ABSTRACT

CILIANO, DAGMARA. Show Me the Function: A Literature Review of Building Understanding through Multiple Function Representations. (Under the direction of Dr. Robin Anderson).

The concept of function is among the most essential ideas in mathematics. The introduction, implementation, and translations between representations of functions are imperative to student learning. This literature review examines the development of deeper knowledge of mathematical ideas through utilization of multiple function representations in mathematics teaching. Findings of this review largely confirmed the importance of mathematical representations in the process of learning. While confirming this importance as a widely held belief in the mathematics education community, further investigation of the available literature suggested that there is no general agreement on the significance of particular representations with reference to their sequencing, translations, or placement in instruction. The thesis addressed these topics by considering 1) function representations in mathematics curriculum and 2) connections and understanding among function representations. Additionally, the procedural and conceptual knowledge through representational competence was investigated across literature examples on students' functional thinking and problem solving.

Show Me the Function: A Literature Review of Building Understanding through Multiple Function Representations

by
Dagmara Ciliano

A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Master of Science

Mathematics Education

Raleigh, North Carolina

2021

APPROVED BY:

Dr. Cyndi Edgington

Dr. Molly Fenn

Dr. Robin Anderson
Chair of Advisory Committee

BIOGRAPHY

Dagmara Ciliano graduated with a Bachelor of Science degree in Mathematics from St. John's University, in New York in 2001. She is certified in Secondary Mathematics Education in the state of New Jersey. After six years of teaching courses in Algebra, Trigonometry, and AP Calculus in New Jersey public schools, she spent an additional year teaching Algebra, AP Calculus, and Advanced Quantitative Reasoning in Massachusetts. During her tenure as a high school teacher, she also worked on Algebra curriculum aligning it to state and district benchmarks. In 2018, she entered the master's program in Mathematics Education at North Carolina State University. She plans to complete the program in the Spring of 2021.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Motivation

The motivation behind this literature review is the role of multiple representations, their sequencing and introduction, and the part they play in understanding the concept of a function. Over the last few decades, countless research papers and scholarly publications suggest that using multiple mathematical representations of functions is valuable to students' conceptual understanding. It is important, however, that students make connections between those various representations (Chigeza, 2013). The competence to transfer between multiple mathematical forms along with the ability to select and apply best suited representations is the heart of understanding mathematical ideas (Knuth, 2000a, 2000b). This view of mathematical representations as tools to advancing a broader view of abstract ideas prompted the focus of this literature review and its attention on multiple representations and understanding of mathematical concepts, specifically with reference to introduction of functions to middle and high school grades Algebra students.

Personal experience as both an educator and a student of mathematics inspired the review of educational research discussing the concept of function from the learning and understanding point of view. What causes a person to comprehend the abstract mathematical concept of a function? Why do some learners see a graph as a pictorial representation of hills and turns while others connect it to increasing and decreasing values? Are procedural drills and algebraic step-by-step manipulations hindering meaningful function analysis and interpretations? These were the type of questions this literature review sought to investigate.

Relevance

The National Council of Teachers of Mathematics (NCTM) (2000) advocates the teaching and learning of representations as one of the essential standards in mathematics curriculum. The utilization of multiple representations expands past the mathematics classroom. Mathematical representations are used to display and communicate social and physical experiences. NCTM specifically recommends that students' use of representations be developed gradually throughout K-12 grades. NCTM warns of the shortcomings of students learning formal representations as an end itself. Mathematical forms should be merely tools used to organize and model ideas. Learners should be able to not only create and apply mathematical representations appropriate for specific problems or situations, but also be fluent in translating between them. The mathematics education research community recognizes functions as one of the most unifying concepts in mathematics (Knuth, 2000b). Meaningful connections initiate representational competence that is essential to building deeper understanding of the idea of function.

This ongoing development of function representations is echoed throughout the Common Core Standards Initiative (2010). In kindergarten, students are introduced to numerical symbols and are expected to start associating geometric shapes with their characteristics. In addition to formal symbols, like Arabic numerals and arithmetic signs, students are encouraged to use verbal representations while communicating those early mathematical ideas (Common Core Standards Initiative, 2010, p. 2). By the end of elementary school students are expected to use number lines to represent arithmetic operations on real numbers. Recognizing and analyzing simple patterns is also included in the elementary school curriculum as well as displaying simple data sets in charts and picture graphs (Common Core Standards Initiative, 2010, pp. 24-25). Although, in elementary school students are not familiar with the concept of function, they practice functional thinking and mathematical forms that will subsequently be utilized to represent functional relationships. In middle school, translating between various mathematical representations starts taking place as students build on the previous knowledge and experiences. Linear relationships are introduced, and students are expected to manipulate linear equations to find solutions to real world scenarios (Common Core Standards Initiative, 2010, p. 47). This progresses the representational fluency by means of algebraic, verbal, and contextual forms. By the time students enter high school, they are ready for the formal introduction to the concept of function, which is greatly reinforced by algebraic and graphical representations (Common Core Standards Initiative, 2010, pp. 67-71). Throughout their mathematical education, students become increasingly fluent in utilizing representations to communicate and better understand mathematical ideas, including the concept of function.

Literature Selection

The topic of this literature review was narrowed down to research and studies concerning both formal and informal representations in mathematics curriculum and instruction. The sources were selected to balance earlier with more recent studies. As such, research in the field of mathematics education from the last four decades was considered. The traditional view of mathematical topics without a distraction of current technical advances as well as the modern approach to classroom instruction were explored. The literature pertaining to specific curriculum guidelines was limited to studies within the United States while research from additional countries, including Australia and South Korea, was considered as pertaining to students' understanding of mathematical concepts regardless of the curriculum requirements. The most common keywords within the resources were *function*, *representation*, *graph*, and *understanding*. The resources were acquired through the means of both electronic and hard copies.

Types of Function Representations Discussed

There are numerous ways to represent a function. Throughout this review formal and informal mathematical representations are remarked. For instance, gesturing and pictorial illustrations, though not formally defined, are utilized by both students and teachers as causal or spontaneous ways to communicate a relationship between quantities. Traditionally, however, the concept of function is accompanied with algebraic or graphical representations. Common Core Standards Initiatives (2010, p.70) narrows down the recommended function representations to algebraic, graphical, numerical (or tabular), and verbal forms. These forms are more likely to be used in formal instructions, in text, or during standardized assessments. Even so, to enrich the discussion on connecting and translating between function representations as an aid in deeper conceptual understanding, this review focused on two of the more casual representations: illustrative and contextual along with the two of the favored forms: algebraic and graphical.

Algebraic representations

Algebraic representations of functions are by far the most favored in the mathematics classrooms. The standardized ways in which a function rule can be algebraically described is undoubtedly beneficial. Expressions can be easily manipulated and often used to translate between other function forms such as tables and graphs. This accessibility of utilizing algebraic forms has some drawbacks nevertheless. One of the most common misconceptions students can have about a function, for instance, is that every function must have an algebraic equation to define it. The overemphasizing of the algebraic forms as the only acceptable answer to a math problem can cause algebra students and the teachers to shy away from other representation forms. An argument can be made that over-relying on algebraic forms of functions can also lead to a purely procedural understanding, with students focusing on following the steps in algorithms to solve for ordered pairs, or find numeric values without linking the functional relationship to its context. Algebraic representations of functions are standardized forms that help students develop conceptual understanding of abstract ideas by making meaningful connections to other mathematical forms.

Contextual representations

The importance of context is mentioned throughout Common Core Standards Initiatives (2010) and advocated by the National Council of Teachers of Mathematics (2000). However, the recognition of context as a function representation is far less common in educational literature. Nevertheless, contextual representations can help students "de-abstract" ideas and thus make them more relatable, and easier to interpret. Whether with simple arithmetic or complicated functions, students can get a better sense of the abstract concepts when put in the real-life scenarios. Aiding addition or subtraction comprehension with the use of counting blocks in elementary schools, or illustrating bacteria growth with a science video, contextualized abstract notions become more approachable. Without context, mathematical representations can become an end in itself rather than instruments used to communicate ideas or tools of problem solving.

Illustrative representations

Pictures, diagrams, videos, dynamic models or physical objects can all aid in illustrating a functional relationship. These are concrete yet casual forms of representations that often serve both as a learning and a teaching tool. Some abstract concepts like that of a conic section, for example, are hard to access with a formal approach. Slicing a physical ice cream cone and revealing a curve of a parabola can allow for a quick understanding of the topic. Illustrative representations of functions, although unconventional, can serve a great purpose of both placing abstract concepts in context and quick accessibility to representations that are not easily portrayed with words or numbers. In today's classroom the use of technology can make illustrative representations take center stage when introducing problems or new concepts. Dynamic software can aid with three dimensional models and pictures or videos can help access real life scenarios not accessible within the classroom. It is worth noting that not all illustrations aid in mathematical understanding. A picture of a famous billionaire alongside a problem introducing exponential function and compound interest, for instance, is hardly aiding with connecting mathematical ideas. An illustration of a series of time lapsed photographs of a bouncing basketball, on the other hand, can place the exponential relation between the height and time variables in context.

Graphical representations

Graphs are widely recognized by the mathematics education community as an essential visual representation of functions. Throughout their mathematics education, students become fluent in both creating graphical representations of functions and interpreting them while relating the shapes of the graphs to the functions' characteristics. Because of their visual nature, graphical representations of function come with an array of misconceptions. One of the most common misinterpretations of graphs is attributing the shapes of curves to the similarly fitting context. Students often analyze a velocity to time graph as the representation of the terrain. A frequent example is the increase in speed portrayed by a positive sloping line being perceived as a hill, while a horizontal line is often misread as a flat surface instead of an instance of constant velocity. Nonetheless, graphing is an essential skill as it provides an accessible and standardized visual display of functional relationships.

Key Terms

It is understood that terminology can have degrees of variance in meaning when used within specific areas of education or mathematics. This section provides an explanation of key terms that were adopted throughout this review with their meaning that is particular to the topics covered.

Representation

Unless otherwise stated, it is implied that by representation specifically "external representation" is meant. External representations are such that can be seen, touched, or heard. These can include algebraic symbols, graphs, or tables among other commonly used mathematical forms. Throughout the review, pictures, diagrams, gestures, videos, verbal explanations, and even physical objects, although less formal, are also considered as external mathematical representations.

Translating

The process of moving from one form of representation to another is considered translating between those forms. This review primarily focuses on translating between external representations of function such as that of a linear equation and its corresponding graph. At times "translation" is also referred to as "moving" or "transferring" between multiple mathematical forms.

Understanding

The review refers mainly to students' conceptual understanding or conceptual knowledge. Procedural understanding or knowledge is at times discussed with relation to conceptual understanding; for example, as a tool to translate between function representations. The nuances between the two categories of knowledge are further discussed in chapter three.

Organization of Paper

The following section contains two chapters discussing mathematical representations throughout the K-12 curriculum and the multiple function representations in regards to mathematics education research literature.

Chapter two starts with a synopsis of the curriculum inclusions of the mathematical representations in early grades through middle school. The connections that students make between mathematical representations are investigated through reviewing of Common Core Standards (2010) along with recommendations of the National Council of Teachers of Mathematics (2000). Conclusions from various educational studies are explored as they relate to utilizing mathematical representations to develop understanding. The importance of building on prior knowledge is considered as a bridge to the formal introduction of functions in secondary grades.

Chapter three explores the connections and understanding that students make among function representations. Four function representations are considered in closer details: algebraic, contextual, illustrative, and graphical. Translating and connecting between function representations as well as their sequencing and introduction (entering) is approached in detail as a focus of the literature review. Educational research within the topic is debated while pointing out both the benefits and drawbacks of the various function representations and their utilization as it pertains to developing functional thinking and knowledge. The chapter closes with considerations of what utilizing multiple function representation affords in cultivating conceptual and procedural understanding.

The review concludes with a summary, discussion about limitations of the research, and recommendations for future studies.

CHAPTER 2

FUNCTION REPRESENTATIONS IN MATHEMATICS THROUGH GRADES

This chapter investigates how the introduction of various representations throughout K - 8 mathematics curriculum impacts students' understanding of functions. Traditionally, the very notion of teaching mathematics relies heavily on a system of symbols and representations that allow the instructor to convey the concepts and ideas at hand. The concept of a function is communicated with the help of algebraic, graphical, contextual, and verbal representations. Algebraic and graphical forms of a function are often supported with tabular and numerical notations (Figure 1), while contextual and verbal forms can be enhanced with physical and illustrative depictions (Figure 2). In a way, all mathematical concepts can only be communicated through various representations: from docile numeric symbols to elaborate graphical illustrations (Bossé & Adu-Gyamfi, 2014). This is especially evident in the area of Algebra, where teachers depend on various ways to articulate abstract mathematical ideas (NCTM, 2000). Whether in a kindergarten class where a stock of cut-out felt apples on a bulletin board acts for a value of eight or a "tipped over 8" looking symbol signifies values increasing to never ending infinity in a high school introductory calculus class; both instructors and students need a representation to help them communicate in a language of mathematics (Driscoll, 1999). The National Council of Teachers of Mathematics (NCTM, 2000) embraces the teaching and learning of representations as one of the crucial standards in mathematics classrooms. NCTM recommends students' use of various representations to display not only mathematical concepts but social and physical phenomena as well. It further reasons that the students' successful use of representations should be developed gradually from the elementary throughout high school years.

x	$y = \lvert x \rvert + 1$	(x, y)
-3	$\lvert -3 \rvert + 1 = 4$	$(-3, 4)$
-1	$\lvert -1 \rvert + 1 = 2$	$(-1, 2)$
0	$\lvert 0 \rvert + 1 = 1$	$(0, 1)$
1	$\lvert 1 \rvert + 1 = 2$	$(1, 2)$
3	$\lvert 3 \rvert + 1 = 4$	$(3, 4)$

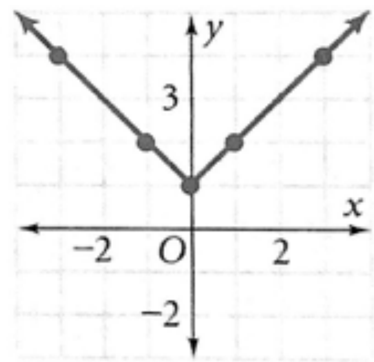


Figure 1. Example of algebraic and graphical forms of a function along with tabular and numerical notations (Bellman, et al., 2009).

Figure 2. Pictures of a dog taken at equal intervals of time supporting contextual and verbal description of distance to time relations (Illustrative Mathematics, 2019).

At young ages, developmentally, children are typically capable of grasping concepts through simple verbal, contextual, and illustrative representations (Chigeza, 2013). Thus, it naturally makes sense to introduce these representations in the early grades of learning mathematics. Throughout early elementary grades, the mathematics curriculum focuses mainly on number sense and two-dimensional geometry (Common Core Standards Initiative, 2010). Because of the simplicity of those topics, students can be introduced to the ideas and concepts via contextual and verbal representations that are often interchangeable. The numerical and symbolic representations are still uncomplicated enough at the elementary school level to have a simple meaning. The felt cutout or a plastic block can symbolize an apple. But so can a letter

"A" or another written symbol of choice. These simple depictions are a foundation to representations of numbers, ideas, and mathematical concepts.

Mathematical Representations in Elementary and Early Middle School Grades

Principles and Standards for School Mathematics (2000) recommends children as young as kindergarten be able to describe qualitative and quantitative changes in variables set in a familiar context. Young students express mathematical ideas and thoughts often in both conventional and unconventional forms. Drawings, gestures, and unconventional symbols help children communicate their thinking (Edwards, et al., 1993). By the end of fifth grade, students should be familiar with using tables and equations involving both arithmetic and algebraic expression in a more detailed manner connecting them to specific mathematical concepts. Numerical and geometric patterns are studied and expressed verbally and symbolically in formal mathematical terms. The means of representing the mathematical ideas behind patterns as early function concepts are crucial to understanding mathematics (NCTM, 2000). It is important, however, that the introduction of the formal representations of these new mathematical ideas is presented as a mere tool of communication and demonstration of concepts (Greeno & Hall 1997). Students should recognize the limitations of various representations and begin transferring between them freely to recognize the advantages of one mathematical form over another and thus build their functional thinking on a deeper, conceptual level. Young students' reasoning with quantities relating to one another initiate familiarity with functions, often by using physical objects, illustrations, and basic symbols (NCTM, 2000).

Practice with physical, textile objects as concrete representations are a crucial foundation for students' use and understanding of formal mathematical symbols in later grades (NCTM, 2000). Although it is certainly undisputed that symbols and representations are important for detailing mathematical knowledge, the form of the symbols, or representations, does not need to be specific at first. The symbols and representations themselves are not the knowledge in itself (Steinbring, 2006). A felt apple can represent an apple in an arithmetic problem at hand, but so can a letter symbol or another physical object. Young students and their teachers often rely on objects, and gestures to communicate their thinking. It is not uncommon nor unconventional to use fingers to aid simple arithmetic or to help visualize a quantity in a school setting and everyday situations alike. In Steinbring's argument, the representations are not equivalent to the idea or knowledge itself. His line of reasoning leads to the belief that students capable of using multiple representations of a single mathematical idea develop a deeper conceptual understanding. Young children, developmentally, are capable of recognizing mathematical representations and the ideas such representations depict need not to be uncomplicated.

As students move along the learning trajectories dictated by set curriculum, more complex mathematical ideas lead to introduction of multiple representations. Often, both students and teachers recognize the introduction of letter symbols in mathematics as the beginning of algebraic thinking (Driscoll, 1999). This introduction usually takes place in middle school grades where algebraic expressions are used alongside verbal, contextual, and illustrative representations in topics ranging from proportional relationships in early middle school through linear equations and developing a concept of a function (Common Core Standards Initiative, 2010, pp. 53, 55). As early as sixth grade, students are introduced to symbolic representations that no longer stand for an unknown quantity in an algebraic expression but rather depict a variable (Common Core Standards Initiative, 2010, pp. 39, 43-45). The understanding of a concept of a variable can be essential to subsequently grasping the formal definition of the function and its many representations (Leinhardt, et al., 1990). Middle school students become increasingly fluent in manipulating various mathematical representations long before they associate them with functions.

Students' Connections to Mathematical Representations

Students in elementary school classrooms do not gravitate to objects like counting chips or felt apples because they are just physical play things but rather because they represent mathematical ideas (Steinbring, 2006). In those early grades, learners start making connections between the representations; both formal mathematical symbols and unconventional forms such as drawings and physical objects. Children as young as elementary school ages clearly distinguish the mere representation of a mathematical concept from the physical form of such representation (Steinbring, 2006). Understandably, young learners develop these connections over time and may not instinctively link physical objects with their symbolic mathematical representations (Uttal, et al., 2009). A balanced approach to the use of various representations, both illustrative and concrete, should account for the benefits and disadvantages of such representations in support of mathematical thinking.

Elementary school children often develop mathematical signs to express mathematical concepts in order to share their thinking (Kaput, 1999). It is not an uncommon opinion among educators and researchers that hands-on activities help young children develop a better understanding of mathematical concepts. When introduced to “real life algebra problems” illustrating quantities changing over time, Kaput argued, young students should be given a chance to use physical models and simulations. In Kaput’s early function introduction tasks, elementary school students created graphical representations of correspondence between two quantities from the data they collected. Kaput’s (1999) research involved a group of 15 third and fourth graders over a 5-week period. The students were part of a program operated by teachers from their school under the assistance of staff members from a graphic software project. Over the course of the study, working with their peers, students logged their own records of running, jogging, and walking, along with computer simulations and creations of graphs and data keeping. This group of students was able to develop a comprehension of a concept of quantity and build a rough understanding of constant speed and time intervals. Students were able to build mathematical understanding by connecting their experiences to functions and representations without any formal introduction to those fundamental mathematical concepts. A strong connection between variables they observed and of what their graphs represented was established. This deeper understanding was argued to come from developing the representations of the relations between the two quantities, rather than through pure manipulations of supplied symbols (Kaput, 1999). Students are more likely to build a deeper knowledge of mathematical representation when they develop a better understanding of the quantities based on practical experience, especially if that experience is set in a familiar context (Kaput, 1999; NCTM, 2000; Panasuk & Beyranevand, 2010). Students were afforded the opportunity to create their own data sets and graphical representations to communicate their analysis subsequently leading to deeper connections between the various mathematical representations.

It is important to note that a mere implementation of hands-on activities without a prior formal introduction to a mathematical concept does not always promise successful student understanding. Action based interactions and real-world models are not a universal remedy nor a guarantee for students to build strong conceptual understanding of correlations and functional thinking (Ellis, 2011). In a “quantitatively-rich” situation, students might be more inclined to focus on searching for patterns and simply isolating numbers instead of making mathematical connections. Ellis (2011) provided an example of middle school students with no prior formal introduction to linear functions. The hands-on activities involved working with arithmetic patterns created by the number of rotations of two different sized simultaneously turning gears. Even after engaging with concrete models and counting the physical rotations of the gears, some students would not link the numerical values to their original context. They described the numerical patterns represented in a table as values going "up by 4" or "up by 5" with no contextual significance. It was often up to the instructor to help bridge the mathematical representations to the desired mathematical concepts being introduced. Mathematical comprehension cannot rely on context alone.

Students should be introduced to ideas separate from any form of representation rather than equating specific representations with the mathematical concepts. This approach is more likely to warrant a deeper development of the use of the signs and symbols as a way of communicating mathematical ideas (Steinbring, 2006). Philosopher and mathematician Charles Sanders Peirce described any representation as meaningless unless someone interprets it (Greeno & Hall, 1997). Although there is some merit in providing students with a formal representation and teaching them how to read it, more benefit comes from affording students the opportunity to develop their own representations based on their own interpretation of a mathematical concept (Greeno & Hall 1997). Students should not simply be given the algebraic signs and other mathematical representations during the initial introduction of new mathematical concepts but rather develop the understanding of the representations as they learn to communicate the ideas at hand (Steinbring, 2006). The initial separation of mathematical idea and its corresponding form rejects the notion of mathematical representation being an end in itself.

From the early elementary school grades The National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the important connections in mathematics that students make between their own real-life experiences outside of the classroom and the formal mathematical representations they learn in school. As students become more fluent in the language of mathematics, the connections between the ideas and their representations are crucial to the understanding of the new ideas and concepts. By middle school, learners interconnect the ideas and build on one another. This linking between concepts and skills allows building on previous knowledge and view of mathematics as an extensive yet coherent subject. In high school, where the formal introduction to functions takes place, students are able to connect concepts and comprehend the multiple representations of the same idea. The connections learners develop between various representations and contexts throughout the mathematics curriculum subsequently become building blocks for better functional thinking.

Representations, understanding, and building on prior knowledge

Unlike simple numbers and symbols that are indeed necessary to simply perform mathematical procedures, most learning should rely on students' deeper conceptual grasp of the concepts (Chigeza, 2013) and build on previously developed representations. The use and introduction of graphical representation of linear functions throughout the curriculum, for instance, can be used to illustrate this argument (Leinhardt, et al., 1990). Students, especially in early years of learning mathematics, learn abstract concepts while relying on more tangible and specific representations. The introduction of ideas that can be represented with two (or more) different representations, as Leinhardt, et al. discussed, does not prevent students from using connections with physical objects and abstract ideas but rather strengthens their understanding from linking between conceptual representations. The use of multiple abstract forms, such as algebraic symbols and graphs, contributes to better understanding of mathematical concepts.

Students can struggle with understanding mathematics when they are not able to make connections between various mathematical representations (Bossé & Adu-Gyamfi, 2014; Chigeza, 2013; Knuth, 2000a). Steinbring's (2006) study, among his other research, pointed out that such struggles stem from students developing conceptual understanding that is not impartial from relying on specific forms of representations. More specifically his research focused on symbolic, verbal, and contextual forms. In his study, Steinbring introduced relatively simple arithmetic tasks to groups of elementary third and fourth grade students. The problem at hand involved children working out a "magic square" (Figure 3) with a constant diagonal sum. Similar magic square puzzles are used in early Algebra learning as they are helpful with developing symbols. Such puzzles encourage students' use of symbols and connecting their meaning to desired solutions (Arcavi, 1994; Driscoll, 1999). With the accounts of the young students' work, Steinbring made numerous observations of the use of contextual, algebraic or symbolic, and verbal representations that young children use to express algebraic operations before being formally introduced to such mathematical concepts. In his argument, the Steinbring reasoned that because students are able to arrive at the use of mathematical signs before being formally assigned their meaning to specific mathematical concepts, such signs themselves are not equivalent to the conceptual understanding of the ideas. The fluency in mathematical symbols and representations does not translate into the understanding of mathematical concepts without making significant connections between them.

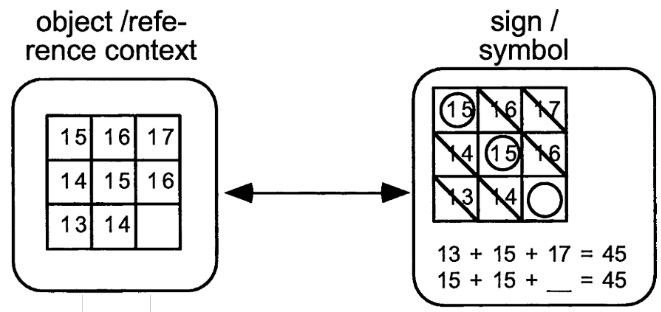


Figure 3. Example of a magic square used in Steinbring's study showing a conceptual relation between a reference object and symbols.

During elementary school, students could spend years operating with letters and symbols that solely stand in for one quantity, for example a letter "A" for a number of apples. With the progression of algebraic ideas in middle school however, letters and symbols not only represent multiple quantities, additionally those quantities are being given their own characteristics depending on their relationship to each other (Common Core Standards Initiative, 2010). This introduces the complicated notion that a familiar symbol (a letter *x* or *y* most likely) could stand for a variable that is no longer a simple unknown but can change its meaning depending on context or a form of representation. Context of the variable, its domain, or its very form, are some of the characteristics of variables that are often left for the student to figure out on their own (Leinhardt, et al., 1990). As Leinhardt and colleagues argued this understanding of variables is crucial to the concept of functional relationships and their representations, and a poor introduction to the very notion of variables can lead to poor conceptual understanding of functions.

Throughout middle school, as students begin to analyze relationships between sets of quantities, they must recognize their correlation to one another (Common Core Standards Initiative, 2010). Building on their familiarity with relationships between corresponding quantities and representing them as ordered pairs not only in a numerical but also in a coordinate form, middle school students start viewing and analyzing the graphical representation of such related quantities in reference to the formal definition of a function (Common Core Standards Initiative, 2010). Students are often asked to translate from equations to graphs, which usually proves to be more straightforward than analyzing a graph to yield an equation and/or other ways of representing the relation between graphically illustrated relationships (Leinhardt, et al., 1990). In a study by Knuth (2000b) in middle school and high school classrooms, designed to show the connections students make between equations and graphs, it became apparent that even those learners that were able to equate the linear equations to their corresponding graphs, often did so on a purely procedural level. In his observations, Knuth noted that oftentimes students simply did not perceive an ordered pair on the graph as a solution to the very linear equation it represents, nor did they connect it to the variables involved. Making connections between multiple representations of one mathematical relation is crucial to the analysis of the solutions.

What can seem like an easy transition between symbolic and verbal or contextual representation to the instructor, can prove a difficult task for students faced with not only acquiring a new concept but learning multiple formal representations to efficiently communicate it. Panasuk and Beyranevand's (2010) study on introducing linear relations in middle school classrooms aligned with numerous previous research findings, including that of Mosley (2005) and Niemie (1996), asserting the benefits of students and teachers using multiple representations while introducing new mathematical concepts. Based on the seventh and eighth grade student interviews, Panasuk reasoned that students could often operate with mathematical symbols and representations without having a concrete understanding of the concepts, in this case of linear relations. In their study, the authors discovered that even those students who were successful at solving simple linear equations could not identify those very same linear relations presented in different forms: verbal, diagram, nor symbolic. Students were expected not only to observe and explain any similarities between the three representations of a fairly uncomplicated situation but ultimately to state that each of the representations seen in *Figure 4* represents the same linear relationship. Panasuk and her co-researcher blamed this lack of proficiency on not being exposed to multiple representations of simple linear relationships with the initial introduction to those concepts. Even when students are familiar with various symbols and can efficiently operate among them, the lack of connecting new representations to previous knowledge can expose the more procedural nature of their proficiency rather than conceptual understanding of the topic (Panasuk & Beyranevand, 2010).

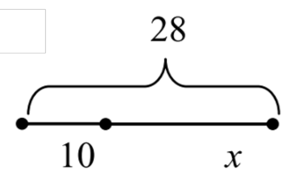
Mark is 10 years older than his sister. He is now 28. How old is his sister now?		$x + 10 = 28$

Figure 4. Linear relationship with one unknown posed in three different representations (Panasuk & Beyranevand, 2010).

As students grow older, and progress through grades, they can grasp more complex concepts and become familiar with more intricate forms of mathematical representations, such as graphs of functions and their corresponding algebraic expressions. In their adolescence, high school students are capable of viewing multiple representations and translating between them (Chigeza, 2013). High school mathematics curriculum focuses heavily on functions and their properties (Common Core Standards Initiative, 2010, pp. 62, 67-71). Students are expected to not only grasp the concept of a function, use proper notation, but also to be able to interpret functions in different contexts as well as use different representations. By the end of middle school grades, students are ready to move past simple linear functions they have seen throughout the late elementary and early middle school grades. In high school, learners are expected to reason with quadratic, exponential, and trigonometric models (Common Core Standards Initiative, 2010).

Representations and a Concept of Function

The concept of function is formally introduced in high school (Common Core Standards Initiative, 2010). High school students are expected to be able to express functions by means of graphs, verbal rules, algebraic expressions, or a recursive rule. Great importance is usually placed on graphical representation to help students illustrate functions while algebraic expressions are emphasized to make sense of function's properties. Before the formal introduction of functions, high school students should be aware of all the mathematical representations used to express the new learned concepts. By ninth grade, students are familiar with plotting points, graphing lines, understanding recursive patterns, and writing equations. To build functional thinking, students must relate those familiar representations to the new learned concepts (NCTM, 2000), translate between various representations (Cunningham, 2005), and recognize strengths and weaknesses of one mathematical form over another (Common Core Standards Initiative, 2010).

While representations more specific to the concept of function (i.e., function notation or mapping) are only introduced in late middle school and high school years (Common Core Standards Initiative, 2010), a lot of time and effort in mathematics education research focuses on younger students in early elementary grades (Leinhardt, et al., 1990). This leads to a lack of research more specific to the introduction of functional thinking to elementary school students (Blanton & Kaput, 2011). There are studies, however, that focus on young students making connections between abstract mathematical concepts and real-life context (i.e., Bell & Janvier, 1981; Dugdale & Kibbey, 1986; Swan, 1980, as cited in Leinhardt, et al., 1990). These studies are often based on problems designed for students to abstract algebraic and functional thinking from contextual representations.

Numerous educational research studies addressing representations and functional thinking use travel (often represented with time-distance or time-velocity relations) or growth over time type problems. Leinhardt et al. (1990) highlighted that these kinds of problems, however, do not always help students instinctively build a conceptual understanding of functions but rather point to figurative understanding of the mere representations used to illustrate the mathematical relationships in context. A contextualized graph involving speed with respect to time (Figure 5) is one such example of graphical representation being perceived as an illustration (Janvier, 1981). Twenty high school students, across all grades, were likely to interpret curves of the graph as the turns of the race track (see illustration *G* in *Figure 5*) (Bell & Janvier, 1981). Leinhardt drew upon earlier research (Malik, 1980) to further argue that, intuitively, young students have some notion of functions that are based on real world observations, things changing over time being a great example. Leinhardt, et al. (1990) concluded that early mathematics education is missing the use of functions and representations as building blocks for supporting the early comprehension of abstract mathematical reasoning.

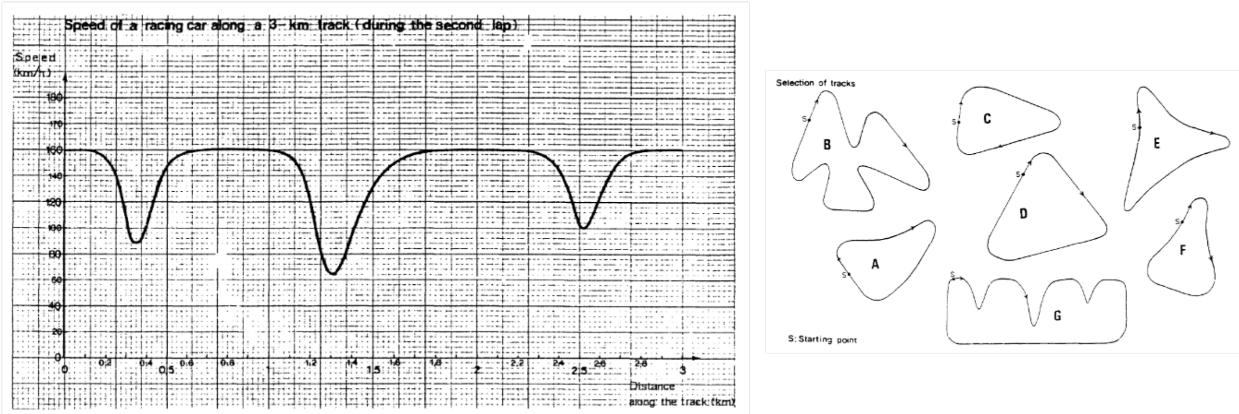


Figure 5. Contextualized interpretation graph for given shapes of a racing track (Bell & Janvier, 1981)

Even when students are fluent with creating graphs, especially in early grades, some can struggle with the interpretations of function through graphical representations (Blanton, et al., 2015; Yerushalmy, 1997). Ainsworth (2006) offered some reasoning as to why younger students might have such difficulties. One of the reasons, Ainsworth argued, is that children simply rely on the visual interpretations of a graph rather than applying the mathematical knowledge to understand the very characteristics of a graph, like shape, extrema, or even the setup of axes and units. The author called on Elby (2000) as an example of where a velocity-time graph of a bicyclist riding over a hill produces a U-shaped graph. Young learners will often choose graphs with a hill shaped curve as a representation simply because they tend to rely on intuitive knowledge. It is too often that children simply perceive graphs as pictures (Elby, 2000). The hill shaped graph (Figure 6), in this case, contextually, reminds them of the very hill that the biker was riding, rather than a mere representation of the relation between velocity and time (Ainsworth, 2006). This spontaneous and unsophisticated perceiving of a function graph as a realistic interpretation is not uncommon among students of all mathematical experience levels.

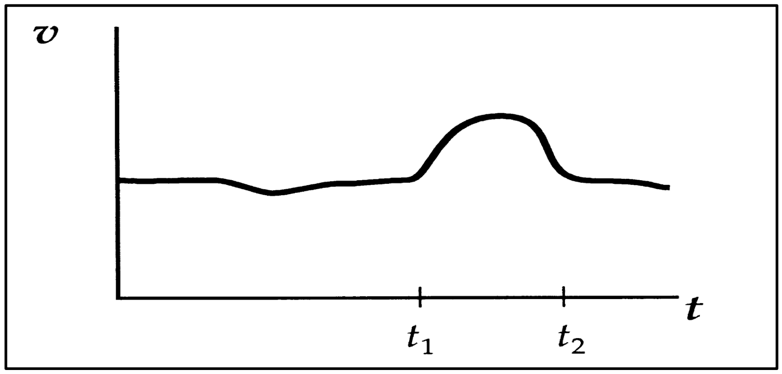


Figure 6. A “hill shaped” velocity versus time graph (Elby, 2000).

The instance of graphing representations of functions and students’ inability to build on their elementary school mathematical knowledge in later grades is a common thread through Leinhardt et al.’s (1990) findings. Although students as early as first grade are introduced to the procedural ways of graphical representation of real-life situations, they are not introduced to the formal definition of functions until secondary grades (Common Core Standards Initiative, 2010). This early introduction to graphical representation without any ties to functions, Leinhardt et al. argued, shows up in later years as a purely procedural understanding of plotting points which consequently results in students struggling with freely transitioning between various forms of function representations, specifically the algebraic expressions and graphs.

When students in algebra classrooms fail to make connections between graphical and algebraic function representations it might be due to a lack of a full grasp of the concept of equivalency rather than the lack of understanding of the concept of function. Van Dyke and Craine (1997) discussed this topic of students’ making connections between the familiar representation and concept of function. When honors algebra students were presented with simple linear functions in algebraic equation form, many were unsure whether the corresponding graph would cross the origin. In the same vein of concerns, students could not pick out a point on the line that related its coordinates as a solution to the linear equation they were given (see

Figure 7). Even though they were proficient in the various forms of function representations, and, according to the authors, understood the importance of the tables, graphs, and equations representing a function, the students perceived the different representations as separate or unrelated. Despite the fact that in high school algebra classes students are already aware that changes in algebraic form do not alter the relation represented by it they must be taught that various forms of representations of the same function are in that same sense equivalent (Van Dyke & Craine, 1997).

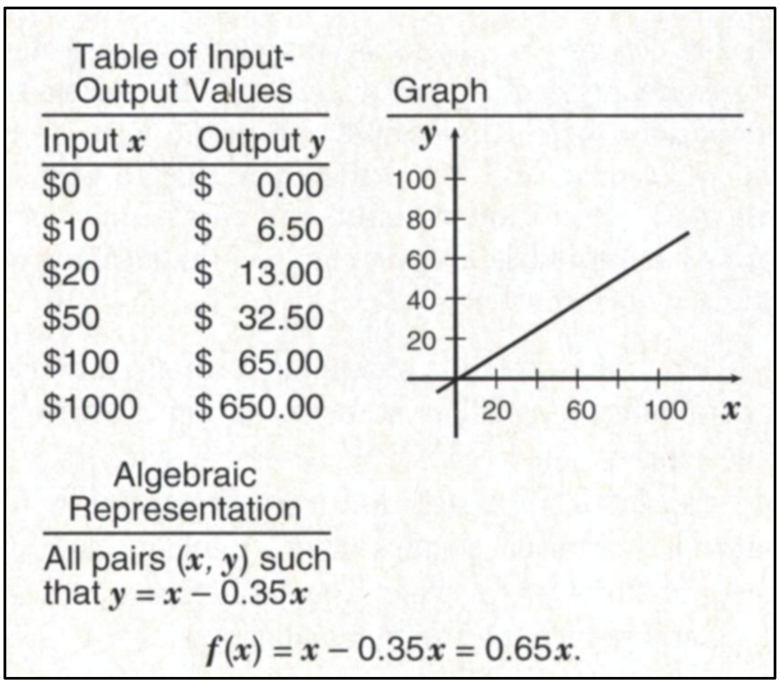


Figure 7. From VanDyke and Craine (1997). A visual aid reminding students of the connections among the various representations.

Another explanation for students selecting algebraic forms with no connections to a graphical representation of a function could simply be due to the math curriculum favoring procedural manipulation of algebraic forms (Knuth, 2000b). In Knuth's (2000b) study of 284 students enrolled in first year algebra through advanced placement pre-calculus classes, the majority of the participants relied heavily on algebraic representation and exact solutions when presented with a linear function problem (Figure 8). Knuth argued that students should have chosen the graphical representation over the algebraic one as a means of solution. Students chose the equation forms as the easier approach even with graphs providing a better tool for the given problem. Knuth argued that their choices were influenced by students' overfamiliarity with manipulations of algebraic forms. Although multiple representations of the same functions are equivalent, they are not necessarily equal in the sense of their benefits depending on the task at hand. Students' inability to choose one representation over the other, often points to their disconnection between the representations and the concept of function.

Given an equation $3.2y + 5.1x = -7.5$ and its graph shown below

a) Explain how you could find a solution to this equation.

b) Could you find a solution by using a method other than the one you used in part (*a*)? Explain your answer.

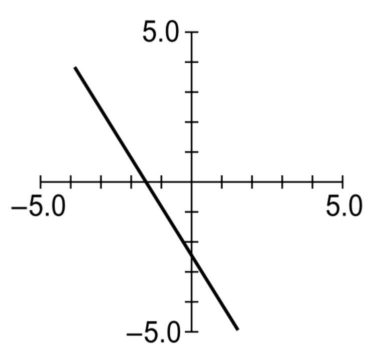


Figure 8. Algebraic and graphical representation for a function from Knuth (2000b) study.

From the beginning of their mathematics instructions, students rely on representations to both communicate and comprehend the concepts learned. Building on their experiences inside and outside of the mathematics classroom, students depend on those experiences to develop an understanding that relies on making connections between different forms of conveying mathematical ideas. The bridging of concepts and their representations becomes especially important with the introduction of a concept of a function; an abstract notion with essential dependence on multiple representations and the bridging between them.

Chapter Two Summary

This chapter explored the introduction and implementation of representation throughout mathematics curriculum as the primary means of communicating abstract ideas. Representations were further considered with the respect of subsequent introduction of the concept of function which usually takes place in middle and secondary grades. Significance of function representations in the curriculum and mathematics instruction was discussed within the context of Common Core Standards (2010) along with the opinions of the National Council of Teachers of Mathematics (2000). Specific examples were examined of educational research studies focusing on children developing mathematical representations (i.e., Kaput, 1999; Steinbring, 2006) and students making connections between multiple function forms derived through hands-on activities (i.e., Ellis, 2011). Several misconceptions that students hold were mentioned, specifically misinterpretations of function graph analysis (Ainsworth, 2006; Elby, 2000). Procedural fluency drawbacks were briefly discussed relating to students working with algebraic and graphical representations of functions (Knuth, 2000b). The chapter closed with the note of the importance of developing representational competence early to afford a better comprehension of function representation in secondary grades. In the next chapter, connections are made between students' understanding of functions and the specific roles each function representation can play.

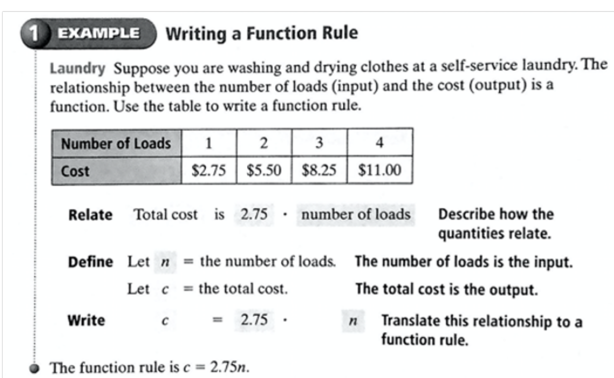
CHAPTER 3

CONNECTIONS AND UNDERSTANDING AMONG FUNCTION REPRESENTATIONS

This chapter begins with the introduction of the concept and definition of function as defined in mathematics curriculum. Traditionally the formal definition of function is introduced in eight grade with a considerable focus on linear relationships. A comprehensive analysis of linear, quadratic, exponential, and trigonometric functions takes place in secondary grades (Common Core Standards Initiative, 2010). Students rely on their previous knowledge of mathematical representations as they learn to express new learned concepts with algebraic and graphical representations. Relevant research and examples of studies support both the benefits and drawbacks of those representations. Knuth (2000) and Panasuk (2011), for instance, pointed to students' preference of algebraic equations while Wagner's research (1981) showed students' struggles with algebraic rules tied to poor understanding of the symbolic role of variables. Contextual and illustrative representations can serve in a supportive role to more formal representations by placing functions in context and thus making an abstract concept far more approachable. Although less present in secondary classrooms, the importance of illustrative and contextual representations is advocated by both Common Core Standards (2010) and the National Council of Teachers of Mathematics (2000). No matter the level of representational competence, the literature overwhelmingly stresses the significance of the ability to meaningfully translate between function forms. As the connections and translations between the discussed function representations are explored, this chapter concludes with a discussion of conceptual and procedural understanding as it relates to the topic addressed within this review.

Concept and Definition of Function

While much of the field of mathematics is a study of abstract thinking, that becomes especially pronounced with the proper introduction of relations and functions. The formal definition of a function is usually introduced in introductory algebra class, in late middle school grades or early high school courses (Common Core Standards Initiative, 2010). Most Algebra textbooks define a function as "*a relation that assigns exactly one output value for each input value*" adding that "*for each input there is only one corresponding output*" (Bellman, et al., 2009, p. 27). Typically, such an introduction is followed by an example of a functional relationship between two quantities illustrated by an algebraic expression, table of values, verbal description, familiar context, or graphical representation (Figure 9). Mathematical representations that students have seen over the many years of their mathematical education is expected to aid their understanding of patterns, relations, and functions with "more sophistication" (NCTM, 2000).


1 EXAMPLE **Writing a Function Rule**

Laundry Suppose you are washing and drying clothes at a self-service laundry. The relationship between the number of loads (input) and the cost (output) is a function. Use the table to write a function rule.

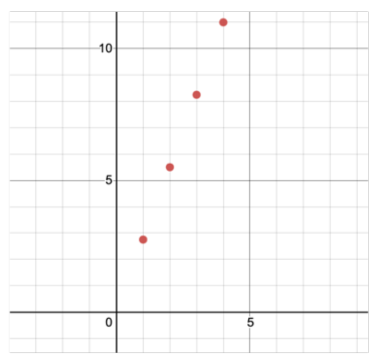
Number of Loads	1	2	3	4
Cost	\$2.75	\$5.50	\$8.25	\$11.00

Relate Total cost is 2.75 · number of loads **Describe how the quantities relate.**

Define Let n = the number of loads. **The number of loads is the input.**
Let c = the total cost. **The total cost is the output.**

Write c = 2.75 · n **Translate this relationship to a function rule.**

The function rule is $c = 2.75n$.



n	$2.75n$
1	2.75
2	5.5
3	8.25
4	11

Figure 9. Example following a formal definition of function in high school Algebra textbook (Bellman, et al., 2009) and a dynamic software generated graph (desmos.com).

Although students are not formally introduced to the definition of function until high school, they have worked with the notion of functions throughout the elementary and middle school years. The National Council of Teachers of Mathematics (NCTM, 2000) outlines students' use of various representations to model real life situations. As early as elementary school, patterns are studied that model linear relationships and those patterns are often represented with both graphical and verbal or contextual representations. By sixth grade, students model and solve problems using variables, tables, and graphs that often represent functional relationships. Throughout middle school grades students are presented with real life problems (often referred to as "word problems") that illustrate linear relationships set in context. Intuitively, even younger students are aware of relationships with correlating variables, such as things growing over time. This section will present a synthesis of the educational research and literature approach to mathematical representations with specific connections to not just underlying functional relationships but the very concept and definition of function.

Types of Function Representations

By high school, students use mathematical representations to reason with abstract entities of functions. Representations are used to illustrate mathematical concepts along with facilitating, examining, and analyzing mathematical relationships (NCTM, 2000). It is important to distinguish that this paper focuses on what is known as "external representations"; that is to say, mathematical representations that can be simply put on paper, or observed in the physical world around us, and/or described in words. In contrast, "internal representations" are those that learners form mentally, using their imagination to connect concrete, physical objects to the mathematical interpretations (Lingefjärd & Jonaki, 2016). While both external and internal representations are closely connected in developing mathematical comprehension, external representations take standard forms of graphs, equations or tables.

The following section focuses on four main such external representations most commonly used in the classrooms to help students reason with functions: algebraic, contextual, illustrative, and graphical. Each representation is defined by their role in the mathematics curriculum and accompanied by a short overview of its placement in current mathematics standards. The corresponding subsections provide a synopsis of research study examples and findings focusing on each function representation, its benefits and downfalls.

Algebraic Representations of Function

Algebraic representations are often the first and preferred form of function depiction by students (Panasuk, 2011) and their instructors (Knuth, 2000b). An algebraic representation of a function is typically a symbolic expression or algebraic equation that defines a rule of a function. A glance over elementary and middle school curriculums (Common Core Standards Initiative, 2010) might provide some explanation as to why algebraic or symbolic forms are so preferable. In elementary schools, by third grade, students are expected to recognize arithmetic patterns and connect them with algebraic operations thus introducing the belief that many if not all mathematical concepts are meant to be represented with algebraic forms (Common Core Standards Initiative, 2010, p.23). In fourth and fifth grade, students are introduced to using letter symbols, soon referred to as variables, to stand in for an unknown quantity in algebraic equations (Common Core Standards Initiative, 2010, pp. 29, 35). In middle school, as they are expected to start analyzing quantitative relationships between variables, students learn to write equations representing those relationships (Common Core Standards Initiative, 2010, pp. 43, 49, 54-55). It is worth mentioning that middle school curriculum encourages making connections between symbolic forms, graphs, and tables that relate to a given equation, and those are mostly representing linear relationships (Common Core Standards Initiative, 2010, p. 52). By the time students are formally introduced to the concept of function, they have spent years learning how to approach many math problems with an algebraic or symbolic form.

Before the formal introduction to the definition of function, functional relationships and their algebraic representations are woven throughout the mathematics curriculum. In elementary school students are mostly focused on arithmetic operations. By third grade, the curriculum draws attention to arithmetic patterns, including those in addition and multiplication tables (Common Core Standards Initiative, 2010, p. 23). Before entering middle school, students generate and analyze patterns with the expectations of producing a general rule (Common Core Standards Initiative, 2010, pp. 29, 33). Although students can describe patterns verbally or with an aid of graphs and illustrations, more often than not, arriving with an algebraic expression is rewarded and considered the soundest form. Countless "real-world" situations and relationships that students become very familiar with throughout their early mathematics education years are modeled by linear functions that they will not be formally introduced to until eighth grade.

It has been argued that students working with functions place a significant preference on solutions in algebraic forms (Knuth, 2000a; Panasuk, 2000; Yerushalmy & Schwartz, 1993). One of the underlying reasons for this issue is the overwhelming number of practice problems and examples used in mathematics classrooms being aimed at algebraic solutions as an end goal in problem solving (Yerushalmy & Schwarz, 1993). This over-reliance on algebraic expressions stems from years of focusing on purely arithmetic operations. Panasuk and Beyranevand (2011) outlined the issues stemming from a sudden shift from arithmetic to algebraic procedures in early middle school curriculum. As students transition from arithmetic to algebraic thinking, the approach to problem solving becomes somewhat procedural, the authors argued. The very process of practiced steps that students so heavily rely on while solving simple and multistep equations, if applied to algebraic expressions tied to functional relationships takes away the awareness of the relationships between the variables (Panasuk & Beyranevand, 2011). This is especially evident with expansive coverage of linear functions. A significant part of the curriculum is devoted to algebraic manipulation of linear equations with the focus on their forms: standard, point-slope, and slope-intercept. This excessive technical manipulation of terms within an algebraic expression of linear functions can cause detachment from the implication of the symbols and quantities they reference (Arcavi, 1994).

Algebraic forms of functions are a convenient form of representation as they can be easily manipulated and used to transfer between other forms of representations. Given an equation of a linear function, for example, Algebra students are taught how to fill in the table of values, find contextual solutions, or graph and analyze function behavior. Moreover, algebraic expression is usually the form that can be used with a graphing utility calculator or dynamic software. It comes as no surprise that algebraic representations are much favored throughout the math curriculum and in mathematics classrooms. What is overlooked is the reinforcement of the idea that not every functional relationship can be expressed using a single equation or moreover, some functions cannot be described with algebraic forms at all (Bossé & Adu-Gyamfi, 2014). A frequent misconception among algebra students is that a function cannot exist without an algebraic form that describes it (Bossé & Adu-Gyamfi, 2014) or a definite rule (Vinner, 1983). Because of the convenience and accessibility of algebraic representations as well as their overly utilized use, function equations can seem like an end in itself rather than a form of a broader concept.

The very idea of a variable along with algebraic representations frequently presents students with struggles as they start to develop connections between various ways of depicting mathematical concepts (Driscoll, 1999). The concept of a variable is closely connected to the idea of a function (Leinhardt, et al., 1990). Wagner's study (1981) on reasoning with variables in the context of function revealed that many students hold a simple misconception that different symbols (letters) used as a variable in an equation of a function can change that function's characteristic. Two of the equations in Wagner's example were $7w + 22 = 109$ and $7n + 22 = 109$. Students that declared the two function equations with changed variables as two different functions, clearly did not recognize the algebraic equation as merely a form of representing the functional relationship between those variables, regardless of the used symbols. Leinhardt et al. examined Wagner's study and concluded that those students saw variables as concrete objects, and not as representations of concepts. Just as with other mathematical forms, students must understand that a symbol, such as a variable in an algebraic function equation, is a representation of a notion and not an object on its own.

Fluency in algebraic forms is encouraged across mathematics curriculum as it is beneficial in problem solving, and mathematical representations among various contexts (NCTM, 2000). Algebraic equations are also prioritized as they are often used as a form of solutions in standardized test problems. Algebraic symbols allow students to easily manipulate the equations used to illustrate the functional relationship rules. While the technical manipulation of equations in process can seem to be very procedural, it affords the students quick access to solutions and thus providing them with the ability to easily transfer between various representations. A tabular representation can often not be achieved without an equation and numerical manipulation. A graph of a function, in today's classroom, more often than not, is created with the aid of technology that requires an algebraic approach. But most importantly, algebraic equations as function representations afford the student to abstract from context (Saul, 2001). Algebraic representation is in many ways the most flexible in its affordances of abstraction and connectivity to other mathematical forms.

Contextual Representations of Functions

Mathematics students should be afforded the opportunity to relate mathematical concepts in context (NCTM, 2000). The curriculum connects mathematics to other content areas, and real-world situations. Young children, from kindergarten through second grade, learn mathematics predominantly through connecting it to their daily experiences (NCTM, 2000). By the end of middle school, students learn to apply important mathematical concepts to other taught subjects like biology or social studies (NCTM, 2000). High school students become fluent in explaining mathematical applications in real world scenarios Common Core Standards Initiative, 2010, p. 58). Students at all grade levels are expected to make sense of mathematical results in context and consider the benefits of mathematical representations used in different settings (Common Core Standards Initiative, 2010). Connecting mathematical ideas to context outside of mathematics helps students perceive those concepts as an interconnected area of study rather than a disjointed collection of abstract concepts and tedious procedures.

As students learn to utilize multiple representations of a function, graphical or algebraic, those forms can become an end in themselves thus causing students to struggle with drawing conclusions that connect the representations back to their original context (Driscoll, 1999). Ellis (2011) example that was discussed in Chapter 2, outlines such a situation where students lost sense of the contextual setting even when working with physical models. Students worked with sets of different sized gears and generated algebraic rules to the correlations between the number of rotations between them. After abstracting the algebraic and tabular forms from context those same students lost the connection between the representation of a functional relationship and the original problem (Ellis, 2011). Placing the concept of function in context helps with conceptualizing the relationship between quantities. Additionally, the links between the representation formats and their context need to be reinforced past the introduction of the problem.

Clues from context provide a way to more deeply understand and analyze a function. For example, in an algebraic form functions illustrating temperature rising by 2 degrees each hour or a jar filling up with 2 marbles each day can look all but the same: $y = 2x + b$. Only when "de-abstracted", or contextualized, the function can be analyzed on a deeper level. The independent and dependent variables in the equation take on different meanings when put in context. The range of one allows negative numbers and fractions while the other, reasonably speaking, can only include positive even integers. Constructing a graphical representation for each situation requires different planning and thinking. Discussion of continuity or discrete values as well as domain and range boundaries take on a different meaning when placed in a real-world scenario. Contextual representations allow a deeper analysis of functions and stimulate the choice of function representations (McKendree, 2002).

It should be noted that contextual representations could hinder students' understanding of a concept if not set up in an easily accessible or recognizable scenario (Greeno & Hall, 1997). Common and often used examples come to mind of problems involving driving a car. Although the scenario of driving a car is broadly familiar to most, it might not be very helpful to young children. Even though most kids had spent at least some time inside a car, they themselves are not capable of driving one. Thus, realistic speeds, or discussions of distances between cities or time it takes to drive places might be lost on the students. Unfamiliar or overly complex real-life scenarios could hold learners back rather than facilitate better understanding of quantities and variables.

With a relatable context, the concept of function, abstract in its formal definition, can feel more familiar and intuitive. Mathematical problems in specific contexts facilitate meaningful student engagement which allows meaningful mathematical activities that can broaden the conceptual understanding (Meyer, 2001) of functions. Meyer (2001) cited Bell (1996) as one of the many researchers that firmly believe the algebraic concepts should always be introduced in real-world settings and those settings should be familiar to students. Setting up a math problem in an authentic situation that is familiar to young learners, or better yet, context that is accessible within the classroom setting (i.e., using students' arm spans to create a scatter plot) should prove to be more beneficial.

For a large part of Algebra education, students become skilled at new mathematical concepts as they move through stages of abstraction over time (Meyer, 2001). With time, students can become fluent in recognizing and describing functions with multiple forms. Formal mathematical representations, whether algebraic, graphical, or numerical, each offer advantages in problem solving and communicating function characteristics. A graph, table, or equation of an exponential function can aid in modeling a growing bacteria population, an ever-favorite textbook scenario. Numbers get easily substituted, values found, and the shape of the exponential curve can tell students a lot about rapidly increasing values. Those formal representations can have their limitations in understanding the real-life situation they model. Real life bacteria populations cannot grow to infinitely large numbers the way its modeling curve might suggest. The drawback to abstract forms of a function is the context lost.

Illustrative Representations of Functions

Illustrative representations are less formally recognized in the curriculum than other mathematical representations of functions such as graphs or equations. When referring to illustrative representation, in this literature review, it is meant to be exactly that: a visual aid (apart from a graph) demonstrating a functional relationship. Pictures or videos, diagrams, gestures, and drawings are used to set up real-life scenarios that represent a function rule and so they provide a form of function representation. By their very nature, illustrative forms of functions go hand in hand with contextual representations and support students' grasp of relationships between variables in familiar settings. Illustrative representations such as photographs can be beneficial with functions set in scenarios that are not immediately relatable to students (Figure 10). The benefits of illustrating functional relationships tend to be neglected by the time students progress to upper elementary or middle school grades. From scolding children for using their fingers while counting to focusing on abstract notations, illustrative representations are at times presented as "childish" and perceived as an inferior form of communicating mathematical concepts (Boaler, et al., 2016).

6. In science class Phylis used a light sensor to measure the intensity of light (in lumens per square meter, or lux) that passes through layers of colored plastic. The table below shows her readings.

Light Experiment

Number of layers	0	1	2	3	4	5	6
Intensity of light (lux)	431	316	233	174	128	98	73

a. Write an exponential equation to model Phylis's data. Let x represent the number of layers, and let y represent the intensity of light in lux.

b. What does your r-value represent?

c. If Phylis's sensor cannot register readings below 30 lux, how many layers can she add before the sensor stops registering?

Figure 10. Exponential function practice problem in context with accompanying illustration (Murdock, et al., 2002).

Pictures that accompany mathematical problems are a way for students to place a concept in context. Additionally, students can use their own drawings or diagrams to illustrate and convey their mathematical understanding (Woleck, 2001). Almost every idea in mathematics can be supported with an illustration (Boaler, et al., 2016). It is nearly impossible, however, to present one illustrative representation that will appeal to every student or help them deepen their understanding (Panasuk, 2011); student-initiated depictions often complement formal representations. Sketching out mathematical ideas is beneficial to students at any grade level; it helps with understanding of problems, conveying ideas, communicating their reasoning, and deepening conceptual understanding (Boaler, 2016).

It should be remarked that not every picture supplementing a problem or concept plays the same role in aiding the understanding of mathematical ideas. Elia and Philippou (2004) described a research study that addressed the effect of pictures depending on their purpose in solving problems. The authors categorized illustrations into four roles: decorative, representational, informative, and organizational. As one might quickly guess, the decorative pictures had little to do with helping students understand or solve a problem in any more efficient way. An example given was of a computational problem that asked to find the number of boys and girls in a classroom. The picture of a boy and a girl that supplemented the problem had no effect on how students approach the solution. It is worth mentioning that these kinds of illustrations are often found in textbooks and lesson presentations. The representational, informational, and organizational illustrations (see *Figure 11* for Elia's examples) were all very helpful to students. Among other benefits, representational images helped with placing the problem in context, informational pictures supported communicating with understanding of the problem set up, and organizational illustrations guided analytical reasoning. Elia and Philippou (2004) concluded that the use of illustrative representations hinges on the connection between the picture and the mathematical problem or idea, as well as students' abilities.

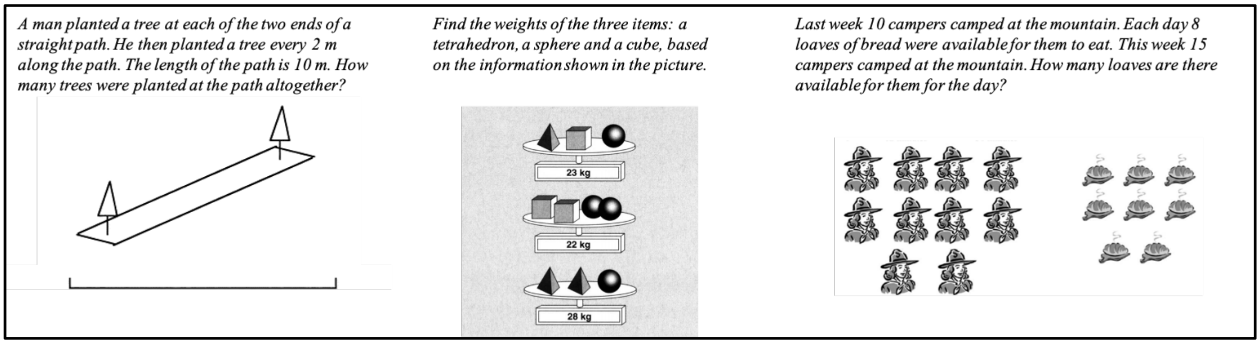


Figure 11. Representational (Booth & Thomas,1999), informational (Olson, 1998), and organizational (Misailidou, 2003) illustrations from Elia and Philippou (2004) paper.

Most common form of communication within the mathematics classroom is undoubtedly verbal. Teachers speak while delivering the class lesson, introducing new concepts, or launching a problem. Students are encouraged to explain their thinking, clarify their process, or ask follow up questions (NCTM, 2000). Gesturing naturally accompanies speaking in everyday life. Within the mathematics classroom, gesturing has beneficial effects on learning (Cook, et al., 2017). Cook's research focused on the instructor's gesturing while explaining a mathematics problem. To avoid experiment bias in the teacher's delivery, Cook's team developed a computer animated avatar that served the role of a teacher during the experiments. The study concluded that students that were exposed to the avatar (teacher) that made meaningful gestures and facial expressions learned more and were able to solve the given problems faster than those that listened to an avatar that did not gesture. Although there is limited research on gestures that are specific to the idea of functions and their representations, it is undeniable that relevant gesturing can be helpful in conveying mathematical ideas connected to functional thinking (Cook, et al., 2012). Most students while learning about increasing and decreasing values of function, for instance, have seen their teacher gesture dynamically the shape of the function graph. Whether students motion to represent a geometric shape or use gesture to represent the slope of a line, gesturing has a definite place among illustrative mathematical representations.

Illustrative representations in the area of mathematics education are closely tied to the new and emerging electronic tools (i.e., 3-D digital dynamic tools, virtual reality headsets, educational applets), as well as currently widely used technological resources (i.e., graphing calculators, tablets) (Common Core Standards Initiative, 2010, pp. 7, 50, 69-72, 81; NCTM, 2000). Nearly any form of representations can be coupled with dynamic software, graphing calculators, countless applets, and easily accessible online resources. Students are able to use electronic technologies to access illustrative representations of ideas or problems in any aspect of their role. Whether helping with problems set in context outside of the mathematical spectrum or graphing complicated functions, both students and teachers can find instantaneous aid with searching for images, video explanations via internet search engines, creating graphs, tables, or creating simulations with the help of various software tools. National Council of Teachers of Mathematics as well as Common Core Standards, both recognize the importance of electronic technologies as an access to representations that can be meaningfully explored throughout the mathematics curriculum (Common Core Standards Initiative, 2010; NCTM, 2000).

Graphical Representations of Functions in Cartesian Plane

One of the essential components of high school mathematics is graphing on the Cartesian plane. This is especially evident in Algebra curriculum. Common Core Standards Initiative (2010) recognizes a graph of a function as a beneficial way for students to model functions that can aid in analyzing the function's properties. Graphical representations are recommended for illustrating the very relationships that a given function models. The Standards also include using graphs to manipulate mathematical expressions as well as a visual representation of solutions to an equation in form of a coordinate pair for a single equation and/or points of intersections corresponding to a system of equations (Common Core Standards Initiative, 2010, pp. 54-55, 66). Graphing is fundamental to the concept of function and so the elements of graphing are embedded throughout the current high school standards and mathematics curriculum.

By the time students are formally introduced to the concept of function they are already fairly familiar with graphs. In early elementary years students work with number lines to model arithmetic operations and by the end of third grade they are familiar with representing data in bar charts (Common Core Standards Initiative, 2010, pp. 20, 24-25). Coordinate system is used in the geometry aspect of plotting points to create two dimensional figures. By the end of middle school students are able to represent real world problems by plotting points in the coordinate plane, and interpret the coordinates in context. Students utilize graphs to analyze relationships between corresponding terms of arithmetic patterns which will subsequently be called on as examples of linear functions (Common Core Standards Initiative, 2010, pp. 35, 38). When functions are first introduced in pre-algebra classes, the idea of representing patterns by plotting points in the coordinate plane is well established.

Research has shown, however, that even students that are proficient in creating and reading graphs can fall short in analyzing those graphs as a representation of a function (Ainsworth, 2006; Leinhardt, et al., 1990; Roth & Bowen, 2001). Leinhardt, et al. (1990) through broad literature research have outlined several misconceptions students have while working with graphical representations of functions. While some of the misconceptions are connected to poor understanding of the concept of variable others stem from difficulties with the function notation itself. Additionally, it is not uncommon for learners to believe that functions can only be linear and continuous. Most interestingly, among the various misconceptions that Leinhardt, et al. have pointed out, one common uncertainty among students is what actually constitutes a function as described by its formal definition. This rudimentary confusion shows how great the disconnection can be between the familiar mathematical representation such as a graph and the fairly new concept the representation merely portrays: a function.

Because of students' familiarity with graphing on Cartesian planes, it is easy to fall into an assumption that functions, especially linear functions, when represented as a graph are, in fact, easily accessible. Teachers often rely on students' prior knowledge in hopes to build on their mathematical experiences while learning a new concept. Arcavi's (2003) paper discussing the role of visualizing in mathematics provided a great example on students' perception of lines. Arcavi recognized some of the difficulties learners can encounter when analyzing seemingly simple graphical representations. The author argued that even an uncomplicated graphical representation can take on different meanings when placed in different contexts. His observations are supported with a simple diagram of three parallel lines. As seen in *Figure 12*, three plain parallel lines can be perceived very distinctly on their own or superimposed on a Cartesian plane. Those well versed in Parallel coordinates, will also consider the three lines yet differently when they are placed with parallel coordinates axis. Three parallel lines on a plane without context will most likely recall geometrical properties while parallel lines in a Cartesian plane can bring on characteristics associated with linear functions: slope, intercepts, or lack of common solution between them. Students' perception of a graphical representation, by this argument, is rooted in their conceptual understanding (Arcavi, 2003). The ability to analyze graphical representation of a function cannot simply rely on proficiency in point plotting or awareness of coordinate plane structure.

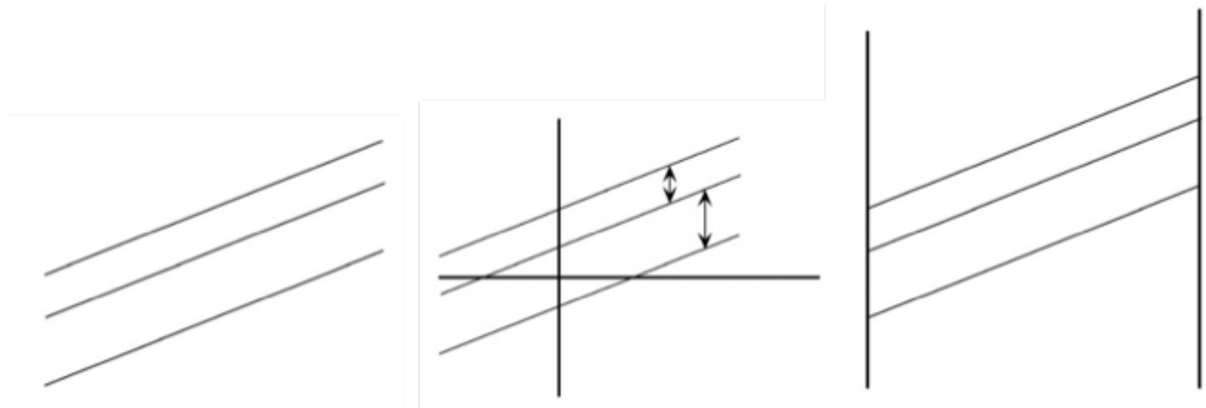


Figure 12. Three parallel lines: on a plane, on a Cartesian plane, and on Parallel Axes Representation (Arcavi, 2003).

There are numerous research papers that attempt to articulate why students can struggle with graphical representations of functions (Kaput, 1999; Leinhardt, et al., 1990; NCTM, 2000; Vinner, 1983). Vinner's (1983) work with high performing Algebra students in grades tenth and eleventh provided an insight into the connections between students' understanding of function definition and its graphical representation. In his study, Vinner introduced the idea of the cognitive connection students make between the learned definition of a function and the graph or image of a function that was provided as a formal graphical representation. A geometry student, for instance, that has been only shown examples of isosceles triangles where the base was always presented as a horizontal segment might not recognize an isosceles triangle that is somewhat tilted on a textbook page. In a similar manner, an Algebra student might not recognize a graph as a function representation if said graph does not look like the "concept image" they were taught to associate with a function (Vinner, 1983). Vinner defined a "concept image" as a combination of a mental (or internal) visualization of an idea that is combined with some acquired knowledge of its properties. Analyzing 146 students' conceptions of both function definition and image of a function, Vinner concluded that the initial image of a function along with a formal definition had a great impact on how students perceived and analyzed graphs or descriptions of functions that did not match perfectly their perception of what they were thought a function graph should look like. *Figure 13* is an example of a function graph that many students in Vinner's studies struggled with as they "claimed that a graph of a function should be symmetrical, persistent, always increasing or always decreasing, reasonably increasing, etc." (Vinner, 1983, p. 303). The misconception of the initial image of a function and what that might look like for any given function does not necessarily have to prohibit students from successfully working with functions and/or analyzing their graphs at a satisfactory level. The connection students make between functions and graphs, Vinner noted, can depend on far too many explanations that might not be easily recognizable to the researcher. One of the recommendations the author clearly stated is the possibility of reinforcing the initial concept image and definition of the function throughout the curriculum and not just at the formal introduction of the concept.

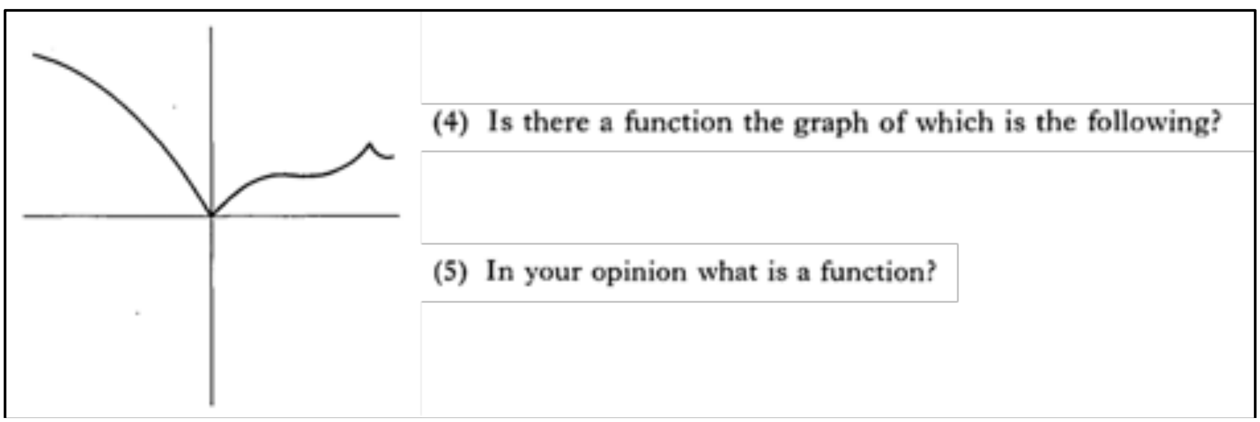


Figure 13. Function graph with sample questions from Vinner's (1983) study.

Even students who excel at creating graphs with clear connections to the functional relationships they represent might not be able to interpret a graph as it relates to a real-world situation (Bell & Janvier, 1981). In their review of educational research, Bell and Janvier felt that most high school grade students were lacking the ability to interpret features of a graph as it related to real life situations. This brings up a question of the contextual level of understanding graphical representations of functions. As mentioned in Chapter 2, the features of the graph ccan take over the very situation the graph is supposed to represent thus creating the disconnection between representation and the very context that stimulates it.

No matter the nature of individual student confusions with graphs, the benefits of graphical representations of functions can hardly be denied. The visual display of graphs can tell a lot about the relationship between variables with just a simple glance (Arcavi, 2003). Without numerical manipulations, or verbal explanations, a graph can convey characteristics of a function quickly and efficiently. Graphs are an accessible and standardized way of efficiently communicating features of a function both as an abstract idea or a contextual model. Graphical representations are also likely for students to encounter in other subjects like social studies or science (Leinhardt, et al. 1990) thus equipping them with tools they can apply in other areas of their studies. Function graphs are part of visual reasoning and illustrating mathematical ideas that supports students' understanding (Boaler, et al., 2016). Moreover, graphical illustrations of functions are often a link between multiple forms of mathematical representations.

Function Representations and Understanding

Mathematics is a field of interrelated ideas and so it is crucial that students make the connections between not only mathematical concepts but their various corresponding representations. Students that are capable of making meaningful connections between mathematical concepts and their forms, both abstract and contextual, develop lasting, conceptual understanding (NCTM, 2000). It has been argued throughout educational research that using multiple function representations strengthens students' understanding of the concept of function (Chigeza, 2013; Cunningham, 2005; Knuth, 2000a; Leinhardt, et al., 1990). As early as elementary school, students discover simple concepts and start connecting between ideas to advance and broaden their mathematical thinking (NCTM, 2000). As mentioned in Chapter 2, Blanton and Kaput (2011) brought up research studies concluding that elementary school children are capable of reasoning about recursive patterns, covarying relationships, and functions using multiple representations. By fifth grade students should be able to analyze patterns and relationships, find ordered pairs from corresponding terms and graph them on a coordinate plane. Before entering high school, the curriculum ensures that students are familiar with function representations (i.e., graphs, tables, equations), terms and vocabulary (i.e., *variables*, *domain*, and *range*), and can apply and build on previous understandings of concepts to work with functional relationships in a formal setting (Common Core Standards Initiative, 2010). In high school, Algebra and Pre-Calculus students are prepared to see the likeness in mathematical concepts that might seem contextually different but share similar representations (NCTM, 2000). Translating between various representations fosters understanding of functions (Cunnigham, 2005).

Sequencing, Translations, and Connections between Function Representations

The next section discusses the links and transitions between representations of functions. With orienting the introduction and sequencing of mathematical forms and the development of connections through meaningful translations between them, each corresponding subsection provides illustrations from literature and curriculum. Although not a lot of research has been reported on the specificity of optimal introduction or sequencing of particular function forms, a lot has been said about the importance of meaningful connections between the representations. Representational competence, knowing what representation is best fitting for a specific task or solution, is essential to function understanding (Knuth, 2000b). The following subsections provide an overview of the importance of sequencing and translating of representations as illustrated with notable examples from educational research and teaching resources.

Entering

Although it is brought up in numerous publications, educational research offers very little in terms of how a new concept should be introduced to students for optimal learning experience. The entry point is the starting point of introduction into mathematical topics, more particularly entry to the concept of functions, and it is very much connected to their representations. Leinhardt and colleagues (1990), in their extensive review of literature, discovered that although the entry point is crucial to how the topics would be further presented and scaffolded, there is no ideal beginning point. Stylianou's (2010) research on instructional approach to the topic of functions in problem solving revealed that individual teachers can have very different perceptions of the role of function representations. Some teachers saw the function representations as an entry point to the topic, while others used it as an explanation tool to aid students with their reasoning. Additionally, a number of teachers stated that representations are used as means of assessment to measure students' problem solving progress. The role of entry point of function representations with respect to both the academic research and classroom observations stand point is scarcely defined in educational studies.

Stylianou's (2010) research is worth mentioning as it focuses on the teachers' perception of mathematical representations in the classroom. Eighteen middle school teachers volunteered to participate in the study that largely emphasized teachers' notions of the representations in the teaching of mathematics. After they worked with math tasks that involved various representations, Stylianou observed that the teachers had varied views on the role of representations. Stylianou concluded that the two main reasons for those mixed views are the lack of a distinct definition of representations in the classroom and teachers' tendency to disconnect representations, especially those apart from algebraic, from the mathematical ideas themselves. Teachers overwhelmingly treated visual and graphical representations as secondary to the algebraic and numeric forms of solutions to the given math problems. It was not clear whether the teachers that recognized representations as an entry point to the mathematical problem solving also construed the representations more broadly, but most of participants of Stylianou's study perceived mathematical representation to be essential in the middle school mathematics curriculum.

In consideration of the entry point of function representations, the concept of "task launching" should not be overlooked. The starting point of a mathematical task is what teachers and the educational field refer to as "launching". A carefully developed task setup can impact students' achievements in the lesson. Similarly, a well-designed entry point to a new topic can impact students' comprehension of the broader mathematical ideas such as functions (Jackson, et al., 2012). From an instructional stand point, the introduction to a task has an underlying goal of students' engagement and peer to peer discussions that help facilitate the lesson. In the eagerness for classroom participation, the task launch design should not neglect the opportunities for mathematical reasoning (González & Eli, 2015). Specific function forms can help with the launching of the mathematical problem and facilitate the connections between the representations and concepts.

With the progressive use of technology and electronic devices, mathematics classrooms can become more engaged, hands-on, discovery-based environments (NCTM, 2000). It is not surprising that the entry point for introducing math problems, lessons, along with formal mathematical concepts is often carried out with the aid of technology that renders visual representations. Whether it is in elementary or high school classrooms, from private educational blogs to board funded subscriptions, both students and teachers find benefits in supplementing instruction and enriching the learning process with contextual and illustrative forms of representations, often supported by technology.

The integration of technology into the mathematics classroom has been shown to have multifaceted benefits (Gunpinar & Pape, 2018). Technological tools can provide instant access to multiple function representations that otherwise are time consuming with a traditional pen and paper approach. At times, conveyance of tools such as graphing calculators or online applets are argued to be far from beneficial to conceptualizing mathematical ideas as they are said to provide illustrations that are meaningless without proper analysis (Hollebrands, 2017). In the case of multiple function representations however, such tools provide access to mathematical forms that are often impossible to create in the restraints of the classroom. The time saving component, although valuable, is outshined by the technological possibilities of introducing mathematical problems in ways that would simply prove impossible without the wide availability of electronic devices. Without software generated two- or three-dimensional graphs of complicated functions (i.e., parametric functions) or illustrations of functions contextually inaccessible within the classroom setting (i.e., observing a water fountain arch) (see *Figure 14*) students, and their instructors, would solely rely on "internal representations" or mental pictures that are far more difficult to utilize in a launch of a task, or introduction to a new concept. Today's mathematics classrooms are rich with technological resources.

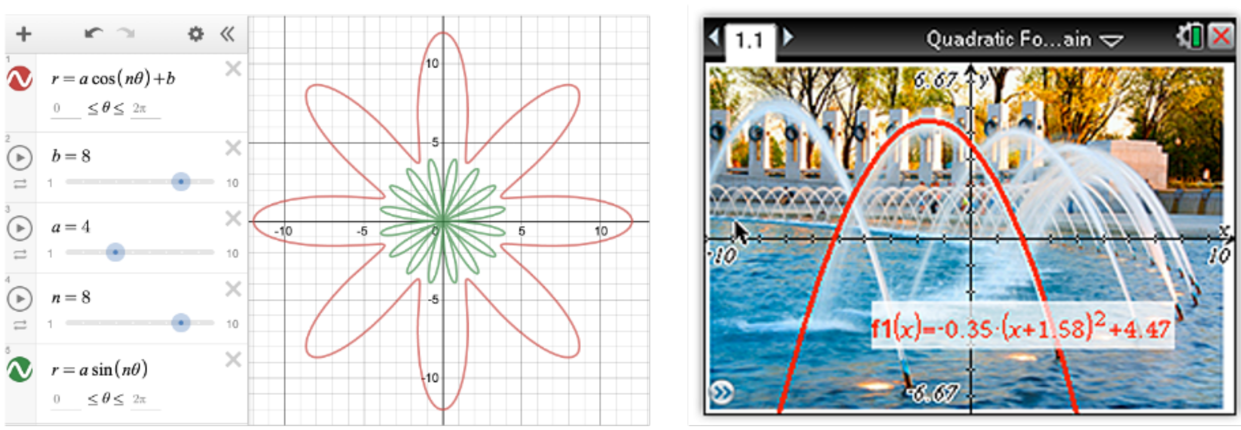


Figure 14. Graphs of parametric equations generated with dynamic software (learn.desmos.org) and a quadratic equation fitting a parabola shape of a water fountain constructed with TI*nspire* graphic calculator (education.ti.com).

Among many such well-known, popular, and peer respected resources, one worth mentioning is the Three-Act Math approach developed by Dan Meyer (2021). Meyer's approach to mathematical tasks consists of three parts: an engaging Act One, informational Act Two, and finally solution with Act Three. The engaging part, or Act One, most often consist of a video, animation, or dynamic software demonstration of a problem thus setting the visual (illustrative) representations as an entry. A great example of an illustrative representation as an entry point is Incredible Shrinking Dollar Task from Three-Act Math series (Meyer, 2021). The problem is introduced with a video of a person copying a dollar bill while reducing its size by 75%. Each subsequent copy is also reduced by 75%. The Act One video provides a contextual as well as illustrative representation of the problem. (see *Figure 15*). Setting up an illustrative representation of a concept as an entry point to understanding is far removed from more traditional textbook approach to functions that traditionally start with numerical (tabular) or algebraic (equations) forms of functions (Leinhardt, et al., 1990). The research is currently lacking on how beneficial this entry point approach is to students' reasoning or understanding of abstract mathematical concepts, what is clear though is that it creates a more relatable approach to students of all academic levels (Bell & Janvier, 1981; Janvier, 2011; Leinhardt et al., 1990). The introduction, or "entry" to a new concept or a challenging mathematical task, is not only important to setting up for optimal understanding of the topic, but it can dictate the sequencing of lessons and learning.

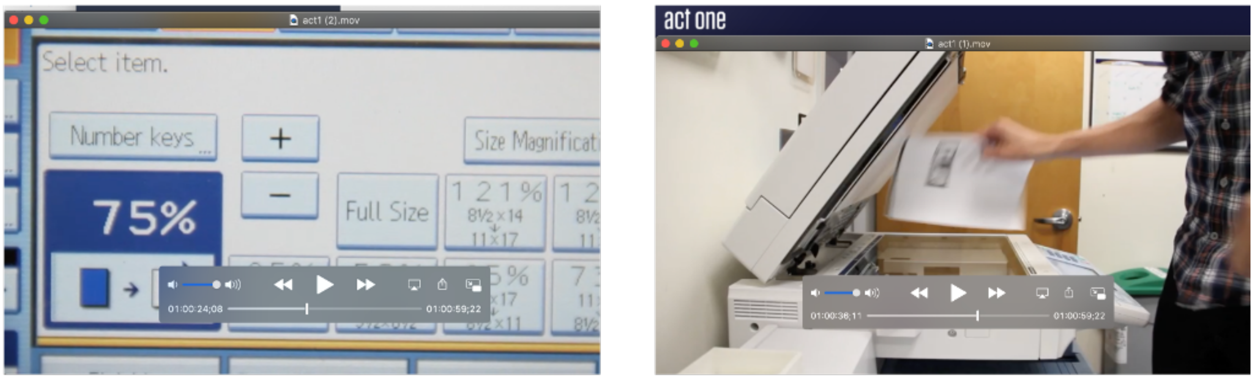


Figure 15. Screenshots of an introductory video for Incredible Shrinking Dollar task as an illustrative representation entry for exponential function task set in context. (https://mrmeyer.com/threeacts/shrinkingdollar/).

Sequencing

The significance of multiple representations in Algebra, and specifically within the topic of functions is undeniable. The effectiveness of representations relies not only on the information that is illustrated but also on the sequence in which they are presented (Ainsworth, 2006). Traditionally, as previously mentioned in this chapter, the introduction to the formal concept of function is often immediately followed by several of its representations: numerical values, algebraic equations, coordinate table, and graphical representation (see example in *Figure 9*). Such immediate display of various representations of the same concept assumes the advantage of students' prior knowledge and classroom experiences (Ainsworth, 2006). Although at this point in their mathematics education students are fluent in the numerical, algebraic, or graphical forms, they might not necessarily relate them to the newly learned, and quite abstract, notions. Plotting points, for instance, might not instinctively translate to visualizing a functional relationship from a graph. Sequencing of the various representations in connection to the concept of function is crucial to students' understanding of the concept of function as a broad idea with implications beyond the mathematics classroom.

The function presentation in math problems is typically sequenced from algebraic equations, to ordered pairs (often in forms of filling in a table of values), to a graph (Leinhardt, et al., 1990). Bell and Janvier (1981) and Ellis (2011) changed up that traditional sequencing of function representations and argued about the respective benefits. Bell and Janvier (1981) rejected the benefits of conventional methods of constructing a table of values to plot a graph of a function. In their opinion, such an approach denied students the full picture of a function as a relation between variables. A graph of a function is much more than just a set of ordered pairs (Bell & Janvier, 1981). In a way, graph tells a story that cannot be conveyed by tables or numerical expressions. Ellis's (2011) approach was to focus on tabular representations before students were introduced to the graphical forms. Moreover, students collected the data during a hands-on activity and only after analyzing their numerical values they were guided towards algebraic and graphical representations. Ellis's study showed benefits of delaying graphical representation while being motivated by the benefits of contextual and situational settings. Bell and Janvier focused on graphical representations that were easily accessible through means of technology which highlighted the benefits of graphs displaying function behavior. These two case studies bring up a notion that sequencing of function representations is important to the perception of a concept of function as a whole, although there might not be a set order of sequencing that guarantees best possible learning.

The sequencing of function representations, and the benefits of the order in which they are presented are related to building on students' prior knowledge, their experience with Algebra topics, and the desired outcomes of individual tasks. Some researchers go as far as to argue that prompting all representations at the same time yields important benefits (Ainsworth, 2006; Leinhardt et al., 1990). With representations presented in a sequence, Ainsworth (2006) points out, the need for added representations is often left to the students. Additionally, without simultaneous presentations of representations, teachers are set in the instructional dilemma of allowing students to come up with their own additional forms of representations or the teacher revealing them as deemed appropriate (Driscoll, 1999). No matter the stance on the appropriate sequence of multiple representations, the most common thread within educational research directed at Algebraic understanding is the ultimate goal of students being able to translate or move between the function representations in a meaningful way. Careful sequencing of function representations aids in translating between them with understanding.

Translating

Translating between function representations is certainly much more than just freely moving between multiple forms. When students successfully translate (or transfer) between different representations they are able to recognize the same function in multiple forms and spot the alterations from one representation to another. Additionally, translating between representations means being able to construct another representation of a function from a given form (Leinhardt, 1990). When students transition meaningfully between forms of representations it shows their deeper understanding of functions. Knuth (2000) claimed that a big part of understanding a function is not just the competence to move between the different forms but also knowing which representation is best suited for its context.

In Algebra classrooms, one of the regularly occurring translations between various forms is from algebraic to graphical forms. From the time students are introduced to functions, more specifically linear functions in middle school, they are drilled on using equation forms to fill out tables, plot points, and draw graphs (Common Core Standards Initiative, 2010; Knuth, 2000, Leinhardt, et al., 1990, Yerushalmy, 1991). It has been argued, however, that this one directional passage through forms exposes the type of fluency in translations between representations that is removed from conceptual understanding of functions (Cunningham, 2005; Knuth, 2000; Panasuk & Beyranevand, 2010). Nonetheless, this practice in translation from algebraic, through numerical and tabular, to graphical representations presents benefits to mathematics students. Transitions from algebraic and graphical representations provide the ease to maneuver between various forms of not just linear but subsequently other types of functions.

Students that excel in transferring from algebraic to graphical representations can struggle with moving in the other direction: graph to equation. The case study of over 170 high school students by Knuth (2000a) revealed difficulties translating between those two function representations. More than half of the students in Knuth (2000a) analysis were enrolled in advanced Algebra courses, which were described as "best mathematics students" in the study, including some enrolled in pre-calculus and advanced calculus classes. The problems presented in the study were somewhat direct; each task involving linear equations and their corresponding graphs (see *Figure 2.6* from Knuth (2000b)). More than three fourths of the group chose an algebraic approach and fewer than a third of the students chose the graphical representation as means of solutions. Knuth concluded that students do not see graphical representation as a process to find solutions. Even students that used graphs, did so as a step in their algebraic approach. Unquestionably, the translation from symbolic to graphical form of a function is one of the most common tasks in classroom practice, testing, and Algebra instruction.

One of the customary means of investigating students' ability to translate between various function representations is matching the algebraic expression to its corresponding graphical form. Plethora of textbooks, workbooks, standardized and teacher made tests, supply that very kind of questions to check for students' proficiency in function notations and representations. It has been indicated throughout research that the nature of these questions (see example in *Figure 16*) examines mostly students' abilities to manipulate algebraic forms. Students select the corresponding graphs by inspecting them with a point plotting approach rather than seeing the graphical representation as a whole (Leinhardt, 1990).

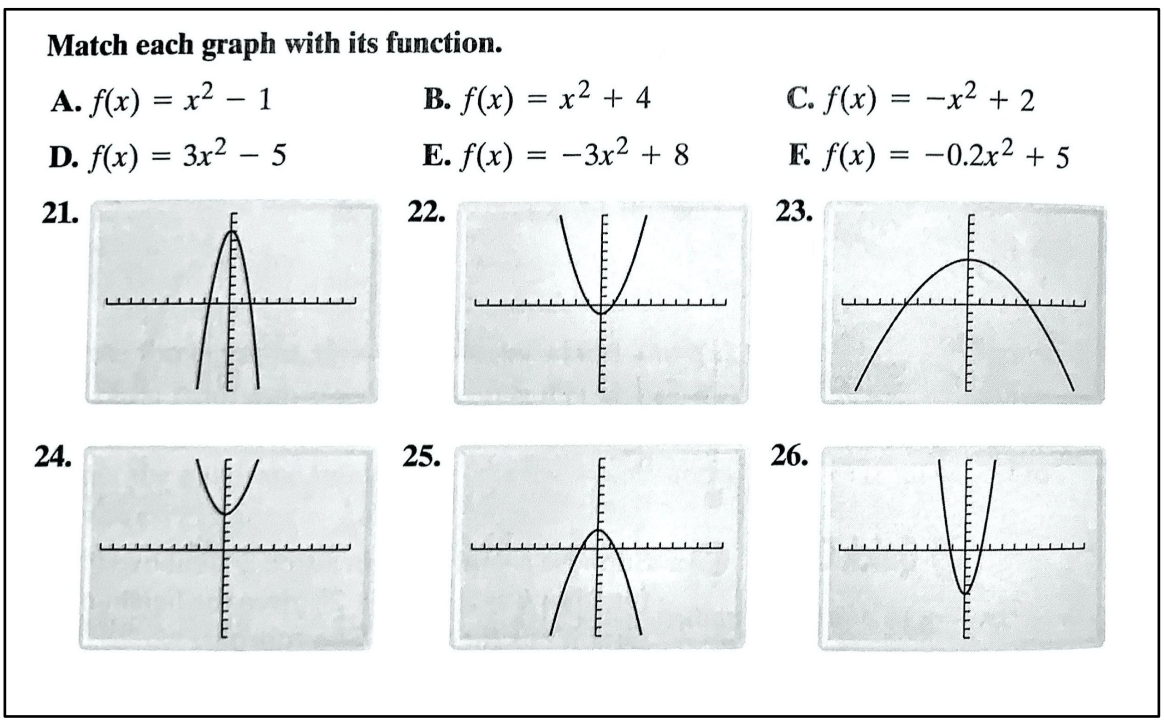


Figure 16. Translating representations example problem in high school Algebra textbook (Bellman, et al., 2009).

Yerushalmy (1991) conducted a study where, in contrast to more typical translations, transferring from a graph of a function to its equation was considered. Thirty-five middle school students participated in pre-algebra lessons over a three-month period. The emphasis was to study students' ability to move between function forms and did not focus on students' proficiency in constructing the representations on their own. The author designed problems where students were provided with computer generated graphical presentations and tasked with matching them to one of the four possible equations of linear functions. The visual representations were purposely set for students not to rely on the usual non-relevant information the researcher believed are commonly used to read off a graph, such as angles between the line and the horizontal axis or reliance on uniform scales of the coordinate system (Figure 17). The goal of the task was to examine students' ability to analyze the properties of given functions rather than its graph. The majority of the students were successful with the task and had no problems moving from graphical to algebraic forms as long as the graphs were not confusing. For example, working with graphs that were designed to be purposely misleading, in terms of scaling of the units on the coordinate system, students had more difficulty with choosing the correct corresponding equation. Yerushalmy argued that even with the graph to equation translation students rely on algebraic to numeric transfers which he argued are more instructionally efficient. The translation between function representation, as in Yerushalmy's case, can be simply motivated by the procedural experience favored in the classroom instruction.

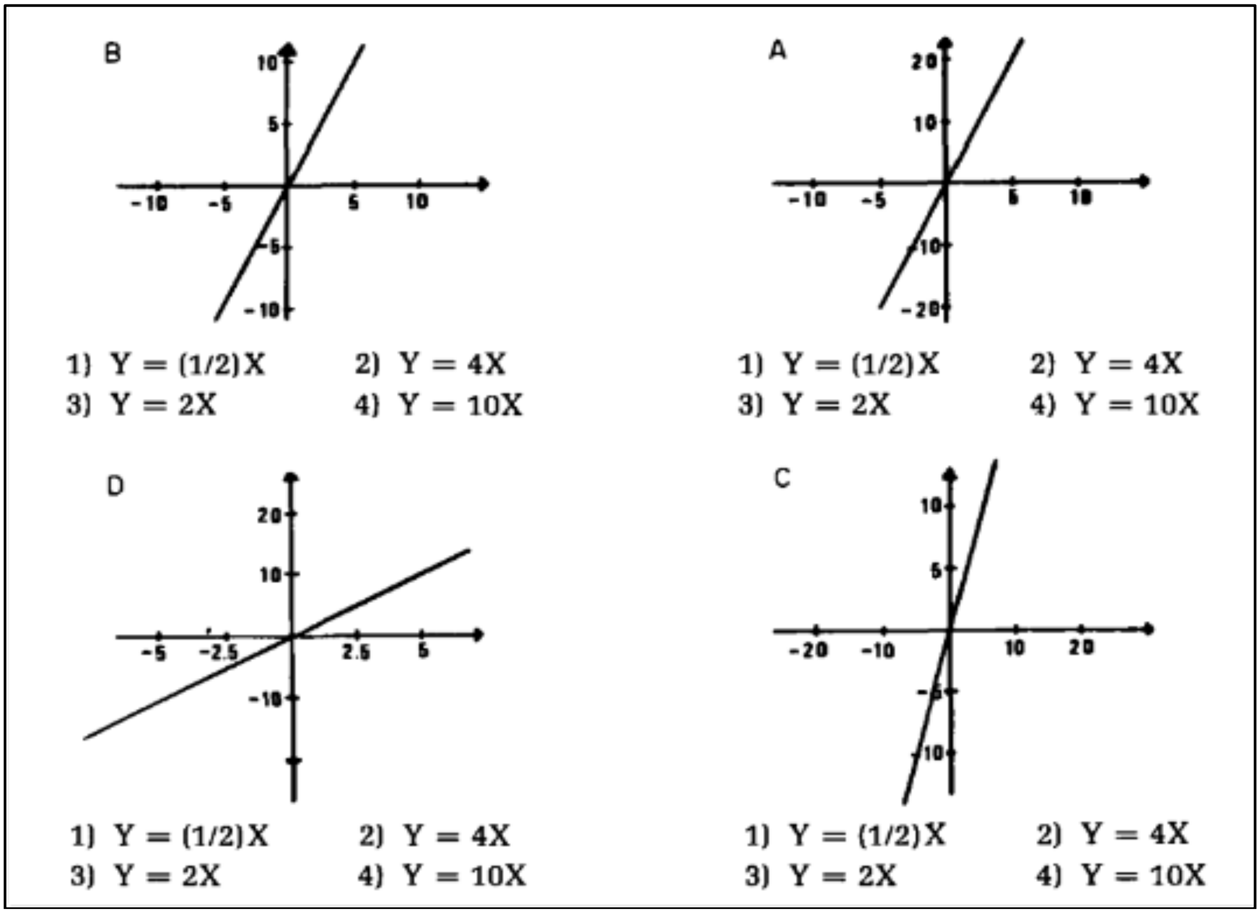


Figure 17. Task from Yerushalmy's (1991) study with two different pictures of the same function and two identical pictures of different functions.

Students become significantly more familiar with symbolic, numerical, and graphical forms in mathematics than other, less commonly practiced representations: illustrative, contextual, or verbal. Long before the introduction of functions, the curriculum gives learners plenty of opportunities to practice working with algebraic expressions, number lines, and the Cartesian plane. Simple equations and substitutions are practiced to find solutions while graphs are seen as a representation of real-life situations, even if not yet formally assigned to functional reasoning. Ainsworth (2006) called on numerous research sources (e.g., Anzai, 1991; Schoenfeld, et al., 1993) that remark on students' difficulties with translating between representations. In his work, Ainsworth noted that those translating difficulties are less pronounced when students are already familiar with the representations at hand. This observation aligns with previously mentioned studies of Yerushalmy (1991), Leinhardt (1990), and Knuth (2000), where students were most comfortable with translating between algebraic and graphical forms of function representations.

There has been growing agreement in the mathematics education field towards students' abilities to translate between representations being greatly beneficial to the comprehension of functions (Cunnigham, 2005; Knuth, 2000a). Cunnigham (2005) in his extensive review of literature found that there have been few studies that involved learners transferring between function representations with the focus of such translation having a direct effect on students' understanding. What the author did figure is that in order for students to develop a deep understanding of functions they must be afforded sufficient opportunities to work with problems that require translations between various forms: algebraic, numeric, and graphic. Students develop an appreciation for functional relationships when they can transfer between various representations while making meaningful connections between them.

Connecting

The use of multiple representations has been long thought to be beneficial to students' understanding of the concept of function. As discussed earlier in this section, both entering and sequencing of function representation is important albeit uncertain of their optimal placement. Both the National Council of Teachers of Mathematics (2000) and Common Core Standards Initiative (2010) recommend that learners, throughout the grades, use and translate between multiple forms of mathematical concepts. The connections students make between the various representations is what supports the growth of the core and characteristics of functional relationships (Even, 1998). The mere use and practice of transferring between function representations are not in themselves a guarantee for better understanding unless students recognize and make the connections between them.

The literature does not offer many insights into the complexity of working with different representations while emphasizing on the connectedness between said function forms. One of the studies worth mentioning is Even's (1998) work with college students. Although the subjects of this study were older than the learners' that this review is generally concerned with, the curriculum involved, namely quadratic functions, was well within the bounds of general high school Algebra courses. The study involved 152 college students from education fields; some were prospective secondary math teachers. The students were familiar with quadratic functions and have studied and used them since high school and throughout college courses. The problem presented involved an algebraic form of a quadratic equation in a standard form: $y = ax^2 + bx + c$. The task (*Problem 1* in *Figure 18*) asked for the number of real solutions to the equation $ax^2 + bx + c = 0$ given two possible solutions (Figure 18). This problem, according to the author, was purposely designed in a "non-standard" set up to expose the connections students make between function representations. Only 14% of the students were able to answer the question correctly and did so by referring to a graph either mentally or by sketching parabolas (Figure 19). Even notes that although the students were very familiar with quadratic functions, they did not make the connection to the graphical representations after first being presented with a quadratic expression in its abstract form. A handful of students who were shown or suggested to use a graph still did not make a connection between the two representations even after translating between them. To the learners that linked the abstract quadratic expression and its corresponding graph the problem became quite straightforward and easy to solve. Analyzing the parabolas with the two possible solutions provided, left little ambiguity as to the number of real solutions, or zeros of the function. The ability to make connections between different function representations, Even concluded, reveals a deeper level of understanding and knowledge of functions.

PROBLEM 1
If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

Figure 18. Non-standard problem addressing function knowledge (Even, 1998).

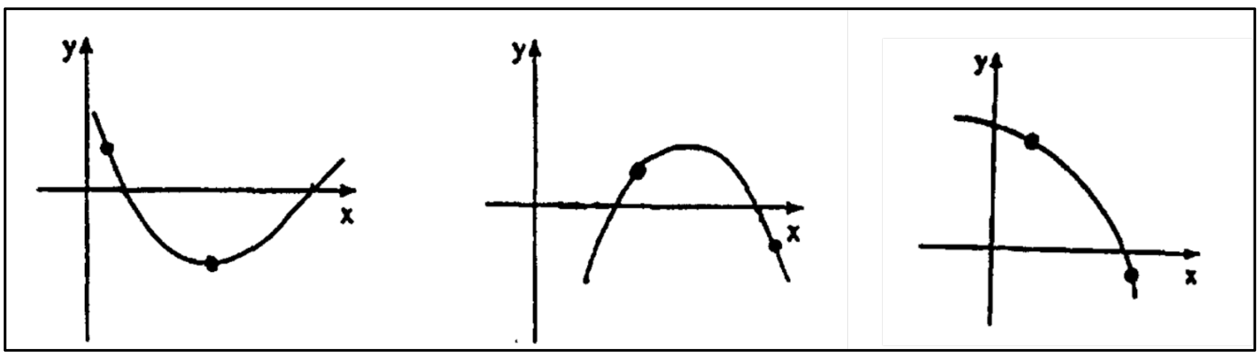
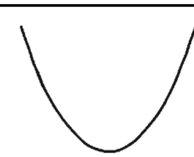


Figure 19. Graphic approach to Problem 1 (Even, 1998).

In a study of 400 students in grades eighth through tenth over a five-month period, Ronda (2015) investigated students linking the various representations of functions. The educational research on students making connections between multiple function forms is somewhat limited, and studies relevant to the topic of connectedness are mostly focused on algebraic and graphical forms. What makes one of the examples in Ronda's study notable is the attention to linking tabular representations with graphs (Figure 20). After a meticulous breakdown of the responses, the researcher was able to pinpoint the specific connections within students' problem-solving strategies. The three approaches included point plotting, analyzing general trends of the listed *y* values, and recognizing the curve as a representation of a quadratic function thus looking for constant second differences in *y* values within the given tables. The two latter strategies, it might be argued, show a more sophisticated connectedness as students look for global properties of a function and/or use properties of the specific function type. Students that were able to answer the question correctly, across all grades, overwhelmingly used point wise approach. The percentage of students linking tabular and graphical representations in a broader understanding of functions definitely increased in grades tenth from eight and ninth. The majority of students across the study could only connect the representation by relying on individual plotted points or ordered pairs. Similar to Even's (1998) conclusion, Ronda (2015) determined that specific function type and previous knowledge affect students making connections between multiple function representations.

Which table of values can be part of the graph on the right? Circle the letter corresponding to your choice and explain what you did to get your answer.



a.

x	1	2	3	4	5
y	0	8	12	10	4

b.

x	1	2	3	4	5
y	1	2	8	16	32

c.

x	1	2	3	4	5
y	0	2	4	6	8

d.

x	1	2	3	4	5
y	8	5	3	2	2

Figure 20. Matching the shape of a graph (without Cartesian Plane context) and tables of values task (Ronda, 2015).

Contextual representations aided with hand-on activities help students make connections between numerical and algebraic forms and other function representations. In Ellis's (2011) studies, pre-algebra students were not familiar with graphical representations of quadratic functions. Seven eighth grade students explored the relationship between lengths of the sides of rectangles and their areas. Students created tables and developed a functional relationship between the height and area of the rectangles. The author deliberately refrained from initially allowing students graphical forms of illustrating the relationships between the variables that were modeled with quadratic equations. Ellis thus afforded her students a deeper understanding of the covariations between the quantities they were studying by avoiding a purely procedural view of graphs by point plotting. This, Ellis argues, enabled the students to interpret their later constructed graphs by reasoning about the values of each point in context. Students were able, for example, to predict the shapes of parabolas based on the contextual experience they had from studying the "growing rectangles" using dynamic software. Ellis concluded that students made seamless connections between various function representations. Additionally, she argued, students used the representations that were most helpful to communicate the function studied. By allowing her students to initially link between the numerical data and physical models, students subsequently saw the connections between the functional relationship at hand and the ways of representing it.

Making meaningful connections between multiple representations goes beyond translating between the different function forms. Whether transferring from illustrative to algebraic form, or contextual to graphical representations (as described through examples), students must be able to see the characteristics and properties of the functions across the different representations (Ronda, 2015). When students are able to link between function representations, they show a broader grasp of mathematical ideas and provide a window into their understanding of a concept of function.

Conceptual and Procedural Understanding and Function Representations

Much of educational research in mathematics focuses on students' understanding of concepts. Throughout literature, mathematical knowledge is often sorted into two categories: conceptual and procedural understanding. Although each of those categories can be perceived literally by their corresponding name, it is useful to specify a simple characterization of each. Procedural understanding is often narrowly defined as a "step-by-step" method of accomplishing a task (Hiebert & Lefevre, 1986). This notion of procedural understanding (or procedural knowledge) is sometimes perceived as a lesser component in supporting the development of understanding of an abstract concept. Students can find themselves following steps of a process without giving much thought to the end result. When students manipulate forms of linear equations, for instance, without the realization of the equivalency between them, the procedure takes the president to the concept at hand. It is important to recognize a more broad view of procedural knowledge: one that requires a level of skills and strategies to successfully construct, sequence, and carry out the steps of a given algorithm (Stylianou, 2020). Conceptual understanding, conversely, is routinely thought of as a deeper grasp of mathematical concepts, where students comprehend ideas that are interconnected, and build their knowledge on the relations between them (Hilbert, 1986). For example, the ability to rewrite a linear equation in order to better suit it to a solution, or manipulate an expression to analyze its characteristics, shows the connection students make to the mathematical concept studied. Conceptual understanding (or conceptual knowledge), by definition, is more internal and thus more difficult to verbalize (Rittle-Johnson & Schneider, 2014). Procedural fluency and conceptual knowledge are essential to mathematical competence (NCTM, 2014). Comprehensive knowledge of functional relationships depends on both procedural and conceptual understanding.

As continually mentioned, function representations cannot be an end in themselves. Each representation tells a different story about the function it depicts. While graphs illustrate a function behavior in both global and local sense, equations provide abstracted form with endless contextual implications. Every function representation offers an important step in the process of functional thinking. When first introduced to the concept of function, learners tend to see it as a computational process (Ronda, 2015). Whether by filling tables of values or finding solutions, the idea of a function as a whole can be lost on procedural drills. As students translate between multiple representations and make relevant connections between them, they consider functions as more than procedures but a mathematical idea in itself; an object they can comprehend in a conceptual way.

With multiple function representations discussed throughout this review, it is remarked that the connections and translations between the representations do not always equate with function comprehension. Knuth (2000a; 2000b) study showed that even students effortlessly translating between graphical and numerical representations at times showed the procedural connection at best. When function values are presented in the tabular form, the connection to the graph or function equation can be simply tested by checking a few of the concrete values or points (DeBock et al., 2015). The connections and translations students make between function representations are often procedural. However, that procedural approach does not automatically equate to poor understanding. Procedural fluency is advocated in mathematics classrooms (NCTM, 2014). The aspiration is to expand the mathematical knowledge beyond procedural proficiency towards more adoptable comprehension of functions (Knuth, 2000a). When the process of translating between function representations becomes conceptualized, deeper connections are made.

Even though The National Council of Teachers of Mathematics (NCTM, 2000) distinctly recommends for students as early as third grade to translate among mathematical representations in problem solving, elementary grades and early middle school mathematics classroom instructions often allow students to focus on one form of representation tied to an idea or concept. This lack of using multiple representations embodying a singular concept could be a reason for students' struggles with conceptual understanding of functions later. Of course, in those early grades there is some assumption of young children's inability to grasp more complex concepts, like correlations between quantities which are at the core of formal function definitions (Blanton & Kaput, 2011). The low confidence in children's ability to translate between multiple representations of concepts at times results in focusing on singular representations which subsequently can hinder linking ideas and building between them thus leading to a less conceptual comprehension.

Some of the more recent educational research focuses on multiple function representations and their role in the classroom with emphasis on introduction of functional thinking to children as young as first and second grade (Blanton & Kaput, 2011). Over the decades, numerous researchers have agreed that the conceptual understanding of functions in later grades greatly depends on the successful implementation of multiple representations in earlier grades (i.e., Leinhardt, et al., 1990). Learners use multiple representations to reason about concepts as well as procedures. Those are the tools they subsequently use to develop a broader idea of function (Stylianou, 2020). Building on the procedural mastery gained in those early years subsequently allows students in middle and high school grades to develop deeper conceptual understanding and functional thinking.

Implementing multiple representations in younger grades along with functional thinking is emphasized in Blanton and Kaput's (2011) research. Their work drew conclusions from an extensive study examining elementary school students' functional thinking. The five-year research stemmed from professional development projects and graduate courses for elementary school teachers. Drawing from the data, the authors concluded that students are indeed capable of functional analysis much earlier than their formal introduction to function relationships. As early as first grade children are able to reason about co-varying quantities, and communicate how two qualities can correspond. Early formal introduction to functional thinking and development of mathematical representations to communicate patterns and relationships is crucial to development of conceptual thinking in middle and high school grades (Blanton & Kaput, 2011). Multiple representations are essential to conceptual understanding and should be introduced early along with functional reasoning.

The predicament of focusing on procedural fluency of a new mathematical concept prior to the concrete understanding of it is ever present in educational research; with some authors arguing that there is little to no place for procedural fluency (i.e., Chigeza, 2013). There have been ongoing discussions among researchers and teachers alike about the correlations between the procedural and conceptual understanding (Rittle-Johnson, et al., 2014). With a more complex perception of procedural knowledge, there is no doubt of their dependency on each other. When processes of manipulating equations, or creating graphical forms, are rooted in mathematical reasoning, connections are made that are undeniably linked to conceptualizing the process. This is especially true for function representations and their support of the grasp of functions themselves. When students work with multiple representations, they communicate their understanding by linking between representations which, by definition, reveals their conceptual knowledge (Stylianou, 2020). Students cannot meaningfully translate and make conceptual connections between the various function representations without some level of procedural competency.

Chapter Three Summary

This chapter investigated specific function representations. The role of algebraic, contextual, illustrative, and graphical forms was examined within the frame of middle school and secondary mathematics curriculum. The connections between representations were further explored through investigating the relevant educational research. Examples of studies pertaining to links between specific function representations in support of understanding suggested overreliance on algebraic forms and narrow view of graphs as a depiction of individual ordered pairs (i.e., De Bock et al., 2015; Knuth, 2000; Ronda, 2015). Correspondingly, Even's study (1998), among others, determined that students who were able to make meaningful connections between function representations displayed a more comprehensive understanding of the concept of function. The chapter concluded with a discussion of the importance of both procedural and conceptual understanding of mathematical ideas. While procedural fluency can at times be presented as a less significant part of building a strong understanding of functions, utilizing multiple representations requires practical competency.

CHAPTER 4

SUMMARY AND DISCUSSION

The motivation behind this literature review was to examine the use of multiple function representations throughout mathematics curriculum and the utilization of the various mathematical forms in the development of functional thinking alongside the understanding of the concept of function. The review has examined research taking place over the 40-year period from 1980 to 2020. The literature considered a vast range of mathematics education topics from symbol sense to functions analysis. The research involved a wide range of students' ages along with their mathematical abilities: from learners as young as kindergarten through graduate education students. This chapter covers the relevant mathematical education research findings, conclusions, and implications of the studies discussed in chapters two and three.

This chapter is organized into three sections. The first section offers a synopsis of the literature reviewed in relation to the topics covered in chapters two and three. The second section discusses the boundaries found in the review. The last and final section offers thoughts on the need for future research.

Summary

This review was rooted in educational studies, research findings, and articles focusing on the broad topic of external mathematical representations. More particular attention was placed on topics of function representations, their connection to the understanding of the concept of function, and development of functional thinking. Chapter two considered mathematical representations, their introduction and placement, throughout the curriculum from kindergarten through middle school. This allowed for a discussion of students' familiarity with function representations at the formal introduction of function definition, which usually takes place in high school grades. In chapter three, specific function representations were explored. The four main function forms: algebraic, graphical, illustrative, and contextual, were investigated with their place in classroom instruction. The connections and translations between those representations were examined as a foundation for students' development of a rich understanding of functions and their characteristics. The affordance of utilizing various mathematical representations was connected to building procedural and conceptual knowledge of functions.

Representations in Curriculum

Chapter two explored various mathematical representations and their place in the curriculum. Supported largely by the National Council of Teachers of Mathematics (NCTM, 2000) and embraced by the Common Core Standards Initiative (2010) is the belief that students' use of mathematical representations should be built on gradually through the grades. Young children's understanding of mathematical concepts is supported by the use of representations. This finding is supported by multiple studies including that of Chigeza (2013) and Edwards et al., (1993). Driscoll's (1999) *Guide for Teachers* often points to the representations both students' and their instructors use to cultivate algebraic reasoning and communicate their thinking in both problem solving and building better understanding of abstract concepts.

Overwhelmingly, the literature agrees on the point of mathematical representations not being an end in themselves. This belief is ever present in the work of Greeno and Hall (1997), Leinhardt, et al. (1990), and Steinbring (2006). Chapter three goes into a more detailed analysis of this principle specifically around function representations. As students utilize graphical representations to work with functions, for instance, it is important that the introduction of such graphs is not disconnected from the concept. Leinhardt, et al. (1990) specifically pointed out the limited utilization of graphical representations in the context of functional relationship between variables in the early grades. The author argued that graphs and functions are only explored with such connection in mathematics in high school. This disconnect from graph and the function it represents can misleadingly enforce the idea of representation being a standalone object. Uttal, et al. (2009) pointed to the fact that young children may not intuitively connect physical objects or informal representations to the mathematical concepts they represent, while Steinbring (2006) argued such connections are naturally made by elementary school learners. Most mathematics educators and researchers agree that representations are not an idea in themselves even if they disagree on the methods by which this belief should be enforced in the classroom.

Mathematical representations, both in a broader sense, and more specific to the concept of function, find their place throughout the K-12 curriculum. Knuth (2000a; 2000b) studies focused on the algebraic and graphical representations of functions. He argued that some heavy focus is placed on algebraic representations in middle school and high school grades thus dwarfing other function representations. This general concern is shared by both Panasuk and Beyranevand (2011) and Arcavi (1994). In their work, while arguing for more arithmetic processes and less procedural algebraic manipulations respectively, the authors concluded that the classroom instruction's focus on symbolic forms takes away from perceiving representations as a tool for understanding abstract concepts. The multiple representations of mathematics concepts including functions are well established throughout the curriculum, although their role in algebra instruction and development of function understanding can be at times far from definite.

Connections and Understanding

Throughout the review, various function representations were discussed. The main four function representations were explored in detail in chapter three: algebraic, contextual, illustrative, and graphical. Numerous literature pieces discussed algebraic forms and their graphical connections (Arcavi, 2003; Knuth, 2000a; Knuth 2000b; Panasuk & Beyranevand, 2011; Yerushalmy & Schwartz, 1993) including students' struggles with bridging between them in support of concept understanding. Illustrative and contextual representations, while interesting in their practical role in the classroom, are less present in the educational research. Boaler's (2016) and Meyer's (2020) use of illustrative representations were addressed with a narrower focus placed on classroom participation. While student engagement certainly supports comprehension development, less emphasis is placed on illustrative and contextual representations in their cognitive role. It should be noted that none of the research included in this review discussed one function representation in isolation from other mathematical forms, thus reinforcing the idea of the necessity of multiple representations in education and mathematics.

The review in its majority addressed external representations, however some attention was given to mental connections students make when working with functions. Vinner's (1993) research pointed out the internal image of the function students create when they are first introduced to the idea. In Vinner's view the "concept image" presented to students while learning a new mathematical idea has to be reinforced and revisited past the initial establishment of a new mathematical form. This is an important point, as students develop the connections between various representations with the individual representation forms evolving with the more complex ideas (i.e., line graph versus trigonometric curve). As expected, an internal perception student can have is difficult to measure or analyze. Leinhardt's research (Leinhardt et al., 1990) discussed the great connections that students hold between the mental images and what a function should look like. The variance between learner's internal idea of function and its representation was brought up in the literature with connections to students' misconception about graphical, tabular, and algebraic representations (i.e., Bell & Janvier, 1998; Bossé & Adu-Gyamfi, 2014; Vinner, 1993). Although internal representations undeniably play a significant role in students' understanding of functions, it is only through external representations that the knowledge can be shared, and as such this literature review focused mainly on external function representations.

Without excessive emphasis on classroom instruction, chapter three addressed the entering, sequencing, and translating between the function representation in support of student understanding. Leinhardt et al. (1990) pointed out the importance of introduction of mathematical concepts while discussing the impact of appropriate "entry point" in subsequent presentations of various concepts and their mathematical representations. This brought up the notion of teachers' and instructors' role in the sequencing of multiple representations as a tool in learning a new concept or expanding on an idea. Stylianou's (2010) study provided an insightful view the teachers themselves have about specific mathematical representations including preference to algebraic over graphical representations. Knuth (2000a, 2000b) research focused on translations between algebraic and graphical forms and the connections students make as they relate individual ordered pairs to the function in context. Transitions between tabular and graphical representations were also addressed with support of Ronda's (2015) study with connection to Ellis's (2011) work with similar content. Translating between various representations is shown to have a significant influence on students' deeper understanding of functions.

It would not be appropriate to omit the use of technology in today's classroom when discussing translating and introducing function representations. In chapter three, the use of electronic advances was interwoven within the topics of translating between function representations, illustrative forms of concepts, and entering or launching new ideas and problems. The example of Meyer's (2020) Three-Act Math activities was mentioned to demonstrate the use of videos as both illustrative and contextual function representations while Boaler's (2016) work supported the argument of informal representations playing an important role in building mathematical knowledge. Graphic utility calculators (education.ti.com) and dynamic software (desmos.com) were mentioned as both the conveyance tools (Hollebrands, 2017) and access to representations otherwise unattainable within the constraints of the classroom environment. In a current era of fairly easy access to technology in and outside of the classroom, the use of electronic advancement is both inevitable and beneficial.

The development of students' knowledge of functions through utilization of multiple representations was presented in chapter three within categories of conceptual and procedural understanding. The writings of Rittle-Johnson and Schneider (2014) provided a thoughtful insight into the education research field's view of procedural and conceptual understandings and their interconnectivity. As several researchers (DeBock, et al., 2015; Knuth, 2000a, 2000b) pointed out, students can over rely on procedural competence which can hinder their conceptual grasp of functions. This is especially evident in dealing with numerical forms and algebraic equations manipulations where practice with step-by-step algorithms can outweigh the thoughtful, conceptual process of mathematical thinking. With authors like Chigeza (2013) arguing for overuse of purely procedural approach and National Council of Teachers of Mathematics (NCTM, 2014) advocating procedural fluency as a building block to conceptual knowledge the literature provided a balanced argument for a two-directional relationship between the procedural and conceptual understanding of functions.

This literature review focused on multiple function representations and the role they play in aiding students' understanding. The investigation of numerous research studies and writings from the mathematics education field found recurring conclusions of undeniable significance of utilizing multiple representations not only pertaining to the concept of functions but with benefits to advance mathematical thinking across K-12 grades. Math educators and researchers alike agree that sequencing of function representations allows for building on students' prior knowledge and experience. It has been shown that developing conceptual understanding emerges through procedural fluency and translating between algebraic, graphical, tabular, and numerical forms along with benefits of illustrative and contextual representations. The subject matter, theme, and the literature reviewed were not without some limitations which are discussed in the next section.

Limitations of the Review

The limitations of the review fall into two categories: boundaries chosen to narrow down the vast topic of mathematical representations (both of educational and mathematical implication) and the mathematics educational research lacking literature specific to the issues of interest. To narrow down the nearly boundless depth of the topic of external function representations the boundaries of the review by topic were set by examining research pertaining to the kindergarten through high school grades and selecting learner-centered studies focusing on mathematical representations that aid function comprehension. This section discusses the literature limitations that were closely related to the topics covered.

Student age group limitations

The literature review considered the topic of function representations throughout the K-12 curriculum. Because the formal introduction of functions typically does not take place until eighth or ninth grade, there were two separations within the theme of representations specific to functions. One, a lot of mathematics education studies concerned with mathematical representations revolve around young learners. In those early years, students are not familiar with the concept of functions, even if they have experienced working with functional relationships. And two, since children become familiar with function definition in high school grades, consequently the research focuses on functional thinking and understanding with few connections to the specific role of various representations. And so even though functions and their representations frequently find their place in educational research, fewer writings address the functions with younger learners or focus on the role of representations as an introduction to the concept of functions in later grades.

Several studies discussed in the review addressed symbol sense and the development of mathematical representations. Steinbring's (2006) writings provided a great insight into the development of formal mathematical symbols to communicate ideas in early elementary grades. This early development of forms and symbols is essential to representation competency in later grades. Boaler's (2016) work advocated for rich illustrations accompanying mathematical problems including informal ways of gesturing. These are just two of a few examples of literature that provide great insight into the cognitive and intuitive ways young students can enhance their mathematical thinking and communication. The literature lacks the continuity of such studies with older learners. The development of representations specific to problems solving involving functions in older grades is rather underexplored in research studies.

Learner-centered considerations

The field of education offers countless studies and research that explores efficient and effective ways of mathematical instruction. This literature review was bound to a learner-centered view. That is to say, the review was not meant to explore better ways of teaching functional thinking via multiple representations but rather was set to investigate the roles that those representations play in students' understanding of functions. This approach limited the education literature resources; far more studies approach the concept of functions from the pedagogical angle rather than the learning point of view.

Many of the research examples used in the review referenced the topic of the instructor's role in the students' learning via multiple representations. The teacher's role was most frequently mentioned while discussing translations between different forms. Ellis's (2011) study, for example, specifically addressed the teacher's role in helping students bridge between numerical and graphical representations of functions while investigating linear relationships through engaging in hands-on activities. Although it could be interpreted as both a hindering occurrence to students' understanding as well as a positive guidance from the teacher, Ellis (2011) kept the general tone of the discussion of the teacher's involvement in students' learning as positive. It is generally understood that the teacher often takes on a guiding role, aiding and encouraging students' transitioning between mathematical forms. Some of the literature, however, places the teacher in the center of discussions on learning and understanding of concepts that involves multiple representations. The overall tone of such studies can leave an impression of the author shifting from investigation of student understanding to the instruction itself inhibiting the learning process. Stylianou’s (2010) writing, for instance, discussed teachers' perceptions of representations and the effects they can have on sequencing and transferring between the function forms. Boaler (2016) talked of teachers forbidding students from using informal representations in the classroom, such as gesturing or finger counting. Although such observations are linked to students being provided with learning opportunities, they offer little insight into the roles of such representations in developing understanding of concepts. It was at times proving difficult to find educational literature on function representations and students’ understanding that did not overly focus on the teacher’s role.

The literature often pointed to the teacher influencing the students’ use of certain function representations to fit their own preferences. Quite often such preference was placed on algebraic and equation forms of functions. Works of Knuth (2000a), Panasuk (2000), and Yerushalmy and Schwartz (1993) were among the literature pointing out students’ significant preference on algebraic representations of functions. While such preference can surely have multiple underlying causes, it is not often discussed whether it stems from a conceptual grasp of the concept of function. Stylianou (2010) argued that the instructors are to blame for students’ preferences. Such an argument, inevitably, did not add to the examination of students’ understanding. The writings of Bossé & Adu-Gyamfi (2014) and Vinner (1983), still, pointed out the consequence of overlying on algebraic representations. They argued that such reliance could lead to students’ misbelief that every function must have an algebraically defined rule. This literature review sought out studies that focused on the role the mathematical representations can have on students’ understanding of functions regardless of the quality of instructions or teachers’ shortfalls in mathematical knowledge.

Future Research

In summary, the exploration of the current literature provided many insights into the extensive field of educational research in the area of mathematical representations, alas not without room for new studies to fill the gaps discussed in a previous section. Suggestions for future research can be outlined in three main points:

- Additional research on the development and impact of mathematical representations in middle and high school settings,
- More investigating of the insight into functional thinking in elementary school children along with more formal definition of function,
- Further studies on the use of modern technology and electronic advances in mathematics classrooms with a particular focus around its influence on students' understanding of the concept of function.

REFERENCES

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and instruction*, 16(3), 183-198

Anzai, Y. (1991). Learning and use of representations for physics expertise. *Toward a general theory of expertise: Prospects and limits*, 64-92.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the learning of Mathematics*, *14*(3), 24-35.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational studies in mathematics*, *52*(3), 215-241.

Bell, A. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bernardz, C. Kieran, & L.Lee (Eds.), *Approaches to Algebra. Perspectives to Research and Teaching* (pp. 167–187). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the learning of mathematics*, 2(1), 34-42.

Bellman, A. E., Bragg, S. C., & Handlin, W. G. (2009). Prentice Hall mathematics, Algebra 1. Boston, MA, New Jersey: Pearson Prentice Hall.

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), 511-558.

Blanton M.L., Kaput J.J. (2011) Functional Thinking as a Route Into Algebra in the Elementary Grades. In: Cai J., Knuth E. (eds) In *Early Algebraization.* Advances in Mathematics Education (pp. 27-45). Springer, Berlin, Heidelberg.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as Understanding: The Importance of Visual Mathematics for our Brain and Learning. *Journal of Applied & Computational Mathematics*, *05*(05). https://doi.org/10.4172/2168-9679.1000325

Booth, R. D., & Thomas, M. O. (1999). Visualization in mathematics learning: Arithmetic problem-solving and student difficulties. *The Journal of Mathematical Behavior*, *18*(2), 169-190.

Bossé, M. J., & Adu-Gyamfi, K. (2014). Processes and reasoning in representations of linear functions. *International Journal of Science and Mathematics Education*, *12*(1), 167–192.

Chigeza, P. (2013). Translating Between and Within Representations: Mathematics as Lived Experiences and Interactions. *Mathematics Education Research Group of Australasia*.

Cook, S. W., Friedman, H. S., Duggan, K. A., Cui, J., & Popescu, V. (2017). Hand gesture and mathematics learning: lessons from an avatar. *Cognitive science*, *41*(2), 518-535.

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and cognitive processes*, *27*(4), 594-610.

Cunningham, R. F. (2005). Algebra teachers' utilization of problems requiring transfer between algebraic, numeric, and graphic representations. *School Science and Mathematics*, 105(2), 73-81.

De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' Understanding of Proportional, Inverse, and Affine Functions: Two Studies on the Role of External Representations. *International Journal of Science and Mathematics Education*, *13*(1), 47-69.

Driscoll, M. (1999). Fostering Symbol Sense, *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10* (pp. 115-123). Portsmouth, NH: Heinenmann

Dugdale, S., & Kibbey, D. (1986). Interpreting graphs [Computer program]. *Pleasantville, NY: Sunburst Communications*.

Edwards, C., Gandini, L., & Forman, G. (Eds.). (1993). *The Hundred Languages of Children : The Reggio Emilia Approach to Early Childhood Education.* Ablex Publishing Corp.

Elby, A. (2000). What students' learning of representations tells us about constructivism. *The Journal of Mathematical Behavior*, 19(4), 481-502.

Elia, I., & Philippou, G. (2004). The Functions of Pictures in Problem Solving. *28th International Group for the Psychology of Mathematics Education*, *2*, 327–334.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In *Early Algebraization* (pp. 215-238). Springer, Berlin, Heidelberg.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105-121.

González, G., & Eli, J. A. (2017). Prospective and in-service teachers' perspectives about launching a problem. *Journal of Mathematics Teacher Education*, *20*(2), 159–201.

Greeno, J. G., & Hall, R. P. (1997). Practicing representation: Learning with and about representational forms. *Phi Delta Kappan*, *78*, 361-367.

Gunpinar, Y., & Pape, S. (2018). Teachers' Instructional Practices within Connected Classroom Technology Environments to Support Representational Fluency. *Journal of Computers in Mathematics and Science Teaching*, *37*(1), 27–52.

Hiebert, J. (1986). *Conceptual and Procedural Knowledge: The Case of Mathematics.* Hillsdale, NJ: Erlbaum.

Hiebert, J. & Lefevre, P. (1986). *Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Illustrative Mathematics. (2019). Retrieved from https://www.illustrativemathematics.org

Jackson, K. J., Shahan, E. C., Gibbons, L. K., & Cobb, P. A. (2012). Launching complex tasks. *Mathematics Teaching in the Middle School*, *18*(1), 24-29.

Janvier, C. (1981). Use of situations in mathematics education. *Educational Studies in Mathematics*, *12*(1), 113-122.

Kaput, J. J. (1999). Teaching and Learning a New Algebra With. *Mathematics Classrooms That Promote Understanding*, (Ccm), 1–35.

Knuth, E. J. (2000a). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500–508.

Knuth, E. J. (2000b). Understanding Connections between Equations and Graphs. *The Mathematics Teacher*, *93*(1), 48–53.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of educational research, 60(1), 1-64.

Lingefjärd, T., & Ghosh, J. B. (2016). Learning mathematics as an interplay between internal and external representations. *Far East Journal of Mathematical Education*, *16*(3), 271-297.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. International Journal of Mathematics Education in Science and Technology, 11(4), 489-492.

McKendree, J., Small, C., Stenning, K., & Conlon, T. (2002). The role of representation in teaching and learning critical thinking. *Educational Review*, *54*(1), 57–67.

Meyer, D. (2020). *3acts* – Dy/Dan. https://blog.mrmeyer.com/category/3acts

Meyer, M. (2001) Representation in Realistic Mathematics Education. In A.A. Cuoco & F. Corcio, *Teaching Mathematical Modeling and Skills of Representation: the roles of representation in school mathematics* (pp.238-250). Reston, Va: National Council of Teachers of Mathematics.

Misailidou, C. (2003). Diagnostic tests for ratio and proportion. Learning, Teaching and Assessment. Faculty of Education, University of Manchester (on line). Available: http://www.education.man.ac.uk/lta/cm/

Moseley, B. (2005). Students' early mathematical representation knowledge: The effects of emphasizing single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics*, *60*(1), 37-69.

Murdock, J., Kamischke, E. & Kamischke, E. (2002). *Discovering algebra : an investigative approach*. Emeryville, CA: Key Curriculum Press.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics [NCTM]. (2014). *Procedural fluency in mathematics: A position of the National Council of Teachers of Mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*

Niemi, D. (1996). Assessing conceptual understanding in mathematics: Representations, problem solutions, justifications, and explanations. *The Journal of Educational Research*, *89*(6), 351-363.

Olson, J. (1998). Guess the Weight! *Teaching Children Mathematics, 5*(2), 104-104. Retrieved from http://www.jstor.org/stable/41197114

Panasuk, R. M., & Beyranevand, M. L. (2010). Algebra Students' Ability to Recognize Multiple Representations and Achievement. *International Journal for Mathematics Teaching and Learning*.

Rittle-Johnson, B., & Schneider, M. (2014). *Developing Conceptual and Procedural Knowledge of Mathematics*. (R. Cohen Kadosh & A. Dowker, Eds.), *Oxford Handbook of Numerical Cognition* (Vol. 1, pp. 1118–1134). Oxford University Press.

Ronda, E. (2015). Growth points in linking representations of function: a research-based framework. *Educational Studies in Mathematics*, *90*(3), 303–319.

Saul, M. (2001) Algebra: What Are We Teaching? In A.A. Cuoco & F. Corcio, *Teaching Mathematical Modeling and Skills of Representation: the roles of representation in school mathematics* (pp.35-43). Reston, Va: National Council of Teachers of Mathematics.

Steinbring, H. (2006). What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction. *Educational studies in mathematics*, *61*(1-2), 133-162.

Stylianou, D. A. (2020). Problem Solving in Mathematics with Multiple Representations. *Handbook of Learning from Multiple Representations and Perspectives*.

Uttal, D. H., O'Doherty, K., Newland, R., Hand, L. L., & DeLoache, J. (2009). Dual representation and the linking of concrete and symbolic representations. *Child Development Perspectives*, *3*(3), 156–159.

Van Dyke, F., & Craine, T. V. (1997). Equivalent representations in the learning of algebra. *The Mathematics Teacher*, 90(8), 616-619.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*(3), 293–305. https://doi.org/10.1080/0020739830140305

Wagner, S. (1981). Conservation of equation and function under transformations of variable. *Journal for research in mathematics education*, 12(2), 107-118.

Woleck, K. R. (2001). Litent to Their Pictures: An Investigation of Children's Mathematical Drawings. In A.A. Cuoco & F. Corcio, *Teaching Mathematical Modeling and Skills of Representation: the roles of representation in school mathematics* (pp.215-227). Reston, Va: National Council of Teachers of Mathematics.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, *7*(1), 42–57.

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education*, *28*(4), 431-466.

Yerushalmy, M., & Schwartz, J. L. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. *Integrating research on the graphical representation of functions*, 41-68.