ABSTRACT

PAQUETTE, CHRISTOPHER T. Root Bounds for Pham Multivariate Polynomial Systems and Their Applications. (Under the direction of Hoon Hong.)

This thesis is on the bounds of roots of Pham systems which are type of multivariate polynomial system of equations. In Chapter 2 we review the previous works on root bounds for univariate polynomials and multi-polynomial system. In the subsequent chapters, we present original contributions. A summary of results is as follows:

1. Chapter 3: We derive improved versions of the Lagrange bound for univariate polynomial equations that trade a small or no increase in complexity for higher accuracy.

2. Chapter 4: We derive root size bounds for Pham Systems. We end up using univariate root size bounds in the process.

3. Chapter 5: We derive quality (overestimation) bounds on our root size bounds for Pham systems from Chapter 4.

4. Chapter 6: We improve our root size bounds for Pham systems. We prove in certain cases this improvement approaches the root size.

5. Chapter 7: We define root spread and derive a root spread bound for Pham systems.

6. Chapter 8: We derive ways in which the roots of a Pham system can be related to the roots of its associated derivative system, across several definitions of relationship.

Root Bounds for Pham Multivariate Polynomial Systems
and Their Applications

by
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DEDICATION

To my family. My father, who passed away before he could see this finished, my mother, who helped me through difficult times, and my sister who was always willing to listen.

BIOGRAPHY

Chris, the oldest of two children, was born in Durham, North Carolina and raised just down the road in Cary, North Carolina for his formative years. Later he would move to Mooresville, North Carolina where he would spend the rest of his childhood. He graduated with a major in Mathematics and a minor in Economics. He then started graduate studies in Mathematics at North Carolina State University, eventually working under the direction of Professor Hoon Hong.

ACKNOWLEDGEMENTS

I would like to begin by thanking Professor Hoon Hong for all of his help as my advisor over the years. For being here with me through all these downs and ups. Without his continuous guidance, support, and ideas this dissertation would never have come to fruition.

It has been a pleasure and an honor to work with my committee. I would like to thank Professor Erich Kaltofen. His questions about my approach helped me refine my tests and improve the nature of my samples to better test my conjectures and have more meaningful discussion of my theorems. In addition he taught me computer algebra, before which I had had to teach myself piecemeal. His class was deeply helpful for learning what there was to learn, and for applying this more complete knowledge to my work. I would like to thank Professor Agnes Szanto. Her questions about why I chose the definition of derivative I did, and if there was one with better geometric meaning were extremely helpful. They resulted in ideas I would never have had, and directions I would never have looked into without them. Thanks to this perspective I have endeavored to have a more solid understanding of the choices I made, including their positives and negatives. I would like to thank Professor Cynthia Vinzant. Her questions about the history of Pham systems made me look into a rich history I would not have explored in depth. I would be a poorer person if I had not been challenged to look into where these things I have spent years working with came from. In addition she taught me algebraic geometry, without which I would have been hopelessly lost about many of the whys and hows.

I would also like to extend my thanks to all of my colleagues. Without support from them this would never have been possible.

I would like to thank my family, for believing in me and always encouraging me to be the best that I could. And for keeping me from ever giving up through it all. I would like to thank my mother, for being there for all my life. And for stepping in to fill a role that she had never had to before when my dad unexpectedly passed away on the cusp of both my sister and I finally finishing our college careers. I would like to thank my father, who was always there for me to the very end. I will never be able to express how much he meant to me, and how much he will be missed.

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INTRODUCTION

Finding bounds on the roots of polynomials has been a fundamental problem in mathematics for centuries. Many root size bounds for univariate polynomials have been found over this time period, with widely varying complexities and qualities. Activity includes the works of Lagrange from the mid 1700's to Batra et al who published an improvement to one of Lagrange's bounds in 2017. There are also other types of bounds on the roots of univariate polynomials, such as Gauss-Lucas theorem, where the roots of a polynomial provide a bound on where the roots of the derivative of that polynomial can be. These pursuits can and have been extended into finding bounds on the roots of multivariate polynomial systems of equations.

For multivariate polynomial systems of equations many things which are true for univariate polynomials do not generalize. For example Bezout's theorem provides an upper bound on the number of roots of a multivariate polynomial system of equations if that number is finite. But there is no guarantee it is finite, or that that number is achieved. Yet in the case that the number is finite and equal to the Bezout number there are known root size bounds for multivariate polynomial systems of equations.

But this requirement being met is not immediately obvious in general. One way around this is to restrict yourself to a subset of multivariate polynomial systems where it is known to hold. We have chosen to work with Pham systems. Pham systems are 'nice': they exhibit more univariate-like behavior than general multivariate polynomial systems.

This dissertation will focus on bounding the roots of Pham systems in several ways, finding results analogous to known results for univariate polynomials. As univariate polynomials are a special case of Pham systems we start by first deriving a root size bound for univariate polynomials that improves Lagrange. We then move on to deriving a root size bound for general Pham systems, show that result has a quality bounded by the coefficient sizes, then improve it. Next we consider a different method of bounding the roots, a root spread bound. Finally, we define a 'derivative' for Pham systems and using this derive bounds on the roots of derivative systems which use the roots of the original system.

The thesis is structured as follows.

1. Chapter 2: We review the previous works on root bounds for univariate polynomials and multi-polynomial system.

2. Chapter 3: We derive improved versions of the Lagrange bound for univariate polynomial equations that trade a small or no increase in complexity for higher accuracy.

3. Chapter 4: We derive root size bounds for Pham Systems. We end up using univariate root size bounds in the process.

4. Chapter 5: We derive quality (overestimation) bounds on our root size bounds for Pham systems from Chapter 4.

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7. Chapter 8: We derive ways in which the roots of a Pham system can be related to the roots of its associated derivative system, across several definitions of relationship.

CHAPTER

2

REVIEW

2.1 Univariate Polynomials

2.1.1 Root Size Bounds for Univariate Polynomials

To be entirely unambiguous we must define what we mean by a root size bound for a univariate polynomial.

Definition 1 (Root Size)**.** *The* root size *of f, written as $\overline{R}_f$, is defined by*

$$\overline{R}_f \;=\; \max_{\substack{x\in\mathbb{C}\\ f(x)=0}} \;||x||_\infty$$

where $||x||_\infty = |x|$.
We will often write $\overline{R}$ when the intended f is clear from the context.

We will use the following toy polynomial as a running example throughout this section.

Example 1 (Running example)**.**

$$f = x^2 + 2x + 9$$

Example 2. *For the running example (Example 1), we have*

$$\overline{R} = 3$$

Definition 2 (Root Size Bound)**.** *A function* $B : \mathbb{C}[x] \longrightarrow \mathbb{R}^{\geq 0}$ *is called a* root size bound *if*

$$\underset{f\in\mathbb{C}[x]}{\forall} \quad \overline{R} \ \leq \ B(f)$$

There are many known univariate root size bounds (see [29], [16], [5], [39], [21], and more), we shall limit our more in depth review to a selection of relevant root size bounds.

From now on, let

$$f = x^d + a_1 x^{d-1} + \cdots + a_d x^0$$

Theorem 1 (Cauchy Root Size Bound [11])**.**

$$B^C = 1 + \max_{1\leq j\leq d} |a_j|$$

This root size bound by Cauchy is an example of one of the oldest known root size bounds. It is also a root size bound that does not match the geometry of the roots. That is, in general $tB^C(f(x)) \neq t^d B^C(f(x/t))$. The disadvantage of this type of root size bound is that to optimize the bound you must first optimize the scaling of the polynomial, increasing the complexity of the computation.

Example 3. *For the running example (Example 1), we have*

$$B^C = 1 + \max\{2, 9\} = 1 + 9 = 10$$

Theorem 2 (Apocryphal Root Size Bound [35] page 220)**.**

$$B^A = 2 \ \max_{1\leq j\leq d} |a_j|^{\frac{1}{j}}$$

This is another root size bound. This bound has the advantage of scaling, unlike the Cauchy root size bound from Theorem 1. For the rest of this section and our own work we shall consider root size bounds which scale like B^A.

Example 4. *For the running example (Example 1), we have*

$$B^A = 2 \ \max\left\{2^{\frac{1}{1}}, 9^{\frac{1}{2}}\right\} = 2 \ \max\{2, 3\} = 6$$

Theorem 3 (Fujiwara Root Size Bound [19])**.**

$$B^F = 2 \ \max\left\{|a_1|, |a_2|^{\frac{1}{2}}, \ldots, |a_{d-1}|^{\frac{1}{d-1}}, \left|\frac{a_d}{2}\right|^{\frac{1}{d}}\right\}$$

The Fujiwara bound is worth noting as it is simple to understand and compute. It is also a clear improvement of Theorem 2.

Example 5. *For the running example (Example 1), we have*

$$B^F = 2\ \max\left\{2^{\frac{1}{1}}, \left(\frac{9}{2}\right)^{\frac{1}{2}}\right\} = 2\ \max\left\{2, \frac{3}{\sqrt{2}}\right\} = \frac{6}{\sqrt{2}}$$

Note that as $B^A = a_d^{\frac{1}{d}}$ *the Fujiwara root size bound is smaller than the apocryphal root size bound in this case.*

Theorem 4 (Lagrange Root Size Bound [26])**.**

$$B^L = s_1 + s_2$$

where s_1 *and* s_2 *are the two largest ones among*

$$|a_j|^{\frac{1}{j}}, \quad j = 1, \ldots, d$$

Example 6. *For the running example (Example 1), note* $s_1 = 3$ *and* $s_2 = 2$*. Thus,*

$$B^L = 3 + 2 = 5$$

Proof. Lagrange did not provide a proof. The most recent proof is by Batra-Mignotte-Stefanescu JSC [4]. They verify s_1+s_2. The following proof is simpler and more motivating (deriving instead of verifying).

$$f(x) = 0$$

Solve for x^d, take the absolute value of both sides, then apply the triangle inequality.

$$\Longrightarrow \quad r^d \le \sum_{j=1}^{d} |a_j|\, r^{d-j} \quad \text{where } r = |x|$$

By dividing by r^d, we have

$$\Longleftrightarrow \quad 1 \le \sum_{j=1}^{d} \left(\frac{|a_j|^{\frac{1}{j}}}{r}\right)^j$$

By pulling s_1 out from the sum and noting $|a_j|^{\frac{1}{j}} \le s_2$ for all other j, we have

$$\Longrightarrow \quad 1 \le \left(\frac{s_1}{r}\right)^k - \left(\frac{s_2}{r}\right)^k + \sum_{j=1}^{d} \left(\frac{s_2}{r}\right)^j \quad \text{where } s_1 = |a_k|^{\frac{1}{k}}$$

We consider two cases.

1. $r \le s_1$. We are done.

2. $r > s_1$. Note

$$1 \leq \left(\frac{s_1}{r}\right)^k - \left(\frac{s_2}{r}\right)^k + \sum_{j=1}^{d} \left(\frac{s_2}{r}\right)^j$$

By reindexing the sum, we have

$$\implies 1 \leq \left(\frac{s_1}{r}\right)^k - \left(\frac{s_2}{r}\right)^k + \sum_{j=0}^{d-1} \left(\frac{s_2}{r}\right)^{d-j}$$

We may now approximate the sum as a geometric series. Thus,

$$\implies 1 \leq \left(\frac{s_1}{r}\right)^k - \left(\frac{s_2}{r}\right)^k + \frac{\frac{s_2}{r}}{1 - \frac{s_2}{r}}$$

By rewriting $\left(\frac{s_1}{r}\right)^k - \left(\frac{s_2}{r}\right)^k$, we have

$$\iff 1 \leq \left(\frac{s_1}{r} - \frac{s_2}{r}\right) \sum_{j=0}^{d-k-1} \left(\frac{s_2}{r}\right)^{d-k-1-j} \left(\frac{s_2}{r}\right)^j + \frac{\frac{s_2}{r}}{1 - \frac{s_2}{r}}$$

Recall $r > s_2$. Thus,

$$\implies 1 \leq \left(\frac{s_1}{r} - \frac{s_2}{r}\right) \sum_{i=0}^{d-k-1} \left(\frac{s_2}{r}\right)^j + \frac{\frac{s_2}{r}}{1 - \frac{s_2}{r}}$$

We may now approximate the sum as a geometric series. Thus,

$$\implies 1 \leq \frac{\frac{s_1}{r} - \frac{s_2}{r}}{1 - \frac{s_2}{r}} + \frac{\frac{s_2}{r}}{1 - \frac{s_2}{r}}$$

By combining, we have

$$\iff 1 \leq \frac{\frac{s_1}{r}}{1 - \frac{s_2}{r}}$$

Simplify,

$$\iff 1 \leq \frac{s_1}{r - s_2}$$

As $r - s_2 > 0$, we have

$$\iff r - s_2 \leq s_1$$

Move s_2 to the right-side.

$$\iff r \leq s_1 + s_2$$

□

Recently Batra et al. [4] improved Lagrange's bound.

Theorem 5. *Define* $f(x) := x^d + a_1 x^{d-1} + \cdots + a_d x^0$ *where* $a_d \neq 0$. *Define* π *to be the* *lexicographically greatest such that* $a_{\pi_1}^{\frac{1}{\pi_1}} \geq \cdots \geq a_{\pi_d}^{\frac{1}{\pi_d}}$. *Finally, define*

$$B^B \quad := \max \left\{ \frac{s_1 + s_2 + \sqrt{(s_1+s_2)^2 - 4s_2(s_1 - s_2)^2}}{2}, \left(s_1^{\pi_1 - 1} s_2^0 + \cdots + s_1^0 s_2^{\pi_1 - 1} \right)^{1/(\pi_1 - 1)} \right\}$$

Then we have

$$B^B \leq B^L$$

Example 7. *For the running example (Example 1), note* $\pi_1 = 2$, *then we have*

$$B^B = \max \left\{ \frac{5 + \sqrt{17}}{2}, \ 5 \right\} = 5$$

Theorem 6 (Univariate Lower Bound)**.** *Let* f *be a polynomial with complex coefficients and degree* d. *Then we have the following lower root size bound.*

$$L^U = \max_{1 \leq j \leq d} \left(\frac{|a_j|}{\binom{d}{j}} \right)^{\frac{1}{j}}$$

This theorem is an obvious consequence of Vieta's formulae.

Example 8. *For the running example (Example 1), we have*

$$L^U = \max \left\{ \frac{2}{2}, \frac{3}{1} \right\} = 3$$

2.1.2 Quality of the Root Size Bounds for Univariate Polynomials

There are many quality results for root size bounds. To look further into this see [41], [13], [3], [24], [4].

We shall represent the quality of a root size bound with the overestimation. We define overestimation to be $\log_2 \frac{B}{\overline{R}}$. Thus to approximate our overestimation we shall use a lower bound on $\overline{R}$.

Theorem 7.

$$\log_2 \frac{B^A}{\overline{R}} \leq 1 + \log_2 \max_{1 \leq j \leq d} \binom{d}{j}$$

Example 9. *Referring back to our f from Example 1, we have*

$$\max_{1 \leq j \leq 2} \binom{d}{j} = 2$$

Thus the overestimation bound of the root size bound is

$$\log_2 \frac{B^A}{\overline{R}} \leq 1 + \log_2 2 = 1 + 1 = 2$$

Recall $\overline{R} = 3$. Thus the overestimation of the root size bound is

$$\log_2 \frac{6}{3} = \log_2 2 = 1$$

Proof.

$$\log_2 \frac{B^A}{\overline{R}} \leq \log_2 \frac{2 \max\limits_{1\leq j\leq d} |a_i|^{\frac{1}{j}}}{\max\limits_{1\leq j\leq d} \left(\frac{|a_j|}{\binom{d}{j}}\right)^{\frac{1}{j}}} \leq \log_2 \frac{2 \max\limits_{1\leq j\leq 2} \binom{d}{j} \max\limits_{1\leq j\leq d} |a_j|^{\frac{1}{j}}}{\max\limits_{1\leq j\leq d} |a_j|^{\frac{1}{j}}} = 1 + \log_2 \max_{1\leq j\leq 2} \binom{d}{j}$$

□

2.1.3 Relationship Between the Roots of Polynomials and Their Derivatives

Theorem 8 (Rolle's Theorem for Univariate Polynomials [1])**.** *Let f be a function continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2). If $f(x_1) = f(x_2) = 0$ then $f'(z) = 0$ for some $z \in (x_1, x_2)$.*

Example 10. *Let*

$$f(x) = x^2 - x - 2$$

Then

$$x_1 = -1, \quad x_2 = 2$$

Note

$$f'(x) = 2x - 1$$

Thus, $f'(z) = 0 \implies z = 1/2$. Note

$$1/2 \in (-1, 2)$$

There are many generalized versions of Rolle's theorem. A history can be found in [30]. More recent results include [18].

Theorem 9 (Marden's Theorem for Univariate Polynomials [40])**.** *Let f be a 3rd degree polynomial, and let the zeroes y_1, y_2, and y_3 of f be non-collinear. There is a unique ellipse inside the triangle with vertices y_1, y_2, and y_3 which is tangent to the sides at their midpoints. The foci of that ellipse are the zeroes of the derivative f'.*

As with Rolle's theorem there exist generalizations of Marden's theorem. These results include a generalization to polynomials with 3 unique roots that may have multiplicity higher than 1 [27], and a generalization for when the roots of f are the equivalent of the verticies of a regular n-gon [37].

Theorem 10 (Gauss-Lucas Theorem for Univariate Polynomials [28])**.** *If f is a nonconstant polynomial with complex coefficients then all roots of f' belong to the convex hull of the set of roots of f.*

Example 11. *Let*

$$f = x^3 + 4\,x^2 + 2\,x + 1$$

We shall graph the roots of f and f'. Thus, we have

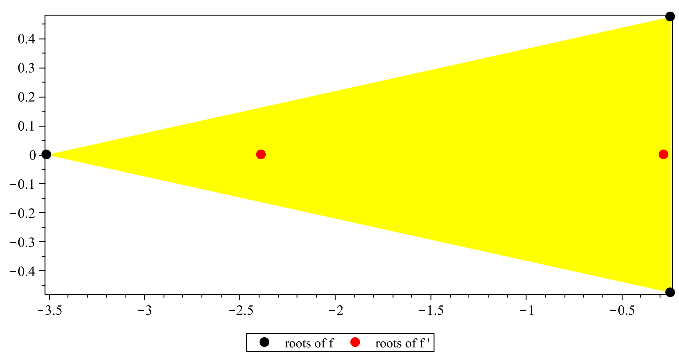


Figure 2.1: Demonstration of Gauss-Lucas theorem, showing the roots of a polynomial (in black), their convex hull (in yellow), and the roots of the derivative (in red).

Where the roots of f are in black, the roots of f' are in red, and the convex hull of the roots of f is filled in with yellow. In this example the containment of the red dots inside the yellow triangle is clear.

Proof. In $\mathbb{C}$ any polynomial $f(x)$ can be written as a product of linear factors. So $f(x) = \sum_{j=1}^{d} (x - \alpha_j)$ where a_0 is the leading coefficient. Note that $\alpha_1, \ldots, \alpha_n$ are the roots of f.
Let z be a complex number such that $f(z) \neq 0$, then we apply the logarithmic derivative.

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{d} \frac{1}{z - \alpha_j}$$

Let z be a zero of f' and recall $f(z) \neq 0$ then

$$\begin{aligned}
\sum_{j=1}^{d} \frac{1}{z-\alpha_j} &= 0 \\
\implies \sum_{j=1}^{d} \left(\frac{1}{z-\alpha_j}\right)\left(\frac{\bar{z}-\bar{\alpha_j}}{\bar{z}-\bar{\alpha_j}}\right) &= 0 \\
\implies \sum_{j=1}^{d} \left(\frac{\bar{z}-\bar{\alpha_j}}{|z-\alpha_j|^2}\right) &= 0 \\
\implies \sum_{j=1}^{d} \left(\frac{\bar{z}}{|z-\alpha_j|^2}\right) &= \sum_{j=1}^{d} \left(\frac{\bar{\alpha_j}}{|z-\alpha_j|^2}\right) \\
\implies \sum_{j=1}^{d} \left(\frac{1}{|z-\alpha_j|^2}\right)\bar{z} &= \sum_{j=1}^{d} \left(\frac{\bar{\alpha_j}}{|z-\alpha_j|^2}\right)
\end{aligned}$$

Let $r_j = \frac{1}{|z-\alpha_j|^2}$ and $R = \sum_{j=1}^{d} r_j$. Then, we have $R\bar{z} = \sum_{j=1}^{d} \bar{\alpha_j} r_j$. Thus $z = \sum_{j=1}^{d} \alpha_j \left(\frac{r_j}{R}\right)$. r_j are nonnegative real numbers for all j. Thus, $\sum_{j=1}^{d} \left(\frac{r_j}{R}\right) = 1$ and so z is a convex combination of the α_j by definition.

We must now cover the case where $f(z) = 0$. Suppose $f(z) = 0 = f'(z)$ then by definition z is a convex combination of the roots $\alpha_1, \ldots, \alpha_d$, as $z = 1 \cdot \alpha_j$ for some j. □

There exist several generalizations of Gauss-Lucas. A selection of these generalizations can be found in [12], [14], and [6].

2.1.4 Newton Identities for Univariate Polynomials

Later we will review a generalized version of the Newton Identities, so we finish our review of univariate polynomials by recalling the classical Newton Identities for univariate polynomials.

Theorem 11 (Newton Identities for Univariate Polynomials [31])**.** *There are two cases.*

For $j \leq d-1$, we have the following.

$$P_k = \sum_{j=1}^{k-1} (-1)^{j+1} \gamma_j P_{k-j} + (-1)^{k+1} k \gamma_k$$

For $j \geq d$, we have the following.

$$P_k = \sum_{j=1}^{d} (-1)^{j+1} \gamma_j P_{k-j}$$

where $\gamma_k = \sum_{1 \leq \pi_1 \leq \cdots \leq \pi_k \leq d} x_{\pi_1} \ldots x_{\pi_k}$ *and* $P_k = \sum_{j=1}^{n} x_j^k$, $k = 1, 2, 3, \ldots$

2.2 Multivariate Polynomial Systems

Unlike the univariate case, the number of roots of multivariate polynomial systems of equations are not determined solely by their degree vector. For this dissertation we shall restrict our domain to multivariate polynomial systems that have exactly the Bezout number of roots, that is, $d_1 \cdots d_n$, where d_i is the degree of F_i.

Example 12 (Running example)**.**

$$F_1 = x_1^2 + x_1 \cdot x_2 + x_2^2 + 4x_1 + 4x_2 + 1$$
$$F_2 = x_1^2 + x_1 \cdot x_2 + 4x_2^2 + x_1 + 4x_2 + 1$$

2.2.1 Root Size Bounds for Multivariate Polynomial Systems

There are fewer known root size bounds for multivariate polynomial systems than there are known root size bounds for univariate polynomials. Additionally, the root size bounds which are known are often pessimistic.
One such root size bound for multivariate polynomial systems is by Canny.

Theorem 12 (Canny Root Size Bound [8] page 70)**.**

$$U^C \;=\; \left(\left(1 + \sum_{1 \leq i \leq n} (d_i - 1) \right) 3\, a \right)^{n \left(1 + \sum_{1 \leq i \leq n} (d_i - 1) \right)^n}$$

where

$$a_{i(e)} \in \mathbb{Z},\ \textit{and}\ a = ||F||_\infty$$

Example 13. *For the running example (Example 12), we have*

$$a = 4,\ n = 2,\ d_1 = d_2 = 2$$

Thus,

$$U^C = ((1+2)\ 12)^{2(1+2)^2} = 36^{18}$$

Another root size bound for multivariate polynomial systems is by Yap.

Theorem 13 (Yap Root Size Bound [42] page 345)**.** *If $(\alpha_1, \ldots, \alpha_n)$ is a zero of F, a multivariate polynomial system with finitely many complex roots, $a_{i(e)} \in \mathbb{Z}$, and $\underset{i}{\forall}\ |\alpha_i| \neq 0$ then*

$$U^Y \ = \ \left(2^{3/2} N K\right)^{\Omega}\ 2^{(n+1)\ d_1 \cdots d_n}$$

where

$$\begin{aligned} N &= \binom{1 + \sum\limits_{1 \le i \le n} d_i}{n} \\ K &= \max\left\{\sqrt{n+1}, ||F_1||_2, \ldots, ||F_n||_2\right\} \\ \Omega &= \prod_{1 \le i \le n} d_i + \prod_{1 \le j \le n} \prod_{\substack{1 \le i \le n \\ i \neq j}} d_i \end{aligned}$$

Example 14. *For the running example (Example 12), we have*

$$n = 2,\ d_1 = d_2 = 2,\ N = 10,\ K = 8,\ \Omega = 8$$

Thus,

$$U^Y = \left(2^{3/2} \cdot 80\right)^8\ 2^{12}$$

The last root size bound for multivariate polynomial systems we shall cover is the DMM root size bound.

Theorem 14 (DMM Root Size Bound [17])**.** *Let F be a multivariate polynomial system such that D_D is the number of roots in $(\mathbb{C}^*)^n$ and $a_{i(e)} \in \mathbb{Z}$, then*

$$U^{DMM} \ = \ 2^{D_D} \varrho C$$

where

$$\begin{aligned} D_D &= \textit{Number of roots of } F \le \prod_{1 \le i \le n} d_i \\ \varrho &= \prod_{i=1}^{n} (\#\varphi_i)^{\Phi_i} \end{aligned}$$

$$C = \prod_{i=1}^{n} ||F_i||_\infty^{\Phi_i}$$

where $(\#\varphi_i)$ *is the number of lattice points in* φ_i*, the convex hull of the support of* F_i *and* Φ_i *is the mixed volume of the polytopes* $\varphi_0, \ldots, \hat{\varphi}_i, \ldots, \varphi_n$*.*

Example 15. *For the running example (Example 12), we have*

$$n = 2,\ D_D = 4,\ \#\varphi_1 = \varphi_2 = 6,\ \Phi_1 = \Phi_2 = 2,\ \varrho = 6^4,\ ||F_1||_\infty = ||F_2||_\infty = 4,\ C = 4^4$$

Thus,

$$U^{DMM} = 2^4 \cdot 6^4 \cdot 4^4$$

2.2.2 Other Root Bounds

As for univariate polynomials, root size bounds are not the only way to bound the roots of multivariate polynomials. Work here is relatively sparse. It includes a multidimensional version of Rolle's theorem [20].

2.3 Multivariate Pham Systems

Definition 3 (Pham system)**.** *We say that* $F \in \mathbb{C}[x]^n$ *is a* Pham *system if it has the following form*

$$\begin{aligned} F_1 &= x_1^{d_1} + \sum_{|e|<d_1} a_{1,(e)}\ x^e \\ &\vdots \\ F_n &= x_n^{d_n} + \sum_{|e|<d_n} a_{n,(e)}\ x^e \end{aligned}$$

where we used the following short-hands: $x = (x_1, \ldots, x_n)$*,* $e = (e_1, \ldots, e_n)$*,* $x^e = x_1^{e_1} \cdots x_n^{e_n}$ *and* $|e| = e_1 + \cdots + e_n$*. These short hands will continue to be used for ease of reading.*

Example 16.

$$\begin{aligned} F_1 &= x_1^2 - i \cdot x_1 - x_2 - 2 \\ F_2 &= x_2^2 - x_1 - x_2 + i \end{aligned}$$

Remark 1. *Note that the coefficient subscript scheme runs in the opposite direction compared to the coefficient subscript scheme used for univariate polynomials. This is intentional, as this* *new scheme is a more straightforward mapping between the monomials and their coefficients. The univariate coefficient subscript scheme would require more thought to decode from the degree vector of each monomial in the Pham system, and so was less suitable.*

Remark 2. *We shall review a brief history of Pham systems.*

- *Named as a tribute to Frédéric Pham's earlier work with a related concept in [38].*
- *Mourrain and Pan in [34] had Pham systems appear as a special class of polynomial system where they were able to derive a significant improvement to the computation time of roots.*
- *Laureano Gonzalez-Vega in [23] found a quantifier elimination algorithm which allows quantifier elimination for Pham systems with parametric coefficients.*
- *Pardo and Martin in [36] derived another improvement to the computation time of the roots of Pham systems.*
- *Most recently Dratman et al in [15] examined robust algorithms for solving Pham systems. They showed that all robust algorithm have complexity at least polynomial in the number of roots of the Pham system. They found a specific algorithm which was quadratic in the number of roots of the Pham system.*

Remark 3. *One real-world application of Pham systems is in physics. They appear when applying the numerical polynomial homotopy continuation method to the two-dimensional nearest-neighbor ϕ^4 model, as seen in [33], and [32].*

2.3.1 Gröbner Bases and Pham Systems

We would like there to be a unique representative for every set of Pham systems with the same set of roots. Fortunately, it is known that under certain conditions we will have exactly this. To talk about these conditions we must first get some background on what a Gröbner basis is. We will then discuss how this relates to Pham systems.

Definition 4 (Gröbner Basis)**.** *Fix a monomial order $>$ and let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. A* Gröbner basis *for I is a collection of nonzero polynomials*

$$\{F_1, \ldots, F_n\} \subset I$$

such that the ideal of leading terms of $F_1, \ldots, F_n$ generates the ideal of leading terms of I.

It is clear from definition that all Pham systems are a Gröbner basis under any total degree ordering. Pham systems always being a Gröbner basis under any total degree ordering is useful on it's own (see for instance [9]). However, we have more.

Definition 5 (Minimal Gröbner Basis)**.** *A* Minimal Gröbner basis (MGB) *is a Gröbner basis with the additional properties that*
(1) The leading coefficient of F_j is one, for all $j \in \{1, \ldots, n\}$.
(2) For each i and j, $i \neq j$, no leading monomial of F_i is divisible by the leading monomial of F_j.

Proposition 15. *Every Pham System is a MGB under any total degree ordering.*

This does not get us to having a unique representative for every set of Pham systems with the same solution set yet, as is clear from every Pham system being a MGB. This implies that we need an even stronger condition.

Definition 6 (Reduced Gröbner Basis)**.** *A* Reduced Gröbner basis (RGB) *is a Gröbner basis with the additional properties that*
(1) The leading coefficient of F_j is one, for all $j \in \{1, \ldots, n\}$.
(2) For each i and j, $i \neq j$, no term of F_i is divisible by the leading monomial of F_j.

Proposition 16. *There is a one-to-one correspondence between ideals and varieties (counting multiplicities).*

Proposition 17. *There is a one-to-one correspondence between ideals and RGBs.*

Together these tell us that RGBs are a unique representative of a variety (counting multiplicities). That is, there is one and only one RGB for each variety (counting multiplicities).

Proposition 18. *A Pham System F where $d_1 = \cdots = d_n$ is a RGB under any total degree ordering.*

Proposition 19. *Pham Systems where $d_1 = \cdots = d_n$ have a one-to-one correspondence with their varieties (counting multiplicities).*

Proposition 20. *Every possible solution set, counting multiplicities, of a Pham system has one and only one representative Pham system which is also a RGB.*

Remark 4. *Note that this is not the same as saying every possible set of complex numbers. Not every possible set of complex numbers has an associated Pham system to which it is the solution set. In fact, not every possible set of complex numbers has a multivariate polynomial system that it is the solution to.*

2.3.2 Generalized Newton Identities for Pham Systems

Generalizing the Newton identities into Pham systems has been the subject of several studies. Starting in 1981 we have work by Aizenberg and Kytmanov [2]. Further work includes [22], [7], and [25].

Theorem 21 (Generalized Newton Identities [22], Theorem 3.2). *Let $\delta = (d_1, \ldots, d_n)$, $\beta \in \mathbb{Z}^n$ a multi-index such that $||\beta|| < ||\delta||$, and e^i be the degree index for F_i, Then*

$$T_{\delta-\beta} + \sum_{||\beta||<||\sum\limits_{i=1}^{n} e^i||<||\delta||} a_{1(e^1)} \cdots a_{n(e^n)} T_{e^1+\cdots+e^n-\beta} = \sum_{\sum\limits_{i=1}^{n} e^i=\beta} \left(\Gamma_{e^1,\ldots,e^n} - \prod_{i=1}^{n} d_i \right) a_{1(e^1)} \cdots a_{n(e^n)}$$

where

$$\Gamma_{[e^1,\ldots,e^n]} = \begin{vmatrix} e_1^1 & \ldots & e_n^1 \\ \vdots & & \vdots \\ e_1^n & \ldots & e_n^n \end{vmatrix}$$

where the subscript represents which x_i you are finding the exponent of.

CHAPTER

3

IMPROVED ROOT SIZE BOUNDS FOR UNIVARIATE POLYNOMIALS

Two of the simplest root size bounds for univariate polynomials are the apocryphal root size bound of $2s_1$, which uses only the largest coefficient, and the root size bound by Lagrange of $s_1 + s_2$, which uses the largest two coefficients. This leads to a natural question. Can the root size bound by Lagrange be improved to a root size bound using some linear combination of all of the coefficients? This chapter will provide a positive answer to that question.

Notation 1. *We will use the following notations.*

1. $f = x^d + a_1 x^{d-1} + \cdots + a_d x^0$
2. $c_j = |a_j|^{\frac{1}{j}}$
3. $s_1 \geq \cdots \geq s_d$ *is a sorted list of* $c_1, \ldots, c_d$

Example 17. *Let*

$$f = x^2 + 2x + 9$$

Then,

$$c_1 = 2$$
$$c_2 = 3$$

Thus,

$$s_1 = 3$$
$$s_2 = 2$$

3.1 Main Results

Theorem 22 (Improved Lagrange Root Size Bound)**.** *Let f be a polynomial of degree d. Then the following B^I is a root size bound.*

$$B^I = \sum_{j=1}^{d} \frac{s_j - s_{j+1}}{q_j}$$

where each q_j is the only nonnegative root of the polynomial equation $z^j + \cdots + z^1 - 1 = 0$, and $s_{d+1} = 0$.

Remark 5. *An important note is like the Lagrange bound this bound is independent of degree, allowing for the q_j's to be precomputed. This allows rapid and relatively accurate computation of the bound for high-degree super-sparse polynomials.*

Example 18. *Let $f = x^3 - 3x^2 - 4x - 1$ then*

$$\begin{array}{ccc} s_1 = 3 & s_2 = 2 & s_3 = 1 \\ q_1 = 1 & q_2 > 0.618 & q_3 > 0.543 \end{array}$$

Thus,

$$\overline{R} \leq \frac{3-2}{1} + \frac{2-1}{.618} + \frac{1}{.543} < 4.460$$

Note

$$B^L = s_1 + s_2 = 5$$

Finally, by solving we find

$$\overline{R} < 4.049$$

Thus we have the following inequalities.

$$\overline{R} < 4.049 < 4.460 < 5$$

Corollary 23 (Two Term Improved Lagrange Root Size Bound)**.** *Let f be a polynomial of degree d. Then the following B^{Icor} is a root size bound.*

$$B^{Icor} = s_1 + \left(\frac{1}{q_d} - 1\right) s_2$$

Example 19. *Recall f from Example 18. By recalling $s_1 = 3,\ s_2 = 2$ and $q_3 > 0.543$, we have*

$$B^{Icor} < 3 + \left(\frac{1}{0.543} - 1\right) 2 < 4.684$$

Theorem 24 (Quality)**.** *We have*

$$B^I \leq B^{Icor} \leq B^L$$

3.2 Proofs

3.2.1 Proof of Theorem 22

For ease of understanding this proof will be done using multiple smaller lemmas.

Lemma 25. *Let $g(z) = z_1^1 + \cdots + z_d^d$. Then we have*

$$\underset{x\in\mathbb{C}\backslash\{0\}}{\forall}\ 0 = f(x) \implies 1 \leq g\left(\frac{c}{|x|}\right)$$

Proof. Let $x \in \mathbb{C}\backslash\{0\}$. Assume $0 = f(x)$. We will show that $1 \leq \left(\frac{c_1}{|x|}\right)^1 + \cdots + \left(\frac{c_d}{|x|}\right)^d$.
Since $0 = f(x)$, we have

$$0 = x^d + a_1 x^{d-1} + \cdots + a_d x^0$$

By dividing through by x^d, we have

$$0 = 1 + a_1 x^{-1} + \cdots + a_d x^{-d}$$

By moving the constant coefficient 1 to the LHS, we have

$$-1 = a_1 x^{-1} + \cdots + a_d x^{-d}$$

By taking the absolute value of both sides, we have

$$|-1| = \left|a_1 x^{-1} + \cdots + a_d x^{-d}\right|$$

By applying the triangular inequality, we have

$$1 \leq |a_1|\,|x|^{-1} + \cdots + |a_d|\,|x|^{-d}$$

Rewriting so that $|x|$ is a denominator, we have

$$1 \leq \left(\frac{|a_1|^{\frac{1}{1}}}{|x|}\right)^1 + \cdots + \left(\frac{|a_d|^{\frac{1}{d}}}{|x|}\right)^d$$

By recalling $c_j = |a_j|^{\frac{1}{j}}$, we have

$$1 \leq \left(\frac{c_1}{|x|}\right)^1 + \cdots + \left(\frac{c_d}{|x|}\right)^d$$

By recalling $g(z) = z_1^1 + \cdots + z_d^d$, we have

$$1 \leq g\left(\frac{c}{|x|}\right)$$

□

Remark 6. *Following Lagrange, we hope to find a root size bound that is a linear combination of c_js. Note if $z_1, \ldots, z_d$ are nonnegative real numbers, $1 \leq z_1^1 + \cdots + z_d^d$ implies that $1 \leq z_1 + \cdots + z_d$. This is illustrated by the following figure where $d = 2$.*

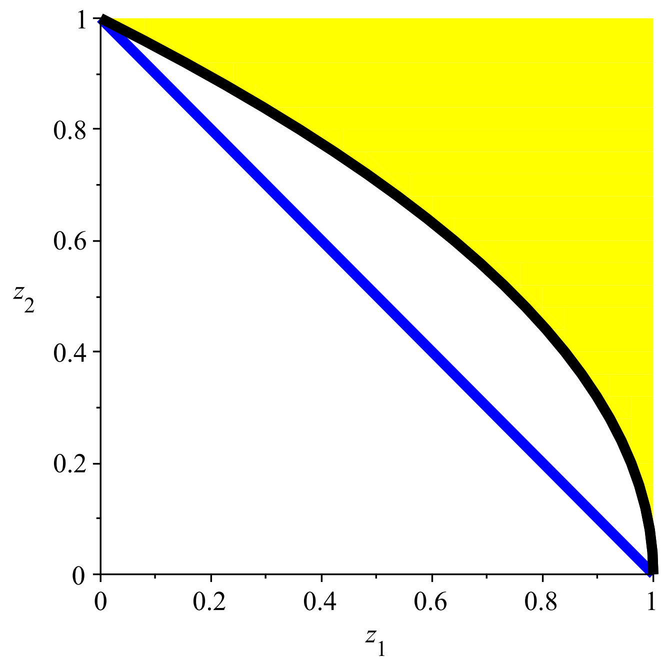


Figure 3.1: Graph of the line $1 = z_1 + z_2^2$ (in black), and the line $1 = z_1 + z_2 + z_3$ (in blue), showing that the linear surface is below the powered one.

From this, we immediately see $|x| \le c_1 + \cdots + c_d$. However, this is worse than the Lagrange bound. Thus, we see that we need to do something else. We will use a different coordinate system.

Notation 2 (Coordinate transformation)**.**

1. *π is a permutation of $1, \ldots, d$ such that $s_j = c_{\pi_j}$*
2. *$u_j = \frac{s_j}{|x|}$*
3. *$\delta_j = u_j - u_{j+1}$ where $u_{d+1} = 0$*

Lemma 26. *Let $h(z) = (z_1 + \cdots + z_d)^{\pi_1} + \cdots + z_d^{\pi_d}$. Then we have*

$$1 \le g\left(\frac{c}{|x|}\right) \iff 1 \le h(\delta)$$

Proof. Immediate from the following repeated rewriting

$$1 \le g\left(\frac{c}{|x|}\right)$$

$$
\begin{aligned}
&\iff 1 \leq \left(\frac{c_1}{|x|}\right)^1 + \cdots + \left(\frac{c_d}{|x|}\right)^d \\
&\iff \ 1 \leq \left(\frac{c_{\pi_1}}{|x|}\right)^{\pi_1} + \cdots + \left(\frac{c_{\pi_d}}{|x|}\right)^{\pi_d} \\
&\iff \ 1 \leq \left(\frac{s_1}{|x|}\right)^{\pi_1} + \cdots + \left(\frac{s_d}{|x|}\right)^{\pi_d} \\
&\iff \ 1 \leq u_1^{\pi_1} + \cdots + u_d^{\pi_d} \\
&\iff \ 1 \leq (\delta_1 + \cdots + \delta_d)^{\pi_1} + \cdots + \delta_d^{\pi_d} \\
&\iff \ 1 \leq h(\delta)
\end{aligned}
$$

□

Now we will carry out a linear overestimation of h. First we will convert $1 < h(z)$ into set notation and note an important fact.

Notation 3 (Set notation)**.**

1. *$A = \left\{z \in \mathbb{R}^d_{\geq 0} : 1 < h(z)\right\}$*
2. *Let q_j^π be the positive real number such that $h\Big(0,\ \ldots, 0, q_j^\pi, 0,\ \ldots, 0\Big) = 1$*
3. *$H = \text{ConvexHull}\left\{0, q_\pi^1 e_1, \ldots, q_\pi^d e_d\right\}$*

Lemma 27. *$H^c \supseteq A$*

Proof.

1. Note that as $\underset{j}{\forall}\ z_j \geq 0$, h is a convex function over $\mathbb{R}^d_{\geq 0}$.
2. Hence A^c is convex in $\mathbb{R}^d_{\geq 0}$ because sublevel sets of convex functions are convex.
3. Note that $0 \in A^c$.
4. Note that $q_1^\pi e_1, \ldots, q_d^\pi e_d \in A^c$.
5. Then $H \subseteq A^c$.
6. So $H^c \supseteq A$.

□

Lemma 28. *Let $w(z) = \frac{z_1}{q_1^\pi} + \cdots + \frac{z_d}{q_d^\pi}$. Then we have*

$$1 \leq h(\delta) \implies 1 \leq w(\delta)$$

Proof.

Remark 7. *We will start with a graphical proof to show the intuition. For simplicity we will let* $\deg(f) = 2$.

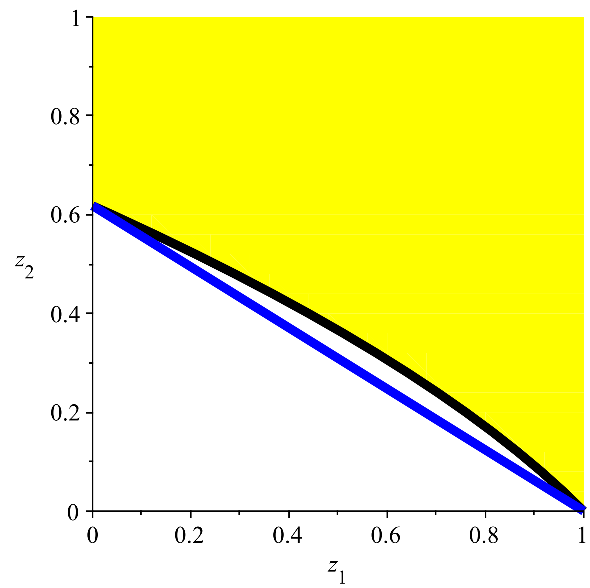


Figure 3.2: Graph in the coordinate system δ of the surface $1 = z_1 + z_2$ (in black), the area above this surface (in yellow), and the surface $1 = \frac{z_1}{q_1} + \frac{z_2}{q_2}$ (in blue) showing that in this coordinate system $1 = z_1 + z_2$ is above $1 = \frac{z_1}{q_1} + \frac{z_2}{q_2}$.

The black curve is $h(z)$*. The yellow shaded region is the region above* $h(z)$*. The blue line is* $w(z)$*. Unfortunately the curve* $h(z)$ *is difficult to work with directly. Instead we note that as the blue line is entirely below the yellow-shaded region it is also a bound.*
In higher dimensions $h(z)$ *is a hypersurface and* $w(z)$ *is a hyperplane below that hypersurface.*

Now we prove it algebraically. We will divide the proof into two cases.

1. Case: $1 < h(z)$

   (a) $z \in A$.

   (b) $z \in H^c$ by Lemma 27.

   (c) $z \notin H$.

   (d) $z \notin \text{ConvexHull}\{0, q_1^\pi e_1, \ldots, q_d^\pi e_d\}$.

(e) $\nexists_{\substack{\lambda_0+\cdots+\lambda_d=1 \\ \lambda_0,\ldots,\lambda_d\geq 0}} z = \lambda_0 \cdot 0 + \lambda_1 q_1^\pi e_1 + \cdots + \lambda_d q_d^\pi e_d$

(f) $\nexists_{\substack{\lambda_1+\cdots+\lambda_d\leq 1 \\ \lambda_1,\ldots,\lambda_d\geq 0}} z = \lambda_1 q_1^\pi e_1 + \cdots + \lambda_d q_d^\pi e_d$

(g) $\exists_{\substack{\lambda_1+\cdots+\lambda_d> 1 \\ \lambda_1,\ldots,\lambda_d\geq 0}} z = \lambda_1 q_1^\pi e_1 + \cdots + \lambda_d q_d^\pi e_d$

(h) $\exists_{\substack{\lambda_1+\cdots+\lambda_d> 1 \\ \lambda_1,\ldots,\lambda_d\geq 0}} z = [\lambda_1 q_1^\pi, \cdots, \lambda_d q_d^\pi]$

(i) By applying $w(z) = \frac{z_1}{q_1^\pi} + \cdots + \frac{z_d}{q_d^\pi}$, we have

$$w(z) = \frac{z_1}{q_1^\pi} + \cdots + \frac{z_d}{q_d^\pi}$$

(j) By recalling $z = [\lambda_1 q_1^\pi, \ldots, \lambda_d q_d^\pi]$, we have

$$w(z) = \frac{\lambda_1 q_1^\pi}{q_1^\pi} + \cdots + \frac{\lambda_d q_d^\pi}{q_d^\pi}$$

(k) By simplifying, we have

$$w(z) = \lambda_1 + \cdots + \lambda_d$$

(l) By recalling $\lambda_1 + \cdots + \lambda_d > 1$, we have

$$1 < w(z)$$

2. Case: $1 = h(z)$

(a) By continuity,

$$1 = h(z) \leq w(z)$$

(b) Thus,

$$1 \leq w(z)$$

$\square$

Lemma 29. *Consider the permutation π. Then for all j*

$$q_j^\pi \geq q_j$$

where q_j^π is the unique positive real root of $z^{\pi_j} + \cdots + z^{\pi_1} - 1 = 0$ and $q_j = q_j^I$ where I is the identity permutation.

Proof.

1. Let $u_j^\pi(z) = z^{\pi_j} + \cdots + z^{\pi_1} - 1$

2. Then $u_j^I(z) \geq u_j^\pi(z)$ for all $1 \leq j \leq d$ and $0 \leq z \leq 1$

   (a) Recall $u_j^\pi(z)$,
   $$u_j^\pi(z) = \sum_{k=1}^{j} z^{\pi_k} - 1$$

   (b) By letting $e_1 < \cdots < e_j$ be a permutation of $\pi_1, \ldots, \pi_j$, we have
   $$u_j^\pi(z) = \sum_{k=1}^{j} z^{e_k} - 1$$

   (c) By noting $k \leq e_k$, we have
   $$z^k \geq z^{e_k} \text{ for all } 0 \leq z \leq 1$$

   (d) Thus,
   $$u_j^I(z) \geq u_j^\pi(z)$$

3. $q_j^I \leq q_j^\pi$

   (a) By letting $z = q_j^I$ in item 2. we have,
   $$0 = u_j^I(q_j^I) \geq u_j^\pi(q_j^I)$$

   (b) Note,
   $$u_j^\pi(1) \geq 0$$

   (c) Thus, by Intermediate Value Theorem
   $$q_j^I \leq q_j^\pi$$

$\square$

Proof of Theorem 22. With the above lemmas we may prove the theorem. Observe,

$$\begin{aligned} & 0 = f\,(x) \\ & \text{By Lemma 26, we have} \\ \implies & h\,(\delta) \geq 1 \end{aligned}$$

By Lemma 28, we have

$$\implies w(\delta) \geq 1$$

By recalling $w(\delta) = \dfrac{\delta_1}{q_1^\pi} + \cdots + \dfrac{\delta_d}{q_d^\pi}$, we have

$$\iff \frac{\delta_1}{q_1^\pi} + \cdots + \frac{\delta_d}{q_d^\pi} \geq 1$$

By recalling $\delta_j = u_j - u_{j+1}$ and $u_{\delta+1} = 0$, we have

$$\iff \frac{u_1 - u_2}{q_1^\pi} + \cdots + \frac{u_{d-1} - u_d}{q_{d-1}^\pi} + \frac{u_d}{q_d^\pi} \geq 1$$

By recalling $u_j = \dfrac{s_j}{|x|}$, we have

$$\iff \frac{\frac{s_1}{|x|} - \frac{s_2}{|x|}}{q_1^\pi} + \cdots + \frac{\frac{s_{d-1}}{|x|} - \frac{s_d}{|x|}}{q_{d-1}^\pi} + \frac{\frac{s_d}{|x|}}{q_d^\pi} \geq 1$$

Simplifying, we have

$$\iff \frac{s_1 - s_2}{q_1^\pi} + \cdots + \frac{s_{d-1} - s_d}{q_{d-1}^\pi} + \frac{s_d}{q_d^\pi} \geq |x|$$

By recalling Lemma 29, we have

$$\implies \frac{s_1 - s_2}{q_1} + \cdots + \frac{s_{d-1} - s_d}{q_{d-1}} + \frac{s_d}{q_d} \geq |x|$$

□

Proposition 30.

$$1 \leq \frac{q_j^\pi}{q_j} \leq 2$$

Proof. From Lemma 29 we have $1 \leq \frac{q_j^\pi}{q_j}$, and by [10] and [24] we have that $1/2 \leq q_j^\pi \leq 1$, for all j and π. Thus, $\frac{q_j^\pi}{q_j} \leq \frac{1}{1/2} = 2$. □

Example 20 (Accuracy loss)**.** *We would like to know in practice how much information we are losing by approximating q_j^π with q_j^I. Thus, let $\pi = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$. We have the following graph, where q_j^I is in red and q_j^π is in blue.*

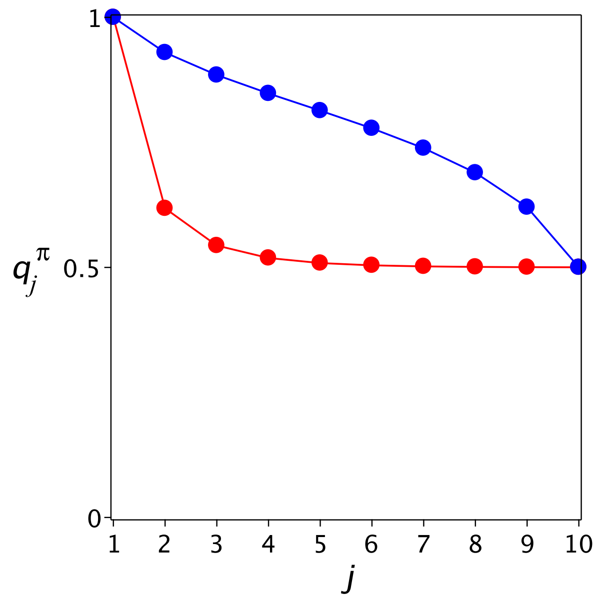


Figure 3.3: Graph showing both q_j^π and q_j^I for j from 1 to 10, where $\pi = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$. The blue curve is q_j^π and the red curve is q_j^I.

As is clear, in all cases the starting value and ending value are the same. As the figure demonstrates in the middle q_j^I drops significantly more sharply than q_j^π at first, and then begins to decline very slowly until $q_d^I = q_d^\pi$.

3.2.2 Proof of Theorem 24

1. We have $B^I = \sum_{j=1}^{d} \frac{s_j - s_{j+1}}{q_j}$, we want to show $B^I \leq B^L$.

$$B^I = \sum_{j=1}^{d} \frac{s_j - s_{j+1}}{q_j}$$

By writing out the sum, we have

$$B^I = \frac{s_1}{q_1} + \frac{s_2}{q_2} - \frac{s_2}{q_1} + \cdots + \frac{s_d}{q_{d-1}} - \frac{s_d}{q_d}$$

By pulling out each s_j, we have

$$B^I = s_1\left(\frac{1}{q_1}\right) + s_2\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \cdots + s_d\left(\frac{1}{q_{d-1}} - \frac{1}{q_d}\right)$$

By noting $q_j \geq q_{j+1}$ and $s_2 \geq s_j$ for all $2 \leq j \leq d$, we have

$$B^I \leq s_1 \left(\frac{1}{q_1}\right) + s_2 \left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \cdots + s_2 \left(\frac{1}{q_{d-1}} - \frac{1}{q_d}\right)$$

By factoring, we have

$$B^I \leq s_1 \left(\frac{1}{q_1}\right) + s_2 \left(\frac{1}{q_2} - \frac{1}{q_1} + \cdots + \frac{1}{q_{d-1}} - \frac{1}{q_d}\right)$$

By noting $q_1 = 1$, we have

$$B^I \leq s_1 + s_2 \left(\frac{1}{q_2} - \frac{1}{1} + \cdots + \frac{1}{q_{d-1}} - \frac{1}{q_d}\right)$$

By cancelation, we have

$$B^I \leq s_1 + \left(\frac{1}{q_d} - 1\right) s_2$$

As $\lim_{d \to \infty} \left(\frac{1}{q_d} - 1\right) = 1$, we have

$$B^I \leq s_1 + s_2$$

Thus,

$$B^I \leq B^L$$

Remark 8. *Corollary 23 is a consequence of this proof.*

In summary, there are many univariate root size bounds. Two of those root size bounds are $2\ s_1$ and $s_1 + s_2$. Considering these bounds one naturally wonders if there is an improved bound of the form $p_1\ s_1 + \cdots + p_d\ s_d$ where $p_j \neq 0$. We have derived such a bound, as it is a trivial rewriting of B^I, and the proof requires an entirely different approach than the proof of the prior bounds.

CHAPTER

4

ROOT SIZE BOUNDS FOR PHAM SYSTEMS

We start by recalling the definition of a Pham system.

Definition 7 (Pham system)**.** *We say that $F \in \mathbb{C}[x]^n$ is a* Pham *system if it has the following form*

$$
\begin{aligned}
F_1 &= x_1^{d_1} + \sum_{|e|<d_1} a_{1,(e)}\ x^e \\
&\ \vdots \\
F_n &= x_n^{d_n} + \sum_{|e|<d_n} a_{n,(e)}\ x^e
\end{aligned}
$$

where we used the following short-hands: $x = (x_1, \ldots, x_n)$, $e = (e_1, \ldots, e_n)$, $x^e = x_1^{e_1} \cdots x_n^{e_n}$ and $|e| = e_1 + \cdots + e_n$. These short hands will continue to be used for ease of reading.

We will use the following toy Pham system throughout this chapter.

Example 21 (Running example)**.**

$$
\begin{aligned}
F_1 &= x_1^2 - i \cdot x_1 - x_2 - 2 \\
F_2 &= x_2^2 - x_1 - x_2 + i
\end{aligned}
$$

Definition 8 (Root Size)**.** *The* root size *of F, written as $\overline{R}_F$, is defined by*

$$\overline{R}_F \;=\; \max_{\substack{x\in\mathbb{C}^n\\ F(x)=0}} \;||x||_\infty$$

We will often write $\overline{R}$ when the intended F is clear from the context.

Example 22. *For the running example (Example 38), we have*

$$\overline{R}_F \approx 1.998$$

Notation 4. *To convey our results concisely we will use the following notations.*

1. $b_{i,j} = \sum_{|e|=j} \left|a_{i,(e)}\right| \qquad (b_{i,d_i} = 1) \qquad\qquad c_{i,j} = \left(\frac{b_{i,j}}{b_{i,d_i}}\right)^{\frac{1}{d_i-j}}$
2. $s_{i,1} \geq \cdots \geq s_{i,d_i}$ *is a sorted list of* $c_{i,0}, \ldots, c_{i,d_i-1}$

Example 23. *For the running example (Example 38), we have*

$$\begin{aligned} b_{1,0} &= 2 & \quad b_{2,0} &= 1\\ b_{1,1} &= 2 & \quad b_{2,1} &= 2\\ b_{1,2} &= 1 & \quad b_{2,2} &= 1 \end{aligned}$$

$$\begin{aligned} c_{1,0} &= \sqrt{2} & \quad c_{2,0} &= 1\\ c_{1,1} &= 2 & \quad c_{2,1} &= 2 \end{aligned}$$

Thus,

$$\begin{aligned} s_{1,1} &= 2 & \quad s_{2,1} &= 2\\ s_{1,2} &= \sqrt{2} & \quad s_{2,2} &= 1 \end{aligned}$$

4.1 Main Results

Theorem 31 (Improved Lagrange Root Size Bound for Pham Systems)**.** *The following U^I is a root size bound.*

$$U^I \;=\; \max_{1\leq i\leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j}\right)$$

where again each q_j is the only positive root of $z^j + \cdots + z^1 - 1 = 0$, and $s_{i,d_i+1} = 0$

Remark 9. *An important note is like the univariate Improved Lagrange bound this bound is independent of degree, allowing for the q_j's to be precomputed. This allows rapid and relatively accurate computation of the bound for high-degree super-sparse Pham systems.*

Example 24. *For the running example (Example 38), we have*

$$U^I \leq \max\{2.875,\ 2.619\} = 2.875$$

Theorem 32 (Lagrange Root Size Bound for Pham Systems)**.** *The following U^L is a root size bound.*

$$U^L \ = \ \max_{1\leq i\leq n} \left(s_{i,1} + s_{i,2}\right)$$

Example 25. *For the running example (Example 38), we have*

$$U^L = \max\left\{s_{1,1} + s_{1,2},\ s_{2,1} + s_{2,2}\right\} = \max\left\{2 + \sqrt{2},\ 2 + 1\right\} = 2 + \sqrt{2} < 3.415$$

Theorem 33 (Apocryphal Root Size Bound for Pham Systems)**.** *The following U^A is a root size bound.*

$$U^A \ = \ 2\ s_{*,1}$$

where $s_{*,1} = \max\limits_{1\leq i\leq n} s_{i,1}$

Example 26. *For the running example (Example 38), we have*

$$U^A = 2\max\{s_{1,1},\ s_{2,1}\} = 2\max\{2,\ 2\} = 4$$

4.2 Proofs

The proofs shall be done in two parts. First we shall derive a general setup useful for proving all three theorems, then we shall split into a second part which is different for each theorem.

Part 1:

Let $\rho \in \mathbb{C}^n$ be such that $F(\rho) = 0$ and $\overline{R} = ||\rho||_\infty$. Let k be such that $\overline{R} = |\rho_k|$.

Then, from $F_k(\rho) = 0$, we have

$$0 = \rho_k^{d_k} + \sum_{|e|<d_k} a_{k,(e)}\rho^e$$

By moving $\rho_k^{d_k}$ to the left hand side and taking absolute values of both sides, we have

$$\left|\rho_k^{d_k}\right| = \left|\sum_{|e|<d_k} a_{k,(e)}\rho^e\right|$$

From the triangular inequality, we have

$$\left|\rho_k^{d_k}\right| \le \sum_{|e|<d_k} \left|a_{k,(e)}\rho^e\right|$$

By distributing the absolute values, we have

$$|\rho_k|^{d_k} \le \sum_{|e|<d_k} |a_{k,(e)}||\rho_1|^{e_1}\cdots|\rho_n|^{e_n}$$

Since $|\rho_k| \ge |\rho_1|, \ldots, |\rho_n|$, we have

$$|\rho_k|^{d_k} \le \sum_{|e|<d_k} |a_{k,(e)}||\rho_k|^{|e|}$$

Recalling $\overline{R} = |\rho_k|$, we have

$$\overline{R}^{d_k} \le \sum_{|e|<d_k} |a_{k,(e)}|\overline{R}^{|e|}$$

By collecting in $\overline{R}$, we have

$$\overline{R}^{d_k} \le \sum_{0\le j<d_k} \left(\sum_{|e|=j} \left|a_{k,(e)}\right|\right) \overline{R}^j$$

Recall $b_{k,j} = \sum\limits_{|e|=j} \left|a_{k,(e)}\right|$ for $0 \le j < d_k$. Then we have

$$\overline{R}^{d_k} \le \sum_{0\le j<d_k} b_{k,j}\overline{R}^j$$

Suppose that all $b_{k,j}$ are zero. Then all $a_{k,(e)}$ are zero. Hence $F_k = x_k^{d_k}$. Thus $\rho_k = 0$. Hence $\overline{R} = 0$. Since $U^* = 0$ we obviously have $\overline{R} = 0 = U^*$. Thus from now on assume that some $b_{k,j}$ are non-zero.

Consider the univariate polynomial

$$g\left(r\right) = r^{d_k} - \sum_{0\le j<d_k} b_{k,j}r^j$$

where r is a new variable. We will try to bound $\overline{R}$ from above by studying the graph of g. Observe

1. $g\left(\overline{R}\right) \le 0$.

2. The leading coefficient of g is positive. Thus the graph of g becomes eventually positive for sufficiently large r.

3. The coefficients of g have exactly one sign variation. Thus by Descartes rule of signs, the polynomial g has one and only one positive real root. Let us call it r^*.

Thus we conclude $\overline{R} \leq r^*$.
Thus, it suffices to find a root size bound of g.

At this point we reach the second part of the proofs. The outcome depends on which univariate root size bound we choose to bound g with.

Part 2:

Proof. Proof of Theorem 31.
We will use the improved Lagrange bound, obtaining

$$\overline{R} \leq \sum_{j=1}^{d_i} \frac{s_{k,j} - s_{k,j+1}}{q_j}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, as

$$\overline{R} \leq \max_{1 \leq i \leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right) = U^I$$

□

Proof. Proof of Theorem 32.
We will use the Lagrange bound, obtaining

$$\overline{R} \leq s_{k,1} + s_{k,2}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, as

$$\overline{R} \leq \max_{1 \leq i \leq n} (s_{i,1} + s_{i,2}) = U^L$$

□

Proof. Proof of Theorem 33.
We will use the apocryphal bound, obtaining

$$\overline{R} \leq 2\ s_{k,1}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, as

$$\overline{R} \leq 2\ s_{*,1} = U^A$$

□

In summary, as a step between univariate polynomials and general multivariate polynomial systems it makes sense to have a root size bound specialized for Pham systems. We have derived several such bounds and left a framework that can convert any univariate root size bound into a related root size bound for Pham systems.

CHAPTER

5

QUALITY OF THE ROOT SIZE BOUNDS FOR PHAM SYSTEMS

In the previous chapter we derived three root size bounds for Pham systems. But having only a root size bound is of limited use. Therefore, we would like to know how good they are, that is their quality. We shall represent the quality of a root size bound with the overestimation. We define overestimation to be $\log_2 \frac{U^*}{\underline{R}}$. As before, we first need definitions and appropriate notations, some new and some recalled.

Definition 9 (Lower Root Size)**.** *The* lower root size *of F, written as $\underline{R}_F$, is defined by*

$$\underline{R}_F = \min_{\substack{x \in \mathbb{C}^n \\ F(x)=0}} ||x||_\infty$$

We will often write $\underline{R}$ when the intended F is clear from the context.

We will use the following toy Pham system throughout this chapter.

Example 27 (Running example)**.**

$$F_1 = x_1^2 - i \cdot x_1 - x_2 - 2$$
$$F_2 = x_2^2 - x_1 - x_2 + i$$

Example 28. *For the running example (Example 38), we have*

$$\underline{R} \approx 1.202$$

Notation 5. *To convey our results concisely we will use the following notations.*

1. $b_{i,j} = \sum_{|e|=j} \left|a_{i,(e)}\right| \qquad (b_{i,d_i} = 1) \qquad\qquad c_{i,j} = \left(\frac{b_{i,j}}{b_{i,d_i}}\right)^{\frac{1}{d_i-j}} \qquad\qquad \underline{c}_{i,j} = \left(\frac{b_{i,d_i-j}}{b_{i,0}}\right)^{\frac{1}{d_i-j}}$

2. $s_{i,1} \geq \cdots \geq s_{i,d_i}$ *is a sorted list of* $c_{i,0}, \ldots, c_{i,d_i-1}$

3. $\underline{s}_{i,1} \geq \cdots \geq \underline{s}_{i,d_i}$ *is a sorted list of* $\underline{c}_{i,0}, \ldots, \underline{c}_{i,d_i-1}$

Example 29. *For the running example (Example 38), we start by recalling the* $b_{i,j}$ *which were computed in Example 23.*

$$\begin{aligned} b_{1,0} &= 2 \qquad & b_{2,0} &= 1 \\ b_{1,1} &= 2 \qquad & b_{2,1} &= 2 \\ b_{1,2} &= 1 \qquad & b_{2,2} &= 1 \end{aligned}$$

Then we compute,

$$\begin{aligned} \underline{c}_{1,0} &= \tfrac{1}{\sqrt{2}} \qquad & \underline{c}_{2,0} &= 1 \\ \underline{c}_{1,1} &= 1 \qquad & \underline{c}_{2,1} &= 2 \end{aligned}$$

Thus,

$$\begin{aligned} \underline{s}_{1,1} &= 1 \qquad & \underline{s}_{2,1} &= 2 \\ \underline{s}_{1,2} &= \tfrac{1}{\sqrt{2}} \qquad & \underline{s}_{2,2} &= 1 \end{aligned}$$

5.1 Main Results

In this section we begin with theorems, then convey experimental results, then finally comparisons to other known root size bounds.

Theorem 34 (Overestimation Bound of the Improved Lagrange Root Size Bound for Pham Systems)**.** *We have*

$$\log_2 \frac{U^I}{\overline{R}} \quad \leq \quad \log_2 \max_{1\leq i\leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right) \; + \; \log_2 \min_{1\leq i\leq n} \sum_{j=1}^{d_i} \frac{\underline{s}_{i,j} - \underline{s}_{i,j+1}}{q_j}$$

where again each q_j *is the only positive root of* $z^j + \cdots + z^1 - 1 = 0$*, and* $s_{i,d_i+1} = \underline{s}_{i,d_i+1} = 0$*.*

Example 30. *For the running example (Example 38), we have*

$$\begin{aligned}
\log_2 \frac{U^I}{\overline{R}} &\le \log_2 \max\{2.875, 2.619\} + \log_2 \min\{1.438, 2.619\} \\
&= \log_2(2.875) + \log_2(1.438) \\
&\le 1.524 + .525 \\
&= 2.049
\end{aligned}$$

Theorem 35 (Overestimation Bound of the Lagrange Root Size Bound for Pham Systems)**.** *We have*

$$\log_2 \frac{U^L}{\overline{R}} \quad \le \quad \log_2 \max_{1\le i\le n}(s_{i,1} + s_{i,2}) \;+\; \log_2 \min_{1\le i\le n}(\underline{s}_{i,1} + \underline{s}_{i,2})$$

Example 31. *For the running example (Example 38), we have*

$$\begin{aligned}
\log_2 \frac{U^L}{\overline{R}} &\le \log_2 \max\left\{2+\sqrt{2}, 2+1\right\} + \log_2 \min\left\{1+\frac{1}{\sqrt{2}}, 2+1\right\} \\
&= \log_2(2+\sqrt{2}) + \log_2\left(1+\frac{1}{\sqrt{2}}\right) \\
&\le 1.772 + 0.772 \\
&= 2.544
\end{aligned}$$

Corollary 36 (Overestimation Bound of the Lagrange Root Size Bound for Large Constant Coefficients)**.** *Suppose that $\left|a_{i,(0,\ldots,0)}\right|$ is sufficiently large so that $s_{i,1} = c_{i,0}$ for all i. Then we have*

$$\log_2\left(\frac{U^L}{\overline{R}}\right) \;\le\; \log_2\left((1+w)\left(1+w^{\frac{1}{d-1}}\right)\right)$$

where $w = \max\limits_{1\le i\le n} \frac{s_{i,2}}{s_{i,1}}$ and $d = \max\limits_{1\le i\le n} d_i$.

Example 32. *Let F be the following.*

$$\begin{aligned}
F_1 &= x_1^3 + 5\, x_1x_2 + 25\, x_1 + 8000 \\
F_2 &= x_2^3 - x_1x_2 - x_1 - 8000
\end{aligned}$$

Then clearly $s_{i,1} = c_{i,0}$ for all i.
Note

$$w = 1/4, \quad d = 3$$

Then we have,

$$\begin{aligned}\log_2\left(\frac{U^L}{\overline{R}}\right) &\le \log_2\left(\frac{5}{4}\left(1+\left(\frac{1}{4}\right)^{\frac{1}{3-1}}\right)\right)\\ &= \log_2\left(\frac{5}{4}\cdot\frac{3}{2}\right)\\ &= \log_2\frac{15}{8}\\ &\le 0.907\end{aligned}$$

Theorem 37 (Overestimation Bound of the Apocryphal Root Size Bound for Pham Systems)**.** *We have*

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \le 2 + d\ \log_2\frac{s_{*,1}}{c_{*,0}}$$

where $s_{*,1} = \max_{1\le i\le n} s_{i,1}$, $d = \max_{1\le i\le n} d_i$, $c_{*,0} = \max_{1\le i\le n}\ c_{i,0}$

Example 33. *For the running example (Example 38), we have*

$$\begin{aligned}\log_2\left(\frac{U^A}{\overline{R}}\right) &\le 2 + 2\log_2\left(\frac{2}{2^{1/2}}\right)\\ &= 2 + 2\log_2 2^{1/2}\\ &= 2+1\\ &= 3\end{aligned}$$

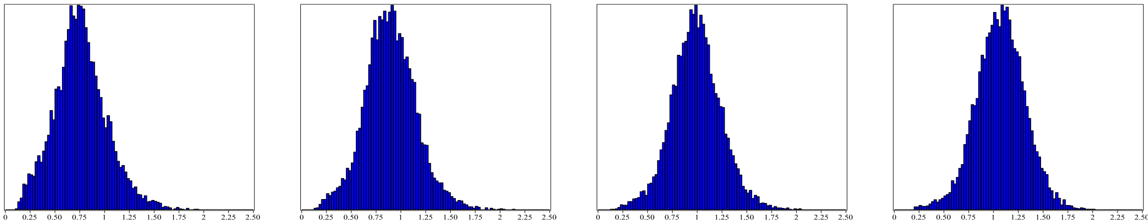
5.1.1 Experimental Quality

- Figure 5.1. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 10 using independently identically uniformly distributed coefficients.

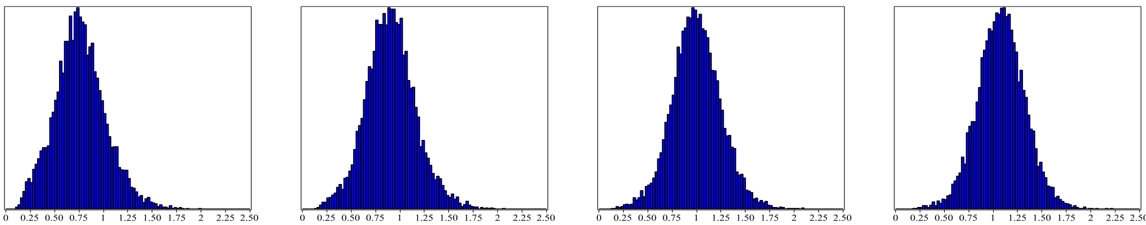
- Figure 5.2. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the rationals between -10 and 10 using independently identically



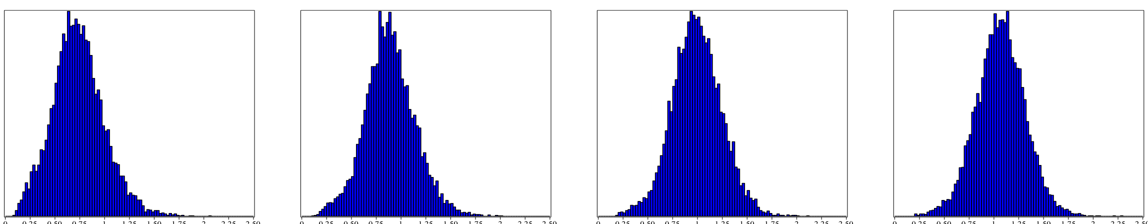
$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 5.1: Overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed integer coefficients between -10 and 10.



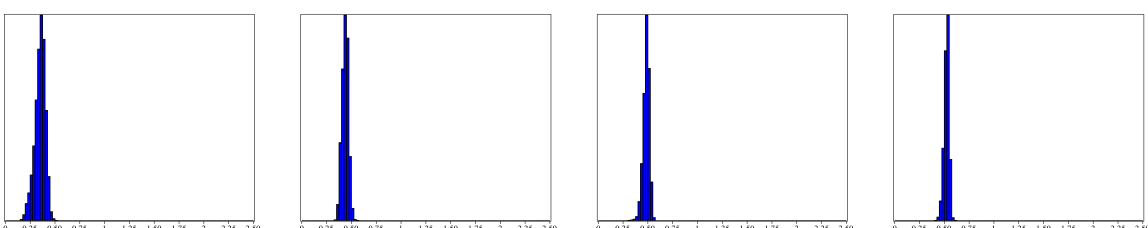
$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 5.2: Overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed rational coefficients between -10 and 10.



$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 5.3: Overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed integer coefficients between -10 and 10, except the constant coefficient, which is defined to be zero.



$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 5.4: Overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed integer coefficients between -10 and 10, except the constant coefficient, which is defined to be 'large'.

uniformly distributed coefficients. The numerator was chosen between -44810 and 44810 with divisor 4481.

Remark 10. *Figure 5.2 and Figure 5.1 are close enough to identical the differences are accounted for by noise, thus the behavior for rational coefficients and integer coefficients appears to be the same.*

- Figure 5.3. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined to be 0.

Remark 11. *As d goes up and the domain the coefficients are chosen from remains constant the independent identical uniform distribution case shown in figure 5.1 approaches the $c_{*,0} = 0$ case shown in figure 5.3.*

- Figure 5.4. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

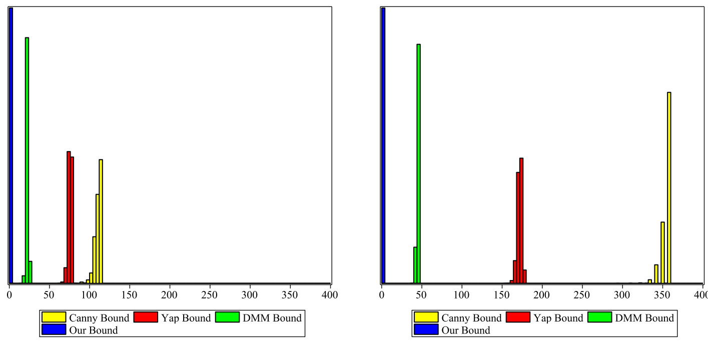
  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined such that $c_{i,0} = 20 \max_{1 \leq j < d} c_{i,j}$. Systems that would result in a function with $c_{i,0} = 0$ were regenerated until a system that did not have $c_{i,0} = 0$ was found.

5.1.2 Comparison to Other Bounds

We will now discuss experimental comparisons of the overestimation of U^I to the overestimation of the Canny, Yap, and DMM root size bounds. We begin with the following graphs.

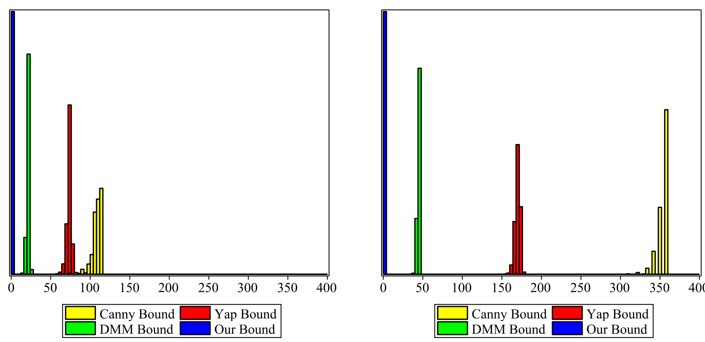
- Figure 5.5. The figure compares the overestimation of U^I for random Pham systems with a fixed degree vector and number of variables with the overestimation of known multivariate root size bounds. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution.



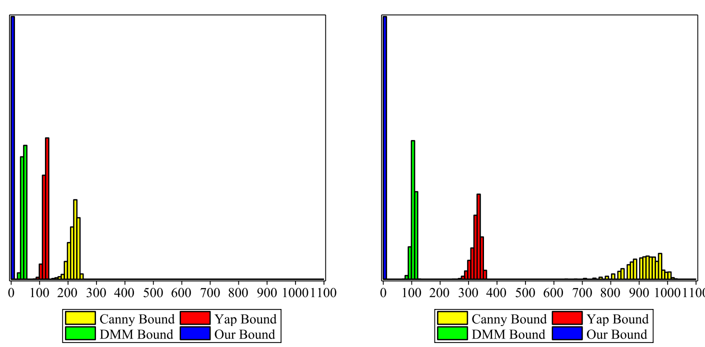
$d = 2,\ n = 2$ $\qquad$ $d = 3,\ n = 2$

Figure 5.5: Overestimation of $U^I(F)$ (in blue), $U^{DMM}(F)$ (in green), $U^Y(F)$ (in red) and $U^C(F)$ (in yellow) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10.



$d = 2,\ n = 2$ $\qquad$ $d = 3,\ n = 2$

Figure 5.6: Overestimation of $U^I(F)$ (in blue), $U^{DMM}(F)$ (in green), $U^Y(F)$ (in red) and $U^C(F)$ (in yellow) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10, except the constant coefficient, which is defined to be zero.



$d = 2,\ n = 2$ $\qquad$ $d = 3,\ n = 2$

Figure 5.7: Overestimation of $U^I(F)$ (in blue), $U^{DMM}(F)$ (in green), $U^Y(F)$ (in red) and $U^C(F)$ (in yellow) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10, except the constant coefficient, which is defined to be 'large'.

We must start our discussion by reiterating that our root size bounds are specialized for Pham systems, and so can take advantage of their structure. In exchange they do not hold for non-Pham multivariate polynomial systems. The other root size bounds are root size bounds for all multivariate polynomial systems with integer coefficients and $d_1 \cdots d_n$ many roots and so do not and cannot account for this structure. Thus it is expected that our root size bounds will perform significantly better within their more limited scope.

From best to worst in terms of overestimation it went U^I, the DMM root size bound, the Yap root size bound, then the Canny root size bound for both the $d = 2$ and $d = 3$ cases.

Note that some subsets of this order are known for all multivariate polynomial systems of equations with integer coefficients and $d_1 \cdots d_n$ many roots. Specifically the DMM root size bound is known to always be smaller than the Canny root size bound. On the other hand, the Canny and Yap root size bounds are known to be incomparable.

- Figure 5.6. The figure compares the overestimation of U^I for random Pham systems with a fixed degree vector and number of variables with the overestimation of known multivariate root size bounds. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined to be 0.

  This case is very similar to the independently identically uniformly distributed coefficient case for all tested bounds.

- Figure 5.7. The figure compares the overestimation of U^I for random Pham systems with a fixed degree vector and number of variables with the overestimation of known multivariate root size bounds. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined such that $c_{i,0} = 20 \max_{1 \leq j < d} c_{i,j}$. Systems that would result in a function with $c_{i,0} = 0$ were regenerated until a system that did not have $c_{i,0} = 0$ was found.

  Note this particular comparison is not particularly fair to the other bounds. U^I is specifically good in this case. Thus the other bounds are negatively affected by the large $c_{*,0}$ while our bound is improved by it.

Now that we have seen all of these comparisons we are willing to conjecture that U^I will always be at least as good as the other bounds.

Proposition 38. *For every Pham system F, we have $U^I \leq U^C$, where the Canny Bound is denoted U^C.*

First recall the Canny Bound, recorded earlier in this dissertation as Theorem 12.

Proof. We will start with the case $d = 1$.

$$U^I = \max_{1\leq i\leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right) \leq 2 \max_{1\leq i\leq n} s_{i,1} = 2s_{*,1} \leq 2\ a \leq 3\ a \leq (3\ a)^n$$

So we are done with the $d = 1$ case.

Let $d \geq 2$ and recall

$$U^I = \max_{1\leq i\leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right) \leq 2 \max_{1\leq i\leq n} s_{i,1}$$

Note that for each F_i there are exactly $\binom{n+j-1}{n-1}$ terms of total degree j

$$U^I \leq 2 \left(\binom{n+j-1}{n-1}\ a \right)^{\frac{1}{d_i - j}}$$

As Pham systems are monic the base is greater than 1, so we may drop the exponent, and drop the -1 from the choose.

$$U^I \leq 2 \binom{n+j}{n}\ a$$

By noting $j \leq d$, we have

$$U^I \leq 2 \binom{n+d}{n}\ a$$

By applying the well-known inequality $\binom{n+d}{n} \leq \left(\frac{e(n+d)}{n}\right)^n$, we have

$$U^I \leq 2 \left(\frac{e(n+d)}{n} \right)^n\ a$$

There are two cases.

Case 1: $n \geq d$

$$U^I \leq 2 \left(\frac{e(2n)}{n} \right)^n\ a$$

By simplifying, we have

$$U^I \leq 2^{n+1}\ e^n\ a$$

By noting $2^{n+1} < D_C^{nD_C^n}$, $e^n < 3^{nD_C^n}$, and $a \geq 1$, we have

$$U^I < (3\ D_C\ a)^{nD_C^n}$$

Thus,

$$U^I < U^C$$

Case 2: $n < d$

$$U^I \leq 2\ \left(\frac{e(2d)}{n}\right)^n\ a$$

By simplifying and dropping the n^n in the denominator, we have

$$U^I < 2^{n+1}\ e^n\ d^n\ a$$

By noting $2^{n+1} < 3^n D_C^n$, $d^n < D_C^n$ $e^n < 3^n$, $a \geq 1$, we have

$$U^I < (3\ D_C\ a)^{nD_C^n}$$

Thus,

$$U^I < U^C$$

□

Proposition 39. *For every Pham system F, we have $U^I \leq U^Y$, where the Yap Bound is denoted U^Y.*

First we recall the Yap root size bound, recorded earlier in this dissertation as Theorem 13.

Proof. By noting $K \geq 1$, $K >\ s_{*,1}$, and $\Omega > 1$, we have

$$s_{*,1} \leq K^\Omega.$$

Thus,

$$2\ s_{*,1} \leq 2^{3/2} K^\Omega$$

By noting that $N \geq 1$ and adding more terms, we have

$$2\ s_{*,1} \leq 2^{3\Omega/2}\ K^\Omega\ N^\Omega\ 2^{(n+1)\ d_1 \cdots d_n}$$

Thus,

$$U^I \leq U^Y$$

$\square$

Conjecture 40. *For every Pham system F, we have $U^I \leq U^{DMM}$.*

A proof is made difficult by the DMM root size bound's use of lattice points, which are difficult to usefully bound below.

5.2 Proofs

5.2.1 Overestimation Bound of the Root Size Bound for Pham Systems

Lemma 41. *The following L^I is a lower root size bound.*

$$L^I = \max_{1\leq i\leq n} \frac{1}{\sum_{j=1}^{d_i} \frac{\underline{s}_{i,j}-\underline{s}_{i,j+1}}{q_j}}$$

Lemma 42. *The following L^L is a lower root size bound.*

$$L^L = \max_{1\leq i\leq n} \frac{1}{\underline{s}_{i,1}+\underline{s}_{i,2}}$$

Lemma 43. *The following L^A is a lower root size bound.*

$$L^A = \frac{1}{2}\max_{1\leq i\leq n} \frac{1}{\underline{s}_{i,1}}$$

As with the proofs of Theorems 31, 32, and 33, the proofs shall be done in two parts. First we shall derive a general setup useful for proving all three lemmas, then we shall split into a second part which is different for each lemma.

Part 1:

We will prove that $L^* \leq \underline{R}$. Let us first deal with a special case. Suppose that $a_{1,(0,\ldots,0)} = \cdots = a_{n,(0,\ldots,0)} = 0$. Then one sees immediately that $L^* = 0$ and $\underline{R} = 0$. Thus we have $L^* \leq \underline{R}$. Hence, from now on, let k be such that $a_{k,(0,\ldots,0)} \neq 0$. Let $\rho \in \mathbb{C}^n$ such that $F(\rho) = 0$ and $||\rho||_\infty = \underline{R}$. Then, we have

$$0 = \rho_k^{d_k} + \sum_{|e|<d_k} a_{k,(e)}\rho^e$$

By moving $a_{k,(0,\ldots,0)}$ to the left hand side and taking absolute values of both sides, we have

$$\left|a_{k,(0,\ldots,0)}\right| \;=\; \left|\rho_k^{d_k} + \sum_{0<|e|<d_k} a_{k,(e)}\rho^e\right|$$

From the triangular inequality, we have

$$\left|a_{k,(0,\ldots,0)}\right| \;\leq\; \left|\rho_k^{d_k}\right| + \sum_{0<|e|<d_k} \left|a_{k,(e)}\rho^e\right|$$

By distributing the absolute values, we have

$$\left|a_{k,(0,\ldots,0)}\right| \;\leq\; |\rho_k|^{d_k} + \sum_{0<|e|<d_k} |a_{k,(e)}||\rho_1|^{e_1}\cdots|\rho_n|^{e_n}$$

Since for k we have $\underline{R} \geq |\rho_1|, \ldots, |\rho_n|$, we have

$$\left|a_{k,(0,\ldots,0)}\right| \;\leq\; \underline{R}^{d_k} + \sum_{0<|e|<d_k} \left|a_{k,(e)}\right| \underline{R}^{|e|}$$

By collecting in $\underline{R}$, we have

$$\left|a_{k,(0,\ldots,0)}\right| \;\leq\; \underline{R}^{d_k} + \sum_{0<j<d_k} \sum_{|e|=j} \left|a_{k,(e)}\right| \underline{R}^{j}$$

By recalling $b_{k,j} = \sum\limits_{|e|=j} \left|a_{k,(e)}\right|$ for $0 \leq j < d_k$ and $b_{k,d_k} = 1$, we have

$$b_{k,0} \;\leq b_{k,d_k}\, \underline{R}^{d_k} + \sum_{0<j<d_k} b_{k,j}\underline{R}^j = \sum_{0<j\leq d_k} b_{k,j}\underline{R}^j$$

Recall that $b_{k,0} \neq 0$. Then $\underline{R} > 0$.
By dividing both side by $b_{k,0}\, \underline{R}^{d_k}$, we have

$$\left(\frac{1}{\underline{R}}\right)^{d_k} \;\leq \sum_{0<j\leq d_k} \frac{b_{k,j}}{b_{k,0}} \left(\frac{1}{\underline{R}}\right)^{d_k-j}$$

By reindexing, we have

$$\left(\frac{1}{\underline{R}}\right)^{d_k} \;\leq \sum_{0\leq j<d_k} \frac{b_{k,d_k-j}}{b_{k,0}} \left(\frac{1}{\underline{R}}\right)^{j}$$

Consider the univariate polynomial

$$g(r) = r^{d_k} - \sum_{0 \le j < d_k} \frac{b_{k,d_k-j}}{b_{k,0}} r^j$$

where r is a new variable. We will try to bound $\frac{1}{\underline{R}}$ from above (equivalently bound $\underline{R}$ from below) by studying the graph of g.
Observe

1. $g\left(\frac{1}{\underline{R}}\right) \le 0$.
2. The leading coefficient of g is positive. Thus the graph of g becomes eventually positive for sufficiently large r.
3. The coefficients of g have exactly one sign variation. Thus by Descartes rule of signs, the polynomial g has one and only one positive real root. Let us call it r^*.

Thus we conclude $\frac{1}{\underline{R}} \le r^*$, equivalently $\underline{R} \ge \frac{1}{r^*}$.
Thus, it suffices to find a root size bound of g.

At this point we reach the second part of the proofs. As before, the outcome depends on which univariate root size bound we choose to bound g with.

Part 2:

Proof of Lemma 41. We will use the improved bound, obtaining

$$\frac{1}{\underline{R}} \le \sum_{j=1}^{d_k} \frac{\underline{s}_{k,j} - \underline{s}_{k,j+1}}{q_j}$$

equivalently

$$\underline{R} \ge \frac{1}{\sum_{j=1}^{d_k} \frac{\underline{s}_{k,j} - \underline{s}_{k,j+1}}{q_j}}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, hence we have

$$\underline{R} \ge \max_{\substack{1 \le i \le n \\ b_{i,0} \ne 0}} \frac{1}{\sum_{j=1}^{d_i} \frac{\underline{s}_{i,j} - \underline{s}_{i,j+1}}{q_j}}$$

When $b_{k,0} = 0$, we have $\underline{s}_{k,1} = \infty$ and in turn $\frac{1}{\sum_{j=1}^{d_k} \frac{\underline{s}_{k,j} - \underline{s}_{k,j+1}}{q_j}} = 0$. Thus we can drop safely the condition $b_{k,0} \neq 0$, obtaining

$$\underline{R} \geq \max_{1 \leq i \leq n} \frac{1}{\sum_{j=1}^{d_i} \frac{\underline{s}_{i,j} - \underline{s}_{i,j+1}}{q_j}}$$

$\square$

Proof of Lemma 42. We will use the Lagrange bound, obtaining

$$\frac{1}{\underline{R}} \leq \underline{s}_{k,1} + \underline{s}_{k,2}$$

equivalently

$$\underline{R} \geq \frac{1}{\underline{s}_{k,1} + \underline{s}_{k,2}}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, hence we have

$$\underline{R} \geq \max_{\substack{1 \leq i \leq n \\ b_{i,0} \neq 0}} \frac{1}{\underline{s}_{i,1} + \underline{s}_{i,2}}$$

When $b_{i,0} = 0$, we have $\underline{s}_{i,1} = \infty$ and in turn $\frac{1}{\underline{s}_{i,1} + \underline{s}_{i,2}} = 0$. Thus we can drop safely the condition $b_{i,0} \neq 0$, obtaining

$$\underline{R} \geq \max_{1 \leq i \leq n} \frac{1}{\underline{s}_{i,1} + \underline{s}_{i,2}}$$

$\square$

Proof of Lemma 43. We will use the apocryphal bound, obtaining

$$\frac{1}{\underline{R}} \leq 2\ \underline{s}_{k,1}$$

equivalently

$$\underline{R} \geq \frac{1}{2\ \underline{s}_{k,1}}$$

The right hand side is difficult to compute from the coefficients of F since it is not easy to determine k. Thus we will trade-off accuracy for efficiency, by taking the maximum, hence we have,

$$\underline{R} \geq \frac{1}{2} \max_{\substack{1 \leq i \leq n \\ b_{i,0} \neq 0}} \frac{1}{\underline{s}_{i,1}}$$

When $b_{i,0} = 0$, we have $\underline{s}_{i,1} = \infty$ and in turn $\frac{1}{2\,\underline{s}_{i,1}} = 0$. Thus we can drop safely the condition $b_{i,0} \neq 0$, obtaining

$$\underline{R} \geq \frac{1}{2} \max_{1 \leq i \leq n} \frac{1}{\underline{s}_{i,1}}$$

$\square$

5.2.2 Overestimation Bound of the Improved Lagrange Root Size Bound for Pham Systems

Note

$$\frac{U^I}{\overline{R}} \leq \frac{U^I}{\underline{R}}$$

Thus,

$$\frac{U^I}{\overline{R}} \;\leq\; \frac{\max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right)}{\max\limits_{1 \leq i \leq n} \frac{1}{\sum\limits_{j=1}^{d_i} \frac{\underline{s}_{i,j} - \underline{s}_{i,j+1}}{q_j}}}$$

By taking logarithm and simplifying, we have

$$\log_2 \left(\frac{U^I}{\overline{R}} \right) \;\leq\; \log_2 \max_{1 \leq i \leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right) \;+\; \log_2 \min_{1 \leq i \leq n} \sum_{j=1}^{d_i} \frac{\underline{s}_{i,j} - \underline{s}_{i,j+1}}{q_j}$$

5.2.3 Overestimation Bound of the Lagrange Root Size Bound for Pham Systems

Note

$$\frac{U^L}{\overline{R}} \leq \frac{U^L}{\underline{R}}$$

Thus,

$$\frac{U^L}{\overline{R}} \;\leq\; \frac{\max\limits_{1 \leq i \leq n} s_{i,1} + s_{i,2}}{\max\limits_{1 \leq i \leq n} \frac{1}{\underline{s}_{i,1} + \underline{s}_{i,2}}}$$

By taking logarithm and simplifying, we have

$$\log_2 \left(\frac{U^L}{\overline{R}} \right) \;\leq\; \log_2 \max_{1 \leq i \leq n} (s_{i,1} + s_{i,2}) \;+\; \log_2 \min_{1 \leq i \leq n} (\underline{s}_{i,1} + \underline{s}_{i,2})$$

5.2.4 Overestimation Bound of the Lagrange Root Size Bound for Pham Systems with Large Constant Coefficients

While a corollary of Theorem 35 neither the result nor is the proof is obvious, and so Corollary 36 requires proof.

Suppose that $\left|a_{i,(0,\ldots,0)}\right|$ is sufficiently large so that $s_{i,1} = c_{i,0}$ for all i.
Then we have $s_{i,1} \geq s_{i,2} \geq c_{i,1}, \ldots, c_{i,d_i-1}$.
Since $b_{i,d_i} = 1$, we have

$$s_{i,1}^{d_i} = b_{i,0} \qquad s_{i,2}^{d_i-1} \geq b_{i,1} \qquad \cdots \qquad s_{i,2}^{1} \geq b_{i,d_i-1}$$

Let $w_i = \frac{s_{i,2}}{s_{i,1}}$. Note the following inequalities and simplifications.

$$\begin{array}{lccccccc}
\underline{c}_{i,0} & = & \left(\frac{b_{i,d_i}}{b_{i,0}}\right)^{\frac{1}{d_i-0}} & = & \left(\frac{1}{s_{i,1}^{d_i}}\right)^{\frac{1}{d_i-0}} & = & \left(\frac{1}{s_{i,1}^{d_i}}\frac{s_{i,2}^{0}}{s_{i,1}^{0}}\right)^{\frac{1}{d_i}} & = & \frac{1}{s_{i,1}} w_i^{\frac{0}{d_i}} \\
\underline{c}_{i,1} & = & \left(\frac{b_{i,d_i-1}}{b_{i,0}}\right)^{\frac{1}{d_i-1}} & \leq & \left(\frac{s_{i,2}^{1}}{s_{i,1}^{d_i}}\right)^{\frac{1}{d_i-1}} & = & \left(\frac{1}{s_{i,1}^{d_i-1}}\frac{s_{i,2}^{1}}{s_{i,1}}\right)^{\frac{1}{d_i-1}} & = & \frac{1}{s_{i,1}} w_i^{\frac{1}{d_i-1}} \\
\underline{c}_{i,2} & = & \left(\frac{b_{i,d_i-2}}{b_{i,0}}\right)^{\frac{1}{d_i-2}} & \leq & \left(\frac{s_{i,2}^{2}}{s_{i,1}^{d_i}}\right)^{\frac{1}{d_i-2}} & = & \left(\frac{1}{s_{i,1}^{d_i-2}}\frac{s_{i,2}^{2}}{s_{i,1}^{2}}\right)^{\frac{1}{d_i-2}} & = & \frac{1}{s_{i,1}} w_i^{\frac{2}{d_i-2}} \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
\underline{c}_{i,d_i-1} & = & \left(\frac{b_{i,1}}{b_{i,0}}\right)^{\frac{1}{1}} & \leq & \left(\frac{s_{i,2}^{d_i-1}}{s_{i,1}^{d_i}}\right)^{\frac{1}{1}} & = & \left(\frac{1}{s_{i,1}^{1}}\frac{s_{i,2}^{d_i-1}}{s_{i,1}^{d_i-1}}\right)^{\frac{1}{1}} & = & \frac{1}{s_{i,1}} w_i^{\frac{d_i-1}{1}}
\end{array}$$

Since $w_i \leq 1$, we have

$$\frac{1}{s_{i,1}} w_i^{\frac{0}{d_i}} \;\geq\; \frac{1}{s_{i,1}} w_i^{\frac{1}{d_i-1}} \;\geq\; \frac{1}{s_{i,1}} w_i^{\frac{2}{d_i-2}} \;\geq\; \cdots \;\geq\; \frac{1}{s_{i,1}} w_i^{\frac{d_i-1}{1}}$$

Thus we have

$$\begin{aligned}
\underline{s}_{i,1} &\leq \frac{1}{s_{i,1}} w_i^{\frac{0}{d_i}} = \frac{1}{s_{i,1}} \\
\underline{s}_{i,2} &\leq \frac{1}{s_{i,1}} w_i^{\frac{1}{d_i-1}}
\end{aligned}$$

Next we recall the following inequality.

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \max_{1 \leq i \leq n} \left(s_{i,1} + s_{i,2}\right) + \log_2 \min_{1 \leq i \leq n} \left(\underline{s}_{i,1} + \underline{s}_{i,2}\right)$$

Note $s_{i,2} = s_{i,1}w_i$ and recall $\underline{s}_{i,1} \leq \frac{1}{s_{i,1}}$, and $\underline{s}_{i,2} \leq \frac{1}{s_{i,1}} w_i^{\frac{1}{d_i - 1}}$. Thus,

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \max_{1 \leq i \leq n} (s_{i,1} + s_{i,1}w_i) + \log_2 \min_{1 \leq i \leq n} \left(\frac{1}{s_{i,1}} + w_i^{\frac{1}{d_i - 1}} \frac{1}{s_{i,1}} \right)$$

By pulling out the $s_{i,1}$'s where possible, we have

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \max_{1 \leq i \leq n} (s_{i,1}(1 + w_i)) + \log_2 \min_{1 \leq i \leq n} \left(\frac{1}{s_{i,1}} \left(1 + w_i^{\frac{1}{d_i - 1}} \right) \right)$$

Take $\max\limits_{1 \leq i' \leq n} w_{i'}$ first. Note this is the important trick, as it will allow us to apply the outer min and max.

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \max_{1 \leq i \leq n} \left(s_{i,1} \left(1 + \max_{1 \leq i' \leq n} w_{i'} \right) \right) + \log_2 \min_{1 \leq i \leq n} \left(\frac{1}{s_{i,1}} \left(1 + \max_{1 \leq i' \leq n} w_{i'}^{\frac{1}{d_{i'} - 1}} \right) \right)$$

Now we may separate the i and i' components. Thus,

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \left(\left(1 + \max_{1 \leq i' \leq n} w_{i'} \right) \max_{1 \leq i \leq n} (s_{i,1}) \right) + \log_2 \left(\left(1 + \max_{1 \leq i' \leq n} w_{i'}^{\frac{1}{d_{i'} - 1}} \right) \min_{1 \leq i \leq n} \left(\frac{1}{s_{i,1}} \right) \right)$$

By applying log properties, we have

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \left(1 + \max_{1 \leq i' \leq n} w_{i'} \right) + \log_2 \max_{1 \leq i \leq n} (s_{i,1}) + \log_2 \left(1 + \max_{1 \leq i' \leq n} w_{i'}^{\frac{1}{d_{i'} - 1}} \right) + \log_2 \min_{1 \leq i \leq n} \left(\frac{1}{s_{i,1}} \right)$$

Note we may rewrite the min as a max in the denominator. Thus,

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \left(1 + \max_{1 \leq i' \leq n} w_{i'} \right) + \log_2 \max_{1 \leq i \leq n} (s_{i,1}) + \log_2 \left(1 + \max_{1 \leq i' \leq n} w_{i'}^{\frac{1}{d_{i'} - 1}} \right) + \log_2 \left(\frac{1}{\max\limits_{1 \leq i \leq n} s_{i,1}} \right)$$

By applying log properties and simplifying, we have

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \left(1 + \max_{1 \leq i' \leq n} w_{i'} \right) \left(1 + \max_{1 \leq i' \leq n} w_{i'}^{\frac{1}{d_{i'} - 1}} \right)$$

Note $w = \max\limits_{1 \leq i' \leq n} w_{i'}$ and $d = \max\limits_{1 \leq i \leq n} d_i$. Thus,

$$\log_2 \frac{U^L}{\overline{R}} \leq \log_2 \left((1 + w) \left(1 + w^{\frac{1}{d-1}} \right) \right)$$

5.2.5 Overestimation Bound of the Apocryphal Root Size Bound for Pham Systems

Note

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq \log_2\left(\frac{U^A}{\underline{R}}\right)$$

Thus,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \;\leq\; \log_2 2 \max_{1\leq i\leq n}\left(s_{i,1}\right) \;+\; \log_2 2 \min_{1\leq i\leq n}\left(\underline{s}_{i,1}\right)$$

Recall $s_{*,1} = \max\limits_{1\leq i\leq n} s_{i,1}$ and substitute $\min\limits_{1\leq i\leq n} \underline{s}_{i,1} \leq \max\limits_{1\leq i\leq n} \min\limits_{0<j\leq d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{-1}{j-0}}$.

Remark 12. *This substitution is the key step in this proof, as it will eventually allow us to write in terms of $c_{i,j}$ and $c_{i,0}$. Without this step further simplification is quite difficult. Unfortunately this step also represents significant information loss for some Pham systems.*

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \;\leq\; 2 + \log_2 s_{*,1} \;+ \log_2 \max_{1\leq i\leq n} \min_{0<j\leq d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{-1}{j-0}}$$

Combine logs,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq 2 + \log_2\left(\frac{s_{*,1}}{\max\limits_{1\leq i\leq n} \min\limits_{0<j\leq d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j-0}}}\right)$$

Write the denominator in terms of $c_{i,*}$, recalling $c_{i,j} = b_{i,j}^{\frac{1}{d_i-j}}$ and $b_{i,d_i} = 1$,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq 2 + \log_2\left(\frac{s_{*,1}}{\max\limits_{1\leq i\leq n} \min\left\{\min\limits_{0<j<d_i} \left(\frac{c_{i,0}^{d_i-0}}{c_{i,j}^{d_i-j}}\right)^{\frac{1}{j-0}}, \left(\frac{c_{i,0}^{d_i-0}}{1}\right)^{\frac{1}{d_i-0}}\right\}}\right)$$

Distribute the exponents,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq 2 + \log_2\left(\frac{s_{*,1}}{\max\limits_{1\leq i\leq n} \min\left\{\min\limits_{0<j<d_i} \frac{c_{i,0}^{d_i/j}}{c_{i,j}^{d_i/j-1}}, \frac{c_{i,0}^{d_i/d_i}}{1}\right\}}\right)$$

Simplify by pulling the $s_{*,1}$ inside the max min,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq\ 2+\log_2\left(\frac{1}{\max\limits_{1\leq i\leq n}\ \min\left\{\min\limits_{0<j<d_i}\ \frac{c_{i,0}^{d_i/j}}{s_{*,1}^{d_i/j}}, \frac{c_{i,0}}{s_{*,1}}\right\}}\right)$$

Note the exponents of $c_{i,0}$ and $s_{*,1}$ are the same for the first component of the min,

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq\ 2+\log_2\left(\frac{1}{\max\limits_{1\leq i\leq n}\ \min\left\{\min\limits_{0<j<d_i}\ \left(\frac{c_{i,0}}{s_{*,1}}\right)^{d_i/j}, \frac{c_{i,0}}{s_{*,1}}\right\}}\right)$$

As $c_{i,0} \leq s_{*,1}$, we have $\min\left\{\min\limits_{0<j<d_i}\ \left(\frac{c_{i,0}}{s_{*,1}}\right)^{d_i/j}, \frac{c_{i,0}}{s_{*,1}}\right\} = \left(\frac{c_{i,0}}{s_{*,1}}\right)^{d_i}$, thus

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq\ 2+\log_2\left(\frac{1}{\max\limits_{1\leq i\leq n}\left(\frac{c_{i,0}}{s_{*,1}}\right)^{d_i}}\right)$$

Let $d = \max\limits_{1\leq i\leq n} d_i$ then

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq\ 2+\log_2\left(\frac{1}{\max\limits_{1\leq i\leq n}\left(\frac{c_{i,0}}{s_{*,1}}\right)^{d}}\right)$$

Unfortunately the above step is necessary, as without it there is no way to further simplify. Thus, information is lost for Pham systems where there is a significant difference in the size of the largest d_i compared other d_is.

Note that d has no relation with i, so we pull it out of the max

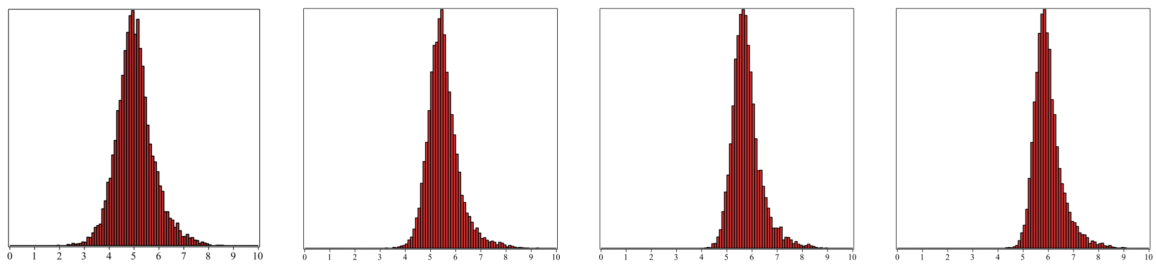
$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq\ 2+\log_2\left(\frac{1}{\left(\frac{\max\limits_{1\leq i\leq n} c_{i,0}}{s_{*,1}}\right)^{d}}\right)$$

By recalling $c_{*,0} = \max\limits_{1\leq i\leq n} c_{i,0}$, we have

$$\log_2\left(\frac{U^A}{\overline{R}}\right) \leq 2 + d\ \log_2\frac{s_{*,1}}{c_{*,0}}$$

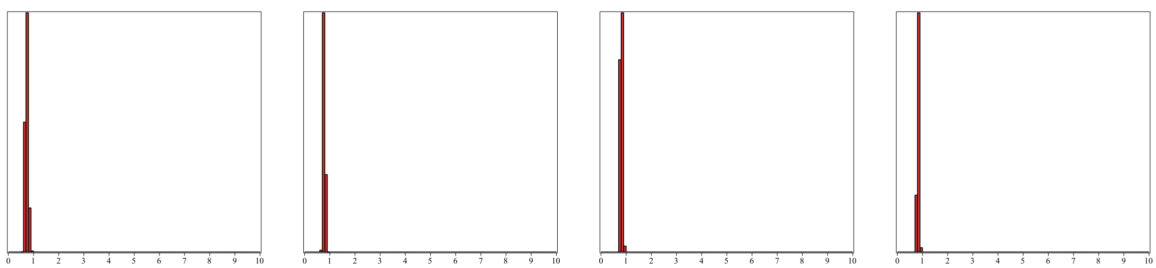
5.3 Experiments

Now that we have bounds on the overestimation of our root size bounds we would like to experimentally check the overestimation of that overestimation, defined as $\log_2 \frac{2^{\text{Overestimation Bound}}}{2^{\text{Overestimation}}}$. We shall check the overestimation of the overestimation of the U^I root size bound.



$d = 3,\ n = 2$ $d = 4,\ n = 2$ $d = 5,\ n = 2$ $d = 6,\ n = 2$

Figure 5.8: Overestimation of the overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10.



$d = 3,\ n = 2$ $d = 4,\ n = 2$ $d = 5,\ n = 2$ $d = 6,\ n = 2$

Figure 5.9: Overestimation of the overestimation of $U^I(F)$ where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10, except the constant coefficient, which is defined to be 'large'.

Figure 5.8. The figure compares the overestimation of the overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation of the overestimation while the vertical axis represents frequency.

Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 10 using independently identically uniformly distributed coefficients.

Remark 13. *In practice the overestimation of the overestimation is pessimistic compared to* *the overestimation, as the lower root size bounds we have are lower bounds for $\underline{R}$, and $\overline{R} \geq \underline{R}$. In cases where $\overline{R} >> \underline{R}$ the overestimation is particularly pessimistic.*

Figure 5.9. The figure compares the overestimation of the overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation of the overestimation while the vertical axis represents frequency.

Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined such that $c_{i,0} = 20 \max_{1 \leq j < d} c_{i,j}$. Systems that would result in a function with $c_{i,0} = 0$ were regenerated until a system that did not have $c_{i,0} = 0$ was found.

Remark 14. *In this case the constant coefficient is somehow dominant, and so $\underline{R}$ will normally be closer to $\overline{R}$ than in the uniform distribution case.*

In summary, when giving a root size bound it is useful to be able to bound the quality in terms of the coefficient size, number of variables, and the degree. We have given such quality bounds for our root size bounds. In the case where the constant coefficient, $c_{*,0}$ is always the largest for every equation we have that the quality is bound above by 2, which does not depend on the number of variables, the degree, or the coefficient size beyond the requirement.

CHAPTER

6

IMPROVEMENT OF THE ROOT SIZE BOUNDS FOR PHAM SYSTEMS

In chapter 4 we derived root size bounds for Pham systems. A natural question to arise is how can we improve those root size bounds? We will start this chapter with an exploration of the special case where our Pham system is a univariate polynomial, and move on to exploring improving the root size bounds for more general Pham systems.

We begin with definitions.

Definition 10 (Bounding Sequence)**.** *Let F be a Pham system, and let $U_{i,*}(F) = (U_{i,0}(F), U_{i,1}(F), \ldots)$ be the sequence recursively defined by*

$$U_{i,j}(F) = \begin{cases} U(F) & \text{if } j = 0 \\ \left(\sum_{|e|<d_i} \left| a_{i,(e)} \right| U_{i,j-1}(F)^{|e|} \right)^{1/d_i} & \text{if } j > 0 \end{cases}$$

Where $U(F)$ is any root size bound of F. We will often write $U_{i,j}$ when the intended F is clear from the context.

Example 34. *Let*

$$F_1 = x_1^2 - 10x_1 - x_2 - 1$$
$$F_2 = x_2^2 - x_1 - x_2 - 1$$

By letting $U_{i,0} = U^I$*, we have*

$$U_{1,*}(F) = (11.61803, 11.34894, 10.92760, 10.69729, \ldots)$$
$$U_{2,*}(F) = (11.61803, 4.92301, 4.15595, 4.01043, \ldots)$$

Definition 11 ($U_{i,*}$ Limit)**.** *Let* F *be a Pham system, and let* $U_{i,*}(F) = (U_{i,0}(F), U_{i,1}(F), \ldots)$*. Then we define* $\hat{U}_i$ *to be the following.*

$$\hat{U}_i = \lim_{j \to \infty} U_{i,j}$$

Example 35. *Let*

$$F_1 = x_1^2 - 10x_1 - x_2 - 1$$
$$F_2 = x_2^2 - x_1 - x_2 - 1$$

Then we have the following.

$$\hat{U}_1(F) \approx 10.47024$$
$$\hat{U}_2(F) \approx 3.92348$$

Graphically it looks like the following

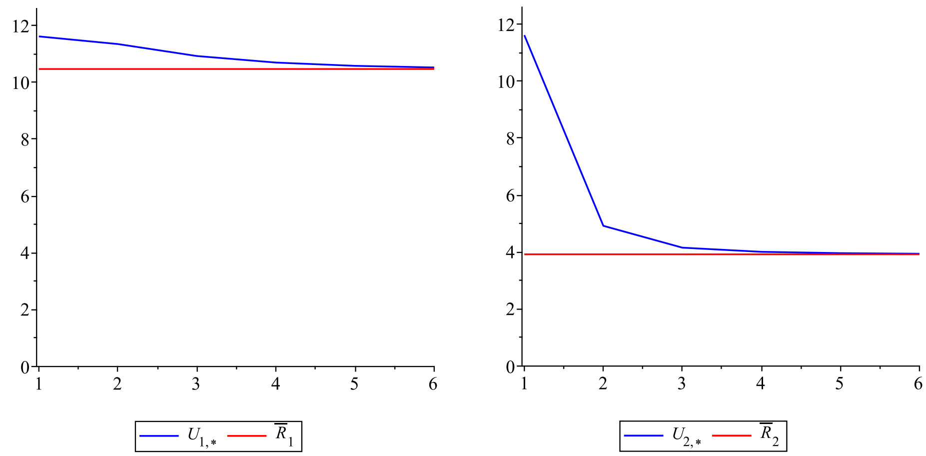


Figure 6.1: $U_{i,j}$ for j from 0 to 5, where $U_{i,0} = U^I$ (in blue), and $\overline{R}_i$ (in red). Note $\overline{R}_1 \neq \overline{R}_2$ and $\hat{U}_1 \neq \hat{U}_2$.

A question to arise is does $\hat{U}_i = \overline{R}_i$ for all Pham systems? Tragically, the answer is no.

Example 36. *Let*

$$F_1 = x_1^2 + x_1 + x_2 + 1$$
$$F_2 = x_2^2 + x_1 + x_2 + 1$$

Then we have the following figures.

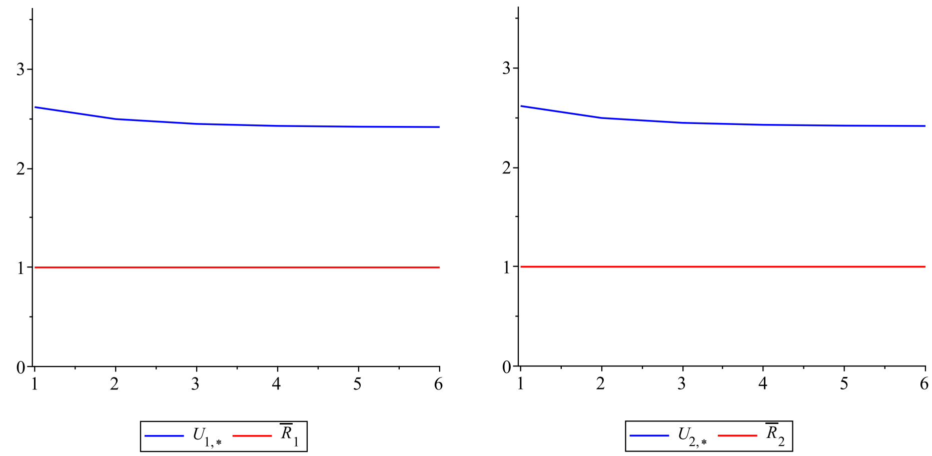


Figure 6.2: $U_{i,j}$ for j from 0 to 5, where $U_{i,0} = U^I$ (in blue), and $\overline{R}_i$ (in red). Note that $U_{i,j}$ does not approach $\overline{R}_i$.

where $U_{i,0} = U^I$. Note which root size bound is chosen for $U_{i,0}$ does not affect $\hat{U}_i$.

However $\hat{U}_i = \overline{R}_i$ is possible.

Example 37. *Let*

$$F_1 = x_1^2 - x_1 - x_2 - 1$$
$$F_2 = x_2^2 - x_1 - x_2 - 1$$

Then we have the following figures.

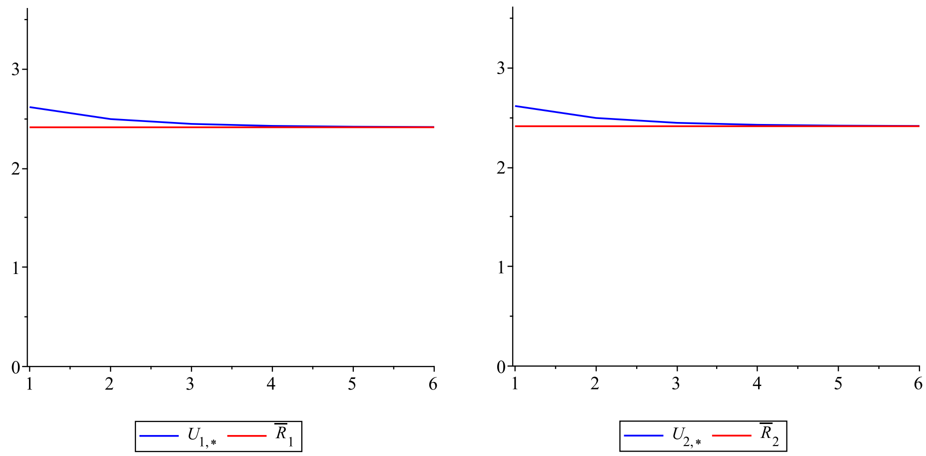


Figure 6.3: $U_{i,j}$ for j from 0 to 5, where $U_{i,0} = U^I$ (in blue), and $\overline{R}_i$ (in red). Note that $U_{i,j}$ approaches $\overline{R}_i$.

where $U_{i,0} = U^I$*. Note which root size bound is chosen for* $U_{i,0}$ *does not affect* $\hat{U}_i$*.*

6.1 Main Results

It is sensible to begin with the special case of our Pham system being a univariate polynomial, as this may hint at what will hold true for larger polynomial systems.

Theorem 44 (Improvement of the Root Size Bound for Univariate Polynomials)**.** *Let* $f = x^d - \sum_{e<d} a_e x^e$*. If* $\underset{e}{\forall}\ a_e \geq 0$ *then* $\hat{U} = \overline{R}$*.*

In other words, U_* will approach the root size when our Pham system is a univariate polynomial whose non-leading coefficients are all non-positive.

Theorem 45 (Improvement of the Root Size Bound for Pham Systems)**.** *If* $\underset{i}{\forall}\ F_i = x_i^d - g(x)$ *where* $g(x) = \sum_{|e|<d} a_e x^e$ *and* $\underset{|e|<d}{\forall} a_e = a_{1,(e)} = \cdots = a_{n,(e)} \geq 0$ *then* $\hat{U}_i = \overline{R}_i$*.*

That is to say, if the non-leading terms of the Pham system are the same for every F_i then we have $\lim_{j\to\infty} U_{i,j} = \overline{R}_i$.

6.2 Proofs

6.2.1 Improvement of the Root Size Bound for Univariate Polynomials

Proof. Note $\hat{U}$ exists.

By definition

$$U_j = \left(|a_{d-1}|\, U_{j-1}^{d-1} + \cdots + |a_0|\right)^{1/d}$$

Thus,

$$\lim_{j\to\infty} U_j = \lim_{j\to\infty} \left(|a_{d-1}|\, U_{j-1}^{d-1} + \cdots + |a_0|\right)^{1/d}$$

As powering to a positive exponent is a monotonically increasing function, we have

$$\lim_{j\to\infty} U_j = \left(\lim_{j\to\infty} |a_{d-1}|\, U_{j-1}^{d-1} + \cdots + \lim_{j\to\infty} |a_0|\right)^{1/d}$$

By basic limit rules, we have

$$\lim_{j\to\infty} U_j = \left(|a_{d-1}| \lim_{j\to\infty} U_{j-1}^{d-1} + \cdots + |a_0|\right)^{1/d}$$

Recalling the definition of $\hat{U}$, we have

$$\hat{U} = \left(|a_{d-1}| \lim_{j\to\infty} U_{j-1}^{d-1} + \cdots + |a_0|\right)^{1/d}$$

Note $\lim\limits_{j\to\infty} U_{j-1} = \hat{U}$. Thus,

$$\hat{U} = \left(|a_{d-1}|\, \hat{U}^{d-1} + \cdots + |a_0|\right)^{1/d}$$

As $\underset{e}{\forall}\ a_e \geq 0$, we have

$$\hat{U} = \left(a_{d-1}\hat{U}^{d-1} + \cdots + a_0\right)^{1/d}$$

By definition,

$$\overline{R} = \left(a_{d-1}\overline{R}^{d-1} + \cdots + a_0\right)^{1/d}$$

Next we recall

$$\hat{U} \geq \overline{R}$$

Thus,

$$\hat{U} = \overline{R}$$

$\square$

Now that we have a univariate result we will generalize the main idea of the proof into a related proof for multivariate Pham systems.

6.2.2 Improvement of the Root Size Bound for Pham Systems

Proof. Note $\hat{U}_i$ exists.

By definition

$$U_{i,j} = \max_{1\le i\le n} \left(|a_{d-1,0,\dots,0}|\, U_{i,j-1}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

But as $g(x)$ is the same for all i, we have $U_k = U_{1,k} = \cdots = U_{n,k}$. Thus,

$$U_j = \left(|a_{d-1,0,\dots,0}|\, U_{j-1}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

Thus

$$\lim_{j\to\infty} U_j = \lim_{j\to\infty} \left(|a_{d-1,0,\dots,0}|\, U_{j-1}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

As powering to a positive exponent is a monotonically increasing function, we have

$$\lim_{j\to\infty} U_j = \left(\lim_{j\to\infty} |a_{d-1,0,\dots,0}|\, U_{j-1}^{d-1} + \cdots + \lim_{j\to\infty} |a_{0,\dots,0}|\right)^{1/d}$$

By basic limit rules, we have

$$\lim_{j\to\infty} U_j = \left(|a_{d-1,0,\dots,0}| \lim_{j\to\infty} U_{j-1}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

Recalling the definition of $\hat{U}$, we have

$$\hat{U} = \left(|a_{d-1,0,\dots,0}| \lim_{j\to\infty} U_{j-1}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

Note $\lim\limits_{j\to\infty} U_{j-1} = \hat{U}$. Thus,

$$\hat{U} = \left(|a_{d-1,0,\dots,0}|\, \hat{U}^{d-1} + \cdots + |a_{0,\dots,0}|\right)^{1/d}$$

As $\underset{e}{\forall}\, a_e > 0$, we have

$$\hat{U} = \left(a_{d-1,0,\dots,0}\hat{U}^{d-1} + \cdots + a_{0,\dots,0}\right)^{1/d}$$

Remark 15. *Note so far in this proof very little differs from the univariate argument. The remaining part is where more work was needed.*

Note for any solution set $(x_1, \dots, x_n)$ we have

$$x_1^d = \cdots = x_n^d$$

Thus,

$$|x_1| = \cdots = |x_n|$$

Thus we have equality in the following.

$$\overline{R} = \left(a_{d-1,0,\ldots,0}\overline{R}^{d-1} + \cdots + a_{0,\ldots,0}\right)^{1/d}$$

Finally, we recall

$$\hat{U} \geq \overline{R}$$

Thus,

$$\hat{U} = \overline{R}$$

As this was true independent of i, we have

$$\hat{U}_i = \overline{R}_i$$

□

6.3 Experiments

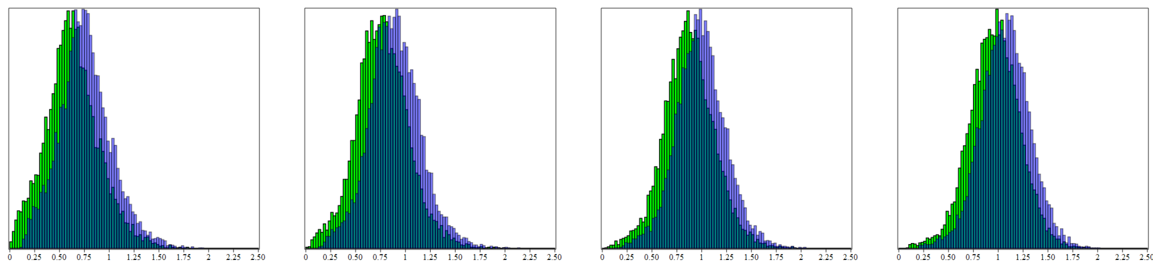
- Figure 6.4. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 10 using independently identically uniformly distributed coefficients. The blue bars shows the quality of U^I while the green bars shows the quality of $\max_{1 \leq i \leq n} U_{i,10}(F)$ where $U_{i,0} = U^I$.

  Remark 16. *There appears to be a general across-the-board reduction in the overestimation, so the bound is improved. The histogram of* $\max_{1 \leq i \leq n} U_{i,10}(F)$ *seems to keep a similar shape to the histogram of* $U_{i,0}$.

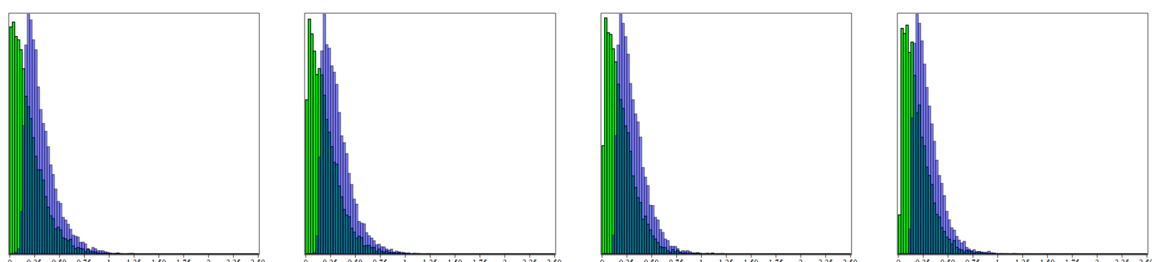
- Figure 6.5. The figure compares overestimation for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation while the vertical axis represents frequency.

  Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 0 using independently identically



$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 6.4: Overestimation of $\max_{1 \leq i \leq n} U_{i,10}(F)$ (in green) and $U^I(F)$ (in blue) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10.



$d = 3, \ n = 2$ $d = 4, \ n = 2$ $d = 5, \ n = 2$ $d = 6, \ n = 2$

Figure 6.5: Overestimation of $\max_{1 \leq i \leq n} U_{i,10}(F)$ (in green) and $U^I(F)$ (in blue) where F are Pham systems with independently identically uniformly distributed coefficients between 0 and 10.

uniformly distributed coefficients. The blue bars shows the quality of U^I while the green bars shows the quality of $\max_{1 \le i \le n} U_{i,10}(F)$ where $U_{i,0} = U^I$.

Remark 17. *There is clear improvement from the prior experiment with random coefficient signs, as expected. This case does not suffer information loss from using the absolute value and triangular inequality.*

In summary, it is always useful to be able to improve results you already have. In light of this we have shown a method for improving root size bounds, and have shown that it can be used to find an upper bound on the size of each coordinate of the roots. In certain cases we have that the improvement scheme approaches the root size.

CHAPTER

7

ROOT SPREAD BOUNDS FOR PHAM SYSTEMS

In chapter 4 we derived root size bounds for Pham systems. In this chapter we will formulate a definition for "root spread". We will then derive root spread bounds for Pham systems.

Definition 12 (Root Spread)**.** *The* root spread *of F is denoted by S and defined as*

$$S = \frac{\frac{\overline{R}-\underline{R}}{2}}{\frac{\overline{R}+\underline{R}}{2}}$$

Remark 18. *This definition can, of course, be written more succinctly as*

$$S = \frac{\overline{R} - \underline{R}}{\overline{R} + \underline{R}}$$

The long-form definition was chosen to emphasis that the root spread is found by taking the average of the difference of $\overline{R}$ and $\underline{R}$ divided by the average of the sum of $\overline{R}$ and $\underline{R}$, which is more natural.

We will use the following toy Pham system throughout this chapter.

Example 38 (Running example).

$$\begin{aligned} F_1 &= x_1^2 - i \cdot x_1 - x_2 - 2 \\ F_2 &= x_2^2 - x_1 - x_2 + i \end{aligned}$$

Example 39. *For the running example (Example 38), we have*

$$S \approx \frac{1.998 - 1.202}{1.998 + 1.202} \leq 0.2488$$

7.1 Main Results

Theorem 46 (Improved Lagrange Root Spread Bound for Pham Systems). *We have*

$$S \leq \frac{1-u}{1+u}$$

where $u = \frac{1}{\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i} \frac{s_{i,j}-s_{i,j+1}}{q_j}\right) \min\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i} \frac{\underline{s}_{i,j}-\underline{s}_{i,j+1}}{q_j}\right)}$

Example 40. *For the running example (Example 38), we have*

$$0.2421 \leq u$$

Thus,

$$S \leq \frac{1 - 0.2421}{1 + 0.2421} \leq 0.6102$$

Theorem 47 (Lagrange Root Spread Bound for Pham Systems). *We have*

$$S \leq \frac{1-v}{1+v}$$

where $v = \frac{1}{\max\limits_{1\leq i\leq n}(s_{i,1}+s_{i,2}) \min\limits_{1\leq i\leq n}\left(\underline{s}_{i,1}+\underline{s}_{i,2}\right)}$

Example 41. *For the running example (Example 38), we have*

$$0.1715 \leq v$$

Thus,

$$S \leq \frac{1 - 0.1715}{1 + 0.1715} \leq 0.7073$$

Theorem 48 (Apocryphal Root Spread Bound for Pham Systems)**.** *We have*

$$S \leq \frac{1 - \frac{1}{4}\zeta^d}{1 + \frac{1}{4}\zeta^d}$$

where

$$\zeta = \frac{c_{*,0}}{s_{*,1}} \qquad c_{*,0} = \max_{1 \leq i \leq n} c_{i,0} \qquad d = \max_{1 \leq i \leq n} d_i$$

Remark 19. *As* $\left(\frac{c_{*,0}}{s_{*,1}}\right) \leq 1$, *we have that* $\frac{1-\zeta}{1+\zeta} \geq \frac{3}{5}$. *It is clear that* $\lim_{c_{*,0} \to \infty} S \to 0$. *Thus, this bound is not useful for cases with relatively large* $c_{*,0}$. *On the other hand, when* $c_{*,0}$ *is relatively small this bound will prove competitive with Theorem 47.*

Example 42. *For the running example (Example 38), we have*

$$0.1250 \leq \zeta$$

Thus,

$$S \leq \frac{1 - 0.1250}{1 + 0.1250} \leq 0.7778$$

7.2 Proofs

Proof of Theorem 46. Note

$$S = \frac{\frac{\overline{R} - \underline{R}}{2}}{\frac{\overline{R} + \underline{R}}{2}} = \frac{\overline{R} - \underline{R}}{\overline{R} + \underline{R}} = \frac{1 - \frac{\underline{R}}{\overline{R}}}{1 + \frac{\underline{R}}{\overline{R}}}$$

Thus in order to bound S from above, we need to bound $\frac{\underline{R}}{\overline{R}}$ from below. We have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{L^I}{U^I}$$

By applying the definitions of U^I and L^I, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\max_{1 \leq i \leq n} \frac{1}{\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j}}}{\max_{1 \leq i \leq n} \left(\sum_{j=1}^{d_i} \frac{s_{i,j} - s_{i,j+1}}{q_j} \right)}$$

By rewriting the numerator, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\frac{1}{\min\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{\underline{s}_{i,j}-\underline{s}_{i,j+1}}{q_j}\right)}}{\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{s_{i,j}-s_{i,j+1}}{q_j}\right)}$$

By simplifying, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{1}{\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{s_{i,j}-s_{i,j+1}}{q_j}\right)\min\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{\underline{s}_{i,j}-\underline{s}_{i,j+1}}{q_j}\right)}$$

By recalling $u = \frac{1}{\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{s_{i,j}-s_{i,j+1}}{q_j}\right)\min\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{d_i}\frac{\underline{s}_{i,j}-\underline{s}_{i,j+1}}{q_j}\right)}$, we have

$$\frac{\underline{R}}{\overline{R}} \geq u$$

Hence,

$$S \leq \frac{1-u}{1+u}$$

$\square$

Proof of Theorem 47. Note

$$S = \frac{\frac{\overline{R}-\underline{R}}{2}}{\frac{\overline{R}+\underline{R}}{2}} = \frac{\overline{R}-\underline{R}}{\overline{R}+\underline{R}} = \frac{1-\frac{\underline{R}}{\overline{R}}}{1+\frac{\underline{R}}{\overline{R}}}$$

Thus in order to bound S from above, we need to bound $\frac{\underline{R}}{\overline{R}}$ from below. We have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{L^L}{U^L}$$

By applying the definitions of U^L and L^L, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\max\limits_{1\leq i\leq n}\frac{1}{\underline{s}_{i,1}+\underline{s}_{i,2}}}{\max\limits_{1\leq i\leq n}\left(s_{i,1}+s_{i,2}\right)}$$

By rewriting the numerator, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\frac{1}{\min\limits_{1 \leq i \leq n} \left(\underline{s}_{i,1} + \underline{s}_{i,2}\right)}}{\max\limits_{1 \leq i \leq n} \left(s_{i,1} + s_{i,2}\right)}$$

By simplifying, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{1}{\max\limits_{1 \leq i \leq n} \left(s_{i,1} + s_{i,2}\right) \min\limits_{1 \leq i \leq n} \left(\underline{s}_{i,1} + \underline{s}_{i,2}\right)}$$

By recalling $v = \frac{1}{\max\limits_{1 \leq i \leq n} (s_{i,1} + s_{i,2}) \min\limits_{1 \leq i \leq n} \left(\underline{s}_{i,1} + \underline{s}_{i,2}\right)}$, we have

$$\frac{\underline{R}}{\overline{R}} \geq v$$

Hence,

$$S \leq \frac{1 - v}{1 + v}$$

□

Proof of Theorem 48. Recall

$$S = \frac{1 - \frac{\underline{R}}{\overline{R}}}{1 + \frac{\underline{R}}{\overline{R}}}$$

Thus in order to bound S from above, we need to bound $\frac{\underline{R}}{\overline{R}}$ from below.
Note

$$\frac{\underline{R}}{\overline{R}} \geq \frac{L^A}{U^A}$$

By applying the definitions of U^A and L^A, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\frac{1}{2} \max\limits_{1 \leq i \leq n} \frac{1}{\underline{s}_{i,1}}}{2\ s_{*,1}}$$

By converting the numerator into $b_{i,j}$ notation, we have

$$\frac{\underline{R}}{\overline{R}} \geq \frac{\frac{1}{2} \max\limits_{1 \leq i \leq n} \min\limits_{0 < j \leq d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j-0}}}{2\ s_{*,1}}$$

By recalling $b_{i,d_i} = 1$ and $c_{i,j}^{d_i - j} = b_{i,j}$, we have

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \frac{\max\limits_{1 \leq i \leq n} \min \left\{ \min\limits_{0 < j < d_i} \left(\frac{c_{i,0}^{d_i - 0}}{c_{i,j}^{d_i - j}} \right)^{\frac{1}{j-0}}, \left(\frac{c_{i,0}^{d_i - 0}}{1} \right)^{\frac{1}{d_i - 0}} \right\}}{s_{*,1}}$$

By simplifying, we have

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \frac{\max\limits_{1 \leq i \leq n} \min \left\{ \min\limits_{0 < j < d_i} \frac{c_{i,0}^{d_i/j}}{c_{i,j}^{d_i/j - 1}}, c_{i,0} \right\}}{s_{*,1}}$$

Make the RHS smaller by replacing $c_{i,j}$ in the denominator of the min with $s_{*,1}$ and combine the now redundant minimums.

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \frac{\max\limits_{1 \leq i \leq n} \min\limits_{0 < j \leq d_i} \frac{c_{i,0}^{d_i/j}}{s_{*,1}^{d_i/j - 1}}}{s_{*,1}}$$

As $s_{*,1}$ has nothing to do with i or j, pull the $s_{*,1}$ inside the max min.

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \max_{1 \leq i \leq n} \min_{0 < j \leq d_i} \frac{c_{i,0}^{d_i/j}}{s_{*,1}^{d_i/j}}$$

By rewriting, we have

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \max_{1 \leq i \leq n} \min_{0 < j \leq d_i} \left(\frac{c_{i,0}}{s_{*,1}} \right)^{d_i/j}$$

As $\frac{c_{i,0}}{s_{*,1}} \leq 1$ we have $\min\limits_{0 < j \leq d_i} \left(\frac{c_{i,0}}{s_{*,1}} \right)^{d_i/j} = \left(\frac{c_{i,0}}{s_{*,1}} \right)^{d_i}$. Thus,

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \max_{1 \leq i \leq n} \left(\frac{c_{i,0}}{s_{*,1}} \right)^{d_i}$$

By recalling $d = \max\limits_{1 \leq i \leq n} d_i$, we have

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \max_{1 \leq i \leq n} \left(\frac{c_{i,0}}{s_{*,1}} \right)^{d}$$

Note d has nothing to do with i. Thus,

$$\frac{R}{\overline{\overline{R}}} \geq \frac{1}{4} \left(\frac{\max\limits_{1 \leq i \leq n} c_{i,0}}{s_{*,1}} \right)^{d}$$

By recalling $c_{*,0} = \max\limits_{1 \le i \le n} c_{i,0}$, we have

$$\frac{R}{\overline{\overline{R}}} \ge \frac{1}{4} \left(\frac{c_{*,0}}{s_{*,1}} \right)^d$$

By recalling $\zeta = \frac{c_{*,0}}{s_{*,1}}$, we have

$$\frac{R}{\overline{\overline{R}}} \ge \frac{1}{4}\zeta^d$$

Hence,

$$S \le \frac{1 - \frac{1}{4}\zeta^d}{1 + \frac{1}{4}\zeta^d}$$

$\square$

7.3 Experiments

In this section we will define overestimation of the root spread bound to mean $\frac{\text{Root Spread Bound}}{\text{Root Spread}}$. Note the difference between this definition and the one used for the root size bound, where $\log_2$ was taken.

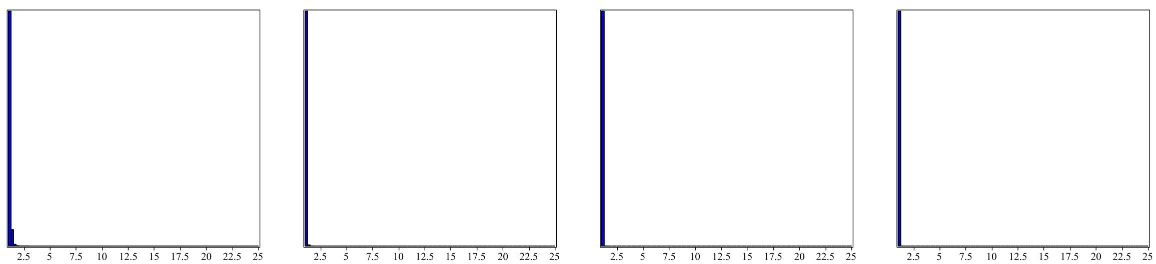
Figure 7.1. The figure compares the overestimation of the root spread for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation of the overestimation while the vertical axis represents frequency.

Given $n = 2$ and d we generated 10,000 Pham systems randomly where each coefficient is randomly chosen from the integers between -10 and 10 using independently identically uniformly distributed coefficients.

Remark 20. *As the lower bound of the overestimation is one, it is easily seen from the figure that the overestimation is low in this case. The largest point of interest is that as the degree increases the overestimation actually goes down. This appears to be because we always choose coefficients between -10 and 10, so the relative size of the constant coefficient is going down.*

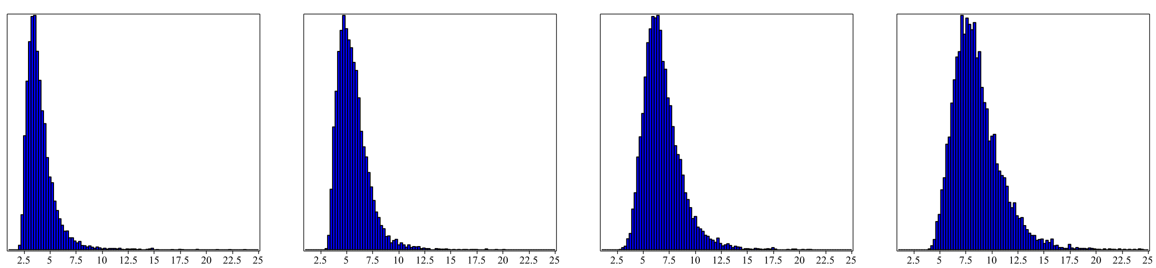
Figure 7.2. The figure compares the overestimation of the root spread for random Pham systems with a fixed degree vector and number of variables. The horizontal axis represents overestimation of the overestimation while the vertical axis represents frequency.

Given $n = 2$ and d we generated 10,000 Pham systems randomly where each non-constant coefficient is randomly chosen from the integers between -10 and 10 using an independent identical uniform distribution. Constant coefficients were defined such that $c_{i,0} = 20 \max\limits_{1 \le j < d} c_{i,j}$. Systems that would result in a function with $c_{i,0} = 0$ were regenerated until a system that did not have $c_{i,0} = 0$ was found.



$d = 3,\ n = 2$ $d = 4,\ n = 2$ $d = 5,\ n = 2$ $d = 6,\ n = 2$

Figure 7.1: Overestimation of $S(F)$ (Theorem 46) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10.



$d = 3,\ n = 2$ $d = 4,\ n = 2$ $d = 5,\ n = 2$ $d = 6,\ n = 2$

Figure 7.2: Overestimation of $S(F)$ (Theorem 46) where F are Pham systems with independently identically uniformly distributed coefficients between -10 and 10, except the constant coefficient, which is defined to be 'large'.

Remark 21. *From earlier we have that as the constant coefficient gets larger S is going to zero. Based on the figures S approaches zero more quickly than the root spread bound does. Thus, the closer S gets to 0 the larger the relative gap will be.*

In summary, the spread of the roots is bounded above, and that bound appears to behave in unexpected ways. That is, the root spread bound appears to get closer to the root spread the smaller the constant coefficient is relative to the other coefficients.

CHAPTER

8

RELATIONSHIP BETWEEN THE ROOTS OF PHAM SYSTEMS AND THEIR DERIVATIVES

We must first define what we mean by *derivative.*

Definition 13 (Derivative of a Pham System)**.** *Given a Pham system F as defined previously the corresponding* derivative system ∂F *is*

$$\begin{aligned}
&\partial_1 F_1 = x_1^{d_1-1} + \frac{1}{d_1}\partial_1 g_1(x_1, ..., x_n) \\
&\vdots \\
&\partial_n F_n = x_n^{d_n-1} + \frac{1}{d_n}\partial_n g_n(x_1, ..., x_n)
\end{aligned}$$

where each F_i and g_i is a polynomial in the variables $x_1, ..., x_n$ and $\deg(g_i(x_1, ..., x_n)) < d_i$ *such that i = 1,...,n.*

Example 43. *Let*

$$F_1 = x_1^3 + (1+i)\,x_1^2 + x_2^2 - i\,x_1 - x_2 - 2$$
$$F_2 = x_2^3 - x_1^2 + (2+2\,i)\,x_2^2 - x_1 - x_2 - i$$

Then

$$\partial_1 F_1 = x_1^2 + \frac{2(1+i)}{3}\,x_1 - \frac{i}{3}$$
$$\partial_2 F_2 = x_2^2 + \frac{2(2+2\,i)}{3}\,x_2 - \frac{1}{3}$$

Remark 22. *Note ∂F is monic. This does not affect the roots but is needed for the derivative system to be a Pham system.*

Remark 23. *We chose this definition of derivative as it has the following useful properties:*

1. *The derivative is also a Pham system.*

2. *Taking the derivative then permuting the x_is or permuting the x_is then taking the derivative will have the same outcome. This is ideal for applications as the choice of which variable is the i-th is often arbitrary.*

While other definitions of derivative might have all of these properties this was the one we chose to work with.

Theorem 49. *The average of the roots of F is the same as the average of the roots of ∂F, counting multiplicity.*

Example 44.

$$F_1 = x_1^3 + (1+i)\,x_1^2 + x_2^2 - i\,x_1 - x_2 - 2$$
$$F_2 = x_2^3 - x_1^2 + (2+2\,i)\,x_2^2 - x_1 - x_2 - i$$

Let F be the Pham system $F = \{F_1, F_2\}$ then the average values of the roots of F are

$$\left(\frac{-1}{3} + \frac{-1}{3}\,i\right), \left(\frac{-2}{3} + \frac{-2}{3}\,i\right)$$

As a point is the average if and only if it is the average coordinate-wise it is sufficient to observe the x_1 and x_2 coordinates. The following plots show the overlap of the coordinates of the averages of the roots.

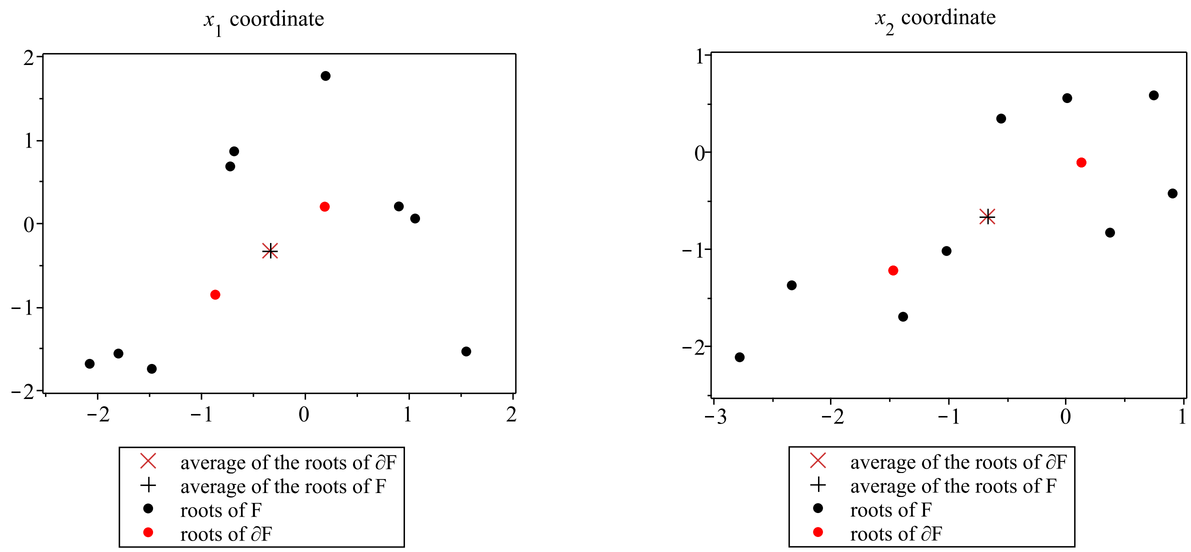


Figure 8.1: Coordinates of the roots and the coordinate-wise average of the roots of F (in black) and ∂F (in red) showing the overlap of the averages.

The following example shows Theorem 49 does not hold for non-Pham multivariate polynomial systems, even if they have no roots at infinity.

Example 45.

$$F_1 = x_1^2 + x_1x_2 + 3x_2^2 + 4x_2 + 2$$
$$F_2 = 3x_1^2 + 3x_1x_2 + 4x_2^2 + 2x_1 + 4x_2 + 3$$

Let F be the multivariate system $\{F_1, F_2\}$. The following graphs show that the coordinates of the averages of the roots do not coincide.

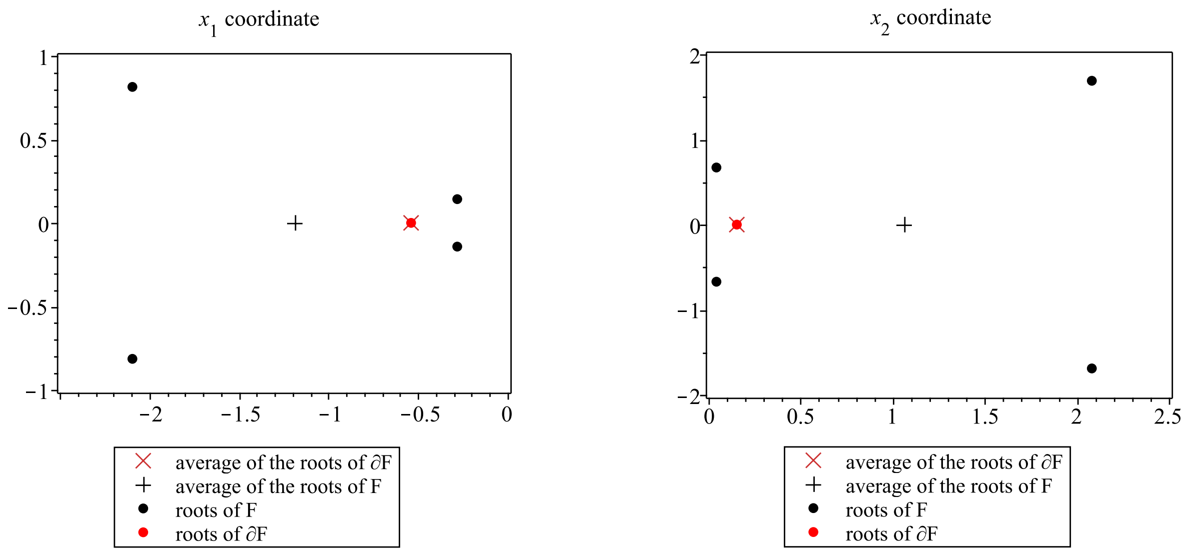


Figure 8.2: Coordinates of the roots and the coordinate-wise average of the roots of F (in black) and ∂F (in red) showing the failure to overlap of the averages.

If instead we look at the system $\{F_2, F_1\}$ *the roots of the original system are the same as the roots of* F*, but we have* $\partial F = \{\frac{\partial F_2}{\partial x_1}, \frac{\partial F_1}{\partial x_2}\}$*. Then the following graphs show that the coordinates of the averages of the roots still do not coincide.*

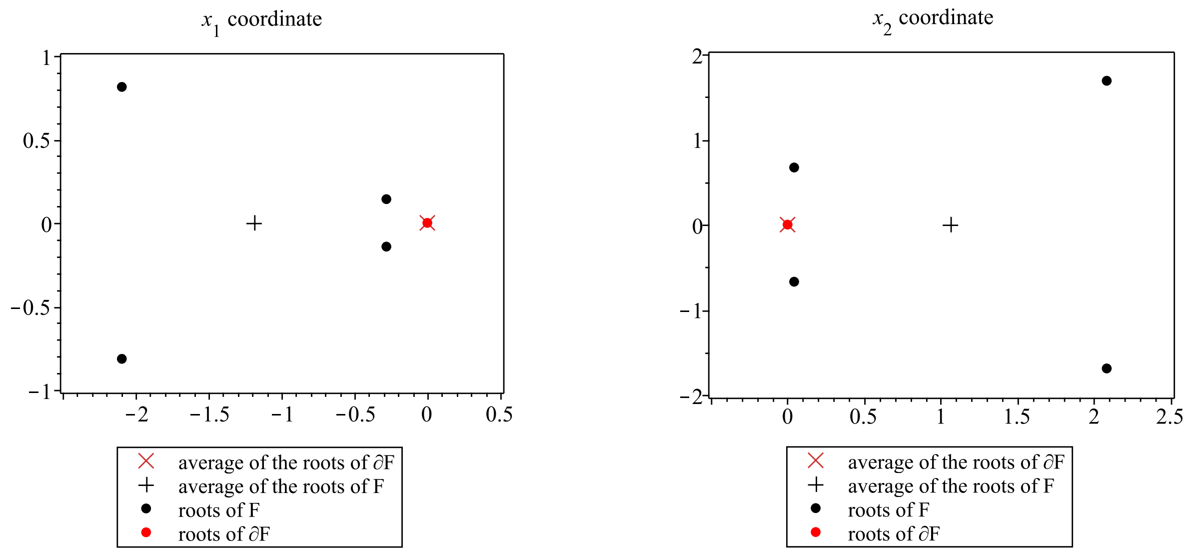


Figure 8.3: Coordinates of the roots and the coordinate-wise average of the roots of F (in black) and ∂F (in red) showing the failure to overlap of the averages.

8.1 Main Results

For this chapter we must have a definition for relationship to make sense of what we mean when we say the roots of F and the roots of ∂F have a relationship. Each definition of relationship we use will have a different subsection dedicated to it.

8.1.1 Convex Hull

In this subsection we say the roots of F and the roots of ∂F have a relationship if the *i-th* coordinate of the roots of ∂F are contained in the convex hull of the *i-th* coordinate of the roots of F. This is Gauss-Lucas theorem in the special case that F is a univariate polynomial, and so can be said to be a direct analog of Gauss-Lucas theorem for Pham systems.

Conjecture 50. *For $1 \leq i \leq n$, the i-th coordinate of the roots of the derivative system ∂F are contained inside the convex hull formed by the i-th coordinate of the roots of the original system F.*

This conjecture is false.

Example 46 (Counterexample:).

$$F_1 = x_1^3 + 20\, x_1\, x_2$$
$$F_2 = x_2^3 + 5\, x_1^2 + 3\, x_1\, x_2$$

Let F be the Pham system $F = \{F_1, F_2\}$. The plots are as follows.

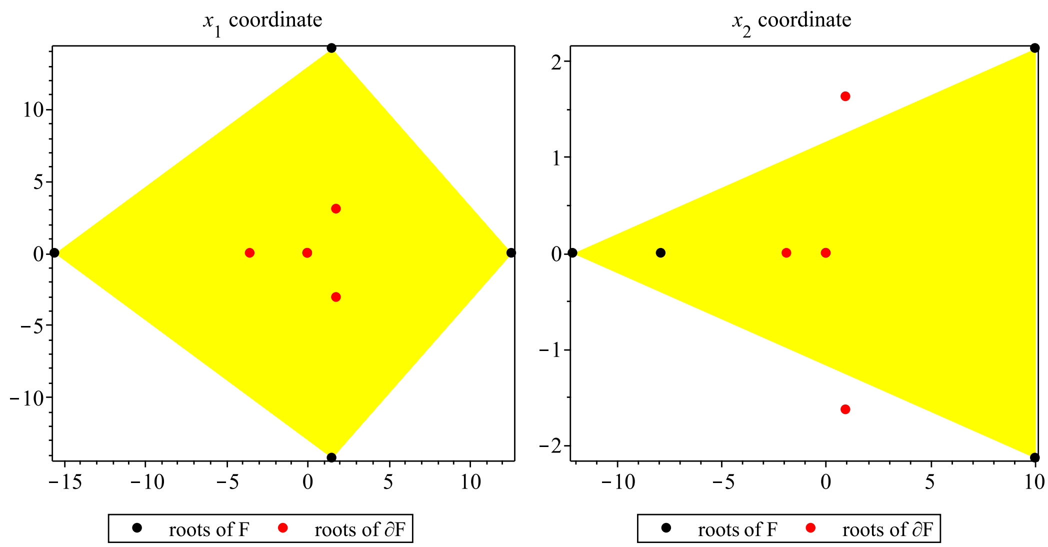


Figure 8.4: The coordinates of the roots of F (in black) their coordinate-wise convex hulls (in yellow), and the coordinates of the roots of ∂F (in red). Note that in the graph on the right some of the red dots are outside the yellow triangle.

Note that while all of the x_1 roots of ∂F are contained in the convex hull of the x_1 roots of F the same is not true for the x_2 roots, and so the conjecture is false.

We have shown that the i-th coordinate of the roots of a derivative system are not always contained in the convex hull of the i-th coordinate of the roots of its Pham system. Thus we will now find sufficient conditions for which the i-th coordinate of the roots of the derivative system will be contained in the convex hull of the i-th coordinate of the roots of its Pham system.

Corollary 51. *Given a 2nd degree n variable Pham system F, the derivative system ∂F has only one root, and each i-th coordinate of this root is contained in the convex hull of the i-th coordinate of the roots of F.*

This is straightforward from Theorem 49.

Next we will need the following definition.

Definition 14 (Triangular Decoupled-Derivative Pham system)**.** *We say that* $F \in \mathbb{C}[x]^n$ *is a* Triangular Decoupled-Derivative Pham *system if it has the following form*

$$\begin{aligned} F_1 &= g_1(x_1) \\ F_2 &= g_2(x_2) + h_2(x_1) \\ &\vdots \\ F_n &= g_n(x_n) + h_n(x_1, \ldots, x_{n-1}) \end{aligned}$$

where $\deg(g_i) = d_i$ *and* $\deg(h_i) < d_i$

Example 47. *The following* F *it a triangular decoupled-derivative Pham system.*

$$\begin{aligned} F_1 &= x_1^3 - x_1^2 - x_1 - 1 \\ F_2 &= x_2^3 + x_1 + x_2 + 16 \end{aligned}$$

Theorem 52. *If* F *is a triangular decoupled-derivative Pham system then the convex hull of the* i*-th coordinate of the roots of* F *contains the* i*-th coordinate of the roots of* ∂F *for* $1 \leq i \leq n$*.*

Example 48. *We would like to verify the theorem with an example. Referring back to* F *from Example 47, we have the following figures.*

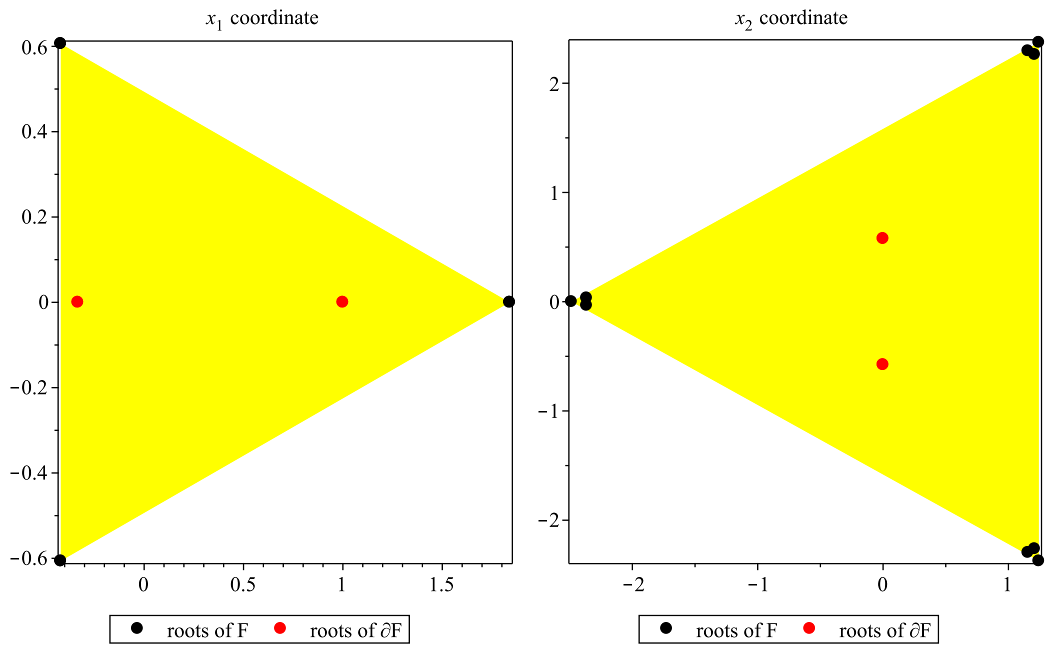


Figure 8.5: The coordinates of the roots of F (in black) their coordinate-wise convex hulls (in yellow), and the coordinates of the roots of ∂F (in red). In all cases the red dots are inside the yellow objects.

Thus we have verified the theorem for F, as coordinates of the roots of ∂F are contained in the yellow triangles, which are the convex hulls of the coordinates of the roots of F.

Proof.

Observe

$$F_1(x_1, \ldots, x_n) = g_1(x_1)$$

This is a univariate polynomial and so we may apply Gauss-Lucas and so we are done.

Suppose that the convex hull for the *i-th* coordinate of the roots of F containing the *i-th* coordinate of the roots of ∂F is true for $1 \leq i \leq k-1$. Then consider

$$F_k(x_1, \ldots, x_n) = g_k(x_k) + h_k(x_1, \ldots, x_{k-1})$$

The x_1 through x_{k-1} roots can be found through sequential substitution, so $h_k(x_1, \ldots, x_{k-1}) = C_{\ell,k}$ where ℓ is the multi-index of the particular choice of x_1 through x_{k-1} roots.

WLOG choose $C_{1,k} = C_k$. Then

$$F_k(x_1, \ldots, x_n) = F_k(x_k) = g_k(x_k) + C_k.$$

$F_k(x_k)$ is a univariate polynomial, so Gauss-Lucas theorem must hold between $F_k(x_k)$ and $F'_k(x_k)$.

Observe that
$$\frac{\partial}{\partial x_k} F_k(x_1, \ldots, x_n) = \frac{d}{dx_k} g_k(x_k) = g'_k(x_k) = F'_k(x_k)$$

Thus the earlier substitution does not change ∂F, and as $C_{1,k}$ is by definition a solution set of F for x_1 through x_{k-1} the x_k coordinates of the roots of F_k are all also roots of F. Hence the convex hull for the *i-th* coordinate of the roots of F contains the *i-th* coordinate of the roots of ∂F for $1 \leq i \leq k$, and so by induction we are done. □

8.1.2 Separate Planes

In the previous section we considered a definition of relationship involving the convex hull. Now we will instead define relationship using a larger convex object. Specifically, we will say the roots of F and the roots of ∂F have a relationship if the smallest circle containing all of the *i-th* coordinate of the roots of F contains all of the *i-th* coordinate of the roots of ∂F for $1 \leq i \leq n$. As we are now dealing with circles we may compare magnitudes, and thus we can apply our results from Chapter 4.

Theorem 53 (Separate Planes)**.** *Let i be arbitrary. We have $\overline{R}_{F,i} \geq \overline{R}_{\partial F,i}$ if*

$$\frac{c_{i,0}^{d_i}}{10^{d_i}\, d_i} \geq \; c_{*,0}^{d_i - 1} \max_{\substack{1 \leq \mu \leq n \\ j \neq 0}} c_{\mu,j}$$

Remark 24. *The 10^{d_i} and d_i from Theorem 53 have no special meaning and arise as a result of strengthening the starting condition, $\overline{R}_{F,i} \geq \overline{R}_{\partial F,i}$, done in the proof for simplification.*

Example 49. *Let*

$$F_1 = x_1^2 + x_2 - 40000$$
$$F_2 = x_2^2 + x_1 + 40000$$

Then

$$d_1 = d_2 = 2$$

$$c_{1,0} = c_{2,0} = c_{*,0} = 200$$

$$\max_{\substack{1 \leq \mu \leq n \\ j \neq 0}} c_{\mu,j} = 1$$

Thus the condition is

$$\frac{200^2}{200} \overset{?}{\geq} 200 \cdot 1$$

Which simplifies into

$$200 \geq 200$$

Thus the condition is met. We would like to verify the theorem in this case. We present the following graph of the complex plane where the coordinates of the roots of F are black and the coordinates of the roots of ∂F are red.

The blue circle marks the smallest circle centered at the average of the i-th coordinate of the roots of F containing all of the i-th coordinate of the roots of F. It is clear in both cases that the red dot is contained inside the blue circle. Thus we have verified the theorem in this case.

Definition 15 (Triangular Pham system)**.** *We say that $F \in \mathbb{C}[x]^n$ is a* Triangular Pham system *if it has the following form*

$$\begin{aligned} F_1 &= x_1^{d_1} + h_1(x_1) \\ F_2 &= x_2^{d_2} + h_2(x_1, x_2) \\ &\vdots \\ F_n &= x_n^{d_n} + h_n(x_1, \ldots, x_n) \end{aligned}$$

where $\deg(h_i) < d_i$

Example 50.

$$\begin{aligned} F_1 &= x_1^2 + x_1 + 1 \\ F_2 &= x_2^3 - x_1\, x_2 + x_1 + 1 \\ F_3 &= x_3^3 + x_1\, x_3 + x_2\, x_3 + 1 \end{aligned}$$

Theorem 54. *Given a Triangular Pham system where $d_i \in \{1, 2, 3\}$ the i-th coordinate of the roots of ∂F are contained inside the smallest circle centered at the average value of the roots which contains all of the i-th coordinate of the roots of F.*

Example 51. *We would like to verify the theorem for the Triangular Pham system from Example 50. We present the following graph of the complex plane where the coordinates of the roots of F are black and the coordinates of the roots of ∂F are red:*

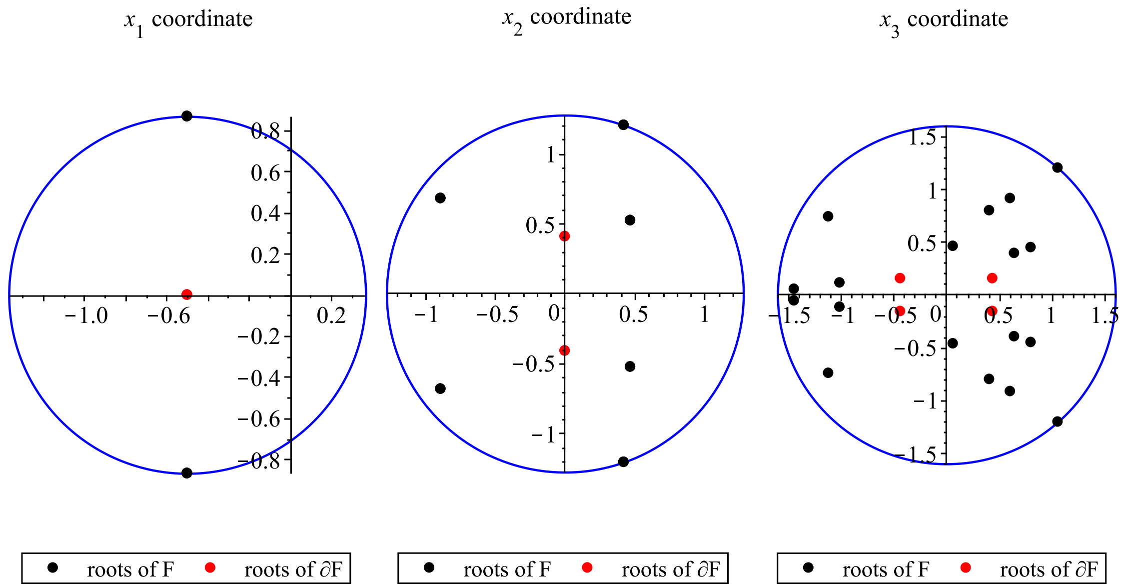


Figure 8.6: The coordinates of the roots of F (in black) their coordinate-wise maximum sizes (in blue), and the coordinates of the roots of ∂F (in red). In all cases the red dots are inside the blue circle.

It is clear that the red dots are contained inside the blue circle that marks the smallest circle centered at the average of the *i-th* coordinate of the roots of F containing all of the *i-th* coordinate of the roots of F. Thus we have verified the theorem in this case. Note that the x_1 and x_2 roots have multiplicity greater than 1. In fact, every x_1 root has multiplicity 9 and every x_2 root has multiplicity 3. This is a natural result of the triangular structure.

Proof. Note there is nothing to contain if $d_i = 1$ so it is vacuously true, and $d_i = 2$ has already been proven. Thus we only need to prove this for $d_i = 3$.

As the Pham system is triangular we have that F_1 is a univariate polynomial. Thus by Gauss-Lucas Theorem we have that the circle containing all of the x_1 roots of F must contain all of the x_1 roots of ∂F.

Suppose that the circle centered at the average value of the roots of F containing all of the *k-th* coordinate of the roots of F contains the *k-th* coordinate of the roots of ∂F for $1 \leq k \leq i-1$. WLOG let every coordinate of the roots of F be centered at 0. This will enable us to use larger magnitude as an equivalent for containment, simplifying notation.
Let the degree of $F_i = 3$.
Note $F_i(x_1, \ldots, x_n) = x_i^3 + h_i(x_1, \ldots, x_i) = F_i(x_1, \ldots, x_i)$.
Let $\overline{F_i}(x_i) = F_i(\alpha_{1,j}, \ldots, \alpha_{(i-1),j}, x_i)$ where $\alpha_{m,j}$ is the *m-th* coordinate of the *j-th* root of F. $\overline{F_i}$

is a univariate polynomial, and hence by Gauss-Lucas theorem the roots of $\frac{\partial \overline{F_i}}{\partial x_i}$ are contained in the convex hull of the roots of F. For every choice of m, we have

$$\overline{R}_{\overline{F_i}} \geq \overline{R}_{\frac{\partial \overline{F_i}}{\partial x_i}}$$

Thus we have containment of the union of the roots of $\frac{\partial \overline{F_i}}{\partial x_i}$ inside the convex hull of the union of the roots of $\overline{F_i}$ for all possible m's. The union of the convex hulls of the roots of $\overline{F_i}(x_i)$ is equivalent to the convex hull of the *i-th* coordinates of the roots of F. Hence,

$$\overline{R}_{F,i} = \overline{R}_{\overline{F_i}}$$

$d_i = 3$, thus the derivatives are degree 2 polynomials. For $1 \leq k \leq i-1$

$$\overline{R}_{F,k} \geq \overline{R}_{\partial F,k}$$

Thus, we have the constant term of some $\frac{\partial \overline{F_i}}{\partial x_i}$ is larger than the constant term of $\frac{\partial F_i}{\partial x_i}$ for all choices of x_1 through x_{i-1}, and thus that

$$\overline{R}_{\frac{\partial \overline{F_i}}{\partial x_i},i} \geq \overline{R}_{\frac{\partial F_i}{\partial x_i},i}$$

$\square$

Proposition 55. *Given a Triangular Pham system the i-th coordinate of the roots of F are not all contained inside the smallest circle centered at the average value of the roots of F which contains all of the i-th coordinate of the roots of ∂F.*

This is the equivalent of saying that when the system is translated to be centered at 0 the largest root of the original system is greater than or equal to the smallest root of the derivative system in every coordinate.

Proof. Let F be a Triangular Pham System. WLOG let the coordinates of the roots of F have average 0.

As the system is triangular we have that F_1 is a univariate polynomial, and hence the circle containing all of the x_1 roots of F must contain all of the x_1 roots of ∂F. Thus the smallest *i-th* coordinate of the roots of ∂F is not larger than $\overline{R}_{F,i}$.

Suppose that the circles centered at the origin containing all of the *k-th* coordinate of the roots of ∂F do not contain all of the *k-th* coordinate of the roots of F for $1 \leq k \leq i-1$.

Consider $F_i(x_1, \ldots, x_n) = x_i^{d_i} + h_i(x_1, \ldots, x_i) = F_i(x_1, \ldots, x_i)$.
and $\hat{F}_i(x_i) = F_i(\alpha_{1,j}, \ldots, \alpha_{(i-1),j}, x_i)$ where $\alpha_{\ell,j}$ is the *ℓ-th* coordinate of the *j-th* root of F.

Consider the roots of the derivative system of $\hat{F}$, $\partial\hat{F}$, $\beta_{1,j}$ through $\beta_{k,j}$. and the x_i roots of ∂F, $\gamma_{1,j}$ through $\gamma_{k,j}$
Define the constant term of $\partial\hat{F}$ to be $\hat{c}_j$ for each $\hat{F}$ and define the constant term of $\frac{\partial F_i}{\partial x_i}$ for after the *j-th* set of x_1 through x_{i-1} roots of F are input into F_i to be c_j. By the inductive assumption $\max(\hat{c}_j) \geq \min(c_j)$.
$\forall m$, let $B \geq \beta_{m,j}$, and $\gamma \leq \gamma_{m,j}$
By Vieta's formulas $-1^n \max(\hat{c}_j) = \prod_{m=1}^{n} \beta_{m,j} \leq B^n$ and $-1^n \min(c_j) = \prod_{m=1}^{n} \gamma_{m,j} \geq \gamma^n$
Thus as $\max(\hat{c}_j) \geq \min(c_j)$, $B \geq \gamma$. □

Conjecture 56. *The i-th coordinate of the roots of ∂F are contained inside the smallest circle centered at the average value of the roots of F which contains all of the i-th coordinate of the roots of F.*

Remark 25. *Initial testing was done with a mix of 200 thousand 2 variable degree 3 and degree 4 Pham systems where the coefficients were chosen using an identical independent uniform distribution. Later testing has included the following.*

- *100,000 2 variable degree 4 Pham systems where integer coefficients were chosen between -10 and 10 using an identical independent uniform distribution, except the constant coefficient which was defined to be zero. This case was a reasonable choice for searching for counterexamples as it is the case where the quality of the root size bounds for Pham system is not defined. This gives the roots less known structure.*

- *100,000 2 variable degree 4 Pham systems where the first system had coefficients chosen between -10,000 and 10,000 using an identical independent uniform distribution and the second system had coefficients chosen between -10 and 10 using an identical independent uniform distribution, except the constant coefficient in both cases, which was defined to be zero. Similar to the above case, this case also considered that different coefficient sizes between the equations can cause distortion favoring one variable.*

- *75,000 2 variable degree 3 Pham systems where the coefficients were chosen using a binomial distribution. In the univariate case this is known to cause the roots to cluster around the real line. This case was a reasonable choice for searching for counterexamples as ∂F would not have the same coefficient distribution, and so is likely to have a different structure.*

8.1.3 Same Plane

In the previous sections we considered a definition of relationship involving the convex hull and the smallest circle containing all of the *i-th* coordinates of the roots of F. Now we will define relationship to involve a similar convex object. We will say the roots of F and the roots of ∂F have a relationship if the smallest circle centered at the average containing the minimum of the infinity norms of the roots of F contains all of the coordinates of all of the roots of ∂F. This is another way of saying we are going to put the largest coordinate of each root of F and all of the coordinates of all of the roots of ∂F on the same graph, then compare the smallest result for F with the largest result for ∂F.

Theorem 57 (Same Plane)**.** *We have $\underline{R}_F \geq \overline{R}_{\partial F}$ if*

$$\forall_i \; c_{i,0} \geq 4 \max_{1 \leq k \leq d_i - 1} c_{i,k}$$

Example 52. *Let*

$$F_1 = x_1^3 + 5\; x_1 x_2 + 25\; x_1 + 8000$$
$$F_2 = x_2^3 - x_1 x_2 - x_1 - 64$$

It is easy to verify that the condition holds for this system. We would like to verify the theorem for this example. We present the following graph of the complex plane where all of the coordinates of the roots of F are black and all of the coordinates of the roots of ∂F are red.

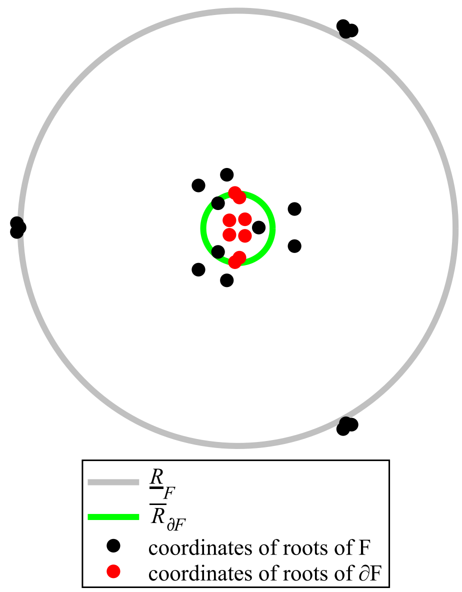


Figure 8.7: The coordinates of the roots of F (in black), $\underline{R}_F$ (in grey), and the coordinates of the roots of ∂F (in red), and $\overline{R}_{\partial F}$ (in green). All red dots are contained inside the grey circle, and there are black dots inside the green circle.

The grey circle is the circle centered at the origin, the average of the roots of F, of radius $\underline{R}_F$, as all of the red points are inside the circle Theorem 57 is verified for this example. Note that even though the grey circle is far outside the green circle, some of the coordinates of the roots of F are still inside the green circle, so the results are not separated.

Example 53. *Let*

$$F_1 = x_1^3 + 5\,x_1x_2 + 25\,x_1 + 8000$$
$$F_2 = x_2^3 - x_1x_2 - x_1 - 8000$$

It is easy to verify that the condition holds for this system. We present the following graph of the complex plane where all of the coordinates of all of the roots of F are black and all of the coordinates of all of the roots of ∂F are red.

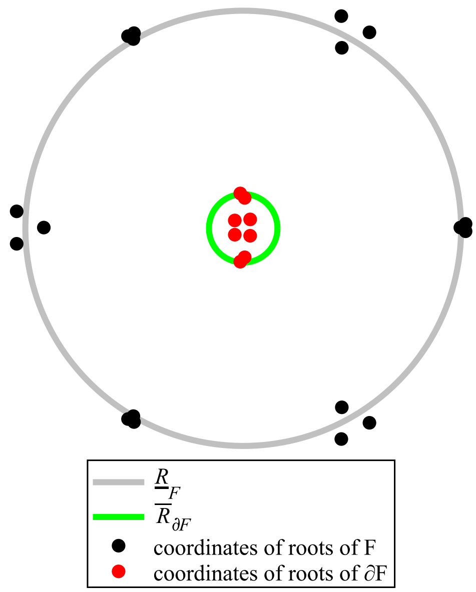


Figure 8.8: The coordinates of the roots of F (in black), $\underline{R}_F$ (in grey), and the coordinates of the roots of ∂F (in red), and $\overline{R}_{\partial F}$ (in green). All red dots are contained inside the grey circle, and there are no black dots inside the green circle.

The grey circle is the circle centered at the origin, the average of the roots of F, of radius $\underline{R}_F$ and the green circle is the circle centered at the origin with radius $\overline{R}_{\partial F}$.
Note in this example the coordinates of the roots are fully separated as the green circle contains none of the black roots.

Example 54. *Let*

$$F_1 = x_1^3 + 5\,x_1x_2 + 25\,x_1 + 4$$
$$F_2 = x_2^3 - x_1x_2 - x_1 - 4$$

It is easy to verify that the condition does not hold for this Pham system, graphing in the same way as in example 52 we will see $\underline{R}_F < \overline{R}_{\partial F}$, and hence we see that the C1 from 57 is not always true.

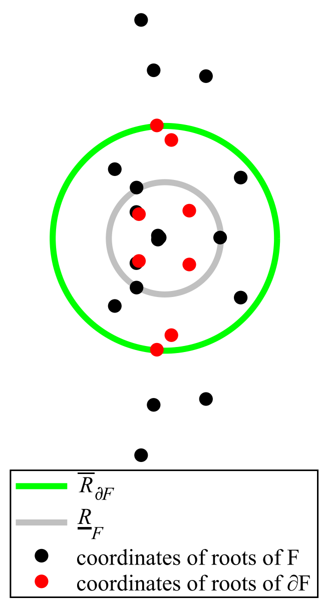


Figure 8.9: The coordinates of the roots of F (in black), $\underline{R}_F$ (in grey), and the coordinates of the roots of ∂F (in red), and $\overline{R}_{\partial F}$ (in green). All red dots are not contained inside the grey circle, so $\overline{R}_{\partial F}$ is not always greater than $\underline{R}_F$.

8.2 Proofs

For the proofs we will need the following lemmas.

Lemma 58. *Let $\partial F = (H_1, \ldots, H_n)$. Then we have*

$$H_i = x_i^{d_i - 1} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ |e| \leq d_i - 2}} \frac{e_i + 1}{d_i} a_{i,(e_1,\ldots,e_i+1,\ldots,e_n)}\, x^e$$

Proof. It is elementary, but error-prone. Thus we will go slowly step by step.

$$H_i = \frac{1}{d_i} \frac{\partial}{\partial x_i} F_i$$

By applying the definition of F_i, we have

$$H_i = \frac{1}{d_i} \left(\frac{\partial}{\partial x_i} x_i^{d_i} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ |e| \leq d_i - 1}} a_{i,(e)}\, \frac{\partial}{\partial x_i} x_1^{e_1} \cdots x_n^{e_n} \right)$$

By evaluating $\frac{\partial}{\partial x_i}$, we have

$$H_i = \frac{1}{d_i}\left(d_i\ x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ |e| \leq d_i - 1}} e_i\ a_{i,(e)}\ x_1^{e_1} \cdots x_i^{e_i-1} \cdots x_n^{e_n} \right)$$

Note $e_i \geq 1$ as we cannot have a negative exponent. Thus,

$$H_i = \frac{1}{d_i}\left(d_i\ x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ e_i \geq 1 \\ |e| \leq d_i - 1}} e_i\ a_{i,(e_1,\ldots,e_n)}\ x_1^{e_1} \cdots x_i^{e_i-1} \cdots x_n^{e_n} \right)$$

By distributing the $1/d_i$, we have

$$H_i = x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ e_i \geq 1 \\ e_1+\cdots+e_n \leq d_i - 1}} \frac{e_i}{d_i}\ a_{i,(e_1,\ldots,e_n)}\ x_1^{e_1} \cdots x_i^{e_i-1} \cdots x_n$$

Now reindex our sum such that $e_i \to e_i' + 1$.

$$H_i = x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_i'+1,\ldots,e_n \geq 0 \\ e_i'+1 \geq 1 \\ e_1+\cdots+e_i'+1+\cdots+e_n \leq d_i - 1}} \frac{e_i'+1}{d_i}\ a_{i,(e_1,\ldots,e_i'+1,\ldots,e_n)}\ x_1^{e_1} \cdots x_i^{e_i'} \cdots x_n^{e_n}$$

As i is no longer used, let $i' = i$. Thus,

$$H_i = x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_i+1,\ldots,e_n \geq 0 \\ e_i+1 \geq 1 \\ e_1+\cdots+e_i+1+\cdots+e_n \leq d_i - 1}} \frac{e_i+1}{d_i}\ a_{i,(e_1,\ldots,e_i+1,\ldots,e_n)}\ x_1^{e_1} \cdots x_i^{e_i} \cdots x_n^{e_n}$$

By simplifying the sum indices, we have

$$H_i = x_i^{d_i-1} + \sum_{\substack{e_1,\ldots,e_i+1,\ldots,e_n \geq 0 \\ e_i \geq 0 \\ e_1+\cdots+e_i+\cdots+e_n \leq d_i - 2}} \frac{e_i+1}{d_i}\ a_{i,(e_1,\ldots,e_i+1,\ldots,e_n)}\ x_1^{e_1} \cdots x_n^{e_n}$$

Finally, remove the redundant $e_i + 1 \geq 0$ and simplify.

$$H_i = x_i^{d_i - 1} + \sum_{\substack{e_1,\ldots,e_n \geq 0 \\ |e| \leq d_i - 2}} \frac{e_i + 1}{d_i} \; a_{i,(e_1,\ldots,e_i+1,\ldots,e_n)} \; x_1^{e_1} \cdots x_n^{e_n}$$

$\square$

Lemma 59. *We have*

$$U_{\partial F}^{I} \leq 2 \max_{1 \leq i \leq n} \max_{1 \leq k \leq d_i - 1} \left(\sum_{\substack{e_1,\ldots,e_n \geq 0 \\ e_1 + \cdots + e_n = k}} \frac{e_i}{d_i} \; \left| a_{i,(e_1,\ldots,e_n)} \right| \right)^{\frac{1}{d_i - k}}$$

Proof. By definition, we have

$$U_{\partial F}^{I} \; = \; \max_{1 \leq i \leq n} \left(\sum_{j=1}^{d_i - 1} \frac{t_{i,j} - t_{i,j+1}}{q_j} \right)$$

where $t_{i,1} \geq \cdots \geq t_{i,d_i - 1}$ are in decreasing order among $\frac{e_i}{d_i} c_{i,j}$ for $1 \leq j < d_i - 1$. Unfortunately this cannot be easily worked with as written, so we shall approximate. Note

$$U_{\partial F}^{I} \leq U_{\partial F}^{A}$$

Then by Theorem 33 and Lemma 58, we have

$$U_{\partial F}^{I} \; \leq \; 2 \max_{1 \leq i \leq n} \max_{0 \leq k \leq d_i - 2} \left(\sum_{\substack{e_1,\ldots,e_n \geq 0 \\ e_1 + \cdots + e_n = k}} \frac{e_i + 1}{d_i} \; \left| a_{i,(e_1,\ldots,e_i+1,\ldots,e_n)} \right| \right)^{\frac{1}{d_i - 1 - k}}$$

Reindex such that $e_i + 1 \rightarrow e_i'$.

$$= \; 2 \max_{1 \leq i \leq n} \max_{0 \leq k \leq d_i - 2} \left(\sum_{\substack{e_1,\ldots,e_i' - 1,\ldots,e_n \geq 0 \\ e_1 + \cdots + e_i' - 1 + \cdots + e_n = k}} \frac{e_i'}{d_i} \; \left| a_{i,(e_1,\ldots,e_i',\ldots,e_n)} \right| \right)^{\frac{1}{d_i - 1 - k}}$$

Rename the dummy symbol e_i' with e_i.

$$= 2 \max_{1\le i\le n} \max_{0\le k\le d_i-2} \left(\sum_{\substack{e_1,\ldots,e_i-1,\ldots,e_n\ge 0\\ e_1+\cdots+e_i-1+\cdots+e_n=k}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-1-k}}$$

By rearranging the sum index and the exponent, we have

$$= 2 \max_{1\le i\le n} \max_{0\le k\le d_i-2} \left(\sum_{\substack{e_1,\ldots,e_i-1,\ldots,e_n\ge 0\\ e_1+\cdots+e_n=k+1}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-(k+1)}}$$

Reindex $k+1$ with $k^{'}$.

$$= 2 \max_{1\le i\le n} \max_{0\le k^{'}-1\le d_i-2} \left(\sum_{\substack{e_1,\ldots,e_i-1,\ldots,e_n\ge 0\\ e_1+\cdots+e_n=k^{'}}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-k^{'}}}$$

Rearrange the sum index.

$$= 2 \max_{1\le i\le n} \max_{1\le k^{'}\le d_i-1} \left(\sum_{\substack{e_1,\ldots,e_i-1,\ldots,e_n\ge 0\\ e_1+\cdots+e_n=k^{'}}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-k^{'}}}$$

By renaming the dummy symbol k' with k, we have

$$= 2 \max_{1\le i\le n} \max_{1\le k\le d_i-1} \left(\sum_{\substack{e_1,\ldots,e_i-1,\ldots,e_n\ge 0\\ e_1+\cdots+e_n=k}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-k}}$$

Since $\frac{0}{d_i} = 0$, we have

$$= 2 \max_{1\le i\le n} \max_{1\le k\le d_i-1} \left(\sum_{\substack{e_1,\ldots,e_i,\ldots,e_n\ge 0\\ e_1+\cdots+e_n=k}} \frac{e_i}{d_i} \left|a_{i,(e_1,\ldots,e_n)}\right| \right)^{\frac{1}{d_i-k}}$$

Simplify the presentation.

$$= \;\; 2 \max_{1\le i\le n} \; \max_{1\le k\le d_i-1} \left(\sum_{\substack{e_1,\dots,e_n\ge 0 \\ e_1+\cdots+e_n=k}} \frac{e_i}{d_i} \; \left|a_{i,(e_1,\dots,e_n)}\right| \right)^{\frac{1}{d_i-k}}$$

□

8.2.1 Averages of the Roots of Pham Systems and Their Derivative Systems

Proof. Note that ∂F is also a Pham system. Let $\delta = \{d_1, ..., d_n\}$ be the index of the degrees of the F_i's where each d_i is the degree of the polynomial F_i. Further let $p_i = (0,...,0,1,0,...,0)$ where the 1 is in the i-th place, e^i a multi-index of the degrees of the x_j's in the polynomial F_i and $a_{i,(e^i)}$ are the coefficients of the term with degree index e_i in the i-th equation, while $b_{i(e^i)}$ are the coefficients of the term of ∂F with degree index e_i. Recall $\Gamma_{[e^1,...,e^n]}$ denotes the determinant

$$\Gamma_{[e^1,\dots,e^n]} = \begin{vmatrix} e_1^1 & \dots & e_n^1 \\ \vdots & & \vdots \\ e_1^n & \dots & e_n^n \end{vmatrix}$$

where the superscripts represent which F_i you are considering and the subscript which x_i variable you are finding the exponent of. Then by [22] we know that the sum $T_{1,i}$ of the first power of the i-th coordinate of the roots of the Pham system can be written as

$$\begin{aligned} T_{1,i} &= \sum_{e^1+\ldots+e^n=\delta-p_i} (\Gamma_{[e^1,\dots,e^n]} - \prod_{k=1}^{n} d_k) \cdot a_{1,(e^1)} \cdots a_{n,(e^n)} \\ &= ((d_i-1)\prod_{k\neq i} d_k - \prod_{k=1}^{n} d_k) \cdot a_{1,(d_1p_1)} \cdots a_{i,((d_i-1)p_i)} \cdots a_{n,(d_np_n)} \\ &= -a_{i,(0\ldots(d_i-1)\ldots0)} \prod_{k\neq i} d_k \end{aligned}$$

because $e^1 + \cdots + e^n = \delta - p_i$ is equivalent to $e^j = d_jp_j, j \neq i$, and $e^i = (d_i-1)p_i$ and all coefficients $a_{j,(d_jp_j)}$ are 1 by definition. Similarly the sum $T'_{1,i}$ of the i-th coordinate ∂F is

$$T'_{1,i} = \frac{-(d_i-1)a_{i,(0,\dots,(d_i-1),\dots,0)} \prod_{k\neq i}(d_k-1)}{d_i}$$

Thus the sum of the roots for ∂F is

$$T'_{1,i} = \frac{-(d_i - 1)a_{i,(0,\ldots,(d_i-1),\ldots,0)} \prod_{k\neq i}(d_k - 1)}{d_i} = \frac{-a_{i,(0,\ldots,(d_i-1),\ldots,0)} \prod_{k=1}^{n}(d_k - 1)}{d_i}$$

The number of roots of a Pham system is known to be $\prod_{k=1}^{n} d_k$ thus the average of the i-th coordinate of the roots of F is

$$\frac{T_{1,i}}{\prod_{k=1}^{n} d_k} = \frac{-a_{i,(0,\ldots,(d_i-1),\ldots,0)} \prod_{k\neq i}(d_k)}{\prod_{k=1}^{n} d_k} = \frac{-a_{i,(0,\ldots,(d_i-1),\ldots,0)}}{d_i}$$

Similarly the average of the i-th coordinate of the roots of ∂F is

$$\frac{T'_{1,i}}{\prod_{k=1}^{n}(d_k - 1)} = \frac{-a_{i,(0,\ldots,(d_i-1),\ldots,0)} \prod_{k=1}^{n}(d_k - 1)}{d_i \prod_{k=1}^{n}(d_k - 1)} = \frac{-a_{i,(0,\ldots,(d_i-1),\ldots,0)}}{d_i}$$

$$\implies \frac{T_{1,i}}{\prod_{k=1}^{n} d_k} = \frac{T'_{1,i}}{\prod_{k=1}^{n}(d_k - 1)}$$

As this is true for every such i, the average of the roots is the same. □

8.2.2 Separate Planes

Lemma 60. *Given a Pham system F then $\overline{R}_{F,i} \geq \overline{R}_{\partial F,i}$ if $U_F^I \leq 1$ and $\Xi \leq \frac{1}{d_i}\left(\frac{c_{i,0}}{5}\right)^{d_i}$ where* $\Xi = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq d_i-1}} c_{i,j}$

Proof of Lemma 60. This proof is long as a sacrifice to answer natural questions about where this condition arises from. We will prove Lemma 60 by repeated rewriting and strengthening (emphasized in red color) the condition $\overline{R}_{F,i} \geq \overline{R}_{\partial F,i}$. Let us begin!

$\overline{R}_{F,i} \geq \overline{R}_{\partial F,i}$

Using our bounds from Lemma 41 and Theorem 31, we have

$\Longleftarrow \quad L_{F,i}^I \geq U_{\partial F,i}^I$

Strengthen the condition further by noting $L_{F,i}^A \leq L_{F,i}^I$. Thus,

$\Longleftarrow \quad L_{F,i}^I \geq U_{\partial F,i}^I$

From Lemma 43 and Lemma 59, we have

$$\iff \frac{1}{2}\min_{1\le k\le d_i}\left(\frac{b_{i,0}-\sum\limits_{1\le j\le d_i-1} b_{i,j}}{1+\sum\limits_{1\le j\le d_i-1} b_{i,j}}\right)^{\frac{1}{k}} \ge 2 \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} \left(\sum_{\substack{e_1,\ldots,e_n\ge 0\\ |e|=\ell}} \frac{e_\mu}{d_\mu}\ \left|a_{\mu,(e)}\right|\right)^{\frac{1}{d_\mu-\ell}}$$

Since $\frac{e_i}{d_i}<1$ we may drop it to strengthen the condition. Thus,

$$\Longleftarrow \frac{1}{2}\min_{1\le k\le d_i}\left(\frac{b_{i,0}-\sum\limits_{1\le j\le d_i-1} b_{i,j}}{1+\sum\limits_{1\le j\le d_i-1} b_{i,j}}\right)^{\frac{1}{k}} \ge 2 \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} \left(\sum_{\substack{e_1,\ldots,e_n\ge 0\\ |e|=\ell}} \left|a_{\mu,(e)}\right|\right)^{\frac{1}{d_\mu-\ell}}$$

Now we can rewrite RHS in terms of $b_{\mu,\ell}$ and move the 2 over. Thus,

$$\iff \min_{1\le k\le d_i}\left(\frac{b_{i,0}-\sum\limits_{1\le j\le d_i-1} b_{i,j}}{1+\sum\limits_{1\le j\le d_i-1} b_{i,j}}\right)^{\frac{1}{k}} \ge 4 \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{1}{d_\mu-\ell}}$$

Now rewrite min as a for all statement to get the following.

$$\iff \forall_{1\le k\le d_i}\left(\frac{b_{i,0}-\sum\limits_{1\le j\le d_i-1} b_{i,j}}{1+\sum\limits_{1\le j\le d_i-1} b_{i,j}}\right)^{\frac{1}{k}} \ge 4 \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{1}{d_\mu-\ell}}$$

By moving the denominator to the RHS, we have

$$\iff \forall_{1\le k\le d_i}\left(b_{i,0}-\sum_{1\le j\le d_i-1} b_{i,j}\right)^{\frac{1}{k}} \ge 4 \left(1+\sum_{1\le j\le d_i-1} b_{i,j}\right)^{\frac{1}{k}} \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{1}{d_\mu-\ell}}$$

By raising everything to the k, we have

$$\iff \forall_{1\le k\le d_i}\left(b_{i,0}-\sum_{1\le j\le d_i-1} b_{i,j}\right) \ge 4^k \left(1+\sum_{1\le j\le d_i-1} b_{i,j}\right) \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{k}{d_\mu-\ell}}$$

We want $b_{i,0}$ alone on the RHS. Thus, we move $\sum\limits_{1\le j\le d_i-1} b_{i,j}$ to the RHS.

$$\iff \forall_{1\le k\le d_i} b_{i,0} \ge \sum_{1\le j\le d_i-1} b_{i,j} + 4^k \left(1+\sum_{1\le j\le d_i-1} b_{i,j}\right) \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{k}{d_\mu-\ell}}$$

By rewriting $\forall_k$ as a max, we have

$$\iff b_{i,0} \ge \sum_{1\le j\le d_i-1} b_{i,j} + \left(1+\sum_{1\le j\le d_i-1} b_{i,j}\right) \max_{1\le k\le d_i}\left(4^k \max_{\substack{1\le\mu\le n\\ 1\le\ell\le d_\mu-1}} b_{\mu,\ell}^{\frac{k}{d_\mu-\ell}}\right)$$

As $U_F^I \leq 1$ we have $c_{i,j} \leq 1$ thus we have $b_{i,j} \leq 1$. Thus,

$$\Longleftarrow \quad b_{i,0} \geq \sum_{1 \leq j \leq d_i - 1} b_{i,j} + \left(1 + \sum_{1 \leq j \leq d_i - 1} 1 \right) \max_{1 \leq k \leq d_i} \left(4^k \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} b_{\mu,\ell}^{\frac{k}{d_\mu - \ell}} \right)$$

By simplifying, we have

$$\Longleftrightarrow \quad b_{i,0} \geq \sum_{1 \leq j \leq d_i - 1} b_{i,j} + d_i \max_{1 \leq k \leq d_i} \left(4^k \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} b_{\mu,\ell}^{\frac{k}{d_\mu - \ell}} \right)$$

By splitting the index space for k, we have

$$\Longleftrightarrow \quad b_{i,0} \geq \sum_{1 \leq j \leq d_i - 1} b_{i,j} + d_i \max \left\{ \max_{1 \leq k \leq d_i - 1} \left(4^k \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} b_{\mu,\ell}^{\frac{k}{d_\mu - \ell}} \right), 4^{d_i} \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} b_{\mu,\ell}^{\frac{d_i}{d_\mu - \ell}} \right\}$$

Replace b_* with c_*.

$$\Longleftrightarrow \quad c_{i,0}^{d_i} \geq \sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} + d_i \max \left\{ \max_{1 \leq k \leq d_i - 1} \left(4 \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} c_{\mu,\ell} \right)^k, \left(4 \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} c_{\mu,\ell} \right)^{d_i} \right\}$$

Let $\Xi = \max_{\substack{1 \leq \mu \leq n \\ 1 \leq \ell \leq d_\mu - 1}} c_{\mu,\ell}$, then we have

$$\Longleftrightarrow \quad c_{i,0}^{d_i} \geq \sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} + d_i \max \left\{ \max_{1 \leq k \leq d_i - 1} \left(4^k \; \Xi^k \right), 4^{d_i} \; \Xi^{d_i} \right\}$$

By taking the d_i-th root of both sides, we have

$$\Longleftrightarrow \quad c_{i,0} \geq \left(\sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} + d_i \max \left\{ \max_{1 \leq k \leq d_i - 1} \left(4^k \;\; \Xi^k \right), 4^{d_i} \; \Xi^{d_i} \right\} \right)^{\frac{1}{d_i}}$$

By applying the triangle inequality, we have

$$\Longleftarrow \quad c_{i,0} \geq \left(\sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} \right)^{\frac{1}{d_i}} + d_i^{\frac{1}{d_i}} \max \left\{ \max_{1 \leq k \leq d_i - 1} 4^{\frac{k}{d_i}} \;\; \Xi^{\frac{k}{d_i}}, 4 \; \Xi \right\}$$

Since $\dfrac{k}{d_i} < 1$ we may drop it from the exponent of the 4.

$$\Longleftarrow \quad c_{i,0} \geq \left(\sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} \right)^{\frac{1}{d_i}} + 4 \; d_i^{\frac{1}{d_i}} \max \left\{ \max_{1 \leq k \leq d_i - 1} \;\; \Xi^{\frac{k}{d_i}}, \Xi \right\}$$

Recalling $U_{I,F} \leq 1$ we have $c_{\mu,\ell} < 1$. Thus,

$$\Longleftrightarrow \quad c_{i,0} \geq \left(\sum_{1 \leq j \leq d_i - 1} c_{i,j}^{d_i - j} \right)^{\frac{1}{d_i}} + 4 \; d_i^{\frac{1}{d_i}} \max \left\{ \max_{1 \leq k \leq d_i - 1} \Xi^{\frac{1}{d_i}}, \Xi \right\}$$

As the max across k is unused we may drop it. Thus,

$$\iff \quad c_{i,0} \geq \left(\sum_{1\leq j\leq d_i-1} c_{i,j}^{d_i-j}\right)^{\frac{1}{d_i}} + 4\, d_i^{\frac{1}{d_i}} \max\left\{\Xi^{\frac{1}{d_i}}, \Xi\right\}$$

As $\Xi^{\frac{1}{d_i}} \geq \Xi$, we have

$$\iff \quad c_{i,0} \geq \left(\sum_{1\leq j\leq d_i-1} c_{i,j}^{d_i-j}\right)^{\frac{1}{d_i}} + 4\, d_i^{\frac{1}{d_i}}\, \Xi^{\frac{1}{d_i}}$$

Recalling $c_{i,j} < 1$, we have

$$\Leftarrow \quad c_{i,0} \geq \left(\sum_{1\leq j\leq d_i-1} c_{i,j}\right)^{\frac{1}{d_i}} + 4\ (d_i\ \Xi)^{\frac{1}{d_i}}$$

Take the largest possible c in the first sum, Ξ, we have,

$$\Leftarrow \quad c_{i,0} \geq \left(\sum_{1\leq j\leq d_i-1} \Xi\right)^{\frac{1}{d_i}} + 4\ (d_i\ \Xi)^{\frac{1}{d_i}}$$

We may now evaluate the sum and increase the multiplier by 1 to d_i.

$$\Leftarrow \quad c_{i,0} \geq\ (d_i\ \Xi)^{\frac{1}{d_i}} + 4\ (d_i\ \Xi)^{\frac{1}{d_i}}$$

By combining like terms, we have

$$\iff \quad c_{i,0} \geq\ 5\ (d_i \Xi)^{\frac{1}{d_i}}$$

Solve for Ξ.

$$\iff \quad \frac{1}{d_i}\left(\frac{c_{i,0}}{5}\right)^{d_i} \geq \Xi$$

□

Proof of Theorem 53. Let F be a Pham system and $t = \frac{1}{2c_{*,0}}$. We can apply Lemma 60 to F by scaling by t when $t\ \Xi \leq \frac{1}{d_i}\left(\frac{t\ c_{i,0}}{5}\right)^{d_i}$ holds. Thus we will prove Theorem 53 by repeated rewriting and strengthening (emphasized in red color) the condition $t\ \Xi \leq \frac{1}{d_i}\left(\frac{t\ c_{i,0}}{5}\right)^{d_i}$.

$$t\ \Xi \leq\ \frac{1}{d_i}\left(\frac{t\ c_{i,0}}{5}\right)^{d_i}$$

We begin by getting all of the t's on the RHS.

$$\iff \quad \Xi \leq\ \frac{1}{d_i}\left(\frac{c_{i,0}}{5}\right)^{d_i} t^{d_i-1}$$

By recalling $t = \dfrac{1}{2c_{*,0}}$, we have

$$\iff \quad \Xi \leq \frac{1}{d_i} \left(\frac{c_{i0}}{5}\right)^{d_i} \frac{1}{2^{d_i-1} c_{*,0}^{d_i-1}}$$

By combining the 2^{d_i-1} and the 5^{d_i}, we have

$$\iff \quad \Xi \leq \frac{1}{d_i} \left(\frac{c_{i,0}}{10}\right)^{d_i-1} \left(\frac{c_{i,0}}{5}\right) \frac{1}{c_{*,0}^{d_i-1}}$$

Rearrange by combining the $1/5$ and $1/d_i$.

$$\iff \quad \Xi \leq \frac{c_{i,0}}{5\ d_i} \left(\frac{c_{i,0}}{10}\right)^{d_i-1} \frac{1}{c_{*,0}^{d_i-1}}$$

Since $\zeta_{i,0} \geq 1$ and $1 < 5 < 10$, we have

$$\iff \quad \Xi \leq \frac{1}{d_i} \left(\frac{c_{i,0}}{10}\right)^{d_i} \frac{1}{c_{*,0}^{d_i-1}}$$

By moving $c_{*,0}^{d_i-1}$ to the LHS, we have

$$\iff \quad c_{*,0}^{d_i-1}\ \Xi \leq \frac{1}{d_i} \left(\frac{c_{i,0}}{10}\right)^{d_i}$$

By recalling $\Xi = \max\limits_{\substack{1 \leq \mu \leq n \\ j \neq 0}} c_{\mu,j}$, we have

$$\iff \quad c_{*,0}^{d_i-1} \max_{\substack{1 \leq \mu \leq n \\ j \neq 0}} c_{\mu,j} \leq \frac{1}{d_i} \left(\frac{c_{i,0}}{10}\right)^{d_i}$$

Rewrite.

$$\iff \quad \frac{c_{i,0}^{d_i}}{10^{d_i}\ d_i} \geq c_{*,0}^{d_i-1} \max_{\substack{1 \leq \mu \leq n \\ j \neq 0}} c_{\mu,j}$$

$\square$

8.2.3 Same Plane

Proof of Main result (Theorem 57). We will derive it by repeated rewriting and strengthening (emphasized in red color) the condition $\underline{R}_F \geq \overline{R}_{\partial F}$. Let us begin!

Using our bounds from Lemma 41 and Theorem 31, we have

$$\Longleftarrow \quad L_F^I \geq U_{\partial F}^I$$

Strengthen the condition further by noting $L_F^A \leq L_F^I$. Thus,

$$\Longleftarrow \quad L_F^A \geq U_{\partial F}^I$$

Thus, from Lemma 43 and Lemma 59 we have the following.

$$\iff \quad \frac{1}{2} \max_{1\le i\le n} \min_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; 2 \max_{1\le i\le n} \max_{1\le k\le d_i-1} \left(\sum_{\substack{e_1,\ldots,e_n\ge 0\\ |e|=k}} \frac{e_i}{d_i} \left|a_{i,(e)}\right| \right)^{\frac{1}{d_i-k}}$$

Since $\dfrac{e_i}{d_i} < 1$ we may drop it to strengthen the condition. Thus,

$$\Leftarrow \quad \frac{1}{2} \max_{1\le i\le n} \min_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; 2 \max_{1\le i\le n} \max_{1\le k\le d_i-1} \left(\sum_{\substack{e_1,\ldots,e_n\ge 0\\ |e|=k}} \left|a_{i,(e)}\right| \right)^{\frac{1}{d_i-k}}$$

By rewriting the RHS in terms of $b_{i,k}$, we have

$$\iff \quad \frac{1}{2} \max_{1\le i\le n} \min_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; 2 \max_{1\le i\le n} \max_{1\le k\le d_i-1} b_{i,k}^{\frac{1}{d_i-k}}$$

Multiply by 2 and pull the resulting 4 inside.

$$\iff \quad \max_{1\le i\le n} \min_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; \max_{1\le i\le n} \max_{1\le k\le d_i-1} 4b_{i,k}^{\frac{1}{d_i-k}}$$

Strengthen the condition by making it true for all i instead of just for the maximum.

$$\Leftarrow \quad \forall_{i} \min_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; \max_{1\le k\le d_i-1} 4b_{i,k}^{\frac{1}{d_i-k}}$$

Rewrite min as a for all statement to get the following.

$$\iff \quad \forall_{i} \forall_{1\le j\le d_i} \left(\frac{b_{i,0}}{b_{i,j}}\right)^{\frac{1}{j}} \;\ge\; \max_{1\le k\le d_i-1} 4b_{i,k}^{\frac{1}{d_i-k}}$$

By raising everything to the j and moving $b_{i,j}$ to the RHS, we have

$$\iff \quad \forall_{i} \forall_{1\le j\le d_i} b_{i,0} \;\ge\; \max_{1\le k\le d_i-1} 4^j b_{i,j}\, b_{i,k}^{\frac{j}{d_i-k}}$$

As the LHS has no js we may rewrite the condition.

$$\iff \quad \forall_{i} b_{i,0} \;\ge\; \max_{\substack{1\le j\le d_i\\ 1\le k\le d_i-1}} 4^j b_{i,j}\, b_{i,k}^{\frac{j}{d_i-k}}$$

Split the index space for j on the RHS.

$$\iff \quad \forall_{i} b_{i,0} \;\ge\; \max\left\{ \max_{\substack{1\le j\le d_i-1\\ 1\le k\le d_i-1}} 4^j b_{i,j}\, b_{i,k}^{\frac{j}{d_i-k}},\; \max_{\substack{j=d_i\\ 1\le k\le d_i-1}} 4^j b_{i,j}\, b_{i,k}^{\frac{j}{d_i-k}} \right\}$$

Simplify,

$$\iff \forall_i\; b_{i,0} \;\geq\; \max\left\{ \max_{\substack{1\leq j\leq d_i-1\\1\leq k\leq d_i-1}} 4^j b_{i,j}\; b_{i,k}^{\frac{j}{d_i-k}},\; \max_{1\leq k\leq d_i-1} 4^{d_i} b_{i,d_i}\; b_{i,k}^{\frac{d_i}{d_i-k}} \right\}$$

By recalling $b_{i,d_i} = 1$, we have

$$\iff \forall_i\; b_{i,0} \;\geq\; \max\left\{ \max_{\substack{1\leq j\leq d_i-1\\1\leq k\leq d_i-1}} 4^j b_{i,j}\; b_{i,k}^{\frac{j}{d_i-k}},\; \max_{1\leq k\leq d_i-1} 4^{d_i} b_{i,k}^{\frac{d_i}{d_i-k}} \right\}$$

Write everything in terms of $c_{i,*}$.

$$\iff \forall_i\; c_{i,0}^{d_i} \;\geq\; \max\left\{ \max_{\substack{1\leq j\leq d_i-1\\1\leq k\leq d_i-1}} 4^j c_{i,j}^{d_i-j} c_{i,k}^{j},\; \max_{1\leq k\leq d_i-1} 4^{d_i} c_{i,k}^{d_i} \right\}$$

Take the d_i-th root of both sides.

$$\iff \forall_i\; c_{i,0} \;\geq\; \max\left\{ \max_{\substack{1\leq j\leq d_i-1\\1\leq k\leq d_i-1}} 4^{\frac{j}{d_i}} c_{i,j}^{1-\frac{j}{d_i}} c_{i,k}^{\frac{j}{d_i}},\; \max_{1\leq k\leq d_i-1} 4 c_{i,k} \right\}$$

Since $\dfrac{j}{d_i} \leq 1$ we may drop it from $4^{\frac{j}{d_i}}$ to strengthen the condition. Thus,

$$\Longleftarrow \forall_i\; c_{i,0} \;\geq\; 4\max\left\{ \max_{\substack{1\leq j\leq d_i-1\\1\leq k\leq d_i-1}} c_{i,j}^{1-\frac{j}{d_i}} c_{i,k}^{\frac{j}{d_i}},\; \max_{1\leq k\leq d_i-1} c_{i,k} \right\}$$

Separate the maximums and pull $c_{i,j}$ out from the max across k.

$$\iff \forall_i\; c_{i,0} \;\geq\; 4\max\left\{ \max_{1\leq j\leq d_i-1} c_{i,j} \left(\max_{1\leq k\leq d_i-1} \frac{c_{i,k}}{c_{i,j}} \right)^{\frac{j}{d_i}},\; \max_{1\leq k\leq d_i-1} c_{i,k} \right\}$$

As k and j run across the same indices $\max \dfrac{c_{i,k}}{c_{i,j}} \geq 1$. Thus,

$$\Longleftarrow \forall_i\; c_{i,0} \;\geq\; 4\max\left\{ \max_{1\leq j\leq d_i-1} c_{i,j} \max_{1\leq k\leq d_i-1} \frac{c_{i,k}}{c_{i,j}},\; \max_{1\leq k\leq d_i-1} c_{i,k} \right\}$$

Simplify,

$$\iff \forall_i\; c_{i,0} \;\geq\; 4\max\left\{ \max_{1\leq j\leq d_i-1} \max_{1\leq k\leq d_i-1} c_{i,k},\; \max_{1\leq k\leq d_i-1} c_{i,k} \right\}$$

Max across j is no longer being used, so we may drop it.

$$\iff \forall_i\; c_{i,0} \;\geq\; 4\max\left\{ \max_{1\leq k\leq d_i-1} c_{i,k},\; \max_{1\leq k\leq d_i-1} c_{i,k} \right\}$$

Maximums are identical, so we may merge them.

$$\iff \forall_i\; c_{i,0} \;\geq\; 4 \max_{1\leq k\leq d_i-1} c_{i,k}$$

$\square$

In summary, Gauss-Lucas theorem for univariate polynomials is well known. Generalizing into Pham systems first required defining derivative. The chosen definition is also a Pham system, along with several other valuable properties. We then looked at several different ways to relate the roots of the derivative with the roots of the original Pham system, to find an analog for Gauss-Lucas theorem. A general conjecture was given and proven for a certain family of Pham systems.

BIBLIOGRAPHY

[1] Alexander Abian. An ultimate proof of Rolle's theorem. *The American Mathematical Monthly*, 86(6):484–485, 1979.

[2] L. A. Aizenberg and A. M. Kytmanov. Multidimensional analogues of Newton's formulas for systems of nonlinear algebraic equations and some of their applications. *Siberian Mathematical Journal*, 22(2):180–189, 1981.

[3] Prashant Batra. On the quality of some root-bounds. In *Revised Selected Papers of the 6th International Conference on Mathematical Aspects of Computer and Information Sciences - Volume 9582*, MACIS 2015, pages 591–595, Berlin, Heidelberg, 2015. Springer-Verlag.

[4] Prashant Batra, Maurice Mignotte, and Doru Stefanescu. Improvements of Lagrange's bound for polynomial roots. *Journal of Symbolic Computation*, 82:19 – 25, 2017.

[5] M. Bidkham and E. Shashahani. An annulus for the zeros of polynomials. *Applied Mathematics Letters*, 24(2):122 – 125, 2011.

[6] Mahmood Bidkham and Sara Ahmadi. Generalization of the GaussLucas theorem for bicomplex polynomials. *Turkish Journal of Mathematics*, 41(6):1618 – 1627, 2017.

[7] Emmanuel Briand and Laureano Gonzalez-Vega. Multivariate Newton sums: identities and generating functions. *Communications in Algebra*, 30(9):4527–4547, 2002.

[8] John F. Canny. *The complexity of robot motion planning*. PhD thesis, M.I.T., 1988.

[9] Eduardo Cattani, Alicia Dickenstein, and Bernd Sturmfels. Computing multidimensional residues. In *Algorithms in algebraic geometry and applications*, pages 135–164. Springer, 1996.

[10] Augustin-Louis Cauchy. Sur la résolution des équations numériques et sur la théorie de lélimination. *Oeuvres Completes, Ser*, 2:87–161, 1829.

[11] Augustin Louis Baron Cauchy. *Exercises de mathématiques*, volume 3. Bure frères, 1828.

[12] Thomas Craven and George Csordas. The Gauss-Lucas theorem and Jensen polynomials. *Transactions of the American Mathematical Society*, 278(1):415–429, 1983.

[13] Matthias Dehmer and Yury Robertovich Tsoy. The quality of zero bounds for complex polynomials. *PLoS ONE*, 7(7):e39537, jul 2012.

[14] J. L. Díaz-Barrero and J. J. Egozcue. A generalization of the Gauss-Lucas theorem. *Czechoslovak Mathematical Journal*, 58(2):481–486, 2008.

[15] Ezequiel Dratman, Guillermo Matera, and Ariel Waissbein. Robust algorithms for generalized Pham systems. *computational complexity*, 18(1):105–154, 2009.

[16] J. L. Daz-Barrero and J. J. Egozcue. Bounds for the moduli of zeros. *Applied Mathematics Letters*, 17(8):993 – 996, 2004.

[17] Ioannis Z. Emiris, Bernard Mourrain, and Elias P. Tsigaridas. The DMM bound: multivariate (aggregate) separation bounds. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 243–250, 2010.

[18] J.-Cl. Evard and F. Jafari. A complex Rolle's theorem. *The American Mathematical Monthly*, 99(9):858–861, 1992.

[19] Matsusaburô Fujiwara. Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. *Tohoku Mathematical Journal, First Series*, 10:167–171, 1916.

[20] Massimo Furi and Mario Martelli. A multidimensional version of Rolle's theorem. *The American Mathematical Monthly*, 102(3):243–249, 1995.

[21] Le Gao and Narendra Kumar Govil. Annular bounds for the zeros of a polynomial. *International Journal of Mathematics and Mathematical Sciences*, 2018:1–7, 2018.

[22] Maria-Jose Gonzalez-Lopez and Laureano Gonzalez-Vega. *Grobner bases and applications*, volume 251 of *London Mathematical Society Lecture Note Series*, chapter Newton Identities in the multivariate case: Pham Systems, pages 351–366. Cambridge University Press, 1998.

[23] Laureano Gonzalez-Vega. A special quantifier elimination algorithm for Pham systems. *Contemporary Mathematics*, 253:115–134, 2000.

[24] Aaron Herman and Hoon Hong. Quality of positive root bounds. *Journal of Symbolic Computation*, 74:592 – 602, 2016.

[25] Alexey A Kytmanov. Analogs of recurrent newton formulas. *Russian Mathematics*, 53(10):34–44, 2009.

[26] Joseph-Louis Lagrange. Traité de la résolution des équations numériques. *Mémoires de l' Académie Royale des Sciences et des Belle-Lettres de Berlin*, 1769.

[27] Ben-Zion Linfield. On the relation of the roots and poles of a rational function to the roots of its derivative. *Bulletin of the American Mathematical Society*, 27(1):17–21, 1920.

[28] F. Lucas. Propriétés géométriques des fractionnes rationnelles. *CR Acad. Sci. Paris*, 77:431–433, 1874.

[29] Morris Marden. The geometry of the zeros of a polynomial in a complex variable. *New York*, 1949.

[30] Morris Marden. The search for a Rolle's theorem in the complex domain. *The American Mathematical Monthly*, 92(9):643–650, 1985.

[31] D. G. Mead. Newton's identities. *The American Mathematical Monthly*, 99(8):749–751, 1992.

[32] Dhagash Mehta, Tianran Chen, Jonathan D Hauenstein, and David J Wales. Communication: Newton homotopies for sampling stationary points of potential energy landscapes, 2014.

[33] Dhagash Mehta, Jonathan D. Hauenstein, and Michael Kastner. Energy-landscape analysis of the two-dimensional nearest-neighbor φ^4 model. *Phys. Rev. E*, 85:061103, Jun 2012.

[34] Bernard Mourrain and Victor Y. Pan. Solving special polynomial systems by using structured matrices and algebraic residues. In *Foundations of Computational Mathematics*, pages 287–304. Springer, 1997.

[35] A. M. Ostrowski. *Solution of equations and systems of equations*. Academic Press, New York, 1966.

[36] Luis Miguel Pardo and Jorge San Martın. Deformation techniques to solve generalised Pham systems. *Theoretical computer science*, 315(2-3):593–625, 2004.

[37] James L Parish. On the derivative of a vertex polynomial. In *Forum Geom*, volume 6, pages 285–288, 2006.

[38] Frédéric Pham. Formules de Picard-Lefschetz généralisées et ramification des intégrales. *Bulletin de la Société Mathématique de France*, 93:333–367, 1965.

[39] Josep Rubió-Massegú. Note on the location of zeros of polynomials, 2011.

[40] Jörg Siebeck. Über eine neue analytische Behandlungweise der Brennpunkte. *Journal für die reine und angewandte Mathematik*, 1864.

[41] A. Van der Sluis. Upperbounds for roots of polynomials. *Numerische Mathematik*, 15(3):250–262, 1970.

[42] Chee-Keng Yap. *Fundamental problems of algorithmic algebra*. Oxford University Press, 2000.