ABSTRACT

YU, XINGJIAN. Subinvariance and Supersolvability of Leibniz Algebra. (Under the direction of Kailash Misra and Ernest Stitzinger.)

In 1965, Leibniz algebras were first discovered by A. Bloh[Blo65], as a generalization of Lie algebras introduced by Marius Sophus Lie[O'C00]. Leibniz algebras satisfy all definitions of Lie algebras except the antisymmetry property, hence there are left and right Leibniz algebras. In this dissertation, all Leibniz algebras are assumed to be left Leibniz algebras. Mathematicians are curious which results of Lie algebras also holds for Leibniz algebras since it is clear all Lie algebras are Leibniz algeras.

In this dissertation, we are going to look at some results of subinvariance and supersolvability of Lie algebras, that were studied in [Sch51] and [Sal13], and see if they have analogues for Leibniz algebras. For example, in chapter 2, we proved that the radical and nilradical of a subinvariant subalgebra are the intersections of the subalgebra with the radical and nilradical of the containing Leibniz algebra. Another major result we also showed is that the radical and nilradical of an ideal is the ideal of the algebra, and this result has been used in chapter 3 also. In chapter 3, we proved a series of result involving c-ideals, Frattini ideals, and supersolvable Leibniz algebras.

Subinvariance and Supersolvability of
Leibniz Algebra

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DEDICATION

To my parents.

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CHAPTER

1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

In the 1870s, to study infinitesimal transformations, Marius Sophus Lie introduced the concept of Lie algebras[O'C00]. Leibniz algebras are a generalization of Lie algebras. They were first discovered in 1965 by A. Bloh, who called them D-algebras[Blo65]. Later, they were studied by Loday in [Lod93]. Leibniz algebras satisfy all definitions of Lie algebras except the antisymmetry property. Because of that, there are left and right Leibniz algebras. In this dissertation, all Leibniz algebras are assumed to be left Leibniz algebras. They are defined as follows[Dem14]:

Definition 1.1.1. A (left) Leibniz algebra $\mathbf{A}$ is a vector space over a field $\mathbb{F}$, equipped with a bilinear map (multiplication)

$$[\,,\,] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$$

satisfying the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in \mathbf{A}$.

It is clear all Lie algebras are Leibniz algeras. The natural question to ask is: which results of Lie algebras also hold for Leibniz algebras? Some Lie algebra results have already be shown to hold for Leibniz algebras. For example, the Cartan's criterion for Leibniz algebra has been shown by [Alb06]:

Theorem 1.1.1. *Let $\boldsymbol{A}$ be a Leibniz algebra. Then $\boldsymbol{A}$ is solvable if and only if trace$(L_a L_b) = 0$ for all $a \in \boldsymbol{A}^2$ and all $b \in \boldsymbol{A}$. (L_a stands for left multiplication by a.)*

In this dissertation, we are going to look at some results of subinvariance and supersolvability of Lie algebras, and see if they have analogues for Leibniz algebras.

The theory of subinvariant subgroups has been developed by Wielandt [Sch51]. We have studied the subinvariance of Lie algebras from Schenkman [Sch51], and explored some of its results for Leibniz algebras.

The supersolvability results we studied comes from the work of Salemkar[Sal13], where the writers has expanded the work done by a number of other researchers, such as [Tow09].

Throughout this thesis, we have tried to include examples as much as possible for better illustrations.

For better references, we listed a brief summary of each section of each chapters below.

In section 2 of this chapter, we give some preliminary definitions that are used for this dissertation. They are either from [Dem14], or as a Leibniz analogue of Lie definitions from [Sch51].

Chapter 2 includes the Leibniz analogues of Lie algebra results from [Sch51]:

In section 1, we introduce some elementary properties and basic results of subinvariant Leibniz algebras. For example, the property of subinvariance is preserved under homomorphisms. We also show that the composition factors from B to $K \cap B$ appear among those from **A** to K.

In section 2, we prove that a subinvariant subalgebra is the sum of an ideal and a nilpotent algebra.

In section 3, we prove that in characteritic zero, the radical and nilradical of a subinvariant subalgebra are the intersections of the subalgebra with the radical and nilradical of the containing Leibniz algebra. Another major result we have also showed is that the radical and nilradical of an ideal is the ideal of the algebra, and this result has been used in chapter 3 also.

In section 4, we construct a "counterexample" of some of the theorems in section 3 in characteristic p case.

In section 5, we show that the algebra $\{B, C\}$ generated by two subinvariant subalgebras B and C of a Leibniz algebra **A** is subinvariant in **A**; and that the factors in a composition series from $\{B, C\}$ to B occur among those in a composition series from C to $B \cap C$.

In section 6, we prove a few results relating a subalgebra U of a Leibniz algebra **A**, the smallest subinvariant subalgebra containing U, and the largest subinvariant subalgebra contained in U.

In section 8, we show that if B is a subinvariant subalgebra in **A** with centralizer 0, the centralizer of the nilpotent residual of B is contained in B.

In section 9, we list a few results leading to the derivation tower theorem of Lie algebra, followed with a proof that the theorem will hold for Leibniz algebra also, as the Leibniz algebra is indeed Lie with the given constraints of the Lie theorem.

Chapter 3 includes the Leibniz results from [Sal13]:

In section 1, we give some additional definitions that are used in chapter 3 only.

In section 2, we prove some basic Leibniz results that are needed for the supersolvable algebras in the next section. These includes results relating c-ideals, the nilradical, and the Frattini ideal of a Leibniz algebra.

In section 3, we prove some major results of supersolvable Leibniz algebras.

1.2 Preliminaries

Recall we have defined a (left) Leibniz algebra in the introduction. Given any Leibniz algebra **A**, we denote Leib(**A**) = span $\{[a,a] | a \in \mathbf{A}\}$.

Definition 1.2.1. If I is a subspace of a Leibniz algebra **A** such that $I\mathbf{A}+\mathbf{A}I \subseteq I$, then I is an **ideal** or **invariant subalgebra** of **A**. We denote this by $I \lhd \mathbf{A}$.

Remark. The sum and intersection of two ideals of a Leibniz algebra is an ideal. However, unlike the cases for Lie algebras, the product of two ideals may **not** be an ideal. The following is an example from [Dem14].

Example 1.2.1. Let $\mathbf{A} = \text{span}\{x,a,b,c,d\}$ with non-zero multiplications $[a,b] = c, [b,a] = d, [x,a] = a = -[a,x], [x,c] = c, [x,d] = d, [c,x] = d, [d,x] = -d$. Let $I = \text{span}\{a,c,d\}$ and $J = \text{span}\ \{b,c,d\}$. Then I, J are ideals of **A**, but $[I,J] = \text{span}\{c\}$ is not an ideal.

Definition 1.2.2. $CL^l_{\mathbf{A}}(H) = \{x \in \mathbf{A} | [x,h] = 0, \forall h \in H\}$ is the **left centralizer** of H, and $CL_{\mathbf{A}}(H) = \{x \in \mathbf{A} | [x,h] = [h,x] = 0, \forall h \in H\}$ is the **centralizer** of H. Clearly, $CL_{\mathbf{A}}(H)(CL^l_{\mathbf{A}}(H))$ is a subalgebra of **A** and, if H is an ideal of **A**, $CL_{\mathbf{A}}(H)$ $(CL^l_{\mathbf{A}}(H))$ is an ideal of **A**.

Definition 1.2.3. The **left center** of **A** is denoted by $Z^l(\mathbf{A}) = \{x \in \mathbf{A} | [x,a] = 0 \forall a \in \mathbf{A}\}$ and the **right center** of **A** is denoted by $Z^r(\mathbf{A}) = \{x \in \mathbf{A} | [a,x] = 0 \forall x \in \mathbf{A}.\}$. The **center** of **A** is $Z(\mathbf{A}) = Z^l(\mathbf{A}) \cap Z^r(\mathbf{A})$.

Remark. For simplicity, we will denote span$\{\dots\}$ with $< \cdots >$.

If B and C are subalgebras of **A** such that $\mathbf{A} = B + C$ and $B \cap C = 0$, we will write $\mathbf{A} = B \dotplus C$ when B or C is an ideal. If, furthermore, B, C are both ideals of **A**, we will write $\mathbf{A} = B \oplus C$.

We next give the example of an $n-$dimensional cyclic Leibniz algebra that are going to be used quite often for other examples in this dissertation. This can be found in [Dem14]. Cyclic Leibniz algebras has been studied extensively in [Bat14].

Example 1.2.2. Let $\mathbf{A}$ be an $n-$dimensional Leibniz algebra over a field of characteristic zero, generated by a single element a. Then $\mathbf{A} = < a, a^2, \ldots, a^n >$ and we have $[a, a^n] = \alpha_1 a + \cdots + \alpha_n a^n$ for some $\alpha_1, \ldots \alpha_n \in \mathbb{F}$.

By the Leibniz identity we have $0 = [a, [a^n, a]] = [[a, a^n], a] + [a^n, [a, a]] = [\alpha_1 a + \cdots + \alpha_n a^n, a] = \alpha_1 [a, a]$ which implies that $\alpha_1 = 0$. Hence $\mathbf{A}^2 = \text{span}\{a^2, \ldots, a^n\} = \text{Leib}(\mathbf{A})$.

Remark. It is obvious that all cyclic Leibniz algebras are solvable. In particular, when $[a, a^n] = 0$, the algebra is nilpotent.

Definition 1.2.4. M is a **maximal ideal** of $\mathbf{A}$ if and only if the quotient algebra $\mathbf{A}/M$ is simple or one dimensional. In other words, M cannot be properly contained in another proper ideal of $\mathbf{A}$.

For a Leibniz algebra $\mathbf{A}$. The left normalizer of H in $\mathbf{A}$ is defined by $N^l_{\mathbf{A}}(H) = \{x \in \mathbf{A} | [x, a] \in H, \forall a \in H\}$. In other words, H acts like a left ideal in $N^l_{\mathbf{A}}(H)$. The right normalizer is defined by $N^r_{\mathbf{A}}(H) = \{x \in \mathbf{A} | [a, x] \in H, \forall a \in H\}$. The normalizer is defined by $N_{\mathbf{A}}(H) = N^l_{\mathbf{A}}(H) \cap N^r_{\mathbf{A}}(H)$. Note that the left normalizer and the normalizer are both subalgebras, but the right normalizer may not be.

Definition 1.2.5. A **normal series** of $\mathbf{A}$ is a series of algebras starting with (0) and ending with $\mathbf{A}$, each algebra contained in the next, whereas each algebra is an ideal of the next algebra .

Remark. If B and C are members of a normal series of $\mathbf{A}$ such that $B \subseteq C$, then by a **normal series from** B **to** C we mean a chain $B = B_r \lhd B_{r-1} \lhd \cdots \lhd B_0 = C$. For a normal series $B = B_n \lhd \ldots B_i \lhd B_{i-1} \lhd \cdots \lhd B_0 = \mathbf{A}$. The quotient algebras B_{i-1}/B_i are called the **factors** of a normal series.

Remark. Any normal series such that each algebra is a maximal ideal in the next algebra is called a **composition series**. A **composition series from** B **to** C is defined similarly. The quotient algebras are called the **composition factors** or **simple factors**.

Definition 1.2.6. B is a **subinvariant subalgebra** of a Leibniz algebra $\mathbf{A}$ (or B is subinvariant in $\mathbf{A}$) if B is a subalgebra of A, and $B = B_r \lhd B_{r-1} \lhd \cdots \lhd B_0 = \mathbf{A}$, that is, there exists a normal series from B to $\mathbf{A}$.

For better illustrations, we have an example of a subalgebra that is subinvariant, followed by a subalgebra that is not subinvariant.

Example 1.2.3. Let $\mathbf{A} =< x, y, z >$ with non-zero multiplications $[x, y] = z$ (Heisenberg Leibniz algebra) and all other products are zero. Let $B =< x >$, then B is a subalgebra of $\mathbf{A}$ since $[x, x] = 0 \in B$. We now have the following chain of ideals.

$$B =< x > \lhd < x, z > \lhd \mathbf{A}$$

so B is a subinvariant subalgebra of $\mathbf{A}$. Note that B is not an ideal of $\mathbf{A}$, since $[x, y] = z \notin B$.

Example 1.2.4. Let $\mathbf{A} =< x, y, z >$ with non-zero multiplications $[x, z] = z$. Let $B =< x >$, then B is a subalgebra of $\mathbf{A}$ since $[x, x] = 0 \in B$. But B is not subinvariant. Suppose B is subinvariant, then there exists an ideal C of $\mathbf{A}$ of the form $< x, ay + bz >$ with the chain of ideals:

$$B =< x > \lhd < x, ay + bz > \lhd \mathbf{A}$$

Then, $< x > \lhd < x, ay + bz > \implies [x, ay + bz] = bz \in < x >$, which implies $b = 0$. Hence $C =< x, ay >$

Now, $< x, ay > \lhd \mathbf{A} \implies [x, z] = z \in < x, ay >$, which clearly is not true, so our assumption is false.

Definition 1.2.7. For a Leibniz algebra $\mathbf{A}$, the series of ideals

$$\mathbf{A} \supseteq \mathbf{A}^2 = [\mathbf{A}, \mathbf{A}] \supseteq \mathbf{A}^3 = [\mathbf{A}, \mathbf{A}^2] \supseteq \dots$$

is called the **lower central series** of $\mathbf{A}$. And, the series of ideals

$$\mathbf{A} \supseteq \mathbf{A}^{(1)} = [\mathbf{A}, \mathbf{A}] \supseteq \mathbf{A}^{(2)} = [\mathbf{A}^{(1)}, \mathbf{A}^{(1)}] \supseteq \dots$$

is called the **derived series** of $\mathbf{A}$.

Definition 1.2.8. A Leibniz algebra $\mathbf{A}$ is **solvable** if $\mathbf{A}^{(m)} = 0$ for some integer $m \geq 0$.

Definition 1.2.9. A Leibniz algebra $\mathbf{A}$ is **nilpotent** of class c if each product of $c + 1$ elements is 0 and some product of c elements is not 0. An element is **left normed** if it is of the form $[a_1, [a_2, [\dots [a_{n-1}, a_n] \dots]]]$. We will write such an element as $[a_1 a_2 \dots a_n]$

Remark. (Prop 4.2 of [Dem14]) It is known that any product of length k can be written as the linear combination of terms consisting of the original elements that are left normed of length k.

Definition 1.2.10. The maximum solvable ideal of a Leibniz algebra $\mathbf{A}$ is called the radical of $\mathbf{A}$, denoted by $\mathrm{Rad}(\mathbf{A})$ or $R(\mathbf{A})$. The maximum nilpotent ideal of a Leibniz algebra $\mathbf{A}$ is called the nilradical of $\mathbf{A}$, denoted by $\mathrm{Nil}(\mathbf{A})$ or $N(\mathbf{A})$.

Remark. A nilpotent Leibniz algebra is always solvable. Hence the nilradical is always contained in the radical.

CHAPTER

2

SUBINVARIANCE OF LEIBNIZ ALGEBRA

In Chapter 1, we introduced the preliminary information for this dissertation. In this chapter, we are going to explore the work of Schenkman on subinvariant Lie algebras[Sch51] and see if the results will hold for Leibniz algebras. We assume all Leibniz algebras are finite dimensional through out this chapter.

2.1 Elementary Properties of Subinvariant Subalgebras

In this section, we have listed some elementary results of subinvariant subalgebras for Leibniz algebras as easy consequences from its definition and set theory. The proofs are similar to [Sch51].

Theorem 2.1.1. *Let $\boldsymbol{A}$ be a Leibniz algebra. Let K and B be subinvariant in $\boldsymbol{A}$, C subinvariant in K and let $\boldsymbol{A}^*$ be any subalgebra of $\boldsymbol{A}$. Then*

1. *C is subinvariant in $\boldsymbol{A}$.*
2. *$K \cap \boldsymbol{A}^*$ is subinvariant in $\boldsymbol{A}^*$.*
3. *If $K \subseteq \boldsymbol{A}^*$, then K is subinvariant in $\boldsymbol{A}^*$.*
4. *$K \cap B$ is subinvariant in $\boldsymbol{A}$.*

5. *If T is a homomorphism of $\boldsymbol{A}$ onto $T(\boldsymbol{A})$, then $T(K)$ is subinvariant in $T(\boldsymbol{A})$.*

6. *Conversely, if $\bar{F}$ is subinvariant in $\bar{\boldsymbol{A}}$, and if F is the complete inverse image of $\bar{F}$ under T, then F is subinvariant in $\boldsymbol{A}$.*

7. *$\boldsymbol{A}^*$ is subinvairant in $\boldsymbol{A}$, If and only if $\boldsymbol{A}$ is nilpotent.*

Proof. 1. This follows directly from the definition of subinvariant subalgebra. For $C = C_s \lhd C_{s-1} \lhd \dots C_1 \lhd C_0 = K = K_r \lhd \cdots \lhd K_1 \lhd K_0 = \mathbf{A}$.

2. We have $K = K_r \lhd \cdots \lhd K_1 \lhd K_0 = \mathbf{A}$. Let $K_i^* = K_i \cap \mathbf{A}^*$ for $i = 0, 1, 2, \dots, r$. Let $x_j^* \in K_j^*$, $x_{j+1}^* \in K_{j+1}^*$. Then $[x_j^*, x_{j+1}^*] \in K_{j+1} \cap \mathbf{A}^* = K_{j+1}^*$ since $K_{j+1} \lhd K_j$, and $\mathbf{A}^*$ is a subalgera. Hence $K_{j+1}^* \lhd K_j^*$ for $j = 0, 1, \dots, r-1$. Therefore, $K \cap \mathbf{A}^*$ is subinvariant in $\mathbf{A}^*$ with $K \cap \mathbf{A}^* = K_r^* \lhd \cdots \lhd K_1^* \lhd K_0^* = \mathbf{A}^*$.

3. This follows directly from (2), for $K = K \cap \mathbf{A}^*$.

4. This follows directly from (1) and (3). $K \cap B \subseteq B \xrightarrow{(3)} K \cap B$ is subinvariant in B. Also, B subinvariant in $\mathbf{A} \xrightarrow{(1)} K \cap B$ is subinvariant in $\mathbf{A}$.

5. We have $K = K_r \lhd \cdots \lhd K_1 \lhd K_0 = \mathbf{A}$. Let $T(x_j) = y_j \in T(K_j), T(x_{j+1}) = y_{j+1} \in T(K_{j+1})$. Then $[y_j, y_{j+1}] = [T(x_j), T(x_{j+1})] = T([x_j, x_{j+1}]) \in T(K_{j+1})$ since $[x_j, x_{j+1}] \in K_{j+1}$. Hence $T(K)$ is subinvariant in $T(\mathbf{A})$ with $T(K) \lhd \cdots \lhd T(K_1) \lhd T(K_0) = T(\mathbf{A})$.

6. If $\bar{F} = \bar{F}_r \lhd \cdots \lhd \bar{F}_1 \lhd \bar{F}_0 = \bar{\mathbf{A}}$. Let F_i be the complete inverse image of $\bar{F}_i$ for $i = 0, 1, \dots, r$, and let $f_i \in F_i$. Then since $T([f_i, f_{i+1}]) = [T(f_i), T(f_{i+1})] \in \bar{F}_{i+1}$, it follows that $[f_i, f_{i+1}] \in F_{i+1}$, hence $F_{i+1} \lhd F_i$. Therefore F is subinvariant in $\mathbf{A}$ with $F = F_r \lhd \cdots \lhd F_1 \lhd F_0 = \mathbf{A}$.

7. We first prove the if direction. Theorem 4.16 of [Dem14] says that nilpotent Leibniz algebras satisfy the normalizer condition, which means every proper subalgebra of the nilpotent Leibniz algebra $\mathbf{A}$ is properly contained in its normalizer. So, for an arbitrary proper subalgebra H of $\mathbf{A}$, we always have the chain of subalgebras starting from $B_r = H$, followed by its normalizer $B_{r-1} = N_{\mathbf{A}}(H)$, then $B_{r-2} = N_{\mathbf{A}}(B_{r-1})$,etc. The chains have elements that are properly increasing. In other words, Any proper subalgebra of a nilpotent Leibniz algebra is subinvariant.

   For the only if direction. Suppose $\mathbf{A}^*$ is subinvariant in $\mathbf{A}$. Then by definition, there must be a subalgebra V of $\mathbf{A}$ such that $\mathbf{A}^*$ is a proper ideal of V. Hence V is the normalizer of $\mathbf{A}^*$ in $\mathbf{A}$ and $\mathbf{A}^*$ satisfies the normalizer condition, which is equivalent to say $\mathbf{A}$ is nipotent by Theorem 4.16 of [Dem14].

□

Lemma 2.1.1. *Let* $\mathbf{A}$ *be a Leibniz algebra. If* M *is a maximal ideal of* $\mathbf{A}$ *and if* B *is subinvariant in* $\mathbf{A}$, *then either* $B \subseteq M$ *or* $B \cap M$ *is a maximal ideal of* B *and* $B/(B \cap M) \cong \mathbf{A}/M$

Proof. Let $\pi : \mathbf{A} \mapsto \mathbf{A}/M = \bar{\mathbf{A}}$. Then $\bar{\mathbf{A}}$ is simple. Then either $\pi : B \mapsto 0 \in \bar{\mathbf{A}}$, i.e., $B \subseteq M$ or $\pi : B \mapsto \bar{B} = (B+M)/M \cong \mathbf{A}/M$. For if $B + M \neq \mathbf{A}$, B subinvariant in $\mathbf{A}$ implies that there exists at least one ideal $B' \neq \mathbf{A}$ such that $B \lhd B' \lhd \mathbf{A}$. Therefore, $\bar{B'} = (B'+M)/M$ is an ideal of $\bar{\mathbf{A}}$, contradicting the fact $\bar{\mathbf{A}}$ is simple. From the second isomorphism theorem, $(B+M)/M \cong B/(B\cap M)$. $B \cap M$ is a maximal ideal of B since $B/(B \cap M) \cong \mathbf{A}/M$ is simple. □

Example 2.1.1. Let $\mathbf{A} =< x, y, z >$ with non-zero multiplications: $[x,z] = z + y, [x, y] = y$. Then, $M =< y, z >$ is a maximal ideal of $\mathbf{A}$. $B =< z > \lhd < y, z > \lhd \mathbf{A}$. Then $B \subseteq M$.

We have a following theorem as a consequence of the lemma above.

Theorem 2.1.2. *If* K *and* B *are subinvariant subalgebras of* $\mathbf{A}$, *then the factors in the composition series from* $K \cap B$ *to* B *appear among those in a composition series from* K *to* $\mathbf{A}$.

Proof. K subinvariant in $\mathbf{A} \implies K = K_r \lhd \cdots \lhd K_1 \lhd K_0 = \mathbf{A}$ and for $i = 1, 2, \ldots, r$, K_i is a maximal ideal of K_{i-1}. Similarly, B subinvariant in $\mathbf{A} \implies B = B_s \lhd \cdots \lhd B_1 \lhd B_0 = \mathbf{A}$ and for $i = 1, 2, \ldots, s$, B_i is a maximal ideal of B_{i-1}. By Theorem 2.1.1(4), $K \cap B$ is subinvariant in $\mathbf{A}$, so we will have $K \cap B = K_r \cap B \lhd \cdots \lhd K_1 \cap B \lhd K_0 \cap B = B = B_s \lhd \cdots \lhd B_1 \lhd B_0 = \mathbf{A}$. Hence the factors are of the form $(K_i \cap B)/(K_{i+1} \cap B) \cong K_i/K_{i+1}$. □

2.2 A subinvariant subalgebra as the sum of an ideal and a nilpotent algebra

In this section, we give some key Leibniz results that will be used throughout the rest of this chapter.

Definition 2.2.1. The nilpotent residual $\mathbf{A}^{\omega} = \cap_{i=1}^{\infty} \mathbf{A}^i$ of a Leibniz algebra $\mathbf{A}$ is the intersection of the terms in the lower central series. Since dim$\mathbf{A}$ is finite, $\mathbf{A}^{\omega} = \mathbf{A}^t$ for some t.
Theorem 4.4 of [Bar11] stated that if B is a subinvariant subalgebra of $\mathbf{A}$, then $B^{\omega} \lhd \mathbf{A}$. We have provided a different proof that introduces a useful function.

Theorem 2.2.1. *If* B *is a subinvariant subalgebra of a Leibniz algebra* $\mathbf{A}$, *then* $B^{\omega} = \cap_{i=1}^{\infty} B^i$ *is an ideal of* $\mathbf{A}$.

Proof. $B^{\omega} = B^p$ for some integer p since $\mathbf{A}$ is finite dimensional.

Since B is subinvariant in $\mathbf{A}$, we have the following sequence:

$$B^{\omega} = B^{s} \lhd \cdots \lhd B^{2} \lhd B = B_{t} \lhd \cdots \lhd B_{1} \lhd B_{0} = \mathbf{A}$$

From above we noticed that:

$$\begin{aligned}
&B\mathbf{A} + \mathbf{A}B \subseteq B_1\mathbf{A} + \mathbf{A}B_1 \subseteq B_1 \\
&B(B\mathbf{A} + \mathbf{A}B) + (B\mathbf{A} + \mathbf{A}B)B \subseteq BB_1 + B_1B \subseteq B_2B_1 + B_1B_2 \subseteq B_2 \\
&\ldots
\end{aligned}$$

We define a new function f on subalgebras by:

1. $f(B, C) = BC + CB$
2. $f^2(B, C) = f(B, f(B, C)) = B(BC + CB) + (BC + CB)B$
3. $f^3(B, C) = f(B, f^2(B, C) = B(B(BC + CB) + (BC + CB)B) + (B(BC + CB) + (BC + CB)B)B$

Define inductively, we have:
$f^{n+1}(B, C) = f(B, f^n(B, C))$.

Assume $f(B^n, \mathbf{A}) \subseteq f^n(B, \mathbf{A})$. Then,

$$\begin{aligned}
f(B^{n+1}, \mathbf{A}) &= B^{n+1}\mathbf{A} + \mathbf{A}B^{n+1} \\
&= (BB^n)\mathbf{A} + \mathbf{A}(BB^n) \\
&= B(B^n\mathbf{A}) + B^n(B\mathbf{A}) + (\mathbf{A}B)B^n + B(\mathbf{A}B^n) \\
&= B(B^n\mathbf{A} + \mathbf{A}B^n) + (B^n(B\mathbf{A}) + (\mathbf{A}B)B^n) \\
&\subseteq Bf(B^n, \mathbf{A}) + f(B^n, B\mathbf{A}) + f(B^n, \mathbf{A}B) \\
&\subseteq Bf^n(B, \mathbf{A}) + f^n(B, B\mathbf{A}) + f^n(B, \mathbf{A}B) \\
&\subseteq f^{n+1}(B, \mathbf{A})
\end{aligned}$$

It is clear that $f^n(B, B\mathbf{A})$ and $f^n(B, \mathbf{A}B)$ are contained in $f^n(B, B\mathbf{A} + \mathbf{A}B) = f^{n+1}(B, \mathbf{A})$. Also, $Bf^n(B, \mathbf{A}) \subseteq Bf^n(B, \mathbf{A}) + f^n(B, \mathbf{A})B = f^{n+1}(B, \mathbf{A})$.

Using results above, we notice that:

$$B^{\omega}\mathbf{A} + \mathbf{A}B^{\omega} = f(B^{\omega}, \mathbf{A}) = f(B^{s+t}, \mathbf{A}) \subseteq f^{s+t}(B, \mathbf{A}) \subseteq B^{\omega}$$

Hence the result. $\square$

It is natural to ask if the converse of a theorem holds. In this case, it does not, which can be shown by the following counterexample.

Example 2.2.1. Let $\mathbf{A} =< u, n, k, n^2 >$ be a Leibniz algebra over a field of characteristic zero, with non-zero products $[u, n] = u, [n, u] = -u + k, [u, n^2] = k, [n, k] = -k$. Let $B =< n, k >$, it is easy to see that B is a subalgebra of $\mathbf{A}$. $B^\omega =< k >$, which is an ideal of $\mathbf{A}$. Clearly, B is not an ideal of $\mathbf{A}$ since $[u, n] = u \notin \mathbf{A}$. If B is subinvariant, we have $B =< n, k > \lhd < n, k, \alpha u + \beta n^2 > \lhd \mathbf{A}$. $< n, k > \lhd < n, k, \alpha u + \beta n^2 >$ implies that $[n, \alpha u + \beta n^2] = -\alpha u + \alpha k \in < n, k > \implies \alpha = 0$. Now, $< n, k, \beta n^2 > \lhd \mathbf{A}$, but $[u, n] = u \notin < n, k, \beta n^2 >$, which is a contradiction.

Corollary 2.2.2. *If B is a subinvariant subalgebra of a Leibniz algebra $\boldsymbol{A}$ such that $B = [B, B]$, then B is an ideal of $\boldsymbol{A}$.*

Proof. In this case, $B^\omega = B$, and it follows from the preceeding theorem. □

The next theorem shows that any arbitrary Leibniz algebra can be expressed as the sum of an ideal and a nilpotent subalgebra. We need the following definition for the proof of the theorem.

Definition 2.2.2. A Cartan subalgebra of the Leibniz Algebra $\mathbf{A}$ is a nilpotent subalgebra C such that $C = N_{\mathbf{A}}(C)$.

Theorem 2.2.3. *If $\boldsymbol{A}$ is a Leibniz algebra, then $\boldsymbol{A} = \boldsymbol{A}^\omega + H$, where H is a nilpotent algebra and $\boldsymbol{A}^\omega = \cap_{i=1}^{\infty} \boldsymbol{A}^i$.*

Proof. Let C be a Cartan subalgebra of $\mathbf{A}$. $\mathbf{A}^\omega \lhd \mathbf{A}$. So, by cor 6.3 of [Bar11], $(C + \mathbf{A}^\omega)/\mathbf{A}^\omega$ is a Cartan subalgebra of $\mathbf{A}/\mathbf{A}^\omega = \bar{\mathbf{A}}$.
Clearly, $\mathbf{A}/\mathbf{A}^\omega$ is nilpotent. Then by Lemma 2.2 of [Bar11], if $(C + \mathbf{A}^\omega)/\mathbf{A}^\omega \neq \bar{\mathbf{A}}$, the normalizer of it in $\bar{\mathbf{A}}$, $N_{\bar{\mathbf{A}}}((C + \mathbf{A}^\omega)/\mathbf{A}^\omega) \neq (C + \mathbf{A}^\omega)/\mathbf{A}^\omega$, which countradicts the fact that $(C + \mathbf{A}^\omega)/\mathbf{A}^\omega$ is a Cartan subalgebra of $\bar{\mathbf{A}}$.
Hence, $(C + \mathbf{A}^\omega)/\mathbf{A}^\omega = \bar{\mathbf{A}} \implies C + \mathbf{A}^\omega = \mathbf{A}$ so C is the desired H. □

Example 2.2.2. Let $\mathbf{A} =< x, y, z >$ with non-zero products: $[x, y] = y, [y, x] = -y, [x, x] = z$. Then it is easy to see $\mathbf{A}^\omega =< y >$ and $H =< x, z >$. Clearly $\mathbf{A} = \mathbf{A}^\omega + H$ and H is nilpotent as $H^3 = 0$.

We have the following result as a direct consequence of the above two theorems.

Theorem 2.2.4. *If B is a subinvariant subalgebra of a Leibniz algebra $\boldsymbol{A}$, then $B = B^\omega + H$, where $B^\omega = \cap_{i=1}^{\infty} B^i$ is an ideal of $\boldsymbol{A}$, and H is a nilpotent algebra.*

2.3 The radical and nilradical of a subinvariant subalgebra

In this section, we assume the Leibniz algebras are over a field of characteristic zero. To prove the corresponding Leibniz results from Schenkman, we first proved a few theorems that are needed. To show our first main theorem, we need to first introduce some Lie algebra results from [Jac62].

Definition 2.3.1. Let A be an associative algebra. Then A is nilpotent if there exists a positive integer k such that every product of k elements of A is 0. For a finite dimensional associative algebra A, the radical of A is the maximal nilpotent ideal of A.

Theorem 2.3.1. *(Cor 2 on P.45 of [Jac62]) Let L be a linear Lie algebra, R the radical of L, L^* the associative envelope of L and T the (nilpotent) radical of L^*. Then $L \cap T$ = all nilpotent elements of R and $[R, L] \subseteq T$.*

Remark. Example of an associative envelope: If $L = sl(n,\mathbb{F})$ the special linear Lie algebra generated by $n \times n$ matrices. Then, the associative envelope L^* of L is the associative algebra generated by $n \times n$ matrices, $gl(n,\mathbb{F})$.

Theorem 2.3.2. *Let $\boldsymbol{A}$ be a Leibniz algebra over a field $\mathbb{F}$ of characteristic zero, $R(\boldsymbol{A})$ be the radical of $\boldsymbol{A}$, $N(\boldsymbol{A})$ be the nil radical of $\boldsymbol{A}$. Then $\boldsymbol{A}R(\boldsymbol{A}) + R(\boldsymbol{A})\boldsymbol{A} \subseteq N(\boldsymbol{A})$.*

Proof. Let $T(\mathbf{A}) = \{T_x | x \in \mathbf{A}\}$ denote the vector space of left multiplications by elements of $\mathbf{A}$. Then $T : \mathbf{A} \mapsto T(\mathbf{A}), T(x) = T_x$ is a homomorphism and $T(\mathbf{A})$ is a Lie algebra under commutation. Also, $T : R(\mathbf{A}) \mapsto T(R(\mathbf{A}))$, the elements of $T(R(\mathbf{A}))$ are left multiplications by elements of $R(\mathbf{A})$ on $\mathbf{A}$, then $T(R(\mathbf{A}))$ is contained in the radical of $T(\mathbf{A})$. By the theorem above, $[T(R(\mathbf{A})), T(\mathbf{A})] \subseteq R(T(\mathbf{A})^*)$. Hence there exists an n such that

$$[T_{s_n}, T_{r_n}].[T_{s_{n-1}}, T_{r_{n-1}}]\dots[T_{s_1}, T_{r_1}] = 0$$

where $s_i \in \mathbf{A}$, $r_i \in R(\mathbf{A})$ or $s_i \in R(\mathbf{A})$, $r_i \in \mathbf{A}$. Hence, $T_{s_n r_n} \dots T_{s_1 r_1} = 0$. Alternatively, $s_n r_n(\dots(s_1 r_1)x)\dots) = 0$ for all $x \in \mathbf{A}$. Hence $s_n r_n(\dots(s_1 r_1)(s_0 r_0))\dots) = 0$. Hence $(\mathbf{A}R(\mathbf{A})+R(\mathbf{A})\mathbf{A})^{n+1} = 0$, which means $\mathbf{A}R + R\mathbf{A}$ is nilpotent. Since $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A}$ is an ideal in $\mathbf{A}$, $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq N(\mathbf{A})$. □

Example 2.3.1. Let $\mathbf{A}$ =span$\{x, y, z\}, [x, z] = z$ and all other products are zero. Then, $R(\mathbf{A}) = \mathbf{A}$ since $\mathbf{A}$ is solvable. $N(\mathbf{A}) = \text{span}\{y, z\}$. Then, $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} = \mathbf{A}^2 + \mathbf{A}^2 = \text{span}\{z\} \subseteq N(\mathbf{A})$.

Theorem 2.3.3. *Let $\boldsymbol{A}$ be a Leibniz algebra over a field $\mathbb{F}$ of characteristic zero, $R(\boldsymbol{A})$ be the radical of $\boldsymbol{A}$, N($\boldsymbol{A}$) be the nil radical of $\boldsymbol{A}$. Then, $\boldsymbol{A}^2 \cap R(\boldsymbol{A}) = \boldsymbol{A}R(\boldsymbol{A}) + R(\boldsymbol{A})\boldsymbol{A} \subseteq N(\boldsymbol{A})$*

Proof. $\mathbf{A} = R(\mathbf{A}) + S$ is the Levi decomposition for Leibniz algebra, where $R(\mathbf{A})$ is the radical of $\mathbf{A}$, and S is a semisimple subalgebra of $\mathbf{A}$. Then $\mathbf{A}^2 = (R(\mathbf{A}) + S)(R(\mathbf{A}) + S) = (\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A}) + S^2$ and $\mathbf{A}^2 \cap R(\mathbf{A}) = \mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A}$ since $S^2 \cap R(\mathbf{A}) = 0$ and $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq R(\mathbf{A})$. □

Example 2.3.2. Let $\mathbf{A} =< a, a^2, a^3 >$, i.e., the three dimensional cyclic Leibniz algebra with $[a, a^3] = a^3$. Then, $R(\mathbf{A}) = \mathbf{A}$ since $\mathbf{A}$ is solvable. $\mathbf{A}^2 =< a^2, a^3 >= N(\mathbf{A})$. Then, $\mathbf{A}^2 \cap R(\mathbf{A}) = \mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq N(\mathbf{A})$.

Corollary 2.3.4. *Let $R(\boldsymbol{A})$ be the radical of $\boldsymbol{A}$, $N(\boldsymbol{A})$ the nil-radical of $\boldsymbol{A}$ and $N(R)$ the nil-radical of $R(\boldsymbol{A})$. Then $N(R) = N(\boldsymbol{A})$.*

Proof. By Theorem 2.3.2, $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq N(\mathbf{A})$. On the other hand, $N(\mathbf{A}) \subseteq R(\mathbf{A}) \implies N(\mathbf{A})$ is a nilpotent ideal of $R(\mathbf{A}) \implies N(\mathbf{A}) \subseteq N(R)$. So, $\mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq N(\mathbf{A}) \subseteq N(R)$. Also, $\mathbf{A}N(R) + N(R)\mathbf{A} \subseteq \mathbf{A}R(\mathbf{A}) + R(\mathbf{A})\mathbf{A} \subseteq N(R)$, which implies that $N(R)$ is a nilpotent ideal of $\mathbf{A}$. Hence $N(R) \subseteq N(\mathbf{A})$. Putting together, we have $N(R) = N(\mathbf{A})$. □

Example 2.3.3. Let $\mathbf{A} = sl(2, \mathbb{C}) \oplus C$, where $C =< a, a^2, a^3 >$ is the non-nilpotent cyclic algebra with $[a, a^3] = a^3$, and multiplication between the two summands are always 0.

It is easy to see that the radical $R(\mathbf{A}) = C$, and the nilradical $N(\mathbf{A}) =< a^2, a^3 >$. Then, $N(R) = N(\mathbf{A})$
.

The following lemma is corresponding to lemma 2 from Schenkman.

Lemma 2.3.1. *A solvable subinvariant subalgebra of a Leibniz algebra of characteristic 0 is contained in the radical of the algebra.*

Proof. Let $\mathbf{A}$ be a Leibniz algebra with radical $R(\mathbf{A})$ and subinvariant subalgebra S of $\mathbf{A}$. Then $\bar{\mathbf{A}} = \mathbf{A}/R(\mathbf{A})$ is a semisimple Lie algebra, hence $\bar{A}$ is the direct sum of simple Lie ideals and $\bar{S} = (S + R(\mathbf{A}))/R(\mathbf{A})$ is the direct sum of some of the simple ideals in $\bar{\mathbf{A}}$. But $\bar{S}$ is solvable , which yields that $\bar{S} = 0$ and $S \subseteq R(\mathbf{A})$. □

Example 2.3.4. Let $\mathbf{A} = sl(2, \mathbb{C}) \oplus C$, where $C =< a, a^2, a^3 >$ is the non-nilpotent cyclic algebra with $[a, a^3] = a^3$, and multiplication between the two summands are always 0. Let $B =< a^2 >$, then B is subinvariant with the chain of ideals $B =< a^2 > \lhd < a^2, a^3 > \lhd \mathbf{A}$. Then, B is solvable and contained in the radical of $\mathbf{A}$, which is C.

The next theorem is a more general Leibniz case for Lemma 2.1 of [Sal13], which is also what triggered the investigation of this section.

Theorem 2.3.5. *Let I be an ideal in a Leibniz algebra $\boldsymbol{A}$, $R(I)$ and $N(I)$ be the radical and nil-radical of I. Then, $R(I), N(I) \lhd \boldsymbol{A}$ and $\boldsymbol{A}R(I) + R(I)\boldsymbol{A} \subseteq N(I)$.*

Proof. Let $x \in I$ and C be the cyclic subalgebra generated by $x \in \mathbf{A}$. Let $\bar{I} = I + C$,$R(\bar{I})$ be the radical of $\bar{I}$. Since $R(I)$ is subinvariant in $\bar{I}$ with $R(I) \lhd I \lhd \bar{I}$, lemma 2.3.1 yields that $R(I) \subseteq R(\bar{I})$. Hence $R(I) \subseteq R(\bar{I}) \cap I$. Since $R(\bar{I})$ and I are both ideals of $\bar{I}$, $R(\bar{I}) \cap I$ is a solvable ideal of I hence

$R(\bar{I})\cap I\subseteq R(I)$. Thus $R(I)=R(\bar{I})\cap I$. Hence $R(I)$ is an ideal of $\bar{I}$ and $R(I)x+xR(I)\subseteq R(I)$. Since this holds for all $x\in\mathbf{A}$, $R(I)$ is an ideal of $\mathbf{A}$.

Now, let $\overline{R(I)}=R(I)+C$. Then $\overline{R(I)}$ is solvable since $R(I)$ and C are both solvable. Hence $\overline{R(I)}^2$ is nilpotent (characteristic 0). $R(I)\overline{R(I)}+\overline{R(I)}R(I)\subseteq\overline{R(I)}^2\cap R(I)$, which is a nilpotent ideal of $R(I)$. Thus $R(I)\overline{R(I)}+\overline{R(I)}R(I)$ is contained in the nilradical, $N(R)$ of $R(I)$, which is also the nilradical of I by Corollary 2.3.4. Hence $xR(I)+R(I)x\subseteq N(I)$ for all $x\in\mathbf{A}$. Since $N(I)\subseteq R(I)$, it follows that $N(I)$ is an ideal of $\mathbf{A}$. Hence, $\mathbf{A}R(I)+R(I)\mathbf{A}\subseteq N(I)$. □

Example 2.3.5. Let $\mathbf{A}=sl(2,\mathbb{C})\oplus C$, where $C=<a,a^2,a^3>$ is the non-nilpotent cyclic algebra with $[a,a^3]=a^3$, and multiplication between the two summands are always 0. Let $I=C$, then the radical of I, $R(I)=I$ and the nilradical of I, $N(I)=<a^2,a^3>$. It is clear $R(I),N(I)$ are ideals of $\mathbf{A}$ and $\mathbf{A}R(I)+R(I)\mathbf{A}=<a^2,a^3>=N(I)$.

The following lemma is corresponding to lemma 4 from Schenkman.

Lemma 2.3.2. *Let M be a nilpotent subinvariant subalgebra of $\boldsymbol{A}$, a Leibniz algebra over a field of characteristic 0. Then $M\subseteq N(\boldsymbol{A})$, the nil radical of $\boldsymbol{A}$.*

Proof. We have $M_n=M\lhd M_{n-1}\lhd\cdots\lhd\mathbf{A}=M_0$. Let N_j be the nilradical of M_j. Since M is a nilpotent ideal of M_{n-1}, $M\subseteq N_{n-1}$. Also, by Theorem 2.3.5, N_i is a nilpotent ideal of M_{i-1}, which means $N_i\subseteq N_{i-1}$. Thus, we have $M=M_n\subseteq N_{n-1}\subseteq N_{n-2}\subseteq\cdots\subseteq N_1\subseteq N(\mathbf{A})$, the nilradical of $\mathbf{A}$.

□

Example 2.3.6. Let $\mathbf{A}$=span$\{x,y,z\}$ with non-zero products $[x,z]=y,[x,y]=y,[x,x]=z$. Then, $M=<z>$ is a nilpotent subinvariant subalgebra with a chain of ideals: $<z>\lhd<y,z>\lhd\mathbf{A}$. It is obvious that M is contained in the the nilradical of $\mathbf{A}$, $<y,z>$.

The following theorem is corresponding to Theorem 6 from Schenkman.

Theorem 2.3.6. *Let $\boldsymbol{A}$ be a Leibniz algebra over a field of charateristic 0, with radical $R(\boldsymbol{A})$ and nilradical $N(\boldsymbol{A})$. Let B be a subinvariant subalgebra of $\boldsymbol{A}$ with radical $R(B)$ and nilradical $N(B)$. Then $R(B)=B\cap R(\boldsymbol{A})$ and $N(B)=B\cap N(\boldsymbol{A})$.*

Proof. $B\cap R(\mathbf{A})$ is a solvable ideal of B since $R(\mathbf{A})\lhd\mathbf{A}$ and hence $B\cap R(A)\subseteq R(B)$. $R(B)$ is a solvalbe subinvariant subalgebra of $\mathbf{A}$, hence by lemma 2.3.1 $R(B)\subseteq R(\mathbf{A})$. So $R(B)\subseteq B\cap R(\mathbf{A})$. Thus $R(B)=B\cap R(\mathbf{A})$. Similarly, $N(B)=B\cap N(\mathbf{A})$. □

We are going to define a Leibniz algebra that we'll use in the examples of the above theorem.

Let $\mathbb{F}^2=\text{Span}\{x,y\}$ whereas $x=[1\ 0]^T$, $y=[0\ 1]^T$. Let $\mathbf{A}=gl_2(\mathbb{F})\oplus\mathbb{F}^2$, whereas $gl_2(\mathbb{F})$ is the Lie algebra of 2×2 matrices over a field $\mathbb{F}$. We define the non-zero products as follows:

1. $[a,b]=ab-ba$ (commutator bracket) if $a,b\in gl_2(\mathbb{F})$

2. $[a,u]=au$ (matrix multiplication) if $a\in gl_2(\mathbb{F})$, $u\in\mathbb{F}^2$.

3. $[u,a]=0$ if $a\in gl_2(\mathbb{F})$, $u\in\mathbb{F}^2$.

Note that this is not a Lie algebra because $[a,u]\neq-[u,a]$.

Example 2.3.7. Let $\mathbf{A}=gl_2(\mathbb{F})\oplus\mathbb{F}^2$. Let $B_3=B=<x>, B_2=<x,y>=, B_1=<I>\oplus<x,y>, B_0=\mathbf{A}$, whereas $I=2\times2$ identity matrix. Then, $B=B_3\lhd B_2\lhd B_1\lhd B_0=\mathbf{A}$ is a chain of ideals.

$N(B)=<x>, N(\mathbf{A})=<x,y>$ and $B\cap N(\mathbf{A})=N(B)$.

Similarly, $R(B)=<x>, R(\mathbf{A})=<I>\oplus<x,y>$ and $B\cap R(\mathbf{A})=R(B)$.

Example 2.3.8. Folllowing the example above, let $B_1^*=sl_2(\mathbb{F})\oplus\mathbb{F}^2$, whereas $sl_2(\mathbb{F})$ is the Lie algebra of 2×2 matrices of trace zero. Then, $B=B_3\lhd B_2\lhd B_1^*\lhd B_0=\mathbf{A}$ is another chain of ideals. This shows the chain of ideals need not be unique for a subinvariant subalgebra.

The theorem can also be used to determine if a subalgebra is subinvariant, as the following example showed:

Example 2.3.9. Let $\mathbf{A}=gl_2(\mathbb{F})\oplus\mathbb{F}^2$. Let $B=<I>$. Then, $N(B)=B$, and $N(\mathbf{A})=<x,y>$. Hence $B\cap N(\mathbf{A})=0\neq B$. So, B is not subinvariant in $\mathbf{A}$.

2.4 A counterexample in charateristic p case

In this section,we are going to discuss a counterexample to show the following results at characteristic zero do not hold at characteristic $p>2$.

1. Lemma 2.3.1: A solvable subinvariant subalgebra is contained in the radical of the algebra

2. Lemma 2.3.2: A nilpotent subinvariant subalgebra is contained in the nilradical of the algebra

3. Theorem 2.3.5: If I is an ideal in $\mathbf{A}$, $R(I)$ and $N(I)$ are the radical and nilradical of I, then $R(I)$ and $N(I)$ are ideals in $\mathbf{A}$ and $\mathbf{A}R(I)+R(I)\mathbf{A}$ is contained in $N(I)$.

4. Theorem 2.3.6: Let B be subinvariant in $\mathbf{A}$. $R(B)$ and $N(B)$ are the radical and nilradical of $\mathbf{A}$. $R(B)$ and $N(B)$ are the radical and nilradical of B. Then $R(B)=R(\mathbf{A})\cap B$ and $N(B)=N(\mathbf{A})\cap B$.

This example was originally construced in p.75 of [Jac62], and was also studied by [Boy20]. Note that this example are in case of Lie algebras, since all Lie algebras are Leibniz algebras. We first introduce the definition of characterstic ideal.

Definition 2.4.1. [Boy20] Let $\mathbf{A}$ be a Leibniz algebra. A linear map $D : \mathbf{A} \mapsto \mathbf{A}$ is a derivation if $D([x, y]) = [D(x), y] + [x, D(y)]$. Let Der($\mathbf{A}$) be the Lie algebra of all derivations of $\mathbf{A}$. An ideal I is characteristic if it is invariant under derivations, i.e., $D(I) \subseteq I$ for all $D \in$ Der($\mathbf{A}$).

Example 2.4.1. Let $\mathbb{F}$ be a field of characteristic $p > 2$. Let C be the commutative associative cyclic algebra with basis $\{1, x, x^2, \ldots, x^{p-1}\}$. Hence $x^p = 0$ and $\dim(C) = p$. It is clear that the maximal nilpotent ideal of C is N with basis $\{x, x^2, \ldots, x^{p-1}\}$. Then $\dim(C/N) = 1$. Now, we define a linear transformation D on the basis as $D(x) = 1, \ldots, D(x^t) = t x^{t-1}, \ldots$ and extend this linearly to all of C. It is easy to see that D is a derivation (since D acts like taking the derivative of x^i) and D does not leave N invariant: $D(N)$ has basis $\{1, x, x^2, \ldots, x^{p-2}\}$.

Let L be a simple Lie algebra over $\mathbb{F}$ and let $M = L \otimes C$. Then M is a Lie algebra with multiplication $[(a \otimes z), (b \otimes y)] = [a, b] \otimes (zy)$ on basis elements for L and C, and elements in M are linear combinations of basis elements $b \otimes z$. Since $(L \otimes N)^p = L^p \otimes N^p = 0$, $T = L \otimes N$ is a nilpotent ideal of M. Since $M/(L \otimes N) \cong L \otimes \{1\} \cong L$, T is both the radical and nilradical of M. Define D on the basis for M by $D(b \otimes z) = b \otimes D(z)$. It is easy to see that D is a derivation of M. Since $D(g \otimes x) = g \otimes 1$ is not in T, the radical and nilradical are not characterisic in M, as showed in [Sel57]

We continue this example to show other results that fail at characteristic p.

Now let P be the vector space direct sum of $< D >$ and M, i.e., $P =< D > \oplus M$, with product $[D + m, D + n] = D(m) - D(n) + [m, n]$. Note the product is always in M. It is easy to see P is a Lie algebra.

It is easy to see that the only ideal of P is M, since any ideals of M is not characteristic, they cannot be ideals in P. Also, M is not solvable nor nilpotent since $M/(L \otimes N) \cong L \otimes \{1\} \cong L$ is simple. Hence the radical and nilradical of P are both $< 0 >$.

For lemma 2.3.1, T is solvable by definition, and it is subinvariant in P, as $T \lhd M \lhd L$. But T is not contained in the radical of P, which is $< 0 >$. So 1 fails.

For lemma 2.3.2, T is nilpotent by definition, and it is subinvariant in P, as $T \lhd M \lhd L$. But T is not contained in the nilradical of P, which is $< 0 >$. So 2 fails.

For theorem 2.3.5, M is the only ideal in P. Hence its radical and nilradical T cannot be ideals in P. So 3 fails.

For theorem 2.3.6, T is subinvariant in P.The radical and nilradical of T is itself since T is nilpotent. T can not equal to the intersection of T and the radical/nilradical of P since the latter is zero. So 4 fails.

2.5 The sum theorems

In this section, we studied the properties of the sum of subinvariant subalgebras.

Theorem 2.5.1. *Let* $\mathbf{A}$ *be a Leibniz algebra over a field of characteristic 0. Let* B *and* C *be subinvariant subalgebra of* $\mathbf{A}$*. If* $\{B, C\}$ *denote the subalgebra generated by* B *and* C*, then* $\{B, C\}$ *is subinvariant in* $\mathbf{A}$*.*

Proof. If both B and C are nilpotent, then by Lemma 2.3.2, B, C and consequently $\{B, C\}$ are contained in $N(\mathbf{A})$. Then by Theorem 2.1.1 (7), $\{B, C\}$ is subinvariant in $N(\mathbf{A})$, which means it is subinvariant in $\mathbf{A}$, so the theorem holds.

In general, we have $B^{\omega} \neq 0$, or $C^{\omega} \neq 0$. By Theorem 2.2.1, B^{ω}, C^{ω} are ideals of $\mathbf{A}$. Hence $K = \{B^{\omega}, C^{\omega}\}$, the algebra generated by B^{ω}, C^{ω} is an ideal of $\mathbf{A}$. Let $\bar{B} = (B+K)/K$, $\bar{C} = (C+K)/K$. Then $\bar{B}$ and $\bar{C}$ are nilpotent in $\bar{\mathbf{A}} = \mathbf{A}/K$. So by the first part of the proof, $\{\bar{B}, \bar{C}\}$ is subinvariant in $\bar{\mathbf{A}}$. But $\{\bar{B}, \bar{C}\} = \{B/K, C/K\} = \{B, C\}/K = \overline{\{B, C\}}$. For if H_i stands for B or C, $\overline{\{B, C\}} = \sum[\ldots[H_1, H_2], \ldots, H_n]/K = \sum[\ldots[H_1/K, H_2/K], \ldots, H_n/K] = \sum[\ldots[\bar{H}_1, \bar{H}_2], \ldots, \bar{H}_n] = \{\bar{B}, \bar{C}\}$. Hence, $\{\bar{B}, \bar{C}\} = \overline{\{B, C\}}$ is subinvariant in $\bar{\mathbf{A}}$. It follows from Theorem 2.1.1(6), $\{B, C\}$ is subinvariant in $\mathbf{A}$. Hence the result. □

Example 2.5.1. Let $\mathbf{A} =< x_1, x_2, x_3, x_4 >$ with non-zero products $[x_1, x_3] = x_4, [x_3, x_2] = x_4$. Let $B =< x_1 >, C =< x_2 >$. Then B, C are subinviariant in $\mathbf{A}$ with chain of ideals: $B =< x_1 > \lhd < x_1, x_4 > \lhd \mathbf{A}$ and $C =< x_2 > \lhd < x_2, x_4 > \lhd \mathbf{A}$. Now, The subalgebra generated by B and C is $< x_1, x_2 >$, which is subinvariant with a chain of ideals $< x_1, x_2 > \lhd < x_1, x_2, x_4 > \lhd \mathbf{A}$.

Theorem 2.5.2. *Let* $\mathbf{A}$ *be a Leibniz algebra over a field of characteristic 0. Let* B *be subinvariant in* $\mathbf{A}$*. Let* c *be any element of* $\mathbf{A}$ *and let* B^* *be the smallest algebra containing* B *such that* $cB^* + B^*c \subseteq B^*$*. Then* B^* *is subinvariant in* $\mathbf{A}$*.*

Proof. If $c \in B$, then $B^* = B$. So suppose $c \notin B$.

Recall that we have defined the function f on subspaces B and C in the proof of Theorem 2.2.1. We can extend it to act on the elements of $\mathbf{A}$ by:

1. $f(x, y) = xy + yx$
2. $f^2(x, y) = f(x, f(x, y)) = x(xy + yx) + (xy + yx)x$
3. $f^3(x, y) = f(x, f^2(x, y)) = x(x(xy + yx) + (xy + yx)x) + (x(xy + yx) + (xy + yx)x)x$

Define inductively, we have:
$f^{n+1}(x,y) = f(x, f^n(x,y))$. Also, we denote $f^0(x,y) = y$, then $f^1(x,y) = f(x, f^0(x,y)) = f(x,y)$.

We claim that $B^* = \{\sum f^i(c,B)\}$, where B stands for all elements of B.

It is clear $B \subseteq B^*$ as $f^0(c,B) = B$. Also, it is easy to see that B^* is invariant under $L_c + R_c$: $(L_c + R_c)(f^i(c,B)) = cf^i(c,B) + f^i(c,B)c = f^{i+1}(c,B)$. Hence $B^* \subseteq \{\sum f^i(c,B)\}$.

On the other hand, $B \subseteq B^*$, and since $cB^* + B^*c = f(c,B^*) \subseteq B^*$, $f^k(c,B) \subseteq f^k(c,B^*) \subseteq B^*, \forall k$. It follows that $B^* \supseteq \{\sum f^i(c,B)\}$. Consequently, $B^* = \{\sum f^i(c,B)\}$.

If now B is nilpotent, then $B \subseteq$N($\mathbf{A}$) $= N$. But N is an ideal of $\mathbf{A}$, so by how it is constructed, $B^* \subseteq N$ and consequently B^* is subinvariant in N by Theorem 2.1.1(7). Hence B^* is subinvariant in $\mathbf{A}$.

If B is not nilpotent, then by Theorem 2.2.1, $B^\omega \neq 0$ is a non-zero ideal of $\mathbf{A}$. Let $\bar{\mathbf{A}} = \mathbf{A}/B^\omega$, $\bar{B} = B/B^\omega$, $\overline{B*} = B^*/B^\omega$, and $\bar{c} = (c+B^\omega)/B^\omega$. Then, $\overline{B*} = \{\sum f^i(c,B)\}/B^\omega = \{\sum f^i((c+B^\omega)/B^\omega, B/B^\omega)\} = \{f^i(\bar{c},\bar{B}\}$. But $\bar{B}$ is in $N(\bar{\mathbf{A}})$, and hence so is $\{f^i(\bar{c},\bar{B})\} = \overline{B*}$. It follows that $\bar{B}^*$ is subinvariant in $\bar{\mathbf{A}}$. Hence B^* is subinvariant in $\mathbf{A}$ by Theorem 2.1.1(6). $\square$

Example 2.5.2. Let $\mathbf{A} =< x_1, x_2, x_3, x_4 >$ with non-zero products $[x_1, x_3] = x_4, [x_3, x_2] = x_4$. Let $B =< x_1 >, c =< x_2 >$. Then B is subinvariant in $\mathbf{A}$ as showed above. Then, the smallest subalgebra containing B such that $cB^* + B^*c \subseteq B$ is $B^* =< x_1, x_2 >$, which is also subinvariant as shown in the last example.

From Theorem 2.1.2, we know that if K and B are subinvariant subalgeras of a Leibniz algebra $\mathbf{A}$, then the factors in a composition series from $K \cap B$ to B occur among those in a composition series from K to $\{K, B\}$. If we add the qualifying statement "not counting multiplicities of one-dimensional factors" and assume that the base field is of characteristic zero, then the converse of this, namely that the factors in a composition series from K to $\{K, B\}$ occur among those in a composition series from $K \cap B$ to B. To show this we need the following lemma first.

Lemma 2.5.1. *If $\mathbf{A}$ is a Leibniz algebra over a field of characteristic 0, then $\mathbf{A} = \mathbf{A}^{(\omega)} + R(\mathbf{A})$,where $\mathbf{A}^{(\omega)} = \cap_{i=1}^{\infty} \mathbf{A}^{(i)}$ is the intersection of the terms in the derived series, and $R(\mathbf{A})$ is the radical of $\mathbf{A}$.*

Proof. $\mathbf{A}/R(\mathbf{A})$ is semisimple. Therefore, $\mathbf{A}/R(\mathbf{A}) = [\mathbf{A}/R(\mathbf{A}), \mathbf{A}/R(\mathbf{A})] = \ldots (\mathbf{A}/R(\mathbf{A}))^{(\omega)}$. Also, $\mathbf{A}/R(\mathbf{A}) = (\mathbf{A}/R(\mathbf{A}))^{(\omega)} = (\mathbf{A}^{(\omega)} + R(\mathbf{A}))/R(\mathbf{A})$. Hence, $\mathbf{A} = \mathbf{A}^{(\omega)} + R(\mathbf{A})$. $\square$

Theorem 2.5.3. *If B and C are subinvariant in $\mathbf{A}$,a Leibniz algebra over a field of characteristic 0, then the factors in a composition series from B to $\{B, C\}$ occur among those in a composition series from $B \cap C$ to C, not counting multiplicities of one-dimensional factors.*

Proof. Note that the factors in a composition series are either one dimensional or simple and non-abelian. So it suffices to show the occurrence of non-abelian factors.

It is easy to see that the composition factors are never one dimensional or abelian between a Leibniz algebra and its radical, and are always one dimensional for a solvable Leibniz algebra.

By corollory 2.2.2, $B^{(\omega)} = \cap_{i=1}^{\infty} B^{(i)} \lhd \mathbf{A}$:

Since $B^{(\omega)} = B^{(r)}$ for some r, and $B^{(r)} \lhd B, B$ subinvariant in $\mathbf{A} \implies B^{(\omega)}$ is subinvariant in $\mathbf{A}$. Also,$[B^{(\omega)}, B^{(\omega)}] = [B^{(r)}, B^{(r)}] = B^{(r)} = B^{(\omega)}$, hence $B^{(\omega)} \lhd \mathbf{A}$.

Let $\bar{\mathbf{A}} = \mathbf{A}/B^{(\omega)}, \bar{B} = B/B^{(\omega)}, \bar{C} = (C + B^{(\omega)})/B^{(\omega)}.\overline{\{B,C\}} = \{B,C\}/B^{(\omega)}$, and $\overline{(B \cap C)} = (B \cap C + B^{(\omega)})/B^{(\omega)}$. Note that as in the proof of Theorem 2.5.1, $\overline{\{B,C\}} = \{\bar{B}, \bar{C}\}$, and $\overline{(B \cap C)} = \bar{B} \cap \bar{C}$. Then the composition factors from $\{B,C\}$ to B are the same as those from $\overline{\{B,C\}}$ to $\bar{B}$. (Isomorphism theorem). Also, the factors from $\bar{C}$ to $\overline{(B \cap C)} = \bar{B} \cap \bar{C}$ occur among those from C to $B \cap C$. Hence it suffices to show the theorem for $\bar{B}, \bar{C}$ in $\bar{\mathbf{A}}$.

In this case, $\bar{B}$ and $\bar{B} \cap \bar{C} \subseteq \bar{B}$ are solvable. By the preceding lemma, $\bar{C} = \bar{C}^{(\omega)} + R(\bar{C})$, $\bar{C}^{(\omega)} = \cap_{i=1}^{\infty} C^{(i)}$. Then, $R(\bar{C})$ and $\bar{B} \subseteq R(\bar{\mathbf{A}})$. Hence,$\overline{\{B,C\}} = \{\bar{B}, \bar{C}\} = \{\bar{B}, \bar{C}^{(\omega)}, R(\bar{C})\} \subseteq \{\bar{C}^{(\omega)}, R(\bar{\mathbf{A}})\} = \bar{C}^{(\omega)} + R(\bar{\mathbf{A}})$ since $\bar{C}^{(\omega)}, R(\bar{\mathbf{A}}) \lhd \bar{A}$. Now, the non-abelian composition factors from $\bar{C}^{(\omega)} + R(\bar{\mathbf{A}})$ to $R(\bar{\mathbf{A}})$ are the same as those from $(\bar{C}^{(\omega)} + R(\bar{\mathbf{A}}))/R(\bar{\mathbf{A}})$ to $R(\bar{\mathbf{A}})/R(\bar{\mathbf{A}}) = 0$, and are contained in $\bar{C}^{(\omega)}$ (all abelian ones are in $R(\bar{\mathbf{A}})$), since $(\bar{C}^{(\omega)} + R(\bar{\mathbf{A}}))/R(\bar{\mathbf{A}})$ is a homomorphic image of $\bar{C}^{(\omega)}$. It follows that the non-abelian composition factors from $\{\bar{B}, \bar{C}\}$ to $\bar{B}$ must occur among those from $\bar{C}^{(\omega)}$ to 0 (since $\{\bar{B}, \bar{C}\} \subseteq \bar{C}^{(\omega)} + R(\bar{\mathbf{A}})$), consequently in $\bar{C}$; and hence between $\bar{C}$ and $\overline{B \cap C}$, since $\overline{B \cap C}$ is solvable, and therefore in $R(\bar{C})$. (All composition factors from $\overline{B \cap C}$ to 0 are abelian)

There are two possible cases for one dimensional factors.

- There exists one or more one dimensional factors between $\bar{C}$ and $\bar{B} \cap \bar{C}$; in this case the theorem is obviously true.

- If none of the factors from $\bar{C}$ to $\bar{B} \cap \bar{C}$ are one dimensional, then $R(\bar{C}) \subseteq \bar{B} \cap \bar{C}$. Since $\bar{B} \cap \bar{C}$ is solvable and subinvariant, $\bar{B} \cap \bar{C} \subseteq R(\bar{C})$. Consequently, $R(\bar{C}) = \bar{B} \cap \bar{C}$. In this case, $\{\bar{B}, \bar{C}\} = \{\bar{B}, \bar{C}^{(\omega)}, R(\bar{C})\} = \bar{B} + \bar{C}^{(\omega)}$, since $\bar{C}^{(\omega)} \lhd \bar{\mathbf{A}}$. Then, $\bar{B} = R(\bar{B} + \bar{C}^{(\omega)})$. Let $\bar{S} = R(\bar{B} + \bar{C}^{(\omega)})$. Then, $\bar{B} \subseteq \bar{S}$ since $\bar{B}$ is solvable. On the other hand, if $\bar{s} \subseteq \bar{S}$, then $\bar{s} = \bar{b} + \bar{c}, \bar{b} \in \bar{B}, \bar{c} \in \bar{C}^{(\omega)}$. But then since $\bar{b} \in \bar{S}, \bar{c} \in \bar{S}$ hence $\bar{c} \in R(\bar{C})$ since $\bar{c}$ is solvable. Hence $\bar{S} \subseteq \bar{B}$. So, $\bar{B} = R(\bar{B} + \bar{C}^{(\omega)}) = R\{\bar{B}, \bar{C}\}$, and the factors from $\{\bar{B}, \bar{C}\}$ to $\bar{B}$ are non-abelian. Hence the result.

□

Example 2.5.3. Let $\mathbf{A} =< x_1, x_2, x_3, x_4 >$ with non-zero products $[x_1, x_3] = x_4, [x_3, x_2] = x_4$. Let $B =< x_1 >, C =< x_2 >$. Then the only composition factor from $\{B,C\}$ to B is $< x_1, x_2 > / < x_1 > \cong < x_2 >$, and the only composition factor from C to $B \cap C$ is $< x_2 > / \{0\} \cong < x_2 >$.

2.6 The greatest subinvariant subalgebra contained in, and the least subinvariant subalgebra containing, an arbitrary algebra.

If U is an arbitrary subalgebra of a Leibniz algebra $\mathbf{A}$, let U_* denote the algebra generated by all the subinvariant subalgebras of $\mathbf{A}$ contained in U; and let U^* denote the intersection of all the subinvariant subalgebras of $\mathbf{A}$ containing U. Then by Theorem 2.1.1(4), U^* is minimal such that $U^* \lhd\lhd \mathbf{A}$ and $U^* \supseteq U$. If the base field of the algebra is of charactersitic zero, then by Theorem 2.5.1, U_* is maximal such that $U_* \lhd\lhd U$. Note that $U_* \subseteq U \subseteq U^*$.

We then have the following results.

Lemma 2.6.1. *If U is any subalgebra of a Leibniz algebra $\mathbf{A}$ over a field of characteristic zero, then U_* is an ideal of U.*

Proof. Let $b \in U$. If U_1 is the minimal subalgebra such that $U_* \subseteq U_1$ and $bU_1 + U_1 b \subseteq U_1$ then $U_1 \lhd\lhd \mathbf{A}$ by Theorem 2.5.2. Clearly, $U_1 \subseteq U$, hence $U_1 \subseteq U_*$. Thus, $bU_* + U_* b \subseteq U_*, \forall b \in U$, which implies that U_* is an ideal of U. □

Example 2.6.1. Let $\mathbf{A} =< x_1, x_2, x_3, x_4 >$ with non-zero products $[x_1, x_3] = x_4, [x_3, x_2] = x_4$. Let $U =< x_1, x_2 >$. Then U is subinvariant in $\mathbf{A}$ with the chain of ideals $U =< x_1, x_2 > \lhd < x_1, x_2, x_4 > \lhd \mathbf{A}$. Since U is subinvariant, $U_* = U$ is an ideal of U.

Theorem 2.6.1. *If U and V are subalgebras of a Leibniz algebra $\mathbf{A}$ over an arbitrary field and if $\{U, V\} = U + V$, then $\{U^*, V^*\} = U^* + V^*$.*

Proof. Since $U^{*\omega}$ and $V^{*\omega}$ are ideals of $\mathbf{A}$, it suffices to show $U^* = U + U^{*\omega}$ and $V^* = V + V^{*\omega}$. By its definition, $U^* \supseteq U + U^{*\omega}$. On the other hand, $(U + U^{*\omega})/U^{*\omega}$ is a subalgebra of the nilpotent Leibniz algebra $U^*/U^{*\omega}$. Hence by Theorem 2.1.1(7), $(U + U^{*\omega})/U^{*\omega}$ is subinvariant in $U^*/U^{*\omega}$, which by Theorem 2.1.1(5), is subinvariant in $\mathbf{A}/U^{*\omega}$ since U^* is subinvariant in $\mathbf{A}$. Then, by Theorem 2.1.1(1) and (6), $U + U^{*\omega}$ is subinvariant in $\mathbf{A}$ hence $U + U^{*\omega} \supseteq U^*$. It follows that $U^* = U + U^{*\omega}$. Similarly, $V^* = V + V^{*\omega}$ and the proof is complete. □

Example 2.6.2. Let $\mathbf{A} = B \oplus C$, where $B = \{b, b^2 >$ is the cyclic Leibniz algebra with $[b, b^2] = b^2$, $C = \{c, c^2 >$ is the cyclic Leibniz algebra with $[c, c^2] = c^2$, and $[x, y] = 0 = [y, x]$ for all $x \in B, y \in C$. Let $U = \{b - b^2 >$, then U is a subalgebra that is not an ideal. Similarly, $V =< c - c^2 >$ is a subalgebra that is not an ideal. Hence, $U^* = B$, and $V^* = C$. Then, $\{U, V\} = U + V$, and $\{U^*, V^*\} = \{B, C\} = B + C$.

We now give a generalization of Theorem 2.5.1 to the case where one of the algebras need not be subinvariant.

Theorem 2.6.2. *Let B be subinvariant in a Leibniz algebra $\mathbf{A}$ over a field of characteristic zero. Let V be any subalgebra of $\mathbf{A}$, and let $K = \{B, V\}$. Then the composition factors from K^* to K_* are part of those from V^* to V_*.*

Proof. By Lemma 2.6.1, K_* is an ideal of K. Therefore, since $V \subseteq K$, $\{K_*, V\} = K_* + V$. Then by the preceding theorem, $\{K_*, V^*\} = K_* + V^*$. Since by definition, $(K_*)^* =$ the intersection of all subinvariant subalgebras of $\mathbf{A}$ containing K_*, which is simply K_*.

It follows that the composition factors from $K_* + V^*$ to K_* are the same as those from V^* to $V^* \cap K_*$. For by theorem 2.1.2, the composition factors from V^* to $V^* \cap K_*$ occur among those from $K_* + V^*$ to K_*. On the other hand, if two subalgebras of a series from $K_* + V^*$ to K_* have the same intersections with V^*, then by their forms, they must be the same algebras and hence there exists a one to one correspondance between the composition factors of the two series, $K_* + V^*$ to K_*, and V^* to $V^* \cap K_*$.

Now, $V \subseteq K$, hence $V_* \subseteq V^* \cap K_*$. Also, $B \subseteq K_*$, $V \subseteq V^*$; consequently $K = \{B, V\} \subseteq \{K_*, V^*\} = K_* + V^*$, a subinvariant subalgebra of $\mathbf{A}$, hence by its definition, $K^* \subseteq K_* + V^*$. We then have that $V_* \subseteq V^* \cap K_* \subseteq V^*$ and $K_* \subseteq K^* \subseteq K_* + V^*$. Hence the composition factors from K^* to K_* occurs among those from $K_* + V^*$ to K_*, which is the same as the composition factors from V_* to $V^* \cap K_*$ as part of those from V^* to V_*. □

Recall a Cartan subalgebra of the Leibniz Algebra $\mathbf{A}$ is a nilpotent subalgebra C such that $C = N_{\mathbf{A}}(C)$. We also need a lemma to prove the next theorem.

Lemma 2.6.2. *Let b be an element of the Leibniz algebra $\mathbf{A}$ over an arbitrary field, and let $\mathbf{A}_1$ be the preserved space of l_b (left multiplication by b). If $b \in B$, B subinvariant in $\mathbf{A}$, then $\mathbf{A}_1$ is contained in B.*

Proof. $\mathbf{A}_1 = (l_b)^k \mathbf{A}_1$, which for k large enough is in B, since $b \in B$, and B is subinvariant in $\mathbf{A}$. □

Theorem 2.6.3. *If K is a Cartan subalgebra of a Leibniz algebra $\mathbf{A}$ over an arbitrary field, then,*

1. *$K^* = \mathbf{A}$*
2. *K_* need not be an ideal of $\mathbf{A}$.*

Proof.
1. Suppose not. Then there exists some $V = K^*$ such that $K \subseteq V$ and $V \lhd\lhd \mathbf{A}$. Then there must be a V_1 such that $V \subseteq V_1 \lhd \mathbf{A}$, i.e., V_1 is a proper ideal of $\mathbf{A}$. If K is a Cartan subalgebra $\mathbf{A}$, K is also a Cartan subalgebra of V_1. Then, by Theorem 6.6 of [Bar11], $V_1 + N_{\mathbf{A}}(K) = V_1 + K = \mathbf{A}$. But this cannot happen since $K \subseteq V \subseteq V_1$, and V_1 is a proper ideal of $\mathbf{A}$. Hence $K^* = \mathbf{A}$.

2. To show (2), we consider an example of a Leibniz algebra.

   Let $\mathbf{A} = \{a_1, a_2, b, k\}$ with non-zero multiplications: $[k, a_1] = a_1, [k, a_2] = a_2, [b, a_2] = a_1$. It is easy to see this is a Leibniz algebra by checking the Leibniz identity holds. $K = \{b, k\}$ is a Cartan subalgebra. For $K^2 = 0$ and it is easy to check $N_{\mathbf{A}}(K) = K$. Now, by the lemma above, K_* cannot contain k, else $\{a_1, a_2\} \in K$. Hence, $K_* =< b >$ since $K_* = \{b\} \lhd \{b, a_1\} \lhd \{b, a_1, a_2\} \lhd \mathbf{A}$. But K_* is not an ideal of $\mathbf{A}$, for $[b, a_2] = a_1 \notin K_*$.

□

Using results from this and last sections, we have the following corollary.

Corollary 2.6.4. *Let U be a subalgebra of a Leibniz algebra $\mathbf{A}$ of char 0. Then the subalgebra of U that is generated by all subinvariant subalgebras of $\mathbf{A}$ that are contained in U is*

1. *Subinvariant in $\mathbf{A}$. (Theorem 2.5.1)*

2. *An ideal in U. (Lemma 2.6.1)*

3. *Not necessarily an ideal in $\mathbf{A}$ (Theorem 2.6.3).*

2.7 The centralizer of a subinvariant subalgebra

The results in this section are valid for an arbitrary field.

Lemma 2.7.1. *If H is a nilpotent Leibniz algebra with center Q, and if $P \neq (0)$ is an ideal of H, then $(P \cap Q) \neq (0)$.*

Proof. For some n, $[\overbrace{H, \ldots, [H}^{\text{n-1 times}}, P]\ldots] \neq (0)$, and $[\overbrace{H, \ldots, [H}^{\text{n times}}, P]\ldots] = (0)$, since H is nilpotent. Therefore, $[\overbrace{H, \ldots, [H}^{\text{n-1 times}}, P]\ldots] \subseteq Z^r(H)$, the right center of H. On the other hand, when multiplied by H on the right, i.e., $[[\overbrace{H, \ldots, [H}^{\text{n-1 times}}, P]\ldots], H]$, we know any product of length n can be left normalized, which means the product is eventually contained in $H^n = (0)$, hence it is also contained in the left center$Z^l(H)$. So, $[\overbrace{H, \ldots, [H}^{\text{n-1 times}}, P]\ldots] \subset Q$, the center of H. But $[\overbrace{H, \ldots, [H}^{\text{n-1 times}}, P]\ldots] \subseteq P$, since P is an ideal. Therefore, $(P \cap Q) \neq (0)$. □

Example 2.7.1. Let $H =< x, y, z >$ with non-zero products $[x, x] = z$. Then the center of H is $Q =< y, z >$. Let $P =< y >$, then P is an ideal of H and $P \cap Q =< y >$.

Lemma 2.7.2. *Let* $\mathbf{A}$ *be a Leibniz algebra over an arbitrary field and let* $\mathbf{A}^{\omega} = \cap_{k=1}^{\infty}\mathbf{A}^{k}$*. If* Z *is the centralizer of* $\mathbf{A}^{\omega}$ *in* $\mathbf{A}$*, and if* Z *is not contained in* $\mathbf{A}^{\omega}$*, then* $\mathbf{A}$ *has center not equal to zero.*

Proof. Since $\mathbf{A}^{\omega}$ is an ideal of $\mathbf{A}$, so is Z. By Theorem 2.2.3, $\mathbf{A} = \mathbf{A}^{\omega} + H$, where H is a nilpotent subalgebra. Let $\mathbf{A}_1 = Z + H$. Then, $\mathbf{A}_1$ is a subalgebra of $\mathbf{A}$, since $Z \lhd \mathbf{A}$ and H is a subalgebra of $\mathbf{A}$. Hence, $\mathbf{A}_1 = \mathbf{A}_1^{\omega} + H_1$, where H_1 is a nilpotent subalgebra and $\mathbf{A}_1^{\omega} \lhd \mathbf{A}_1$. But $\mathbf{A}_1^{\omega} = (Z+H)^{\omega} \subseteq Z$, since H is nilpotent and $Z \lhd A_1$.

If $Z \subseteq \mathbf{A}_1^{\omega}$, then $Z \subseteq \mathbf{A}^{\omega}$, since $\mathbf{A}_1^{\omega} \subseteq \mathbf{A}^{\omega}$. But this contradicts the hypothesis that Z is not contained in $\mathbf{A}^{\omega}$. It follows that there exists a $z_1 \in Z$ such that $z_1 \notin \mathbf{A}_1^{\omega}$. Since $Z \subseteq \mathbf{A}_1 = \mathbf{A}_1^{\omega} + H_1$, we can write $z_1 = a_1 + h_1$, where $a_1 \in \mathbf{A}_1^{\omega}$, $h_1 \in H_1$. Now, $h_1 \neq 0$ otherwise $z_1 \in \mathbf{A}_1^{\omega}$, contradicts how we choose z_1. Hence, $h_1 = z_1 - a_1 \in Z$, since $z_1 \in Z$ and $a_1 \in \mathbf{A}_1^{\omega} \subseteq Z$. It follows that $H_1 \cap Z \neq 0$.

Now, $H_1 \cap Z \lhd H_1$, since $Z \lhd \mathbf{A}$. The center of H_1, denoted by Q, is not zero since H_1 is nilpotent. Consequently by the previous Lemma, $P = (H_1 \cap Z) \cap Q \neq (0)$. Finally, $P \subseteq Z(\mathbf{A})$. For $\mathbf{A} = \mathbf{A}^{\omega} + H$, where $H \subseteq \mathbf{A}_1 = \mathbf{A}_1^{\omega} + H_1$, and hence $H \subseteq \mathbf{A}^{\omega} + H_1$, since $\mathbf{A}_1^{\omega} \subseteq \mathbf{A}^{\omega}$. Consequently, $\mathbf{A} = \mathbf{A}^{\omega} + H_1$. But $P \subseteq CL_{\mathbf{A}}(\mathbf{A}^{\omega}) = Z$, and $P \subseteq Z(H_1) = Q$. Therefore, $P \subseteq Z(\mathbf{A})$. □

Example 2.7.2. Let $\mathbf{A} = <x, y, z>$ with non-zero products $[x, z] = z$. Then $\mathbf{A}^{\omega} = <z>$. The centralizer of $\mathbf{A}^{\omega}$ in $\mathbf{A}$ is $Z = <y, z>$, which is not contained in $\mathbf{A}^{\omega}$. From the lemma, we know that the center of $\mathbf{A} = Z(\mathbf{A}) \neq 0$, as $Z(\mathbf{A}) = <y>$.

Theorem 2.7.1. *Let* $\mathbf{A}$ *be a Leibniz algebra over an arbitrary field. If* B *is subinvariant in* $\mathbf{A}$ *and if* B *has centralizer* (0)*, then the centralizer of* $B^{\omega} = \cap_{k=1}^{\infty} B^k$ *in* $\mathbf{A}$ *is contained in* B^{ω}.

Proof. By the countrapositive of the previous lemma, if B has centralizer (0), the center of $B = Z(B)$ is also (0). Then $CL_B(B^{\omega}) = B \cap CL_{\mathbf{A}}(B^{\omega}) = B \cap Z$ is contained in B^{ω}. Suppose $Z \nsubseteq B^{\omega}$, then Z is not contained in B since $Z \subset B$ implies $Z = Z \cap B = CL_B(B^{\omega}) \subseteq B^{\omega}$. Since B^{ω} is an ideal of $\mathbf{A}$, and Z is an ideal of $\mathbf{A}$ we have $K = B + Z$ a subalgebra of $\mathbf{A}$ properly containing B. Since $B \lhd\lhd \mathbf{A}$, B is a subinvariant subalgebra of K. Therefore, B is an ideal of some subinvariant subalgebra B_1 of K and $B^{\omega} \subset B_1^{\omega}$. Since $Z \nsubseteq B^{\omega}$, there exists $z \in B_1 \cap Z$ and $z \notin B$. There are four possibilities: Case(1): $z^2 = 0$; Case(2):$z^2 \notin B$; Case(3): $z^2 \in B - B^{\omega}$; or Case(4): $z^2 \in B^{\omega}$.

Case(1): If $z^2 = 0$, then choose $C = \text{span}\,\{z, B\}$ which is a subalgebra properly containing B since $z \in B_1$ and $B \lhd B_1$. Clearly $B^{\omega} \subseteq C^{\omega}$. Since $z^2 = 0$, $z \in B_1$, and $B \lhd B_1$, we have $(z+B)(z+B) \subseteq B^2 + (B \cap Z) = B^2 + CL_B(B^{\omega}) \subseteq B^2 + B^{\omega} \subseteq B^2$ which implies $C^2 = B^2$. Now, $(z+B)(z+B)(z+B) \subseteq (z+B)B^2 \subset B^3$ since as shown above $zB \subseteq B^2$ and by Leibniz identity $zB^2 \subseteq (zB)B + B(zB) \subseteq B^2B + BB^2 \subseteq B^3$. This implies that $C^3 = B^3$. Since $C^{\omega} = C^t$ for some $t \in \mathbb{Z}_{\geq 2}$, repeating this process we get $C^{\omega} = C^t = B^t = B^{\omega}$. Since $z \in C \cap Z = CL_C(C^{\omega})$ and $z \notin C^{\omega} = B^{\omega} \subset B$, by the lemma above, we have $Z(C) \neq \{0\}$. Since $CL_{\mathbf{A}}(B) \supseteq CL_C(B) \supseteq CL_C(C) = Z(C)$, we get $CL_{\mathbf{A}}(B) \neq \{0\}$ which is a contradiction.

Case(2): If $z^2 \notin B$, then $z^4 = (z^2)^2 = 0$ since $z^2 \in Leib(\mathbf{A})$ which is an abelian ideal of $\mathbf{A}$. Since $z^2 \in B_1$ and $B \lhd B_1$, we have $C = \text{span}\{z^2, B\}$ a subalgebra. Clearly $B^\omega \subseteq C^\omega$. Then as in Case(1), we get $B^\omega = C^\omega$ which leads to a contradiction.

Case(3): In this case $z^2 \in B$, but $z^2 \notin B^\omega$. Since $z \in Z$ which is an ideal in $\mathbf{A}$, $z^2 \in B \cap Z = CL_B(B^\omega) \subseteq B^\omega$ which is a contradiction.

Case(4): Suppose $z^2 \in B^\omega$. As in case(1) consider the subalgebra $C = \text{span}\ \{z, B\}$ properly containing B. Since $B^\omega \lhd B$, $(z + B)(z + B) \subseteq B^2 + B^\omega \subseteq B^2$. This implies $C^2 = B^2$. Repeating this process as in case(1), we get $C^\omega = B^\omega$ which leads to a contradiction.

Hence we have $CL_{\mathbf{A}}(B^\omega) = Z \subset B^\omega$ proving the theorem.

□

Example 2.7.3. Let $\mathbf{A} =< a, a^2, a^3 >$ be the cyclic algebra with $[a, a^3] = a^3$. Let $B =< a^2 >$, which is subinvariant in $\mathbf{A}$. $B^\omega = 0$, so the centralizer of B^ω in $\mathbf{A}$ is $\mathbf{A}$ itself, hence it is not contained in B^ω, then from the theorem above, we know that the centralizer of B is not zero, as $CL_{\mathbf{A}}(B) =< a^2, a^3 >$.

2.8 Derivation Tower Theorem

For a Lie algebra L, we can construct a series of derivation algebras. Let $D_0 = L$, and $D_1 = D(L)$, the derivation algebra of L, $D_2 = D(D_1)$, the derivation algebra of D_1. Likewise, we can construct a series of algebras, where each term D_{i+1} is the derivation algebra of the previous term D_i. In his paper[Sch51], Schenkman showed that we can get a "better" construction if we restrict L to be the algebra of center zero.

With the above restriction, L is now isomorphic to its inner derivations ad$L$, which is an ideal in $D(L)$. With the help of the following lemma, now we can have $D_0 \cong \text{ad}L \cong L$. Also, since the centralizer of D_0 in D_1 is zero, the center of D_1 is also zero (the center is always contained in any centralizer). Thus, D_1 is an ideal of D_2. If we repeat this process, we can get a series of algebras, whereas each term is embedded in the next. Note that from the lemma, it is always true that $D_i \lhd D_{i+1}$. The formal definition is noted after the lemma.

Lemma 2.8.1. *[Sch51] Let L be a Lie algebra with center zero, and let D_1 be the derivation algebra of L, D_0 the algebra of inner derivations of L. Then*

1. *L is isomorphic to D_0;*
2. *D_0 is an ideal of D_1;*
3. *the centralizer of D_0 in D_1 is zero.*

Definition 2.8.1. [Sch51] If L is a Lie algebra with center zero, let $D_0 = L, D_1 = D(L)$, the derivation algebra of $L, \ldots, D_n = D(D_{n-1}), \ldots$, then the series of algebras $D_0, D_1, D_2, \ldots$ is called the tower of derivation algebra of L.

When we take the derivation of a Lie algebra, we sometimes get a larger algebra (dimension-wise), though we do not have to get a larger algebra, even if the center is 0. Such algebras are called complete. If the center is not 0, it might be smaller. So we would think that the dimension of the terms in the tower of derivation algebras would keep growing. However, it has been shown by Schenkman in the following theorem that the series actually has an upper bound, which is dependent on the Lie algebra itself. In other words, $D_s = D_{s+1}$ for some s.

Theorem 2.8.1. *[Sch51] Let L be a Lie algebra over an arbitrary field and let the center of L be zero, so that $L = D_0$. Then, $D_0, D_1, D_2, \ldots D_n$, the tower of derivation algebras of L, ends finitely, and $dim D_n \leq d + c$, where d is the dimension of the derivation algebra of $L^{\omega} = \cap_{i=1}^{\infty} L^i$, and c is the dimension of the center of L^{ω}.*

L^i here refers to the i-th term of the lower central series for a Lie algebra.

A natural question to ask is, can we get an analogous result for Leibniz algebras? To find the answer, we want to construct a tower for Leibniz algebras. Let us start with an arbitrary Leibniz algebra **A**. Let $D(\mathbf{A})$ denote the derivation algebra of **A**. **A** is mapped to $D(\mathbf{A})$ by $L(x) = L_x$,i.e., we are mapping an element to the left multiplication operation by that element $\forall x \in \mathbf{A}$. Then $L(\mathbf{A})$ is an ideal in $D(\mathbf{A})$ and $D(\mathbf{A})$ is a Lie algebra. To construct the rest of the tower, the same technique as in the Lie case can be used. As before, we want **A** to be embedded in $D(\mathbf{A})$, thus we take the case that the left center of A is 0. Then, as in the Lie case, we get each derivation algebra embedded in the next. Thus we can define the tower of derivations for a Leibniz algebras as we did in Lie. Does this series terminate? The answer is yes, as shown by the following theorem.

Theorem 2.8.2. *Let $\mathbf{A}$ be a Leibniz algebra over an arbitrary field and let the left center of $\mathbf{A}$ be zero, so that $\mathbf{A} = D_0$ and each D_i has left center zero. Then, $D_0, D_1, D_2, \ldots D_n$, the tower of derivation algebras of $\mathbf{A}$, ends finitely, and $dim D_n \leq d + c$, where d is the dimension of the derivation algebra of $\mathbf{A}^{\omega} = \cap_{i=1}^{\infty} \mathbf{A}^i$, and c is the dimension of the center of $\mathbf{A}^{\omega}$.*

$\mathbf{A}^i$ here refers to the lower central series for a Leibniz algebra.

Proof. $Z^l(\mathbf{A}) = 0 \implies \text{Leib}(\mathbf{A}) = 0 \implies$ **A** is Lie. So the derivation tower series also become the derivation tower series for a Lie algebra, whose construction is based by Lemma 2.8.1, which has be shown by Schenkman as well. Also, $Z(\mathbf{A}) \subseteq Z^l(\mathbf{A}) \implies Z(A) = 0$. The result follows from Schenkman's theorem. □

CHAPTER

3

SUPERSOLVABLE LEIBNIZ ALGEBRAS

3.1 Additional preliminaries

In this section, we are going to give some definitions that are used in this chapter only. Throughout this section, all Leibniz algebras will be finite dimensional over a field $\mathbb{F}$ of characteristic 0. Let **A** be a Leibniz algebra with a subalgebra H. We define the following notations:

$Soc(\mathbf{A})$ the socle of **A**, which is the sum of all minimal ideals of **A**.

$Asoc(\mathbf{A})$ the abelian socle of **A**, which is the sum of all minimal abelian ideals of **A**.

$F(\mathbf{A})$ the Frattini subalgebra of **A**, which is the intersection of all maximal subalgebra of **A**. Note that it is an ideal of **A** in the characteristic 0 case. (Theorem 2.10 of [Bat13])

$\phi(\mathbf{A})$ the Frattini ideal of **A**, which is the largest ideal that is contained in the Frattini subalgebra.

3.2 C-ideals and preliminary results

After studying the results of solvability and supersolvability in terms of c-ideals in [Tow09], Salemkar investigated further the influence of c-ideality of subalgebras on the structure of finite dimensional Lie algebras [Sal13]. In this section, we studied the Leibniz analogue of some preliminary results that are vital for the next section. Some results are identical for Lie and Leibniz algebras with similar proofs, some results require certain modifications.

The following is a lemma that we are going to use often in our proofs.

Lemma 3.2.1. *(Lemma 1.9 of [Bat13]) If B is a minimal ideal of $\mathbf{A}$, then either $[B,\mathbf{A}]=0$ or $[b,a]=-[a,b]$ for all $a\in\mathbf{A}$ and all $b\in B$.*

The following corollary is a direct result from the above lemma.

Corollary 3.2.1. *$CL_{\mathbf{A}}(Soc(\mathbf{A}))=CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$; $CL_{\mathbf{A}}(Asoc(\mathbf{A}))=CL^{l}_{\mathbf{A}}(Asoc(\mathbf{A}))$.*

Proof. $CL_{\mathbf{A}}(Soc(\mathbf{A}))\subseteq CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$ always, so we only need to show $CL_{\mathbf{A}}(Soc(\mathbf{A}))\supseteq CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$. We can denote $Soc(\mathbf{A})=\oplus_{i=1}^{n}S_i$, where each S_i is a minimal ideal of $\mathbf{A}$. Let $x\in CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$, then for any $a_i\in S_i$, $[x,a_i]=0$. Then, by the lemma above, either $[a_i,x]=0$ or $[a_i,x]=-[x,a_i]=0$. Hence x annihilates a_i on both sides, which means $x\in CL_{\mathbf{A}}(Soc(\mathbf{A}))$. Similarly, $CL_{\mathbf{A}}(Asoc(\mathbf{A}))=CL^{l}_{\mathbf{A}}(Asoc(\mathbf{A}))$. □

We included the equality to the left centralizer for the following proposition. This equality is needed for the proof of corollary 3.2.4 .

Proposition 3.2.1. *Let $\mathbf{A}$ be a Leibniz algebra over a field of arbitrary characteristic with $\phi(\mathbf{A})=0$. Then $Asoc(\mathbf{A})=N(\mathbf{A})=CL_{\mathbf{A}}(Soc(\mathbf{A}))=CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$. In particular, if $\mathbf{A}$ is solvable, then $CL_{\mathbf{A}}(N(\mathbf{A}))=CL^{l}_{\mathbf{A}}(N(\mathbf{A}))=N(\mathbf{A})$. Here, $N(\mathbf{A})$ is the nilradical of $\mathbf{A}$.*

Proof. $Asoc(\mathbf{A})=N(\mathbf{A})=CL_{\mathbf{A}}(Soc(\mathbf{A}))$ is shown in Theorem 2.4 of [Bat13]. From the corollary above, we get $CL_{\mathbf{A}}(Soc(\mathbf{A}))=CL^{l}_{\mathbf{A}}(Soc(\mathbf{A}))$.

On the other hand, $\mathbf{A}$ solvable means that $Asoc(\mathbf{A})=Soc(\mathbf{A})$ since all minimal ideals are abelian now. (Minimal ideals of $\mathbf{A}$ are irreducible modules of $\mathbf{A}$, by Theorem 3.2 of [Dem14], all irreducible modules of $\mathbf{A}$ are of dimension 1 hence abelian). Then, $N(\mathbf{A})=Asoc(\mathbf{A})=CL_{\mathbf{A}}(Soc(\mathbf{A}))=CL_{\mathbf{A}}(Asoc(\mathbf{A}))=CL_{\mathbf{A}}(N(\mathbf{A}))=CL^{l}_{\mathbf{A}}(N(\mathbf{A}))$ by the corollary above.

□

Definition 3.2.1. Like in Lie theory, let $\mathbf{A}$ be a finite dimensional Leibniz algebra, and H a subalgebra of $\mathbf{A}$. Then H is called a **c-ideal** of $\mathbf{A}$ if there is an ideal K of $\mathbf{A}$ such that $\mathbf{A}=H+K$ and $H\cap K$ is contained in the core of H (with respect to $\mathbf{A}$), which is denoted by $H_{\mathbf{A}}$, that is, the largest ideal of $\mathbf{A}$ contained in H.

Note that by definition, all ideals are c-ideals, since for any ideal $I\lhd\mathbf{A}$, $\mathbf{A}=I+\mathbf{A}$ and $I\cap\mathbf{A}=I\subseteq I_{\mathbf{A}}=I$. The following is an example of a c-ideal that is not an ideal.

Example 3.2.1. Let $\mathbf{A} = < x, y, z >$ with $[x, y] = z, [x, z] = -y$ and other products zero. Let $M = < x >$, which is a subalgebra but clearly not an ideal of $\mathbf{A}$. Then $\mathbf{A} = M + K$, where $K = < y, z > \lhd \mathbf{A}$. We can see that $M \cap K = 0 \subseteq$ core of $M = 0$. Hence M is a c-ideal of $\mathbf{A}$.

Lemma 3.2.2. *Let H be a c-ideal of a Leibniz algebra $\mathbf{A}$. Then:*

1. *There is an ideal N of $\mathbf{A}$ such that $\mathbf{A} = H + N$ and $H \cap N = H_{\mathbf{A}}$;*
2. *If K is an ideal of $\mathbf{A}$ contained in H, then H/K is a c-ideal of $\mathbf{A}/K$.*

Proof.
1. Since H is a c-ideal of $\mathbf{A}$, there exists an ideal K of $\mathbf{A}$ such that $\mathbf{A} = H + K$ and $H \cap K \subseteq H_{\mathbf{A}}$. We can define the ideal $N := K + H_{\mathbf{A}}$. Then, $H + N = H + K + H_{\mathbf{A}} = H + K = \mathbf{A}$, and $H \cap N = H \cap (K + H_{\mathbf{A}}) = H \cap K + H \cap H_{\mathbf{A}} = H_{\mathbf{A}}$.
2. By part 1, H a c-ideal of $\mathbf{A}$ implies that there exits an ideal N of $\mathbf{A}$ such that $\mathbf{A} = H + N$ and $H \cap N = H_{\mathbf{A}}$.

   Then, $\mathbf{A}/K = (H + N)/K = H/K + N/K$, whereas N/K is an ideal of $\mathbf{A}/K$ and $H/K \cap N/K = (H \cap N)/K = H_{\mathbf{A}}/K$ is the core of H/K. Hence the result.

□

Example 3.2.2. Let $\mathbf{A} = < x, y, z >$, with non-zero products $[x, z] = z + y$, and $[x, y] = y$.

1. Let $H = < x, y >$, then H is a c-ideal, with its core $H_{\mathbf{A}} = < y >$. Let $N = < y, z >$, then N is an ideal of $\mathbf{A}$, and $\mathbf{A} = H + N$, with $H \cap N = < y >= H_{\mathbf{A}}$.
2. Let $K = < y >$, then K is an ideal of $\mathbf{A}$ such that $K \subseteq H$. Let $H/K = \bar{H}$, $\mathbf{A}/K = \bar{\mathbf{A}}$. Then, it is easy to see $\bar{H}$ is a c-ideal of $\bar{\mathbf{A}}$, with $\bar{H}_{\bar{\mathbf{A}}} = \bar{0}$.

Lemma 3.2.3. *Let N be a minimal ideal of a Leibniz algebra $\mathbf{A}$. If N has a maximal subalgebra R that is a c-ideal in $\mathbf{A}$, then $dimN = 1$.*

Proof. Since R is a c-ideal of $\mathbf{A}$, by the above lemma, there exists an ideal K of $\mathbf{A}$ such that $\mathbf{A} = R + K$ and $R \cap K = R_{\mathbf{A}} = 0$ since R is contained in a minimal ideal N. Hence $\mathbf{A} = R \dot{+} K$.

Now, $N = N \cap \mathbf{A} = N \cap (K \dot{+} R) = (N \cap K) + R$, and $N \cap K$ is an ideal of $\mathbf{A}$ contained in N. But N is a minimal ideal, so either $N \cap K = N$ and $R = R \cap N = R \cap (N \cap K) \subseteq R \cap K = 0 \implies R = 0$, or $N \cap K = 0$ and $N = R$, a contradiction. Either way, dim$N$=1. □

In order to give a non-trivial example, we give an example of the contrapositive of the above lemma.

Example 3.2.3. Let $\mathbf{A} = sl(2,\mathbb{C}) \oplus \mathbb{C}^2$, whereas $\mathbb{C}^2 =< x, y >$. Let $N =< x, y >$. Then, N is a minimal ideal of $\mathbf{A}$ with dimension 2. Any maximal subalgebra of N is of the form $M =< ax + by >, a, b \in \mathbb{F}$. Since M is not an ideal of $\mathbf{A}$, and $\dim M = 1$, the core of M, $M_{\mathbf{A}} = 0$. But the only ideals of $\mathbf{A}$ are $\mathbf{A}, 0, sl(2,\mathbb{C}), \mathbb{C}^2$. So, $\mathbf{A} = M + \mathbf{A}$, and $M \cap \mathbf{A} = M \nsubseteq M_{\mathbf{A}}$, so M is not a $c-$ideal.

Next, We have listed a few results that are needed to prove Lemma 3.2.5.

Corollary 3.2.2. *The Frattini subalgebra of a nilpotent Leibniz algebra **A** equals the derived subalgebra of **A**.*

Proof. Like always, we denote the Frattini subalgebra by $\phi(\mathbf{A})$, and the derived subalgebra by $\mathbf{A}^2$. By Lemma 2.5 of [Tow73], $\phi(\mathbf{A}) \subseteq \mathbf{A}^2$. So we only need to show $\mathbf{A}^2 \subseteq \phi(\mathbf{A})$.

Let M be a maximal subalgebra of $\mathbf{A}$. Since $\mathbf{A}$ is nilpotent, M is an ideal in $\mathbf{A}$ by Theorem 4.16 of [Dem14]. Then,M is a maximal ideal of $\mathbf{A}$, which means $\mathbf{A}/M$ is either one dimensional or simple. But $\mathbf{A}/M$ cannot be simple since $\mathbf{A}$ is nilpotent. Hence $\mathbf{A}/M$ is one dimensional and has no proper subalgebras. Thus, $\mathbf{A}^2 \subseteq M$. Since M is an arbitrary maximal subalgebra of $\mathbf{A}$, $\mathbf{A}^2 \subseteq \phi(\mathbf{A})$. Hence $\mathbf{A}^2 = \phi(\mathbf{A})$.

□

Remark. Conversely, if $\mathbf{A}^2 = \phi(\mathbf{A})$, then all maximal subalgebras are ideals of $\mathbf{A}$ and hence $\mathbf{A}$ is nilpotent by Theorem 4.16 of [Dem14].

Lemma 3.2.4. *(Lemma 4.1 of [Tow73]) If C is a subalgebra of **A**, and B is an ideal of **A** contained in $\phi(C)$, then $B \subseteq \phi(\mathbf{A})$.*

Corollary 3.2.3. *If C is a nilpotent ideal of a Leibniz algebra **A**, then $\phi(C) \subseteq \phi(\mathbf{A})$.*

Proof. C is a nilpotent ideal of $\mathbf{A} \implies \phi(C) = C^2$ is an ideal of $\mathbf{A}$. So by the above lemma, $\phi(C) \subseteq \phi(\mathbf{A})$. □

Lemma 3.2.5. *Let **A** be a Leibniz algebra, and K be a nilpotent ideal of **A** with $K \cap \phi(\mathbf{A}) = 0$. Then there is a subalgebra H of **A** such that $\mathbf{A} = K + H$.*

Proof. Let $K^2 = [K, K]$ denote the derived subalgebra of K. Then, by cor 3.2.2, $K^2 = \phi(K)$. Also, since K is nilpotent, by cor 3.2.3, $\phi(K) \subseteq \phi(\mathbf{A})$. So, $K^2 \subseteq K \cap \phi(\mathbf{A}) = 0$, hence K is abelian.

Lemma 7.2 of [Tow73] says that if B is an abelian ideal of an algebra A such that $B \cap \phi(A) = 0$, then there exists a subalgebra C of A such that $A = B + C$. □

Example 3.2.4. Let $\mathbf{A} =< x, y >$ with non-zero product $[x, y] = y$. $K =< y >$ is an nilpotent ideal of $\mathbf{A}$. Then $\phi(\mathbf{A}) = 0$, so $K \cap \phi(\mathbf{A}) = 0$. Thus, $H =< x >$ is the subalgebra of $\mathbf{A}$ such that $\mathbf{A} = K + H$.

Proposition 3.2.2. *Let K be an ideal of a solvable Leibniz algebra $\boldsymbol{A}$. Then $N(K/K\cap\phi(\boldsymbol{A})) = N(K)/(K\cap \phi(\boldsymbol{A}))$. In particular, $N(\boldsymbol{A}/\phi(\boldsymbol{A})) = N(\boldsymbol{A})/\phi(\boldsymbol{A})$.*

Proof. Since $N(K)/K\cap\phi(\mathbf{A})$ is a nilpotent ideal of $K/K\cap\phi(\mathbf{A})$, $N(K)/K\cap\phi(\mathbf{A})$ is always contained in $N(K/K\cap\phi(\mathbf{A}))$. So we only need to show $N(K/K\cap\phi(\mathbf{A}))\subseteq N(K)/K\cap\phi(\mathbf{A})$.

We can write $N(K/K\cap\phi(\mathbf{A}))$ as $M/K\cap\phi(\mathbf{A})$ for some ideal M of $\mathbf{A}$ such that M contains $K\cap\phi(\mathbf{A})$.

Since $N(K/K\cap\phi(\mathbf{A}))$ is nilpotent, by Theorem 5.5 of [Bar11], M is a nilpotent ideal of $\mathbf{A}$. Therefore, $M\subseteq N(K)$, which implies that $N(K/K\cap\phi(\mathbf{A})) = M/K\cap\phi(\mathbf{A})$ is contained in $N(K)/K\cap\phi(\mathbf{A})$.

The last part is just the special case for $K=\mathbf{A}$. □

Example 3.2.5. Let $\mathbf{A} =< a, a^2, a^3 >$ be the non-nilpotent cyclic Leibniz algebra, with non-zero products $[a, a^3] = a^3$. Then $\mathbf{A}$ has two maximal subalgebras $< a^2, a^3 >$ and $< a-a^3, a-a^3 >$. Hence, $\phi(\mathbf{A}) =< a^2 - a^3 >$. Let $K =< a^3 >$ then $K\cap\phi(\mathbf{A}) =< 0 >$. Then, it is clear $N(K/K\cap\phi(\mathbf{A})) = N(K)/K\cap\phi(\mathbf{A})$.

Proposition 3.2.3. *Let $\mathbf{A}$ be a Leibniz algebra and K be a nonzero solvable ideal of $\mathbf{A}$ such that $K\cap\phi(\boldsymbol{A}) = 0$. Then the nil radical of K is the direct sum of minimal abelian ideals of $\mathbf{A}$ which are contained in K.*

Proof. Let $\bar{\mathbf{A}} = \mathbf{A}/\phi(\mathbf{A})$, $\bar{K} = (K+\Phi(\mathbf{A}))/\phi(\mathbf{A})$. Then $N(\bar{K})$ is a nonzero nilpotent ideal of $\bar{\mathbf{A}}$ by Theorem 2.3.5. Then, Prop 3.2.1 implies that $N(\bar{K})\subseteq N(\bar{\mathbf{A}}) = Asoc(\bar{\mathbf{A}})$ since $\phi(\bar{\mathbf{A}}) = 0$. Let X be the direct sum of all minimal abelian ideals of $\bar{\mathbf{A}}$ that are contained in $\bar{K}$, and Let Y be the direct sum of all minimal abelian ideals of $\bar{\mathbf{A}}$ that intersects $\bar{K}$ at 0. Then it is clear that $X\neq 0$, $Asoc(\bar{\mathbf{A}}) = X\oplus Y$, and $\bar{K}\cap Y = 0$. Hence, $N(\bar{K}) = N(\bar{K})\cap(X\oplus Y) = (N(\bar{K})\cap X) + (N(\bar{K})\cap Y) = X + 0 = X$.

Now, let $\bar{N}$ be an arbitrary minimal abelian ideal of $\bar{\mathbf{A}}$ with $\bar{N}\subseteq\bar{K}$, and N be the pre-image of $\bar{N}$ in $\mathbf{A}$. (i.e., $\bar{N} = N/\phi(\mathbf{A})$.) It is easy to see that $K\cap N$ is a nonzero ideal of $\mathbf{A}$, hence it contains a minimal abelian ideal T of $\mathbf{A}$. Then, $T\cap\phi(\mathbf{A})\subseteq K\cap N\cap\phi(\mathbf{A}) = K\cap\phi(\mathbf{A}) = 0$, which yields that $\bar{N} = (T+\phi(\mathbf{A}))/\phi(\mathbf{A})\cong T/T\cap\phi(\mathbf{A})\cong T$. As $N(K)\cong N(K+\phi(\mathbf{A})/\phi(\mathbf{A})) = N(\bar{K})$, and the above correspondance $T\mapsto\bar{N}$ is a bijection bewteen the minimal ideals of $\mathbf{A}$ in $N(K)$ and the minimal ideals of $\bar{\mathbf{A}}$ in $N(\bar{K})$ and $N(K)\cong N(\bar{K})$ = direct sum of minimal abelian ideals of $\bar{\mathbf{A}}$ contained in $N(\bar{K})\cong$ direct sum of minimal abelian ideals of $\mathbf{A}$ contained in $N(K)$, the result holds. □

Example 3.2.6. Let $\mathbf{A} = sl(2,\mathbb{C})\oplus B$, where $B =< x, y, z >$ with non-zero products $[x, z] = z$. Clearly, $\phi(\mathbf{A}) = 0$. Let $K =< x, y, z >$, then $N(K) =< y, z >=< y >\oplus< z >$.

Definition 3.2.2. Let $\mathbf{A}$ be a Lie(Leibniz) algebra over a field $\mathbb{F}$ of arbitrary characteristic. Then, $\mathbf{A}$ is **supersolvable** if it admits a complete flag made up of ideals.

Remark. Note that by [Dem14], if $\mathbb{F}$ is algebraically closed of characteristic 0, solvable implies supersolvable.

Proposition 3.2.4. *Let $\boldsymbol{A}$ be a Leibniz algebra over a field of arbitrary characteristic, and K be an ideal of $\boldsymbol{A}$ having a series $0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K$ such that K_i is ideal in $\boldsymbol{A}$ and dim$(K_i/K_{i-1}) = 1$ for all $1 \leq i \leq n$. Then $\boldsymbol{A}/C^l_{\boldsymbol{A}}(K)$ is supersolvable.*

Proof. Let $T : x \mapsto T_x$ acting on K, whereas T_x is the left multiplication by x. Then T is a homomorphism from $\mathbf{A}$ into the derivation algebra of K, $Der(K)$. Using the chain of ideals, we have a basis for K such that T_x can be represented in upper triangular form.

The kernel of T is the left centralizer of K in $\mathbf{A}$,$C^l_{\mathbf{A}}(K)$. Then, $\mathbf{A}/C^l_{\mathbf{A}}(K) \cong Im(T)$, which is supersolvable as is seen by the matrix representation of $Im(T)$. □

Example 3.2.7. Let $\mathbf{A} =< x_1, x_2, x_3, x_4 >$ be a Leibniz algebra with non-zero products $[x_1, x_3] = x_4, [x_3, x_2] = x_4$. Let $K =< x_2, x_3, x_4 >$ with the series of ideals $0 = K_0 \subseteq K_1 =< x_4 >\subseteq K_2 =< x_4, x_3 >\subseteq< x_2, x_3, x_4 >= K_4 = K$. Then, the left centralizer of K in $\mathbf{A}$ is $CL^l_{\mathbf{A}}(K) =< x_4, x_2 >$. Denote $\bar{\mathbf{A}} = \mathbf{A}/CL^l_{\mathbf{A}}(K) \cong< \bar{x}_1, \bar{x}_3 >$, whereas $\bar{x}_i = (x_i + CL^l_{\mathbf{A}}(K))/CL^l_{\mathbf{A}}(K)$. Then, $\bar{\mathbf{A}}$ is supersolvable with the series of ideals $< \bar{x}_1 >\subseteq< \bar{x}_1, \bar{x}_3 >$ as $[\bar{x}_1, \bar{x}_3] = \bar{x}_4 = \bar{0}$.

Corollary 3.2.4. *Let $\boldsymbol{A}$ be a solvable Leibniz algebra over a field of characteristic zero, and $N(\boldsymbol{A})$ possesses a series $\phi(\boldsymbol{A}) = R_0 \leq R_1 \cdots \leq R_n = N(\boldsymbol{A})$ such that R_i is ideal in $\boldsymbol{A}$ and dim$(R_i/R_{i-1}) = 1, 1 \leq i \leq n$. Then $\boldsymbol{A}$ is supersolvable.*

Proof. Since the field is of characteristic zero, by Theorem 2.10 of [Bat13], $\phi(\mathbf{A})$ is an ideal of $\mathbf{A}$. Let $\bar{\mathbf{A}} = \mathbf{A}/\phi(\mathbf{A})$. Then the nil radical of $\bar{\mathbf{A}}$, denoted by $\bar{N} = N(\mathbf{A})/\phi(\mathbf{A})$ satisfies the hypothesis of prop 3.2.4. That is,

$$0 = \bar{R_0} \subseteq \bar{R_1} \subseteq \cdots \subseteq \bar{R_n} = \bar{N}(\bigstar)$$

whereas $\bar{R_i} = R_i/\phi(\mathbf{A})$, and $\bar{R_i} \lhd \bar{\mathbf{A}}$, dim$(\bar{R_i}/\bar{R}_{i-1}) = 1$. Hence, $\bar{\mathbf{A}}/C^l_{\bar{\mathbf{A}}}(\bar{N})$ is supersolvable. But $\phi(\bar{\mathbf{A}}) = 0$, so by Prop 3.2.1, $C^l_{\bar{\mathbf{A}}}(\bar{N}) = \bar{N}$. So, $\bar{\mathbf{A}}/C^l_{\bar{\mathbf{A}}}(\bar{N}) = \bar{\mathbf{A}}/\bar{N} = \mathbf{A}/N(\mathbf{A})$ is supersolvable.

$\mathbf{A}/N(\mathbf{A})$ is supersolvable implies that there exists a flag of ideals from $N(\mathbf{A})$ to $\mathbf{A}$. This together with ($\bigstar$) means that there exists a flag of ideals from $\phi(\mathbf{A})$ to $\mathbf{A}$. Hence $\bar{\mathbf{A}} = \mathbf{A}/\phi(\mathbf{A})$ is supersolvable. By Theorem 6 of [Bur15], $\mathbf{A}$ is supersolvable.

□

Example 3.2.8. Let $\mathbf{A} =< x, y, z >$ with non-zero products $[x, z] = z$. Then $\phi(\mathbf{A}) =< z >$, $N(\mathbf{A}) =< y, z >$. Since $\phi(\mathbf{A}) \subseteq N(\mathbf{A})$ and dim$N(\mathbf{A})/\phi(\mathbf{A}) = 1$, we know $\mathbf{A}$ is supersolvable, with the chain of ideals $< z >\subseteq< y, z >\subseteq \mathbf{A}$.

3.3 Main results

Salemkar did research on supersolvable Lie algebras and c-ideals as inspired by Tower. Next, we reviewed the main results from [Sal13] and investigated the Leibniz analogues.

Theorem 3.3.1. *Let $\mathbf{A}$ be a solvable Leibniz algebra such that every maximal subalgebra of $N(\mathbf{A})$ is c-ideal in $\mathbf{A}$. Then, $\mathbf{A}$ is supersolvable.*

Proof. By Theorem 6 of [Bur15], $\mathbf{A}/\phi(\mathbf{A})$ supersolvable implies that $\mathbf{A}$ is supersolvable, so we may assume $\phi(\mathbf{A}) = 0$. Then, by Prop 3.2.1, $N(\mathbf{A}) = Asoc(\mathbf{A}) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, whereas each B_i is a minimal abelian ideal of $\mathbf{A}$.

We want to show that $\dim B_i = 1, \forall i$. Suppose B_i^* is any maximal subalgebra of B_i. Then, $M := B_i^* + (B_1 \oplus \cdots \oplus \hat{B}_i \oplus \cdots \oplus B_n)$ is a maximal subalgebra of $N(\mathbf{A})$. By hypothesis, M is a c-ideal, also from Lemma 3.2.2(1), there exists an ideal N of $\mathbf{A}$ such that $\mathbf{A} = M + N$ and $M \cap N = M_{\mathbf{A}} = B_1 \oplus \cdots \oplus \hat{B}_i \oplus \cdots \oplus B_n$. But then, $\mathbf{A}$ can be written as $(M - M \cap N) + N = B_i^* \dotplus N$, and $B_i^* \cap N = 0 \subseteq (B_i^*)_{\mathbf{A}} = \{0\}$. Therefore, B_i^* is a c-ideal in $\mathbf{A}$, and by Lemma 3.2.3, $\dim B_i = 1$.

Now, we define $K_i = B_1 \oplus B_2 \oplus \cdots \oplus B_i$ for $i = 1, \ldots, n$. Then $N(\mathbf{A})$ posseses a chain of ideals: $\phi(\mathbf{A}) = 0 = K_0 \leq K_1 \leq \cdots \leq K_n = N(\mathbf{A})$, whereas $K_i \lhd \mathbf{A}$, $\dim K_i/K_{i-1} = 1$. Then, by Cor 3.2.4, $\mathbf{A}$ is supersolvable. □

Here is an example of the above theorem.

Example 3.3.1. Let $\mathbf{A} = < x, y, z >$ with non-zero products $[z, y] = 2y$. Then $\mathbf{A}$ is a solvable Leibniz algebra as $\mathbf{A}^{(2)} = 0$. It is easy to see that $N(\mathbf{A}) = < x, y >$. We can check that each maximal subalgebra of $N(\mathbf{A})$ is a c-ideal:

1. $< x >$ and $< y >$ are ideals of $\mathbf{A}$ hence are c-ideals.

2. $M = < ax + by >$ where a, b are non-zero scalars.

   $z(ax + by) = 2by \notin M$ so M is not an ideal of $\mathbf{A}$ hence the core of M, $M_{\mathbf{A}} = 0$.

   Let $K = < y, z >$, which is an ideal of $\mathbf{A}$. Then, $M + K = \mathbf{A}$ since $ax = ax + by - by = ax$ is in $M + K$, so x is also. $M \cap K = 0 = M_{\mathbf{A}}$. Hence M is a c-ideal.

Now, all maximal subalgebras of $N(\mathbf{A})$ are c-ideals of $\mathbf{A}$. By the theorem above, $\mathbf{A}$ is supersolvable. $\mathbf{A}$ is supersolvable indeed, as it has a chain of ideals $< x > \subseteq < x, y > \subseteq \mathbf{A}$.

The following corollary is a direct result from Theorem 3.3.1. (since all ideals are c-ideals).

Corollary 3.3.2. *Let **A** be a solvable Leibniz algebra, and every maximal subalgebra of $N(\mathbf{A})$ is an ideal of **A**. Then **A** is supersolvable.*

Remark. The assumption that **A** is solvable is vital for the corollary above. Salemkar provided an example in his paper, which also works for Leibniz algebras. Let H be a nilpotent Leibniz algebra, K a simple Leibniz algebra, and $\mathbf{A} = H \oplus K$. Then $N(\mathbf{A}) = H$, and each maximal subalgebra of $N(\mathbf{A})$ is an ideal of **A**, but **A** is not solvable hence cannot be supersolvable.

Salemkar claimed that if **A** is a solvable Lie algebra, K is an ideal of **A** such that $\mathbf{A}/K$ is supersolvable, then $N(K)$ is a direct sum of one dimensional minimal abelian ideals, which implies that any maximal subalgebra of $N(K)$ is an ideal in **A**. We gave a counterexample here to show this statement is indeed not true, followed by a lemma that will be used in place of the claim.

Example 3.3.2. Let $V =< x, y >$, T be a linear transformation on V with $T(x) = x, T(y) = 2y$. Then $\mathbf{A} =< x, y, T >$ is a Lie algebra with $[T, x] = x, [T, y] = 2y$ be the only non-zero brackets. Now, **A** is solvable, as $\mathbf{A}^{(2)} = 0$. Let $K =< x, y >$, then K is nilpotent, hence $N(K) = K = N(\mathbf{A}) =< x > \oplus < y >$, whereas $< x >$ and $< y >$ are minimal abelian ideals. Also, $\mathbf{A}/K \cong< T >$, which is trivially supersolvable. Let $I =< x + y >$, then I is a maximal subalgebra of K, but $[T, x + y] = x + 2y$, which means I is not an ideal in **A**.

Lemma 3.3.1. *Let **A** be a Leibniz algebra. Let $K \lhd \mathbf{A}$ such that A/K is supersolvable. $N = N(K)$ is the direct sum of one-dimensional abelian ideals of **A**. Then, $\mathbf{A}/CL_K(N)$ is supersolvable.*

Proof. $\mathbf{A}/CL^l_{\mathbf{A}}(N) \cong \{T_a|_N\}$, where $T_a|_n$ is the left multiplication operator by elements of **A** acting on N. So, the set $\{T_a|_N\}$ is a Lie algebra. Then, $T_a|_N$ are simuntaneously diagnolizable. Hence, $\mathbf{A}/CL^l_{\mathbf{A}}(N)$ is a set of abelian Lie algebra. Since N is the direct sum of one dimensional abelian ideals, by Cor 3.2.1, $CL^l_{\mathbf{A}}(N) = CL_{\mathbf{A}}(N)$. Also, $CL_K(N) = K \cap CL_{\mathbf{A}}(N)$, hence $\mathbf{A}/CL_K(N) \cong \mathbf{A}/K \cap CL_{\mathbf{A}}(N) \cong$ a subalgebra of $\mathbf{A}/K \oplus A/CL_{\mathbf{A}}(N)$. Since $\mathbf{A}/K$ is supersolvable, and $A/CL_{\mathbf{A}}(N)$ is abelian, $\mathbf{A}/CL_K(N)$ is supersolvable. □

Example 3.3.3. Let $\mathbf{A} =< x, y, z >$ with non-zero product $[x, z] = z$. Let $K =< y, z >\lhd \mathbf{A}$. Then $\mathbf{A}/K \cong< x >$ is supersolvable trivially, and $N = N(K) = K =< y > \oplus < z >$. So $CL_K(N) = K$, and $\mathbf{A}/CL_K(N) = \mathbf{A}/K$, which is supersolvable.

Lemma 3.3.2. *Let **A** be a solvable Leibniz algebra. Let $K \lhd \mathbf{A}$ such that A/K is supersolvable. $N = N(K)$ is the direct sum of one-dimensional abelian ideals of **A**. Then, **A** is supersolvable.*

Proof. N is abelian, hence $N \subseteq CL_K(N)$. Also $N \supseteq CL_K(N)$ (proof at the end), so $N = CL_K(N)$. Hence $\mathbf{A}/N \cong \mathbf{A}/CL_K(N)$ is supersolvable (by Lemma 3.3.1) and N possesses a flag of ideals of **A**. Hence **A** is supersolvable.

To show $N \supseteq CL_K(N)$, suppose not. Then, there exists an ideal $B \supseteq N$ such that $B/N = (CL_K(N)+N)/N$. Then, $B^2 = 0$ which means B is nilpotent, hence $B \subseteq N$, a contradiction. $\square$

Example 3.3.4. Let $\mathbf{A} =< x, y, z >$ with non-zero products $[x, y] = y, [x, z] = 2z$. Let $K =< y, z > \lhd \mathbf{A}$. $\mathbf{A}/K \cong< x >$ is supersolvable trivially, and $N(K) = K =< y > \oplus < z >$. Then by the lemma, $\mathbf{A}$ is supersolvable with the flag of ideals $< y > \subseteq < y, z > \subseteq \mathbf{A}$.

Theorem 3.3.3. *Let* $\mathbf{A}$ *be a solvable Leibniz algebra. Then* $\mathbf{A}$ *is supersolvable if and only if there is an ideal* K *of* $\mathbf{A}$ *such that* $\mathbf{A}/K$ *is supersolvable and all maximal subalgebras of* $N(K)$ *are ideals of* $\mathbf{A}$*.*

Proof. The "only if" direction is straightforward. For if $\mathbf{A}$ is supersolvable, then there exists an ideal K of $\mathbf{A}$ such that $\dim K = 1$. Then $\mathbf{A}/K$ is supersolvable and the maximal subalgebra of $N(K)$ is 0. Thus, the result holds.

Conversely, we use induction on $\dim\mathbf{A}$. The result is trivial if $\dim\mathbf{A} = 1$. We consider two cases:

Case 1. $\phi(\mathbf{A}) \neq 0$. Suppose $K \cap \phi(\mathbf{A}) \neq 0$, then $\dim\mathbf{A}/(K \cap \phi(\mathbf{A})) < \dim\mathbf{A}$. By Prop 3.2.2, as $\mathbf{A} \mapsto \mathbf{A}/(K \cap \phi(\mathbf{A}))$, $N(K) \mapsto N(K)/(K \cap \phi(\mathbf{A})) = N(K/(K \cap \phi(\mathbf{A})))$. Therefore, every maximal subalgebra of $N(K/(K \cap \phi(\mathbf{A})))$ comes from a maximal subalgebra of $N(K)$, which is an ideal of $\mathbf{A}$. Hence, all maximal subalgebras of $N(K/(K \cap \phi(\mathbf{A})))$ is an ideal in $\mathbf{A}/(K \cap \phi(\mathbf{A}))$. Also, $\frac{\mathbf{A}/(K\cap\phi(\mathbf{A}))}{K/(K\cap\phi(\mathbf{A}))} \cong \mathbf{A}/K$ is supersolvable. So by induction, $\mathbf{A}/(K \cap \phi(\mathbf{A}))$ is supersolvable, and therefore by Theorem 6 of [Bur15], $\mathbf{A}$ is supersolvable.

Now, suppose $K \cap \phi(\mathbf{A}) = 0$. Let N be a minimal abelian ideal of $\mathbf{A}$ such that $N \subseteq \phi(\mathbf{A})$ hence $K \cap N = 0$. If $\mathbf{A} = K + N$, Lemma 2.1 of [Tow73] implies that $\mathbf{A} = K$. Hence every maximal subalgbera of $N(\mathbf{A}) = N(K)$ is an ideal of $\mathbf{A}$. By Cor 3.3.2, $\mathbf{A}$ is supersolvable. Thus, we suppose that $K + N$ is a proper ideal of $\mathbf{A}$. Since $N((K + N)/N) \cong N(K) \cong (N(K) + N)/N$, every maximal subalgebra of $N((K + N)/N)$ can be written as $(M + N)/N$, whereas M is a maximal subalgebra of $N(K)$. So by hypothesis of the theorem, all maximal subalgebras of $N((K+N)/N)$ are ideals of $\mathbf{A}/N$. Hence $\mathbf{A}/N$ is supersolvable by induction hypothesis. Since $N \subseteq \phi(\mathbf{A})$, by Theorem 6 of [Bur15], $\mathbf{A}$ is supersolvable.

Case 2.$\phi(\mathbf{A}) = 0$. By Prop 3.2.1, $N(\mathbf{A}) = Asoc(\mathbf{A})$ is the direct sum of all minimal abelian ideals. If $N(K) = N(\mathbf{A})$, the result holds by Cor 3.3.2. Otherwise, suppose there exists a minimal abelian ideal N of $\mathbf{A}$ with $N \cap K = 0$. Then, $\mathbf{A}/N$ is supersolvable.(Proof at the end) Since $\mathbf{A}/K$ is supersolvable by induction hypothesis, and $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{A}/K \oplus \mathbf{A}/N$, $\mathbf{A}$ is also supersolvable. Let $\pi : \mathbf{A}/(K \cap N) \mapsto \mathbf{A}/K \oplus \mathbf{A}/N$ by $\pi(x + (K \cap N) = (x + K) + (x + N)$. Then $\ker\pi = K \cap N = 0$ by assumption. So, $\mathbf{A} \cong$ a subalgebra of $\mathbf{A}/K \oplus \mathbf{A}/N$.

To show $\mathbf{A}/N$ is supersolvable: We have the following carnonical map: $\mathbf{A} \mapsto \mathbf{A}/N$ and $K \mapsto (K + N)/N \cong K/(K \cap N) \cong K$. Therefore, every maximal subalgebra of $N((K + N)/N)$ comes from a maximal subalgebra of $N(K)$, which is an ideal of $\mathbf{A}$. Hence, all maximal subalgebras of $N((K+N)/N)$

is an ideal in $\mathbf{A}/N$. Also,$\mathbf{A}/K$ supersolvable implies that $\frac{\mathbf{A}/N}{K+N/N}$ is supersolvable. So by induction hypothesis, $\mathbf{A}/N$ is supersolvable. $\square$

Theorem 2.6 of [Bar11] States that for any solvable Leibniz algebra $\mathbf{A}$, $\mathbf{A}^2$ is nilpotent. The above theorem together with this result immediately gives the following corollary.

Corollary 3.3.4. *Let $\mathbf{A}$ be a solvable Leibniz algebra. If all maximal subalgebra of $\mathbf{A}^2$ are ideals in $\mathbf{A}$, then $\mathbf{A}$ is supersolvable.*

To illustrate the above theroem and corollary, we have the following example.

Example 3.3.5. Let $\mathbf{A} =< x, y, z >, [x, z] = z$ and all other products are zero. Then,$\mathbf{A}$ is supersolvable with the chain of ideals $< z >\subsetneq< x, z >\subsetneq< x, y, z >$. Let $K =< x, z >$. Then, $\mathbf{A}/K \cong< y >$ which is trivially supersolvable. $N(K) =< z >$ hence all maximal subalgebra of $N(K) =< 0 >$, which is a 0 ideal. Hence the theorem holds.

$\mathbf{A}^2 =< z >$, hence all maximal subalgebra of $\mathbf{A}^2$ is of dimension 0 hence they are all 0 ideals, and we know $\mathbf{A}$ is supersolvable from the discussion above.

Theorem 3.3.5. *Let $\mathbf{A}$ be a solvable Leibniz algebra. Then $\mathbf{A}$ is supersolvable if and only if there is an ideal K of $\mathbf{A}$ such that $\mathbf{A}/K$ is supersolvable and all maximal subalgebras of $N(K)$ are c-ideals of $\mathbf{A}$.*

Proof. The only if direction is trivial by Theorem 3.3.3, since ideals are c-ideals.

Conversely, if $K \cap \phi(\mathbf{A}) \neq 0$, then $\mathbf{A}/(K \cap \phi(\mathbf{A}))$ satisfies the hypothesis of the theorem.(Same argument as in case 1 of the proof of Theorem 3.3.3). Then, by induction and Theorem 6 of [Bur15], $\mathbf{A}/(K \cap \phi(\mathbf{A}))$ is supersolvable hence $\mathbf{A}$ is supersolvable.

Assume $K \cap \phi(\mathbf{A}) = 0$. By Prop 3.2.3, $N(K) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, whereas B_i are minimal abelian ideals of $\mathbf{A}$. By similar arguements as in the proof of Theorem 3.3.3, $\dim B_i = 1$. By Lemma 3.3.2, $\mathbf{A}$ is supersolvable.

$\square$

Theorem 3.3.6. *Let $\mathbf{A}$ be a Leibniz algebra with a solvable ideal K. Then $\mathbf{A}$ is supersolvable if and only if $\mathbf{A}/K$ is supersolvable and for every maximal subalgebra M of $\mathbf{A}$, either $N(K) \subseteq M$ or $M \cap N(K)$ is a maximal subalgebra of $N(K)$.*

Proof. We want to prove the only if direction first. Assume $\mathbf{A}$ is supersolvable and let M be a maximal subalgebra of $\mathbf{A}$ that does not contain $N(\mathbf{A})$. Then M is a proper subalgebra of $M + N(K)$, which implies that $\mathbf{A} = M + N(K)$ since M is maximal. It follows that $\frac{dimN(K)}{dimM \cap N(K)} \simeq \frac{M+N(K)}{N(K)} \simeq \frac{\mathbf{A}}{M} = 1$. Hence $M \cap N(K)$ must be a maximal subalgebra of $N(K)$.

Now we prove the "if" direction. By properties of carnonical maps, we can see the assumptions are satisfied by $\mathbf{A}/(K \cap \phi(\mathbf{A}))$. Hence if $K \cap \phi(\mathbf{A}) \neq 0$, by induction $\mathbf{A}/(K \cap \phi(\mathbf{A}))$ is supersolvable, so $\mathbf{A}$ is supersolvable by Theorem 6 of [Bur15]. Assume $(K \cap \phi(\mathbf{A}) = 0$. Then, by prop 3.2.3, $N(K) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is a direct sum of minimal abelian ideals B_i. We want to shown $\dim B_i = 1 \forall i$, then the result follows from lemma 3.3.2. Since $B_i \cap \phi(\mathbf{A}) = 0$, there exists a maximal subalgebra M_i of $\mathbf{A}$ such that $\mathbf{A} = B_i \dot{+} M_i$. Then, $N(K) = B_i \dot{+} (N(K) \cap M_i)$, and by hypothesis, $N(K) \cap M_i$ is a maximal subalgebra of $N(K)$. So, $\dim B_i$ =$\dim N(K)$– $\dim N(K) \cap M_i = 1$. $\square$

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