ABSTRACT

FREEDMAN, BENJAMIN NATHANIEL. Existence and Qualitative Description of Solutions to Nonlinear Boundary Value Problems. (Under the direction of Jesús Rodríguez.)

The focus of this work is to study the existence and qualitative properties of solutions to nonlinear boundary value problems. We analyze boundary value problems on continuous time intervals in the context of differential equations as well as ones arising in the context of difference equations. Criteria for the existence of solutions involves nonlinearities satisfying certain size, growth, or geometric constraints. In each of these sets of results, our analysis of nonlinear problems is guided by properties of a set of corresponding linear problems. In certain cases, the corresponding linear problem is invertible. However, many of the results that follow concern cases where it is not. These cases are much more delicate to analyze mathematically.

We start our analysis by looking at a set of nonlinear scalar problems in the differential equation setting. That is, problems on $[0,1]$ of the form

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=G(x)(t)$$

subject to the generalized boundary conditions

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=\phi_i(x)$$

where ω_{ij} represents a function of bounded variation for all $1\le i\le n$ and $1\le j\le n$. We assume here that the associated linear problem is invertible, and results involve applications of classical fixed point theorems from functional analysis.

Next, we focus on discrete-time systems on $[a,b]\cap\mathbb{Z}$ of the form

$$x(k+1)-A(k)x(k)+\psi(x)(k)=f(x)(k)$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1}C_j x(j)+\eta(x)=\phi(x).$$

We give a set of results that can be used when the associated linear problem is invertible, and these results again rely on applications of fixed point theorems. Then we give results that can be applied when this is not the case, and make use of topological degree theoretic arguments.

We then investigate nonlinear perturbations of the classical Legendre boundary value problem.

That is, problems on $(-1,1)$ of the form,

$$[(1-t^2)x'(t)]' + \mu x(t) = \varepsilon f(x(t))$$

subject to the condition that the following limits exist and are finite

$$\lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t)$$

$$\lim_{t\to -1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t).$$

First, we fix $\varepsilon = 1$ and investigate the cases where the associated linear problem is and is not invertible. Finally, we allow ε to vary and investigate the case of weakly nonlinear problems. Results in this case do not require invertibility of the corresponding linear problem and involve an application of the Lyapunov-Schmidt procedure as well as the implicit function theorem for Banach spaces. We provide a qualitative description of the solutions for sufficiently small values of ε.

Finally, we investigate a set of weakly nonlinear problems on infinite intervals both in the differential equations and discrete-time settings. First we present the results concerning the continuous setting, then present the results pertaining to the discrete-time analogs. In both cases, our analysis involves a projection scheme somewhat similar to the Lyapunov-Schmidt procedure as well as an application of the implicit function theorem for Banach spaces. Solutions guaranteed using this framework emanate from a certain solution of the corresponding linear problem.

Existence and Qualitative Description of Solutions to Nonlinear Boundary Value Problems

by
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DEDICATION

To my parents Phil and Gail.

BIOGRAPHY

Benjamin Nathaniel Freedman was born in Philadelphia, Pennsylvania on February 18, 1992 to his incredible parents Gail and Phil Freedman. Ben has one younger brother, Sam Freedman and four older half-brothers, Josh, Chris, Jeff, and Pete. Ben attended Elwood Elementary in the Philadelphia public school system before being homeschooled by his mother from 3rd through 5th grade. After 5th grade, Ben moved with his family to a Media, Pennsylvania. Ben attended middle school and high school in the Rose Tree Media School district. In 2014, Ben graduated from Bucknell University with a bachelor of science in Mathematics. Despite starting undergrad as a pre-med chemistry major, Ben switched to a math major his Junior year and decided to pursue a Ph.D. in mathematics at N.C. State University after graduating from Bucknell.

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CHAPTER

1 INTRODUCTION

This paper is devoted to the study of nonlinear boundary value problems in both the continuous and discrete settings. In particular, we analyze the existence and qualitative properties of solutions. The size and geometric properties of nonlinearities involved as well as their interplay with corresponding linear problems determine the conditions under which we can guarantee solutions in different cases.

In chapter 2, we analyze nonlinear scalar problems in the continuous setting with nonlinearities in the boundary conditions as well as the dynamics. Results of this type were studied by [3] and [29] , but were limited in that they could only be used for second-order problems with a very specific structure. Results in chapter 2 pertain to a larger set of problems on $[0,1]$ of the form

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=G(x)(t)$$

subject to the generalized boundary conditions

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=\phi_i(x)$$

where ψ, G, η_i and ϕ_i for $1\le i\le n$ are nonlinear maps and $\omega_{ij}:[0,1]\to\mathbb{R}$ is a function of bounded variation for all $1\le i\le n$ and $1\le j\le n$. Expressing the linear boundary conditions in this way allows for a lot of flexibility, as any bounded linear functional on $\mathscr{C}[0,1]$ can be represented uniquely by a Riemann-Stieltjes integral with respect to a function of bounded variation. We split the nonlinearities in this way in order to impose qualitatively different conditions on each component. For $1\le i\le n$, the functions $a_i:[0,1]\to\mathbb{R}$ are continuous and $a_n(t)\neq 0$ for all $t\in[0,1]$. We assume in this chapter that the corresponding linear problems are invertible, or rather that

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)=h(t)$$

subject to

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)=v$$

has a unique solution for any continuous function h and $v\in\mathbb{R}^n$. We impose size and growth conditions on the nonlinearities involved, and utilize fixed point theorems to provide our main results.

In chapter 3, we focus on discrete-time systems on $[a,b]\cap\mathbb{Z}$ of the form

$$x(k+1)-A(k)x(k)+\psi(x)(k)=f(x)(k)$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1}C_j x(j)+\eta(x)=\phi(x).$$

where C_j is an $n\times n$ real-valued matrix for all $a\leq j\leq b+1$. The matrix $A(k)$ is an $n\times n$ real-valued matrix for all integers $k\in[a,b]$, and ψ and f represent nonlinearities in the dynamics and η and ϕ represent nonlinearities in the boundary conditions. We investigate cases where the associated linear problem is invertible as well as others where it is not. Cases where it is not invertible involve degree theoretic arguments as we impose certain geometric conditions on nonlinearities involved, and provide advantages over similar results in [20] and [37] by allowing for nonlinearities in the boundary conditions which grow significantly large in magnitude.

In chapter 4, we study nonlinear perturbations of the classical Legendre boundary value problem. That is, problems on $(-1,1)$ of the form

$$[(1-t^2)x'(t)]'+\mu x(t)=\varepsilon f(x(t))$$

subject to the condition that the following limits exist and are finite

$$\lim_{t\to-1^+}x(t),\qquad \lim_{t\to1^-}x(t)$$

$$\lim_{t\to-1^+}x'(t)\qquad \lim_{t\to1^-}x'(t).$$

Here $f:\mathbb{R}\to\mathbb{R}$ is nonlinear and continuous and $\varepsilon\in\mathbb{R}$. We first investigate the case where $\varepsilon=1$, then allow ε to vary in order to determine when we can guarantee solutions for sufficiently small values of

ε. In these cases, the solutions guaranteed emanate from a particular solution to the corresponding linear problem. Results involve an application of the Lyapunov-Schmidt procedure as well as the implicit function for Banach spaces.

In chapter 5, we investigate weakly nonlinear boundary value problems on infinite intervals first in the continuous case then in the context of discrete-time systems. Existing work on this topic ([15], [31], [32]) is significantly more limited than the results we now present since it requires that the corresponding linear problems be invertible. In the continuous case, we analyze problems on$[0,\infty)$ of the form

$$x'(t) - A(t)x(t) = h(t) + \varepsilon f(t, x(t))$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t))dt$$

where A is a continuous $n \times n$ matrix-valued function on $[0,\infty)$, f and g are continuously differentiable maps from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$, and Γ is a bounded linear map from the space of bounded, continuous functions on $[0,\infty)$ into $\mathbb{R}^n$. In the discrete case, we consider problems on $k = 0, 1, 2, \ldots$ of the form

$$x(k+1) - A(k)x(k) = h(k) + \varepsilon f(k, x(k))$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = u + \varepsilon \sum_{k=0}^{\infty} g(k, x(k))$$

where C_k is an $n \times n$ real-valued matrix for all nonnegative integers k. Denoting the nonnegative integers by $\mathbb{Z}_+$, we have that the maps $f : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, ε is a real parameter, and $A(k)$ is a nonsingular $n \times n$ real-valued matrix for all $k \in \mathbb{Z}_+$. In both cases, we provide a framework which allows us to guarantee solutions for sufficiently small values of ε where the solutions guarantees emanate from a particular solution to the corresponding linear problem.

CHAPTER

2

ON THE SOLVABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS SUBJECT TO GENERALIZED BOUNDARY CONDITIONS

2.1 Preliminaries

In this paper, we study nonlinear scalar boundary value problems which we approach by reformulating as an operator equation of the form

$$\mathscr{L}x = H(x) \tag{2.1}$$

where $\mathscr{L}$ is a linear operator, H is a nonlinear operator and both map from a subspace of $\mathscr{C}$ into $\mathscr{C} \times \mathbb{R}^n$ where $\mathscr{C}$ denotes the space of continuous functions on $[0,1]$. Suppose that $\mathscr{L}$ has an inverse, and $H = \Psi + G$ where Ψ and G satisfy two separate conditions. The strategy we will employ is to first give conditions under which $\mathscr{L} - \Psi$ is guaranteed an inverse. That is, we give conditions under which we can uniquely solve the equation

$$\mathscr{L}x - \Psi(x) = y \tag{2.2}$$

for any point y in the space that $\mathscr{L}$ and Ψ map into. Given a result of this type, we then study conditions under which (2.1) has a (possibly non-unique) solution by studying the operator $(\mathscr{L}-\Psi)^{-1}G$ and determining conditions under which it has a fixed point. This will rely on a Schauder's fixed point theorem argument. Throughout this chapter, when we refer to solution we mean a classical solution to the n-th order problems we consider. That is, an n-times continuously differentiable function that satisfies both the differential equation and the boundary conditions.

The literature concerning the study of boundary value problems is extensive. In [35] and [38] the authors analyze boundary value problems subject to linear constraints. The use of projection schemes appears in [22], [27], [35] and [38]. In [2], and [3] the authors obtain existence results based on a global inverse function theorem. The work in [3] and [29] analyzes problems in the continuous setting where nonlinearities are present in both the dynamics and boundary conditions and these nonlinearities satisfy size and growth conditions. However, these results require a corresponding linear problem with a very specific structure which does not allow for flexibility if there are linear terms that don't fit into this structure. This has the effect of making the size and growth conditions impossible to satisfy in many cases. In the results in this paper, we allow for more flexibility in terms of what can be considered part of the linear problem and therefore can establish existence of solutions for many problems for which this closely related previous work is inconclusive.

2.2 Main Results

We consider nonlinear differential equations on the interval $[0,1]$ of the form

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=G(x)(t) \tag{2.3}$$

subject to the boundary conditions

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=\phi_i(x) \tag{2.4}$$

for $1\leq i\leq n$.

Here $\psi:\mathbb{R}\to\mathbb{R}$, the maps η_i and ϕ_i for $1\leq i\leq n$ are nonlinear real-valued maps from $\mathscr{C}=(C[0,1],\mathbb{R},\|\cdot\|_\infty)$ into $\mathbb{R}$ where $\|\cdot\|_\infty$ denotes the supremum norm. Further $G:\mathscr{C}\to\mathscr{C}$ is a continuous map, $a_0,a_1,\ldots,a_n\in\mathscr{C}$ and $a_n(t)\neq 0$ for all $t\in[0,1]$. We use $\mathscr{C}^n$ to denote the subspace of $\mathscr{C}$ consisting of all n-times continuously differentiable functions on $[0,1]$. For $1\leq i\leq n$, $1\leq j\leq n$, $\omega_{ij}:[0,1]\to\mathbb{R}$ is a real-valued function of bounded variation. We will determine conditions under which we can guarantee at least one solution in $\mathscr{C}^n$ to (2.3)$-$(2.4).

To do so, we first consider a closely related problem. That is, we seek conditions under which we can uniquely solve

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=h(t)$$

subject to

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=v_i$$

for $1\le i\le n$ and for any $h\in\mathscr{C}$ and $v\in\mathbb{R}^n$.

Define $L:\mathscr{C}^n\to\mathscr{C}$ by

$$[Lx](t)=a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)$$

and $B:\mathscr{C}^n\to\mathbb{R}^n$ by

$$B(x)=\begin{pmatrix}\sum_{j=1}^n\int_0^1 x^{(j-1)}(t)d\omega_{1j}(t)\\ \sum_{j=1}^n\int_0^1 x^{(j-1)}(t)d\omega_{2j}(t)\\ \cdots\\ \sum_{j=1}^n\int_0^1 x^{(j-1)}(t)d\omega_{nj}(t)\end{pmatrix}.$$

The map $\mathscr{L}:\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ will be defined as

$$\mathscr{L}=\begin{pmatrix}L\\ B\end{pmatrix}.$$

Before proceeding, we state the following remark which illustrates an important special case of (2.3)−(2.4) that can be dealt with using the framework of this section.

Remark 1. *Let $t_0\in[0,1]$, $\beta\in\mathbb{R}$ and let the function $\omega:[0,1]\to\mathbb{R}$ be the step function*

$$\omega(x)=\begin{cases}0, & t<t_0,\\ \beta, & t\ge t_0,\end{cases}$$

So for any $x\in\mathscr{C}$, the Riemann-Stieltjes of x with respect to ω is given by

$$\int_0^1 x(t)d\omega(t)=\beta x(t_0).$$

Therefore, the boundary value problem (2.3)−(2.4) *includes problems of the form*

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=G(x)(t)$$

subject to the multipoint boundary conditions

$$\sum_{i=1}^{q} B_i\,\bar{x}(t_i)+\begin{bmatrix}\eta_1(x)\\ \eta_2(x)\\ \cdots\\ \eta_n(x)\end{bmatrix}=\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\\ \cdots\\ \phi_n(x)\end{bmatrix}$$

where

$$\bar{x}(t)=\begin{bmatrix}x(t)\\ x'(t)\\ \cdots\\ x^{(n-1)}(t)\end{bmatrix}$$

$t_i\in[0,1]$*, and* B_i *is a real-valued* $n\times n$ *matrix for all* $1\le i\le q$.

It is well known from the theory of linear differential equations that $\ker(L)$ is n-dimensional. Without loss of generality, choose a basis $\{u_1,u_2,\dots,u_n\}$ for the kernel of L such that $\|u_1\|+\|u_2\|+\cdots+\|u_n\|\le 1$ and let

$$u=\begin{bmatrix}u_1\\ u_2\\ \cdots\\ u_n\end{bmatrix}$$

Suppose that the functions of bounded variation $\omega_{ij}:[0,1]\to\mathbb{R}$ for $i,j\in\{1,2,\dots,n\}$ appearing in the boundary conditions are such that the $n\times n$ real-valued matrix $\mathscr{B}=[Bu_1|Bu_2|\dots|Bu_n]$ is invertible.

Note that one set of relevant examples where this condition on $\mathscr{B}$ is the case of a linear regular Sturm-Liouville problem perturbed away from an eigenvalue. Results in [3] and [29] analyzed nonlinear perturbations of such problems. One specific example is when L is of the form

$$[Lx](t)=x''(t)+\mu x(t)$$

B is of the form

$$Bx=\begin{bmatrix}\alpha x(0)\\ \gamma x(1)\end{bmatrix}$$

where $\mu\ne n^2\pi^2$ for any positive integer n. For more details regarding Sturm-Liouville theory in general, the reader can consult [4].

Defining the constant B_0 as:

$$B_0 = \|\mathscr{B}^{-1}\|$$

then we have that for any $v \in \mathbb{R}^n$

$$\|u^T \mathscr{B}^{-1} v\| \leq B_0 |v|.$$

As a matter of notation, define $\eta : \mathscr{C} \to \mathbb{R}^n$ by $\eta(x) = \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \\ \cdots \\ \eta_n(x) \end{bmatrix}$.

In the following theorem, we give conditions under which we can solve a very closely related nonlinear problem.

Theorem 1. *The map $\left(L|_{\ker(B)}\right)$ is a bijection from $\mathscr{C}^n$ onto $\mathscr{C}$ and its inverse is continuous. Suppose the map $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant K_1, $\eta : \mathscr{C} \to \mathbb{R}^n$ is Lipschitz with constant K_2 and*

$$A_0 K_1 + B_0 K_2 < 1$$

where $A_0 = \left\|\left(L|_{\ker(B)}\right)^{-1}\right\|$. Then for each pair $h \in \mathscr{C}$, $v \in \mathbb{R}^n$, the boundary value problem

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\, d\omega_{ij}(t) + \eta_i(x) = v_i, \qquad 1 \leq i \leq n$$

has a unique n-times continuously differentiable solution.

Proof. Suppose $\mathscr{L}(x_0) = 0$ for some $0 \neq x_0 \in \mathscr{C}^n$ and $\{u_1, \ldots, u_n\}$ be the basis we chose for $\ker(L)$ above. Since $x_0 \in \ker(L)$ there exists a unique set of constants $c_1, \ldots, c_n \in \mathbb{R}$ with $c_i \neq 0$ for some $1 \leq i \leq n$ such that $x_0 = \sum_{i=1}^n c_i u_i$. Since $x_0 \in \ker(B)$ we have that

$$\begin{aligned} 0 &= B x_0 \\ &= B\left(\sum_{i=1}^{n} c_i u_i\right) \\ &= \sum_{i=1}^{n} c_i B u_i \end{aligned}$$

contradicting the fact that $\{Bu_1,\dots,Bu_n\}$ is a linearly independent set in $\mathbb{R}^n$. Therefore, $x_0=0$ and we conclude that $\mathscr{L}:\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ is one-to-one.

Let $h\in\mathscr{C}$ and $v\in\mathbb{R}^n$. By the general theory of linear scalar ODEs it is well known that the solution $Lx=h$ has at least one solution in $\mathscr{C}^n$. Let $x_p\in\mathscr{C}^n$ be the particular solution to this equation given by variation of parameters. That is,

$$x_p(t)=\sum_{k=1}^{n}u_k(t)\int_0^t \frac{h(s)W_k[u_1,\dots,u_n](s)}{a_n(s)W[u_1,\dots,u_n](s)}ds$$

where W_k denotes the determinant of the matrix obtained by replacing the k^{th} column of the matrix whose determinant is W with e_n (the standard basis vector with a 1 in the n^{th} slot and $0s$ everywhere else).

By our assumption that $\{Bu_1,\dots,Bu_n\}$ is a basis for $\mathbb{R}^n$, there exists a unique set of constants $d_1,\dots,d_n$ such that

$$\sum_{i=1}^{n}d_iBu_i=v-B(x_p).$$

Then we have that

$$L\left(x_p+\sum_{i=1}^{n}d_iu_i\right)=h+0=h$$

and further that

$$\begin{aligned}B\left(x_P+\sum_{i=1}^{n}d_iu_i\right)&=B(x_p)+B\left(\sum_{i=1}^{n}d_iu_i\right)\\&=B(x_p)+\sum_{i=1}^{n}d_iBu_i\\&=B(x_p)+(v-B(x_p)).\end{aligned}$$

Therefore, $\mathscr{L}:\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ is a bijection with

$$\mathscr{L}^{-1}(h,v)^T=\left(L|_{\ker(B)}\right)^{-1}h+u^T\mathscr{B}^{-1}v$$

where $\left(L|_{\ker(B)}\right)$ is the map L restricted to the kernel of B. Consequently, we have that $\|\mathscr{L}^{-1}\|\le\max\{A_0,B_0\}$ where A_0 is an upper bound on the norm of the continuous linear map $\left(L|_{\ker(B)}\right)^{-1}$.

We now define $\Psi:\mathscr{C}\to\mathscr{C}\times\mathbb{R}^n$ by $\Psi(x)=\begin{bmatrix}-\psi(x)\\-\eta(x)\end{bmatrix}$. Note that since the map from $\mathbb{R}$ to $\mathbb{R}$ given by $t\mapsto\psi(t)$ is Lipschitz with constant K_1, then the map from $\mathscr{C}$ to $\mathscr{C}$ defined by $x\mapsto\psi\circ x$ is Lipschitz with constant K_1. For each pair $(h,v)\in\mathscr{C}\times\mathbb{R}^n$, we define the map $H_{(h,v)}:\mathscr{C}\to\mathscr{C}$ by

$$H_{(h,v)}(x)=\mathscr{L}^{-1}\Psi(x)+\mathscr{L}^{-1}[(h,v)^T].$$

Let $x_1,x_2\in\mathscr{C}$. Then we have that

$$\begin{aligned}\|H_{(h,v)}(x_1)-H_{(h,v)}(x_2)\| &= \|\mathscr{L}^{-1}\Psi(x_1)-\mathscr{L}^{-1}\Psi(x_2)\|\\ &\le A_0\|\psi(x_1)-\psi(x_2)\|+B_0\|\eta(x_1)-\eta(x_2)\|\\ &\le (A_0K_1+B_0K_2)\|x_1-x_2\|.\end{aligned}$$

Then $H_{(h,v)}$ is a contraction on $\mathscr{C}$ and so by Banach's fixed point theorem it has a unique fixed point $x_0\in\mathscr{C}$. Since $\mathscr{L}^{-1}$ maps into $\mathscr{C}^n$, we have that $x_0\in\mathscr{C}^n$. Therefore, there exists a unique $x_0\in\mathscr{C}^n$ satisfying $\mathscr{L}(x)-\Psi(x)=(h,v)^T$. Since $h\in\mathscr{C}$ and $v\in\mathbb{R}^n$ were arbitrary, we conclude that the operator $(\mathscr{L}-\Psi):\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ is invertible. From this it follows that for each pair $(h,v)\in\mathscr{C}\times\mathbb{R}^n$, the boundary value problem

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=h(t)$$

subject to

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=v_i, \qquad 1\le i\le n$$

has exactly one solution.

□

The following lemma establishes an important result regarding the map $\mathscr{L}^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}$. The importance of this lemma will become apparent when we provide our conditions for the solvability of (2.3)−(2.4).

Lemma 1. *The operator $\mathscr{L}^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}$ is compact.*

Proof. Let $M>0$ and define $S=\{(h,v)\in\mathscr{C}\times\mathbb{R}^n:\|h\|+|v|\le M\}$. Let $(h,v)\in S$. Then

$$\begin{aligned}\|\mathscr{L}^{-1}(h,v)\| &\le \left\|\mathscr{L}^{-1}\right\|(\|h\|+|v|)\\ &\le \max\{A_0,B_0\}(\|h\|+|v|)\\ &\le \max\{A_0,B_0\}M.\end{aligned}$$

Therefore the set $\{\mathscr{L}^{-1}(S)\}$ is bounded. We now wish to show this set forms an equicontinuous set of functions.

Let $\varepsilon>0$, and let $\delta=\frac{\varepsilon}{\max\{A_0,B_0\}M}$. Then for any $(h,v)\in\mathscr{C}\times\mathbb{R}^n$ and $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta$,

$$
\begin{aligned}
&\left|\left[\mathscr{L}^{-1}(h,v)\right](t_1)-\left[\mathscr{L}^{-1}(h,v)\right](t_2)\right| \\
&\qquad\qquad = \left|\left[\left(L|_{\ker(B)}\right)^{-1} h\right](t_1)-\left[\left(L|_{\ker(B)}\right)^{-1} h\right](t_2)+u^T(t_1)\mathscr{B}^{-1}v(t_1)-u^T(t_2)\mathscr{B}^{-1}v(t_2)\right| \\
&\qquad \le \left|\left[\left(L|_{\ker(B)}\right)^{-1} h\right](t_1)-\left[\left(L|_{\ker(B)}\right)^{-1} h\right](t_2)\right|+\left|u^T(t_1)\mathscr{B}^{-1}v(t_1)-u^T(t_2)\mathscr{B}^{-1}v(t_2)\right| \\
&\qquad \le \max\{A_0,B_0\}M|t_1-t_2| \\
&\qquad < \varepsilon.
\end{aligned}
$$

Therefore the set of functions $\{\mathscr{L}^{-1}(S)\}$ is equicontinous and we conclude that $\mathscr{L}^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}$ is compact by the Arzelá–Ascoli theorem. □

Recall that in the proof of theorem 1, we established that the operator $\mathscr{L}-\Psi$ is a bijection from $\mathscr{C}^n$ onto $\mathscr{C}\times\mathbb{R}^n$. We now state important properties of $(\mathscr{L}-\Psi)^{-1}$ under the conditions of theorem 1. The proof of the first corollary follows directly from corollary 2.3.2 in [8].

Corollary 1. *Suppose the conditions of theorem 1 hold. Then the map $(\mathscr{L}-\Psi)^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}^n$ is Lipschitz continuous with constant*

$$
K\equiv\frac{\max\{A_0,B_0\}}{1-(A_0K_1+B_0K_2)}.
$$

Corollary 2. *Under the conditions of theorem 1, the map $(\mathscr{L}-\Psi)^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}^n$ is compact. This follows from the fact that we can write*

$$
(\mathscr{L}-\Psi)^{-1}=\mathscr{L}^{-1}\left(\Psi\circ(\mathscr{L}-\Psi)^{-1}+I\right).
$$

Therefore it is clear from this representation that $(\mathscr{L}-\Psi)^{-1}$ is compact as the composition of a compact operator with a continuous one.

We now proceed to establish conditions for the solvability of the boundary value problem

$$
a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\cdots+a_0(t)x(t)+\psi(x(t))=G(x)(t)
$$

subject to the boundary conditions

$$
\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t)+\eta_i(x)=\phi_i(x)
$$

for $1 \leq i \leq n$.

We define $\phi : \mathscr{C} \to \mathbb{R}^n$ by $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_n \end{bmatrix}$ and $\mathscr{G} : \mathscr{C} \to \mathscr{C} \times \mathbb{R}^n$ by $\mathscr{G} = \begin{bmatrix} G \\ \phi \end{bmatrix}$.

In doing so, we are now ready to state sufficient conditions under which we can guarantee the existence of at least one solution to the nonlinear boundary value problem (3)−(4).

Before stating theorem 2, define M_0 as the norm of the unique solution to the boundary value problem

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) + \psi(x(t)) = 0$$

subject to the boundary conditions

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t) + \eta_i(x) = 0$$

for $1 \leq i \leq n$.

Theorem 2. *Suppose the map $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant K_1 and $\eta : \mathscr{C} \to \mathbb{R}^n$ is Lipschitz with constant K_2. Suppose that*

$$A_0 K_1 + B_0 K_2 < 1$$

and that there exists a constant M such that for $\|x\| \leq M$, $\|\mathscr{G}(x)\| \leq K^{-1}(M - M_0)$. Then there exists a solution to the boundary value problem

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) + \psi(x(t)) = G(x)(t)$$

subject to the boundary conditions

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t) + \eta_i(x) = \phi_i(x)$$

for $1 \leq i \leq n$.

Proof. Note that the map $(\mathscr{L} - \Psi)^{-1}\mathscr{G} : \mathscr{C} \to \mathscr{C}$ is compact as the composition of a compact operator with a continuous one. Define $B=\{z\in\mathscr{C}:\|z\|\leq M\}$. Let $x\in B$. Then

$$\begin{aligned}\|(\mathscr{L}-\Psi)^{-1}\mathscr{G}(x)\| &\leq \|(\mathscr{L}-\Psi)^{-1}\mathscr{G}(x)-M_0\|+M_0\\ &\leq K\|\mathscr{G}(x)\|+M_0\\ &\leq K(K^{-1}(M-M_0))+M_0\\ &=M.\end{aligned}$$

Since $(\mathscr{L}-\Psi)^{-1}\mathscr{G}(B)\subseteq B$ and B is clearly closed, bounded, and convex we have that $(\mathscr{L}-\Psi)^{-1}\mathscr{G}$ has at least one fixed point in $\mathscr{C}$ by Schauder's fixed point theorem. That is, there exists at least one $x_0\in\mathscr{C}$ such that $(\mathscr{L}-\Psi)^{-1}\mathscr{G}(x_0)=x_0$. Since $(\mathscr{L}-\Psi)^{-1}$ maps into $\mathscr{C}^n$, we have that x_0 must be an element of $\mathscr{C}^n$. This is equivalent to there existing at least one $x_0\in\mathscr{C}^n$ such that $\mathscr{L}(x_0)-\Psi(x_0)=\mathscr{G}(x_0)$. □

The following corollary is immediate:

Corollary 3. *Suppose the map $\psi:\mathbb{R}\to\mathbb{R}$ is Lipschitz with constant K_1 and $\eta:\mathscr{C}\to\mathbb{R}^n$ is Lipschitz with constant K_2. If*

$$A_0K_1+B_0K_2<1$$

and

$$\lim_{\|x\|\to\infty}\frac{\|\mathscr{G}(x)\|}{\|x\|}=0$$

then the boundary value problem (2.3)−(2.4) *has a solution.*

In the next section, we consider advantages this framework provides us in cases where we attempt to analyze problems that are seemingly well-suited to using the framework outlined in [3] and [29]. In these papers, the authors also gave criteria for the existence of solutions to boundary value problems in the differential equation setting with nonlinearities in both the dynamics and the boundary conditions.

2.3 Examples

To view advantages of using this framework as opposed to previous results, consider another set of special cases of the general boundary value problem (2.3)−(2.4). That is, problems already in self-adjoint form. This is a necessity if we are to attempt to use the analysis of [3] and [29].

Remark 2. *Consider differential equations on* $[0,1]$ *of the form:*

$$(p(t)x'(t))'+q(t)x(t)+\psi(x(t))=G(x(t)) \tag{2.5}$$

subject to the boundary conditions

$$\begin{aligned}
&\alpha x(0)+\beta x'(0)+\sum_{j=1}^{2}\int_0^1 x^{(j-1)}(t)\,d\omega_{1j}(t)+\eta_1(x)=0\\
&\gamma x(1)+\delta x'(1)+\sum_{j=1}^{2}\int_0^1 x^{(j-1)}(t)\,d\omega_{2j}(t)+\eta_2(x)=0
\end{aligned}\tag{2.6}$$

where $\psi:\mathbb{R}\to\mathbb{R}$ *is Lipschitz,* $\alpha^2+\beta^2\neq 0$, $\gamma^2+\delta^2\neq 0$, η_1 *and* η_2 *are nonlinear Lipschitz functions from* $\mathscr{C}^2$ *into* $\mathbb{R}$. *The function* $\omega_{ij}:[0,1]\to\mathbb{R}$ *is a function of bounded variation for* $i=1,2$ *and* $j=1,2$. *We assume that* p,p', *and* q *are continuous,* $p(t)>0$ *for all* $t\in[0,1]$. *We assume the map* G *is a continuous function from* $\mathscr{C}$ *into* $\mathscr{C}$ *satisfying*

$$\lim_{\|x\|\to\infty}\frac{\|G(x)\|}{\|x\|}=0.$$

Using results from [3] and [29], we would have to treat the linear integral boundary conditions appearing in (2.6) *as part of the nonlinear component of the problem. Let* $\hat{K}_2$ *be the Lipschitz constant of the map defined by*

$$x\mapsto\begin{bmatrix}\sum_{j=1}^{2}\int_0^1 x^{(j-1)}(t)\,d\omega_{1j}(t)+\eta_1(x)\\ \sum_{j=1}^{2}\int_0^1 x^{(j-1)}(t)\,d\omega_{2j}(t)+\eta_2(x)\end{bmatrix}$$

with respect to the norms used in those previous papers. Suppose $\{u_1,u_2\}$ *is a basis for the solution space of*

$$(p(t)x'(t))'+q(t)x(t)=0$$

and without loss of generality suppose that

$$\left(\int_0^1|u_1(t)|^2\,dt\right)^{\frac{1}{2}}+\left(\int_0^1|u_2(t)|^2\,dt\right)^{\frac{1}{2}}\leq 1.$$

Further, suppose the 2×2 *matrix* $\hat{\mathscr{B}}=\begin{bmatrix}\alpha u_1(0)+\beta u_1'(0), & \alpha u_2(0)+\beta u_2'(0)\\ \gamma u_1(1)+\delta u_1'(1), & \gamma u_2(1)+\delta u_2'(1)\end{bmatrix}$ *is invertible. Then if*

$$\left(\sup_{v\in\mathbb{R}^2}|\hat{\mathscr{B}}^{-1}v|\right)\hat{K}_2\geq 1$$

it would be impossible to establish the existence of solutions to (2.5)−(2.6) *using any of the results appearing in [3] or [29].*

When we formulate (2.5)−(2.6) *within the framework presented in section 3, it is clear that if the map*

$$x \mapsto \begin{bmatrix} (p(t)x'(t))' + q(t)x(t) \\ \alpha x(0) + \beta x'(0) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\, d\omega_{1j}(t) \\ \gamma x(1) + \delta x'(1) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\, d\omega_{2j}(t) \end{bmatrix}$$

is a bijection from its domain onto $\mathscr{C} \times \mathbb{R}^2$ *then the boundary value problem would have a solution provided the Lipschitz constants for* η_1, η_2*, and* ψ *are sufficiently small. The magnitude of the linear integral boundary conditions is completely irrelevant.*

We would now like to point out that the map G that we have just described can be generated in a variety of ways. For example G could be of the form

$$G(x)(t) = g(x(t))$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous or of the type

$$G(x)(t) = \int_0^1 k(t, x(s))\, ds$$

where $k : \mathbb{R}^2 \to \mathbb{R}$.

In addition to the advantages that we have just discussed, we would like to point out that if the nonlinearities η_i appearing above are of the form

$$\eta_i = \sum_{j=1}^{N} f_{i,j}(x(t_j))$$

then in order to use results appearing in [3] or [29] it must be assumed that the operator G is compact. This restriction is no longer present when using the results that we have just presented.

CHAPTER

3

ON NONLINEAR BOUNDARY VALUE PROBLEMS IN THE DISCRETE SETTING

3.1 Preliminaries

Let $\mathbb{Z}$ denote the set of integers and $a, b \in \mathbb{Z}$. We consider nonlinear discrete-time systems on $[a, b] \cap \mathbb{Z}$ of the form

$$x(k+1) - A(k)x(k) + \psi(x)(k) = f(x)(k) \tag{3.1}$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1} C_j x(j) + \eta(x) = \phi(x). \tag{3.2}$$

Throughout this paper we will be working over the integers and subsets of the integers. Therefore, for $\alpha, \beta \in \mathbb{R}$ where $\alpha < \beta$ we will use the notation $[\alpha, \beta]$ to denote the set $\{x \in \mathbb{Z} : \alpha \le x \le \beta\}$. Let a and b be integers and let X denote the space of functions from $[a, b+1]$ into $\mathbb{R}^n$. Let Y denote the space of functions from $[a, b]$ into $\mathbb{R}^n$. We endow X and Y with the norm $\|\cdot\|$ defined by

$$\|x\| = \max |x(k)|$$

where the max ranges over the domain of the function. In (3.1)-(3.2) above, C_j is an $n \times n$ real-valued matrix for all $a \le j \le b+1$. The matrix $A(k)$ is an $n \times n$ real-valued matrix for all integers $k \in [a, b]$.

For $\Lambda \in \mathbb{R}^{n\times n}$, we use the norm

$$|||\Lambda||| = \sup_{|v|=1} |\Lambda v|$$

and for $\mathbb{X} : [a,b] \to \mathbb{R}^{n\times n}$, define the norm by

$$||\mathbb{X}|| = \max_i |||\mathbb{X}(i)|||$$

where i ranges over the domain of the matrix-valued function. For $(h,v) \in Y \times \mathbb{R}^n$, we use the norm

$$||(h,v)|| = ||h|| + |v|.$$

The functions ψ and f are nonlinear maps from X to Y while η and ϕ are nonlinear maps from X to $\mathbb{R}^n$. We seek conditions under which we can find at least one solution to $(3.1)-(3.2)$ in X. In the first section, we formulate our boundary value problem as an operator equation of the form $\mathscr{L}x = \mathscr{G}(x)$ where $\mathscr{L}$ is linear and $\mathscr{G}$ is nonlinear. Results in the first section require the map $\mathscr{L}$ being invertible, and rely on fixed point theorems as well as a global inverse function theorem. Conditions are imposed that restrict the size and impose growth conditions for the nonlinearities involved in both the dynamics and boundary conditions. For the use of similar framework in the analysis of discrete problems, please see [26] and [1] for systems or [3] and [29] for scalar problems. For use of this type of argument in the continuous setting, the reader is referred to [12], [3], and [29].

The second framework provides geometric conditions that can be used to guarantee solvability in certain cases and doesn't require a certain operator to be invertible. Results rely on useful properties of the Brouwer degree of a continuous map between finite dimensional spaces. We impose geometric conditions on the nonlinearities in this section. The reader can reference [20] and [37] for this type of argument in the context of discrete systems. The work presented here is an improvement in certain cases over results presented in these papers as here we allow for more generality in both the dynamics and the boundary conditions. For the use of projection methods in the setting of discrete problems see [10], [11], [19], [22], [28], [1], [34], and [37].

3.2 Main Results

3.2.1 Lipschitz/Growth Conditions

Define the map $L : X \to Y$ by

$$[Lx](k) = x(k+1) - A(k)x(k)$$

and the map $B : X \to \mathbb{R}^n$ by

$$Bx = \sum_{j=a}^{b+1} C_j x(j).$$

Now define $\mathscr{L} : X \to Y \times \mathbb{R}^n$ by $\mathscr{L} = \begin{pmatrix} L \\ B \end{pmatrix}$. We begin by stating some general theory regarding linear difference equations. Throughout section 3.2.1, we make the assumption that the $n \times n$ matrix Γ defined by

$$\Gamma = \sum_{j=a}^{b+1} C_j \Phi(j, a)$$

is invertible where $\Phi(k,l)$ is defined for any $l \in [a, b]$, $k > l$ by

$$\Phi(k,l) = A(k-1)A(k-2)\cdots A(l), \qquad k > l$$

and $\Phi(k,l) = I$ if $k = l$. There is a large class of discrete boundary value problems where this assumption on the linear problem is satisfied. For any case where the matrix A is constant, this condition is satisfied provided

$$\sum_{j=a}^{b+1} C_j e^{Aj}$$

is nonsingular. One specific but important case where this condition may be satisfied is in the study of the periodic behavior of discrete dynamical systems. This implies that $\mathscr{L}$ is a bijection from X onto $Y \times \mathbb{R}^n$. Equivalently, for any $h \in Y$ and $v \in \mathbb{R}^n$ there exists a unique solution to the linear boundary value problem

$$x(k+1) - A(k)x(k) = h(k)$$

subject to

$$\sum_{j=a}^{b+1} C_j x(j) = v$$

and this solution can be represented as,

$$x(k) = \Phi(k,a)\Gamma^{-1}\left\{ v - \sum_{j=a+1}^{b+1} C_j \sum_{l=a}^{j-1} \Phi(j, l+1)h(l) \right\} + \sum_{j=a}^{k-1} \Phi(k, j+1)h(j).$$

For a proof of this fact, please consult lemma 2.1 in [26].

Define the constants A_1 and A_2 by:

$$A_1 = \left(\left\| \Phi(\cdot,a)\Gamma^{-1} \sum_{j=a+1}^{b+1} C_j \sum_{l=a}^{j-1} \Phi(\cdot,l+1) \right\| + \sup_{k\in[a,b+1]} \left\| \left| \sum_{j=a}^{k-1} \Phi(k,j+1) \right| \right\| \right)$$
$$A_2 = \left\| \Phi(\cdot,a)\Gamma^{-1} \right\|.$$

Then for any $h \in Y$, $k \in [a, b+1]$,

$$\left| -\Phi(k,a)\Gamma^{-1} \sum_{j=a+1}^{b+1} C_j \sum_{l=a}^{j-1} \Phi(j,l+1)h(l) + \sum_{j=a}^{k-1} \Phi(k,j+1)h(j) \right| \le A_1 \|h\|$$

and for any $v \in \mathbb{R}^n$,

$$\|\Phi(\cdot,a)\Gamma^{-1} v\| \le A_2 |v|.$$

Therefore it is clear from the representation of $\mathscr{L}^{-1}$ above that for any $(h,v) \in Y \times \mathbb{R}^n$ we have

$$\left\| \mathscr{L}^{-1}(h,v) \right\| \le A_1 \|h\| + A_2 |v|$$

and consequently

$$\|\mathscr{L}^{-1}\| \le \max\{A_1, A_2\}.$$

We now connect results concerning this closely related linear problem to the solvability of the nonlinear boundary value problem (3.1)-(3.2).

Define the maps $\Psi : X \to Y \times \mathbb{R}^n$ by

$$\Psi(x) = \begin{bmatrix} -\psi(x) \\ -\eta(x) \end{bmatrix}$$

and $\mathscr{F} : X \to Y \times \mathbb{R}^n$ by

$$\mathscr{F}(x) = \begin{bmatrix} f(x) \\ \phi(x) \end{bmatrix}.$$

The following result is the discrete time analog of a theorem appearing in [12]. Since [12] is devoted to differential equations, we have provided a proof of the theorem that follows for the sake of completeness.

Theorem 3. *Suppose that $\psi : X \to Y$ is Lipschitz continuous with constant K_1 and $\eta : X \to \mathbb{R}^n$ is Lipschitz continuous with constant K_2. Suppose further that $K^* = A_1K_1 + A_2K_2 < 1$. Then $\mathscr{L} - \Psi$ is a* *bijection from X onto $Y \times \mathbb{R}^n$ and $(\mathscr{L} - \Psi)^{-1} : Y \times \mathbb{R}^n \to X$ is Lipschitz continuous with constant*

$$K = \frac{\max\{A_1, A_2\}}{1 - K*}.$$

Suppose further there exists a positive number M such that for $\|x\| \leq M$

$$\|\mathscr{F}(x)\| \leq K^{-1}\left(M - \left\|(\mathscr{L} - \Psi)^{-1}(0)\right\|\right).$$

Then the boundary value problem

$$x(k+1) - A(k)x(k) + \psi(x)(k) = f(x)(k)$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1} C_j x(j) + \eta(x) = \phi(x).$$

has a solution in X.

Proof. Let $(h, v) \in Y \times \mathbb{R}^n$ and define

$$H_{(h,v)} : X \to X$$

by $H_{(h,v)}(x) = \mathscr{L}^{-1}\Psi(x) + \mathscr{L}^{-1}(h, v)$. Let $x_1, x_2 \in X$. Then

$$\begin{aligned} \|H_{(h,v)}(x_1) - H_{(h,v)}(x_2)\| &= \|\mathscr{L}^{-1}\Psi(x_1) - \mathscr{L}^{-1}\Psi(x_2)\| \\ &\leq A_1\|\psi(x_1) - \psi(x_2)\| + A_2|\eta(x_1) - \eta(x_2)| \\ &\leq A_1K_1 + A_2K_2 < 1. \end{aligned}$$

So $H_{(h,v)}$ is a contraction on X and therefore is guaranteed a unique fixed point $x_0 \in X$ by Banach's fixed point theorem. That is, for any pair $(h, v) \in Y \times \mathbb{R}^n$ there is a unique $x_0 \in X$ such that

$$(\mathscr{L} - \Psi)(x_0) = (\mathscr{L} - \Psi)(x_0) = (h, v)^T.$$

By corollary 2.3.2 appearing in [8], the map $(\mathscr{L} - \Psi)^{-1}$ is Lipschitz continuous with constant

$$K = \frac{\max\{A_1, A_2\}}{1 - K*}.$$

Now let $S = \{z \in X : \|z\| \leq M\}$ and let $x \in S$. Then

$$\begin{aligned}\|(\mathscr{L}-\Psi)^{-1}\mathscr{F}(x)\| &\leq \|(\mathscr{L}-\Psi)^{-1}\mathscr{F}(x)-(\mathscr{L}-\Psi)^{-1}(0)\|+\|(\mathscr{L}-\Psi)^{-1}(0)\| \\ &\leq K\|\mathscr{F}(x)\|+\|(\mathscr{L}-\Psi)^{-1}(0)\| \\ &\leq K(K^{-1}\left(M-\left\|(\mathscr{L}-\Psi)^{-1}(0)\right\|\right)+\left\|(\mathscr{L}-\Psi)^{-1}(0)\right\| \\ &= M.\end{aligned}$$

Note that S is compact and convex. We have now shown that $(\mathscr{L}-\Psi)^{-1}\mathscr{F}(S) \subset S$ and therefore by Brouwer's fixed point theorem $(\mathscr{L}-\Psi)^{-1}\mathscr{F}$ is guaranteed at least one fixed point in S. Since $(\mathscr{L}-\Psi)$ is invertible, this is equivalent to guaranteeing at least one solution in S to the boundary value problem (3.1)−(3.2). □

The following remark points out that under certain cases, the theorem above can be used to guarantee a unique solution to the boundary value problem (3.1)-(3.2).

Remark 3. *Note that theorem 1 implies that if $K^* < 1$ then the boundary value problem*

$$x(k+1) - A(k)x(k) + \psi(x)(k) = h(k)$$

subject to

$$\sum_{j=a}^{b+1} C_j x(j) + \eta(x) = v$$

has a unique solution for any pair $(h, v) \in Y \times \mathbb{R}^n$.

In the following remark, we point out that framework in this section can be used in certain cases to establish existence of solutions to scalar boundary value problems in the discrete setting. For the convenience of the reader, we present a general scalar boundary value problem and reformulate this problem as the equivalent system in the form (3.1)-(3.2).

Remark 4. *Let $a, b, n \in \mathbb{Z}$ with $a < b$ and $n > 0$. Consider discrete scalar boundary value problems on $[a, b+n]$ of the form*

$$a_n(k)x(k+n) + a_{n-1}(k)x(k+(n-1)) + \cdots + a_0(k)x(k) + \psi(x)(k) = f(x)(k)$$

subject to the boundary conditions

$$\sum_{j=1}^{n} c_{i,j}(a)x(j+a-1) + \sum_{j=1}^{n} c_{i,j}(a+1)x(j+a) + \cdots + \sum_{j=1}^{n} c_{i,j}(b+1)x(j+b) + \eta_i(x) = \phi_i(x)$$

for $1 \leq i \leq n$.

Here $a_0,\ldots,a_n \in \hat{Y} = \{f : [a,b] \to \mathbb{R}\}$ *with* $a_n(k) \neq 0$ *for all* $k \in [a,b]$ *The maps* ψ *and* f *are function valued and* η_i *and* ϕ_i *are maps from* $\hat{X} = \{f : [a,b+n] \to \mathbb{R}\}$ *into* $\mathbb{R}$. *Suppose we seek conditions under which we can guarantee at least one solution to this scalar problem in the space* $\hat{X}$. *Then define* $y : [a,b+1] \to \mathbb{R}^n$ *by*

$$y(k) = \begin{bmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(k+(n-1)) \end{bmatrix},$$

the $n \times n$ *matrices* C_l *for* $a \le l \le b+1$ *by*

$$C_l = \left[c_{i,j}(l)\right]$$

the maps $\hat{\psi}, \hat{g} : X \to Y$ *by*

$$\hat{\psi}([y_1, y_2, \ldots, y_n]^T) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \psi(y_1) \end{bmatrix}, \quad \hat{g}([y_1, y_2, \ldots, y_n]^T) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ g(y_1) \end{bmatrix}$$

and $\hat{\eta}, \hat{\phi} : X \to \mathbb{R}^n$ *by*

$$\hat{\eta}([y_1, y_2, \ldots, y_n]^T) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \eta(y_1) \end{bmatrix}, \quad \hat{\phi}([y_1, y_2, \ldots, y_n]^T) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \phi(y_1) \end{bmatrix}.$$

Define for all $k \in [a,b]$, *the* $n \times n$ *matrix* $A(k)$ *is defined by*

$$A(k) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & \cdots & & \cdots \\ -\frac{a_0}{a_n}(k) & -\frac{a_1}{a_n}(k) & -\frac{a_2}{a_n}(k) & \cdots & -\frac{a_{n-1}}{a_n}(k) \end{bmatrix}.$$

Then we could formulate this scalar boundary value problem as a first order system. That is, solving the scalar problem above is equivalent to finding a solution $y \in X$ *to the system*

$$y(k+1) - A(k)y(k) + \hat{\psi}(y)(k) = \hat{f}(y)(k)$$

subject to

$$\sum_{l=a}^{b+1} C_l y(l) + \hat{\eta}(y) = \hat{\phi}(y).$$

In the introduction, we referred to existing results regarding existence of solutions to nonlinear discrete scalar boundary value problems. Two of note are [3] and [29], which analyze nonlinearly perturbed Sturm-Liouville equations in the discrete setting. The following set of examples illustrates that framework in this section can be used to establish existence of solutions to problems for which the results of these previous papers is inconclusive.

Example 1. *Consider the following boundary value problem on* $[a-1, b-1]$*,*

$$\Delta(p(k-1)\Delta x(k-1)) + q(k)x(k) + \psi(x)(k) = f(x)(k)$$

subject to

$$\alpha x(a-1) + \beta \Delta x(a-1) + \sum_{k=1}^{m} \alpha_k x(t_k) = \phi_1(x)$$
$$\gamma x(b) + \delta \Delta x(b) + \sum_{j=1}^{q} \beta_j x(s_j) = \phi_2(x)$$

where $\Delta x(k) = x(k+1) - x(k)$*. If any one of the constants* α_k*,* β_j *for* $k = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots q$ *is sufficiently large in magnitude, we would have no chance of guaranteeing solutions using results appearing in [12] or [29]. However, using the framework presented in this section we could guarantee solutions independent of the sizes of* $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_q$.

3.2.2 Geometric Conditions

In this section, we present results regarding the existence of solutions to (3.1)−(3.2) under conditions independent of the invertibility assumption made in the previous section. In this section, we denote $f - \psi = g$, $\phi - \eta = q$ and rewrite the system (3.1)−(3.2) as

$$x(k+1) - A(k)x(k) = g(x)(k)$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1} C_j x(j) = q(x).$$

The reason for not splitting the nonlinearities like in section 2.1 will become clear when we state our conditions. Throughout this section, we assume that G is generated by a continuous function g

that maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. More specifically

$$g(x)(k) = G(x(k)).$$

We assume in this section that we can split q into the sum of two nonlinear operators q_1 and q_2. Further, suppose there exists a positive number M such that for $\|x\| \leq M$, q_1 satisfies $q_1(-x) = -q_1(x)$. Define $L : X \to Y$ as in last section and the nonlinear maps $N : X \to Y \times \mathbb{R}^n$ by

$$N(x) = \begin{bmatrix} Lx \\ C_{b+1}x(b+1) - q_1(x) \end{bmatrix}$$

and $\mathscr{G} : X \to Y \times \mathbb{R}^n$ by

$$\mathscr{G}(x) = \begin{bmatrix} g(x) \\ q_2(x) - \sum_{l=a}^{b} C_l x(l) \end{bmatrix}$$

Guaranteeing solutions to (3.1)-(3.2) is equivalent to guaranteeing solutions to the nonlinear operator equation $N(x) = \mathscr{G}(x)$.

Lemma 2. *Suppose $\Omega \subset X$ is an open bounded neighborhood of $0 \in X$ such that that $N(x) = 0$ has no solutions on $\partial\Omega$ and $N(x) \neq \lambda\mathscr{G}(x)$ for every $(x, \lambda) \in \partial\Omega \times (0,1)$. Then the equation (3.1) − (3.2) has at least one solution in $\bar{\Omega}$.*

Proof. Consider the map $\mathscr{F} : X \times [0,1] \to Y \times \mathbb{R}^n$ by

$$\mathscr{F}(x, \lambda) = (1-\lambda)N(x) + \lambda(N - \mathscr{G})(x).$$

Since N is odd and has no zeroes on $\partial\Omega$, $d[N, \Omega, 0] \neq 0$ and $\mathscr{F}(x, 0) \neq 0$ for all $x \in \partial\Omega$. Note that $\mathscr{F}(x, \lambda) = 0$ is equivalent to $N(x) = \lambda\mathscr{G}(x)$ which by assumption has no zeroes for $x \in \partial\Omega$, $\lambda \in (0,1)$. Suppose first that there exists an $x_0 \in \partial\Omega$ such that $\mathscr{F}(x_0, 1) = 0$. Then x_0 is a solution in $\partial\Omega$ to (3.1)-(3.2). If not, then $\mathscr{F}$ has no zeroes on $\partial\Omega \times [0,1]$. Therefore by homotopy invariance of the Brouwer degree established in [21],

$$d[(N - \mathscr{G}), \Omega, 0] = d[N, \Omega, 0] \neq 0$$

and so $(N - \mathscr{G})$ must have at least one zero in $\bar{\Omega}$. This of course is equivalent to (3.1)-(3.2) having at least one solution in $\bar{\Omega} \subset X$.

□

We are now ready to state the following theorem giving conditions for the solvability of (3.1)-(3.2).

Theorem 4. *Suppose that for all $v \in \mathbb{R}^n$ and $k \in [a, b]$,*

$$\langle v, A(k)v \rangle \geq |v|^2.$$

Then we can guarantee a solution to

$$x(k+1)-A(k)x(k)=g(x)(k)$$

subject to

$$\sum_{j=a}^{b+1} C_j x(j)=q(x)$$

provided there exists a positive constant $r \leq M$ *such that,*

- *For* $v \in \mathbb{R}^n$ *with* $|v|=r$,

$$\langle v, G(v)\rangle > 0$$

  and

- *For all* $x \in X$ *such that* $|x(b+1)|=r$,

$$|q_1(x)| > |q_2(x)| + \left(\sum_{l=a}^{b+1} |||C_l|||\right) r.$$

,

Proof. Let Ω be the set

$$\Omega=\{u\in X : |u(k)|<r, \ \ \forall k\in[a,b+1]\}$$

and note that Ω is an open, bounded neighborhood around $0\in X$. Further, note that

$$\bar{\Omega}=\{u\in X : |u(k)|\leq r, \ \ \forall k\in[a,b+1]\}$$

so then

$$\partial\Omega=\{u\in\bar{\Omega} : |u(k_0)|=r, \text{ for some } k_0\in[a,b+1]\}.$$

The set Ω is simply the open ball of radius r in X.

First we will show that $N(x)=0$ has no solution on $\partial\Omega$. Suppose to the contrary that there is such a solution and call it x_0. Since $\|x_0\|=r$, there exists some $\hat{k}\in[a,b+1]$ such that $|x_0(\hat{k})|=r$. Suppose that $\hat{k}\in[a,b]$ and that $|x_0(\hat{k}+1)|<r$. Then

$$\begin{aligned}\langle x_0(\hat{k}), A(\hat{k})x_0(\hat{k})\rangle &= \langle x_0(\hat{k}), x_0(\hat{k}+1)\rangle \\ &\leq |x_0(\hat{k})||x_0(\hat{k}+1)| \\ &< r^2,\end{aligned}$$

a contradiction to condition i) above. Therefore, $|x(k)| = r$ for all $k \geq \hat{k}$. In particular $|x_0(b+1)| = r$. By condition iii),

$$\begin{aligned}\left|q_1(x_0) - C_{b+1}x_0(b+1)\right| &\geq |q_1(x_0)| - |||C_{b+1}|||\, r \\ &> 0.\end{aligned}$$

Next we will show that

$$N(x) = \lambda \mathcal{G}(x)$$

has no solutions for $\lambda \in (0,1)$ and $x \in \partial\Omega$. Suppose there does exist such a pair $(y, \bar{\lambda}) \in \partial\Omega \times (0,1)$ satisfying this equation. Then for all $k \in [a,b]$

$$y(k+1) - A(k)y(k) = \bar{\lambda}G(x(k))$$

and further

$$C_{b+1}y(b+1) - q_1(y) = \bar{\lambda}\left[q_2(y) - \sum_{l=a}^{b} C_l\, y(l)\right]$$

Since $y \in \partial\Omega$ there exists some $k_0 \in [a, b+1]$ with $|y(k_0)| = r$. First suppose that $k_0 \in [a,b]$. Since $y \in \bar{\Omega}$, it follows that $\langle y(k_0), y(k) - y(k_0)\rangle \leq 0$ for all $k \in [a, b+1]$. However,

$$\begin{aligned}\langle y(k_0), y(k_0+1) - y(k_0)\rangle &\geq \langle y(k_0), y(k_0+1) - A(k_0)y(k_0)\rangle \\ &= \bar{\lambda}\langle y(k_0), G(y(k_0))\rangle \\ &> 0\end{aligned}$$

which is a contradiction. (Note that the first to the second line follows from i) and the third to fourth line follows from the first part of ii)). Now suppose that $k_0 = b+1$. Then we have from above that

$$|y(k)| < r$$

for all $k \in [a,b]$. Therefore by the second part of condition ii),

$$\begin{aligned}\left|C_{b+1}y(b+1)-q_1(y)\right| &\geq \left|q_1(y)\right| - |||C_{b+1}|||r \\ &> |q_2(y)| + \left(\sum_{l=a}^{b+1} |||C_l|||\right) r - |||C_{b+1}|||r \\ &= |q_2(y)| + \left(\sum_{l=a}^{b} |||C_l|||\right) r \\ &\geq \left|q_2(y) - \sum_{l=a}^{b} C_l\, y(l)\right| \\ &\geq \bar{\lambda}\left|q_2(y) - \sum_{l=a}^{b} C_l\, y(l)\right|\end{aligned}$$

which is a contradiction as well. Therefore, by the previous lemma we can guarantee at least one solution to $(3.1)-(3.2)$ in $\bar{\Omega}$. □

3.3 Examples

Example 2. *As a special case of* (3.1)-(3.2) *are multipoint boundary value problems on* $[a, b+1]$ *of the form*

$$x(k+1) - A(k)x(k) = G(x(k))$$

subject to

$$\sum_{l=a}^{b+1} C_l\, x(l) = q(x(a), q(a+1), \dots, x(b+1))$$

Let N *be the number of integers between* a *and* $b+1$. *Suppose* $q : \mathbb{R}^{Nn} \to \mathbb{R}^n$ *is nonlinear and that* $q = q_1 + q_2$ *where* $-q_1(s_1, s_2, \dots, s_N) = q_1(-s_1, -s_2, \dots, -s_N)$ *for all* $(s_1, s_2, \dots s_n) \in \mathbb{R}^n$ *such that*

$$\max_{i=1,2,\dots,n} |s_i| = M.$$

Further suppose that there exists $r \leq M$ *such that for* $(s_1, s_2, \dots, s_N)$ *satisfying* $|s_N| = r$,

$$|q_1(s_1, s_2, \dots, s_N)| > |q_2(s_1, s_2, \dots, s_N)| + \left(\sum_{l=a}^{b+1} |||C_l|||\right) r.$$

Then if A satisfies the conditions of condition $i)$ *and* $\langle f(v), v\rangle > 0$ *for all* $|v| = r$ *we can guarantee solutions to this two-point boundary value problem. Examples of* $q : \mathbb{R}^{Nn} \to \mathbb{R}^n$ *satisfying this requirement* *include*

$$q(s_1, s_2, \ldots, s_N) = q_2(s_1, s_2, \ldots, s_{N-1}) + \beta |s_N|^2 s_N$$

for β sufficiently large or alternatively

$$q(s_1, s_2, \ldots, s_N) = q_2(s_1, s_2, \ldots, s_{N-1}) + \delta e^{|s_N|} s_N$$

for δ sufficiently large.

Using the framework appearing in [37], we would be unable to guarantee solutions in cases where the magnitude of $q(s_1, \ldots, s_N)$ grows very quickly as one of $s_1, \ldots, s_N$ grows in magnitude. However, results presented in this paper allow us to guarantee solutions in certain cases to problems where this happens to be the case.

Example 3. *Consider discrete systems on $[a, b+1]$ of the form*

$$x(k+1) - A(k)x(k) = G(x(k))$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1} C_j x(j) = q(x)$$

where q itself is an odd function on all of X. Then letting $q_1 = q$ and $q_2 = 0$ using the above construction, we get that condition $iii)$ of theorem 2 is satisfied provided

$$\liminf_{\|x\| \to \infty} \frac{|q(x)|}{\|x\|} > \sum_{l=a}^{b+1} |||C_l|||.$$

Example 4. *We now discuss the case of completely linear boundary conditions. For previous results involving geometric conditions for the solvability of nonlinear discrete time systems subject to linear boundary conditions, the reader is referred to [20]. Here, we consider discrete systems on $[a, b+1]$ of the form*

$$x(k+1) - A(k)x(k) = G(x(k))$$

subject to the multipoint boundary conditions:

$$\sum_{j=a}^{b+1} C_j x(j) = 0$$

where $C_j \in \mathbb{R}^{n \times n}$. Then letting $q_1(x) = \sum_{j=a}^{b+1} C_j x(j)$ and $q_2(x) = 0$ we have that condition $iii)$ is satisfied *provided*

$$\left|\sum_{j=a}^{b+1} C_j x(j)\right| > 0$$

for all $x \in X$ *satisfying* $\|x\| \leq r$ *and* $|x(b+1)| = r$. *Using results appearing in [20], we would be unable to establish existence of solutions for such problems unless* C_{b+1} *is invertible and*

$$\sum_{l=a}^{b} C_{b+1}^{-1} C_l = I.$$

CHAPTER

4

ON NONLINEAR LEGENDRE BOUNDARY VALUE PROBLEMS

4.1 Preliminaries

The classical Legendre eigenvalue-eigenfunction problems consists of finding the scalars μ and functions $x:(-1,1)\to\mathbb{R}$ such that

$$[(1-t^2)x'(t)]'+\mu x(t)=0$$

for all $t\in(-1,1)$ where

$$\lim_{t\to-1^+} x(t), \qquad \lim_{t\to1^-} x(t)$$

$$\lim_{t\to-1^+} x'(t), \qquad \lim_{t\to1^-} x'(t).$$

all exist and are finite. It is well-known that nontrivial solutions of this problem exist if and only if $\mu=k(k+1)$ where k is a nonnegative integer [4]. If $\mu=k(k+1)$, the only solutions are the constant multiples of the k^{th} Legendre polynomial. In this paper, we consider a nonlinear perturbation of the differential equation subject to the same boundary conditions. That is, the existence of finite limits of $x(t)$ and $x'(t)$ at 1 and -1.

Approaches similar to the ones appearing in this paper have been used in a variety of settings in the study of nonlinear boundary value problems. For the use of arguments similar to those in section 4.2.1 in the continuous and discrete cases, the reader is referred to [12], [17], [28], [2], and [29]. For the general theory of projection methods in nonlinear boundary value problems we suggest [32]. For the use of projection methods similar to those in subsections 4.2.2 and 4.2.3, see [10], [24], [25], [34], [36], and [38]. For results involving topological degree theory arguments in the analysis of discrete boundary value problems, the reader may consult [5] and [9].

4.2 Main Results

4.2.1 The Case of Invertible L

Even though in this paper we are mainly interested in the cases where the parameter μ in the equation below is an eigenvalue of the associated linear Legendre equation, we devote this first section to the case where $\mu \neq k(k+1)$ for any nonnegative integer k. We consider boundary value problems on $(-1,1)$ of the form,

$$[(1-t^2)x'(t)]' + \mu x(t) = f(x(t)) \tag{4.1}$$

subject to the condition that the following limits exist and are finite

$$\begin{gathered} \lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t) \\ \lim_{t\to -1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t). \end{gathered} \tag{4.2}$$

Throughout this paper, we assume that $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz. Let $\mathscr{L}^2$ denote the space of functions $\mathscr{L}^2 = (\mathscr{L}^2[-1,1], \|\cdot\|_2)$, X be defined as the subspace of functions in $\mathscr{L}^2$ where the limits appearing in (4.2) exist and are finite and

$$D(L) = \{x \in X : x' \text{ is absolutely continuous and } x'' \in \mathscr{L}^2\}.$$

This implies that $f \circ x \in \mathscr{L}^2$ for all $x \in \mathscr{L}^2$. We seek conditions under which we can guarantee the existence of a solution to the boundary value problem (4.1)-(4.2).

We now present some basic results regarding a closely related linear boundary value problem. If $\mu \neq k(k+1)$ for all k, the equation

$$[(1-t^2)x'(t)]' + \mu x(t) = h(t)$$

has exactly one solution satisfying the condition that the following limits exist and are finite

$$\lim_{t\to-1^+} x(t), \qquad \lim_{t\to 1^-} x(t)$$

$$\lim_{t\to-1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t).$$

Define the map $L : D(L) \to \mathscr{L}^2$ by

$$[Lx](t) = [(1-t^2)x'(t)]' + \mu x(t).$$

Clearly, if $\mu \neq k(k+1)$ for all k then L is a bijection from $D(L)$ onto $\mathscr{L}^2$.

Let P_k denote the k^{th}-degree Legendre polynomial and $p(t) = (1-t^2)$. From general Sturm-Liouville theory, the equation $(px')' + \lambda x = 0$, subject to the condition that the limits in (4.2) exist and are finite, has countably many simple eigenvalues $\lambda_k = k(k+1)$ with corresponding eigenfunctions P_k for $k \geq 0$. It is also well-known that L is self-adjoint and that the graph of L is closed [4]. Further, the unique solution $x_h \in D(L)$ to $Lx = h$ guaranteed above can be represented by the eigenfunction expansion,

$$x_h = \sum_{k=0}^{\infty} \frac{(k+\frac{1}{2})\langle h, P_k\rangle}{[\mu - k(k+1)]} P_k$$

where $\langle\cdot,\cdot\rangle$ denotes the standard $\mathscr{L}^2$ inner product. From this is follows that L^{-1} is continuous and that

$$\left\|L^{-1}\right\| \leq \left(\sum_{k=0}^{\infty} \left|\frac{1}{(\mu - k(k+1))^2(k+\frac{1}{2})}\right|\right)^{1/2}.$$

This information, as well as more on the general theory of Legendre polynomials and the Legendre differential equation can be found in [4].

Before presenting the next corollary, we first must introduce some notation. Let $\mathscr{C}$ denote the space of continuous functions on $[-1,1]$, and $\|\cdot\|_\infty$ denote the standard norm on this space. That is, for a continuous function x on $[-1,1]$,

$$\|x\|_\infty = \max_{t\in[-1,1]} |x(t)|.$$

In the following corollary, we establish the continuity of L^{-1} by giving a bound on its operator norm that will be useful later.

Corollary 4. *There exists $K > 0$ such that for all $h \in Im(L) \subset \mathscr{L}^2$, the unique solution x_h to the equation $Lx = h$ satisfies*

$$\|x_h\|_\infty \leq K\|h\|$$

and

$$\|x_h'\|_\infty \leq K\|h\|.$$

Proof. Define the map $\hat{L} : \hat{D}(L) \to Im(L)$ by

$$[\hat{L}x](t) = [(1-t^2)x'(t)]' + \mu x(t)$$

where $\hat{D}(L)$ consists of the same set of functions as $D(L)$ but is endowed with the norm $\|\cdot\|_{H^2}$ given by

$$\|z\|_{H^2} = \|z\|_\infty + \|z'\|_\infty + \|z''\|.$$

Note that the map $\hat{L}$ is a continuous, linear bijection onto $\mathscr{L}^2$, and that $\hat{D}(L)$ and $Im(L)$ are Banach spaces. Therefore, by a consequence of the open mapping theorem, $\hat{L}^{-1}$ is continuous. This means there exists a $K > 0$ such for any $h \in \mathscr{L}^2$ the unique solution x_h to $Lx = h$ satisfies:

$$K\|h\| \geq \|x_h\|_{H^2} \geq \|x_h\|_\infty$$

and

$$K\|h\| \geq \|x_h\|_{H^2} \geq \|x_h'\|_\infty$$

as required. □

Now we are ready to provide the following lemma which establishes an important property of the map L^{-1}.

Lemma 3. *The map $L^{-1} : Im(L) \to \mathscr{L}^2$ is compact.*

Proof. Consider the map $\tilde{L} : \tilde{D}(L) \to Im(L)$ defined by

$$[\tilde{L}x](t) = [(1-t^2)x'(t)]' + \mu x(t)$$

where $\tilde{D}(L)$ consists of the same set of functions as $D(L)$ but endowed with the norm $\|\cdot\|_\infty$. Note that $\tilde{L}$ is invertible due to the fact that L is invertible. We wish to show that $\tilde{L}$ is compact using the Arzela-Ascoli theorem. Let $M > 0$ and define S to be the set $S = \{z \in Im(L) : \|z\| \leq M\}$. Let $h \in S$ and observe that

$$\begin{aligned} \|\tilde{L}^{-1}h\|_\infty &\leq K\|h\| \\ &\leq KM. \end{aligned}$$

Therefore, the $\tilde{L}^{-1}(S)$ is a uniformly bounded set of functions in $\mathscr{C}$. We now wish to show that this set is equicontinuous.

Let $h \in S$ and let $\varepsilon > 0$. By the previous corollary along with the mean value theorem, for any $h \in \mathscr{L}^2$ $\tilde{L}^{-1}h$ is Lipschitz on S with constant KM. Let $\delta = \varepsilon/KM$ and $|t_1 - t_2| < \delta$. Then we have that

$$\begin{aligned}|\tilde{L}^{-1}h(t_1) - \tilde{L}^{-1}h(t_2)| &\leq KM|t_1 - t_2| \\ &< \varepsilon.\end{aligned}$$

Therefore, $\tilde{L}^{-1}(S)$ is an equicontinuous set of functions in $\mathscr{C}$. By the Arzelá-Ascoli theorem, $\tilde{L}^{-1}: Im(L) \to \bar{D}(L)$ is compact. Therefore, it follows that $L^{-1}: Im(L) \to D(L)$ is a compact operator. □

We now discuss the issue of whether we can guarantee a solution to the nonlinear boundary value problem

$$[(1-t^2)x'(t)]' + \mu x(t) = f(x(t))$$

where x satisfies the condition that limits in (4.2) exist and are finite. Define $F: \mathscr{L}^2 \to \mathscr{L}^2$ by

$$F(x) = f \circ x.$$

It is evident that the boundary value problem (4.1)-(4.2) is equivalent to the operator equation $Lx = F(x)$. Now we are ready to state the following theorem in which we establish our main result of this subsection.

Theorem 5. *Suppose that $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz and that $\mu \neq k(k+1)$ for all nonnegative integers k. Then if*

$$\lim_{|s| \to \infty} \frac{|f(s)|}{|s|} = 0$$

there exists a solution to the boundary value problem

$$[(1-t^2)x'(t)]' + \mu x(t) = f(x(t))$$

subject to the condition that the limits in (4.2) *exist and are finite.*

The proof of this theorem is a standard application of Schauder's fixed point theorem applied to the operator $L^{-1}F$. We omit the details. Note that results in this subsection depended heavily on L having an inverse, which is only the case if we assume $\mu \neq k(k+1)$ for any $k \in \mathbb{N}$. The following subsections analyze situations where L is not invertible.

4.2.2 The Case of Non-Invertible L

We will now assume that $\mu = k(k+1)$ for some $k \in \{0, 1, 2, \ldots\}$. As a consequence of the general Sturm-Liouville theory outlined in the previous section, $\mu = k(k+1)$ implies that the kernel of L is one-dimensional and spanned by P_k. Further, as stated in [13], we have that $h \in \text{Im}(L)$ if and only if

$$\langle h, P_k \rangle = 0.$$

Therefore, it follows that Im(L)=[Ker(L)]$^{\perp}$. In this section we will assume that $\lim_{s\to\infty} f(s)$ and $\lim_{s\to-\infty} f(s)$ exist and are finite. We denote these values by

$$f(\infty) \equiv \lim_{s\to\infty} f(s) \qquad \text{and} \qquad f(-\infty) \equiv \lim_{s\to-\infty} f(s).$$

We employ the Lyapunov-Schmidt procedure. For the readers convenience, we now outline the basic elements of this process.

First define $U : \mathscr{L}^2 \to \mathscr{L}^2$ by

$$[Ux](t) = \left(k + \frac{1}{2}\right)\langle x, P_k \rangle P_k(t).$$

Note that U is a projection onto Ker(L) $= span\{P_k\}$. Define the projection $E : \mathscr{L}^2 \to \mathscr{L}^2$ onto [Ker(L)]$^{\perp}$ =Im(L) by $E = I - U$. Note that the map L restricted to $D(L)\cap$ Im(L) is a bijection onto Im(L)=Im(E). Therefore, it follows that there exists a linear map M :Im(E)$\to D(L)\cap$Im(L) satisfying $LMh = h$ for all $h \in$Im(L) and $MLx = Ex = (I-U)x$ for all $x \in D(L)$. In fact, we can represent this map M with the eigenfunction expansion,

$$[Mh](t) = \sum_{l \neq k} \frac{(l+\frac{1}{2})\langle h, P_l \rangle}{[\mu - l(l+1)]} P_l(t).$$

Note that $M : Im(L) \to Im(L) \cap D(L)$ is a compact operator as a consequence of the argument appearing in lemma 3 along with the fact that $Im(L)$ is a closed subspace of $\mathscr{L}^2$. Using these projections, we analyze the operator equation $Lx = F(x)$ in the following way:

$$Lx = F(x) \Longleftrightarrow \begin{cases} E(Lx - F(x)) = 0 \\ \quad \text{and} \\ (I - E)(Lx - F(x)) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I - U)x - MEF(x) = 0 \\ \quad \text{and} \\ F(x) \in Im(L) \end{cases}$$

$$\Longleftrightarrow \begin{cases} x = Ux + MEF(x) \\ \quad \text{and} \\ \int_{-1}^{1} f(x(t))P_k(t)dt = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x = \alpha P_k + w(x) \\ \quad \text{and} \\ \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt = 0 \end{cases}$$

where $w(x) = MEF(x)$.

Define the constants J_1 and J_2 as follows:

$$J_1 = f(\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt$$
$$J_2 = f(\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt.$$

Note that if $k = 0$, then $J_1 = g(\infty)$ and $J_2 = g(-\infty)$. If $k \geq 1$, then

$$J_1 = \left(\int_{\{t:P_k(t)>0\}} P_k(t)dt\right)[f(\infty) - f(-\infty)]$$
$$J_2 = \left(\int_{\{t:P_k(t)>0\}} P_k(t)dt\right)[f(-\infty) - f(\infty)].$$

Theorem 6. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(-\infty)$ and $f(\infty)$ exist and are finite. Then we can guarantee a solution to the boundary value problem* (4.1)-(4.2) *in either of the following cases:*

(i) *If $k = 0$ and $f(-\infty)f(\infty) < 0$*

(ii) *If* $k \geq 1$ *and* $f(-\infty) \neq f(\infty)$.

Proof. We begin by noting that,

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt$$
$$= \int_{\{t : P_k(t) < 0\}} f[\alpha P_k(t) + w(x(t))] P_k(t) dt + \int_{\{t : P_k(t) > 0\}} f[\alpha P_k(t) + w(x(t))] P_k(t) dt.$$

Since w is bounded, we have by the Lebesgue Dominated Convergence Theorem that

$$\lim_{\alpha \to \infty} \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt = f(\infty) \int_{\{t : P_k(t) > 0\}} P_k(t) dt + f(-\infty) \int_{\{t : P_k(t) < 0\}} P_k(t) dt = J_1$$

and

$$\lim_{\alpha \to -\infty} \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt = f(\infty) \int_{\{t : P_k(t) < 0\}} P_k(t) dt + f(-\infty) \int_{\{t : P_k(t) > 0\}} P_k(t) dt = J_2.$$

Condition iii) implies that $J_1 J_2 < 0$, and we proceed by supposing without loss of generality that $J_2 < 0 < J_1$.

Therefore there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$,

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt > 0 \tag{4.3}$$

and for $\alpha \leq -\alpha_0$,

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt < 0. \tag{4.4}$$

Note that M is a compact linear map from $Im(L)$ onto $D(L) \cap Im(L)$ and E is a projection so w is a nonlinear compact mapping.
Define $H_1 : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2$ by

$$H_1(x, \alpha) = \alpha P_k + w(x)$$

and $H_2 : \mathscr{L}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$H_2(x, \alpha) = \alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t) dt.$$

Let $H : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2 \times \mathbb{R}$ be defined by

$$H(x, \alpha) = (H_1(x, \alpha), H_2(x, \alpha)).$$

Guaranteeing a fixed point for H is equivalent to guaranteeing a solution to (4.1)-(4.2). We endow the space $\mathscr{L}^2 \times \mathbb{R}$ with the norm

$$\|(x,\alpha)\| = \max\{\|x\|, |\alpha|\}.$$

Define

$$r = \sup_{t\in\mathbb{R}} |f(t)|.$$

The existence of r is guaranteed by the continuity of $f : \mathbb{R} \to \mathbb{R}$ along with the fact that $f(\infty)$ and $f(-\infty)$ exist and are finite. Choose $\alpha_0 > r$ so that (4.3) and (4.4) are satisfied and let $\delta = \alpha_0 + r$. As stated in [4], $|P_k(t)| \leq 1$ for all $t \in [-1,1]$. We know that f and ME are bounded, so there exists $b_1 > 0$ such that for any $x \in \mathscr{L}^2$, $\alpha \in \mathbb{R}$,

$$\|H_1(x,\alpha)\| \leq b_1.$$

Let $\mathscr{B}$ be the set

$$\mathscr{B} = \{(x,\alpha) \in \mathscr{L}^2 \times \mathbb{R} : \|x\| \leq b_1, |\alpha| \leq \delta\}.$$

Clearly $\|H_1(x,\alpha)\| \leq b_1$ for all $(x,\alpha) \in \mathscr{B}$ by construction, so it suffices to show that $\|H_2(x,\alpha)\| \leq \delta$ for all $(x,\alpha) \in \mathscr{B}$ in order to show that $H(\mathscr{B}) \subset \mathscr{B}$.
Suppose that $\alpha \in [\alpha_0, \delta]$. Then

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t)\, dt > 0$$

and therefore $H_2(x,\alpha) < \alpha \leq \delta$. Further, since $\left| \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t)\, dt \right| \leq r$ it follows that

$$\alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t)\, dt \geq \alpha_0 - r \geq 0.$$

Therefore if $\alpha \in [\alpha_0, \delta]$ then $|H_2(x,\alpha)| \in [0,\delta]$. Suppose that $\alpha \in [0, \alpha_0)$. Then

$$\begin{aligned} |H_2(x,\alpha)| &= \left| \alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))] P_k(t)\, dt \right| \\ &\leq \alpha_0 + r \\ &= \delta. \end{aligned}$$

Therefore, if $(x,\alpha) \in \mathscr{B}$ and $\alpha \in [0,\delta]$ then $|H_2(x,\alpha)| \leq \delta$.

A symmetric argument can be used to show that if $(x,\alpha) \in \mathscr{B}$ and $\alpha \in [-\delta, 0]$ then $|H_2(x,\alpha)| \leq \delta$.

Therefore, $H(\mathscr{B}) \subset \mathscr{B}$. Since $H : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2 \times \mathbb{R}$ is compact (following from the compactness of M) and $\mathscr{B}$ is closed, bounded, and convex it follows that H is guaranteed a fixed point by Schauder's Fixed Point Theorem. □

4.2.3 The Case of Weak Nonlinearities

In this subsection, assume that our nonlinearity is of the form $\varepsilon f(x(t))$ where ε is a real parameter and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. That is, we now examine boundary value problems of the form

$$[(1-t^2)x'(t)]' + \mu x(t) = \varepsilon f(x(t))$$

subject to the condition that the limits appearing in (4.2) exist and are finite. Due to the fact that we will impose differentiability conditions on the function-valued operator representing our nonlinearity, we consider operators defined on the space of continuous functions. Again let $\mathscr{C}$ denote the space of continuous functions on $[-1,1]$ endowed with the supremum norm and let

$$\mathscr{D} = \left(C^2[-1,1], \|\cdot\|_\infty\right) \subset \mathscr{C}$$

where $C^2[-1,1]$ denotes the set of twice continuously differentiable functions on $[-1,1]$. In this section, denote $\mathscr{L} : \mathscr{D} \to \mathscr{C}$ by

$$[\mathscr{L}x](t) = [(1-t^2)x'(t)]' + \mu x(t)$$

and $F : \mathscr{C} \times \mathbb{R} \to \mathscr{C}$ by

$$[F(x,\varepsilon)](t) = \varepsilon f(x(t)).$$

Suppose again that $\mu = k(k+1)$.

In this section, for $x \in \mathscr{C}$ and $l \in \mathbb{N}$ we denote

$$x_l = \left[\left(l + \frac{1}{2}\right) \int_{-1}^{1} P_l(t)x(t)dt\right].$$

Define the projections $U : \mathscr{C} \to \mathscr{C}$ by

$$[Ux](t) = x_k P_k(t)$$

and $E : \mathscr{C} \to \mathscr{C}$ by $E = I - U$. Note that the map $\mathscr{L}$ restricted to $\mathscr{D} \cap$ Im(L) is a bijection onto Im(L)=Im(E). Therefore, it follows that there exists a linear map M :Im$(E) \to \mathscr{D} \cap$ Im(L) satisfying

$$\mathscr{L}Mh = h$$

for all $h \in \text{Im}(L)$ and

$$M\mathscr{L}x = Ex = (I-U)x$$

for all $x \in \mathscr{D}$. Note that M is simply

$$\left[\mathscr{L}|_{\mathscr{D}\cap\text{Im}(L)}\right]^{-1}$$

and observe further that M is continuous. We note that solving

$$\mathscr{L}x = F(x,\varepsilon)$$

is equivalent to solving the system

$$\begin{cases} (I-U)x - MEF(x,\varepsilon) = 0 \\ \qquad \text{and} \\ U(f\circ x) = 0. \end{cases}$$

Define the map $G : \mathscr{D}\times\mathbb{R} \to \text{Im}(L)\times \text{Ker}(L)$ by

$$G(x,\varepsilon) = \begin{bmatrix} (I-U)x - MEF(x,\varepsilon) \\ U(f\circ x) \end{bmatrix}.$$

It is well known that F is continuously differentiable with respect to x and for any $x \in \mathscr{C}$, $\varepsilon \in \mathbb{R}$,

$$\left(\frac{\partial F}{\partial x}(x,\varepsilon)h\right)(t) = \varepsilon f'(x(t))h(t).$$

From that it follows that

$$\frac{\partial G}{\partial x}(x,\varepsilon)$$

exists for all $(x,\varepsilon) \in \mathscr{C}\times\mathbb{R}$ and is given by

$$\frac{\partial G}{\partial x}(x,\varepsilon)w = \begin{bmatrix} [(I-U) - \varepsilon ME(f'\circ x)]w \\ U(f'\circ x)w \end{bmatrix}.$$

Let $\bar{x} = \alpha P_k$ for $\alpha \in \mathbb{R}$. For $(\bar{x},0)$ and $w \in \mathscr{C}$:

$$\frac{\partial G}{\partial x}(\bar{x},0)w = \begin{bmatrix} (I-U)w \\ U(f'\circ\bar{x})w \end{bmatrix}.$$

Since $F \in C^1$ and M is continuous, it follows that $G \in C^1$. For $w \in \mathscr{C}$, we can decompose w as

$w = u + v$ where

$$u = w_k P_k$$
$$v = w - w_k P_k.$$

With this in mind, we can rewrite the previous expression as

$$\frac{\partial G}{\partial x}(\bar{x},0)(u+v) = \begin{bmatrix} v \\ U(f' \circ \bar{x})(u+v) \end{bmatrix}.$$

Define the maps H_1 :Ker$(L) \to \mathbb{R}$ by

$$H_1(u) = \int_{-1}^{1} P_k(t) f(u(t)) dt,$$

$H_2 : \mathbb{R} \to$Ker(L) by $H_2(\alpha) = \alpha P_k$ and finally $H : \mathbb{R} \to \mathbb{R}$ by $H = H_1 \circ H_2$. That is,

$$H(\alpha) = \int_{-1}^{1} P_k(t) f(\alpha P_k(t)) dt.$$

Therefore for any number in $\mathbb{R}$, H' : Ker$(L) \to \mathbb{R}$ exists and for $\beta \in \mathbb{R}$,

$$[H'(\alpha)](\beta) = \int_{-1}^{1} P_k(t)[f'(\alpha P_k(t))](\beta P_k(t)) dt.$$

We are now ready to give conditions for the solvability of our boundary value problems examined this section.

Theorem 7. *Suppose that there exists $\alpha_0 \in \mathbb{R}$ such that $H(\alpha_0) = 0$ and $H'(\alpha_0) \neq 0$. Then there exists and open neighborhood $I \subset \mathbb{R}$ of 0 such that for any $\varepsilon \in I$ there exists a solution to*

$$[(1-t^2)x'(t)]' + \mu x(t) = \varepsilon f(x(t))$$

satisfying the condition that the limits appearing in (4.2) *exist and are finite.*

Proof. Recall that $G \in C^1$ and let $\bar{x} = \alpha_0 P_k$. Then $(I-U)\bar{x} - MEF(\bar{x},0) = 0$ and

$$\begin{aligned} UF(\bar{x}) &= \int_{-1}^{1} P_k(t) f(\alpha_0 P_k(t)) dt \\ &= H(\alpha_0 P_k(t)) \\ &= 0. \end{aligned}$$

Therefore $G(\bar{x},0) = 0$. We now wish to show that $\frac{\partial G}{\partial x}(\bar{x},0)$ is a bijection from $\mathscr{C}$ onto Im$(L) \times$ Ker(L). Since $\frac{\partial G}{\partial x}(\bar{x},0)$ is linear, in order to show this map is injective it suffices to show that it has a trivial kernel. Suppose that $\frac{\partial G}{\partial x}(\bar{x},0)(u+v)=0$. Then

$$0=v$$

and so

$$0=U(f'\circ\bar{x})u=\left[\int_{-1}^{1}P_k(t)[f'(\alpha_0 P_k(t))]u(t)dt\right]$$

implying that $u=0$ due to our assumption that $H'(\alpha_0)\neq 0$. Note that since $H'(\alpha_0)$ is a nonzero linear map from $\mathbb{R}\to\mathbb{R}$, then it is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. This implies that the map $U(f'\circ\bar{x})$ restricted to Ker(L) is a bijection onto Ker(L). Given $h_1\in$Im(L) and $h_2\in$ Ker(L), we have that

$$\frac{\partial G}{\partial x}(\bar{x},0)(h_1+\hat{h}_2)=(h_1,h_2)$$

where $\hat{h}_2$ is the unique element of Ker(L) that maps to h_2 under $U(f'\circ\bar{x})$. So $\frac{\partial G}{\partial x}(\bar{x},0)$ is surjective and therefore a bijection from $\mathscr{C}$ onto Im(L)×Ker(L). By the implicit function theorem [16], there exists a neighborhood $V_0\subset\mathbb{R}$ of 0 on which there exists a continuous function $\phi: V_0\to\mathscr{D}$ satisfying

$$G(\phi(\varepsilon),\varepsilon)=0$$

for all $\varepsilon\in V_0$. Denoting $\phi(\varepsilon)=x_\varepsilon$ we have that

$$\begin{aligned}0&=G(\phi(\varepsilon),\varepsilon)\\&=G(x_\varepsilon,\epsilon)\\&=\mathscr{L}x_\varepsilon-F(x_\varepsilon,\varepsilon).\end{aligned}$$

In other words for any $\varepsilon\in V_0$ we can guarantee a solution to

$$[(1-t^2)x'(t)]'+\mu x(t)=\varepsilon f(x(t))$$

satisfying the condition that the limits in (4.2) exist and are finite. □

Remark 5. *Let x_ε denote the solution in $\mathscr{D}$ guaranteed by the implicit function theorem to*

$$[(1-t^2)x'(t)]'+\mu x(t)=\varepsilon f(x(t)).$$

Note that

$$\lim_{\varepsilon\to 0}x_\varepsilon=\bar{x}$$

where this limit is in the sense of uniform convergence. That is, solutions guaranteed by the above *theorem are ones that emanate from a certain solution to the linear homogeneous problem.*

4.3 Examples

Example 5. *Consider the boundary value problem*

$$[(1-t^2)x'(t)]' = \varepsilon f(x(t))$$

on $(-1,1)$ *subject to the condition that the limits in* (4.2) *exist and are finite.*

Suppose that there exists a number α_0 *such that* $f(\alpha_0)=0$ *and* $f'(\alpha_0)\neq 0$. *Then since the constant Legendre polynomial is* $P_0(t)=1$, *for* $\alpha\in\mathbb{R}$,

$$\int_{-1}^{1} P_0(t) f(\alpha P_0(t))dt = \int_{-1}^{1} f(\alpha)dt$$

so then $\int_{-1}^{1} P_0(t)f(\alpha_0 P_0(t))dt = 0$. *However, provided* $\beta\neq 0$,

$$\int_{-1}^{1} P_0(t)[f'(\alpha_0 P_0(t))](\beta P_0(t))dt = \beta\int_{-1}^{1} f'(\alpha_0)dt$$

so then $\int_{-1}^{1} P_0(t)[f'(\alpha_0 P_0(t))](\beta P_0(t))dt \neq 0$.

Example 6. *Consider the boundary value problem*

$$[(1-t^2)x'(t)]' + 2x(t) = \varepsilon f(x(t))$$

subject to the condition that the limits in (4.2) *exist and are finite The constant Legendre polynomial is* $P_1(t)=t$, *so the condition in theorem* 7 *is satisfied provided there exists a number* α_0 *satisfying*

$$\int_{-1}^{1} t f(\alpha_0 t)dt = 0$$

and $f(\alpha_0)\neq f(-\alpha_0)$.

CHAPTER

5

ON WEAKLY NONLINEAR BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

5.1 Preliminaries

We first consider continuous nonlinear boundary value problems on the infinite interval $[0,\infty)$ of the form

$$x'(t) - A(t)x(t) = h(t) + \varepsilon f(t, x(t)) \tag{5.1}$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t))\,dt \tag{5.2}$$

where A is a continuous $n \times n$ matrix-valued function on $[0,\infty)$, f and g are continuously differentiable maps from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$, and Γ is a bounded linear map from the space of bounded, continuous functions on $[0,\infty)$ into $\mathbb{R}^n$. Our main focus will be on the case where the bounded, continuous function h and vector $u \in \mathbb{R}^n$ are such that the linear problem

$$x'(t) - A(t)x(t) = h(t) \tag{5.3}$$

subject to

$$\Gamma(x) = u \tag{5.4}$$

has a solution.
Then we will investigate

$$x(k+1) - A(k)x(k) = h(k) + \varepsilon f(k, x(k)) \tag{5.5}$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = u + \varepsilon \sum_{k=0}^{\infty} g(k, x(k)). \tag{5.6}$$

where C_k is an $n \times n$ real-valued matrix for all nonnegative integers k. Denoting the nonnegative integers by $\mathbb{Z}_+$, we have that the maps $f : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, ε is a real parameter, and $A(k)$ is a nonsingular $n \times n$ real-valued matrix for all $k \in \mathbb{Z}_+$.

In our analysis, we use a scheme somewhat similar to the Lyapunov-Schmidt procedure and results are obtained through an application of the implicit function theorem for Banach spaces. We provide a framework which allows us to determine cases when for ε sufficiently small in magnitude, (5.1)-(5.2) has solutions which emanate from a particular solution to (5.3)-(5.4).

5.2 Main Results

5.2.1 Differential Equations

We use $\mathscr{C}$ to denote the space of bounded, continuous functions from $[0, \infty)$ into $\mathbb{R}^n$, and pair this space with the norm $\|x\|_\infty = \sup_{t \geq 0} |x(t)|$. It is clear that $(\mathscr{C}, \|\cdot\|_\infty)$ is a Banach space. We use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|$ for the standard operator norm on the space of $n \times n$ real-valued matrices. Throughout this section, we assume that $\Gamma : \mathscr{C} \to \mathbb{R}^n$ is a bounded linear map and write

$$\|\Gamma\| = \sup_{\|x\|_\infty = 1} |\Gamma(x)|.$$

For previous results establishing existence of solutions to boundary value problems on infinite intervals the reader is referred to [15] in the continuous case and [27], [31], and [32] in the discrete case. **In all of these results, it is assumed that a certain corresponding linear problem is invertible. In these results, we remove this assumption and the situation gets more delicate mathematically.**

Let $\Phi(t)$ denote the fundamental matrix for $x'(t) - A(t)x(t) = 0$ such that $\Phi(0) = I$ and Φ_i denote the i^{th} column of Φ for $1 \leq i \leq n$. As mentioned in the introduction, our analysis will include a discussion of a set of closely related linear problems. Throughout the paper, the reader will see that conditions we will impose on A guarantee that for any $\psi \in \mathscr{C}$, $\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)\psi(s)ds \in \mathscr{C}$.

We define Λ as the $n \times n$ matrix

$$\Lambda = [\Gamma(\Phi_1(\cdot))|\Gamma(\Phi_2(\cdot))|\cdots|\Gamma(\Phi_n(\cdot))].$$

Note that a function $x \in \mathscr{C}$ is a solution to

$$x'(t) - A(t)x(t) = 0$$

subject to

$$\Gamma(x) = 0$$

if and only if $x(0) \in \ker(\Lambda)$. Given $\psi \in \mathscr{C}$ and $w \in \mathbb{R}^n$, we know by variation of parameters that any solution to $x'(t) - A(t)x(t) = \psi(t)$ is of the form

$$x(t) = \Phi(t)x(0) + \Phi(t)\int_0^t \Phi^{-1}(s)\psi(s)ds.$$

Imposing the condition that $\Gamma(x) = w$ we get that

$$\Lambda x(0) = w - \Gamma\left(\Phi(\cdot)\int_0^{\cdot} \Phi^{-1}(s)\psi(s)ds\right).$$

Let p denote the dimension of $\ker(\Lambda)$ for some integer $0 \leq p \leq n$. If $p = 0$, it is clear that (5.3)-(5.4) has a unique solution. The bulk of our results concern the case where $p \geq 1$. In this case, we let W be a matrix whose columns form a basis for $[\ker(\Lambda^T)]^{\perp}$. Note that there exists a solution to the linear boundary value problem

$$x'(t) - A(t)x(t) = \psi(t)$$

subject to

$$\Gamma(x) = w$$

if and only if

$$W^T\left[w - \Gamma\left(\Phi(\cdot)\int_0^{\cdot} \Phi^{-1}(s)\psi(s)ds\right)\right] = 0.$$

Throughout this paper we will mainly be studying the structure of the solution set to (5.1)-(5.2) in the cases when the matrix Λ is singular and the corresponding linear problem (5.3)-(5.4) has a solution, or equivalently where h and u satisfy

$$W^T\left[u-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)h(s)ds\right)\right]=0.$$

Based on the discussion above, it is clear that there exists a solution to the nonlinear boundary value problem

$$x'(t)-A(t)x(t)=h(t)+\varepsilon f(t,x(t))$$

subject to

$$\Gamma(x)=u+\varepsilon\int_0^{\infty}g(t,x(t))dt$$

for $\varepsilon\neq 0$ if there exists $x\in\mathscr{C}$ and $v\in\ker(\Lambda)$ satisfying

$$x(t)=\Phi(t)v+\Phi(t)\int_0^t\Phi^{-1}(s)[h(s)+\varepsilon f(s,x(s))]ds$$

and

$$W^T\left[\int_0^{\infty}g(t,x(t))dt-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)f(s,x(s))ds\right)\right]=0.$$

Remark 6. *It should be observed that the problems we're considering include ones of the form*

$$\dot{x}(t)-A(t)x(t)=\varepsilon f(t,x(t))$$

subject to

$$\int_0^{\infty}B(t)x(t)dt+\sum_{k=0}^{\infty}C_kx(t_k)=\varepsilon\int_0^{\infty}g(t,x(t))dt$$

where B is a function-valued matrix whose entries are integrable functions from $[0,\infty)$ into $\mathbb{R}^n$. and C_k for $k\geq 0$ is an $n\times n$ matrix with

$$\sum_{k=0}^{\infty}\|C_k\|<\infty.$$

.

We now list the following set of conditions which we will impose in our first theorem.

I) There exists positive constants K,α such that

$$\|\Phi(t)\Phi^{-1}(s)\|\leq Ke^{-\alpha(t-s)}$$

for all $t \geq s \geq 0$.

II) For any compact subset $S \subset \mathbb{R}^n$, $\frac{\partial f}{\partial x}$ is uniformly continuous on $[0,\infty) \times S$ and

$$\sup_{t \geq 0} \left\| \frac{\partial f}{\partial x}(t,0) \right\| < \infty.$$

III) For any compact subset $S \subset \mathbb{R}^n$, $\frac{\partial g}{\partial x}$ is uniformly continuous on $[0,\infty) \times S$ and

$$\int_0^\infty \left\| \frac{\partial g}{\partial x}(t,0) \right\| dt < \infty.$$

IV) For all $h \in \mathscr{C}$,

$$\int_0^\infty |g(t,h(t))| dt < \infty.$$

V) There exists an integrable $s : [0,\infty) \to \mathbb{R}$ satisfying

$$\left\| \frac{\partial g}{\partial x}(t,x_1) - \frac{\partial g}{\partial x}(t,x_2) \right\| \leq s(t)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

Note that for $x \in \mathscr{C}$, $v \in \ker(\Lambda)$, $\varepsilon \in \mathbb{R}$, and $t \geq 0$ we have that,

$$\begin{aligned}
&\left| x(t) - \Phi(t)v - \Phi(t) \int_0^t \Phi^{-1}(s)[h(s) + \varepsilon f(s,x(s))] ds \right| \\
&\leq \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + \int_0^\infty \|\Phi(t)\Phi^{-1}(s)\| |h(s) + \varepsilon f(s,x(s))| ds \\
&\leq \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + [\|h\|_\infty + |\varepsilon| \sup_{s \geq 0} |f(s,x(s))|] K \int_0^\infty e^{-\alpha(t-s)} ds \\
&= \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + [\|h\|_\infty + |\varepsilon| \sup_{s \geq 0} |f(s,x(s))|] K \alpha^{-1}.
\end{aligned}$$

Also observe that

$$\begin{aligned}
&\left| W^T \left[\int_0^\infty g(t,x(t)) dt - \Gamma\left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s,x(s)) ds \right) \right] \right| \\
&\leq \|W^T\| \left[\int_0^\infty |g(t,x(t))| dt - \|\Gamma\| \left(\int_0^\infty \|\Phi(t)\Phi^{-1}(s)\| |f(s,x(s))| ds \right) \right] \\
&\leq \|W^T\| \left[\int_0^\infty |g(t,x(t))| dt - \|\Gamma\| \left(\sup_{s \geq 0} |f(s,x(s))| K \int_0^\infty e^{-\alpha(t-s)} ds \right) \right] \\
&= \|W^T\| \left[\int_0^\infty |g(t,x(t))| dt - \|\Gamma\| \left(\sup_{s \geq 0} |f(s,x(s))| K \alpha^{-1} \right) \right] < \infty.
\end{aligned}$$

From this is follows that H given by

$$H((x,v),\varepsilon)=\begin{bmatrix} H_1((x,v),\varepsilon) \\ H_2((x,v),\varepsilon) \end{bmatrix}=\begin{bmatrix} x(t)-\Phi(t)v-\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)[h(s)+\varepsilon f(s,x(s))]ds \\ \\ W^T\left[\int_0^\infty g(t,x(t))dt-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)f(s,x(s))ds\right)\right] \end{bmatrix}$$

is a well-defined map from $\mathscr{C}\times\ker(\Lambda)\times\mathbb{R}$ to $\mathscr{C}\times\mathbb{R}^p$

Our main result will involve an application of the implicit function theorem for Banach spaces [16]. This requires continuous Fréchet differentiability of H.

In the following lemma, for $i=1,2$ we use $\frac{\partial H_i}{\partial(x,v)}$ to denote the partial (Fréchet) derivative of H_i with respect to (x,v).

Lemma 4. *Suppose that $I)-V)$ hold. Then for any $((x,v),\varepsilon)\in\mathscr{C}\times\ker(\Lambda)\times\mathbb{R}$, the bounded linear maps $\frac{\partial H_1}{\partial(x,v)}((x,v),\varepsilon)$ and $\frac{\partial H_2}{\partial(x,v)}((x,v),\varepsilon)$ exist and are given by*

$$\left[\frac{\partial H_1}{\partial(x,v)}((x,v),\varepsilon)\right](\psi,w)(t)=\psi(t)-\Phi(t)w-\varepsilon\left(\Phi(t)\int_0^t\Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)$$

and

$$\left[\frac{\partial H_2}{\partial(x,v)}((x,v),\varepsilon)\right](\psi,w)=W^T\left[\int_0^\infty\frac{\partial g}{\partial x}(t,x(t))\psi(t)dt-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)\right].$$

Further, H_1 and H_2 are continuously (Fréchet) differentiable.

Proof. For $x,\psi\in\mathscr{C}$ and $v,w\in\ker(\Lambda)$ we have that

$$H_1((x+\psi,v+w),\varepsilon)-H_1((x,v),\varepsilon)-\psi(t)+\Phi(t)w+\varepsilon\left(\Phi(t)\int_0^t\Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)$$
$$=\varepsilon\left(\Phi(t)\int_0^t\Phi^{-1}(s)\left[f(s,(x+h)(s))-f(s,x(s))-\frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]ds\right).$$

For $a,b\in\mathbb{R}^n$, let $L(a,b)$ denote the straight line segment connecting a and b. Note that by the mean value theorem, for all $t\geq 0$ we have that

$$\left|f(t,(x+\psi)(t))-f(t,x(t))\right|\leq\sup_{\nu(t)\in L(x(t),(x+\psi)(t))}\left|\frac{\partial f}{\partial x}(t,\nu(t))\psi(t)\right|$$

and

$$\left|g(t,(x+\psi)(t))-g(t,x(t))\right|\leq\sup_{\zeta(t)\in L(x(t),(x+\psi)(t))}\left|\frac{\partial g}{\partial x}(t,\zeta(t))\psi(t)\right|.$$

Then we have that for $t \geq 0$,

$$\left|\left(\int_0^t \Phi(t)\Phi^{-1}(s)\left[f(s,(x+h)(s))-f(s,x(s))-\frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]ds\right)\right|$$
$$\leq \sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\left[\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right]\right\|\left(\int_0^\infty \left\|\Phi(t)\Phi^{-1}(s)\right\| ds\right)\|\psi\|_\infty$$
$$\leq \sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\left[\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right]\right\| K\alpha^{-1}\|\psi\|_\infty$$

and $\sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\left[\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right]\right\| K\alpha^{-1} \to 0$ as $\|\psi\|_\infty \to 0$ by II).

We also have that

$$\Bigg|H_2((x+\psi,v+w),\varepsilon)-H_2((x,v),\varepsilon)-$$
$$W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,x(t))\psi(t)dt-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)\right]\Bigg|$$
$$=\Bigg|W^T\left(\int_0^\infty\left[g(s,(x+\psi)(s))-g(s,x(s))-\frac{\partial g}{\partial x}(s,x(s))\psi(s)\right]ds\right.$$
$$\left.-\Gamma\left(\Phi(t)\int_0^t\Phi^{-1}(s)\left[f(s,(x+\psi)(s))-f(s,x(s))-\frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]ds\right)\right]\Bigg|$$
$$\leq\left(\|W^T\|\int_0^\infty \sup_{\zeta(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial g}{\partial x}(s,\zeta(s))-\frac{\partial g}{\partial x}(s,x(s))\right\| ds\|\psi\|_\infty\right.$$
$$\left.+\|W^T\|\|\Gamma\|\sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right\|\int_0^\infty\left\|\Phi(t)\Phi^{-1}(s)\right\|dt\right)\|\psi\|_\infty$$
$$\leq\|W^T\|\left(\|s\|_{L^1}\|\psi\|_\infty+\|\Gamma\|\sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right\|K\alpha^{-1}\right)\|\psi\|_\infty$$

where $\|\cdot\|_{L^1}$ denotes the standard norm on $L^1[0,\infty)$. Note that

$$\|W^T\|\left(\|s\|_{L^1}\|\psi\|_\infty+\|\Gamma\|\sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right\|K\alpha^{-1}\right)\to 0$$

as $\|\psi\|_\infty \to 0$ by II). Now we will show that the map

$$(x,v)\mapsto\frac{\partial H_i}{\partial(x,v)}$$

is continuous for $i=1,2$. Note that for $\|\psi\|_\infty=1$,

$$\begin{aligned}
&\left\|\left[\frac{\partial H_1}{\partial(x,v)}(x_1,v_1)-\frac{\partial H_1}{\partial(x,v)}(x_2,v_2)\right]\psi\right\|_\infty\\
&=\sup_{t\in[0,\infty)}\left|\left(\int_0^t\Phi(t)\Phi^{-1}(s)\left[\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right]\psi(s)ds\right)\right|\\
&\le\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|\left(\int_0^\infty\left\|\Phi(t)\Phi^{-1}(s)\right\|dt\right)\\
&\le K\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|\alpha^{-1}
\end{aligned}$$

and $K\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|\alpha^{-1}\to 0$ as $\|x_1-x_2\|_\infty\to 0$. We also have that

$$\begin{aligned}
&\left|\left[\frac{\partial H_2}{\partial(x,v)}(x_1,v_1)-\frac{\partial H_2}{\partial(x,v)}(x_2,v_2)\right]\psi\right|\\
&\le\|W^T\|\left(\int_0^\infty\left\|\frac{\partial g}{\partial x}(s,x_1(s))-\frac{\partial g}{\partial x}(s,x_2(s))\right\|ds\right.\\
&\left.+\|\Gamma\|\int_0^\cdot\|\Phi(\cdot)\Phi^{-1}(s)\|\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|ds\Big]\Big)\Big)\right|\\
&\le\|W^T\|\left(\|x_1-x_2\|_\infty\|s\|_{L^1}+K\alpha^{-1}\|\Gamma\|\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|\right).
\end{aligned}$$

Note that $\|W^T\|\left(\|x_1-x_2\|_\infty\|s\|_{L^1}+K\alpha^{-1}\|\Gamma\|\left\|\frac{\partial f}{\partial x}(s,x_1(s))-\frac{\partial f}{\partial x}(s,x_2(s))\right\|\right)\to 0$ as $\|x_1-x_2\|_\infty\to 0$, proving our desired result. □

Remark 7. *The most interesting case and the one we will focus mostly on is the case where Λ is singular. In this case, solving the nonlinear boundary value problem* (5.1)-(5.2) *is equivalent to solving the operator equation $H_1((x,v),\varepsilon)=H_2((x,v),\varepsilon)=0$. For the sake of completeness in our analysis it is worth mentioning the case where Λ is invertible. If Λ is invertible, then* (5.3)-(5.4) *has a unique solution and the matrix W does not exist. The nonlinear boundary value problem* (5.1)-(5.2) *is then equivalent to finding a continuous function x and $v\in\mathbb{R}^n$ satisfying*

$$x(t)-\Phi(t)v-\Phi(t)\int_0^t\Phi^{-1}(s)[h(s)+\varepsilon f(s,x(s))]ds=0$$

where

$$v=\Lambda^{-1}\left[u+\varepsilon\int_0^\infty g(t,x(t))dt-\Gamma\left(\Phi(\cdot)\int_0^\cdot\Phi^{-1}(s)[h(s)+\varepsilon f(s,x(s))]ds\right)\right].$$

Define $\Psi : \mathscr{C} \times \mathbb{R}^{n+1} \to \mathscr{C} \times \mathbb{R}^n$ *by* $[\Psi_1, \Psi_2]^T$ *where*

$$\Psi_1((x,v),\varepsilon)(t) = x(t) - \Phi(t)v - \Phi(t)\int_0^t \Phi^{-1}(s)[h(s) + \varepsilon f(s,x(s))]ds$$

and

$$\Psi_2((x,v),\varepsilon)(t) = v - \Lambda^{-1}\left[u + \varepsilon \int_0^\infty g(t,x(t))dt - \Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)[h(s)+\varepsilon f(s,x(s))]ds\right)\right].$$

and note that $\Psi((\bar{x}, v_0), 0) = 0$ *where* $\bar{x}$ *denotes the unique solution to* $x'(t) - A(t)x(t) = h(t)$ *satisfying* $x(0) = v_0$ *where*

$$v_0 = \Lambda^{-1}\left[u - \Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)h(s)ds\right)\right].$$

Further note that by an analogous argument to the one appearing in the previous lemma, Ψ *is continuously differentiable at each point in* $\mathscr{C} \times \mathbb{R}^{n+1}$ *under conditions* $I) - V)$ *and*

$$\frac{\partial \Psi}{\partial(x,v)}((\bar{x}, v_0), 0)[\psi, w]^T = [\psi(\cdot) + \Phi(\cdot)w, w]^T$$

which is clearly a bijection from $\mathscr{C} \times \mathbb{R}^n$ *to* $\mathscr{C} \times \mathbb{R}^n$. *Therefore by the implicit function theorem for Banach spaces, there exists a solution to* (5.1) - (5.2) *for sufficiently small* ε *and those solutions converge uniformly to* $\bar{x}$ *as* ε *goes to* 0.

Now we shift our focus back to the case where Λ is singular. For the sake of notation, for any $y \in \mathbb{R}^n$ we define the function $x_y(t) = \Phi(t)[y + \Gamma\big(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)f(s,x(s))ds\big)] + \Phi(t)\int_0^t \Phi^{-1}(s)h(s)ds$. We also write

$$\frac{\partial H}{\partial(x,v)} = \begin{bmatrix} \frac{\partial H_1}{\partial(x,v)} \\ \\ \frac{\partial H_2}{\partial(x,v)} \end{bmatrix}.$$

Theorem 8. *Suppose that* $I) - V)$ *hold and that there exists* $y \in \ker(\Lambda)$ *such that*

$$W^T\left[\int_0^\infty g(t,x_y(t))dt - \Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)f(s,x_y(s))ds\right)\right] = 0$$

and $\phi : \ker(\Lambda) \to \mathbb{R}^p$ *given by*

$$\phi(w) = W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,x_y(t))\Phi(t)dt - \Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x_y(s))\Phi(s)ds\right)\right]w$$

is a bijection from $\ker(\Lambda) \subset \mathbb{R}^n$ *onto* $\mathbb{R}^p$. *Then there exists* ε_0 *such that for all* $|\varepsilon| \le \varepsilon_0$, *the boundary* *value problem*

$$x'(t) = A(t)x(t) = h(t) + \varepsilon f(t, x(t))$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t))dt.$$

is guaranteed a solution x_ε. *Moreover* $\|x_\varepsilon - x_y\|_\infty \to 0$ *as* $\varepsilon \to 0$.

Proof. We have shown that H is continuously differentiable. Note that $H_1((x_y, y), 0) = 0 = H_2((x_y, y), 0)$. Suppose that $\frac{\partial H}{\partial(x,v)}((x_y, y), 0)(z, v) = 0$. Then $z(t) = \Phi(t)v$ for all $t \geq 0$ and therefore

$$W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s))\Phi(s)ds - \Gamma\left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s))\Phi(s)ds \right) \right] v = 0$$

implying that $v = 0$. Therefore $\frac{\partial H}{\partial(x,v)}((x_y, y), 0)$ is one-to-one. Let $(\hat{h}, \hat{v}) \in \mathscr{C} \times \mathbb{R}^p$. Then by assumption there exists a unique $w \in \ker(\Lambda)$ satisfying

$$W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s))\Phi(s)ds - \Gamma\left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s))\Phi(s)ds \right) \right] w = \hat{v} - v_*.$$

where v_* denotes the vector

$$v_* = W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s))\hat{h}(s)ds - \Gamma\left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s))\hat{h}ds \right) \right].$$

Therefore

$$\left[\frac{\partial H_1}{\partial(x,v)}((x_y, y), 0) \right] (\hat{h} + \Phi(\cdot)w, w)(t) = \hat{h}(t)$$

and

$$\left[\frac{\partial H_2}{\partial(x,v)}((x_y, y), 0) \right] (\hat{h} + \Phi(\cdot)w, w)(t) = (\hat{v} - v_*) + v_* = \hat{v}$$

and $\frac{\partial H}{\partial(x,v)}((x_y, y), 0)$ is a bijection from $\mathscr{C} \times \ker(\Lambda)$ onto $\mathscr{C} \times \mathbb{R}^p$. Our result follows from the implicit function theorem for Banach spaces. □

In results up to this point, we assume that h is simply an element of $\mathscr{C}$. In the following set of results, we investigate problems where we know that $h \in \mathscr{C} \cap L^1[0, \infty)$. In this case, we impose the following set of conditions.

$I')$ There exists positive constant K such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq K$$

for all $t \geq s \geq 0$.

$II')$ $\frac{\partial g}{\partial x}$ is uniformly continuous on $[0,\infty)\times\mathbb{R}^n$ and

$$\int_0^\infty \left\| \frac{\partial g}{\partial x}(t,0) \right\| dt < \infty.$$

$III')$ For all $h \in \mathscr{C}$,

$$\int_0^\infty |g(t,h(t))| dt < \infty.$$

$IV')$ $\frac{\partial f}{\partial x}$ is uniformly continuous on $[0,\infty)\times\mathbb{R}^n$ and

$$\int_0^\infty \left\| \frac{\partial f}{\partial x}(t,0) \right\| dt < \infty.$$

$V')$ There exists $s \in L^1[0,\infty)$ satisfying

$$\left\| \frac{\partial g}{\partial x}(t,x_1) - \frac{\partial g}{\partial x}(t,x_2) \right\| \leq s(t)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

$VI')$ There exists $h_1 \in L^1[0,\infty)$ such that for every compact subset S of $\mathbb{R}^n$ there exists a constant C satisfying

$$|f(t,x)| \leq C h_1(t)$$

for all $t \geq 0$ and $x \in S$ and

$$|f(t,x_1) - f(t,x_2)| \leq h_1(t)|x_1 - x_2|$$

for all $x_1, x_2 \in S$ and $t \geq 0$.

$VII')$ There exists $h_2 \in L^1[0,\infty)$ such that for any compact subset $S \subset \mathbb{R}^n$,

$$\left\| \frac{\partial f}{\partial x}(k,x_1) - \frac{\partial f}{\partial x}(k,x_2) \right\| \leq h_2(k)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in S$.

Before stating the main theorem in this section, it is worth mentioning for the sake of completeness that if Λ is invertible, an analogous argument to the one appearing in remark 2 holds. This is because Ψ is continuously differentiable on $\mathscr{C}\times\mathbb{R}^{n+1}$ under conditions $I')-VII')$ and satisfies the conditions of the implicit function theorem at the point $((\bar{x},v_0),0)$ where $\bar{x}$ and v_0 are defined the same as in remark 2. Therefore, we can guarantee solutions to $(5.1)-(5.2)$ for ε sufficiently small and these solutions converge uniformly to $\bar{x}$ as the absolute value of ε goes to zero.

Theorem 9. *Suppose that $I')-VII')$ hold and that there exists $y\in\ker(\Lambda)$ such that*

$$W^T\left[\int_0^\infty g(t,x_y(t))dt-\Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)f(s,x_y(s))ds\right)\right]=0$$

and $\phi:\ker(\Lambda)\to\mathbb{R}^p$ defined by

$$\phi(w)=W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,x_y(t))\Phi(t)dt-\Gamma\left(\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x_y(s))\Phi(s)ds\right)\right]w$$

is a bijection from $\ker(\Lambda)\subset\mathbb{R}^n$ onto $\mathbb{R}^p$. Then there exists ε_0 such that for all $|\varepsilon|\le\varepsilon_0$, the boundary value problem

$$x'(t)-A(t)x(t)=h(t)+\varepsilon f(t,x(t))$$

subject to

$$\Gamma(x)=u+\varepsilon\int_0^\infty g(t,x(t))dt.$$

is guaranteed a solution x_ε. Moreover $\|x_\varepsilon-x_y\|_\infty\to 0$ as $\varepsilon\to 0$.

Proof. We wish to show that H is continuously differentiable under this new set of conditions. Recall that

$$\begin{aligned}H_1((x+\psi,v+w),\varepsilon)(t)-H_1((x,v),\varepsilon)(t)-\left[\psi(t)-\Phi(t)w+\varepsilon\left(\Phi(t)\int_0^t \Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)\right]\\=\varepsilon\left(\Phi(t)\int_0^t \Phi^{-1}(s)\left[f(s,(x+\psi)(s))-f(s,x(s))-\frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]\right).\end{aligned}$$

We have that

$$\begin{aligned}&\left\|\Phi(\cdot)\int_0^\cdot \Phi^{-1}(s)\left[f(s,(x+\psi)(s))-f(s,x(s))-\frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]ds\right\|_\infty\\&\le K\int_0^\infty \sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial f}{\partial x}(s,\nu(s))-\frac{\partial f}{\partial x}(s,x(s))\right\|ds\|\psi\|_\infty\\&\le K\|h_2\|_{L^1}\|\psi\|_\infty^2\end{aligned}$$

and $K\|h_2\|_{L^1}\|\psi\|_\infty \to 0$ as $\|\psi\|_\infty \to 0$. Note also that for $\|\psi\|_\infty = 1$,

$$\begin{aligned}
&\left\|\Phi(\cdot)\left(\int_0^{\cdot} \Phi^{-1}(s)\left[\frac{\partial f}{\partial x}(s,x_1(s)) - \frac{\partial f}{\partial x}(s,x_2(s))\right]\psi(s)ds\right)\right\|_\infty \\
&\leq \|\Phi(t)\Phi^{-1}(s)\| \int_0^\infty \left\|\frac{\partial f}{\partial x}(s,x_1(s)) - \frac{\partial f}{\partial x}(s,x_2(s))\right\| ds \\
&\leq K\|h_2\|_{L^1}\|x_1 - x_2\|_\infty \to 0
\end{aligned}$$

as $\|x_1 - x_2\|_\infty \to 0$. We also have that

$$\begin{aligned}
&\Bigg| H_2((x+\psi, v+w),\varepsilon) - H_2((x,v),\varepsilon) - \\
&W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,x(t))\psi(t)dt - \Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)\frac{\partial f}{\partial x}(s,x(s))\psi(s)ds\right)\right]\Bigg| \\
&= \Bigg| W^T\Bigg(\int_0^\infty \left[g(s,(x+\psi)(s)) - g(s,x(s)) - \frac{\partial g}{\partial x}(s,x(s))\psi(s)\right]ds \\
&-\Gamma\left(\Phi(t)\int_0^t \Phi^{-1}(s)\left[f(s,(x+\psi)(s)) - f(s,x(s)) - \frac{\partial f}{\partial x}(s,x(s))\psi(s)\right]ds\right)\Bigg]\Bigg| \\
&\leq \Bigg(\|W^T\|\int_0^\infty \sup_{\zeta(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial g}{\partial x}(s,\zeta(s)) - \frac{\partial g}{\partial x}(s,x(s))\right\| ds\|\psi\|_\infty \\
&+\|W^T\|\|\Gamma\| \sup_{\nu(s)\in L(x(s),(x+\psi)(s))}\left\|\frac{\partial f}{\partial x}(s,\nu(s)) - \frac{\partial f}{\partial x}(s,x(s))\right\| K\int_0^t \left\|\frac{\partial f}{\partial x}(s,\nu(s)) - \frac{\partial f}{\partial x}(s,x(s))\right\| dt\Bigg)\|\psi\|_\infty \\
&\leq \|W^T\|\Big(\|s\|_{L^1}\|\psi\|_\infty + \|\Gamma\|K\|\psi\|_\infty\|h_2\|_{L^1}|\Big)\|\psi\|_\infty.
\end{aligned}$$

and $\|W^T\|\Big(\|s\|_{L^1}\|\psi\|_\infty + \|\Gamma\|\|\psi\|_\infty\|h_2\|_{L^1}|K\Big) \to 0$ as $\|\psi\|_\infty \to 0$. Also note that for $\|\psi\|_\infty = 1$

$$\begin{aligned}
&\left|\left[\frac{\partial H_2}{\partial(x,v)}(x_1,v_1) - \frac{\partial H_2}{\partial(x,v)}(x_2,v_2)\right]\psi\right| \\
&\leq \|W^T\|\Bigg(\int_0^\infty \left\|\frac{\partial g}{\partial x}(s,x_1(s)) - \frac{\partial g}{\partial x}(s,x_2(s))\right\| ds \\
&+\|\Gamma\|\|\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)\|\left\|\frac{\partial f}{\partial x}(s,x_1(s)) - \frac{\partial f}{\partial x}(s,x_2(s))\right\| ds\Big]\Big)\Bigg| \\
&\leq \|W^T\|\Big(\|x_1 - x_2\|_\infty\|s\|_{L^1} + \alpha^{-1}\|\Gamma\|\|x_1 - x_2\|_\infty\|h_2\|_{L^1}\Big).
\end{aligned}$$

It is clear that $\|W^T\|\Big(\|x_1 - x_2\|_\infty\|s\|_{L^1} + \alpha^{-1}\|\Gamma\|\|x_1 - x_2\|_\infty\|h_2\|_{L^1}\Big) \to 0$ as $\|x_1 - x_2\|_\infty \to 0$. Therefore, H_1 and H_2 is continuously differentiable and so H is as well. It follows that H satisfies the conditions of the conditions of the implicit function theorem for Banach spaces by an analogous argument to the one appearing in theorem 1. □

5.2.2 Discrete-Time Systems

In this section, we consider discrete time systems on $k = 0,1,2,\ldots$ of the form

$$x(k+1) - A(k)x(k) = h(k) + \varepsilon f(k, x(k)) \tag{5.7}$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = u + \varepsilon \sum_{k=0}^{\infty} g(k, x(k)). \tag{5.8}$$

Throughout the section, we will denote the set of nonnegative integers as $\mathbb{Z}_+$. Here C_k is an $n \times n$ real-valued matrix for all integers $k \in \mathbb{Z}_+$. The maps $f : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, ε is a real parameter, and $A(k)$ is a nonsingular $n \times n$ real-valued matrix for all $k \in \mathbb{Z}_+$. We seek conditions under which we can guarantee bounded solutions to problems of the form (5.7)-(5.8).

In this section, we use $|\cdot|$ for the Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|$ to refer to the induced operator norm on the space of $n \times n$ real-valued matrices. We denote l_∞ as the space of bounded $\mathbb{R}^n$-valued sequences on $\mathbb{Z}_+$ and $\|x\|_\infty$ as the norm

$$\|x\|_\infty = \sup_{k \geq 0} |x(k)|.$$

We let

$$l_1 = \{y : \mathbb{Z}_+ \to \mathbb{R}^n : \sum_{k=0}^{\infty} |y(k)| < \infty\}$$

and for elements $x \in l_1$, we use the norm

$$\|x\|_1 = \sum_{k=0}^{\infty} |x(k)|.$$

Note that $(l_\infty, \|\cdot\|_\infty)$ and $(l_1, \|\cdot\|_1)$ are Banach spaces.

We start by recalling some general theory in order to discuss the homogeneous linear system

$$x(k+1) - A(k)x(k) = 0$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = 0.$$

The fundamental matrix for the equation $x(k+1) = A(k)x(k)$ is given by

$$\Phi(k, l) = A(k) \cdots A(l)$$

for $k > l$ and $\Phi(k,l) = I$ if $k = l$. For the sake of notation, for $k \in \mathbb{Z}^+$ we denote $\Phi(k,0)$ as simply $\Phi(k)$. In this paper, we focus on the study nonlinear boundary value problems. As we have started to indicate, part of our analysis involves discussing a set of corresponding linear problems. Properties imposed in each context on A will guarantee that t $\sum_{k=0}^{\infty} C_k \Phi(k)$ converges and that $\sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k,l+1)h(l)$ is a vector in $\mathbb{R}^n$ for all h in an appropriate sequence space. Define the matrix Λ by

$$\Lambda \equiv \sum_{k=0}^{\infty} C_k \Phi(k).$$

We know $x \in l_\infty$ is a solution to

$$x(k+1) - A(k)x(k) = 0$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = 0.$$

if and only if $x(0) \in \ker(\Lambda)$.

Given $\psi \in l_\infty$ and $v \in \mathbb{R}^n$, we know by variation of parameters that any solution to

$$x(k+1) - A(k)x(k) = \psi(k)$$

is of the form

$$x(k) = \Phi(k)x(0) + \sum_{l=0}^{k-1} \Phi(k,l+1)\psi(l).$$

Imposing the condition that

$$\sum_{k=0}^{\infty} C_k x(k) = v$$

we have that

$$\Lambda x(0) = v - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k,l+1)\psi(l).$$

Therefore, there exists a unique solution to the linear problem above if and only if $v - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k,l+1)h(l)$ lies in the image of Λ. Let p denote the dimension of $\ker(\Lambda)$. If $p = 0$, it is clear that Let the columns of the matrix W be a basis for $\ker(\Lambda^T)$. We know that $Im(\Lambda) = [\ker(\Lambda^T)]^\perp$, so there exists a unique bounded solution to the linear boundary value problem above if and only if

$$W^T\left[v-\sum_{k=0}^{\infty}C_k\sum_{l=0}^{k-1}\Phi(k,l+1)\psi(l)\right]=0.$$

Throughout this paper we will mainly be studying the structure of the solution set to (5.7)-(5.8) in the cases when the matrix Λ is singular and the corresponding linear problem has a solution, or equivalently where h and u satisfy

$$W^T\left[u-\Gamma\left(\Phi(\cdot)\int_0^{\cdot}\Phi^{-1}(s)h(s)ds\right)\right]=0.$$

With this in mind, we note that the nonlinear boundary value problem

$$x(k+1)-A(k)x(k)=h(k)+\varepsilon f(k,x(k))$$

subject to

$$\sum_{k=0}^{\infty}C_k x(k)=\varepsilon\sum_{k=0}^{\infty}g(k,x(k))$$

is equivalent (for $\varepsilon\neq 0$) to solving

$$\begin{cases} x(k)-\Phi(k)v-\varepsilon\left(\sum_{l=0}^{k-1}\Phi(k,l+1)[h(l)+f(l,x(l))]\right)=0 \\ \quad\text{and} \\ W^T\left[\sum_{l=0}^{\infty}g(l,x(l))-\sum_{k=0}^{\infty}C_k\sum_{l=0}^{k-1}\Phi(k,l+1)f(l,x(l))\right]=0. \end{cases}$$

For the sake of notation, we now state conditions which will be imposed in the first part of our first theorem in this section.

DI) $\sum_{k=0}^{\infty}\|C_k\|<\infty$.

DII) There exists positive constants K,α such that

$$\|\Phi(k,l+1)\|=\|A(k)A(k-1)\cdots A(l)\|\leq Ke^{-\alpha(k-l)}$$

for all $k\geq l\geq 0$.

DIII) For every compact subset S of $\mathbb{R}^n$, $\frac{\partial f}{\partial x}$ exists and is uniformly continuous on $\mathbb{Z}_+\times S$. Further

$$\sup_{k\in\mathbb{Z}_+}\left\|\frac{\partial f}{\partial x}(k,0)\right\|<\infty.$$

DIV) There exists $s \in l_1$ such that

$$\left\| \frac{\partial g}{\partial x}(k,x_1) - \frac{\partial g}{\partial x}(k,x_2) \right\| \leq s_k |x_1 - x_2|$$

for all $k \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

DV) $\frac{\partial g}{\partial x}(k,0)$ exists for all $k \geq 0$ and

$$\sum_{k=0}^{\infty} \left\| \frac{\partial g}{\partial x}(k,0) \right\| < \infty.$$

DVI) For all $(\beta_k) \in l_\infty$,

$$\sum_{k=0}^{\infty} |g(k,\beta_k)| < \infty.$$

Note that for $x \in l_\infty$, $v \in \ker(\Lambda)$, $\varepsilon \in \mathbb{R}$, and $k \geq 0$ we have that,

$$\begin{aligned}
&\left| x(k) - \Phi(k)v - \left(\sum_{l=0}^{k-1} \Phi(k,l+1)[h(l) + \varepsilon f(l,x(l))] \right) \right| \\
&\leq \|x\|_\infty + \sup_{l \geq 0} \|\Phi(l)\| + \sum_{l=0}^{k-1} \|\Phi(k,l+1)\| |h(l) + \varepsilon f(l,x(l))| \\
&\leq \|x\|_\infty + \sup_{l \geq 0} \|\Phi(l)\| + [\|h\|_\infty + |\varepsilon| \sup_{l \geq 0} |f(l,x(l))|] K \sum_{n=0}^{\infty} e^{-\alpha n} \\
&= \|x\|_\infty + \sup_{l \geq 0} \|\Phi(l)\| + [\|h\|_\infty + |\varepsilon| \sup_{l \geq 0} |f(l,x(l))|] K \alpha^{-1}.
\end{aligned}$$

Also observe that

$$\begin{aligned}
&\left| W^T \left[\sum_{l=0}^{\infty} g(l,x(l)) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k,l+1) f(l,x(l)) \right] \right| \\
&\leq \|W^T\| \left[\sum_{l=0}^{\infty} |g(l,x(l))| + \left(\sum_{k=0}^{\infty} \|C_k\| \right) \left(\sum_{l=0}^{k-1} \|\Phi(k,l+1)\| |f(l,x(l))| \right) \right] \\
&\leq \|W^T\| \left[\sum_{l=0}^{\infty} |g(l,x(l))| + \left(\sum_{k=0}^{\infty} \|C_k\| \right) \left(\sup_{l \geq 0} |f(l,x(l))| K \sum_{n=0}^{\infty} e^{-\alpha n} \right) \right] \\
&= \|W^T\| \left[\sum_{l=0}^{\infty} |g(l,x(l))| - \left(\sum_{k=0}^{\infty} \|C_k\| \right) \left(\sup_{l \geq 0} |f(l,x(l))| K \alpha^{-1} \right) \right] < \infty.
\end{aligned}$$

From this is follows that H given by

$$H((x,v),\varepsilon)=\begin{bmatrix} H_1((x,v),\varepsilon) \\ H_2((x,v),\varepsilon) \end{bmatrix}=\begin{bmatrix} x(k)-\Phi(k)v-\left(\sum_{l=0}^{k-1}\Phi(k,l+1)[h(l)+\varepsilon f(l,x(l))]\right) \\ W^T\left[\sum_{l=0}^{\infty}g(l,x(l))-\sum_{k=0}^{\infty}C_k\sum_{l=0}^{k-1}\Phi(k,l+1)f(l,x(l))\right] \end{bmatrix}$$

is a well-defined map from $l_\infty\times\ker(\Lambda)\times\mathbb{R}$ to $l_\infty\times\mathbb{R}^p$

Remark 8. *An example of a case where condition HII) is satisfied is the case where A is a constant matrix whose eigenvalues are each less than 1 in magnitude.*

Like in the section regarding differential equations, our main result in this section will involve an application of the implicit function theorem for Banach spaces. The following lemma establishes the differentiability of the map H above.

Lemma 5. *Assume $HI)-HVI)$ hold. Then the maps $F:l_\infty\to l_\infty$ defined by $F(x)(k)=f(k,x(k))$ and $G:l_\infty\to\mathbb{R}^n$ by $G(x)=\sum_{k=0}^{\infty}g(k,x(k))$ are continuously differentiable.*

Proof. For the sake of notation, for $a,b\in\mathbb{R}^n$ we denote $L(a,b)$ as the straight line segment connecting a and b. Note that by the mean value theorem, for all $k\geq 0$ we have that

$$\left|f(k,(x+\psi)(k))-f(k,x(k))\right|\leq\sup_{\nu(k)\in L(x(k),(x+\psi)(k))}\left|\frac{\partial f}{\partial x}(k,\nu(k))\psi(k)\right|$$

and

$$\left|g(k,(x+\psi)(k))-g(k,x(k))\right|\leq\sup_{\zeta(k)\in L(x(k),(x+\psi)(k))}\left|\frac{\partial g}{\partial x}(k,\zeta(k))\psi(k)\right|.$$

Note that for $h\in l_\infty$,

$$\sup_{k\geq 0}\left|F(x+h)(k)-F(x)(k)-\frac{\partial f}{\partial x}(k,x(k))h(k)\right|\leq\sup_{k\geq 0}\left\|\frac{\partial f}{\partial x}(k,\beta_k)-\frac{\partial f}{\partial x}(k,x(k))\right\|\|h\|_\infty.$$

It is clear that $\sup_{k\geq 0}\left\|\frac{\partial f}{\partial x}(k,\beta_k)-\frac{\partial f}{\partial x}(k,x(k))\right\|\to 0$ as a consequence of $DIII)$. So for any $x\in l_\infty$,

$$\left[\frac{\partial F}{\partial x}(x)h\right](k)=\frac{\partial f}{\partial x}(k,x(k))h(k).$$

For $x_1,x_2\in\mathbb{R}^n$ and $\|h\|=1$,

$$\left\|\left[\frac{\partial F}{\partial x}(x_1)-\frac{\partial F}{\partial x}(x_2)\right]h\right\|_\infty\leq\sup_{k\geq 0}\left\|\frac{\partial f}{\partial x}(k,x_1(k))-\frac{\partial f}{\partial x}(k,x_2(k))\right\|$$
$$\to 0$$

as $\|x_1-x_2\|_\infty\to 0$ as a consequence of $DIII)$. Note that by the mean value theorem, for all $k\geq 0$ we have that there exists $\zeta_k \in L(x(k),(x+h)(k))$ satisfying

$$g(k,(x+h)(k)) - g(k,x(k)) = \frac{\partial g}{\partial x}(k,\zeta_k)h(k).$$

Now observe

$$\begin{aligned}
\left| G(x+h)(k) - G(x)(k) - \sum_{k=0}^{\infty} \frac{\partial g}{\partial x}(k,x(k))h(k) \right| &\leq \sum_{k=0}^{\infty} \left\| \frac{\partial g}{\partial x}(k,\zeta_k) - \frac{\partial g}{\partial x}(k,x(k)) \right\| \|h\|_\infty \\
&\leq \sum_{k=0}^{\infty} s_k \|\zeta - x\|_\infty \|h\|_\infty \\
&= \|s\|_1 \|\zeta - x\|_\infty \|h\|_\infty.
\end{aligned}$$

It is clear that $\|s\|_1\|\zeta - x\|_\infty$ goes to 0 as $\|h\|_\infty \to 0$. So for any $x, h \in l_\infty$,

$$\frac{\partial G}{\partial x}(x)h = \sum_{k=0}^{\infty} \frac{\partial g}{\partial x}(k,x(k))h(k)$$

and for $\|h\|_\infty = 1$ we have that

$$\begin{aligned}
\left| \left[\frac{\partial G}{\partial x}(x_1) - \frac{\partial G}{\partial x}(x_2) \right] h \right| &\leq \sum_{k=0}^{\infty} \left\| \frac{\partial g}{\partial x}(k,x_1(k)) - \frac{\partial g}{\partial x}(k,x_2(k)) \right\| \\
&\leq \|s\|_1 \|x_1 - x_2\|_\infty \ \to 0
\end{aligned}$$

as $\|x_1 - x_2\|_\infty \to 0$. □

Let $\bar{x}(k) = \Phi(k)y + \sum_{l=0}^{k-1}\Phi(k,l+1)h(l)$ for some $y \in \ker(\Lambda)$. Then

$$H((\bar{x},y),0) = \begin{bmatrix} 0 \\ W^T\left[\sum_{l=0}^{\infty} g(l,x_y(l)) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1}\Phi(k,l+1)f(l,x_y(l))\right] \end{bmatrix}$$

and

$$\left[\frac{\partial H}{\partial(x,v)}((\bar{x},y),0)\right](h,w) = \begin{bmatrix} h(k) - \Phi(k)w \\ W^T\left[\sum_{l=0}^{\infty} \frac{\partial g}{\partial x}(l,x_y(l))h(l) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1}\Phi(k,l+1)\frac{\partial f}{\partial x}(l,x_y(l))h(l)\right] \end{bmatrix}.$$

Remark 9. *As mentioned in the introduction, the nonlinear boundary value problem* (5.7)-(5.8) *above can be viewed as an operator equation of the form $\mathscr{L}x = \varepsilon\mathscr{F}(x)$ where $\mathscr{L}$ is linear and $\mathscr{F} = [F,G]^T$ is nonlinear. For the sake of completeness in our analysis, it is worth mentioning that in the case where Λ is invertible, this operator equation can be rewritten as*

$$x - \varepsilon\mathscr{L}^{-1}\mathscr{F}(x) = 0.$$

Let $\Psi : l_\infty \times \mathbb{R} \to l_\infty$ by $\Psi(x,\varepsilon) = x - \varepsilon\mathscr{L}^{-1}\mathscr{F}(x)$ and by the lemma above, $\frac{\partial \Psi}{\partial x}(x)$ exists for all $x \in l_\infty$

and is given by

$$\frac{\partial \Psi}{\partial x}(x) = I - \varepsilon \mathcal{L}^{-1}\frac{\partial \mathcal{F}}{\partial x}(x).$$

It is clear that $\Psi(0,0) = 0$ *and that*

$$\frac{\partial \Psi}{\partial x}(0,0) = I$$

which is clearly a bijection from l_∞ *to* l_∞ *. Therefore, we have by the implicit function for Banach spaces that there exists* $\varepsilon_0 > 0$ *such that for all* $|\varepsilon| < \varepsilon_0$ *there exists a unique solution to the nonlinear boundary value problem* (5.7)-(5.8)*. Moreover, if we denote this solution by* x_ε *we have that* $\|x_\varepsilon\|_\infty \to 0$ *as* $\varepsilon \to 0$*. In the results that follow, we make the assumption that the matrix* Λ *is not invertible.*

For the sake of notation, for any $y \in \mathbb{R}^n$ we define the function $x_y(k) = \Phi(k)y + \sum_{l=0}^{k-1}\Phi(k,l+1)h(l)$.

Theorem 10. *Assume* $DI) - DVI)$ *hold and suppose that there exists* $y \in \ker(\Lambda) \subset \mathbb{R}^n$ *such that*

$$W^T\left[\sum_{l=0}^{\infty} g(l, x_y(l)) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1}\Phi(k,l+1)f(l,x_y(l))\right] = 0$$

and that the map $\Psi : \ker(\Lambda) \to \mathbb{R}^p$ *defined by*

$$\Psi(w) = W^T\left[\sum_{l=0}^{\infty}\frac{\partial g}{\partial x}(l, x_y(l))\Phi(l) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1}\Phi(k,l+1)\frac{\partial f}{\partial x}(l,x_y(l))\Phi(l)\right] w$$

is a bijection from $\ker(\Lambda) \subset \mathbb{R}^n$ *onto* $\mathbb{R}^p$*. Then there exists* ε_0 *such that for all* $|\varepsilon| \le \varepsilon_0$*, the boundary value problem*

$$x(k+1) = A(k)x(k) + \varepsilon f(k, x(k))$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = \varepsilon \sum_{k=0}^{\infty} g(k, x(k)).$$

is guaranteed a solution x_ε*. Further, we have that* $\|x_\varepsilon - x_y\|_\infty \to 0$ *as* $\varepsilon \to 0$*.*

Proof. Note that H is continuously differentiable as a consequence of lemma 1. Then we have that $H((x_y, y), 0) = 0$ so we wish to show that

$$\frac{\partial H}{\partial (x, v)}((x_y, y), 0)$$

is a bijection from $l_\infty \times \mathbb{R}^n$ onto $l_\infty \times \mathbb{R}^p$. Let $\hat{h} \in l_\infty$ and $\hat{w} \in \mathbb{R}^p$. By assumption, there exists a unique $w \in \ker(\Lambda) \subset \mathbb{R}^n$ such that

$$W^T\left[\sum_{l=0}^{\infty}\frac{\partial g}{\partial x}(l,x_y(l))\Phi(l)-\sum_{k=0}^{\infty}C_k\sum_{l=0}^{k-1}\Phi(k,l+1)\frac{\partial f}{\partial x}(l,x_y(l))\Phi(l)\right]w=\hat{w}.$$

Choosing $h(k)=\hat{h}(k)+\Phi(k)w$ we have that

$$\frac{\partial H}{\partial(x,v)}((x_y,y),0)(h,w)=(\hat{h},\hat{w}).$$

Therefore, by the implicit function theorem for Banach spaces (see [16]), there exists a ε_0 such that for all $|\varepsilon|\leq\varepsilon_0$ there exists a bounded solution x_ε to the boundary value problem

$$x(k+1)=A(k)x(k)+\varepsilon f(k,x(k))$$

subject to

$$\sum_{k=0}^{\infty}C_k x(k)=\varepsilon\sum_{k=0}^{\infty}g(k,x(k)).$$

□

In our results up to this point, we assume that h is simply an element of l_∞. In the next set of results, we will investigate problems where we know that h is an element of l_1. We impose the following set of conditions.

DI') $\sum_{k=0}^{\infty}\|C_k\|<\infty$.

DII') There exists positive constant K such that

$$\|\Phi(k,l+1)\|=\|A(k)A(k-1)\cdots A(l)\|\leq K$$

for all $k\geq l\geq 0$.

DIII') There exists $h_1\in l_1$ such that for each compact subset $S\subset\mathbb{R}^n$, there exists a constant C satisfying $|f(k,x)|\leq C h_1(k)$ for all $k\geq 0$, $x\in\mathbb{R}^n$ and $|f(k,x_1)-f(k,x_2)|\leq h_1(k)|x_1-x_2|$ for all $k\geq 0$ and $x_1,x_2\in\mathbb{R}^n$. Further, there exists $h_2\in l_1$ such that

$$\left\|\frac{\partial f}{\partial x}(k,x_1)-\frac{\partial f}{\partial x}(k,x_2)\right\|\leq h_2(k)|x_1-x_2|$$

for all $k\geq 0$ and $x_1,x_2\in\mathbb{R}^n$.

DIV') There exists $s, t \in l_1$ such that

$$\left\| \frac{\partial g}{\partial x}(k, x_1) - \frac{\partial g}{\partial x}(k, x_2) \right\| \leq s_k |x_1 - x_2|$$

and

$$|g(k, x_1) - g(k, x_2)| \leq t_k |x_1 - x_2|$$

for all $k \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

DV') $\frac{\partial g}{\partial x}(k, 0)$ exists for all k and

$$\sum_{k=0}^{\infty} \left\| \frac{\partial g}{\partial x}(k, 0) \right\| < \infty.$$

DVI') For all $(\beta_k) \in l_\infty$,

$$\sum_{k=0}^{\infty} |g(k, \beta_k)| < \infty.$$

Theorem 11. *Suppose that $DI') - DVI')$ hold and that there exists $y \in \ker(\Lambda)$ such that*

$$W^T \left[\sum_{l=0}^{\infty} g(l, x_y(l)) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k, l+1) f(l, x_y(l)) \right] = 0$$

and that the map $\psi : \ker(\Lambda) \to \mathbb{R}^p$ defined by

$$\Psi(w) = W^T \left[\sum_{l=0}^{\infty} \frac{\partial g}{\partial x}(l, x_y(l)) \Phi(l) - \sum_{k=0}^{\infty} C_k \sum_{l=0}^{k-1} \Phi(k, l+1) \frac{\partial f}{\partial x}(l, x_y(l)) \Phi(l) \right] w$$

is a bijection from $\ker(\Lambda) \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Then there exists ε_0 such that for all $|\varepsilon| \leq \varepsilon_0$, the boundary value problem

$$x(k+1) = A(k)x(k) + \varepsilon f(k, x(k))$$

subject to

$$\sum_{k=0}^{\infty} C_k x(k) = \varepsilon \sum_{k=0}^{\infty} g(k, x(k)).$$

is guaranteed a solution x_ε.

Further, we have that $\|x_\varepsilon - x_y\|_\infty \to 0$ as $\varepsilon \to 0$.

Proof. The fact that H_2 is continuously differentiable under this new set of conditions can be shown using an almost identical argument to the one showing H_2 is continuously differentiable. We wish to show that $H_1 : l_\infty \times \mathbb{R}^{n+1} \to l_\infty$ is continuously differentiable.

Recall that

$$\begin{aligned} H_1((x+h, v+w)(k), \varepsilon) - H_1((x, v)(k), \varepsilon) - \left[h(k) - \Phi(k) w - \varepsilon \left(\Phi(k) \sum_{l=0}^{k-1} \Phi^{-1}(l) \frac{\partial f}{\partial x}(l, x(l)) h(l) \right) \right] \\ = \varepsilon \left(\sum_{l=0}^{k-1} \Phi(k,l) \left[f(l, (x+h)(l)) - f(l, x(l)) - \frac{\partial f}{\partial x}(l, x(l)) h(l) \right] \right). \end{aligned}$$

Note further that

$$\begin{aligned} &\left| \left(\sum_{l=0}^{k-1} \Phi(k,l) \left[f(l, (x+h)(l)) - f(l, x(l)) - \frac{\partial f}{\partial x}(l, x(l)) h(l) \right] \right) \right| \\ &\leq \left| \sum_{l=0}^{k-1} \Phi(k,l) \left[\frac{\partial f}{\partial x}(l, \beta_l) - \frac{\partial f}{\partial x}(l, x(l)) \right] h(l) \right| \\ &\leq \sum_{l=0}^{\infty} \left\| \Phi(k,l) \left[\frac{\partial f}{\partial x}(l, \beta_l) - \frac{\partial f}{\partial x}(l, x(l)) \right] \right\| \|h\|_\infty \\ &\leq K \left(\sum_{k=0}^{\infty} \left\| \frac{\partial f}{\partial x}(l, \beta_l) - \frac{\partial f}{\partial x}(l, x(l)) \right\| \right) \|h\|_\infty \\ &\leq K \|h_2\|_1 \|\beta - x\|_\infty \|h\|_\infty \end{aligned}$$

and $K\|h_2\|_1\|\beta - x\|_\infty \to 0$ as $\|h\|_\infty \to 0$. Note also that for $x_1, x_2 \in l_\infty$ that

$$\begin{aligned} \left| \sum_{l=0}^{k-1} \Phi(k,l) \left[\frac{\partial f}{\partial x}(l, x_1(l)) - \frac{\partial f}{\partial x}(l, x_2(l)) \right] h(l) \right| &\leq \sum_{l=0}^{\infty} \left| \Phi(k,l) \left[\frac{\partial f}{\partial x}(l, x_1(l)) - \frac{\partial f}{\partial x}(l, x_2(l)) \right] \right| \\ &\leq K \sum_{l=0}^{\infty} \left| \left[\frac{\partial f}{\partial x}(l, x_1(l)) - \frac{\partial f}{\partial x}(l, x_2(l)) \right] \right| \\ &\leq K \|h_2\|_1 \|x_1 - x_2\|_\infty \\ &\to 0 \end{aligned}$$

as $\|x_1 - x_2\|_\infty \to 0$. Therefore, H_1 is continuously differentiable and therefore H is as well. It follows that H satisfies the conditions of the implicit function theorem by the same argument as the one appearing in the first part of this theorem, establishing our desired result.

□

5.3 Examples

Example 7. *Consider the boundary value problem*

$$\dot{x}(t) - Ax(t) = \varepsilon f(t, x(t))$$

subject to

$$\sum_{k=0}^{\infty} C_k x(t_k) = \varepsilon \int_0^{\infty} g(t, x(t)) dt$$

where $x : \mathbb{Z}^+ \to \mathbb{R}^n$, $f : \mathbb{R}^3 \to \mathbb{R}^2$ *is twice continuously differentiable,* C_k *is an* 2×2 *real-valued matrix and* $t_k \geq 0$ *for all* $k \geq 0$*. We assume that*

$$\Lambda = \sum_{k=0}^{\infty} C_k e^{At_k}$$

is singular. Suppose that the matrix A *is diagonalizable. That is, there exists an invertible matrix*

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

and diagonal matrix

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

satisfying

$$A = PBP^{-1}.$$

Therefore, we have that

$$A^k = PB^k P^{-1}$$

and so

$$e^{At} = P \left[\sum_{k=0}^{\infty} \frac{1}{k!} B^k t^k \right] P^{-1}.$$

As mentioned above, we assume that Λ *is singular, which implies that the second row is a scalar multiple of the first. Suppose that the second row of* Λ *is* κ *times row one for some* $\kappa \in \mathbb{R}$*. It is clear that* Λ *and* Λ^T *have a one-dimensional kernel and that the kernel of* Λ^T *is spanned by the vector* $[-\kappa, 1]^T$*.*

Write g as $g=[g_1,g_2]$. Suppose that there exists $y\in\ker(\Lambda)$ that satisfies for all $t\geq 0$,

$$\begin{aligned}0&=f_1(t,e^{At}y)=f_2(t,e^{At}y)=\frac{\partial f_1}{\partial x}(t,e^{At}y)=\frac{\partial f_2}{\partial x}(t,e^{At}y)\\&=g_1(t,e^{At}y)=g_2(t,e^{At}y)\end{aligned}$$

and

$$-\kappa\int_0^\infty\frac{\partial g_1}{\partial x}(t,e^{At}y)dt\neq\int_0^\infty\frac{\partial g_2}{\partial x}(t,e^{At}y)dt.$$

Under these assumptions, we have

$$\begin{aligned}W^T\left[\int_0^\infty g(t,e^{tA}y)dt-\sum_{k=0}^\infty C_k e^{As_k}\int_0^t e^{At_k}f(s,e^{As}y)ds\right]&=W^T\left[\int_0^\infty(0)dt-\sum_{k=0}^\infty C_k e^{As_k}\int_0^t e^{At_k}(0)ds\right]\\&=0\end{aligned}$$

and that

$$\begin{aligned}&\left|W^T\left[\int_0^\infty\frac{\partial g}{\partial x}(t,e^{tA}y)-\sum_{k=0}^\infty C_k e^{At_k}\int_0^t e^{-sA}y\frac{\partial f}{\partial x}(s,e^{sA}y)dsdt\right]\right|\\&=\left|\int_0^\infty\frac{\partial g_1}{\partial x}(t,e^{At}y)-\kappa\left(\frac{\partial g_2}{\partial x}(t,e^{At}y)\right)dt\right|\\&\neq 0.\end{aligned}$$

Thus for ε sufficiently small in absolute value, we are guaranteed solutions to the nonlinear boundary value problem above.

Alternatively, suppose for the problem above that the rows of Λ are identical, that A is the matrix

$$A=\begin{bmatrix}-\frac{1}{2} & 0\\ 1 & -\frac{1}{2}\end{bmatrix}$$

and that $f:\mathbb{R}^3\to\mathbb{R}^2$ and $g:\mathbb{R}^3\to\mathbb{R}^2$ are given by

$$f(t,x_1,x_2)=\begin{bmatrix}\frac{(x_1-e^{-t/2})^2}{t^6}\\ \frac{(x_1-e^{-t/2})^2+3(x_2-e^{-t/2}(t+1))^2}{t^8}\end{bmatrix}$$

and

$$g(t,x_1,x_2)=\begin{bmatrix}\frac{x_1^2-e^{-t}}{t^2}\\ \frac{5(te^{-t/2}-e^{-t/2}-x_2)}{t^2}\end{bmatrix}.$$

Then $y=[1,-1]\in\ker(\Lambda)$ *satisfies the conditions imposed in theorem 1. That is,*

$$W^T\left[\int_0^\infty g(t,e^{-t/2},e^{-t/2}(t-1))dt+\sum_{k=0}^\infty C_k e^{At_k}\int_0^t e^{-A(s+1)}f(s,e^{-s/2},e^{-s/2}(s-1))ds\,dt\right]=0,$$

and

$$W^T\sum_{k=0}^\infty C_k e^{At_k}\int_0^t e^{-A(s+1)}\frac{\partial f}{\partial x}(s,e^{-s/2},e^{-s/2}(s-1))ds\,dt=W^T\sum_{k=0}^\infty C_k e^{At_k}\int_0^t e^{-A(s+1)}(0)ds\,dt=0$$

so we have

$$\begin{aligned}
&\left|W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,e^{-t/2},e^{-t/2}(t-1))dt-\sum_{k=0}^\infty C_k e^{At_k}\int_0^t e^{-A(s+1)}\frac{\partial f}{\partial x}(s,e^{-s/2},e^{-s/2}(s-1))ds\,dt\right]\right|\\
&=\left|W^T\left[\int_0^\infty \frac{\partial g}{\partial x}(t,e^{-t/2},e^{-t/2}(t-1))dt\right]\right|\\
&=\left|W^T\int_0^\infty\left[\frac{\partial g_1}{\partial x}(t,e^{-t/2},e^{-t/2}(t-1))dt-\frac{\partial g_2}{\partial x}(t,e^{-t/2},e^{-t/2}(t-1))\right]dt\right|\\
&\neq 0.
\end{aligned}$$

Therefore, by results in section 5.2.1 we can guarantee solutions to the nonlinear boundary value problem in this example for ε *sufficiently close to zero.*

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