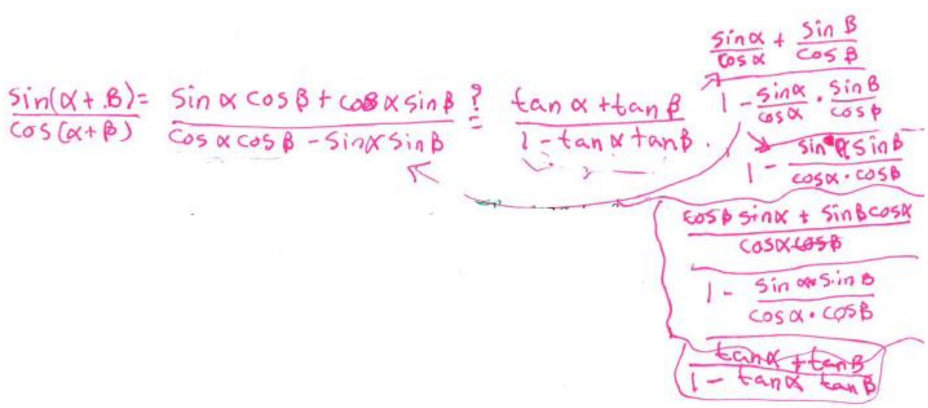

Figure 117

Jessie's Work on the Tangent Angle Sum Identity for Task 6

Since she was unsure how to proceed, she entered a planning phase and looked for identities on the formula sheet that might be helpful. She explained that she was looking for something that she could substitute so that she could simplify and end up with the identity she was trying to show. She did not find anything on the formula sheet that she found helpful, so she progressed to an executing phase in which she returned to working with the identity and trying to simplify on paper.

> And then using um, simplification, like common denominators to... Trying to get from here to there. … to get common denominators at the top, I would multiply to get $\cos\beta$ and $\sin\alpha$. And then the common denominator would be $\cos\alpha\cos\beta$. And then all of that would be over $1 - \sin\alpha\sin\beta$. I'm not sure if multiplying, $\cos\alpha\cos\beta$ would make $\cos^2\alpha\cos\beta$. But I don't see anything that I could cross out. So this is, since it's like multiplication we can separate and have this be $\tan\alpha$ and then $\frac{\sin\beta}{\cos\beta}$. It would be like kind of β, $1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$. So the denominator would match this one. And then the numerator would also be, so crossing this one out, crossing that one would give one and then sine that would give … so we could separate these two because they have the common denominators. And then crossing $\cos\alpha$, we'd get $\sin\beta$, like $\sin\beta$ would cancel.

During this explanation, Jessie worked out the righthand side of Figure 117. She was able to start from the given identity for the tangent angle sum and rewrite it in terms of sines and cosines. She then worked to find common denominators and simplify.

For part 5 of the task, Jessie again began by orienting herself to the task. She began an executing phase and described a similar strategy to the one she has used for the other angle difference identities.

> So I would go with that same idea of like the second part because like for the even-odd $\tan\alpha$, so because the first one was, we could assume that it was positive, then we would negate the second part, which would be, um, so the first part would be $\tan\alpha$. So just like

splitting them in half because they have the same denominator. And so the second part would be negated. Um, so that would give the, since the $-\alpha$ would be like the $-\beta$, then whatever answer we got in the first one would just be second part would be negated, which would be this one that would then make the $-\tan\beta$. Yeah. And then the denominator would be -1 + ...

$\tan(-\theta) = -\tan\theta \qquad -\theta = -\beta$

$$\frac{\tan\alpha}{1-\tan\alpha\tan\beta} \mp \left\{ \frac{\tan\beta}{1-\tan\alpha\tan\beta} \right] = \frac{\tan\alpha - \tan\beta}{1-\tan\alpha\tan\beta}$$

Figure 118

Jessie's Work on the Tangent Angle Difference Identity for Task 6

She continued her explanation during a checking phase, where she clarified how the signs changed from the angle sum identity to the angle difference identity. Her work from both phases is shown in Figure 118.

> Um, because originally it's like that one, but then you're negating the second half of it. So we split it up in the first, this would be kind of like the α, and so negating the β for the, that would need to, but then no, you're not multiplying a negative by the top and the bottom. You just need to multiply by either the top or the bottom.

Since Jessie had used the even/odd identities several times to explain why the signs changed but had been unsure about using it, she was asked to explain her thinking further. She explained her thinking in a checking phase with the work seen in Figure 119.

$$2(x+4) = 2x + 2(4)$$
$$2(x-4) = 2x - \overline{2}(4)$$

Figure 119

Jessie's Work on Negating in Task 6

She explained her thinking as,

> Because like thinking about it, like on expanding would be, for example, versus $2(x - 4)$, and I'm kind of imagining it as if you would say $2(x + 4)$. And then $2x - 2 * 4$. And so, because you just negated the last part, you were just negated negating the last part here as well. But I don't think that would trick, you can really say oh $\tan\alpha - \tan\beta$. I'm not sure that you can like distribute as if it were like an expression or an equation. Cause this is like a value. It's like an expression that represents a number, whereas this would be like a degree. I guess it would be the same cause like that would be a degree and then tangent of that degree versus tangent of this minus tangent of that. It may preserve like the answer at the end, like taking them separately. Yeah, I guess it may be the same time.

She was vacillating on whether she could or could not distribute the negative in the $\tan(\alpha - \beta)$ to get $\tan\alpha - \tan\beta$, so she continued in her checking phase talking through her reasoning using the example in Figure 120.

$\alpha = 60$
$\beta = 30$
$\tan(30°) = \tan(60) - \tan(30)$

Figure 120

Jessie's Example for Negating in Task 6

> I'm not sure. Maybe like if α were $60°$, then β were $30°$, tangent of like the whole value would be technically just be $\tan 30°$. So that $\alpha - \beta$ would just stand for the $30°$ even though like they're separate and then, um, thinking that it should be the same as $\tan 60° - \tan 30°$. So that expansion idea of that minus tangent of this would equal the same thing as just that subtracted and then taking it down. I don't see why it won't be true, but I just never, like, I never distribute it. So if it felt kind of weird to like treat it as if it were an expression. But I didn't see any other way, so I just went with it.

She was still unsure about her simplifications for the angle difference identities, but she could not see another solution, so she used her idea for each of them.

While working on Task 6, Jessie used the resource of the formula sheet and drew upon knowledge of analytic trigonometry. Jessie also used heuristics of substituting values, written algebra, and a simpler case. She was able to identify and use the even/odd identities to simplify the angle difference identities, even though she was unsure whether it was valid to do so. Jessie substituted values on part 1 as her explanation of the identity and part 5 as her explanation of expanding and negating the identity. She used written algebra on all parts of the task to keep track of her work and her thinking. She used a simpler case when describing why she was unsure about her negating for parts 1, 3, and 5.

Task 7: Horizontal Shrink vs. Horizontal Stretch

Jessie’s problem-solving phases as she worked on Task 7 are seen in Figure 121.

	Part 1	
Orienting	x	
Planning		
Executing		x
Checking		

Figure 121

Jessie's Phases on Task: Horizontal Shrink vs. Horizontal Stretch

Jessie quickly oriented herself to the task and entered an executing phase.

> So I know that when like regular graph, say $y = x$ and then $y = 2x$. Then it just gets steeper, and it kinda like shrinks. And when it's like a fraction or a smaller, when you're dividing it, it compresses or like stretches out. Um, so I think it's the same for the graph of like $\sin(2x)$, it's going to like shrink and make the slope steeper for every kinda wave.

She continued her explanation and drew the example in Figure 122 to show the linear equation getting steeper.

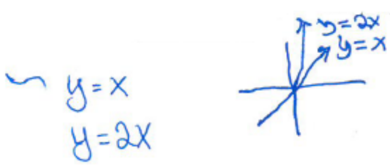

Figure 122

Jessie's Linear Example for Task 7

> So I'm just trying to like compare what happens to regular graphs. So like linear graphs. So when you multiply by a whole number coefficient, then it shrinks it, and it gets closer to the $y - axis$. Whereas if you divide, it kind of stretches out. Um, and I think the same goes for $y = \sin(2x)$. So instead of the sine being like spread out, um, it's going to shrink it up and make it more steep as if it was, um, $y = x$ and $y = 2x$. So it's just gonna kind of do the same thing and compress. And the line just gets steeper.

As she talked through this explanation, she added the diagram in Figure 123 to show sine getting steeper and thus shrinking horizontally.

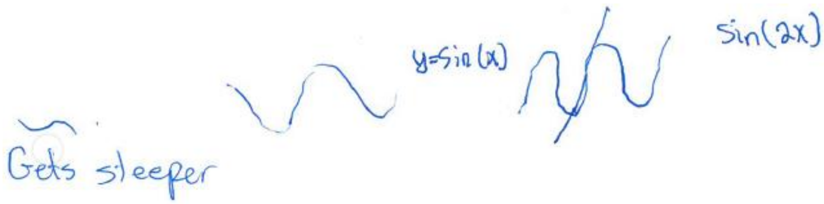

Figure 123

Jessie's Diagram of sine for Task 7

For Task 7, Jessie used heuristics of a simpler case and diagrams. She began with an example of a linear function, which gets steeper with a coefficient of 2. She then generalized that to the sine function and clarified that getting steeper would make it shrink horizontally. She also used diagrams, seen in Figure 122 and Figure 123, to support her explanations.

Task 8: Proving an Identity

On Task 8, Jessie worked through the problem-solving phases, as seen in Figure 124. Jessie had recently done this task as part of one of her mathematics education courses, so her work is based upon what she recalled from the class.

	Part 1		
Orienting	x		
Planning			
Executing		x	
Checking			x

Figure 124

Jessie's Phases on Task 8: Proving an Identity

Jessie oriented herself to the task and immediately moved to an executing phase. She explained,

> I think she is [correct] because if you start from the conclusion, I think just the order in which she did the proof was just mixed up so that she started with a true statement and

> then worked from that by getting equivalent statements, then concluded at the end. So we started with an equivalent statement. We ended up with this. And so that's true. Um, so you could start with $1 = 1$, which we know is true and we can assume, and then you could go ahead and do, you know, multiply by cosine over cosine, which is one. So just keep on producing equivalent statements until you get to where you want to be. And then you can say therefore it's true. … Um, because I don't know if it would be right to start with this one because, like you're, you're assuming what you're trying to prove. And so I would just start to start with an equivalent statement or a true statement, and then work to get there."

She was asked to explain further about the student assuming what she was trying to prove and she explained,

> So when you have a proof, you have like a statement that she's trying to prove. So you want to start with a statement. You can't assume that something's true because you're trying to prove it, and no one should take your word for it when you're just assuming it's true. So you have to start with something that's true and then kind of work your way and say, well, this is the proof for it. And kind of show the steps that you get to the statement to really prove mathematically that it's true.

Jessie recalled from the class that the proof as it was written assumed what the student was trying to prove. However, if it had been written in reverse order, it would have been a valid proof.

Task 9: Angles in a Rectangle

For Task 9, Jessie's progress through the problem-solving phases is seen in Figure 125.

	Part 1								
Orienting	x								
Planning			x		x		x		x
Executing		x		x		x			
Checking								x	

Figure 125

Jessie's Phases on Task 9: Angles in a Rectangle

Jessie began by orienting herself to the task. She began an executing phase in which she marked several things on the given diagram. The initial markings are in blue, as seen in Figure 126. She described what she had marked as,

> So I'm just putting down all the information that I know. So, um, BC would be $3AB$, and then these two sides [of the rectangle] are equal, and these two sides [of the rectangle] are equal. Um, also BP, PQ, QC, those are equal. Um, I just kind of marked the angles to help, um, just highlight them and remember that that's what I'm trying to show.

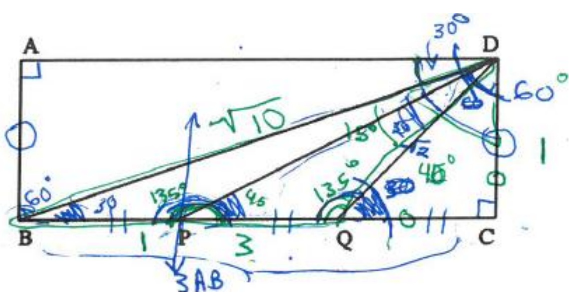

Figure 126

Jessie's Marked Diagram for Task 9

Next, Jessie entered a planning phase, during which she stated, “I'm trying to assume that this [largest triangle] is a 30-60-90.” She then worked silently for several minutes and marked more information on the diagram. During an executing phase, she described what she had been doing as,

> I was trying to use that to kind of section these off. But, um, I couldn't figure out what $\angle DQC$ would be. So these would be the two halves would be a 30-60-90. And then this would be a $30°$, and then the rest of this chunk would be $60°$.
>
> But then I couldn't kind of take this information and use that to show that these two added together would equal that one. But then I tried putting down the fact that $\angle PQT$ was $\angle DQC$ was $180°$. Um. But then I'm not, I want to use the fact that they're all equal to... Cause I know, um, yeah. I'm not sure how I will use, um, that they're equal.
>
> I would assume that this is, there's no indication that it's a 45-45-90, but if I were to try and to see what, this is one, that's one, cause it's like three equal parts and this is three times that one. Um, then this would be $45°$, and PT would then have to be $180° - 45°$. And then since all of this is $90°$, it's $60°$, so $60° - 45°$ would need to be, BQ would be $60° - 45° = 15°$. Um, so $\angle BQD + \angle BPQ$ would be $135° + 150°$. And then, the missing angle would need to be $180° - 150°$, so $\angle DPQ$ would need to be $30°$.

Jessie's work assumed that the largest triangle, ΔDBC, was a 30-60-90 special triangle, and the smallest triangle, ΔDQC, was a 45-45-90 triangle. She then used geometric properties to find the values of other angles in the diagram.

Jessie entered a planning phase of trying to determine what information she had and what information she still needed to find. "I had the side lengths, I have two side lengths and then the angle. Um, and then a side and an angle. I'm thinking... Yeah, not sure I have enough information to do that." She then entered an executing phase of recapping what she had found so far.

> I assumed a 45-45-90 here. Then from that, I kind of find, found like this one is $15°$. This one is $135°$, and I want to say that this is also $135°$, but then that would be, but then that

> wouldn't be true cause D would be $30° + 45° = 45°$. So that is not true. Maybe these two are similar. I'm not sure. The bigger one and then small, that one would have. But then I couldn't do anything with the angles. I'm not sure I have information to kind of use any of the formulas. I feel like there was a much easier strategy that I should have taken.

Jessie also explained why she assumed that ΔDBC was a 30-60-90 triangle, and ΔDQC was a 45-45-90 triangle.

> Cause I think that cause it, it looked like it and I'm not sure. Cause like they're triangles that are x, x, and then $x\sqrt{2}$ or x, x, and $2x$. And then I'm probably just making or just imagining this or because there is, I think there's one that's x, $x\sqrt{2}$ and then $3x$. Um, and I think that's where the 30-60-90 is. But I'm not sure, but the 30-60-90 came from just like noticing that there's a smaller one and then a slightly bigger one and then the 90. That one I flipped over and it kind of looked um equal, [so I thought it was a 45-45-90]. Also, the sides, these are three equal sides. And this is three times this one, so it's one, one and then whatever this size would be.

Jessie recalled that the sides of the two special right triangles have specific ratios, though she was unsure of what those ratios were. She started thinking about using the sides and angles to find more missing information that might be helpful in the planning phase.

> So like 45-45-90. So it would be $\sqrt{2}$. But yeah, I don't know if any of that can help me find the angles. I might use proportions, but I don't think that I can, um, say like $\frac{50°}{15°}$ would be the same as whatever degrees this is over 1. Um, that wouldn't work.

After considering using proportions, but deciding it wouldn't work, Jessie worked quietly for a few minutes. She continued in her planning phase and stated,

> Yeah, I'm not sure. I tried to figure out like alternate interior angles. Like if this is 30°, then I can't remember. But it's something like you have like triangles within triangles, you kind of can use one angle to jump around and assume alternate interior angles. But I don't see where I can do that.

Jessie wanted to continue to work with the angles, but was unsure how to proceed. Jessie entered a checking phase of looking over the work she had done to that point.

> I don't think I was right when I assumed that certain angles. So it's either that's not 30°, or this is not a 45°, or that's not 15°. Because this diagram right here and kind of, I said it because it's like one and one and the fact that two sides are equal means that there should be like alternate interior angles here, but it's not working because this is 15°, but then that's 30°, that's 45°. So I think there was something wrong with my assumptions about certain angles. Cause I feel like I should be able to work with the information that I have. And like each time I try to do something, like alternate interior angles or like properties, the angles don't add up. So I'm thinking that the angles were wrong or some of them. And so what I think like the general idea would be that since this kind of bisects it, the two sides in half then this angle and this angle should be equal because they're like alternate.

Jessie realized that she had reached an inconsistency in her solution, which meant that her initial assumptions were likely not true. She was unsure how to proceed, but suggested that if she were to start over her solution would be to

> I'm thinking that I would have started with the alternate interior angles idea and not assume so much and just stuck with what I had. And then like kind of what the girl did [in Task 8] by starting out with this, like kind of assuming it's true and then working my way backwards with my assumptions. And then rearranging it for the proof.

Though Jessie was unable to reach a solution, she was able to recognize her mistake was likely assuming too much information. She was also able to theorize an alternate path to a solution, but she did not enact this path.

While working on Task 9, Jessie used resources of knowledge of geometry and right triangle trigonometry. She also used the heuristic of substituting values. Jessie began the task by assuming that ΔDBC was a 30-60-90 triangle. This allowed her to substitute values and simplify her calculations. Throughout the task, Jessie used knowledge of geometric relationships with angles. For example, she calculated the values of supplementary angles. She also used knowledge of the Pythagorean Theorem to find the hypotenuse of ΔDBC and ΔDQC. She used knowledge of special right triangles. She was able to correctly identify that ΔDQC was a 45-45-90 triangle because the two legs were of equal length.

Analysis by Domain

As Jessie worked on tasks in different domains, her use of resources and heuristics varied. As shown in Figure 127, Jessie used knowledge of all four trigonometric domains to solve tasks. She also used the calculator in the domains of unit circle trigonometry and trigonometric functions and graphs and used the formula sheet in all four domains. She also frequently relied on the heuristics of substituting values and using simpler cases to guide her thinking.

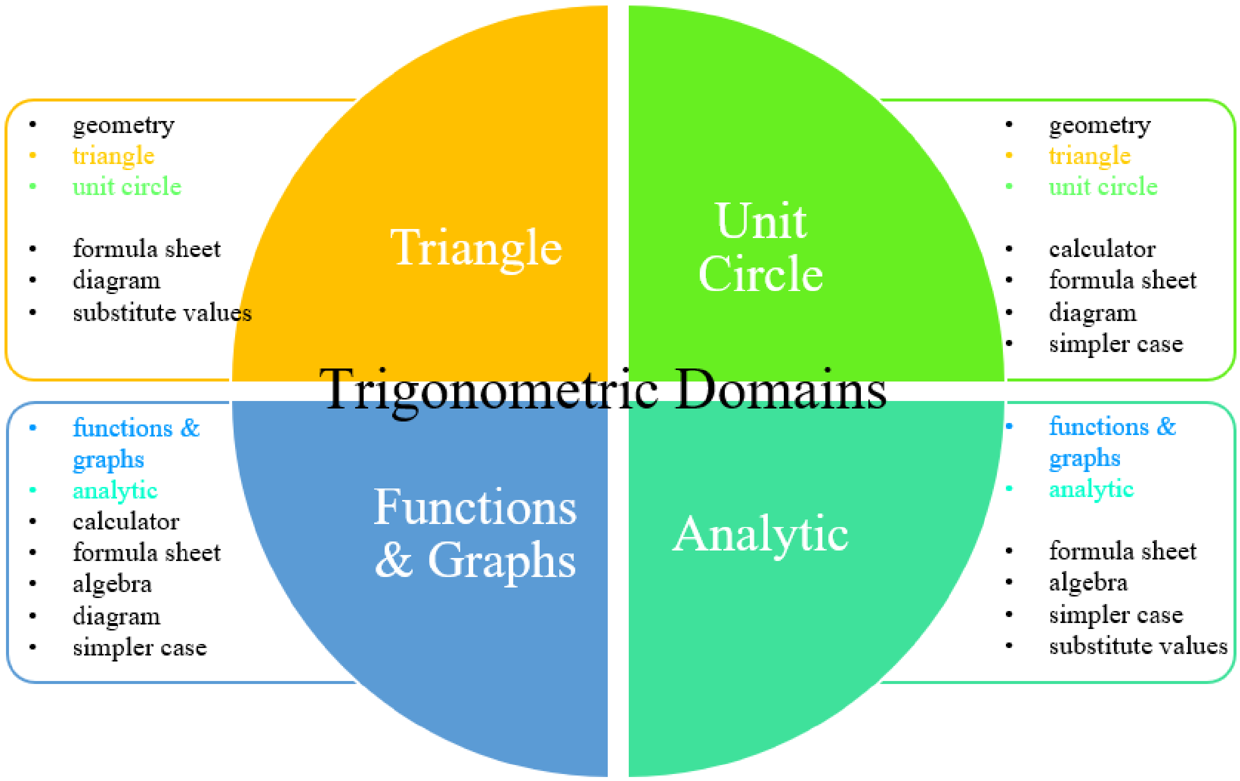

Figure 127

Jessie's Use of Resources and Heuristics Across Domains

Triangle Trigonometry

Tasks 2, 4, and 9 addressed the domain of triangle trigonometry. For Task 2, Jessie completed a single problem-solving cycle for each part of the task. For Tasks 4 and 9, Jessie primarily alternated between planning and executing phases. For both parts 1 and 3 of Task 2, Jessie completed an orienting, planning, and executing phase, but did not complete a checking phase. However, for part 2 of Task 2, Jessie did complete a checking phase after her orienting, planning, and executing phases. On Task 4, Jessie used an orienting phase, two planning phases, and an executing phase for part 1, but on part 2 she jumped from orienting to executing before completing a cycle of orienting, planning, and executing. On Task 9, Jessie completed an initial orienting phase and then alternated between planning and executing three times.

	Task 2									
	Part 1			Part 2				Part 3		
Orienting	x			x				x		
Planning		x			x				x	
Executing			x			x				x
Checking							x			

	Task 4										
	Part 1				Part 2						
Orienting	x				x		x				
Planning		x	x					x		x	
Executing				x		x			x		
Checking											x

	Task 9								
	Part 1								
Orienting	x								
Planning			x		x		x		x
Executing		x		x		x			
Checking								x	

Figure 128

Jessie's Phases on Triangle Trigonometry Tasks

When solving triangle trigonometry tasks, Jessie accessed knowledge of geometry, triangle trigonometry, and unit circle trigonometry. When solving Task 2, Jessie used the Pythagorean Theorem to find the missing side of a triangle necessary to find $\cos\theta$ given $\sin\theta = \frac{8}{17}$. She also used geometry in Task 9 to find the values of angles in the rectangle. On all three of the triangle trigonometry tasks, Jessie used special right triangles. She assumed that the triangles pictured in Task 2 and Task 9 were 30-60-90 triangles and worked on the task based on that assumption. On Task 4, Jessie referenced the triangle she drew to show the value of $\sin\pi$ as a 45-45-90 triangle. Jessie used the unit circle on Task 4 when she was asked to find the values of $\sin\frac{3\pi}{4}$ and $\sin\pi$.

Jessie used the heuristics of diagrams and substituting values to help her solve triangle trigonometry tasks. On both Tasks 2 and 4, Jessie drew diagrams of the unit circle to indicate where the triangles she was working with were located. Jessie also substituted values on both Tasks 2 and 9. For Task 2, Jessie assumed the triangle was a 30-60-90 triangle and substituted the corresponding ratios of sides into the given formula to verify that it worked. For Task 9, Jessie assumed the largest triangle was a 30-60-90 triangle, and the smallest triangle was a 45-45-90 triangle. She then substituted those values into her calculations to find other angles in the figure.

Unit Circle Trigonometry

Tasks 3, 4, and 7 addressed the domain of circle trigonometry. For these tasks, Jessie generally began with a cycle of orienting, planning, and executing before alternating between planning and executing. On Task 3, Jessie initially completed a problem-solving cycle of orienting, planning, and executing. She then completed a planning and checking phase before alternating between planning and executing twice. On Task 4, part 1, Jessie completed an orienting, two planning, and an executing phase. For part 2, she completed phases of orienting, then executing, then back to orienting, before planning, executing, planning, and checking. For Task 7, Jessie only required an orienting and executing phase.

Task 3									
	Part 1								
Orienting	x								
Planning		x		x		x		x	
Executing			x				x		x
Checking					x				

Task 4											
	Part 1				Part 2						
Orienting	x				x		x				
Planning		x	x					x		x	
Executing				x		x			x		
Checking											x

Task 7		
	Part 1	
Orienting	x	
Planning		
Executing		x
Checking		

Figure 129

Jessie's Phases on Unit Circle Trigonometry Tasks

To solve tasks in the domain of unit circle trigonometry, Jessie used knowledge of geometry, triangle trigonometry, and unit circle trigonometry. Jessie used knowledge of geometry on Task 4 to explain why there could not be a right triangle with an obtuse angle. Jessie used both SOH CAH TOA and the law of sines on Task 4. She described how to use SOH CAH TOA to find a missing side or angle in a right triangle. She also suggested using the law of sines to solve for a missing side or angle in a non-right triangle. Jessie used knowledge of the unit circle on Tasks 3 and 4. She drew a diagram of the unit circle when beginning Task 3 and then transitioned to using the unit circle on the formula sheet to evaluate the expressions in the task. On Task 4, Jessie also drew diagrams of the unit circle to explain how to evaluate $\sin \frac{3\pi}{4}$ and $\sin \pi$.

Jessie used heuristics of simpler cases and diagrams to solve tasks in the domain of unit circle trigonometry. For Task 7, Jessie considered the simpler case of a linear function to explain why a function would have a horizontal shrink. For all three tasks, Jessie drew at least one diagram. In Task 3 and Task 4, Jessie drew diagrams of the unit circle to show the location of angles. In Task 7, Jessie drew a sketch of the sine function with a horizontal shrink compared to a sine function without the horizontal shrink.

Trigonometric Functions and Graphs

Tasks 1, 5, and 7 addressed the domain of trigonometric functions and graphs. For Tasks 5 and 7, Jessie generally completed the task with problem-solving cycles of orienting, planning, executing, and checking, but for Task 1, she only used these cycles regularly on part 2. While completing Task 1, Jessie completed part 1 with only phases of orienting and planning. For part 2, she completed four partial or whole problem-solving cycles. For parts 3 and 4, she alternated between executing and either planning or checking phases. For Task 5, Jessie completed a cycle of orienting, planning, and executing for parts 1-3 and a cycle of orienting and executing for part 4. On Task 7, Jessie completed the task with only an orienting and executing phase.

	Task 1																
	Part 1		Part 2														
Orienting	x		x					x				x					
Planning		x		x		x			x				x		x	x	
Executing					x		x			x				x			
Checking											x						x

	Part 3								Part 4									
Orienting	x								x									
Planning						x				x							x	
Executing		x	x		x		x					x	x		x			
Checking				x				x			x			x		x		x

	Task 5										
	Part 1			Part 2			Part 3			Part 4	
Orienting	x			x			x			x	
Planning		x			x			x			
Executing			x			x			x		x
Checking											

	Task 7	
	Part 1	
Orienting	x	
Planning		
Executing		x
Checking		

Figure 130

Jessie's Phases on Trigonometric Functions and Graphs Tasks

To solve tasks in the domain of trigonometric functions and graphs, Jessie drew upon her knowledge of trigonometric functions and graphs and analytic trigonometry. For Task 1, Jessie frequently used both her graphs of sine and cosine and the graphs on the graphing calculator to assist her with determining what transformations would map sine or cosine back onto themselves and onto each other. On Task 5, Jessie was able to identify the general shape of the cosine graph and use that knowledge to find the values of some of the parameters for the fox and rabbit data.

Jessie used knowledge of analytic trigonometry to relate the transformation she had found to identities on Task 1.

Jessie used heuristics of algebra, diagrams, and simpler cases on the trigonometric functions and graphs tasks. Jessie used algebra briefly at the end of Task 1 to show how she was thinking about squaring $\sin^2(x-1)$. On Task 5, Jessie drew a diagram of cosine and sine crossing the y-axis to show why she decided to used cosine for both the fox and rabbit equations. For Task 7, Jessie drew a diagram showing the effects of multiplying by 2 within $\sin(2x)$ and described why this would result in a horizontal shrink. She also used a simpler case on Task 7 when she described the effect of multiplying by 2 in $y = 2x$ and how that would result in a horizontal shrink.

Analytic Trigonometry

Tasks 1, 6, and 8 addressed the domain of analytic trigonometry. Jessie initially began these tasks with a general problem-solving cycle of orienting, planning, and executing, but these cycles became more sporadic as she worked on the longer two tasks. On Task 1, Jessie completed an orienting and planning phase for part 1, then four partial or complete orienting, planning, executing, and checking cycles for part 2. On parts 3 and 4, she primarily alternated between executing and either planning or checking phases. For Task 6, Jessie completed a partial orienting, planning, executing, and checking cycles for each of parts 1 through 3. On part 4, she alternated between executing and either orienting or planning, and on part 5, she progressed from orienting to executing and checking. For Task 8, she used a cycle of orienting, executing, and checking.

	Task 1																
	Part 1		Part 2														
Orienting	x		x					x				x					
Planning		x		x		x			x				x		x	x	
Executing					x		x			x				x			
Checking											x						x

	Part 3								Part 4									
Orienting	x								x									
Planning						x				x							x	
Executing		x	x		x		x					x	x		x			
Checking				x				x			x			x		x		x

	Task 6										
	Part 1			Part 2			Part 3				
Orienting	x			x			x				
Planning		x							x		
Executing			x		x			x		x	
Checking						x					x

	Part 4						Part 5			
Orienting	x		x				x			
Planning					x					
Executing		x		x		x		x		
Checking									x	x

	Task 8		
	Part 1		
Orienting	x		
Planning			
Executing		x	
Checking			x

Figure 131

Jessie's Phases on Analytic Trigonometry Tasks

To solve tasks in the domain of analytic trigonometry, Jessie accessed knowledge of trigonometric functions and graphs and analytic trigonometry. For Task 1, Jessie used the graphs she had created by hand and on the graphing calculator to reason about transformations that would map sine and cosine back to themselves and each other. She also reasoned about whether the same identities would apply for the transformations of sine and cosine. For Task 6, Jessie used the even and odd identities to describe the changes in the identities from the angle sum to angle difference formulas.

Jessie used the heuristics of algebra, substituting values, and simpler cases to solve analytic trigonometry tasks. Jessie used written algebra to keep track of her work and her reasoning for both Tasks 1 and 6. Jessie substituted values when she was reasoning about coordinate transformations in Task 1. For Task 6, Jessie substituted values into the sum and difference identities for sine to determine why the identity changed from an addition to a subtraction. When Jessie was trying to reason about whether she could expand $\tan(\alpha + \beta)$ as $\tan \alpha + \tan \beta$ on Task 6, she used the simpler case of expanding $2(x + 4)$ to $2x + 2(4)$.

Mathematical Understanding for Secondary Teachers

Mathematical Proficiency

Mathematical proficiency consists of six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. Jessie displayed two of the strands: procedural fluency and productive disposition.

Jessie showed procedural fluency on Task 2 by using SOH CAH TOA to solve a right triangle. She was given $\sin \theta = \frac{8}{17}$ and drew a triangle in the unit circle that she labeled with 8 on the opposite side, 17 on the hypotenuse, and x on the adjacent side. She then stated, "And so cosine would be whatever x is over 17." She was able to use the Pythagorean Theorem to solve for x and wrote $\cos \theta = \frac{15}{17}$. Though Jessie did not verbalize her productive disposition explicitly, she was eager and willing to work throughout all nine tasks. She also persisted when the tasks were challenging. For example, she spent over 30 minutes on Task 9, though she was not able to complete the task.

Mathematical Activity

Across the tasks and domains, Jessie demonstrated all three strands of mathematical activity: mathematical noticing, mathematical reasoning, and mathematical creating. The strand of mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Within the strand of mathematical noticing, Jessie demonstrated two strands: form of an argument, and connect within and outside mathematics. On Task 8, which Jessie had previously seen in one of her mathematics education courses, she was able to explain the flaws in the presented proof.

> Um, so you want to start with a statement. You can't assume that something's true because you're trying to prove it, and no one should take your word for it when you’re just assuming it's true. Um, so you have to start with something that's true and then kind of work your way and say, oh, well, this is the proof for it. And kind of show the steps that you get to the statement to really prove mathematically that it's true.

Jessie knew that a proof that assumed what it was trying to prove was not valid. She also recognized on Task 6, that “… one example wouldn't prove that.” Both of these examples show that she had an understanding of the form of a valid argument. Jessie was able to show connections outside of mathematics on Task 5. She described the relationship between the graphs of the numbers of rabbits and foxes as,

> And then I know that the more foxes there are, then the rabbits will decrease like rapidly. And so when the rabbits decrease rapidly than the fox is decreased rapidly and because

they don't have food, but then they, the rabbits increase and then the foxes increase, but then they really depend on each other.

She was able to describe the predator-prey relationship and connect the mathematics to the real-world. Jessie did show a lack of understanding of one strand: symbolic form. She struggled to work with the symbolic trigonometric forms on multiple tasks correctly. For example, on Task 1, she suggested that $\cos^2(x-1) = \cos(x^2 - 2x + 1)$. This incorrectly squared the input of the cosine functions, rather than the output of the cosine function. On Task 6, she suggested that $\tan(60° - 30°) = \tan 30° = \tan 60° - \tan 30°$ and overapplied the distributive property.

Mathematical reasoning consists of three strands: justifying/proving, reasoning when conjecturing and generalizing, and constraining and excluding. Jessie make an attempt at one strand: constraining and excluding, but did not show evidence of the other two strands. On Task 6, Jessie substituted values into the sine difference angle identity to see if she could determine why the sign changed from the sine sum angle identity. She was able to notice that the sign changed from a $+$ to a – in her evaluated solution, but she was not at generalizing this result.

Mathematical creating consists of three strands: representing, defining, and modifying/transforming/manipulating. Jessie demonstrated an understanding of two of the strands: representing and modifying/transforming/manipulating. Jessie demonstrated an understanding of representing when she created diagrams to model situations on several tasks. For example, she created diagrams of triangles on the unit circle for Tasks 2 and 4 and diagrams of graphs on Task 1 and 5. These representations of the problems supported her thinking about the tasks. Jessie demonstrated an understanding of modifying/transforming/manipulating with her algebraic manipulations for Tasks 6 and 9. On Task 6, Jessie was able to manipulate the tangent sum angle identity algebraically. She initially rewrote tangent in terms of sine and cosine but was unable to progress. Thus, she then worked backward from the solution to rewrite it in terms of sine and cosine. On Task 9, Jessie attempted several algebraic manipulations, including the law of sines and Pythagorean Theorem, though she did not find any of them fruitful.

Chapter 5: Cross Case Analysis

This study addressed three research questions across the four cases. These questions include:

Research Question #1: How do secondary pre-service mathematics teachers engage in problem solving phases (e.g., orienting, planning, executing, checking) when solving high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #2: How do secondary pre-service mathematics teachers draw upon problem solving attributes (e.g., resources, heuristics, affect, monitoring) as they solve high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #3: How do secondary pre-service mathematics teachers engage in mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (mathematical noticing, mathematical reasoning, and mathematical creating) as they solve high cognitive demand trigonometric tasks?

The first two research questions will be answered using analysis of the phases and attributes from the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005). The third research question will be answered using strands of the Mathematical Understanding for Secondary Teaching Framework (Heid et al., 2015).

Research Question #1

How do secondary pre-service mathematics teachers engage in problem solving phases (e.g., orienting, planning, executing, checking) when solving high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

The hypothetical phases, as described by Carlson and Bloom (2005), are shown in Figure 132. This graph shows beginning with an orienting phase, denoted by a 1 on the vertical axis, and then following with three cycles of planning, denoted by a 2 on the vertical axis, executing, denoted by a 3 on the vertical axis, and checking, denoted by a 4 on the vertical axis, phases until the task is solved. Though the participants engaged in the phases, none followed this exact hypothetical model for any individual task.

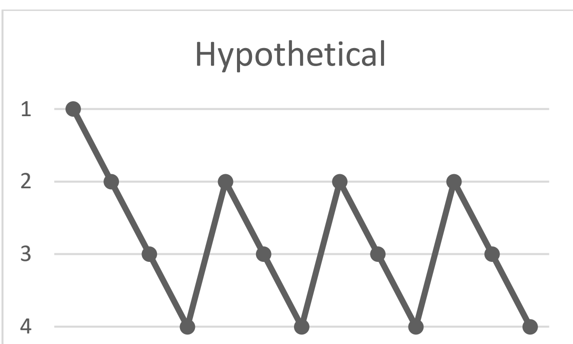

Figure 132

Hypothetical Phases from Carlson and Bloom (2005)

The graphs in Figure 133, Figure 134, Figure 135, and Figure 136 show the actual phases that each participant engaged in as they completed the tasks in each trigonometric domain. Figure 133 shows the phases for all four participants on the domain of triangle trigonometry. The phases of each participant in the domain of unit circle trigonometry is shown in Figure 134. The participants' phases in Figure 135 are for the domain of trigonometric functions and graphs. Finally, Figure 136 shows the phases of each participant in the domain of analytic trigonometry. In each graph, the numbers on the vertical axis represent the phase, with 1 representing an orienting phase, 2 representing a planning phase, 3 representing an executing phase, and 4 representing a checking phase. The graphs show the phases for each participant across all three tasks in the domain.

Triangle Trigonometry

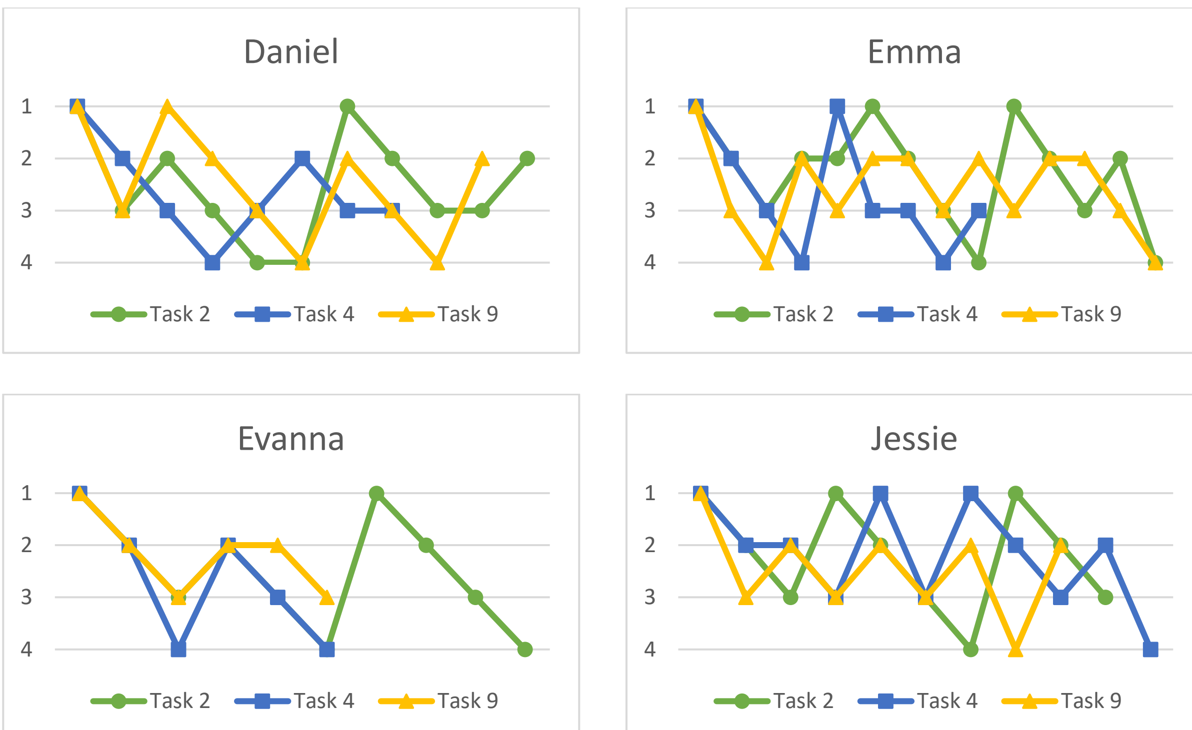

Figure 133

Participants' Phases on Triangle Trigonometry Tasks

The four graphs in Figure 133 show the phases that the participants progressed through as they worked on the tasks in the domain of triangle trigonometry. Across the tasks, all four participants generally began each task with a cycle of orienting, planning, and executing, with some also having a checking phase on their initial cycle. All four participants also ended Task 2 with a cycle of orienting, planning, and executing, with three of the four participants having one additional phase after this cycle. For Task 2, Emma and Evanna concluded with a checking phase, but Daniel concluded with an additional planning phase. On Task 4, all four began with a partial cycle of orienting, planning, executing, and checking, with two participants adding or skipping at least one phase from the cycle. Evanna skipped an executing phase, and Jessie added a planning phase and skipped a checking phase. Daniel, Evanna, and Jessie all began Task 9 with a cycle of orienting, planning, and executing. Emma skipped an initial planning phase and included a checking phase that the others did not have in their initial cycle.

Evanna completed all the triangle trigonometry tasks in a smaller number of phases than the other three participants. This is especially true for Task 4 and 9, on which she used about half the number of phases as the other participants. Another key difference is that Daniel generally followed the hypothesized cycle of orienting, planning, executing, and checking, followed by repeated cycles of orienting, planning, and executing much more than the other participants. Evanna generally only had an initial cycle of orienting, planning, executing, and checking with a few additional phases. Emma and Jessie generally began with a cycle of orienting, planning, executing, and checking, but then lost the cyclic nature of the phases.

Unit Circle Trigonometry

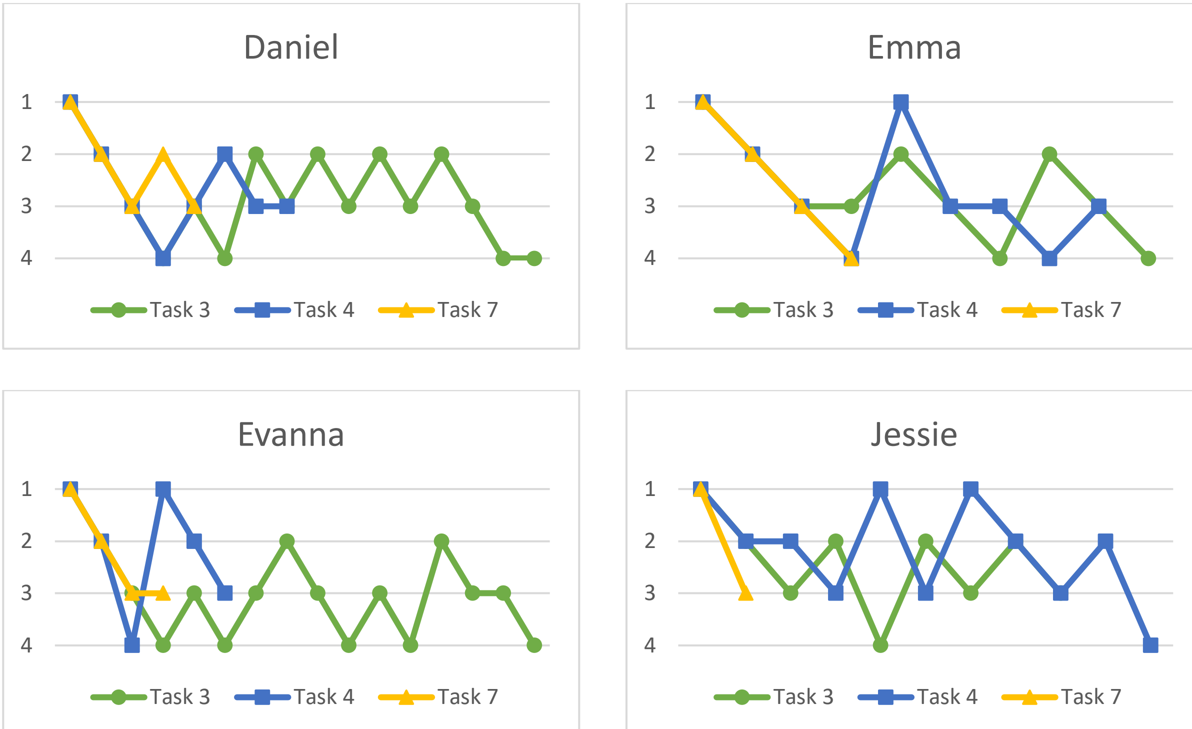

Figure 134

Participants' Phases on Unit Circle Trigonometry Tasks

For all participants, Task 7 required the fewest number of phases. Task 7 contained only a single part and asked participants to explain why $y = \sin(2x)$ is a horizontal shrink of $y = \sin x$, rather than a stretch. Daniel, Emma, and Evanna required the highest number of phases for Task 3, but Jessie required the greatest number of phases on Task 4. Across the tasks, Daniel, Emma, and Evanna generally began with a cycle of orienting, planning, executing, and checking. Jessie skipped at least one of these phases as she began all three of the unit circle trigonometry tasks. Beyond the initial cycle of orienting, planning, executing, and checking, all four participants did not continue with cycles of planning, executing, and checking on the unit circle trigonometry tasks.

Daniel and Jessie both had the majority of their phases as planning and executing. Evanna, however, had the majority of here phases as executing and checking, and Emma had her phases spread more equally across orienting, planning, executing, and checking. Jessie used far fewer checking phases than the others on these three tasks, with only two total checking phases. Daniel used four checking phases, Emma used five checking phases, and Evanna used six checking phases.

Trigonometric Functions and Graphs

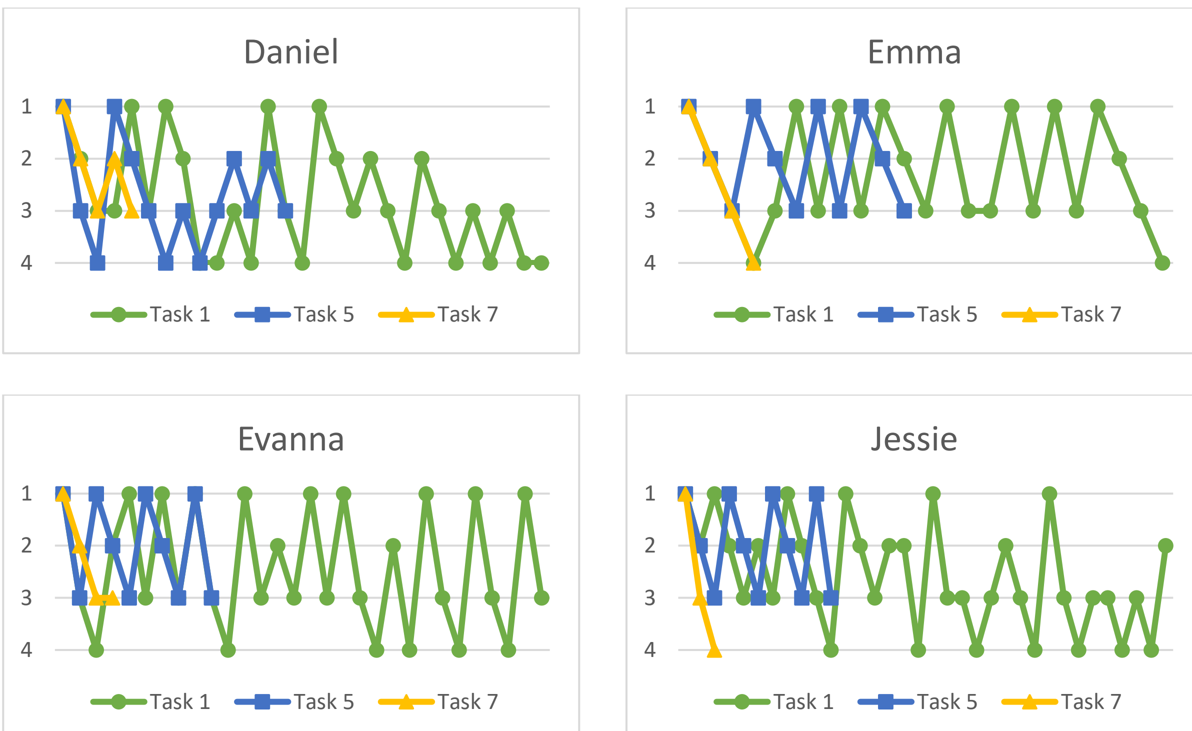

Figure 135

Participants' Phases on Trigonometric Functions and Graphs Tasks

For all participants, Task 7 required the fewest number of phases, and Task 1 required the greatest number of phases to complete. All four participants generally began each task with a cycle of orienting, planning, and executing, with only a few tasks including a checking phase in the initial cycle. After this initial cycle, Daniel only completed another cycle a few times, but

Emma, Evanna, and Jessie all completed multiple orienting, planning, and executing cycles to complete the tasks.

The participants varied in their use of orienting phases at the end of Task 1. Daniel and Jessie each oriented themselves several times in the beginning and middle of the task but stopped orienting themselves at the end of the task. Emma and Evanna continued to orient themselves to the task throughout Task 1. The participants also used checking phases in different ways across the three trigonometric functions and graphs tasks. Daniel used multiple checking phases on each of Tasks 1 and 5, but none on Task 7. Emma used a checking phase at the beginning and end of Task 1 and at the end of Task 7 but did not use any other checking phases on the tasks. Evanna used multiple checking phases throughout Task 1 but did not use any checking phases on either of the other tasks. Jessie used multiple checking phases on Task 1 and a single checking phase at the end of Task 7 but did not use any checking phases on Task 5.

Analytic Trigonometry

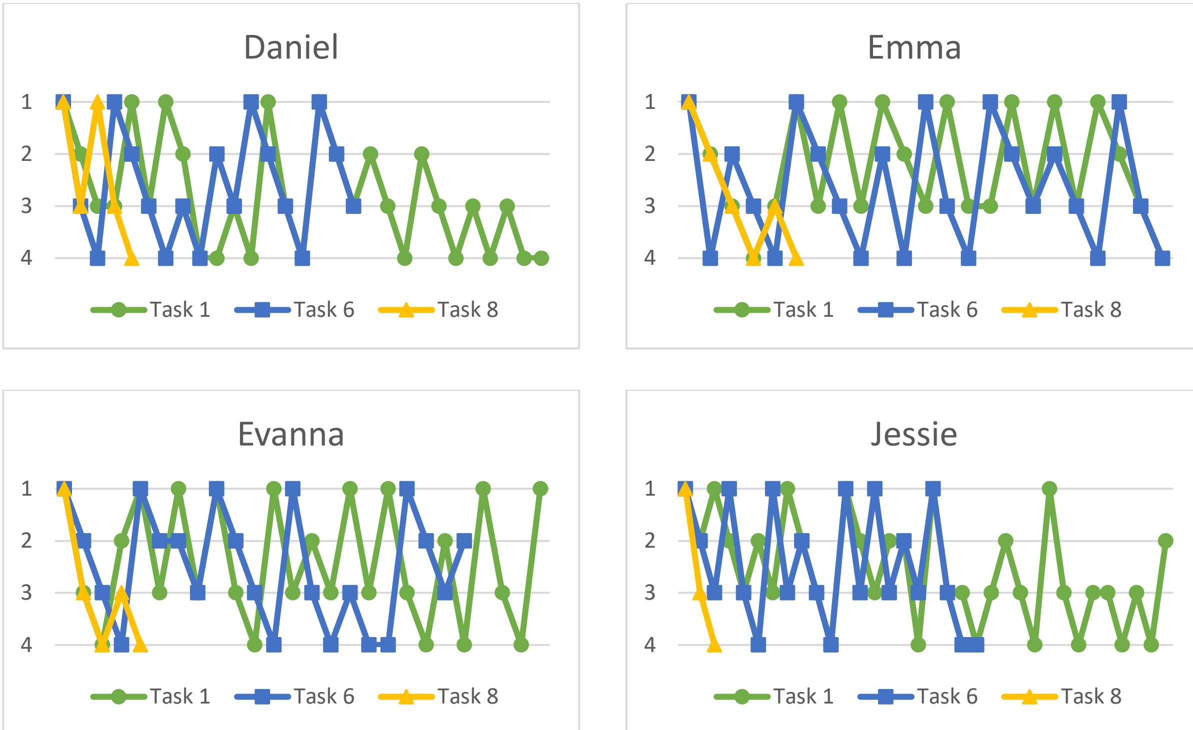

Figure 136

Participants' Phases on Analytic Trigonometry Tasks

Task 8 required very few phases for all of the participants. Tasks 1 and 6 required similar numbers of phases for Emma and Evanna, but Task 6 required fewer phases than Task 1 for Daniel and Jessie. As with the other domains, the participants all generally began each task with a cycle of orienting, planning, executing, and checking. Emma and Evanna generally continued to follow the cycle of orienting, planning, executing, and checking as they completed the longer tasks. However, Daniel and Jessie did not continue to follow the cycle on the longer tasks.

The use of orienting, planning, executing, and checking was spread fairly equally on all tasks for all participants. Unlike the other three trigonometric domains, there was not one phase that was skipped across the task for several or most of the participants. However, Evanna and

Jessie did end Task 1 with an orienting and planning phase, respectively. This is unique since every other task completed concluded with either an executing phase or checking phase.

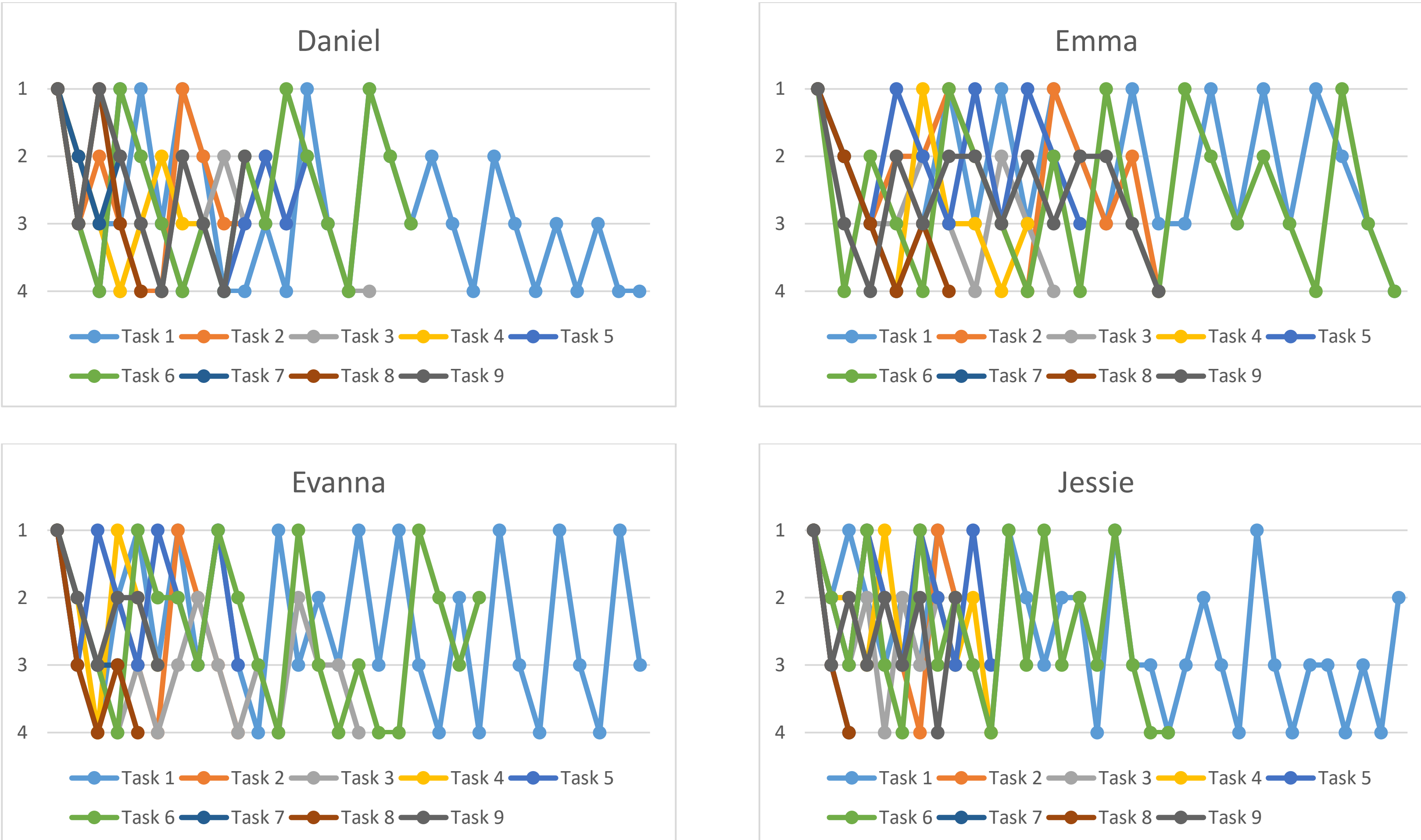

Figure 137

Participants Phases Across Trigonometric Domains

Phases Across Trigonometric Domains

As participants worked across all phases, they generally began each task with a cycle of orienting, planning, executing, and checking. However, they frequently skipped one of the phases in their initial cycle. Across domains, Daniel frequently had cycles of planning, executing, and checking. Jessie frequently alternated between planning and executing phases across domains. Both Emma and Evanna tended to have a longer cycle of three or more of the phases in the hypothetical order, but they both varied in which phase was skipped.

Some tasks provided more opportunities for participants to engage in problem solving cycles than others. In particular, Tasks 3 and 9, which both consisted of a single open-ended part allowed participants to more fully engage in cycles of problem-solving. However, a task being a single open-ended part alone does not ensure that participants will engage in multiple problem-solving cycles, since most participants were able to solve Task 7 with just a single cycle.

Despite the similarities between the phases by some participants, on some tasks, or through some domains, there is no clear pattern to the phases across participants or domains. Given an unlabeled graph, it would be difficult to identify the participant or domain of the task.

Research Question #2

How do secondary pre-service mathematics teachers draw upon problem solving attributes (e.g., resources, heuristics, affect, monitoring) as they solve high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

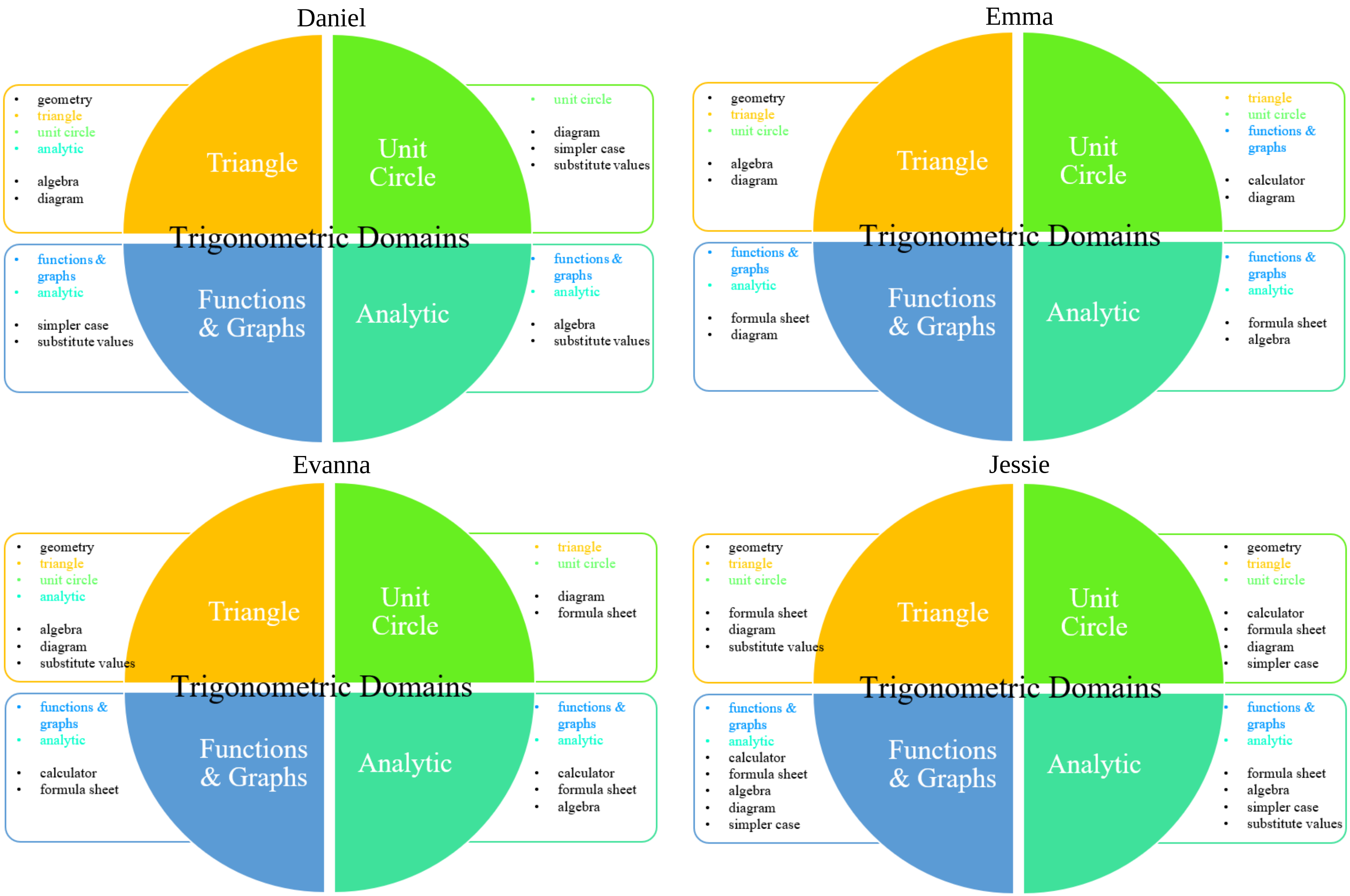

Figure 138

Participants' Resources and Heuristics Across Trigonometric Domains

Figure 138 shows the resources and heuristics used by all participants across all four trigonometric domains. One commonality that appears is that all participants used knowledge of trigonometric functions and graphs and analytic trigonometry in those two domains. All participants used knowledge from the greatest number of domains when solving triangle trigonometry tasks. All four participants used knowledge of geometry, triangle trigonometry, and unit circle trigonometry to solve tasks in the domain of triangle trigonometry. Daniel and Evanna also used knowledge of analytic trigonometry in the domain of triangle trigonometry. The domain of unit circle trigonometry had the greatest variation in the knowledge used to solve the tasks. Daniel used only knowledge of unit circle trigonometry, and Evanna used knowledge of triangle trigonometry and unit circle trigonometry. Emma and Jessie both used knowledge from three domains, with both using knowledge from triangle trigonometry and unit circle trigonometry. Emma also used knowledge of trigonometric functions and graphs, and Jessie used knowledge of geometry. For both domains of trigonometric functions and graphs and analytic trigonometry, all four participants used knowledge from both of those domains to solve the tasks. All of the participants, except for Daniel, used the formula sheet or the calculator in at least one domain. Both Evanna and Jessie used the formula sheet or calculator on at least three of the four domains.

There were a total of four heuristics used across all the tasks: algebra, diagrams, simpler cases, and substituting values. Daniel and Jessie both used all four heuristics across the nine tasks they solved. Evanna used all of the heuristics except for simpler cases, and Emma used just diagrams and algebra. Daniel, Emma, and Jessie tended to use the same heuristics across multiple domains. For example, Daniel used algebra in three different domains, Emma used diagrams in three different domains, and Jessie used a simpler case in three different domains.

However, Evanna only used heuristics in three domains total, with algebra in two domains and diagrams in two domains.

Triangle Trigonometry

For all participants, triangle trigonometry was the domain or one of the domains in which they drew upon knowledge from the greatest number of domains. All four drew upon their knowledge of geometry, triangle trigonometry, and unit circle trigonometry to solve tasks in the domain of triangle trigonometry. Additionally, Daniel and Evanna drew upon knowledge of analytic trigonometry. Geometry was the primary resource drawn upon to solve Task 9 for all four participants. They used knowledge of angle relationships to try to deduce the angle relationship they were looking to find. Triangle trigonometry was used on Tasks 2 and 4 by all participants, but only on Task 9 by the two participants who assumed special right triangles were present in the diagram. All four used knowledge of the unit circle on Task 2 and Task 4. This included finding values on the unit circle and orienting triangles within the unit circle to solve for missing values. Evanna considered using the sum to product identities on Task 9 but decided that would not help her solve the task.

The participants also used similar heuristics across the triangle trigonometry tasks. All four participants used diagrams, and three of the four participants used algebra. All four drew diagrams of triangles in the unit circle on Task 2, and three of the four drew similar diagrams on Task 4. For Task 2, these diagrams generally showed a triangle in the second quadrant and the reference triangle in the first quadrant. Participants then explained why the sine and cosine would have the same value for these two triangles, but different signs because of the quadrant they were located in. For Task 4, everyone but Evanna drew the specific triangle in the unit circle for $\sin\frac{3\pi}{4}$ and Jessie attempted to draw a triangle for $\sin\pi$. The three how used written algebra all used it on Task 9 to record relationships about angles in the rectangle that they noticed. They all wrote a series of equations about angles that summed to 45° or 180° or to other angles. This record of what relationships they had found helped them to reason about the relationship that they were trying to show.

Diagrams and algebra were the only two heuristics used by Daniel and Emma, but Evanna and Jessie both also used the heuristic of substituting values. Both substituted values on Task 9 and Jessie also substituted values on Task 2. For Task 9, both started by assuming one of the triangles in the given diagram was a special right triangle to attempt to simplify their calculations. Jessie ended up finding an inconsistency in her calculations based on her assumption, so knew that it was wrong, but could not determine another strategy. Evanna was also not confident in her strategy about substituting values and abandoned it. Jessie also substituted values on Task 2 to show the given equation was true. She assumed that the given triangle was a 30-60-90 triangle and found the side lengths to substitute into the equation and showed that it was true for those values.

Unit Circle Trigonometry

Unit circle trigonometry had the widest variety of resources and heuristics used across the participants. Daniel only used knowledge of unit circle trigonometry to solve the three tasks. Emma used knowledge of triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs. Evanna used knowledge of triangle trigonometry and unit circle trigonometry. Jessie used knowledge of geometry, triangle trigonometry, unit circle trigonometry. All four used knowledge of the unit circle on Tasks 3 and 4. Also, on Task 4, Emma, Evanna, and Jessie used knowledge of triangle trigonometry, and Jessie used knowledge of geometry. On Task 7, Emma used knowledge of trigonometric functions and graphs, but the other three did not use knowledge from any trigonometric domains. Emma and Jessie used the graphing calculator, and Evanna and Jessie used the formula sheet.

All four used diagrams, but the other heuristics utilized varied. Daniel used a simpler case and substituting values, and Jessie used a simpler case. Emma and Evanna did not use any heuristics other than diagrams. Diagrams were used on Task 3 by Jessie, Task 4 by Daniel, Emma, Evanna, and Jessie, and Task 7 by Daniel, Emma, and Jessie. For Task 3, Jessie drew a partial unit circle before switching to using the one on the formula sheet. The diagrams varied across Task 4. Daniel, Emma, and Jessie drew triangles within the unit circle or partial unit circles. Evanna and Jessie drew obtuse triangles to show why they could not be part of a right triangle. On Task 7, all three who drew diagrams created graphs showing $y = \sin(2x)$ as a horizontal shrink of $y = \sin x$, since the period was shorter for $y = \sin(2x)$. Daniel also used a simpler case on Task 7 when he described the relationship between $y = x^2$ and $y = (2x)^2$ to help explain the relationship between $y = \sin x$ and $y = \sin(2x)$. Similarly, Jessie used the simpler case of a linear function to describe the relationship of a horizontal shrink. Daniel also substituted values to explain why $y = \sin(2x)$ is a horizontal shrink of $y = \sin x$. “So, as you walk along $\sin x$, you get to, you get back to 0 in π seconds. But when you walk along the $\sin(2x)$, you get to 0 in $\frac{\pi}{2}$ seconds because you're walking twice as fast.”

Trigonometric Functions and Graphs

The domain of trigonometric functions and graphs was one of two domains in which all participants used knowledge from the same domains, trigonometric functions and graphs, and analytic trigonometry. All of the participants used knowledge of both domains on Task 1 and knowledge of just trigonometric functions and graphs on Task 5. Only Emma and Evanna used knowledge of trigonometric functions and graphs on Task 7. For Task 1, all participants began by creating a graph of sine and cosine. They then found translations and reflections of sine and cosine back to themselves and each other by considering the graph. Then all related their transformations to identities. For Task 5, all participants were able to calculate several parameters for the equations of the rabbit and fox data presented on a graph. Each participant found a different set of parameters. Jessie found only the vertical shift, Emma found the vertical shift and amplitude, Evanna found the period and vertical shift, and Daniel found all four parameters. On Task 7, Emma and Evanna related the horizontal shrink to decreasing the period of $y = \sin x$, but Daniel and Jessie relied upon other explanations. All of the participants except for Daniel used the formula sheet, and Evanna and Jessie both used the graphing calculator.

The heuristics used by the participants varied considerably across the tasks in this domain. Evanna used no heuristics, and Emma used only diagrams. Daniel used a simpler case and substituting values. Jessie used algebra, diagrams, and a simpler case. All three of the participants who used diagrams created them on Task 7. For this task, they drew graphs of $y = \sin x$ and $y = \sin(2x)$ to show why $y = \sin(2x)$ is a horizontal shrink from the parent function. Daniel and Jessie both also referred to a simpler case on Task 7. Daniel considered $y = x^2$ and $y = (2x)^2$ as a simpler case, and Jessie considered $y = x$ and $y = 2x$ as her simpler case. Both explained that these simpler cases would get steeper vertically, which would result in a horizontal shrink and explained that the same reasoning would apply to the graph of the sine function. Jessie used algebra on Task 1 when she was trying to find the identity that matched her translation between sine and cosine. She identified the translation as shifting 1 to the right or left, depending on which function was shifting onto the other. She was considering the Pythagorean Identity as the identity that match this pair of translations and wrote that $[\sin(x + 1)]^2 + [\cos(x - 1)]^2 = 1$ implies $x^2 + 2x + 1$ and $x^2 - 2x + 1$, so that the $2x$ and $-2x$ terms would cancel out. She decided this would not work, “because I couldn't prove it. So, it may work, but I don't know how to get from here [my answer] to [the Pythagorean Identity]. But it looks like you can, but I couldn’t.”

Analytic Trigonometry

Analytic trigonometry was the other one of the two domains in which all participants used knowledge from the same two domains, trigonometric functions and graphs, and analytic trigonometry. The two domains were both used by all participants on Task 1. On this task, participants all created a graph of sine and cosine, the found transformations mapping sine and cosine back to themselves and each other. After finding the transformations, participants had to find identities that match each transformation. All four participants were able to identify the periodic formulas as the identity for the translations and the even/odd formulas as the identity for the reflections. Daniel, Emma, and Evanna were able to identify the translations from sine to cosine and vice versa as the cofunction identities, but Jessie considered the Pythagorean Identity instead. All four used trigonometric identities in their work to verify the sum and difference angle identities. This included the use of the even and odd identities for sine and cosine as they verified the difference angle identities for sine and cosine, the cofunction identity and Pythagorean Identity as they verified the cosine sum angle identity, and rewriting tangent in terms of sine and cosine as they verified the tangent sum and difference angle identities. Emma and Evanna used knowledge of identities on Task 8 as they created their own versions of the proof. They then compared their proof to the one that the hypothetical student had created. Daniel and Jessie considered the proof presented without creating one of their own and did not use knowledge of analytic trigonometry on Task 8. Emma, Evanna, and Jessie all used the formula sheet as a resource on the analytic trigonometry tasks. Only Evanna used the graphing calculator as a resource to complete these tasks.

The heuristics that were used on the analytic trigonometry tasks also varied across participants. All four participants used algebra on the tasks. Daniel and Jessie both substituted values, and Jessie used a simpler case. Algebra was used by all participants on Task 6 as they verified the sum and difference identities, but this varied from one or two written steps to nearly a full page for a single identity. Especially for the identities that required more than one or two steps, the written algebra was a record of their thinking and progress towards verifying the identity. Emma and Evanna also used written algebra on Task 8 when they worked through the proof for themselves to verify whether or not it was true before attending to the hypothetical student's reasoning. Daniel used substituting values to verify his transformations on Task 1. When he was trying to find the reflection for sine, he worked through reflecting it over various lines with a point from the graph to see if it would end up back on sine. Also, when he was trying to determine whether he had correctly added or subtracted on the translations from sine to cosine and vice versa, he checked his work by plugging in a point on each graph. Jessie used both a simpler case and substituting values on Task 6. She began Task 6 by substituting $\alpha = 30°$ and $\beta = 60°$ into the given sine sum and difference identities. When these were evaluated, she noticed the sign changed in the solution, just as it changed in the identity. She then used the odd identity for sine to explain where the negative had come from in her solution.

> So it was $\sin(-\theta) = -\sin\theta$. … And that alone would end up a negative of the answer. So for the positive one, it was $\cos 30° \sin 60°$, but then because you negated β in the second point, then you would negate the answer. You would just turn the second part and not the α part.

Though the use of substituting values, Jessie was able to reason about the identities.

Attributes Across Trigonometric Domains

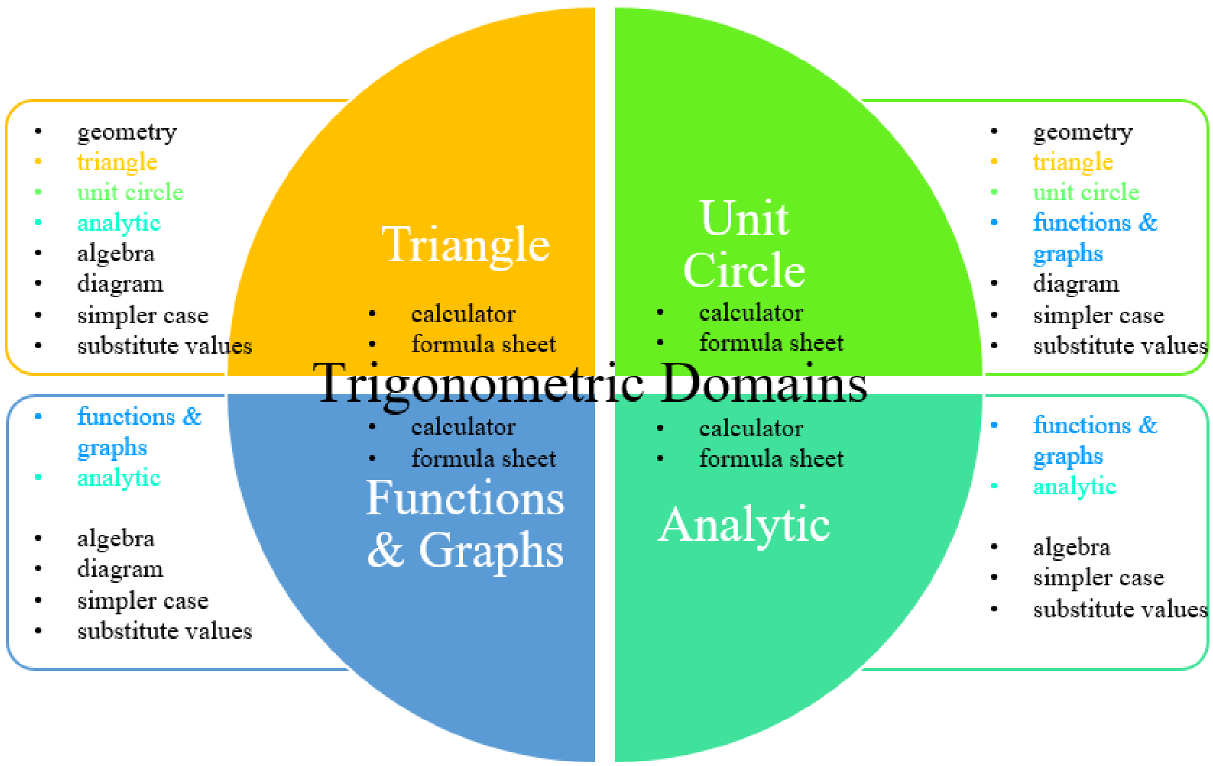

Figure 139

Participants Combined Use of Resources and Heuristics Across Trigonometric Domains

The participants used knowledge from the highest number of domains when solving tasks in the two domains of triangle trigonometry and unit circle trigonometry. Both of these domains were solved using knowledge from geometry, triangle trigonometry, and unit circle trigonometry. Additional knowledge from analytic trigonometry was used to solve tasks in the domain of triangle trigonometry, and knowledge from trigonometric functions and graphs was used to solve tasks in the domain of unit circle trigonometry. For triangle trigonometry and unit circle trigonometry, the connections to other domains happened both on the tasks that connected domains and on the tasks that were just addressing a single domain. Task 4 connected the domains of triangle trigonometry and unit circle trigonometry. Task 7 connected the domains of unit circle trigonometry and trigonometric functions and graphs. Thus, participants were connecting these two domains both when required by the task and when not required by the task. Triangle trigonometry was the only domain that participants used knowledge from another trigonometric domain when the task did not require connecting domains.

For both domains of trigonometric functions and graphs and analytic trigonometry, the only knowledge that was used to solve the tasks was from the domains of trigonometric functions and graphs and analytic trigonometry. All of the participants connected trigonometric functions and graphs and analytic trigonometry on Task 1. This task addressed both domains as participants graphed sine and cosine, then found a series of transformations that mapped the graphs onto themselves, and finally identified the trigonometric identities that matched each transformation. Across the participants, the graphing calculator and formula sheet were used in all four domains as resources.

Across all four domains, the heuristics used by participants were remarkably similar. The heuristics used were algebra, diagrams, simpler cases, and substituting values. Each of these heuristics was used in at least three of the four domains, with simpler cases and substituting values used in all four domains. Additionally, across the participants, all four heuristics were used to solve tasks in the domains of triangle trigonometry and trigonometric functions and graphs. The heuristics missing from the other two domains were algebra from unit circle trigonometry and diagrams from analytic trigonometry.

Research Question #3

How do secondary pre-service mathematics teachers engage in mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (mathematical noticing, mathematical reasoning, and mathematical creating) as they solve high cognitive demand trigonometric tasks?

The complete list of strands within mathematical proficiency and mathematical activity form the Mathematical Understanding for Secondary Teaching (MUST) Framework (Heid et al., 2015) are shown in Figure 140. Mathematical proficiency contains six strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge. All of the strands except for historical and cultural knowledge were addressed in the tasks. Mathematical activity contains three strands: mathematical noticing, mathematical reasoning, and mathematical creating. Each strand of mathematical activity is broken down into further strands. Mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Of these four strands, all were addressed in the tasks except for the structure of mathematical systems. Mathematical reasoning consists of three strands: justifying/proving, reasoning when conjecturing and generalizing, and constraining and extending. Mathematical creating also consists of three strands: representing, defining, and modifying/transforming/manipulating. The strands that were not addressed in the tasks were missing because the tasks were selected to address the four domains, rather than address all the strands.

Mathematical Proficiency	Conceptual Understanding	
	Procedural Fluency	
	Strategic Competence	
	Adaptive Reasoning	
	Productive Disposition	
	Historical and Cultural Knowledge	
Mathematical Activity	Mathematical Noticing	Structure of Mathematical Systems
		Symbolic Form
		Form of an Argument
		Connect Within and Outside Mathematics
	Mathematical Reasoning	Justifying/ Proving
		Reasoning when Conjecturing and Generalizing
		Constraining and Extending
	Mathematical Creating	Representing
		Defining
		Modifying/ Transforming/ Manipulating

Figure 140

Strands of Mathematical Proficiency and Mathematical Activity

The strands that each participant used to solve the trigonometric tasks are shown in Figure 141. The participants demonstrated some common strands. All participants demonstrated the procedural fluency strand of mathematical proficiency. Within mathematical activity, all participants demonstrated at least one strand from each of mathematical noticing, mathematical reasoning, and mathematical creating.

For mathematical proficiency, Evanna demonstrated only the strand of procedural fluency, but other participants demonstrated other strands. Jessie demonstrated an understanding of the two stands of procedural fluency and productive disposition. Emma demonstrated an understanding of four strands: conceptual understanding, procedural fluency, strategic competence, and productive disposition. Daniel demonstrated five of the six strands, including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

Within the three strands of mathematical activity, Emma, Evanna, and Jessie each demonstrated one or two of the strands within each of the strands of mathematical noticing, mathematical reasoning, and mathematical creating. All three demonstrated the strand of connect within and outside mathematics from the mathematical noticing strand and the representing and modifying/transforming/manipulating strands from the mathematical creating strand. Daniel was able to demonstrate nearly all the strands within mathematical noticing, mathematical reasoning, and mathematical creating, only missing the strands of structure of mathematical systems, reasoning when conjecturing and generalizing, and defining.

Daniel

Mathematical Proficiency	Conceptual Understanding	
	Procedural Fluency	
	Strategic Competence	
	Adaptive Reasoning	
	Productive Disposition	
Mathematical Activity	Mathematical Noticing	Symbolic Form
		Form of an Argument
		Connect Within and Outside Mathematics
	Mathematical Reasoning	Justifying/ Proving
		Constraining and Extending
	Mathematical Creating	Representing
		Modifying/ Transforming/ Manipulating

Emma

Mathematical Proficiency	Conceptual Understanding	
	Procedural Fluency	
	Strategic Competence	
	Productive Disposition	
Mathematical Activity	Mathematical Noticing	Symbolic Form
		Connect Within and Outside Mathematics
	Mathematical Reasoning	Justifying/ Proving
	Mathematical Creating	Representing
		Modifying/ Transforming/ Manipulating

Evanna

Mathematical Proficiency	Procedural Fluency	
Mathematical Activity	Mathematical Noticing	Symbolic Form
		Connect Within and Outside Mathematics
	Mathematical Reasoning	Justifying/ Proving
	Mathematical Creating	Representing
		Modifying/ Transforming/ Manipulating

Jessie

Mathematical Proficiency	Procedural Fluency	
	Productive Disposition	
Mathematical Activity	Mathematical Noticing	Form of an Argument
		Connect Within and Outside Mathematics
	Mathematical Reasoning	Constraining and Extending
	Mathematical Creating	Representing
		Modifying/ Transforming/ Manipulating

Figure 141

Participants' Mathematical Proficiency and Mathematical Activity Across Trigonometric Domains

Mathematical Proficiency

The percentage of strands of mathematical proficiency that were demonstrated by the participants varied, as seen in Figure 142. This figure shows what percentage of the five possible strands were demonstrated at any point during the nine tasks by each participant. Each strand was counted as demonstrated or not demonstrated.

Of the five potential strands that were addressed by the tasks, Daniel demonstrated all five, and Emma demonstrated four. However, Jessie only demonstrated two strands, and Evanna only demonstrated one strand. Daniel and Emma demonstrated conceptual understanding. Procedural fluency was the only strand that was demonstrated by all of the participants and was the only strand demonstrated by Evanna. Strategic competence was also only demonstrated by Daniel and Emma. Daniel was the only participant to demonstrate adaptive reasoning. Daniel, Emma, and Jessie demonstrated a productive disposition.

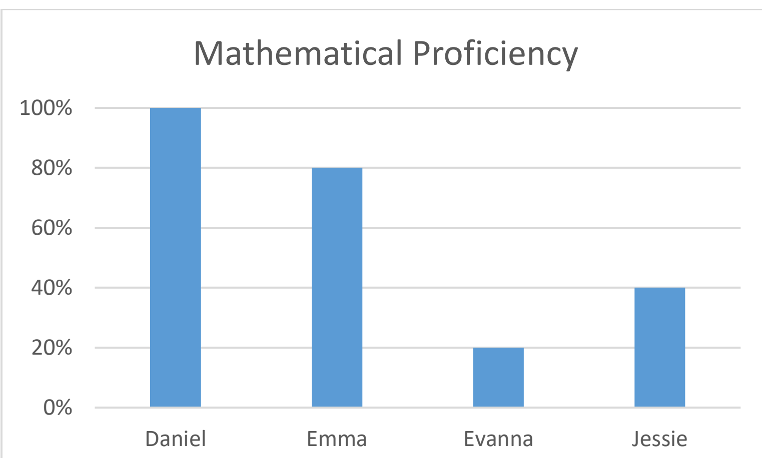

Figure 142

Percent of Mathematical Proficiency Strands Demonstrated by Participant

Only two of the participants, Daniel and Emma, were able to demonstrate conceptual understanding on the tasks. One instance that Daniel showed this was when he used the All Seniors Take Calculus or ASTC mnemonic on Task 2. Even though he initially brought up a memorized trick, he was able to explain that cosine is the x-value, so it has to be negative in the second quadrant. Emma showed conceptual understanding when she was able to connect knowledge across trigonometric domains on Task 4. On this task, she was able to explain that two congruent triangles, one in the first quadrant and one on the second quadrant, would have the same sine value because they would have the same y-coordinate on the unit circle. To show conceptual understanding, they need to demonstrate the "why" of what they were doing.

All four participants were able to demonstrate procedural fluency on different tasks and many showcased procedural fluency on more than one task. Daniel was able to demonstrate procedural fluency when calculating values of parameters for the equations of the rabbits and foxes on Task 5. He described a procedure to read information from the graph and perform calculations for each of the parameters. One example in which Emma demonstrated procedural fluency was on Task 6. For this task, Emma was able to work with the even and odd identities to simplify all of the difference angle identities from the sum angle identities. One task on which Evanna showed procedural fluency was Task 5. Like Daniel, for this task, she was able to calculate values for parameters in the equations modeling the numbers of rabbits and foxes from the data presented on the graph. Jessie was able to demonstrate procedural fluency on Task 2 when she set up an equation using SOH CAH TOA and solved for a missing side. She then used the missing side to find the value of cosine.

Daniel and Emma showed strategic competence. Strategic competence requires both procedural fluency and conceptual understanding. Since Daniel and Emma were the only two to demonstrate both of those strands, they were the only two who should have been able to show strategic competence. Daniel showed strategic competence on Task 6. For this task, he considered several possible identities before choosing the one that he considered to be the easiest and least messy as his first option. Emma demonstrated this strand on Task 9 when she attempted multiple solution paths. She was able to consider whether each path might be helpful and then pursued it if she thought it might help her to find the solution.

Daniel was the only participant who demonstrated adaptive reasoning while solving the tasks. He showed this on Task 2 when he adapted how he viewed the given equation $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$. His initial reasoning about the equation was to consider the Pythagorean Identity and trigonometric functions. Later, he was also able to consider the Pythagorean Theorem and geometric properties to verify the equation.

Three of the four participants, Daniel, Emma, and Jessie, were able to demonstrate a productive disposition. They primarily demonstrated this by their persistence in solving tasks that required multiple attempts. For example, they all spent at least 30 minutes on Task 9 and made multiple attempts to solve the task. However, they also made statements such as, “Now we're going to puzzle again” and “Just to kind of, I'm just trying to come up with something cool. But that wouldn't be too hard to figure it out” that showed they were productively engaged in solving the tasks.

Mathematical Activity

Mathematical activity consists of three strands of mathematical noticing, mathematical reasoning, and mathematical creating. All of the participants demonstrated at least one example of each strand of mathematical noticing, mathematical reasoning, and mathematical creating. The percentage of the nine total mathematical activity strands and the three strands each in mathematical noticing, mathematical reasoning, and mathematical creating that each participant demonstrated are shown in Figure 143. These percentages show whether each participant demonstrated the strand at any point in the interviews. They do not indicate a total number of instances of the strand.

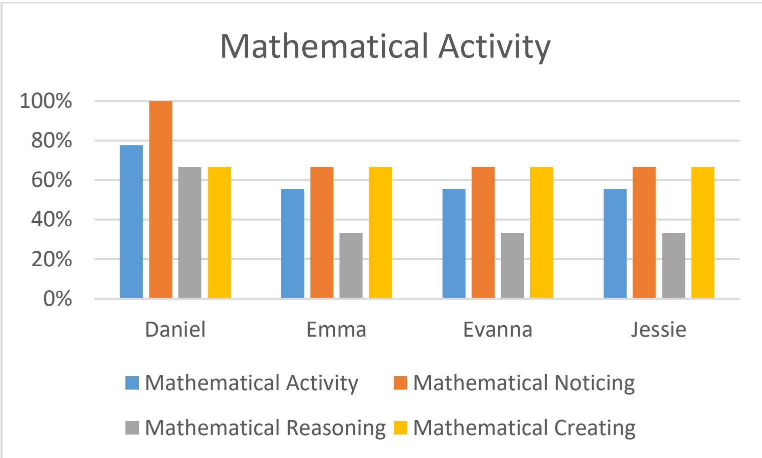

Figure 143

Percent of Mathematical Activity Strands Demonstrated by Participant

The participants consistently demonstrated mathematical noticing. Three of the four participants demonstrated each of the strands of symbolic form and form of an argument. All four participants demonstrated the strand of connect within and outside mathematics. The first strand of mathematical noticing, structure of mathematical systems, was not addressed within the tasks.

For mathematical reasoning, the strand of reasoning when conjecturing and generalizing was not demonstrated by any participants. The strand of justifying/proving was demonstrated by three of the participants, and the strand of constraining and extending was demonstrated by two of the participants.

The last strand of mathematical activity, mathematical creating, was the strand that was most consistently demonstrated across the participants. All four participants demonstrated both strands of representing and modifying/transforming/manipulating. None of the participants demonstrated the strand of defining.

Mathematical Noticing

Mathematical noticing consists of four strands: structure of mathematical systems, symbolic form, form of an argument, and connect within and outside mathematics. Daniel demonstrated the three strands of symbolic form, form of an argument, and connect within and outside mathematics. Emma, Evanna, and Jessie all demonstrated two of the strands, but in different combinations. The strand of structure of mathematical systems was not addressed by any of the tasks and, as a result, was not demonstrated by any of the participants.

Understanding of the symbolic form of trigonometric functions was demonstrated by Daniel, Emma, and Evanna on Task 6. All three were able to work with the sum and difference identities for sine, cosine, and tangent on this task without overapplying the distributive property. For example, Emma wrote $\sin(\alpha - \beta) = \sin(\alpha + -\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$ and then showed how to used the even and odd identities to simplify the $-\beta$s into $\sin\alpha\cos\beta - \cos\alpha\sin\beta$. Jessie demonstrated a lack of understanding of the symbolic form on Task 6. For this task, Jessie suggested one way to simplify $\tan(60° - 30°)$ was to distribute the tangent to get $\tan 60° - \tan 30°$. This over-application of the distributive property showed she did not understand the symbolic form of trigonometric functions.

Daniel and Jessie both demonstrated form of an argument on Task 6 and Task 8. Daniel demonstrated this strand on Task 6. For the tangent sum angle identity on Task 6, Daniel ended up working backward to verify the identity after getting stuck working forward. To formalize his proof, he explained that "… you just read it backwards." Jessie demonstrated an understanding on Task 6 as well, though in a different way. She used a specific case to examine why the sign changed from the sine sum angle identity to the sine difference angle identity but stated, "… one example wouldn't prove that." On Task 8, both recognized that by multiplying both sides of the equation, the hypothetical student was assuming what she was trying to prove. Jessie stated, “so you have to start with something that's true and … show the steps that you get to the statement to really prove mathematically that it's true.” Both Daniel and Jessie demonstrated knowledge that assuming what is to be proving does not create a valid proof. Both were able to extend this knowledge to describe how to turn a proof that assumes what it is trying to prove into a form that makes it valid.

Connect within and outside mathematics was one of the strands that was demonstrated by all four participants. For all four participants, this strand was clearly demonstrated on Task 5. This task specifically asked participants to connect the mathematics of the task to the real-world by providing an explanation of why the graphs of the rabbits and foxes seemed to chase one another. All of the participants provided an explanation that relied upon the predator-prey relationship between the two types of animals, and many provided this explanation before it was explicitly asked in the task. For example, Jessie described,

> So, like in terms of a real-world situation, … there would be few rabbits, so few foxes could feed, and few foxes could survive. And then as the rabbits increase, they have more food. And so more foxes can survive. But then you have a ton of foxes, so they're finishing off the rabbits, so they'd die off, and then the foxes die off and so on. So, kind of in a chase, like the rabbits want to grow, but then the foxes grow. That makes the rabbits decrease.

The other three participants were similarly able to describe this real-world relationship to the data presented in the task and reason about why the two populations did not increase and decrease simultaneously.

Mathematical Reasoning

There are three strands within mathematical reasoning: justifying/proving, reasoning when conjecturing and generalizing, and constraining and extending. Daniel demonstrated two strands: justifying/proving and constraining and extending. Emma, Evanna, and Jessie each demonstrated one strand, either justifying/proving or constraining and extending. None of the participants demonstrated the strand of reasoning when conjecturing and generalizing.

Daniel, Emma, and Evanna demonstrated the strand of justifying/proving. Daniel demonstrated this strand on Task 8 when he explained how he would help the hypothetical student formalize her proof. “I mean, if she did this when it was originally written here, I would just tell her that she was on the right track, but her thinking was backwards.” He recognized that her thinking was sound, but that she needed to reconsider how it was written to make it formal and valid. Emma was able to demonstrate this strand on Task 6 with her proofs of the sum and difference angle identities. Though she did not explicitly explain why her proofs were valid, she was able to create sequences of logically connected mathematical statements starting with the given identity and resulting in the identity she was trying to show. Evanna also demonstrated this strand on Task 6, though her reasoning was more informal than Emma’s proofs. She was able to describe how the even and odd identities would result in the sign changing from the sine and cosine sum angle identities to the sine and cosine difference angle identities.

Both Daniel and Jessie demonstrated the strand of constraining and extending. For Daniel, he considered the effects of constraining and extending the range of the arcsine function on Task 1. He explained that by constraining the arcsine function, it would remain a function, but if it were not constrained, then the graph would continue up and down the y-axis. Jessie attempted to generalize a result she obtained on Task 6 by substituting values. She noticed that when she substituted specific values, the sign changed in her solution just as it did in the provided identities. However, she was not able to explain how to generalize this result and stated, "… one example wouldn't prove that."

Mathematical Creating

Within mathematical creating, there are three strands: representing, defining, and modifying/transforming/manipulating. All four of the participants demonstrated the same two strands: representing and modifying/transforming/manipulating. None of the participants demonstrated the strand of defining.

The strand of representing was demonstrated by all four participants. The primary way that all of the participants demonstrated representing was by creating diagrams to help explain their thinking. On Task 4, for example, Daniel created a diagram of what he called "bow-tie" angles. With this diagram, he was able to explain that all of the angles had the same reference angle in the first quadrant, and thus their sine and cosine values would be the same number with just a sing change based on the quadrant in which they were located. Emma was able to create a representation of her thinking on Task 2. She created a diagram showing two triangles, one in the second quadrant and the reference triangle in the first quadrant. She labeled the angle in the second quadrant as θ and the angle in the first quadrant as $\pi - \theta$ and explained that they would have the same value of cosine, but the one in the second quadrant would be negative since the x-value is negative in the second quadrant. Evanna also created a representation on Task 2. Initially, she drew a triangle that would have been oriented in the first quadrant and solved for the missing side necessary to find the cosine value. When she noticed that the triangle was supposed to be in the second quadrant, she redrew the triangle and realized that he cosine value would be negative, since the x-value would be negative in the second quadrant. Jessie also created representations on Tasks 2 and 4. For both of these tasks, she created diagrams showing the triangles she was referencing within a unit circle.

All four participants were able to demonstrate the strand of modifying/transforming/manipulating. This strand was primarily demonstrated in one of two ways. These two ways were transforming between representations of trigonometric concepts from one domain to another and algebraically manipulating trigonometric expressions. Daniel was able to demonstrate this stand in both of these ways. On Task 1, he was able to transform from a graphical representation of sine and cosine to a function notation of a transformation to an algebraic representation of an identity. On Task 6, he was able to work with complicated algebraic manipulations to expand $\sin^2(\alpha+\beta)$ into $[\sin\alpha\cos\beta+\cos\alpha\sin\beta]^2$ into $\sin^2\alpha\cos^2\beta+2\sin\alpha\cos\beta\cos\alpha\sin\beta+\cos^2\alpha\sin^2\beta$. Emma demonstrated this strand on the same tasks as Daniel. She was able to start with the graph of sine, then identify a transformation that would map sine back to itself, and finally determine an identity that would match her transformation. She also was able to perform algebraic manipulations on Task 1, explaining why the even and odd identities could be used to simplify the difference angle identities from the sum angle identities for sine and cosine. Evanna also demonstrated this strand on the same tasks. Like Daniel and Emma, she was able to transform a graph of sine or cosine into an identity that represented a transformation mapping the function onto itself on Task 1. On Task 6, she was able to perform algebraic manipulations on the tangent sum angle identity to transform it using sines and cosine into the form she was looking for. Jessie demonstrated this strand with algebraic manipulations on Tasks 6 and 9. For Task 6, like the other participants, she was able to manipulate the sum and difference identities algebraically. Specifically, she worked backward from the solution on the tangent sum angle identity by writing it in terms of sine and cosine to get back to a form with tangents in it. On Task 9, Jessie was able to try several algebraic manipulations to find missing sides or angles that might help her to solve the task.

Mathematical Proficiency and Mathematical Activity Across Participants

The strands of mathematical proficiency and mathematical activity that each participant was able to engage in and demonstrate varied. Daniel and Emma were able to engage in the majority of the strands of mathematical proficiency, but Evanna and Jessie did not demonstrate the majority of the strands. Conceptual understanding and strategic competence were only shown by two participants, and adaptive reasoning was only shown by one participant. Some examples of conceptual understanding that were demonstrated included: reasoning across multiple trigonometric domains and explaining why a procedure or mnemonic worked. All participants were able to engage in showing procedural fluency of trigonometric concepts. Procedural fluency was demonstrated across tasks and domains. Some examples of procedural fluency that were demonstrated included: solving a right triangle, evaluating trigonometric expressions using the unit circle, calculating values of parameters for a trigonometric equation from a graph, and verifying the steps of a provided trigonometric proof. Three of the four participants showed a productive disposition to engage in problem-solving on trigonometric tasks. This also showed up across domains. Daniel, Emma, and Jessie were all eager to work on the tasks and persisted when the tasks were challenging.

The percentage of strands of mathematical proficiency that each participant was able to demonstrate across the tasks varied greatly, as seen in Figure 142. All six strands of mathematical proficiency are essential to the ability to teach trigonometric concepts to students. However, only two of the participants demonstrated more than half of the strands that were addressed in the tasks, and two only demonstrated one or two strands. As these pre-service secondary mathematics teachers prepare to teach trigonometric concepts to students, they need opportunities to address the gaps in their mathematical proficiency with trigonometry.

There was more commonality in the strands of mathematical activity than in the strands of mathematical proficiency that participants were able to demonstrate while working on the tasks. All participants were able to demonstrate the strands of connect within and outside mathematics, representing, and modifying/transforming/manipulating. For all participants, the strand of connect within and outside mathematics was demonstrated on Task 5, since the task explicitly asked them to provide a real-world explanation for the trends in the rabbit and fox data. The participants were able to demonstrate representing across tasks and domains. Some examples of the strand of representing that were demonstrated by the participants include: drawing and labeling a triangle for $\sin\theta = \frac{8}{17}$, creating a diagram showing relationships of different angles in the unit circle, and creating a graph of two functions to show their connection. The strand of modifying/transforming/manipulating was primarily shown by transforming from one representation to another on Task 1 and by algebraic manipulations on Task 6. Some examples of the strand of modifying/transforming/manipulating that were shown by the participants include: transforming a representation between two or more domains and performing algebraic manipulations on trigonometric expressions or equations.

For mathematical activity, Daniel was able to demonstrate seven of the nine possible strands, but Emma, Evanna, and Jessie were only able to demonstrate five of the nine possible strands. Understanding and being able to demonstrate all nine strands is important to these participants' understanding of trigonometry. With three of the four participants demonstrating just over half of the strands, they need opportunities to address gaps in their mathematical activity of trigonometry before they are in the classroom teaching trigonometry to students.

Chapter 6: Discussion and Conclusion

The purpose was to examine how pre-service secondary mathematics teachers solve high cognitive demand trigonometry tasks. The work of the participants was analyzed using the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005) and the Mathematical Understanding for Secondary Teaching Framework (Heid et al., 2015). Together these frameworks examined the problem-solving process and mathematics content that the participants used to solve high cognitive demand trigonometry tasks. The study addressed the following three research questions:

Research Question #1: How do secondary pre-service mathematics teachers engage in problem solving phases (e.g., orienting, planning, executing, checking) when solving high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #2: How do secondary pre-service mathematics teachers draw upon problem solving attributes (e.g., resources, heuristics, affect, monitoring) as they solve high cognitive demand trigonometric tasks?

a. Are there differences for triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry?

Research Question #3: How do pre-service secondary mathematics teachers engage in mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and historical and cultural knowledge) and mathematical activity (mathematical noticing, mathematical reasoning, and mathematical creating) as they solve high cognitive demand trigonometric tasks?

This chapter positions the results from this study within the existing literature on the teaching and learning of trigonometry. It also provides implications for the mathematics teacher educators who prepare pre-service secondary mathematics teachers to teach and researchers who study the teaching and learning of trigonometric concepts.

Implications for Mathematics Teacher Educators

Past research has shown that pre-service secondary mathematics teachers have procedural fluency of trigonometry, but lack conceptual understanding (Bryan, 1999; Naidoo & Naidoo, 2009; Topçu et al., 2006; Tuna, 2013). This research agrees with the prior research on pre-service secondary mathematics teachers' knowledge of trigonometry. All of the participants were able to demonstrate procedural fluency when solving high cognitive demand trigonometry tasks, but only two were able to demonstrate a conceptual understanding of trigonometry. The same two participants were able to demonstrate strategic competence and adaptive reasoning. Some examples of the procedural fluency that participants were able to demonstrate on the tasks included using special right triangles to find lengths of sides of a triangle, use the unit circle to evaluate trigonometric expressions, calculating the value of the amplitude and vertical shift from a graph, and rewriting trigonometric expressions. Procedural fluency was present across all four domains and all four participants.

The two participants who were able to demonstrate conceptual understanding, strategic competence, and adaptive reasoning were able to showcase a deeper understanding of the "why" in the procedures they were executing and strategically choose methods to solve the tasks. One example of demonstrating conceptual understanding came from Daniel. He was able to reason about translation on the graph and then write his solution as the identity $\cos(x + 2\pi) = \cos x$. When he identified the periodic identity of $\cos(x + 2\pi n)$ as representing this, he explained, “since it repeats every 2π and you're just shifting everything over to the left.” He understood that it could be shifted any number of 2π, rather than just the one option he had initially written. For strategic competence, both of the participants who demonstrated this strand considered whether a method would be easier to perform or lead them to a solution when they were solving tasks. For example, on Task 9, Emma explained the multiple formulas she had considered using.

> I'm just like, my brain wants to go to law of sines or, which is what I've been doing, or law of cosines but those really don’t help. Like law of sines, I liked that it had more equal signs, so I had the necessity for less variables, but none of my like triangles that I could work with have enough variables that don't go to zero. Um, because it's defined. So that does have some sort of value on the bottom. But because the numerator goes to zero, it doesn't help me find that value. Then you figure out what you want and then law of cosines has too many variables that are missing. Um, I was hoping the sine of some of the angles might like, well I have this, but that's so symbolic that I'm not sure I can get much from it.

She had considered using the law of sine, law of cosines, or SOH CAH TOA, but found that all three had too many unknowns to be able to solve. Her strategic competence is shown by considering and eliminating three possible solution paths.

Since research shows that pre-service secondary mathematics teachers rely primarily on procedural knowledge in trigonometry, they need opportunities to reexamine their understanding of trigonometry as part of their preparation to teach. Rather than adding one more thing to the already limited time that is available in preparation programs, a focus on trigonometry in a mathematics content course can build an understanding of trigonometry concepts while also bridging the gap between algebra and geometry (Calzada & Scariano, 2006). For example, pre-service secondary mathematics could spend a lesson learning how to build the sum and difference identities from segments on the unit circle, as seen in Figure 144.

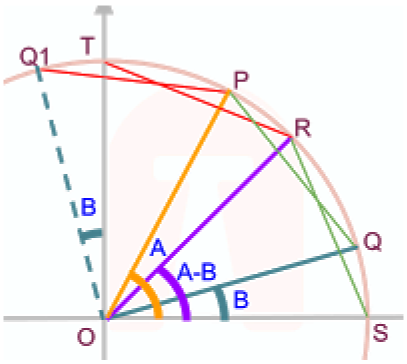

Figure 144

Diagram to Visualize the Sum and Difference Identities

Other activities could include the ones like Moore and Thompson used with pre-service secondary mathematics teachers and in-service secondary mathematics teachers. Moore and colleagues (Moore et al., 2013, 2016; Moore, Paoletti, et al., 2014) found that presenting pre-service secondary mathematics with opportunities to consider the meanings of radians, the unit in the unit circle, and covariational reasoning resulted in a greater understanding of the concepts. Thompson et al. provided opportunities for teachers to consider the input and output of trigonometric functions, which can deepen their understanding of other types of functions. Research has shown that when pre-service secondary mathematics teachers are presented with opportunities to deepen their learning of trigonometric concepts, they are able to build deep conceptual understanding (Moore et al., 2013, 2016; Moore, Paoletti, et al., 2014; Thompson et al., 2007). This deep conceptual understanding would include connecting trigonometry across domains and understanding the relationships between the domains, as well as developing the strands of mathematical proficiency and mathematical activity that the participants were not able to consistently demonstrate. For example, a pre-service secondary mathematics teacher would be able to define sine in all four domains. This would include the right triangle trigonometry definition, the unit circle definition, and the function definition, as well as understand how these all fit together in a way that would make sense to their students.

Allowing pre-service secondary mathematics teachers opportunities to deepen their understanding of trigonometric concepts is essential. This is because pre-service secondary mathematics teachers with deeper content knowledge have been shown to create lesson plans that include more opportunities for students to develop conceptual understanding of mathematics (Cavey & Berenson, 2005; Kahan et al., 2003). Pre-service secondary mathematics teachers need opportunities to deepen their knowledge and confidence in trigonometry before they are in the classroom where they are expected to teach it.

> A major factor, as we see it, is whether we can develop confidence in ourselves as … teachers, to think deeply about making sense in mathematics, so that we may encourage learners to have confidence in their own sense-making and use that confidence to complement the power of supportive experience to have the determination to make sense of problematic new ideas (Chin & Tall, 2012, p. 1-7).

If pre-service secondary mathematics teachers are given opportunities to develop confidence in their own understanding of trigonometry, they will feel more confident to teach trigonometry in their future classrooms.

Implications for High School Mathematics Teachers

Research shows that students have difficulties with trigonometry (Gür, 2009; Kendal & Stacey, 1998; Orhun, 2004; Ross et al., 2011). These difficulties include not retaining knowledge of trigonometry from one school year to the next (Kendal & Stacey, 1998) and having a primarily procedural understanding of trigonometry (Weber, 2005; Yiğit Koyunkaya, 2016). One hypothesis about why these difficulties occur is that students do not comprehend the connections between the trigonometric domains (Markel, 1982; May & Courtney, 2016; Peterson et al., 1998; Thompson, 2008).

Based on the results of this research, teachers should used triangle trigonometry tasks with students to encourage them to connect their knowledge across trigonometric domains. The participants in this study used knowledge from three trigonometric domains and geometry to solve the three triangle trigonometry tasks. Triangle trigonometry was also one of two domains in which all four heuristics were used. Putting these two results together implies that triangle trigonometry tasks provide an accessible way for students to connect domains and use a variety of heuristics.

Since triangle trigonometry invited connections to other domains for the participants in this study, it could be used by high school mathematics teachers to help their students make connections across domains. Tasks that specifically addressed multiple domains supported the participants in connecting knowledge across domains. For example, Task 2 began by connecting the Pythagorean Theorem and the Pythagorean Identity to a right triangle. The task then asked participants to solve a problem using the unit circle. With mindful questioning from the teacher, students could use this task to connect knowledge of geometry with three trigonometric domains: triangle trigonometry, unit circle trigonometry, and analytic trigonometry. Tasks, such as Task 9, that were open-ended also provided opportunities for the participants to connect across domains. Though most of the participants attempted to use their knowledge of geometry and angles, some attempted to use triangle trigonometry and analytic trigonometry as they solved for missing information in the diagram. By providing an open-ended task with multiple solution paths, students can use the connections that they and their classmates see and build a stronger understanding of the connections between trigonometric domains.

Traditionally, students have first learned about trigonometry through triangle trigonometry in a second-year high school mathematics course. Thus, when students learn about unit circle trigonometry and trigonometry functions and graphs in their third-year high school mathematics course and about analytic trigonometry in their fourth-year high school mathematics course, teachers should begin by building upon the students' knowledge of triangle trigonometry. For example, teachers can build the unit circle with students by using their knowledge of special right triangles. Teachers could then build from the unit circle to introduce graphs of trigonometric functions. As a result of this sequencing, students would see connections between the domains of triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs. This sequencing would also help students to move beyond just procedural fluency with trigonometry.

Implications for Future Research

There are multiple possible directions for future research based on the results of this research. Since there were differences in the background of the participants in this study, future research could examine what backgrounds and experiences support pre-service secondary mathematics teachers to have a greater understanding of trigonometry. This could include comparing pre-service secondary mathematics teachers who are double majoring in mathematics and mathematics education, like Daniel, pre-service secondary mathematics teachers who are preparing to teach middle grades, like Jessie, or pre-service secondary mathematics teachers who have taught trigonometry as part of their student teaching. Other background knowledge to investigate, might include knowledge of algebra, functions, and geometry, since those are all related to trigonometry. Knowing what background or experiences and knowledge support pre-service secondary mathematics teachers to understand and problem-solve trigonometry could help design programs or courses for future pre-service secondary mathematics teachers.

One limitation of the study was that the participants might have taken their last trigonometry course several semesters or even years prior to their participation in the study. In order to determine whether having taken a trigonometry course more recently would impact the results, the study could be repeated with participants who had taken trigonometry more recently. Since many pre-service secondary mathematics teachers begin their college mathematics course, one option would be to recruit participants who are in their first year of college and took trigonometry as part of their coursework as a senior in high school. Another option would be to compare pre-service secondary mathematics teachers with college students who have just completed a trigonometry course. Both options would address the limitation of participants not having had a trigonometry course recently.

One possibility would be to design an intervention for pre-service secondary mathematics teachers on trigonometric content that could be taught within a mathematics content course to address connecting trigonometry across the four domains. The intervention could teach trigonometric concepts and support pre-service secondary mathematics teachers to develop strong understandings of each of the four domains as well as their connections. This could include explicitly teaching the pre-service secondary mathematics teachers about different heuristics and problem-solving strategies, such as those used by the participants in the study, and then allowing them to apply those heuristics and problem-solving strategies as they work through trigonometry tasks. The intervention could also support the pre-service secondary mathematics teachers' development of problem-solving strategies. The pre-service secondary mathematics teachers who complete the course could participate in a combination of testing and task-based interviews, like the participants in this study, before and after completing the coursework to measure the impact of the coursework. These pre-service secondary mathematics teachers could also be followed longitudinally into student teaching and their first years in the classroom to determine the impact on their teaching.

Another possibility for future research would be to design a module for in-service secondary mathematics teachers about problem-solving in trigonometry. As part of the module, the in-service secondary mathematics teachers could learn about the results of this research and consider how it impacts the trigonometry tasks they choose to use in their classrooms. There are multiple ways to research the impact of the module. One option would be to research how and why in-service secondary mathematics teachers make choices about what trigonometry tasks to use before and after completing the module. Another option would be to study the students of the teachers who complete the module and analyze their trigonometric learning and knowledge.

Limitations

One limitation of this study was the use of pre-service secondary mathematics teachers who were selected from a convenience sample. Though these participants had a variety of backgrounds, from a freshman to a senior, preparing to teach middle grades and high school, and majoring in only mathematics education or double majoring in mathematics education and mathematics, they may not be representative of all pre-service secondary mathematics teachers since they were selected from a convenience sample. However, the participants demonstrated a range of knowledge of trigonometry and problem-solving strategies when solving trigonometric tasks. Future research should examine a broader range and more purposeful sample of pre-service secondary mathematics teachers.

Another limitation of the study was treating trigonometry as consisting of four distinct domains, triangle trigonometry, unit circle trigonometry, trigonometric functions and graphs, and analytic trigonometry. However, there is a meaningful overlap in content between the four domains. For example, the use of special right triangles shows up in the domains of triangle trigonometry and unit circle trigonometry. Thus, the knowledge of trigonometry across domains cannot be separated, and the pre-service secondary mathematics teachers who participated in the study did use knowledge across domains.

An additional limitation is the nature of qualitative research. Because of the interactions between the individual interviewer and participants, the results are difficult to generalize (Goldin, 2000; Koichu & Harel, 2007). The interviewer may ask questions that influence the thinking and reasoning of the participants, so their work cannot be separated from the interaction with the interviewer. Though the cases presented were chosen to represent the range of work done by all pre-service secondary mathematics teachers who completed the interviews in this study, they may not represent what all pre-service secondary mathematics teachers would do on different tasks or with different interviewers.

Conclusion

The phases that participants progressed through varied by task and domain. One common feature of the phases was that most participants began most tasks with a cycle that matched the hypothetical cycle from Carlson and Bloom (2005). This means that participants began with an orienting phase, then cycled through a phase of planning, executing, and checking. On the tasks that contained more than one part, this cycle was generally seen at the beginning of each part of the task. After this initial cycle, participants phases were much more varied. Other than the initial cycle, there were no clear patterns of phases across participants or domains.

Participants use a variety of resources and heuristics across the tasks. When solving triangle trigonometry tasks, they were able to draw upon knowledge from the domains of triangle trigonometry, unit circle trigonometry, and analytic trigonometry, as well as geometry. Similarly, when solving unit circle trigonometry, the participants were able to draw upon knowledge from the domains of triangle trigonometry, unit circle trigonometry, and trigonometric functions and graphs, as well as geometry. However, for solving tasks in the domains of trigonometric functions and graphs and analytic trigonometry, the participants were only able to draw upon knowledge from those two domains. Participants used a total of four different heuristics across the tasks: algebra, diagrams, simpler cases, and substituting values. All four heuristics were used to solve tasks in the domains of triangle trigonometry and trigonometric functions and graphs. Three of the four heuristics were used in the other two domains, with no algebra in the domain of unit circle trigonometry and no diagrams in the domain of analytic trigonometry.

Participants demonstrated a variety of strands of mathematical proficiency and mathematical activity from the Mathematical Understanding for Secondary Teaching Framework (Heid et al., 2015). Within mathematical proficiency, all four participants were able to demonstrate procedural fluency in trigonometry, three were able to demonstrate a productive disposition for solving trigonometry tasks, two were able to demonstrate conceptual understanding and strategic competence, and only one was able to demonstrate adaptive reasoning. Within mathematical activity, all four participants were able to demonstrate the strands of connect within and outside mathematics, representing, and modifying/transforming/manipulating. The other phases of symbolic form, form of an argument, justifying/proving, and constraining and extending were all demonstrated by either two or three of the participants. This shows that the participants have knowledge of trigonometry and are able to operate mathematically with trigonometry.

Research on trigonometry has consistently shown that it is difficult for students and teachers (Blackett & Tall, 1991; Bressoud, 2010; Gür, 2009; Topçu et al., 2006; Tuna, 2013). Thus, mathematics teacher educators need to support pre-service secondary mathematics teachers in deepening their knowledge of trigonometry. Experiences within mathematics content courses should support pre-service secondary mathematics teachers to build connections between the four trigonometric domains and between the algebra, function, and geometry standards in high school mathematics. Since pre-service secondary mathematics teachers who have a greater knowledge of content have been shown to plan lessons that provide better learning opportunities for students (Cavey & Berenson, 2005; Kahan et al., 2003), the pre-service secondary mathematics teachers who deepen their own understanding of trigonometry will be better prepared to teach trigonometry in their future classrooms.

Future research should continue to examine the trigonometric understanding of pre-service secondary mathematics teachers and in-service secondary mathematics teachers. Past research has shown that trigonometry is difficult for students and teachers (Demir & Heck, 2013; Naidoo & Naidoo, 2009; Wulandari et al., 2018; Yi et al., 2013). Possible avenues for future research include repeating the study with purposefully selected pre-service secondary mathematics teachers from different backgrounds or with students who have taken trigonometry in the last semester or year, designing an intervention for pre-service secondary mathematics teachers to address their content knowledge of trigonometry, and designing a module for in-service secondary mathematics teachers to support their teaching of trigonometry. All of these would build upon this study, while also addressing the scarcity of research on trigonometry.

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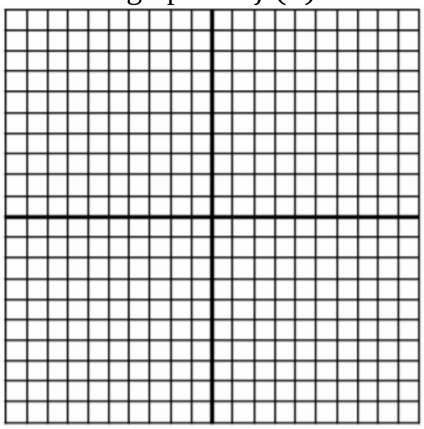
APPENDICES

Appendix A: Interview Tasks by Domain

Interview	Task	Triangle Trigonometry	Unit Circle Trigonometry	Trigonometric Functions and Graphs	Analytic Trigonometric	Connections Between Trigonometric Domains
1	1			X	X	X
1	2	X				
1	3		X			
2	1	X	X			X
2	2			X		
2	3				X	
3	1		X	X		X
3	2				X	
3	3	X				

Appendix B: Interview Tasks

Interview 1
Task 1[1]
a. Sketch graphs of $f(x) = \cos x$ and $g(x) = \sin x$.



b. Find a translation of the plane which maps the graph of f to itself.
Find a reflection of the plane which maps the graph of f to itself.
What trigonometric identities are associated with your translation and reflection?

c. Find a translation of the plane which maps the graph of g to itself.
Find a reflection of the plane which maps the graph of g to itself.
What trigonometric identities are associated with your translation and reflection?

d. Find a reflection and/or translation of the plane which maps the graph of f to the graph of g.
Find a reflection and/or translation of the plane which maps the graph of g to the graph of f.
What trigonometric identities are associated with your translation and reflection?

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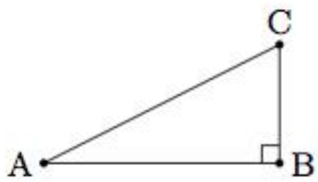
[1] Adapted from https://www.illustrativemathematics.org/content-standards/HSF/TF/A/2/tasks/1698

Interview 1
Task 2[2]



a. In the triangle pictured above show that $\left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1$.

b. Deduce that $\sin^2\theta + \cos^2\theta = 1$ for any acute angle θ.

c. If θ is in the second quadrant and $\sin\theta = \frac{8}{17}$ what can you say about $\cos\theta$? Draw a picture and explain.

[2] Adapted from https://www.illustrativemathematics.org/content-standards/HSF/TF/C/8/tasks/1693

Interview 1
Task 3[3]

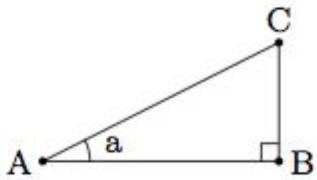
Add a fourth member to the set so that each group of three excludes the fourth. Explain why each group of three excludes the fourth.

$\sin(150^{\circ})$	$\sin(225^{\circ})$
$\cos(120^{\circ})$	**?**

[3] Adapted from http://wodb.ca/is.html

Interview 2
Task 1[4]

Below is a picture of a right triangle with α as the measure of angle A:



Joyce knows that the sine of α is the length of the side opposite A divided by length of the hypotenuse:
$\sin\alpha = \frac{|BC|}{|AC|}$

Joyce says, "the sine of an obtuse angle does not make any sense because I can't make a right triangle with an obtuse angle."

a. Draw a picture and explain how Joyce might define the sine of an obtuse angle.

b. What are $\sin\frac{3\pi}{4}$ and $\sin\pi$? Why?

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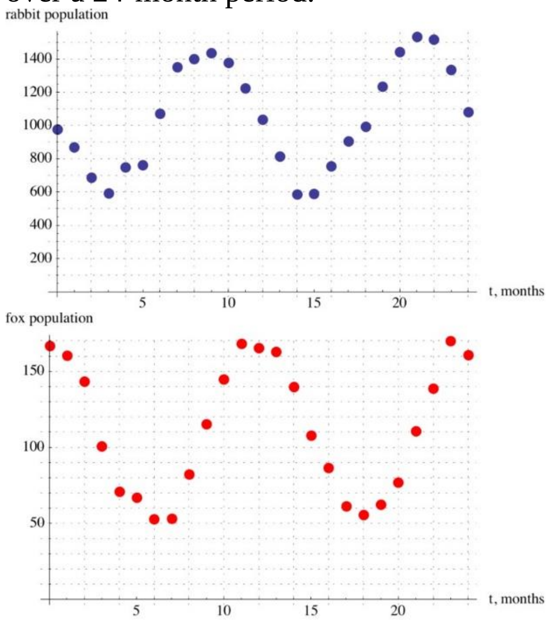
[4] Adapted from https://www.illustrativemathematics.org/content-standards/HSF/TF/A/2/tasks/1692

Interview 2
Task 2[5]

Given below are two graphs that show the populations of foxes and rabbits in a national park over a 24-month period.



a. Explain why it is appropriate to model the number of rabbits and foxes as trigonometric functions of time.

b. Find an appropriate trigonometric function that models the number of rabbits, r(t), as a function of time, with t in months.

c. Find an appropriate trigonometric function that models the number of foxes, f(t), as a function of time, with t in months.

d. Graph both functions on the same coordinate plane and give at least one possible explanation why one function seems to "chase" the other function.

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[5] Adapted from https://www.illustrativemathematics.org/content-standards/HSF/TF/B/5/tasks/817 and https://www.illustrativemathematics.org/content-standards/tasks/816

Interview 2
Task 3[6]

In this task, you will show how all of the sum and difference angle formulas can be derived from a single formula when combined with relations you have already learned.
For the following task, assume that the sum angle formula for sine is true. Namely,
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
a. To derive the difference angle formula for sine, use the even and odd identities to conclude that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

b. To derive the sum angle formula for cosine, use what you learned in a. to show that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

c. Derive the difference angle formula for cosine, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.

d. Derive the sum angle formula for tangent, $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

e. Derive the difference angle formula for tangent, $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

[6] Adapted from https://www.illustrativemathematics.org/content-standards/HSF/TF/C/9/tasks/1116

Interview 3
Task 1[7]

Why is the graph of $y = \sin(2x)$ a horizontal shrink of the graph of $y = \sin(x)$ instead of a horizontal stretch?

[7] Adapted from Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.). (2015). *Mathematical understanding for secondary teachers: A framework and classroom-based situations*. Charlotte, NC: Information Age Publishing.

Interview 3
Task 2[8]

While proving a trigonometric identity a student produced the following sequences of equations.

$$\begin{aligned} \sin x * \cos x * \tan x &= \frac{1}{\csc^2 x} \\ \csc^2 x * \sin x * \cos x * \tan x &= 1 \\ \frac{1}{\sin^2 x} * \sin x * \cos x * \frac{\sin x}{\cos x} &= 1 \\ \cos x * \frac{1}{\cos x} &= 1 \\ 1 &= 1 \end{aligned}$$

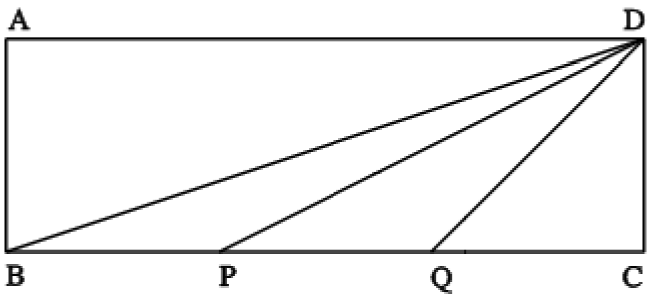
When you ask her about her reasoning, the student replied, “I just treated it the equation like any algebra equation. You know, what you do to one side, you have to do to the other, and then I showed it was the same as $1 = 1$. I know $1 = 1$ is true, so the identity must be true.”

Is the student correct? Why or why not?

[8] Adapted from Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.). (2015). *Mathematical understanding for secondary teachers: A framework and classroom-based situations*. Charlotte, NC: Information Age Publishing.

Interview 3
Task 3[9]

$ABCD$ is a rectangle where $BC = 3AB$. P and Q are points on BC such that $BP = PQ = QC$.



a. Show that: $\angle DBC + \angle DPC = \angle DQC$.

b. Generalize this result.

[9] Adapted from https://nrich.maths.org/1955

Appendix C: Formula Sheet

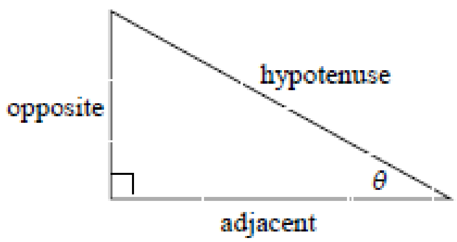
Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that

$$0 < \theta < \frac{\pi}{2} \text{ or } 0^\circ < \theta < 90^\circ.$$



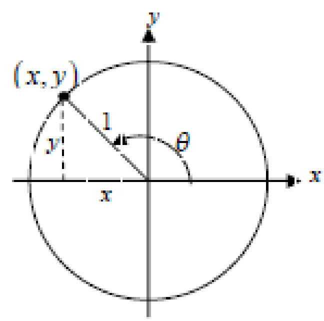
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition

For this definition θ is any angle.



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\csc\theta$, $\theta \neq n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\sec\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\cot\theta$, $\theta \neq n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

Range

The range is all possible values to get out of the function.

$-1 \le \sin\theta \le 1 \qquad \csc\theta \ge 1 \text{ and } \csc\theta \le -1$

$-1 \le \cos\theta \le 1 \qquad \sec\theta \ge 1 \text{ and } \sec\theta \le -1$

$-\infty < \tan\theta < \infty \qquad -\infty < \cot\theta < \infty$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

$$\csc(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cot(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Periodic Formulas

If n is an integer.

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas (alternate form)

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

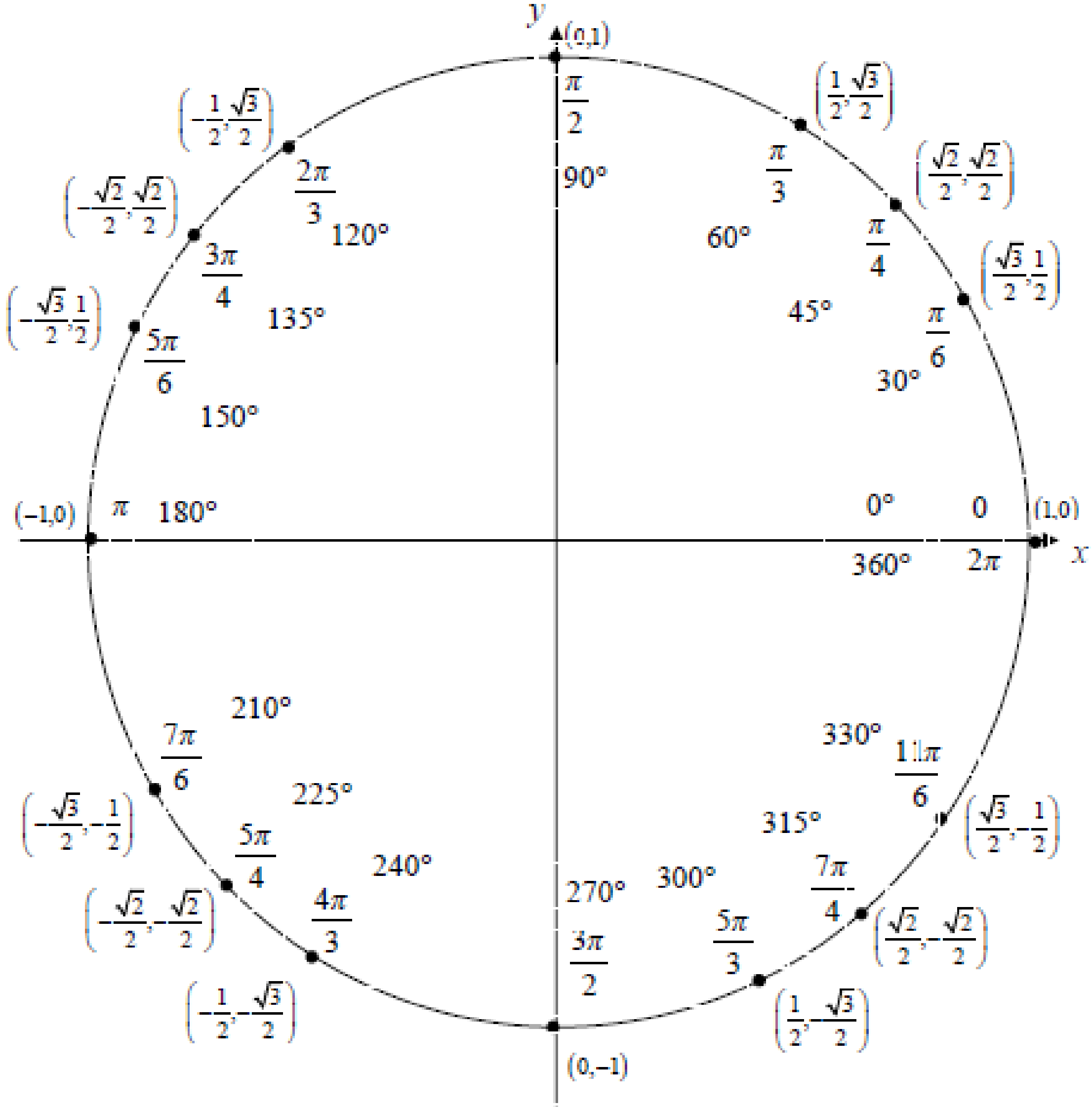
Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Unit Circle



For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Inverse Trig Functions

Definition

$y = \sin^{-1} x$ is equivalent to $x = \sin y$

$y = \cos^{-1} x$ is equivalent to $x = \cos y$

$y = \tan^{-1} x$ is equivalent to $x = \tan y$

Domain and Range

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Inverse Properties

$\cos\left(\cos^{-1}(x)\right) = x \qquad \cos^{-1}\left(\cos(\theta)\right) = \theta$

$\sin\left(\sin^{-1}(x)\right) = x \qquad \sin^{-1}\left(\sin(\theta)\right) = \theta$

$\tan\left(\tan^{-1}(x)\right) = x \qquad \tan^{-1}\left(\tan(\theta)\right) = \theta$

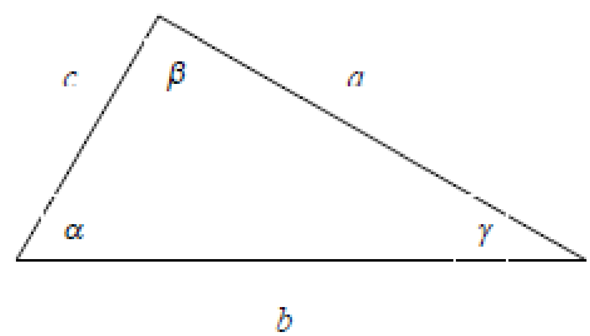
Alternate Notation

$\sin^{-1} x = \arcsin x$

$\cos^{-1} x = \arccos x$

$\tan^{-1} x = \arctan x$

Law of Sines, Cosines and Tangents



Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Mollweide's Formula

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

Appendix D: Interview Protocol

Materials: Pencil, printed tasks, formula sheet (as needed), graphing calculator (as needed)
Permission Form: Ask the participants if (s)he has any questions about the permission form and be sure to collect it.

Introduction:

Say: Hi, I'm _______. (Establish rapport if you haven't already). I really appreciate your taking the time to meet with me today. I am interested in learning about how you think about trigonometry. I will ask you to solve some problems and explain your thinking. It may or may not be similar to what you are doing in class right now. This information is just for me. Your name is not on it and your teacher will not see it.

Roles:

Say: Because I am interested in how you think about trigonometry, it would really help me if you would talk as much as you can. I am not interested in the correct answer; I am interested in how you are thinking about the problems. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. Chances are I probably just missed something you said or did. If I ask a question that does not make sense to you, please tell me.

Recording:

Say: I might take a few notes to help me remember. But, because I don't want to take too many notes, I would like to tape our conversation. The video camera will be focused on what you are writing or doing.

Awareness of materials:

Say: I have a pencil and some paper for you to write on. I also have a (list available materials) available for you to use.

Main Task:

Say: (Read task).

Potential questions to ask if participants struggle to begin or complete tasks:

- What is the problem asking you to do?
- What do you know? How does that help you?
- What do you know about …?
- How does the diagram help you?
- Can you use any of the formulas to help you?

Potential questions to ask as participants complete tasks:

- Can you tell me how you know …?
- Can you explain what you mean by …?
- Why does … make sense?
- How did you decide to …?

Closing:

Say: Thank you. I really appreciate your willingness to take the time to talk with me today.

*If the participant wants to know "how they did" reinforce that you are interested in how they were thinking and they did a really good job talking out loud and sharing their thinking with you.