ABSTRACT

GILMORE, STEVEN. Optimal Feedback Strategies of a Debt Management Problem and Fine Regularity for Nonlocal Balance Laws. (Under the direction of Khai T. Nguyen and Hien T. Tran.)

Abstract. This dissertation examines the mathematical theory of particular nonlinear and nonlocal partial differential equations and their applications.

In the first section, we introduce a model of optimal debt management in a game-theoretic setting in which a borrower services a debt by trading bonds, at a discounted price, with a pool of risk-neutral lenders. The borrower is at risk of going bankrupt at any given time, and there is a debt threshold at which the borrower is forced to declare bankruptcy. In this model, the borrower is viewed as a sovereign state who can reduce the debt by fiscal policy or monetary policy by devaluing their national currency. Both policies induce social costs, and notably, the currency devaluation negatively affects the lenders' confidence in the state's ability to pay the debt back, causing the bond price to fall. The resulting optimal control problem to minimize the cost to the borrower in an infinite time horizon is nonstandard. Indeed, the bond price depends not only on the current debt level but on all future expected values of the debt. Through a vanishing viscosity approach, we show that the corresponding Hamilton-Jacobi-Bellman equations have at least one smooth solution and, as a consequence, optimal fiscal and monetary policies exist. Moreover, we provide results on the behavior of the controls near the bankruptcy threshold. More specifically, we provide conditions under which employing either strategy is non-optimal and, conversely, conditions for which it is optimal to use both strategies until the point of bankruptcy.

In the second section, we study the fine regularity of the Burgers-Poisson equation; a nonlinear dispersive balance law derived to model shallow-water waves. In 2015, existence and uniqueness results to the Cauchy problem for initial data in $\mathbf{L}^1(\mathbb{R})$ were provided. Moreover, the entropy weak solution is in $BV_{loc}(\mathbb{R})$ for all positive time for given initial data. Following recent advancements in the theory of conservation laws, we show the entropy weak solutions belong to a subset of BV functions, the space of special functions of bounded variation (SBV). In particular, we prove that the derivative of a solution consists of only the absolutely continuous part and the jump part.

Optimal Feedback Strategies of a Debt Management Problem
and Fine Regularity for Nonlocal Balance Laws

by
Steven Gilmore

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Applied Mathematics

Raleigh, North Carolina

2021

APPROVED BY:

Alina Chertock	Lorena Bociu
Khai T. Nguyen Co-chair of Advisory Committee	Hien T. Tran Co-chair of Advisory Committee

DEDICATION

Dedicated to my dad, James Gilmore,
for always encouraging me to follow my dreams.

BIOGRAPHY

Steven Gilmore is a curious and loving human born in Melbourne, FL. Growing up in Charlotte, NC, with his parents, Jim and Jennifer, and his sister Kristen, he learned to love math and the outdoors at a young age. In 2014, Steven earned a B.S. in Applied Mathematics from NCSU. He has worked as a software developer, tutor, and part-time musician. In his free time, Steven enjoys cooking, gardening, and playing music. He lives in Raleigh, NC, with his partner Jenna, their bunny Carter, their rooster Tom, and their eight hens (Amelia, Calypso, Little Sue, Joany, Meryl, Rhi-rhi, Stormy, and Stormy).

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my advisor, Prof. Khai Nguyen, for introducing me to the rich theories of optimal control and conservation laws and his kindness and compassion throughout my graduate career. I am grateful for the ideas and laughs shared with our research team over coffee and dinner.

I would like to thank my committee members, Prof. Hien Tran, Prof. Lorena Bociu, and Prof. Alina Chertock, for your mentorship both in and out of the classroom. I am grateful for the knowledge that you've shared throughout my graduate career and all the helpful feedback on this thesis. Thank you to Prof. Mette Olufsen for advising me through my Masters and undergraduate degree and continuing to support me in my endeavors.

In addition, I would like to thank all my teachers and advisors who supported and encouraged me throughout this journey, including the mentorship from Dr. Molly Fenn and Prof. Pierre Gremaud. I am grateful for all my peers in the Department of Mathematics for fostering a welcoming and caring community, with a shout out of love to Zev Woodstock and Stephen Lacina. An especially warm acknowledgment to Mr. Madre. I will always appreciate your tireless sarcasm while making Calculus look effortless and your not-so-subtle push for me to continue on into higher-level math.

I am incredibly grateful to my parents and for my whole family, extended, blended, and chosen. Thank you for the countless ways you have loved me over the years. I am grateful for my dear friend Colin who showed up authentically for me throughout this program, always encouraging me to thrive when I can and survive when I can't. I am grateful for my counselor, Bonnie, for providing soul care services and tools to navigate this life. And I wish to express gratitude from my heart to my partner Jenna for genuinely wanting me to be full of life-energy and supporting me in that journey.

Finally, I am grateful to have a place I call home in Skaruhreh and Tuscarora land and for all the trees and fellow beings who have warmly welcomed me as I am. I am grateful for all who have supported and encouraged me throughout my life.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

The dissertation is devoted to the mathematical theory of certain nonlinear and nonlocal partial differential equations and their applications. It can be summarized in two parts:

(i) we provide a detailed analysis of a system of Hamilton-Jacobi-Bellman equations arising from a model of optimal debt management with bankruptcy risk.

(ii) we study deeper regularity of the Burgers-Poisson equation, a nonlinear dispersive model derived as a simplified model of shallow-water waves.

Chapter 2 introduces necessary preliminary notions related to optimal control theory, functions of bounded variation, and conservation laws.

Chapter 3 is concerned with a problem of stochastic optimal debt management in an infinite time horizon, modeled as a noncooperative game between a sovereign state and a pool of foreign, risk-neutral lenders. We examine and provide well-posedness results for the following system of Hamilton-Jacobi-Bellman (HJB) equations

$$\begin{cases} (r+\rho(x))V \;=\; \rho(x)\cdot B + H(x,V',p) + \dfrac{\sigma^2 x^2}{2}\cdot V''\,, \\ (r+\lambda+v(x))\cdot p - (r+\lambda) \;=\; \rho(x)\cdot[\theta(x)-p] \\ \qquad\qquad\qquad\qquad\qquad\qquad + H_\xi(x,V',p)\cdot p' + \dfrac{(\sigma x)^2}{2}\cdot p''\,. \end{cases} \tag{1.1}$$

Here, the independent variable x is the debt-to-income ratio, and V is the value function for the borrower who is a sovereign state that can decide the devaluation rate of its currency v and the fraction of its income u which is used to repay the debt. The national income of the country is governed by a stochastic process and, at each time the borrower may go bankrupt with probability determined by an instantaneous bankruptcy risk ρ. Moreover, the borrower must declare bankruptcy when the debt-to-income ratio reaches a threshold $x^* > 0$. The salvage function θ determines the fraction of capital that lenders can recover when bankruptcy occurs. To offset the possible loss of part of their investment, the lenders buy bonds at a discounted rate p, which is not given a priori. Instead, it is determined by the expected evolution of the debt-to-income ratio at all future times. Hence, it depends globally on the entire feedback controls u and v. This framework leads to a highly nonstandard optimal control problem, and a "solution" must be understood as a Nash equilibrium, where the strategy implemented by the borrower represents the best reply to the strategy adopted by the lenders, and conversely.

The optimal debt management problem we study is based on a model first introduced in [66]. Through numerical methods, their analysis concluded that using currency devaluation is not optimal unless the government is able to make credible commitments about their future inflation policy. In [15], they performed an analytical study of a slight variant of the model where no currency devaluation is available to the government. The authors construct optimal feedback solutions in the stochastic case and provide an explicit formula for the optimal strategy in the deterministic case. Their analysis also shows how the expected total cost of servicing the total debt together with the bankruptcy cost is affected by different choices of x^*. Interestingly, the possibility of a *"Ponzi scheme"*, where the old debt is serviced by initiating more and more new loans, can be ruled out under a natural assumption. In [64], the analytical study of these models was extended by allowing the possibility of currency devaluation (as in [66]). They show that if the debt-to-income ratio is sufficiently high, then *every* optimal strategy involves employing currency devaluation.

Chapter 3 aims to provide a detailed mathematical analysis of a more general model of debt and bankruptcy. When the currency devaluation is not an option for the borrower ($v \equiv 0$), or the bankruptcy risk

$$\rho(x) \;=\; \begin{cases} 0 & \text{if } x < x^* \\ +\infty & \text{if } x = x^*, \end{cases}$$

our model reduces to the one analyzed in [14] or [64], respectively. We establish an existence result for the system of Hamilton-Jacobi equations (1.1). In turn, this yields the existence of a pair of optimal feedback controls (u^*, v^*) which minimizes the expected cost to the borrower. More precisely, let $L : [0,1[\to [0,+\infty[$ be the cost for the borrower to implement the control strategy $u(\cdot)$ and let $c : [0, v_{\max}[\to [0, +\infty[$ be a social cost resulting by devaluation, i.e., the increasing cost of the welfare and of the imported goods. We will assume that L and c are both non-negative, strictly convex, and $L(0) = c(0) = 0$. Regarding the bankruptcy risk function ρ, we assume that $\rho \geq 0$ is non-decreasing, $\rho(0) = 0$, and $\lim_{x \to x^*-} \rho(x) = +\infty$. We assume that the salvage function θ is non-increasing, positive, and locally Lipschitz. The main result of Chapter 3 is the following existence result (see Theorem 3.2.1).

Theorem 1 (Capuani, Gilmore, Nguyen [24]). *Given the debt-to-income ratio threshold $x^* > 0$ and the functions L, c, ρ and θ satisfying the above assumptions, then the system* (1.1) *admits a solution (V, p) of class C^2 in $[0, x^*]$. This implies that an optimal feedback solution to the related model of debt and bankruptcy exists and is given by*

$$u^*(x) \;=\; \operatorname*{argmin}_{w \in [0,1]} \left\{ L(w) - w \cdot \frac{V(x)}{p(x)} \right\}, \qquad v^*(x) \;=\; \operatorname*{argmin}_{v \in [0, v_{\max}]} \left\{ c(v) - vxV'(x)] \right\}.$$

Since ρ goes to $+\infty$ as x tends to x^*, the system (1.1) is not uniformly elliptic at the boundary $x = 0$ and $x = x^*$. To handle this difficulty, we employ the classical idea of constructing solutions of approximate systems as steady states of corresponding auxiliary parabolic systems. In order to obtain a solution to the original system (1.1), we derive explicit a priori estimates on the derivatives of approximate solutions. Consequently, the devaluation of currency is not optimal when the debt-to-income ratio x is sufficiently small. In addition, we provide lower and upper bounds for the value function V as sub-solution and super-solution to the first equation of (1.1). Relying on these bounds, we show that (see Corollary 3.3.2) when x is sufficiently close to x^*

- *if the risk of bankruptcy ρ slowly approaches to infinity as x tends to x^*, i.e.,*

  $$\int_{x^*-\delta}^{x^*} \rho(t)\, dt < +\infty \qquad for\ some\ \delta > 0,$$

  then the optimal strategy of borrower involves continuously devaluating its currency and making payments;

- *conversely, if the risk of bankruptcy ρ quickly approaches to infinity as x tends to x^*, i.e.,*

  $$\lim_{x \to x^*-} \rho(x)(x^* - x)^2 \;=\; +\infty,$$

  then any action to reduce the debt is not optimal.

In Chapter 4, we study the fine regularity of a nonlinear dispersive equation, called the Burgers-Poisson equation, which is obtained from the Burgers equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G * u]_x$$

where $G(x) = -\frac{1}{2}e^{-|x|}$ and $[G * u](\cdot)$ solves the Poisson equation

$$\varphi_{xx} - \varphi = u .$$

First derived in [80], the Burgers-Poisson equation has been used to model shallow-water waves. Solutions have been shown to exhibit wave-breaking in finite time with smooth initial data. Therefore, we will consider solutions in the weak (distributional) sense.

Based on the vanishing viscosity approach, the existence result for a global weak solution was provided for $u_0 \in BV(\mathbb{R})$ in [44]. However, this approach cannot be applied to the more general case with initial data in $\mathbf{L}^1(\mathbb{R})$. Moreover, there are no uniqueness or continuity results for global weak entropy solutions of (4.1) established in [44]. More recently, the existence and continuity results for global weak entropy solutions of (4.1) were established for $\mathbf{L}^1(\mathbb{R})$ initial data in [53]. The entropy weak solutions are constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers equation and the Lipschitz continuity of solutions to the Poisson equation, approximating solutions satisfy an Oleinik-type inequality for any positive time. As a consequence, the sequence of approximating solutions is precompact and converges in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$. Moreover, using an energy estimate, they show that the characteristics are Hölder continuous, which is used to achieve the continuity property of the solutions. The Oleinik-type inequality gives that the solution $u(t, \cdot)$ is in $BV_{\text{loc}}(\mathbb{R})$ for every $t > 0$. In particular, this implies that the Radon measure $Du(t, \cdot)$ is divided into three mutually singular measures

$$Du(t, \cdot) = D^a u(t, \cdot) + D^j u(t, \cdot) + D^c u(t, \cdot)$$

where $D^a u(t, \cdot)$ is the absolutely continuous measure with respect to the Lebesgue measure, $D^j u(t, \cdot)$ is the *jump part* which is a countable sum of weighted Dirac measures, and $D^c u(t, \cdot)$ is the non-atomic singular part of the measure called the *Cantor part*. For a given $w \in BV_{\text{loc}}(\mathbb{R})$, the Cantor part of Dw does not vanish in general. A typical example of $D^c w$ is the derivative of the Cantor-Vitali ternary function. If $D^c w$ vanishes, then we say the function w is locally in the space of *special* *functions of bounded variation*, denoted by $SBV_{\mathrm{loc}}(\mathbb{R})$. The space of SBV functions was first introduced in [37] and plays an important role in the theory of image segmentation and variational problems in fracture mechanics. Motived by results on SBV regularity for hyperbolic conservation laws ([4, 73, 11, 63]), we show the following (see Theorem 4.0.1).

Theorem 2 (Gilmore, Nguyen [52]). *Let* $u : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ *be the unique* BV_{loc} *weak entropy solution of (4.1) with initial data* $u_0 \in \mathbf{L}^1(\mathbb{R})$*. Then there exists a countable set* $\mathcal{T} \subset \mathbb{R}^+$ *such that*

$$u(t,\cdot) \ \in \ SBV_{\mathrm{loc}}(\mathbb{R}) \qquad \forall t \in \mathbb{R}^+ \setminus \mathcal{T}.$$

As a consequence, the slicing theory of BV functions and the chain rule of Vol'pert [5] implies that the weak entropy solution u is in $SBV_{\mathrm{loc}}([0,+\infty[\times\mathbb{R})$. This result is the first example showing that solutions to a scalar conservation law with a nonlocal source term possess SBV regularity. A common theme in the proofs of recent results on SBV regularity involves an appropriate geometric functional which has certain monotonicity properties and jumps at time t if $u(t,\cdot)$ does not belong to SBV (see e.g., [4]). More precisely, let $\mathcal{J}(t)$ be the set of jump discontinuities of $u(t,\cdot)$. For each $x_j \in \mathcal{J}(t)$, there are minimal and maximal backward characteristics $\xi_j^-(s)$ and $\xi_j^+(s)$ emanating from (t,x_j) which define a nonempty interval $I_j(s) :=]\xi_j^-(s), \xi_j^+(s)[$ for any $s < t$. In this case, the functional $F_s(t)$ defined as the sum of the measures of $I_j(s)$ is monotonic and bounded. Relying on a careful study of generalized characteristics, we show that if the measure $Du(t,\cdot)$ has a non-vanishing Cantor part, then the function F_s jumps up at time t which implies that the Cantor part is only present at countably many t. Due to the nonlocal source, $u(t,\cdot)$ does not necessarily have compact support. Thus, we approach the domain by first looking at compact sets and then "glue" the sections together to recover the full domain.

Chapter 2

Preliminaries and Notation

Let us define some notation used throughout the remaining chapters and recall some preliminary background.

2.1 Notation

Throughout the paper we shall denote by

- For any $\Omega \subset \mathbb{R}^N$, denote by $\partial\Omega$ the boundary of Ω;
- $B_\rho(z)$, the open ball of radius ρ and center z
- $\mathbf{L}^1(\mathbb{R})$, the Lebesgue space of all (equivalence classes of) summable functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^1}$;
- $\mathbf{L}^\infty(\mathbb{R})$, the space of all essentially bounded functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^\infty}$;
- $\mathcal{C}_c^\infty(\mathbb{R})$, the space of all C^∞-functions with compact supports.
- supp u, the essential support of a function $u \in \mathbf{L}^\infty(\mathbb{R})$
- $BV(\Omega)$, the space of functions of bounded variations in Ω;
- $SBV(\Omega)$, the space of special functions of bounded variation in Ω;
- $\mathcal{H}^N$, the N-dimensional Hausdorff measure;
- $\mathcal{L}^N$, the N-dimensional Lebesgue measure;
- $\mathbb{R}^+$, the positive real numbers ($[0, +\infty[$);

2.2 Optimal Control Theory

Given a compact set $U \subset \mathbb{R}^m$, consider the time evolution of a control system

$$\begin{cases} \dot{x}(t) &= f(x, u) \\ x(0) &= x_0 \in \mathbb{R}^N \end{cases} \tag{2.1}$$

where $x : \mathbb{R}^+ \to \mathbb{R}^N$ is the state variable, $u : \mathbb{R}^+ \to U \subset \mathbb{R}^m$ is the control variable, and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is the dynamics. We denote by

$$\mathcal{U}_{ad} = \left\{ u : \mathbb{R}^+ \to U \subset \mathbb{R}^m \ : \ u \text{ is measurable} \right\} .$$

the set of *admissible controls*. To guarantee local existence and uniqueness of solutions, we assume f is Lipschitz continuous with respect to x and continuous with respect to u. Given $x_0 \in \mathbb{R}^N$ and $u \in \mathcal{U}_{ad}$, a *trajectory* is a solution of (2.1), which will be denoted by $y^{x_0,u}(\cdot)$. There are two main ways to assign a control u:

(i) As a function of time: $t \mapsto u(t)$. We say u is an *open loop control*.

(ii) As a function of the state: $x \mapsto u(x)$. We say u is an *feedback control*.

We will consider *optimal control* problems, where we seek to find a control $u(\cdot) \in \mathcal{U}_{ad}$ which is optimal with respect to a given cost function. For example, consider a problem in an infinite horizon with exponential discount with the cost functional

$$J[u, x_0] = \int_0^\infty e^{-\lambda t} \cdot L(y^{x_0,u}(t), u(t)) \, dt \, .$$

Here $\lambda > 0$ is the discount rate and $L : \mathbb{R}^N \times U \to \mathbb{R}^+$ is the running cost. Necessary conditions for optimality were derived by Pontryagin, Boltyanskii, and Gamkrelidze [70] through the methods of calculus of variations. An in-depth discussion of Pontryagin's Maximum principle can be found in [46]. At the same time, Bellman was developing the method of Dynamic Programming [9]. Through this different approach, sufficient conditions for optimality can be phrased in terms of a solution of a partial differential equation. Define the *value function* V by

$$V(x_0) = \inf_{u \in \mathcal{U}_{ad}} \int_0^\infty e^{-\lambda t} \cdot L(y^{x_0,u}(t), u(t)) \, dt \tag{2.2}$$

subject to (2.1). The fundamental idea of dynamic programming is that V satisfies a functional equation called the *Dynamic Programming Principle*.

Proposition 2.2.1. *Suppose that L is a continuous function in both variables. Given $x_0 \in \mathbb{R}^N$ and $\tau > 0$ it holds that*

$$V(x_0) \;=\; \inf_{u \in \mathcal{U}_{ad}} \left\{ \int_0^\tau e^{-\lambda s} \cdot L(y^{x_0,u}(s), u(s))\, ds + e^{-\lambda \tau} V(y^{x_0,u}(\tau)) \right\}.$$

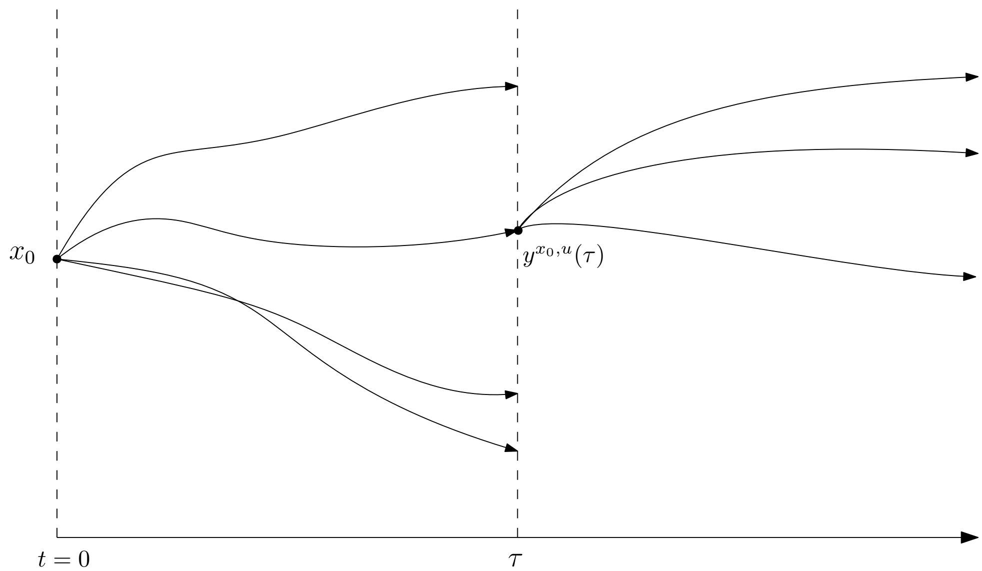

Figure 2.1 The dynamic programming principle yields that the optimization problem on $[0, +\infty[$ can be decomposed into two sub-problems, on the time intervals $[0, \tau]$ and $[\tau, +\infty[$.

In the standard theory, an infinitesimal version of the Dynamic Programming Principle is derived in the form of a first-order nonlinear partial differential equation satisfied by V, called the *Hamilton-Jacobi-Bellman equation*. In general, the value function arising from dynamic optimization problems may be non-smooth. The theory of viscosity solutions, introduced in [32] and [33], provides a framework for studying V in the sense of *viscosity solutions* (see e.g., [23],[25],[8]). Here we present a simple case assuming that V is differentiable.

Corollary 2.2.2. *Suppose that V is differentiable at a point $x \in \mathbb{R}^N$. Then V is a solution of the following Hamilton-Jacobi-Bellman equation*

$$\lambda V(x) \;=\; H(x, \nabla V) \tag{2.3}$$

with the Hamiltonian $H(x,p)$ given by

$$H(x,p) \;=\; \min_{w\in U}\{L(x,w)+p\cdot f(x,w)\}\;.$$

The proof is standard and can be found in, e.g., [8]. If $V(\cdot)$ is known, then the optimal control, in feedback form, can be recovered by solving the minimization problem

$$u^*(x) \;\in\; \arg\min_{w\in U}\{L(x,w)+\nabla V(x)\cdot f(x,w)\}\;.$$

where x is determined by (2.1).

Remark 2.2.3. We examine a special case where $L(x,u)=L(u)$, L is convex and

$$\lim_{|q|\to\infty}\frac{L(q)}{|q|} \;=\; +\infty\,.$$

In that case, it can be shown that V is Lipschitz continuous and by Rademacher's theorem (see e.g., [40], [5], [42]) is differentiable a.e and thus, satisfies (2.3) a.e.

2.3 Stochastic Optimal Control

For a thorough background in stochastic differential equations, we direct the reader to [68], and for stochastic optimal control, see [46],[45],[47].

Definition 2.3.1 (Itô process)**.** Let $(\Omega,\mathcal{F},P)$ be a probability space. Let $w(t)$ be a Brownian motion process. Given functions $\mu(x,t)$ and $\sigma(x,t)$, a (1-dimensional) *Itô process* is a stochastic process $x(t)$ on $(\Omega,\mathcal{F},P)$ of the form

$$x(t) \;=\; x(0)+\int_0^t \mu(x(t))\,ds+\int_0^t \sigma(x(t))\,dw(s)\,. \tag{2.4}$$

such that

$$P\left[\int_0^t \sigma(s,x)^2\,ds<+\infty,\forall t\geq 0\right] \;=\; 1\,.$$

For an Itô process of the form (2.4), the differential form is often used

$$dx \;=\; \mu(x(t))\,dt+\sigma(x(t))\,dw\,. \tag{2.5}$$

We say $x(t)$ is a *strong solution* of (2.5) if, for all $t>0$ the integrals $\int_0^t \mu(x(s))\,ds$ and $\int_0^t \sigma(x(s))\,ds$ exist and (2.4) is satisfied. It's possible to establish an Itô integral version of the chain rule, called the Itô formula (see, e.g. [68, Theorem 4.1.2] or [46, § 5 (3.9)])

Theorem 2.3.2 (Itô formula)**.** *Let $x(t)$ in* (2.5) *be an Itô process. Let $g(t,x) \in C^2(\mathbb{R}^+ \times \mathbb{R})$. Then $Y(t) = g(t, x(t))$ is again an Itô process and*

$$dY \;=\; \frac{\partial g}{\partial t}(t, x(t))\, dt + \frac{\partial g}{\partial x}(t, x(t))\, dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, x(t)) \cdot \sigma(x(t))^2\, dw$$

Definition 2.3.3 (Itô diffusion)**.** A (time-homogeneous) *Itô diffusion* is a stochastic process $x(t)$ satisfying a stochastic differential equation of the form

$$dx \;=\; \mu(x(t))\, dt + \sigma(x(t))\, dw, \qquad t \geq s, \quad x(0) = x_0 \in \mathbb{R} \tag{2.6}$$

where $w(t)$ is a m-dimensional Brownian motion and $\mu : \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ satisfy

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \;\leq\; D|x - y|, \qquad x, y \in \mathbb{R}^n$$

for some constant D and where $|\sigma|^2 = \sum |\sigma_{ij}|^2$.

It is fundamental for many applications that we can associate a second order partial differential operator A to an Itô diffusion $x(t)$. The basic connection between A and $x(t)$ is that A is the *generator* of the process $x(t)$:

Definition 2.3.4. Let $x(t)$ be a time-homogeneous Itô diffusion in $\mathbb{R}^n$. The *generator A of* $x(t)$ is defined by

$$Af(x) \;=\; \lim_{t\downarrow 0} \frac{E\left[f(x(t))\right] - f(y)}{t}, \qquad x \in \mathbb{R}^n\,.$$

We denote by $\mathcal{D}_A$ the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

The relation between A and the functions μ and σ in the stochastic differential equation (2.6) is given by the following result (see [68, Theorem 7.3.3]).

Theorem 2.3.5. *Let $x(t)$ be the Itô diffusion in* (2.6)*. If $V \in C_0^2(\mathbb{R}^n)$ then $f \in \mathcal{D}_A$ and*

$$AV(x) \;=\; \sum_i^N \mu_i(x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^N (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 V}{\partial x_i \partial x_j}\,.$$

The Feynman-Kac formula gives a fundamental link between PDEs and stochastic processes. It offers a method for solving PDEs by simulating random paths of a stochastic process and, conversely, computing expectations of random processes by deterministic methods (see, e.g., [68, Theorem 8.2.1]).

Theorem 2.3.6 (Feynman-Kac formula for boundary value problems)**.** *Let $x(t)$ be an Itô diffusion on $\mathbb{R}^n$ whose generator coincides with a given diffusion operator A. Let $D \subset \mathbb{R}^n$ and let $q(x) \geq 0$ be a continuous function on $\mathbb{R}^n$. Consider the problem of finding $W \in C^2(D) \cap C(\overline{D})$ such that W solves*

$$\begin{cases} Ah(x) - q(x)W(x) \;=\; -\,g(x), & \text{in } D \\ \lim_{x\to y} W(x) \;=\; \varphi(y), & \text{on } \partial D\,. \end{cases}$$

Then if a solution exists, it satisfies the following conditional expectation

$$W(x) \;=\; E\left[\int_0^\tau e^{-\int_0^t q(x(s))\,ds} g(x(t))\,dt + e^{-\int_0^\tau q(x(s))\,ds}\varphi(X_\tau)\right].$$

In the deterministic optimal control setting, controls can be taken in either open loop or in feedback form. In general, these controls do not differ since the state of the system at any time t can be solved for from the initial data and the control used up to time t. In the stochastic case, there are many paths the system state may follow given a control and initial data. Hence, the optimal state depends on the information available to the controller at each time. For that reason, we will focus on controls in feedback form. Consider the stochastic process

$$dx \;=\; \mu(x(t), u(t))\,dt + \sigma(x(t), u(t))\,dw, \qquad x(0) = x_0 \tag{2.7}$$

where $u \in \mathcal{U}_{ad}$ is an admissible (progressively measurable-valued in U) control considered in the feedback sense: $u(t) = \mathbf{u}[x(t)]$. We consider a cost functional

$$J(x_0, \mathbf{u}) \;=\; E\left[\int_0^\infty e^{-\int_0^t q(x(s))ds} L(x(t), u(t))\,dt\right], \tag{2.8}$$

where the running cost $L : \mathbb{R} \times U \mapsto \mathbb{R}$ is a continuous function and $q(x) > 0$ is a continuous function representing variable discount rate. The optimal problem is to find a feedback control $\mathbf{u}^*$ which minimizes (2.8). Let the *value function* be defined as

$$V(x) \;=\; \inf_{u\in\mathcal{U}_{ad}} J(x, u)$$

For a feedback control $\mathbf{u}$, let $A^{\mathbf{u}}$ be the generator of $x(t)$, i.e., the second order differential operator associated with the SDE (2.7),

$$A^{\mathbf{u}}V = \sum_{i}^{N} \mu_i(x, \mathbf{u}) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{j=1}^{N} (\sigma\sigma^T)_{ij}(x, \mathbf{u}) \frac{\partial V}{\partial x_i \partial x_j}$$

Let us assume that Ω is bounded. In this case, the dynamic programming equation (HJB equation) takes the form of a second-order nonlinear PDE with given boundary conditions

$$\begin{cases} q(x)V &= H(x, \nabla_x V) + A^{\mathbf{u}}V, \qquad x \in \Omega \\ V(x) &= g(x)\,, \qquad x \in \partial\Omega\,, \end{cases} \tag{2.9}$$

where the Hamiltonian is given by

$$H(x, p) \;=\; \inf_{v \in U} \left[\mu(x, v) \cdot p + L(x, v)\right]\,. \tag{2.10}$$

If there exists a $c > 0$ such that

$$\sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(x, v)\xi_i\xi_j \;\geq\; c|\xi|^2 \tag{2.11}$$

for all $\xi \in \mathbb{R}^n$ and $v \in U$, then the HJB equation is called *uniformlly elliptic*. In that case, one may expect (2.9) to have a smooth solution, which is unique if Ω is bounded. When (2.11) does not hold, the HJB equation is of *degenerate elliptic type* and one can not expect smooth solutions to (2.9) in general. The following verification theorem [47, Theorem 5.1] is an analog to the dynamic programming principle in the deterministic case.

Theorem 2.3.7 (Verification theorem)**.** *Let $V(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution to the stochastic Hamilton-Jacobi-Bellman equations* (2.9)*. Then for every $x \in \Omega$, it holds that*

(a) $V(x) \leq J(x, \mathbf{u})$ for any admissible feedback control $\mathbf{u}$.

(b) If $\mathbf{u}^*$ is an admissible feedback control such that

$$A^{\mathbf{u}^*}V + L(y, \mathbf{u}^*) = \min_{w \in U} \left[A^w V + L(x, w)\right] \qquad \forall y \in \Omega$$

then $\mathbf{u}^*$ is optimal.

2.4 Measure Theory

We proceed by defining the crucial concepts of geometrical measure theory without rigorously providing the foundations of abstract measure theory (see, e.g., [42], [43] for full background).

Definition 2.4.1. (a) Let μ be a positive measure and ν a real of vector measure on the measure space $(X, \mathcal{E})$. We say that ν is *absolutely continuous* with respect to μ and write $\nu \ll \mu$, if for every $B \in \mathcal{E}$ the following holds:

$$\mu(B) \;=\; 0 \quad \Longrightarrow \quad |\nu|(B) \;=\; 0\,.$$

(b) If μ, ν are positive measures, we say they are *mutually singular* and write $\nu \perp \mu$, if there exists $E \in \mathcal{E}$ such that $\mu(E) = 0$ and $\nu(X \setminus E) = 0$; if μ or ν are real or vector valued, we say that they are *mutually singular* if $|\mu|$ and $|\nu|$ are so.

We consider Radon measures, which are compatible with the topology of the space. The Lebesgue measure coincides with the notion of n-dimensional volume.

Definition 2.4.2 (Radon Measure)**.** Let X be a locally convex space (l.c.s.) metric space, $\mathcal{B}(X)$ its Borel σ-algebra, and consider the measure space $(X, \mathcal{B}(X))$. A set function defined on the relatively compact Borel subsets of X that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset X$ is called a *Radon measure* on X. If $\mu : \mathcal{B}(X) \mapsto \mathbb{R}^m$ is a measure, we say that μ is a *finite Radon measure*.

Definition 2.4.3 (Lebesgue measure)**.** Let $Q_r(x) = \left\{y \in \mathbb{R}^N \;:\; \max_i |x_i - y_i| < r\right\}$ be the open cube with side $2r$ centered at x. Then the outer measure μ defined by

$$\mu(E) \;=\; \inf\left\{\sum_{h=0}^{\infty}(2r_h)^N \;:\; E \subset \bigcup_{h=0}^{\infty} Q_{r_h}(x_h)\right\}, \qquad E \subset \mathbb{R}^N$$

is the *Lebesgue outer measure* denoted by $\mathcal{L}^N$.

The following result provides a way to express the relationship between two measures.

Theorem 2.4.4 (Radon-Nikodým)**.** *Let μ, ν be absolutely continous measures and assume μ is σ-finite. Then there is a unique pair of $\mathbb{R}^m$ valued measures ν^a,ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$, and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in [\mathbf{L}^1(X, \mu)]^m$ such that $\nu^a = f\mu$.*

The Besicovitch covering theorem (see, e.g., [5, Theorem 2.18]) gives a control on the possible overlapping of countable subfamilies of an arbitrary cover of closed balls.

Theorem 2.4.5 (Besicovitch covering)**.** *Let $A \subset \mathbb{R}^N$ be a bounded set, and $\rho : A \mapsto]0, +\infty[$. Then, there is a set $S \subset A$, at most countable, such that*

$$A \subset \bigcup_{x \in S} B_{\rho(x)}(x)$$

and every point of $\mathbb{R}^N$ belongs at most to ξ balls $B_{\rho(x)}(x)$ centered at points of S.

The following Besicovitch derivation theorem (see [5, Theorem 2.22]) compares the density of a vector-valued Radon measure ν with respect to a positive Radon measure μ through a passage to the limit in the quotient between the measures of smaller and smaller balls. It is strictly related to the Radon-Nikodŷm theorem. In particular, it gives a concrete representation of the density ν/μ. It is an important tool to assist in deducing global properties of a measure from local properties.

Theorem 2.4.6 (Besicovitch derivation theorem)**.** *Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^N$ and ν an $\mathbb{R}^m$-valued Radon measure. Then, for μ-a.e. x in the support of μ, the limit*

$$f(x) \ := \ \lim_{\rho \downarrow 0} \frac{\nu(B_\rho(x))}{\mu(B_\rho(x))}$$

exists in $\mathbb{R}^m$. Moreover, the Radon-Nikodŷm decomposition of ν is given by $\nu = f\mu + \nu^s$, where $\nu^s = \nu \llcorner E$ and E is the μ-negligible set

$$E \ = \ (\Omega \setminus \operatorname{supp} \mu) \cup \left\{ x \in \operatorname{supp} \mu \ : \ \lim_{\rho \downarrow 0} \frac{|\nu|(B_\rho(x))}{\mu(B_\rho(x))} \right\}$$

We recall the notion of convolutions and Sobolev spaces (see [18]).

Definition 2.4.7 (Convolution)**.** Let μ be an $\mathbb{R}^m$-valued Radon measure in an open set $\Omega \subset \mathbb{R}^N$. If f is a continuous function, then we call the function

$$[\mu * f](x) \ := \ \int_\Omega f(x-y) d\mu(y)$$

the *convolution between f and μ* whever this makes sense.

Definition 2.4.8 (Weak derivatives)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in \mathbf{L}^1_{loc}(\Omega)$. If there is a $g \in \mathbf{L}^1_{\text{loc}}(\Omega)$ such that for $i \in \{1, \ldots, N\}$,

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx \ = \ - \int_\Omega \phi g \, dx, \qquad \forall \phi \in C_c^\infty(\Omega)$$

then we say that u has a weak i-th derivative given by g. If the weak derivative exists then, it is unique and is denoted by $D^i u$.

The weak derivatives coincide with the classical ones if $u \in C^1(\Omega)$. The following spaces are used when working with functions and their weak derivatives.

Definition 2.4.9 (Sobolev spaces)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set, and $p \in [1, \infty[$. We say that $u \in W^{k,p}$ if $u \in \mathbf{L}^p(\Omega)$ and has weak derivatives up to order k in $\mathbf{L}^p(\Omega)$ for every $i \in \{1, \ldots, N\}$.

Let $\Omega \subset \mathbb{R}$. We recall that $W^{k,p}(\Omega)$ becomes a Banach space $p \in [1, \infty[$ when endowed with the norm defined by

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^{k} \|D^\alpha u\|_{\mathbf{L}^p}^p \right)^{1/p} .$$

When $p = 2$, $W^{k,p}$ becomes a Hilbert space. We will adapt the notation that $H^k(\Omega) = W^{k,2}(\Omega)$.

2.5 Functions of Bounded Variation

We recall some notions of functions of bounded variation and present results that have applications to the field of conservation laws. Throughout this section, let Ω be a generic open set in $\mathbb{R}^N$.

Definition 2.5.1. Let $u \in \mathbf{L}^1(\Omega)$; we say u is a *function of bounded variation* in Ω (denoted by $u \in BV(\Omega)$) if $u \in \mathbf{L}^1(\Omega)$ and the distributional derivative of u is representable by a finite Radon measure Du on Ω, i.e.,

$$-\int_\Omega u \cdot \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega \varphi \, dD_i u \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega), \;\; i = 1, \ldots, N$$

Moreover, u is of *locally bounded variation* on Ω (denoted by $u \in BV_{loc}(\Omega)$) if $u \in \mathbf{L}^1_{loc}(\Omega)$ and u is in $BV(U)$ for all $U \subset\subset \Omega$.

Definition 2.5.2 (Total Variation)**.** Given $u \in \mathbf{L}^1(\Omega)$, the *total variation of u* in Ω is defined as

$$V(u, \Omega) := \sup \left\{ \int_\Omega u(x) \nabla \cdot \varphi(x) \, dx \; : \; \varphi \in C_c^1(\Omega), \, \|\varphi\|_{\mathbf{L}^\infty(\Omega)} \leq 1 \right\} .$$

Some useful properties of the variation of u are listed in the following proposition, including an alternative definition of the space BV (see, e.g., [5, Proposition 3.6]).

Proposition 2.5.3. *Let $u \in \mathbf{L}^1(\Omega)$. Then the space of functions of bounded variation (BV) can be defined as*

$$BV(\Omega) \;=\; \left\{ u \in \mathbf{L}^1(\Omega) \;:\; V(u,\Omega) < +\infty \right\}.$$

In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$ for any $u \in BV(\Omega)$ and $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ with respect to the $\mathbf{L}^1_{\text{loc}}(\Omega)$ topology.

Remark 2.5.4. It holds by definition that $W^{1,1}(\Omega) \subset BV(\Omega) \subset \mathbf{L}^1(\Omega)$. The space $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV} \;:=\; \int_\Omega |u|\, dx + |Du|(\Omega)$$

is a Banach space, but the norm-topology is too strong for many applications.

Definition 2.5.5. Let $u \in [\mathbf{L}^1_{loc}(\Omega)]^m$; we say that u has an *approximate limit at $x \in \Omega$* if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0+} \fint_{B_\rho(x)} |u(y) - z|\, dy \;=\; 0\,. \tag{2.12}$$

The set S_u of points where u does not have an approximate limit is called the *approximate discontinuity set*. For any $x \in \Omega \setminus S_u$, the vector z, uniquely determined by (2.12), is called the *approximate limit of u at x*. We say that x is an *approximate jump point of u* if there exists $a, b \in \mathbb{R}^m$ and $\nu \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0+} \fint_{B^+_\rho(x)} |u(y) - a|\, dy \;=\; 0, \qquad \lim_{\rho \to 0+} \fint_{B^-_\rho(x)} |u(y) - b|\, dy \;=\; 0\,. \tag{2.13}$$

The *set of approximate jump points* is denoted by J_u.

For any $u \in BV(\Omega)$, we write $Du = D^a u + D^s u$ where $D^a u$ is the absolutely continuous part of Du and $D^s u$ is the singular part of Du (both with respect to $\mathcal{L}^N$), as provided by the Radon-Nikodým Theorem (Theorem 2.4.4). Relying on the notions of approximate jump points, we denote the following two measures.

Definition 2.5.6. For any $u \in BV(\Omega)$ the measures

$$D^j u \;:=\; D^s u \llcorner J_u, \qquad D^c u \;:=\; D^s u \llcorner (\Omega \setminus S_u)$$

are called respectively the *jump part* of the derivative and the *Cantor part* of the derivative.

The following theorem compares $|Du|$ with $\mathcal{H}^{N-1}$ and shows, in particular, that $|Du|$ vanishes on any $\mathcal{H}^{N-1}$-negligible set (see [5, Lemma 3.76]).

Theorem 2.5.7. *Let $u \in BV(\Omega)$. Then, for any Borel set $B \subset \Omega$, it holds that*

$$\mathcal{H}^{N-1}(B) \;=\; 0 \;\Longrightarrow\; |Du|(B) \;=\; 0\,.$$

Due to Federer and Vol'pert, see for example [5, Theorem 3.78], the discontinuity set S_u is countably $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. Hence, Du vanishes on $(S_u \setminus J_u)$ and we obtain the following decomposition of Du

$$Du \;=\; D^a u + D^j u + D^c u\,.$$

We will make use of the following properties of $D^j u$ and $D^c u$, stated in 1-dimension.

Proposition 2.5.8. *Let $\Omega \subset \mathbb{R}$, $u \in BV(\Omega)$ and denote by*

$$S \;=\; \left\{ t \in \Omega \;:\; \lim_{\delta \to 0} \frac{|Du|(t-\delta, t+\delta)}{|\delta|} = +\infty \right\} \quad \textit{and} \quad A = \{ t \in \Omega \;:\; Du(\{t\}) \neq 0 \}\,.$$

Then, the jump and Cantor parts of Du can be obtained as the following restrictions:

$$D^j u = D^s u_{|A} \qquad \textit{and} \qquad D^c u = D^s u_{|S \setminus A}\,.$$

Moreover, the Cantor part $D^c u$ vanishes on any sets which are σ-finite with respect to $\mathcal{H}^0$, i.e.,

$$|D^c w|(E) \;=\; 0 \quad \forall \textit{ Borel set } E \textit{ with } \mathcal{H}^0(E) < +\infty\,.$$

We present a simplified version of [5, Theorem 3.108], which shows that the Cantor part of the derivative can be recovered from the corresponding parts of the derivatives of their restrictions.

Theorem 2.5.9. *Let $u(x,y) \in BV(\mathbb{R}^2)$ then $u_x(y) = u(x,y)$ is in BV for almost every $x \in \mathbb{R}$. Moreover, for any $\Omega \subset \mathbb{R}^2$ it holds that*

$$D^c_y u(\Omega) \;=\; \int_{\mathbb{R}} D^c u_x(\Omega_x(y))\, dy \qquad \textit{where} \quad \Omega_x(y) = \{ x \in \mathbb{R} \;:\; (x,y) \in \Omega \}$$

and D^c_y is the distributional derivative of $D^c u$ along $(0,1)$.

Furthermore, we have the following structure result (see e.g. [5, Theorem 3.28])

Proposition 2.5.10. *Let $\Omega \subseteq \mathbb{R}$ and $w \in BV(\Omega)$. Then, for any $x \in A_w$, the left and right hand limits of $w(x)$ exist and*

$$D^j w \;=\; \sum_{x \in A_w} \left(w(x+) - w(x-)\right) \delta_x \,,$$

where $w(x\pm)$ denote the one-sided limits of w at x. Moreover, $D^c w$ vanishes on any sets which are σ-finite with respect to $\mathcal{H}^0$.

Of particular note to the Chapter 4, is the space SBV.

Definition 2.5.11. We way that $u \in BV(\Omega)$ is a *special function with bounded variation* (denoted by $u \in SBV(\Omega)$) if the Cantor part of its derivative $D^c u$ is zero.

The space $SBV(\Omega)$ is a proper subspace of $BV(\Omega)$ if $\Omega \subset \mathbb{R}$. A classic example of a function in $BV(\Omega) \setminus SBV(\Omega)$ is the Cantor-Vitali function, where $\Omega = (0, 1)$.

Lastly, in order to derive bounds on the support of solutions to balance laws in terms of the $\mathbf{L}^1$ norm of their initial data, we will use the following proposition (see e.g., [6, Lemma 3]).

Proposition 2.5.12. *Given $v \in BV(\mathbb{R})$, compactly supported such that, for some constant $B > 0$,*

$$Dv \;\leq\; B \tag{2.14}$$

in the sense of measures, then it holds that

$$\|v\|_{\mathbf{L}^\infty} \;\leq\; \sqrt{2B\,\|v\|_{\mathbf{L}^1}}$$

Proof. Consider $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho \geq 0$ and $\int_{\mathbb{R}} \rho = 1$. Defining the mollifier $\rho_\varepsilon(x) \doteq \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ for any $\varepsilon > 0$ and the approximation

$$v_\varepsilon \;\doteq\; \rho_\varepsilon * v \,.$$

Since v satisfies (2.14), by stand properties of convolutions and the integration-by-parts formula for BV functions, it holds that

$$v_\varepsilon' - B = \rho_\varepsilon' * v - \rho_\varepsilon * B = \rho_\varepsilon * (Dv - B) \leq 0.$$

It can be shown that (see, e.g. [6]) for $v_\varepsilon \in C_0^\infty$,

$$v'_\varepsilon \leq B \quad \Longrightarrow \quad \|v_\varepsilon\|_{\mathbf{L}^\infty} \leq \sqrt{2B\,\|v_\varepsilon\|_{\mathbf{L}^1}}\,.$$

Hence, by taking $\varepsilon \to 0^+$, we have

$$\|v\|_{\mathbf{L}^\infty} \leq \sqrt{2B\,\|v\|_{\mathbf{L}^1}}\,,$$

which completes the proof. □

2.6 Scalar conservation laws

We recall some of the theory and results for a scalar conservation law. In particular, let $u : [0, +\infty[\times\mathbb{R} \to \mathbb{R}$ be some conserved quantity and consider the Cauchy problem

$$u_t + f(u)_x \;=\; 0 \tag{2.15}$$
$$u(0,\cdot) \;=\; u_0(\cdot) \tag{2.16}$$

assuming the flux $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and the initial data u_0 is in $\mathbf{L}^1_{loc}(\mathbb{R})$. A feature of nonlinear systems of the form (2.15) is that, as the following example (see, e.g. [13]) shows, solutions are not necessarily continuous, and thus the notion of solution needs to be extended beyond classical, smooth solutions.

Example 2.6.1. *We consider the following Cauchy problem for Burgers equation*

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x \;=\; 0 \\ u(0,x) \;=\; \dfrac{1}{1+x^2}\,. \end{cases}$$

For $t > 0$ small, the solution can be found using the method of characteristics. Indeed, if u is smooth, then the conservation law is equivalent to

$$u_t + uu_x \;=\; 0\,,$$

and u must be constant along the characteristic lines in the $t-x$ plane:

$$t \;\mapsto\; (t, x + t\bar{u}(x)) \;=\; \left(t, x_0 + \frac{t}{1+x_0^2}\right)$$

When $t < T := 8/\sqrt{27}$, these lines do not intersect and the solution to the Cauchy problem is given implicitly by

$$u\left(t, x + \frac{t}{1+x^2}\right) \;=\; \frac{1}{1+x^2}\,.$$

When $t > T$, the characteristic lines intersect and the above equation no longer defines a single valued solution. The solution can be prolonged for $t > T$ only within a class of discontinuous functions.

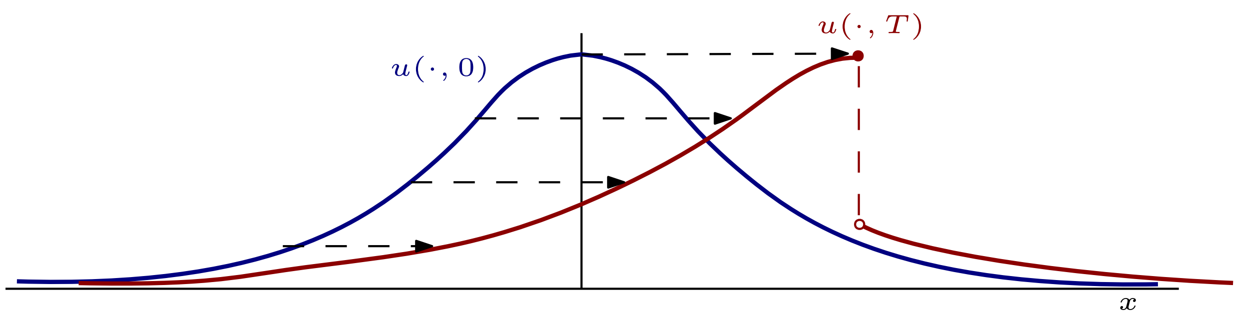

Figure 2.2 As seen in Example 2.6.1, a scalar conservation law may have discontinuous solutions even with smooth initial data.

To account for possible discontinuities in solutions, it is necessary introduce the concept of weak solutions.

Definition 2.6.2 (Distributional solution)**.** Given an open set $\Omega \in \mathbb{R} \times \mathbb{R}$, a measurable function $u : \Omega \mapsto \mathbb{R}$ is a *distributional solution* (2.15) if, for every C^1 function $\phi : \Omega \mapsto \mathbb{R}$ with compact support, it holds that

$$\int\int_{\Omega} \{u\phi_t + f(u)\phi_x\}\; dx\, dt \;=\; 0\,.$$

Remark 2.6.3. A function $u \in C^1$ is a classical solution of (2.15) if and only if u a distributional solution of (2.15).

Definition 2.6.4 (Distributional solution of Cauchy problem)**.** Given an initial condition $u(0,x) = u_0(x) \in \mathbf{L}^1_{loc}(\mathbb{R})$, we say that a function $u : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a *distributional solution* to the Cauchy problem (2.15)-(2.16) if

$$\int_0^T \int_{-\infty}^{\infty} u\varphi_t + f(u)\varphi_x \, dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(0,x)\, dx \;=\; 0\,.$$

It is standard to consider a stronger concept of solution, requiring the continuity of u as a function of time, with values into $\mathbf{L}^1_{loc}(\mathbb{R})$

Definition 2.6.5 (Weak solution of Cauchy problem)**.** Given an initial condition $u(0,x) = u_0(x) \in \mathbf{L}^1_{loc}(\mathbb{R})$, we say that a function $u : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a *weak solution* to the Cauchy problem (2.15)-(2.16) if

(i) the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R})$, i.e.,

$$\|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; L \cdot |t-s| \qquad \forall 0 \leq s \leq t$$

for some constant $L > 0$;

(ii) the initial condition (2.16) holds;

(iii) the restriction of u to the open strip $]0,T[\times\mathbb{R}$ is a *distributional solution* of (2.15)

Remark 2.6.6. Every weak solution is also a solution in the distributional sense to the Cauchy problem (2.15)-(2.16).

With the goal of finding a necessary condition for u to be a weak solution, we define the following characterization of discontinuities of u.

Definition 2.6.7. A function $u \in \mathbf{L}^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ has an *approximate jump discontinuity* at $(\bar{t},\bar{x})$ if there exists $u^\pm$, $\lambda \in \mathbb{R}$ with

$$U(t,x) \;:=\; \begin{cases} u^+ & x > \lambda t \\ u^- & x < \lambda t \end{cases} \tag{2.17}$$

such that the following holds

$$\lim_{r\to 0} \frac{1}{r^2} \int_{-r}^{r} \int_{-r}^{r} |u(\bar{t}+t, \bar{x}+x) - U(t,x)| \; dx \; dt \;=\; 0$$

Using this notion of discontinuity, one can show [13, Theorem 4.1] that a necessary condition for u to be a weak solution for a given conservation law is given by:

Proposition 2.6.8 (Rankine-Hugoniot Condition)**.** *Let u be a bounded distributional solution of* (2.15)*. If u has an approximate jump at $(\bar{t},\bar{x})$ then*

$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \tag{2.18}$$

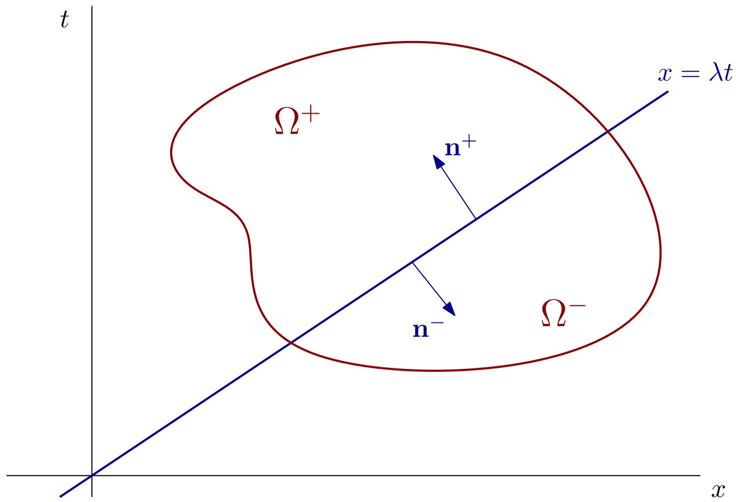

Figure 2.3 In the derivation of the Rankine-Hugoniot condition, the domain is split into two parts and solutions of the conservation law are considered in the distributional sense. The Rankine-Hugoniot condition must hold on the boundary between Ω^+ and Ω^-.

In general, the concept of a weak solution does not guarantee uniqueness. Indeed, in the following example, there are infinitely many possible weak solutions that satisfy the necessary condition (2.18).

Example 2.6.9. *Consider the following Cauchy problem for Burgers equation*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right) & = \ 0 \\ u(0,x) \ = & \begin{cases} 0 & x < 0 \\ 1 & x \geq 0\,. \end{cases} \end{cases}$$

For every $\alpha \in [0,1]$, each solution candidate

$$u^\alpha(t,x) \ = \ \begin{cases} 0, & x < \frac{\alpha}{2}t \\ \alpha, & \frac{\alpha}{2}t < x \leq \frac{\alpha+1}{2}t \\ 1, & x > \frac{\alpha+1}{2}t \end{cases}$$

satisfies (2.18) *and hence there are infinitely many weak solutions to the Cauchy problem.*

To advance the well-posedness theory, it is necessary to look for *admissibility conditions* which single out a unique solution. The goal of these conditions is to find the solution which is most physically relevant.

For the scalar case, it is standard to consider entropy admissible solutions, in the sense of Kruzkov, which guarantees the uniqueness of a solution.

Definition 2.6.10 (Kruzkov entropy solutions)**.** A function $u \in \mathbf{L}^\infty(]0,T[\times\mathbb{R},\mathbb{R})$ is a *weak entropy solution* of (2.15) if u satisfies the following

(i) the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R})$, i.e.,

$$\|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq L \cdot |t-s| \qquad \forall 0 \leq s \leq t$$

for some constant $L > 0$.

(ii) For any $k \in \mathbb{R}$ and any non-negative test function $\phi \in C^1_c(]0,\infty[\times\mathbb{R},\mathbb{R})$ one has

$$\int\int \Big[|u-k|\phi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\,\phi_x\Big]\,dx\,dt \geq 0\,. \tag{2.19}$$

In particular, (2.19) implies that u is a distributional solution of (2.15). From (2.19), we can derive a geometric condition, valid on each line of discontinuity of u (see [13, Theorem 4.4]).

Theorem 2.6.11. *The piecewise constant function U defined in* (2.17) *is a weak entropy solution of* (2.15) *if and only if the Rankine-Hugoniot condition* (2.18) *holds and for every $\alpha \in [0,1]$ it holds that*

$$\begin{cases} f(\alpha u^+ + (1-\alpha)u^-) \geq \alpha f(u^+) + (1-\alpha)f(u^-), & \text{if } u^- < u^+\,, \\ f(\alpha u^+ + (1-\alpha)u^-) \leq \alpha f(u^+) + (1-\alpha)f(u^-), & \text{if } u^- > u^+\,. \end{cases} \tag{2.20}$$

The conditions (2.20) have a geometric interpretation. When $u^- < u^+$, the graph of f should remain above the secant line. When $u^- > u^+$, the graph of f should remain below the secant line.

Remark 2.6.12. The condition (2.20) can be written as a stability condition:

$$\frac{f(u^*)-f(u^-)}{u^*-u^-} \geq \frac{f(u^+)-f(u^*)}{u^+-u^*} \tag{2.21}$$

for every $u^* = \alpha u^+ + (1-\alpha)u^-$, with $\alpha \in]0,1[$.

- In the case where $f'' \geq 0$, (2.21) is equivalent to

$$f'(u^-) \geq \frac{f(u^+)-f(u^-)}{u^+-u^*} \geq f'(u^+)\,, \tag{2.22}$$

which is known as the Lax condition (see [59] and [60]).

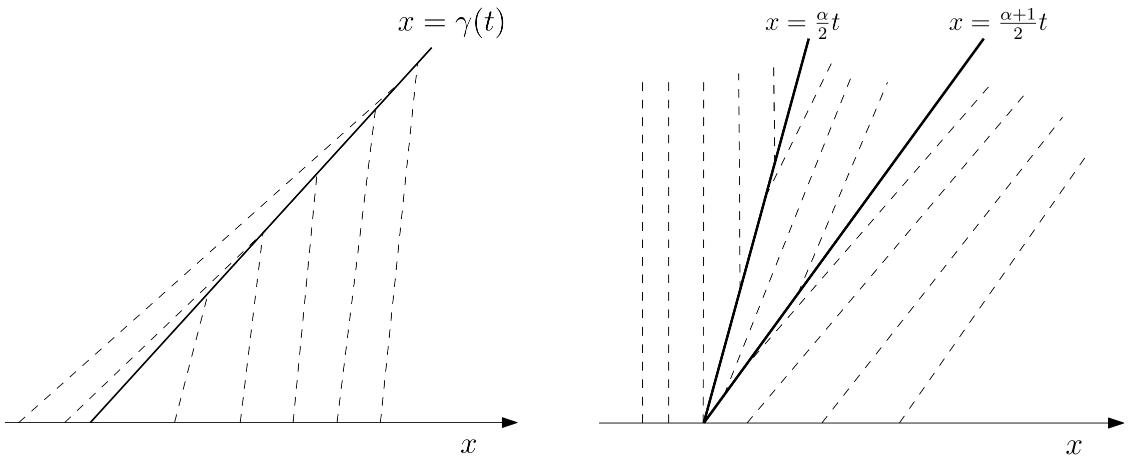

Figure 2.4 Consider a line of discontinuity $x = \gamma(t)$. The Lax condition requires that characteristics on both sides of the jump run towards the line of discontinuity. This situation is exhibited on the left figure. On the other hand, in the solutions constructed in Example 2.6.9 (right figure), neither of the shocks satisfy the admissiblity condition.

- In the case where $f'' > 0$, (2.21) is equivalent to both (2.22) and $u^- > u^+$.

Characteristics associated with classical, Lipschitz solutions provide a principal tool of the theory to understand those solutions. Dafermos [35] extends the notion of characteristics to the framework of weak solutions through differential inclusions [36].

Definition 2.6.13. For any $(t, x) \in]0, +\infty[\times\mathbb{R}$, an absolutely continuous curve $\xi_{(t,x)}(\cdot)$ is called a

(a) *backward characteristic curve* starting from (t, x) if it is a solution of the differential inclusion

$$\dot{\xi}_{(t,x)}(s) \in \left[u\left(s, \xi_{(x,t)}(s)+\right), u\left(s, \xi_{(t,x)}(s)-\right)\right] \qquad a.e.\ s \in [0, t] \tag{2.23}$$

with $\xi_{(t,x)}(t) = x$.

(b) *forward characteristic curve* if $s \in [t, +\infty[$ in (2.23), here denoted by $\xi^{(t,x)}(\cdot)$.

The characteristic curve ξ is called *genuine* if $u(t, \xi(t)-) = u(t, \xi(t)+)$ for almost every t.

Admissible weak solutions of (2.15) possess $\mathbf{L}^1$ stability and $\mathbf{L}^\infty$ monotonicity properties [36][Theorem 6.2.3].

Theorem 2.6.14. *Let u and $\tilde{u}$ be admissible weak solutions of* (2.15) *with initial data u_0 and $\tilde{u}_0$ taking values in a compact interval $[a,b]$. Then there exist $s > 0$, depending on $[a,b]$, such that, for any $t > 0$ and $r > 0$*

$$\int_{|x|<r} [u(x,t) - \tilde{u}(x,t)]^+ \, dx \;\leq\; \int_{|x|<r+st} [u_0(x) - \tilde{u}_0(x)]^+ \, dx\,,$$

and

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{B}_r)} \;\leq\; \|u_0(\cdot) - \tilde{u}_0(\cdot)\|_{\mathbf{L}^1(\mathcal{B}_{r+st})}\,. \tag{2.24}$$

Futhermore, if $u_0(x) \leq \tilde{u}_0(x)$ for a.e. $x \in \mathbb{R}$ then

$$u(x,t) \;\leq\; \tilde{u}(x,t) \qquad a.e.\ (t,x) \in [0,\infty[\times\mathbb{R}$$

It follows from (2.24) that for initial data $u_0 \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, the Cauchy problem (2.15)-(2.16) has a unique weak entropy solution. For $t \in R^+$, consider the map $S_t : \mathbf{L}^1(\mathbb{R}) \to \mathbf{L}^1(\mathbb{R})$ which carries initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ to the admissible weak solution of (2.15) $u(\cdot,t)$, i.e.

$$S_t u_0(\cdot) \;=\; u(\cdot,t)$$

The S defines a contractive semigroup. Indeed, we have the following result (see, e.g., [13, Theorem 6.3]).

Theorem 2.6.15. *Let f be locally Lipschitz continuous. Then there eixsts a continuous semigroup $S : \mathbb{R}^+ \times \mathbf{L}^1(\mathbb{R}) \mapsto \mathbf{L}^1(\mathbb{R})$ such that*

(i) $S_0 \;=\; I,\ S_{t+\tau} \;=\; S_t S_\tau$

(ii) $\|S_t u - S_t v\|_{\mathbf{L}^1} \;\leq\; \|u - v\|_{\mathbf{L}^1}$

(iii) For each $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, the trajectory $t \mapsto S_t u_0$ yields the unique bounded, entropy-admissible, weak solution of the corresponding Cauchy probem (2.15)-(2.16).

(iv) If $u_0(x) \leq v_0(x)$ for all $x \in \mathbb{R}$, then $S_t u_0(x) \leq S_t v_0(x)$ for every $x \in \mathbb{R}$ and $t \geq 0$.

There is a particularly rich theory in the case that f is uniformly convex (see, e.g., [40]). Of note, the entropy solutions satisfy a one-sided jump estimate as shown by Oleinik [67]

Theorem 2.6.16. *Let u be the unique entropy solution of* (2.15)-(2.16) *with initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and uniformly convex flux ($f'' > c \geq 0$). Then $u(\cdot,t) \in BV_{loc}(\mathbb{R})$ and u*

satisfies the one-sided Lipschitz estimate

$$u(y,t) - u(x,t) \;\leq\; \frac{1}{ct} \cdot (y - x)$$

for all $t > 0$ *and* $y > x$.

Analogous results of this section hold in the case where the conservation law has a source term, $g \in C(\mathbb{R}^+ \times \mathbb{R})$. On the other hand, the theory of existence and uniqueness is not complete if the source term is nonlocal. In Chapter 4, we study a specific nonlocal balance law with the goal of shedding light on this outstanding case.

Chapter 3

A sovereign debt management model

This chapter aims to analyze a system of Hamilton-Jacobi-Bellman equations arising from a problem of stochastic optimal debt management in an infinite time horizon, modeled as a non-cooperative game between a sovereign state and a pool of foreign, risk-neutral lenders. In [66], the authors present a model of debt management in which a sovereign state controls both fiscal policy by using income towards repaying the debt, and monetary policy, by devaluating their currency and producing inflation. In the case of devaluating currency, the state is increasing the welfare cost and negatively affecting the trust of foreign investors. At any time, there is a positive probability of the state defaulting on their debt, which depends on the current size of the debt.

There are many motivations for studying how to best balance debt under the risk of bankruptcy. As of 2020, the national debt of the USA is over $27 trillion. The average American holds $90,460 in personal debt, including credit cards, mortgages, and student debt. The financial crisis of 2008 was caused by excessive risk-taking by banks, leading to the values of securities related to U.S. real estate properties to plummet [49]. The crisis sparked what is known as the Great Recession and the European debt crisis involving Greece, Ireland, Italy, Portugal, and Spain. The recession was noted by the emergence of large fiscal deficits across the industrialized world, leading to increases in government debt and debt-to-income (or debt-to-GDP) ratios in the United Kingdom, Japan, and the mentioned Euro area periphery countries [30], [51].

The Euro area periphery economies experienced dramatic spikes in their sovereign yields, where other highly indebted countries did not. A key difference in their debt management strategies was that the US, UK, and Japan could deflate away the burden of nominal debt, while the Euro countries were forced to repay debt solely through fiscal surpluses. These events raised the question as to what role monetary policy should have, if any, in guaranteeing the sustainability of sovereign debt, in view of the existing trade-offs between latter and price stability [66].

In this chapter, we shall examine a model first presented in [66]. In this model, the borrower has a possibility to go bankrupt with an instantaneous bankruptcy risk and must declare bankruptcy when the debt-to-income ratio reaches a threshold $x^* > 0$. To offset the possible loss of part of their investment, the lenders buy bonds at a discounted rate p, which is not given a priori. Rather, it is determined by the expected evolution of the debt-to-income ratio at all future times. Hence it depends globally on the entire feedback controls u and v. This framework leads to a highly nonstandard optimal control problem, and a "solution" must be understood as a Nash equilibrium, where the strategy implemented by the borrower represents the best reply to the strategy adopted by the lenders, and conversely.

In Section 3.1, we will develop the mathematical model and derive the corresponding system of HJB equations. In Section 3.2, we establish an existence result for the system of equations (1.1). In turn, this yields the existence of a pair of optimal feedback controls (u^*, v^*) which minimizes the expected cost to the borrower. The system (1.1) is not uniformly elliptic at the boundary $x = 0$ and $x = x^*$. To handle this difficulty, we employ the classical idea of constructing solutions of approximate systems as steady states of corresponding auxiliary parabolic systems. In order to obtain a solution to the original system (1.1), we derive explicit a priori estimates on the derivatives of approximate solutions. In Section 3.3, we provide lower and upper bounds for the value function V as a sub-solution and super-solution, which in turn yields behavior of the optimal policies near the bankruptcy threshold.

In the mathematical economics literature, some related models of debt and bankruptcy, focusing on quantitative analysis, can be found in [14, 15, 17, 64, 66]. In [17], the borrower has a fixed income, and large debt values determine a bankruptcy risk, which adds uncertainty to the model. In [66], a numerical analysis was performed of a similar model in which the uncertainty comes not from bankruptcy risk but the random evolution of the borrower's income. An analytical study of a slight variant of the model was performed in [15] where no currency devaluation is available to the government. The authors constructed optimal feedback solutions in the stochastic case and provided an explicit formula for the optimal strategy in the deterministic case. Moreover, their analysis also shows how the expected total cost of servicing the total debt together with the bankruptcy cost is affected by different choices of x^*. In [64], the authors continue this study by adding another control in the option for currency devaluation. A stochastic model with no currency devaluation but with uncertainty coming from bankruptcy risk was analyzed in [14].

To conclude this preface, we recount a history of economic literature leading to the model in [66]. The earliest consideration of an economic optimization problem on an unbounded time interval is attributed to [71]. In the framework of [20], [27], models of self-fulfilling debt crises have been considered in several studies (see e.g., [1], [72], [31], [22]). A further extension has been developed, in the tradition of [39], in which sovereign default is instead an optimal government decision (see e.g., [66], [15]). Many past studies have produced qualitative contributions; in contrast, we will consider a fully stochastic approach with quantitative analysis. In modeling optimal default, as in [39], our model is in line with sovereign debt models presented by [2] and [7]. Following the study in [66], we build on these past models by introducing nominal bonds and studying the optimal inflation policy when the government cannot commit to not inflate in the future. Our model differs from others used in sovereign default literature by considering continuous time (see [61], [76]). See [26] for a background in infinite horizon optimal control problems.

3.1 A model of debt with bankruptcy risk

We consider an optimal debt management problem in an infinite time horizon, modeled as a non-cooperative interaction between a borrower and a pool of risk-neutral lenders. In this model, let $v(t)$ be the devaluation rate at time t, regarded as a control. A stochastic process governs the total income Y given by

$$dY \;=\; (\mu + v(t))Y(t)dt + \sigma Y(t)dW$$

where W is a Brownian motion on a filtered probability space, $\sigma > 0$ is the volatility, and μ is the economy's average growth rate. We denote by $X(t)$ the total debt of a borrower, financed by issuing bonds, and

- r = the interest rate paid on bonds, which we will assume coincides with the discount rate;
- λ = the rate at which the borrower pays back the principal.

When an investor buys a bond of unit nominal value, he will receive a continuous stream of payments with intensity $(r + \lambda)e^{-\lambda t}$. If no bankruptcy occurs, the payoff for an investor will thus be

$$\int_0^\infty e^{-rt}(r+\lambda)e^{-\lambda t}\,dt = 1.$$

Otherwise, a lender recovers only a fraction $\theta \in [0,1]$ of his outstanding capital, which is dependent on the total amount of debt at the time of bankruptcy. To offset this possible loss, the investor buys a bond with nominal unit value at a discounted price $p \in [0,1]$. Hence, the total debt evolves according to

$$\dot{X}(t) \;=\; -\lambda X(t) + \frac{(\lambda + r)X(t) - U(t)}{p(t)}$$

where $U(t)$ is the rate of payments that the borrower chooses to make to the lenders at time t. By Itô's theorem [68, Theorem 4.1.2], we derive the stochastic control system for the the debt-to-income (DTI) ratio $x \doteq X/Y$ with dynamics

$$dx(t) \;=\; \left[\left(\frac{\lambda + r}{\bar{p}(t)} - \lambda + \sigma^2 - \mu - v(t)\right) x(t) - \frac{u(t)}{p(t)}\right] dt - \sigma\, x(t)\, dW \tag{3.1}$$

and where $u = U/Y \in [0,1]$ is the portion of the total income allocated to reduce the debt.

In this model, we shall assume that $r > \mu$ and there exists a threshold $x^* > 0$, beyond which bankruptcy immediately occurs. Define T_{x^*} as the time when the borrower's debt reaches to a threshold x^*

$$T_{x^*} \;:=\; \inf\{t > 0 \;:\; x(t) = x^*\}\,. \tag{3.2}$$

On the other hand, it is well-known that there is no way to predict with certainty the time when bankruptcy occurs. Therefore it is natural to consider a variant of the previous model by assuming that the borrower may go bankrupt. More precisely, calling $\mathcal{T}_B$ the random time at which bankruptcy occurs, its distribution is determined as follows. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\left\{\mathcal{T}_B \in [\tau, \tau + \varepsilon] \;\middle|\; \mathcal{T}_B > \tau, x(\tau) = y\right\} \;=\; \rho(y)\cdot\varepsilon + o(\varepsilon)\,, \tag{3.3}$$

where $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0+$ and the smooth increasing function $\rho : [0, x^*[\to [0 + \infty[$ is bankruptcy risk such that $\lim_{x \to x^*} \rho(x) = +\infty$. Letting

$$P(t) \;:=\; \text{Prob.}\{\mathcal{T}_B > t\}, \qquad \forall t \in [0, T_{x^*}[\,,$$

we can compute

$$
\begin{aligned}
P'(t) &= \lim_{\varepsilon\to 0} \frac{P(t+\varepsilon)-P(t)}{\varepsilon} \;=\; \lim_{\varepsilon\to 0} \frac{\text{Prob.}\{\mathcal{T}_B > t+\varepsilon\} - \text{Prob.}\{\mathcal{T}_B > t\}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\text{Prob.}\{\mathcal{T}_B > t+\varepsilon\} - \Big(\text{Prob.}\{\mathcal{T}_B > t+\varepsilon\} + \text{Prob.}\{\mathcal{T}_B \in [t,t+\varepsilon] \cap \mathcal{T}_B > t\}\Big)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{-\text{Prob.}\{\mathcal{T}_B \in [t,t+\varepsilon]\} \cdot \text{Prob.}\{\mathcal{T}_B > t\}}{\varepsilon} \;.
\end{aligned}
$$

Recalling the definition of $P(t)$ and (3.3), it holds that

$$
P'(t) \;=\; \lim_{\varepsilon\to 0} \frac{-P(t)\cdot(\rho(x(t))\cdot\varepsilon + o(\varepsilon))}{\varepsilon} \;=\; -\,\rho(x(t))\cdot P(t)\,.
$$

Hence, the borrower may go bankrupt before T_{x^*} at random time $\mathcal{T}_B$ with probability given by

$$
\text{Prob}\;\{\mathcal{T}_B > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(x(\tau))\,d\tau\right\} & \text{if} \quad t \;<\; T_{x^*}, \\ 0 & \text{if} \quad t \geq T_{x^*}\,. \end{cases} \tag{3.4}
$$

We assume the borrower services his debt by buying bonds at a discounted price p through a pool of risk-neutral lenders. Let $\theta(x(t))$ be the "salvage function" which determines the fraction of the borrower's assets to be recovered by the lenders in the occurrence of bankruptcy at time t. As in [15, 17, 66], the discounted bond price is uniquely determined by the competition of a pool of risk-neutral lenders

$$
\begin{aligned}
p \;=\; E\Big[\int_0^{\mathcal{T}_B} (r+\lambda)\exp\left\{-\int_0^{\tau}(\lambda+r+v(x(s)))\,ds\right\}d\tau \\
+\exp\left\{-\int_0^{\mathcal{T}_B}(r+\lambda+v(x(\tau))d\tau\right\}\cdot\theta(x(\mathcal{T}_B))\Big].
\end{aligned} \tag{3.5}
$$

Given an initial cost x_0, the borrower seeks to find a pair of controls (u,v) which minimize their expected cost, exponentially discounted in time

$$
\text{Minimize } J[x_0,u,v] \;=\; E\left[\int_0^{\mathcal{T}_B} e^{-rt}\cdot\Big[L(u(t))+c(v(t))\Big]\,dt + e^{-r\mathcal{T}_B}\cdot B\right]_{x(0)=x_0}, \tag{3.6}
$$

where

- $L(u)$ is the cost incurred from spending income on debt;

- $c(v)$ is a social cost incurred from devaluating the currency;
- B is the cost incurred upon bankruptcy.

We introduce standard assumptions in the model. Concerning the functions θ, ρ, L and c, we shall assume

(A1) *The map* $\theta : [0, x^*] \to]0, 1]$ *is non-increasing, continuous, and locally Lipschitz in* $[0, x^*[$.

(A2) *The function* $\rho : [0, x^*[\to [0, +\infty[$ *is continuously differentiable for* $x \in [0, x^*[$, *and satisfies*

$$\rho(0) \;=\; 0, \qquad \rho'(x) \;\geq\; 0 \qquad \text{and} \qquad \lim_{x \to x^*} \rho(x) \;=\; +\infty.$$

(A3) *The function* $(L, c) : [0, 1[\times [0, v_{\max}[\to [0, +\infty[\times [0, +\infty[$ *is twice continuously differentiable such that*

$$L'(u),\; c'(v) \;>\; 0, \qquad L''(u),\; c''(v) \;\geq\; \delta_0$$

and

$$L(0) \;=\; c(0) \;=\; 0, \qquad \lim_{u \to 1} L(u) \;=\; +\infty, \qquad \lim_{v \to v_{\max}} c(v) \;=\; +\infty$$

for some constant $\delta_0 > 0$ *and* $v_{\max} \geq 0$.

Remark 3.1.1. When the currency devaluation is not an option for the borrower ($v \equiv 0$), or the bankruptcy risk is non-positive away from the bankruptcy threshold, i.e.

$$\rho(x) \;=\; \begin{cases} 0 & \text{if } x < x^* \\ +\infty & \text{if } x = x^*, \end{cases}$$

this model reduces to the one analyzed in [14] or [64], respectively.

3.1.1 Derivation of HJB equations

The stochastic control system (3.1)-(3.6) is highly non-standard. Here, the discount price $p(\cdot)$ in (3.5) depends on the debt-to-income ratio not only at the present time t but also all the future. Hence, it depends globally on the entire controls u, v. In general, the evolution equation (3.5) with $p(\cdot)$ defined in (3.5) may have no solutions or multiple solutions. Therefore it is quite difficult to study the existence from the viewpoint of open-loop optimal solutions. Indeed, the proposed model can be viewed as a non-cooperative interaction between a borrower and a pool of risk-neutral lenders. Thus we seek to construct equilibrium solutions in feedback form,

$$u = u^*(x), \qquad v = v^*(x) \qquad\qquad \text{for} \quad x \in [0, x^*].$$

Definition: (Stochastic optimal feedback solution). We say that a triple of functions $(u^*(x), v^*(x), p^*(x))$ provides an optimal solution to the problem of optimal debt management (3.1,3.2, 3.6, 3.5) if

(i) Given the function $p^*(\cdot)$, for every initial value $x_0 \in [0, x^*]$ the feedback control (u^*, v^*) with stopping time $\mathcal{T}_B$ as in (3.4) provides an optimal solution to the stochastic control problem (3.6), with dynamics (3.1).

(ii) Given the feedback control $(u^*(\cdot), v^*(\cdot))$, the discounted price p^* satisfies (3.5), where $\mathcal{T}_B$ is the stopping time (3.4) determined by the dynamics (3.1).

Let us introduce the associated Hamiltonian function to (3.1)-(3.5)

$$\begin{aligned} H(x, \xi, p) \;=\; & \min_{(u,v)\in[0,1]\times[0,v_{\max}]} \left\{ L(u) + c(v) - \left(\frac{u}{p} + xv\right)\cdot\xi \right\} \\ & \qquad + \left(\frac{\lambda + r}{p} - \lambda + \sigma^2 - \mu\right) x\,\xi\,, \end{aligned} \tag{3.7}$$

and two functions

$$\tilde{u}(\xi, p) \;=\; \operatorname*{argmin}_{u\in[0,1]} \left\{ L(u) - u\,\frac{\xi}{p} \right\}, \qquad \tilde{v}(x, \xi) \;=\; \operatorname*{argmin}_{v\in[0,v_{\max}]} \left\{ c(v) - vx\xi \right\}. \tag{3.8}$$

Under the assumption **(A3)**, a direct computation yields

$$\begin{aligned} \tilde{v}(x, \xi) \;&=\; \begin{cases} 0 & \text{if} \quad \xi x \le c'(0) \\ (c')^{-1}(\xi x) & \text{if} \quad \xi x > c'(0) \end{cases} \\ \tilde{u}(\xi, p) \;&=\; \begin{cases} 0, & \text{if} \quad \frac{\xi}{p} \le L'(0) \\ (L')^{-1}\left(\frac{\xi}{p}\right), & \text{if} \quad \frac{\xi}{p} > L'(0). \end{cases} \end{aligned} \tag{3.9}$$

In order to compute $u^*(\cdot)$ and $v^*(\cdot)$, we relate the stochastic control problem to a system of second order differential equations utilizing the dynamic programming principle and the Feynman-Kac formula.

(i) Assume that the discount bond price, $p^*(\cdot)$ is given and define the value function V as follows

$$V(x_0) \;=\; \inf_{(u,v)\in\mathcal{U}} J[x_0,u,v], \quad \text{where} \quad \mathcal{U} \;=\; \Big\{(u,v) : [0,x^*]^2 \to [0,1]\times[0,v_{\max}]\Big\}\,.$$

Using the probability density of $\mathcal{T}_B$, (3.4), the cost can be represented as

$$\begin{aligned} J[x_0,u,v] \;=\; E\Bigg[\int_0^\infty \exp\left\{-\int_0^\tau \rho(x(s))+r\,ds\right\}\cdot\Big(L(u(\tau))+c(v(\tau)) \\ +B\cdot\rho(x(\tau))\Big)\,d\tau \;\Big|\; x(t)=x_0\Bigg]. \end{aligned} \tag{3.10}$$

Therefore, by stochastic optimal control theory (Theorem 2.3.7), V solves the following second-order ODE

$$(r+\rho(x))V(x) \;=\; \rho(x)\cdot B + H(x,V'(x),p^*(x)) + \frac{\sigma^2x^2}{2}\cdot V''(x)\,. \tag{3.11}$$

with boundary values

$$V(0) \;=\; 0 \qquad \text{and} \qquad V(x^*) \;=\; B.$$

The optimality condition and (3.9) imply that the feedback strategies are recovered by

$$u^*(x) \;=\; \tilde{u}(V'(x),p(x)) \;=\; \begin{cases} 0, & \text{if} \quad \dfrac{V'(x)}{p(x)} \;\le\; L'(0), \\[2ex] (L')^{-1}\left(\dfrac{V'(x)}{p(x)}\right), & \text{if} \quad \dfrac{V'(x)}{p(x)} \;>\; L'(0), \end{cases} \tag{3.12}$$

and

$$v^*(x) \;=\; \tilde{v}(x,V'(x)) \;=\; \begin{cases} 0, & \text{if} \quad V'(x)x \;\le\; c'(0), \\[2ex] (c')^{-1}\left(V'(x)x\right) & \text{if} \quad V'(x)x \;>\; c'(0)\,. \end{cases} \tag{3.13}$$

(ii) Suppose we know the optimal feedback controls, u^*, v^* and we wish to calculate the discount bond price p^*. Given $x(0) = x_0$, the bond price is given by

$$\begin{aligned} p^*(x_0) \;=\; E\Bigg[& \int_0^{\mathcal{T}_B} (r+\lambda) \exp\left\{ -\int_0^{\tau} \left(\lambda + r + v^*(x(s))\right) ds \right\} d\tau \\ & + \exp\left\{ -\int_0^{\mathcal{T}_B} (r+\lambda+v^*(x(\tau)) d\tau \right\} \cdot \theta(x(\mathcal{T}_B)) \Bigg| \, x(0) = x_0 \Bigg]. \end{aligned}$$

which, considering (3.4), can be represented as

$$\begin{aligned} p^*(x_0) \;=\; E\Bigg[& \int_0^{\infty} \exp\left\{ -\int_0^{\tau} \Big(\rho(x(s)) + r + \lambda + v^*(x(s))\Big) \, ds \right\} \\ & \cdot \Big(r + \lambda + \rho(x(\tau))\theta(x(\mathcal{T}_B))\Big) \, d\tau \Bigg| \, x(0) = x_0 \Bigg]. \end{aligned} \tag{3.14}$$

By the Feynman-Kac formula (Theorem 2.3.6), p^* solves

$$\begin{aligned} & \left(\left[\frac{\lambda + r}{p^*(x(t))} - \lambda + \sigma^2 - \mu - v^*(x(t)) \right] x(t) - \frac{u^*(x(t))}{p^*(x(t))} \right) \cdot (p^*)'(x(t)) - \\ & \Big(r + \lambda + v^*(x(t)) + \rho(x(t))\Big) \cdot p^*(x(t)) + r + \lambda + \rho(x(t)) \cdot \theta(x(t)) \\ & + \frac{(\sigma x)^2}{2} (p^*)''(x(t)) = 0 \,, \end{aligned} \tag{3.15}$$

with $p(0) = 1$ and $p(x^*) = \theta(x^*)$. The equation (3.15) can be written in terms of the Hamiltonian as follows

$$\begin{aligned} (r + \lambda + v^*(x)) \cdot p^*(x) - (r+\lambda) \;=\; & \rho(x) \cdot [\theta(x) - p^*(x)] + \\ & H_{\xi}(x, V'(x), p^*(x)) \cdot (p^*)'(x) + \frac{(\sigma x)^2}{2} \cdot (p^*)''(x) \,. \end{aligned} \tag{3.16}$$

Therefore, finding an optimal solution to the problem of optimal debt management

(3.1)-(3.6) leads to the following system of second order implicit ODEs

$$\begin{cases} (r+\rho(x))V(x) \;=\; \rho(x)\cdot B + H(x,V'(x),p(x)) + \dfrac{\sigma^2x^2}{2}\cdot V''(x)\,, \\ (r+\lambda+v(x))\cdot p(x) - (r+\lambda) \;=\; \rho(x)\cdot[\theta(x)-p(x)] + \\ \qquad\qquad\qquad\qquad H_\xi(x,V'(x),p(x))\cdot p'(x) + \dfrac{(\sigma x)^2}{2}\cdot p''(x)\,, \\ v(x) \;=\; \underset{w\in[0,v_{\max}]}{\operatorname{argmin}}\ \{c(w) - wxV'(x)\}\ , \end{cases} \tag{3.17}$$

with boundary conditions

$$V(0) \;=\; 0, \quad V(x^*) \;=\; B \quad \text{and} \quad p(0) \;=\; 1, \quad p(x^*) \;=\; \theta(x^*). \tag{3.18}$$

To complete this subsection, let us collect some useful properties of Hamiltonian H.

Lemma 3.1.2. *If **(A3)** holds then H is continuous differentiable and its gradient at points $(x,\xi,p)\in[0,+\infty[\times[0,+\infty[\times]0,1]$ can be expressed by*

$$\begin{cases} H_x(x,\xi,p) \;=\; \Big[(\lambda+r) - p(\lambda+\mu+\tilde{v}(x,\xi)-\sigma^2)\Big]\cdot\dfrac{\xi}{p}, \\ H_\xi(x,\xi,p) \;=\; \dfrac{1}{p}\cdot\Big[x\big((\lambda+r) - p(\lambda+\mu+\tilde{v}(x,\xi)-\sigma^2)\big) - \tilde{u}(\xi,p)\Big], \\ H_p(x,\xi,p) \;=\; (\tilde{u}(\xi,p) - x(\lambda+r))\cdot\dfrac{\xi}{p^2}, \end{cases} \tag{3.19}$$

where the functions $\tilde{u},\tilde{v}$ are defined in (3.9). Furthermore,

(i). for all $\xi\in[0,+\infty[$, the function H satisfies

$$\begin{aligned} \left(\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu-v(x))x\right)\cdot\xi \;\le\; & H(x,\xi,p) \\ \le\; & \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right)x\xi\,, \end{aligned} \tag{3.20}$$

and H_ξ satisfies

$$\begin{aligned} \frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu-v(x))x \;\le\; & H_\xi(x,\xi,p) \\ \le\; & \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right)x\,; \end{aligned} \tag{3.21}$$

(ii). for every $(x,p)\in]0,+\infty[\times]0,+\infty[$ the map $\xi\mapsto H(x,\xi,p)$ is concave down and

satisfies

$$H(x,0,p) = 0, \tag{3.22}$$
$$H_\xi(x,0,p) = \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\,. \tag{3.23}$$

Proof. Since $H(x,\cdot,p)$ is defined as the infimum of a family of affine functions, it is concave down. We observe that (3.12) implies

$$H(x,\xi,p) = \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\xi \qquad \text{if } 0 \le \xi \le pL'(0)\,, \tag{3.24}$$

which implies the identities (3.22-3.23). By the concavity property, the map $\xi \mapsto H_\xi(x,\xi,p)$ is monotonically decreasing. Thus (3.23) implies the upper bound in (3.21). Taking $u=0, v=0$ implies the upper bound in (3.20). By (**A3**) it holds that

$$H(x,\xi,p) \ge -\frac{u}{p}\xi + \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu + v\right) x\xi\,. \tag{3.25}$$

taking $u=1$ implies the lower bound in (3.20). Let u^* and v^* denote the minimizers of $\tilde{u}$ and $\tilde{v}$ in (3.9). Using the optimality condition, one computes from (3.7) and (3.12) that

$$\begin{aligned} H_\xi(x,\xi,p) &= -x(v^*)(x,\xi) + \left(\frac{\lambda+r-u^*(\xi,p)}{p} - \lambda + \sigma^2 - \mu\right) x \\ &\ge \left(\frac{\lambda+r-u^*(\xi,p)}{p} - \lambda + \sigma^2 - \mu - v(x)\right) x\,, \end{aligned}$$

where for ξ large enough,

$$u^*(\xi,p) = \operatorname*{arg\,min}_{\omega\in[0,1]} \left\{L(\omega) - \frac{\xi}{p}\omega\right\} = (L')^{-1}\left(\frac{\xi}{p}\right) < 1\,.$$

Observe that, as $\xi \to +\infty$, one has $u^*(\xi,p) \to 1$. The non-increasing property of the map $\xi \mapsto H_\xi(x,\xi,p)$ yields the lower bound in (3.21). □

Corollary 3.1.3. *Suppose the assumption (*__A3__*) holds. Then for all* $(x,\xi,p,v) \in [0,x^*]\times]0,+\infty[\times[\theta_{\min},+\infty[\times[0,v_{\max}]$*, it holds that*

$$|H(x,\xi,p)| \le K_1\cdot\xi, \qquad \textit{and} \qquad |H_\xi(x,\xi,p)| \le K_1,$$

where the constant $\theta_{\min} > 0$ *is defined in* (3.27) *and*

$$K_1 := \max\left\{\left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right)x^*,\ {\theta_{\min}}^{-1} + (\lambda+\mu+v_{\max})\,x^*\right\}. \tag{3.26}$$

Proof. We compute an upper bound of the terms in (3.20)-(3.21)

$$\left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right)x \ \le\ \left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right)x^* \ \le\ K_1\,,$$

and lower bound

$$\left(\frac{(\lambda+r)x-1}{p} + (\sigma^2 - \lambda - \mu - v(x))x\right) \ \ge\ -\left(\frac{1}{\theta_{\min}} + (\lambda+\mu+v_{\max})x^*\right) \ \ge\ -K_1\,.$$

Hence, for any $\xi > 0$, the bounds (3.20)-(3.21) imply the uniform bounds on H and H_ξ. □

3.2 Existence results for the system of HJB equations

In this section, we will construct a solution of the system of Hamilton-Jacobi-Bellman equations (3.17) with boundary conditions (3.18) for a given bankruptcy threshold x^*. In turn, this result yields the existence of optimal feedback controls for the problem of debt management (3.1)–(3.6). Finally, we show how the bankruptcy risk affects the behavior of the optimal feedback control as the debt-to-income ratio tends to x^*. Let us introduce the constant which will be a lower bound of the discount bond price p,

$$\theta_{\min} := \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\}. \tag{3.27}$$

We seek the prove the following existence result.

Theorem 3.2.1. *Under the assumptions* **(A1) - (A3)**, *the system of second order ODEs* (3.17) *with boundary conditions* (3.18) *admits a solution* $(V,p) : [0,x^*] \to [0,B] \times [\theta_{\min}, 1]$ *of class* C^2 *such that* V *is monotone increasing and*

$$v(x) = \arg\min_{w\ge 0}\{c(w) - wxV'\} = 0 \qquad \forall \quad x \in \left[0, \frac{c'(0)}{M^*}\right], \tag{3.28}$$

for a constant M^* *which can be explicitly computed (see Lemma 3.2.4).*

It is well-known (see Theorem 2.3.7 or [67, Theorem 11.2.2]) that having constructed a solution (V, p) to the boundary value problem (3.17)-(3.18), then (u^*, v^*) in (3.12)-(3.13) is an optimal solution to the problem of optimal debt management (3.1)-(3.6). As a consequence of Theorem 3.2.1, we obtain the following result.

Corollary 3.2.2. *Under the same assumptions of Theorem 3.2.1, the debt management problem (3.1)-(3.6) admits an optimal control strategy in feedback form. Moreover, there exists a threshold such that the optimal control strategy does not use currency devaluation for values below that threshold.*

Toward the proof of Theorem 3.2.1, we introduce a system of second order implicit ODEs which approximates (3.17). More precisely, for any given $\varepsilon > 0$, let ρ_ε be a monotone increasing Lipschitz function on $[0, x^*]$ defined by

$$\rho_\varepsilon(x) \ = \ \begin{cases} \rho(x) & \text{if} \quad x \in [0, x_\varepsilon] \\ \dfrac{1}{\varepsilon} & \text{if} \quad x \in [x_\varepsilon, x^*] \end{cases} \qquad \text{with} \qquad x_\varepsilon \ := \ \rho^{-1}\left(\frac{1}{\varepsilon}\right). \tag{3.29}$$

Consider the following system of implicit ODEs

$$\begin{cases} (r + \rho_\varepsilon(x)) \cdot V \ = \ \rho_\varepsilon(x) \cdot B + H(x, V', p + \varepsilon) + \left(\dfrac{\sigma^2 x^2}{2} + \varepsilon\right) \cdot V'' , \\ (r + \lambda + \tilde{v}(x, V')) \cdot p - (r + \lambda) \ = \ \rho_\varepsilon(x) \cdot [\theta(x) - p] \\ \qquad\qquad\qquad + H_\xi(x, V', p + \varepsilon) \cdot p' + \left(\dfrac{(\sigma x)^2}{2} + \varepsilon\right) \cdot p'' , \end{cases} \tag{3.30}$$

with

$$\tilde{v}(x, \xi) \ = \ \begin{cases} 0, & \text{if} \quad \xi x \le c'(0), \\ (c')^{-1}(\xi x), & \text{if} \quad \xi x > c'(0). \end{cases} \tag{3.31}$$

From the assumption **(A3)**, it holds that

$$\tilde{v}(x, \xi) \ \le \ v_{\max}, \qquad |\tilde{v}_x(x, \xi)| \ \le \ \frac{1}{\delta_0} \cdot |\xi| \qquad \text{and} \qquad |\tilde{v}_\xi(x, \xi)| \ \le \ \frac{1}{\delta_0} \cdot |x|. \tag{3.32}$$

We establish an existence result for (3.30) with boundary condition (3.18) by considering the auxiliary parabolic system whose steady states will provide a solution to (3.30). Following [3], we shall construct a compact, convex and positively invariant set of functions $(V, p) : [0, x^*]^2 \mapsto [0, B] \times [\theta_{\min}, 1]$. A topological technique will then yield the existence of solutions $(V_\varepsilon, p_\varepsilon)$ of the system (3.30).

Lemma 3.2.3. *Assume that* **(A1) - (A3)** *hold. Then the system of ODEs (3.30) with boundary conditions (3.18) admits a C^2 solution $(V_\varepsilon, p_\varepsilon) : [0, x^*]^2 \to [0, B] \times [\theta_{\min}, 1]$ such that V_ε is increasing.*

Proof. Given a threshold of bankruptcy $x^* > 0$, we consider the parabolic system with the unknowns $\mathbf{V}(t, x)$ and $\mathbf{p}(t, x)$

$$\begin{cases} \mathbf{V}_t \;=\; -(r + \rho_\varepsilon(x))\mathbf{V} + \rho_\varepsilon(x) \cdot B + H(x, \mathbf{V}_x, \mathbf{p} + \varepsilon) + \Big(\dfrac{\sigma^2 x^2}{2} + \varepsilon\Big)\mathbf{V}_{xx}\,, \\ \mathbf{p}_t \;=\; -(r + \lambda + \widetilde{v}(x, \mathbf{V}_x)) \cdot \mathbf{p} + (r + \lambda) + \rho_\varepsilon(x) \cdot [\theta(x) - \mathbf{p}] + \\ \qquad\qquad H_\xi(x, \mathbf{V}_x, \mathbf{p} + \varepsilon) \cdot \mathbf{p}_x + \Big(\dfrac{(\sigma x)^2}{2} + \varepsilon\Big) \cdot \mathbf{p}_{xx}\,, \end{cases} \tag{3.33}$$

and with the boundary conditions

$$\begin{cases} \mathbf{V}(t, 0) \;=\; 0 \\ \mathbf{V}(t, x^*) \;=\; B \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{p}(t, 0) \;=\; 1 \\ \mathbf{p}(t, x^*) \;=\; \theta(x^*). \end{cases} \tag{3.34}$$

It is well-known (see [3, Theorem 1]) that the parabolic system (3.33) with initial data

$$\mathbf{V}(0, x) \;=\; V_0(x) \qquad \text{and} \qquad \mathbf{p}(0, x) \;=\; p_0(x), \tag{3.35}$$

admits a unique solution $(\mathbf{V}(t, x), \mathbf{p}(t, x)) \in C^2([0, T] \times [0, x^*]) \times C^2([0, T] \times [0, x^*])$ for any $T > 0$. Adopting a semigroup notation, we denote by $S_t(V_0, p_0) = (\mathbf{V}(t, \cdot), \mathbf{p}(t, \cdot))$ the solution to the system (3.33) at time t with initial data (3.35). Consider the closed and convex domain in $C^2([0, x^*]) \times C^2([0, x^*])$

$$\mathcal{D} \;=\; \Big\{(V, p) : [0, x^*]^2 \to [0, B] \times [\theta_{\min}, 1] : (V, p) \in C^2, V' \geq 0, \text{ and (3.18) holds}\Big\}.$$

We claim that $\mathcal{D}$ is positively invariant under the semigroup S_t, namely

$$S_t(\mathcal{D}) \;\subseteq\; \mathcal{D} \qquad \forall\, t \geq 0.$$

Indeed, as in [15], we consider the constant functions $(\mathbf{V}^\pm, \mathbf{p}^\pm)$ defined on $[0, \infty[\times [0, x^*]$ such that

$$(\mathbf{V}^+, \mathbf{p}^+) \;\equiv\; (B, 1) \qquad \text{and} \qquad (\mathbf{V}^-, \mathbf{p}^-) \;\equiv\; (0, \theta_{\min}).$$

Recalling (3.22), one has

$$
\begin{aligned}
-(r+\rho_\varepsilon(x))\mathbf{V}^+ + \rho_\varepsilon(x)\cdot B + H(x, \mathbf{V}_x^+, \mathbf{p}+\varepsilon) + \Big(\frac{\sigma^2 x^2}{2} + \varepsilon\Big)\mathbf{V}_{xx}^+ \\
= \ -(r+\rho_\varepsilon(x))\mathbf{V}^+ + \rho_\varepsilon(x)\cdot B \ = \ -rB \ \le \ 0
\end{aligned}
$$

and

$$
-(r+\rho_\varepsilon(x))\mathbf{V}^- + \rho_\varepsilon(x)\cdot B + H(x, \mathbf{V}_x^-, \mathbf{p}+\varepsilon) + \Big(\frac{\sigma^2 x^2}{2} + \varepsilon\Big)\mathbf{V}_{xx}^- \ = \ \rho_\varepsilon(x)\cdot B \ \ge \ 0.
$$

Therefore $\mathbf{V}^+$ is a super-solution, and $\mathbf{V}^-$ is a sub-solution of the first parabolic equation in (3.33). A standard comparison principle argument [77] yields that

$$
0 \ = \ \mathbf{V}^-(t,x) \ \le \ \mathbf{V}(t,x) \ \le \ \mathbf{V}^+(t,x) \ = \ B \qquad \forall (t,x) \in [0,+\infty[\times[0,x^*].
$$

Similarly, from (**A1**)-(**A3**), (3.27) and (3.31), it holds

$$
\begin{aligned}
-(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot \mathbf{p}^+ + (r+\lambda) + \rho_\varepsilon(x)\cdot[\theta(x) - \mathbf{p}^+] + H_\xi(x, \mathbf{V}_x, \mathbf{p}^+ + \varepsilon)\cdot \mathbf{p}_x^+ \\
+ \left(\frac{(\sigma x)^2}{2} + \varepsilon\right)\cdot \mathbf{p}_{xx}^+ \ = \ -\tilde{v}(x,\mathbf{V}_x) + \rho_\varepsilon(x)\cdot[\theta(x) - 1] \ \le \ 0
\end{aligned}
$$

and

$$
\begin{aligned}
-(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot \mathbf{p}^- + (r+\lambda) + \rho_\varepsilon(x)\cdot[\theta(x) - \mathbf{p}^-] + H_\xi(x, \mathbf{V}_x, \mathbf{p}^- + \varepsilon)\cdot \mathbf{p}_x^- \\
+ \left(\frac{(\sigma x)^2}{2} + \varepsilon\right)\cdot \mathbf{p}_{xx}^- \ = \ -(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot\theta_{\min} + (r+\lambda) \\
+ \rho_\varepsilon(x)\cdot[\theta(x) - \theta_{\min}] \ \ge \ (r+\lambda) - (r+\lambda+v_{\max})\cdot\theta_{\min} \ \ge \ 0.
\end{aligned}
$$

Thus, $\mathbf{p}^+$ is a super-solution and $\mathbf{p}^-$ is a sub-solution of the second parabolic equation in (3.33), and this yields

$$
\theta_{\min} \ = \ \mathbf{p}^-(t,x) \ \le \ \mathbf{p}(t,x) \ \le \ \mathbf{p}^+(t,x) \ = \ 1 \qquad \forall (t,x) \in [0,+\infty[\times[0,x^*].
$$

Setting $\mathbf{W}(t,x) := \mathbf{V}_x(t,x)$ with initial condition $\mathbf{V}(0,x) = V_0(x) \in \mathcal{D}$, we have

$$
\lim_{x\to 0^+} \mathbf{W}(t,x) \ge 0, \quad \text{and} \quad \lim_{x\to x^{*-}} \mathbf{W}(t,x) \ge 0, \quad \forall t \in]0,+\infty[\,,
$$

and, by definition of $\mathcal{D}$,

$$\mathbf{W}(0,x) \ \geq \ 0, \qquad \forall x \in [0,x^*].$$

Differentiating the first equation in (3.30), we obtain that $\mathbf{W}$ solves the following ODE

$$\begin{aligned} \mathbf{W}_t \ = \ & -(r+\rho_\varepsilon)\mathbf{W} + \rho_\varepsilon'(B-\mathbf{V}) \\ & + H_x + H_\xi \mathbf{W}_x + H_p \mathbf{p}_x + \sigma^2 x \mathbf{W}_x + \left(\frac{\sigma^2 x^2}{2} + \varepsilon\right)\mathbf{W}_{xx}. \end{aligned} \tag{3.36}$$

Since

$$(H_x, H_p)(x, 0, \mathbf{p}+\varepsilon) \ = \ 0 \qquad \text{and} \qquad \mathbf{V}(t,x) \ \leq \ B \qquad \forall (t,x) \in [0,+\infty[\times[0,x^*],$$

one can see that the constant function 0 is a sub-solution to (3.36). Thus, by [77],

$$\mathbf{W}(t,x) \ \geq \ 0 \qquad \forall (t,x) \in [0,+\infty[\times[0,x^*]\,,$$

yielding the monotone increasing property of $\mathbf{V}$ with respect to x. From the bounds in Lemma 3.1.2 and the invariance of $\mathcal{D}$, we can apply [3, Theorem 3] to obtain the existence of a steady state solution $(V_\varepsilon, p_\varepsilon) \in \mathcal{D}$ for the system (3.33) which solves (3.30) and (3.18) such that V_ε is monotone increasing. □

In order to obtain a solution to (3.17) from (3.30) by considering the limit $\varepsilon \to 0+$, it is necessary to derive a priori estimates on the derivatives of of the smooth functions $(V_\varepsilon, p_\varepsilon)$.

Lemma 3.2.4. *Under the same assumptions in Lemma 3.2.3, let $(V_\varepsilon, p_\varepsilon)$ be a solution to* (3.30) *and* (3.18). *Then for every $0 < \varepsilon < 1/2$, it holds*

$$\|V_\varepsilon'\|_{\mathbf{L}^\infty([0,x^*])} \ \leq \ M^* \qquad \text{and} \qquad v_\varepsilon(x) \ = \ 0 \qquad \forall x \in \left[0, \frac{c'(0)}{M^*}\right], \tag{3.37}$$

where the constant M^* is explicitly computed by

$$M^* \ := \ \max\left\{M_1, \exp\left(\frac{3K_1 x^*}{2\sigma^2 x_1^2}\right)\cdot\left(\frac{4B}{x^*} + \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right), \exp\left(\frac{2K_1 x^*}{\sigma^2 x_1^2}\right)\cdot\left(\frac{4B}{x^*} + \frac{rB}{K_1}\right)\right\},$$

with K_1 defined in (3.26), and

$$x_1 \;:=\; \min\left\{\frac{1}{6(\lambda+r+\sigma^2)}, \frac{x^*}{2}\right\}, \qquad M_1 \;:=\; 8\left(L\left(\frac{1}{2}\right)+\rho(x_1)\cdot B\right). \tag{3.38}$$

Moreover, for any $\delta \in]0, x^*/2[$, there exists a constant $M_\delta > 0$ such that

$$\|V_\varepsilon''\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)} + \|p_\varepsilon'\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)} + \|p_\varepsilon''\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)} \;\le\; M_\delta. \tag{3.39}$$

Proof. Let $(V_\varepsilon, p_\varepsilon)$ be a solution to (3.30), (3.18) with the properties given in the conclusion of Lemma 3.2.3.

1. Let us first prove (3.37). Let x_1 and M_1 be as in (3.38). From (3.7) and Lemma 3.2.3, for every $(x,\xi) \in [0, x_1] \times [M_1, +\infty[$, it holds

$$\begin{aligned} H(x,\xi,p_\varepsilon+\varepsilon) \;&\le\; \min_{u\in[0,1]}\left\{L(u) - \frac{u}{p_\varepsilon+\varepsilon}\xi\right\} + \left(\frac{\lambda+r}{p_\varepsilon+\varepsilon}+\sigma^2\right)x\xi \\ &\le L\left(\frac{1}{2}\right) + \frac{3(\lambda+r+\sigma^2)x-1}{2(p_\varepsilon+\varepsilon)}\cdot\xi \\ &\le L\left(\frac{1}{2}\right) - \frac{\xi}{8} \;\le\; L\left(\frac{1}{2}\right) - \frac{M_1}{8}. \end{aligned} \tag{3.40}$$

From the definition of ρ_ε in (3.29), it holds that

$$\rho_\varepsilon(x) \;\le\; \rho(x) \qquad \forall x \in [0, x^*], \tag{3.41}$$

and therefore **(A2)** and (3.40) imply that

$$\rho_\varepsilon(x)B + H(x,\xi,p_\varepsilon+\varepsilon) \;\le\; \rho(x_1)B + L\left(\frac{1}{2}\right) - \frac{M_1}{8} \;<\; 0,$$

for all $(x,\xi) \in [0, x_1] \times [M_1, +\infty[$. Since V_ε is non-negative, the first equation of (3.30) yields that

$$V_\varepsilon''(x) \;>\; 0 \qquad \forall x \in]0, x_1],$$

provided $V_\varepsilon'(x) \ge M_1$ for all $x \in [0, x_1]$. Recalling from Lemma 3.2.3 that V_ε' is non-negative, we have that $\|V_\varepsilon'\|_{\mathbf{L}^\infty([0,x_1])}$ is bounded by M_1 or the maximal of V_ε' in $[0, x^*]$ is obtained at (see Figure 3.1)

$$x_m \;:=\; \arg\max_{x\in[0,x^*]} V_\varepsilon'(x) \;>\; x_1\,.$$

Let us establish an upper bound of V'_ε in $[x_1, x^*[$. Since $0 \le x_1 \le \frac{x^*}{2}$, by the mean value theorem, there exists a point $x_2 \in]x_1, \frac{3}{4}x^*[$ such that

$$V'_\varepsilon(x_2) \;=\; \frac{V_\varepsilon(\frac{3}{4}x^*) - V_\varepsilon(x_1)}{\frac{3}{4}x^* - x_1} \;\le\; \frac{B}{\frac{3}{4}x^* - \frac{1}{2}x^*} \;\le\; \frac{4B}{x^*}\,. \tag{3.42}$$

From the first equation of (3.30) we have

$$V''_\varepsilon(x) \;=\; \frac{2}{\sigma^2 x^2 + 2\varepsilon} \cdot \left[rV_\varepsilon(x) + \rho_\varepsilon(x)\left(V_\varepsilon(x) - B\right) - H(x, V'_\varepsilon(x), p_\varepsilon(x) + \varepsilon) \right]. \tag{3.43}$$

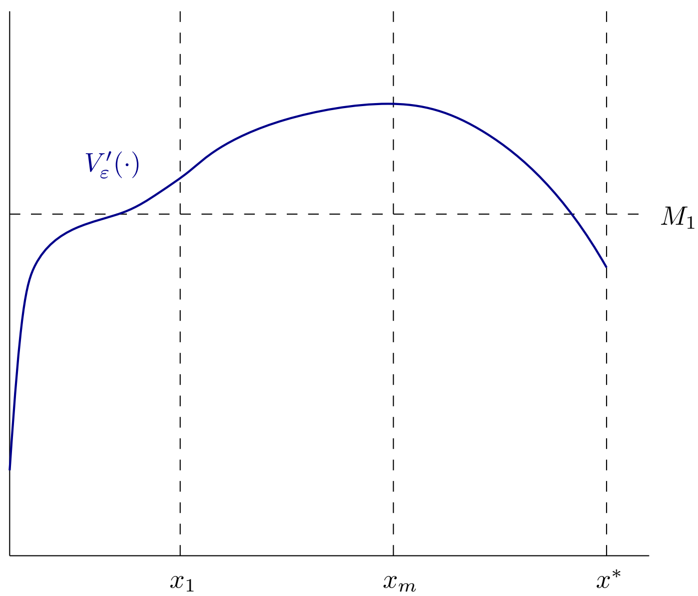

Figure 3.1 The function V'_ε is either bounded near $x = 0$ or does not acheive its maximum near $x = 0$. In this figure, if $V'_\varepsilon(x) > M_1$ for some value of $x \in [0, x_1[$, then the HJB equation for V_ε yields that V'_ε is necessarily increasing until $x = x_1$.

Two cases are considered:

- For any $x \in [x_1, x_2]$, from (3.41), **(A2)**, and the above equality we have

$$\begin{aligned} V''_\varepsilon(x) \;&\ge\; \frac{-2}{\sigma^2 x^2 + 2\varepsilon} \cdot \left(\rho_\varepsilon(x) B + \left|H(x, V'_\varepsilon(x), p_\varepsilon(x) + \varepsilon)\right|\right) \\ &\ge\; \frac{-2}{\sigma^2 x_1^2} \cdot \left(\rho\left(\frac{3x^*}{4}\right) B + \left|H(x, V'_\varepsilon(x), p_\varepsilon(x) + \varepsilon)\right|\right), \end{aligned}$$

where the last inequality holds because $x_2 \leq \frac{3x^*}{4}$. Since $p_\varepsilon(x) \geq \theta_{\min}$, by Lemma 3.1.3 it holds that

$$V''_\varepsilon(x) \;\geq\; \frac{-2}{\sigma^2 x_1^2}\left(\rho\left(\frac{3x^*}{4}\right)B + K_1 \cdot V'_\varepsilon(x)\right),$$

where the constant K_1 is defined in (3.26). Thus, applying Grönwall's inequality in the interval $[x, x_2]$ with $x \in [x_1, x_2]$, we get

$$V'_\varepsilon(x) \;\leq\; \left(V'_\varepsilon(x_2) + \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right)\cdot\exp\left(\frac{2K_1}{\sigma^2 x_1^2}(x_2 - x)\right) - \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right).$$

Recalling (3.42) we obtain that

$$\|V'_\varepsilon\|_{\mathbf{L}^\infty([x_1,x_2])} \;\leq\; \left(\frac{4B}{x^*} + \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right)\cdot\exp\left(\frac{3K_1 x^*}{2\sigma^2 x_1^2}\right). \tag{3.44}$$

- Similarly, for any $x \in [x_2, x^*]$, it holds

$$V''_\varepsilon(x) \;\leq\; \frac{2}{\sigma^2 x_2^2}\left(rB + K_1 \cdot V'_\varepsilon(x)\right) \;\leq\; \frac{2}{\sigma^2 x_1^2}\left(rB + K_1 \cdot V'_\varepsilon(x)\right)$$

  and Grönwall's inequality implies that

$$V'_\varepsilon(x) \;\leq\; \left(V'_\varepsilon(x_2) + \frac{rB}{K_1}\right)\cdot\exp\left(\frac{2K_1}{\sigma^2 x_1^2}(x - x_2)\right) - \frac{rB}{K_1}.$$

  Thus, (3.42) yields

$$\|V'_\varepsilon\|_{\mathbf{L}^\infty([x_2,x^*])} \;\leq\; \left(\frac{4B}{x^*} + \frac{rB}{K_1}\right)\cdot\exp\left(\frac{2K_1 x^*}{\sigma^2 x_1^2}\right). \tag{3.45}$$

Therefore, combining (3.38) with (3.44) and (3.45), we obtain that

$$\|V'_\varepsilon\|_{\mathbf{L}^\infty([0,x^*])} \;\leq\; M^*.$$

Moreover, recalling (3.31), it holds that

$$x \;\leq\; \frac{c'(0)}{\|V'_\varepsilon\|_{\mathbf{L}^\infty([0,x^*])}} \quad\Longrightarrow\quad v_\varepsilon(x) \;=\; 0,$$

and (3.37) follows.

2. For any fixed $0 < \delta < \frac{x^*}{2}$, we will provide uniform bounds on $\|V_\varepsilon''\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$, $\|p_\varepsilon'\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$, and $\|p_\varepsilon''\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$. From (3.43), (3.41) and Lemma 3.1.3, it holds

$$\begin{aligned} |V_\varepsilon''(x)| &\leq \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B+\left|H(x,V_\varepsilon'(x),p_\varepsilon(x)+\varepsilon)\right|\right] \\ &\leq \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B+K_1\cdot V_\varepsilon'(x)\right], \end{aligned}$$

for all $x\in[\delta,x^*-\delta]$. Thus, (3.37) implies that

$$\|V_\varepsilon''\|_{\mathbf{L}^\infty([\delta,x^*-\delta])} \leq \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B+K_1M^*\right]. \tag{3.46}$$

Similarly, from the second equation in (3.30), it holds that

$$|p_\varepsilon''(x)| \leq \frac{2}{\sigma^2\delta^2}\cdot\left[K_1\cdot|p_\varepsilon'(x)|+(r+\lambda+v_{\max}+\rho(x^*-\delta))\right] \quad \forall x\in[\delta,x^*-\delta]. \tag{3.47}$$

By the mean value theorem, there exists a point $x_3\in]\delta,x^*-\delta[$ such that

$$|p_\varepsilon'(x_3)| = \left|\frac{p_\varepsilon(x^*-\delta)-p_\varepsilon(\delta)}{x^*-2\delta}\right| \leq \frac{1-\theta_{\min}}{x^*-2\delta}.$$

Applying Grönwall's inequality to (3.47) in the intervals $[\delta,x_3]$ and $[x_3,x^*-\delta]$, yields

$$|p_\varepsilon'(x)| \leq K_\delta \qquad \forall x\in[\delta,x^*-\delta],$$

for some constant K_δ depending only on δ. Thus, from (3.47) we get

$$|p_\varepsilon''(x)| \leq \frac{2}{\sigma^2\delta^2}\cdot\left[K_1K_\delta+(r+\lambda+v_{\max}+\rho(x^*-\delta))\right] \quad \forall x\in[\delta,x^*-\delta].$$

Combining the two above estimates and (3.46), we obtain (3.39) with the constant

$$M_\delta := \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B+K_1M^*+K_1K_\delta+(r+\lambda+v_{\max}+\rho(x^*-\delta))\right]+K_\delta.$$

This completes the proof. □

We are ready to prove the main result of this section.

Proof of Theorem 3.2.1. The proof is divided into three steps:

Step 1. For any $0 < \varepsilon < 1/2$ sufficiently small, let $(V_\varepsilon, p_\varepsilon)$ be a solution to (3.30) and (3.18) which is constructed in Lemma 3.2.3. Recalling Lemma 3.1.2 and (3.37), we obtain that H and H_ξ are Lipschitz continuous on $[\delta, x^* - \delta] \times [0, M^*] \times [\theta_{\min}, 1]$ for any $\delta \in]0, x^*/2[$. Using the a priori estimates (3.37) and (3.39) in Lemma 3.2.4, and assumptions **(A1)-(A2)**, the system (3.30) implies that the functions V_ε'' and p_ε'' are also uniformly Lipschitz on $[\delta, x^* - \delta]$. Thus, one can apply the Ascoli-Arzelà Theorem to extract a subsequence $(V_{\varepsilon_n}, p_{\varepsilon_n})_{n \geq 0}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that $(V_{\varepsilon_n}, p_{\varepsilon_n})$ converges uniformly to (V, p) in $C^2(]\delta, x^* - \delta[)$ for all $\delta > 0$, where V, p are twice continuously differentiable and solve the system of ODEs (3.30) on the open interval $]0, x^*[$. Moreover, since V_ε' is uniformly bounded by M^* on $[0, x^*]$,

$$\lim_{n\to\infty} \|V_{\varepsilon_n} - V\|_{\mathbf{L}^\infty([0,x^*])} \; = \; 0\,,$$

which implies that

$$V(0) \; = \; \lim_{n\to\infty} V_{\varepsilon_n}(0) \; = \; 0, \qquad V(x^*) \; = \; \lim_{n\to\infty} V_{\varepsilon_n}(x^*) \; = \; B.$$

Step 2. It remains to check the boundary condition (3.18) for p. Let us first show that

$$\lim_{x\to 0^+} p(x) \; = \; 1\,. \tag{3.48}$$

Given $\varepsilon \in]0, \frac{1}{2}[$, we construct a lower bound p^- of p_ε independent of ε in a neighborhood of 0. From the assumption (**A1**), there exists $M > 0$ such that

$$\theta(x) \; \geq \; 1 - Mx \quad \forall x \in [0, x^*/2]. \tag{3.49}$$

Set $\bar{x}_0 := \min\left\{\frac{c'(0)}{M^*}, \frac{x^*}{2}, \frac{1-\theta_{\min}}{M}\right\}$ where M^* is the constant in (3.37). The sub-solution candidate is

$$p^-(x) \; = \; 1 - kx^\gamma,$$

with

$$\gamma \; := \; \min\left\{1, (r+\lambda)\left(\frac{\lambda + r}{\theta_{\min}} + \sigma^2\right)^{-1}\right\} \qquad \text{and} \qquad k \; := \; (1-\theta_{\min}) \cdot \bar{x}_0^{-\gamma}.$$

Note that $p^-(0) = 1 \leq p_\varepsilon(0)$. By choice of k and $\bar{x}_0$, we have

$$p^-(\bar{x}_0) \; = \; 1 - k\bar{x}_0^\gamma \; = \; 1 - (1-\theta_{\min}) \cdot \bar{x}_0^{-\gamma} \cdot \bar{x}_0^\gamma \; = \; \theta_{\min}$$

and from (3.49), it holds for all $x \in [0, \bar{x}_0]$ that

$$p^-(x) \ = \ 1 - (1-\theta_{\min}) \cdot \bar{x}_0^{-\gamma} \cdot x^\gamma \ \leq \ 1 - M\bar{x}_0^{1-\gamma} \cdot x^\gamma \ \leq \ 1 - Mx \ \leq \ \theta(x). \quad (3.50)$$

We claim that p^- is a sub-solution of the second equation in (3.30) in the interval $[0, \bar{x}_0]$. Indeed, recalling (3.37) that $v_\varepsilon(x) \equiv 0$ on the region $x \in [0, \bar{x}_0]$, from Lemma 3.1.2 (i) and (3.50), one estimates

$$\begin{aligned}
(r+\lambda) &- (r+\lambda+v_\varepsilon(x))p^- + \rho_\varepsilon(x)[\theta(x) - p^-(x)] + H_\xi(x, V, p^-) \cdot (p^-)' + \left(\varepsilon + \frac{\sigma^2 x^2}{2}\right)(p^-)'' \\
&\geq \ (r+\lambda)kx^\gamma - \gamma k x^{\gamma-1} H_\xi(x, V, p^-) + \left(\varepsilon + \frac{\sigma^2 x^2}{2}\right)\gamma(1-\gamma)kx^{\gamma-2} \\
&\geq \ (r+\lambda)kx^\gamma - \gamma k x^{\gamma-1} H_\xi(x, V, p^-) \ \geq \ \left[(r+\lambda) - \left(\frac{\lambda+r}{p^-(x)} + \sigma^2\right)\gamma\right] kx^\gamma \\
&\geq \ \left[(r+\lambda) - \left(\frac{\lambda+r}{p^-(\bar{x}_0)} + \sigma^2\right)\gamma\right] kx^\gamma \ = \ \left[(r+\lambda) - \left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right)\gamma\right] kx^\gamma \ \geq \ 0
\end{aligned}$$

for all $x \in [0, \bar{x}_0]$. In turn, we have

$$1 - kx^\gamma \ \leq \ p_\varepsilon(x) \ \leq \ 1 \quad \forall x \in [0, \bar{x}_0], \quad (3.51)$$

and this yields (3.48).

Step 3. To complete the proof, it remains to show the boundary condition

$$\lim_{x \to x^{*-}} p(x) \ = \ \theta(x^*). \quad (3.52)$$

Given $\varepsilon \in]0, 1/2[$ sufficiently small, we will provide an upper bound on the $\mathbf{L}^\infty$ distance of p_ε and θ over $[x^* - \delta, x^*]$ denoted by

$$I_\delta \ := \ \max_{x \in [x^*-\delta, x^*]} \left| p_\varepsilon(x) - \theta(x) \right| \qquad \forall \delta \in [0, x^*/4].$$

For a fixed $0 < \delta < \dfrac{x^*}{4}$, the continuity of p_ε and θ implies that

$$I_\delta \ = \ \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [\theta(x_m) - p_\varepsilon(x_m)], \quad (3.53)$$

for some $x_m \in [x^* - \delta, x^*]$. Assume that $p_\varepsilon(x_m) \neq \theta(x_m)$. Since $p_\varepsilon(0) = \theta(0) = 1$ and

$p_\varepsilon(x^*) = \theta(x^*)$, we can define two points

$$0 \ \le\ x_1^* \ :=\ \max\{x \in [0, x_m[\ :\ p_\varepsilon(x) = \theta(x)\} \ <\ x_m$$

and

$$\frac{3x^*}{4} \ \le\ x_m \ <\ x_2^* \ :=\ \min\{x \in [x_m, x^*]\ :\ p_\varepsilon(x) = \theta(x)\} \ \le\ x^*.$$

Notice that $p_\varepsilon - \theta$ does not change sign in the interval $]x_1^*, x_2^*[$. For simplicity, let us introduce the following function

$$q_\varepsilon(x) \ :=\ \mathrm{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot p_\varepsilon(x).$$

It is clear that

$$|q_\varepsilon'(x)| \ =\ |p_\varepsilon'(x)| \qquad \forall x \in]x_1^*, x_2^*[.$$

Thus, by the second equation in (3.30) we estimate

$$\begin{aligned}
q_\varepsilon''(x) \ &= \mathrm{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot p_\varepsilon''(x) \\
&\le \left(\frac{2}{2\varepsilon + \sigma^2 x^2}\right) \cdot \big[r + \lambda + v_{\max} + \big|H_\xi \cdot p_\varepsilon'(x)\big| - \rho_\varepsilon(x) \cdot \big|p_\varepsilon(x) - \theta(x)\big|\big] \\
&\le \left(\frac{2}{2\varepsilon + \sigma^2 x^2}\right) \cdot \big[r + \lambda + v_{\max} + \big|H_\xi \cdot q_\varepsilon'(x)\big|\big] \quad \forall x \in]x_1^*, x_2^*[\,.
\end{aligned}$$

Set $\bar{x}_1^* := \max\left\{x_1^*, \dfrac{x^*}{2}\right\}$, we then have

$$q_\varepsilon''(x) \ \le\ \frac{8}{(\sigma x^*)^2} \cdot \big(r + \lambda + v_{\max} + K_1 \cdot |q_\varepsilon'(x)|\big) \quad \forall x \in]\bar{x}_1^*, x_2^*[, \tag{3.54}$$

where K_1 is defined in (3.26). Two cases may occur:

Case 1: See Figure 3.2. If $x_m - \bar{x}_1^* > \delta$ then

$$\begin{aligned}
q_\varepsilon(x_m) - q_\varepsilon(x_m - \delta) \ &=\ \mathrm{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [p_\varepsilon(x_m) - p_\varepsilon(x_m - \delta)] \\
&\le\ \mathrm{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [\theta(x_m) - \theta(x_m - \delta)] \ \le\ \Delta_\delta\theta \cdot \delta,
\end{aligned}$$

where $\Delta_\delta\theta = \sup_{x \in [0, x^* - \delta]} \left|\frac{\theta(x+\delta) - \theta(x)}{\delta}\right|$. Thus, by mean value theorem there exists

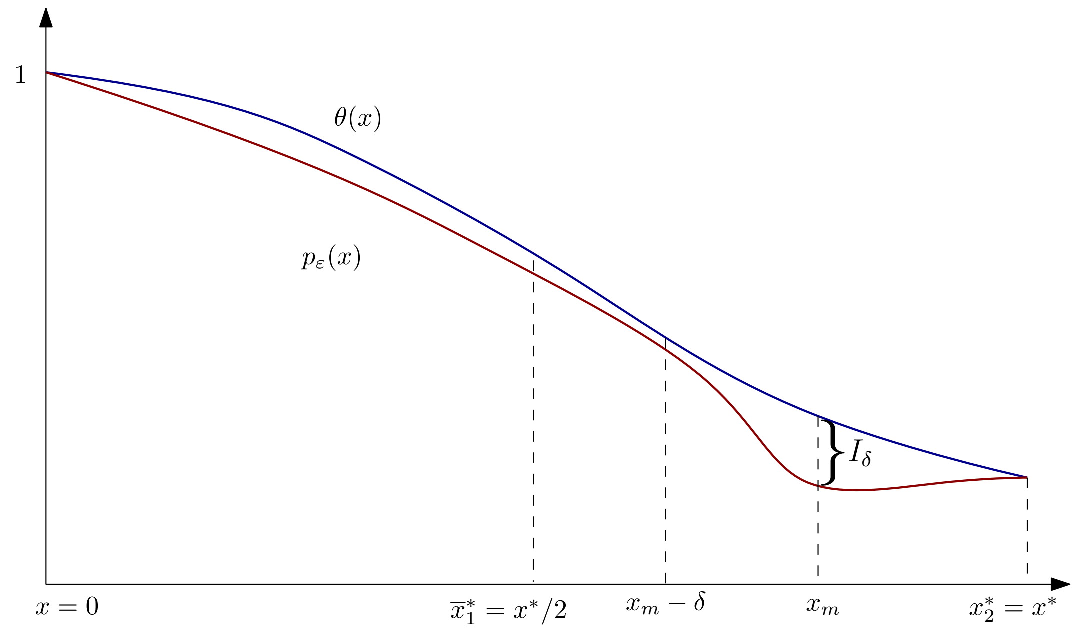

Figure 3.2 In Case 1, p and θ, potentially don't cross for $x \in]0, x_m]$. In this case, $x \mapsto (p_\varepsilon - \theta)(x)$ does not change sign in the interval $[x_m - \delta, x_m]$, allowing for a mean value theorem argument to hold.

$\bar{x} \in \left]x_m - \delta, x_m\right[$ such that

$$q_\varepsilon'(\bar{x}) \;=\; \frac{q_\varepsilon(x_m) - q_\varepsilon(x_m - \delta)}{\delta} \;\leq\; \Delta_\delta \theta.$$

Let g be the solution to the ODE

$$g'(x) \;=\; \frac{8}{(\sigma x^*)^2} \cdot (r + \lambda + v_{\max} + K_1 \cdot g(x)), \qquad g(\bar{x}) \;=\; \Delta_\delta \theta \;\geq\; q_\varepsilon'(\bar{x}).$$

Solving the above equation, one gets

$$g(x) \;=\; \left(\frac{r + \lambda + v_{\max}}{K_1} + \Delta_\delta \theta\right) \cdot \exp\left(\frac{8K_1}{(\sigma x^*)^2} \cdot (x - \bar{x})\right) - \frac{r + \lambda + v_{\max}}{K_1} \;\geq\; 0,$$

for all $x \geq \bar{x}$. In particular, it holds that

$$g'(x) \;=\; \frac{8}{(\sigma x^*)^2} \cdot (r + \lambda + v_{\max} + K_1 \cdot |g(x)|) \qquad \forall x \in \left]\bar{x}, x_2^*\right[.$$

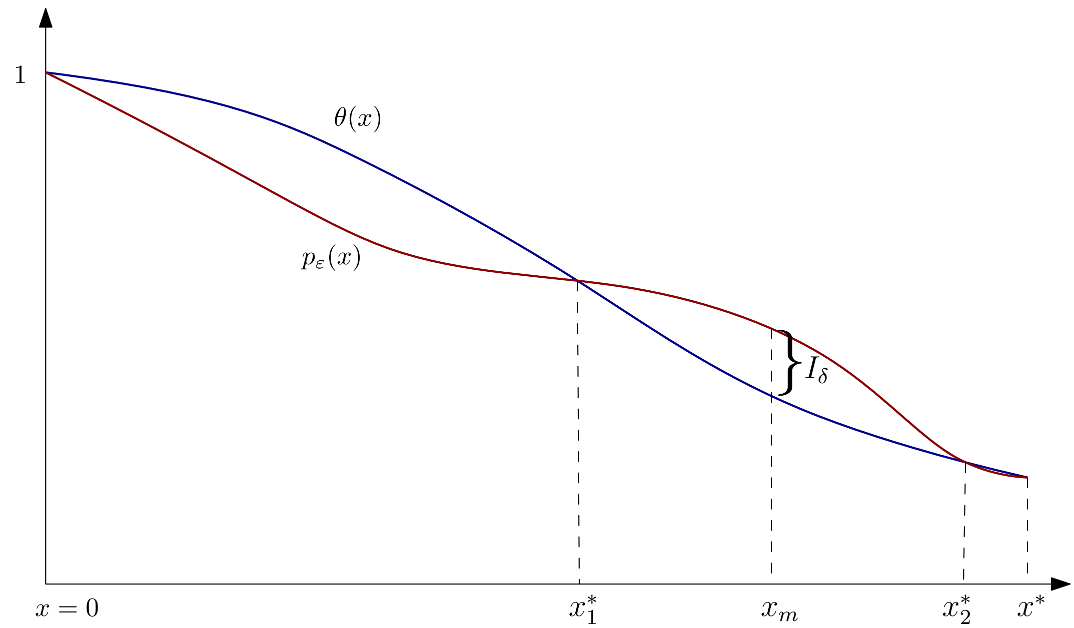

Figure 3.3 In Case 2, p and θ have a non-trivia crossing point, x_1^*. In the sub-case (i), it holds that $I_\delta \leq \theta(x_1^*) - \theta(x_m)$ and the boundary condition holds by the continuity of θ.

A standard comparison argument yields

$$q'_\varepsilon(x) \;\leq\; g(x) \;\leq\; \left(\frac{r+\lambda+v_{\max}}{K_1} + \Delta_\delta\theta\right)\cdot\exp\left(\frac{4K_1}{\sigma^2 x^*}\right) \quad \forall x \in]\bar{x}, x_2^*[.$$

Thus, from (3.53) it holds

$$\begin{aligned} I_\delta &= \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big)\cdot(\theta(x_m) - \theta(x_2^*)) + q_\varepsilon(x_2^*) - q_\varepsilon(x_m) \\ &\leq \sup_{x,y\in[x^*-\delta,x^*]} |\theta(x)-\theta(y)| + \left(\frac{r+\lambda+v_{\max}}{K_1} + \Delta_\delta\theta\right)\cdot\exp\left(\frac{4K_1}{\sigma^2 x^*}\right)\cdot\delta. \\ &\leq \frac{r+\lambda+v_{\max}}{K_1}\cdot\exp\left(\frac{4K_1}{\sigma^2 x^*}\right)\cdot\delta \\ &\qquad + \left[\exp\left(\frac{4K_1}{\sigma^2 x^*}\right)+1\right]\cdot \sup_{x,y\in[x^*-\delta,x^*]}|\theta(x)-\theta(y)|. \end{aligned} \tag{3.55}$$

Case 2: Let us assume that $0 < x_m - \bar{x}_1^* \leq \delta$. Since $\delta < \dfrac{x^*}{4}$ and $x_m \geq \dfrac{3x^*}{4}$, we have that $\bar{x}_1^* > \dfrac{x^*}{2}$ and this yields $\bar{x}_1^* = x_1^*$. Two subcases are considered:

(i) See Figure 3.3. If $q_\varepsilon(x_m) - q_\varepsilon(x_1^*) \geq 0$ then (3.53) implies that

$$\begin{aligned} I_\delta &= \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [\theta(x_m) - \theta(x_1^*)] + q_\varepsilon(x_1^*) - q_\varepsilon(x_m) \\ &\leq |\theta(x_m) - \theta(x_1^*)| \ \leq \sup_{x,y \in [x^*-\delta, x^*]} |\theta(x) - \theta(y)|. \end{aligned} \tag{3.56}$$

(ii) Otherwise, if $q_\varepsilon(x_m) - q_\varepsilon(x_1^*) < 0$ then by mean value theorem there exists $\tilde{x} \in \left]x_1^*, x_m\right[$ such that

$$q_\varepsilon'(\tilde{x}) \;=\; \frac{q_\varepsilon(x_m) - q_\varepsilon(x_1^*)}{x_m - x_1^*} \;<\; 0.$$

With the same argument in case 1, it holds that

$$q_\varepsilon'(x) \;\leq\; \tilde{g}(x) \;\leq\; \frac{r + \lambda + v_{\max}}{K_1} \cdot \exp\left(\frac{4K_1}{\sigma^2 x^*}\right) \qquad \forall x \in]\tilde{x}, x_2^*[\,,$$

where $\tilde{g}$ is the solution to

$$\tilde{g}'(x) \;=\; \frac{8}{(\sigma x^*)^2} \cdot (r + \lambda + v_{\max} + K_1 \cdot \tilde{g}(x)), \qquad \tilde{g}(\tilde{x}) \;=\; 0 \;\geq\; q_\varepsilon'(\tilde{x}).$$

Thus, as in (3.55), it holds

$$I_\delta \;\leq\; \frac{r + \lambda + v_{\max}}{K_1} \cdot \exp\left(\frac{4K_1}{\sigma^2 x^*}\right) \cdot \delta + \sup_{x,y \in [x^*-\delta, x^*]} |\theta(x) - \theta(y)|. \tag{3.57}$$

From (3.55)-(3.57), we obtain that

$$\|p_\varepsilon - \theta\|_{\mathbf{L}^\infty([x^*-\delta, x^*])} \;\leq\; C_1 \cdot \delta + C_2 \cdot \sup_{x,y \in [x^*-\delta, x^*]} |\theta(x) - \theta(y)|,$$

for all $\varepsilon \in \left]0, \frac{1}{2}\right[$, $\delta \in \left]0, \frac{x^*}{4}\right[$ with the constants

$$C_1 \;=\; \frac{r + \lambda + v_{\max}}{K_1} \cdot \exp\left(\frac{4K_1}{\sigma^2 x^*}\right), \qquad C_2 \;=\; \exp\left(\frac{4K_1}{\sigma^2 x^*}\right) + 1.$$

In particular,

$$\|p - \theta\|_{\mathbf{L}^\infty(]x^*-\delta, x^*[)} \;\leq\; C_1 \cdot \delta + C_2 \cdot \sup_{x,y \in [x^*-\delta, x^*]} |\theta(x) - \theta(y)|$$

and the uniform continuity of θ yields (3.52) and completes the proof. $\square$

3.3 Behavior of optimal feedback controls

This section is devoted to the behavior of optimal feedback controls near the bankruptcy threshold x^*. Roughly speaking, let (u^*, v^*, p) be an optimal solution to the problem of optimal debt management (3.1)–(3.6), and V be the corresponding value function. In Theorem 3.2.1 we showed that for sufficiently small initial debt, the optimal strategy does not involve currency devaluation (3.28). In addition, we will show that when the debt-to-income ratio x is close to x^*

- *if the risk of bankruptcy ρ slowly approaches to infinity then the optimal strategy of borrower involves continuously devaluating its currency and making payment, i.e. $u^*(x) > 0$ and $v^*(x) > 0$,*
- *conversely, if the risk of bankruptcy ρ quickly approaches to infinity then any action to reduce the debt is not optimal, i.e. $u^*(x) = v^*(x) = 0$.*

To this aim, we will first establish lower and upper bounds for the value function V and then apply the bounds to analyze the feedback strategies (3.12) and (3.13). Recalling that

$$p \ \in \ [\theta_{\min}, 1] \qquad \text{with} \qquad \theta_{\min} \ = \ \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\},$$

we introduce a non-decreasing function $\beta : [0, x^*[\to [0, +\infty[$ defined by

$$\beta(t) \ = \ \max_{s\in[0,t]} \left[\rho(s)\ln\left(\frac{t}{s}\right)\right] + \frac{\lambda+r}{\theta_{\min}} + \frac{\sigma^2}{2} \ < \ +\infty \qquad \forall t \in [0, x^*[. \tag{3.58}$$

Proposition 3.3.1. *Under the same assumptions in Theorem 3.2.1, let (V, p) be a set of solutions to* (3.17). *Then it holds that*

$$V(x) \ \le \ V_1(x) \ := \ B \cdot \inf_{t\in[x,x^*[} \left[\frac{\beta(t)}{r\ln\left(\frac{t}{x}\right) + \beta(t)}\right] \qquad \forall x \in]0, x^*[. \tag{3.59}$$

In addition if we assume that

$$x^* \ \ge \ \frac{2}{r+\lambda} \qquad \text{and} \qquad \lim_{x\to x^*-} \rho(x)(x^*-x)^2 \ = \ +\infty, \tag{3.60}$$

then there exists $x^\diamond \in [x^*/2, x^*[$ *such that*

$$V(x) \;\geq\; V_2(x) \;:=\; B \cdot \left(1 - \frac{\ln\left(\dfrac{x^*}{x}\right)\left(1 - \dfrac{x}{x^*}\right)}{\ln\left(\dfrac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \right) \qquad \forall x \in [x^\diamond, x^*]. \tag{3.61}$$

Proof. **1.** For any given $x_1 \in]0, x^*[$ and $x_2 \in]0, x_1[$, we seek for an upper bound for V as a super-solution to the first equation of (3.17) of the form

$$V_1(x) \;=\; \begin{cases} B & \text{if} \quad x \in [x_1, x^*], \\ B\left(1 - \alpha \ln\left(\dfrac{x_1}{x}\right)\right) & \text{if} \quad x \in [x_2, x_1], \\ B\left(1 - \alpha \ln\left(\dfrac{x_1}{x_2}\right)\right) & \text{if} \quad x \in [0, x_2], \end{cases}$$

with $\alpha > 0$ satisfying the following relation

$$-r + \alpha \cdot \left(r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1) \right) \;=\; 0. \tag{3.62}$$

It is clear that

$$V_1(0) \;=\; B\left(1 - \alpha \ln\left(\frac{x_1}{x_2}\right)\right) \;\geq\; 0 \;=\; V(0) \qquad \text{and} \qquad V_1(x^*) \;=\; B \;=\; V(x^*).$$

For every $x \in]0, x_2[$, it holds

$$\begin{aligned} &-(r + \rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ &= \; -(r + \rho(x))V_1(x) + \rho(x)B \;\leq\; B \cdot \left[-r + \alpha(r + \rho(x_2)) \ln\left(\frac{x_1}{x_2}\right) \right] \\ &\leq\; B \cdot \left(-r + \alpha \cdot \left(\beta(x_1) + r \ln\left(\frac{x_1}{x_2}\right) \right) \right) \;=\; 0. \end{aligned}$$

Similarly, for every $x \in]x_1, x^*[$, one has

$$\begin{aligned} -(r + \rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ = \; -(r + \rho(x))V_1(x) + \rho(x)B \;=\; -rB \;<\; 0. \end{aligned}$$

On the other hand, for every $x \in]x_2, x_1[$, we compute

$$V_1'(x) \;=\; \frac{B\alpha}{x} \;>\; 0 \qquad \text{and} \qquad V_1''(x) \;=\; -\frac{B\alpha}{x^2},$$

and use Lemma 3.1.2 to obtain

$$\begin{aligned}
-(r+\rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x)& \\
\leq\; -(r+\rho(x))V_1(x) + \rho(x)B + \left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right) x V_1'(x) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x)& \\
=\; B \cdot \left[-r + \alpha \cdot \left((r+\rho(x)) \ln\left(\frac{x_1}{x}\right) + \frac{\lambda+r}{\theta_{\min}} + \frac{\sigma^2}{2}\right)\right]&
\end{aligned}$$

Applying the definitions of $\beta(\cdot)$ (3.58) and α (3.62), we have that

$$\begin{aligned}
-(r+\rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x)& \\
\leq\; B \cdot \left[-r + \alpha \cdot \left(r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1)\right)\right] \;=\; 0.&
\end{aligned}$$

Hence, V_1 is a super-solution of the first equation of (3.17) and

$$V(x) \;\leq\; V_1(x), \qquad \forall x \in [0, x^*].$$

In particular, we have

$$V(x_2) \;\leq\; V_1(x_2) \;=\; B \cdot \left(1 - \frac{r}{r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1)} \cdot \ln\left(\frac{x_1}{x_2}\right)\right),$$

and this implies that

$$V(x_2) \;\leq\; B \cdot \frac{\beta(x_1)}{r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1)}.$$

Since the above inequality hold for every $x_2 \in]0, x_1[$, one obtains (3.59).

2. We seek for a lower bound for V as a sub-solution to the first equation of (3.17) in the interval $[x_1, x^*]$ of the form

$$V_2(x) \;=\; B\left(1 - \alpha_1 \ln\left(\frac{x^*}{x}\right)(x^* - x)\right), \qquad \forall x \in [x_1, x^*],$$

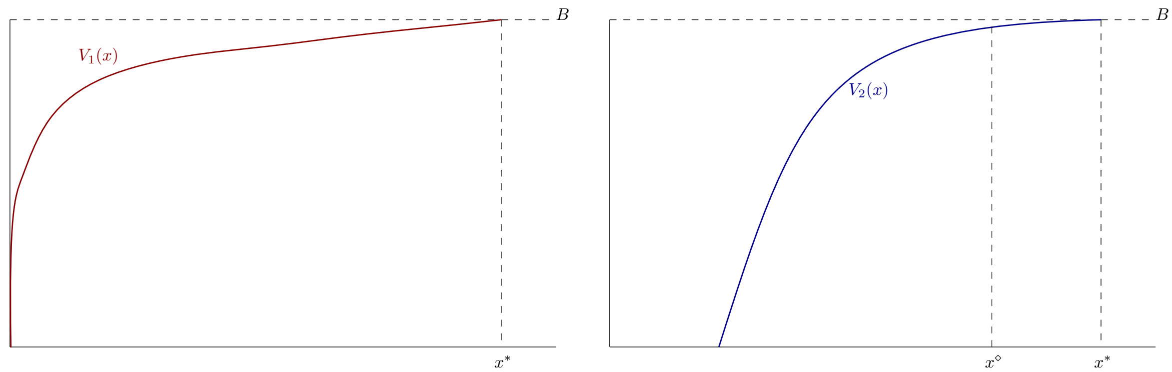

Figure 3.4 Super-solution $V_1(\cdot)$ and sub-solution $V_2(\cdot)$ candidates for the value function V.

with

$$x_1 \in [x^*/2, x^*] \qquad \text{and} \qquad \alpha_1 := \left[\ln\left(\frac{x^*}{x_1}\right)(x^* - x_1)\right]^{-1} \geq \frac{2}{\ln 2 \cdot x^*}. \tag{3.63}$$

Clearly, $V_2(x_1) = 0$. For every $x \in]x_1, x^*[$, we compute

$$V_2'(x) = B\alpha_1 \cdot \left(\ln\left(\frac{x^*}{x}\right) + \frac{x^* - x}{x}\right) \geq 0 \quad \text{and} \quad V_2''(x) = -B\alpha_1\left(\frac{1}{x} + \frac{x^*}{x^2}\right). \tag{3.64}$$

For simplicity, denote the constant

$$C_1 := \lambda + \mu + v_{\max}.$$

Using Lemma 3.1.2 and the first condition of (3.60), we have

$$H(x, V_2'(x), p(x)) \geq -C_1 V_2'(x) x, \qquad \forall x \in [x_1, x^*]$$

which implies that

$$\begin{aligned} &-(r + \rho(x))V_2(x) + \rho(x)B + H(x, V_2'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \\ &\quad \geq -(r + \rho(x))V_2(x) + \rho(x)B - C_1 x V_2'(x) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x). \end{aligned}$$

On the other hand, from (3.64), one has

$$C_1 x V_2'(x) \leq 2B\alpha_1 C_1 (x^* - x) \leq B\alpha_1 C_1 x^*, \qquad \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \geq -B\alpha_1 \sigma^2 x^*,$$

for all $x \in [x_1, x^*]$. Thus,

$$\begin{aligned} -(r+\rho(x))V_2(x) + \rho(x)B + H(x, V_2'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \\ \geq\; B \cdot \left[\alpha_1 \left(\rho(x) \ln\left(\frac{x^*}{x}\right)(x^* - x) - (C_1 + \sigma^2)x^*\right) - r\right]. \end{aligned}$$

From the second condition of (3.60), it holds

$$\lim_{x \to x^*} \rho(x) \ln\left(\frac{x^*}{x}\right)(x^* - x) \;=\; +\infty,$$

and therefore there exists $x^\diamond \in]x^*/2, x^*[$ sufficiently close to x^* such that

$$\frac{2}{\ln 2 \cdot x^*} \cdot \left(\rho(x) \ln\left(\frac{x^*}{x}\right)(x^* - x) - (C_1 + \sigma^2)x^*\right) - r \;\geq\; 0 \qquad \forall x \in [x^\diamond, x^*[.$$

In particular, if we choose $x_1 = x^\diamond$ then (3.65) and (3.63) imply that

$$-(r+\rho(x))V_2(x) + \rho(x)B + H(x, V_2'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \;\geq\; 0,$$

for all $x \in [x^\diamond, x^*[$. Since

$$V_2(x^\diamond) \;=\; 0 \;\leq\; V(x^\diamond) \quad \text{and} \qquad V_2(x^*) \;=\; B \;\leq\; V(x^*),$$

V_2 is a sub-solution to the first equation of (3.17) in $[x^\diamond, x^*]$ and thus

$$V(x) \;\geq\; V_2(x), \qquad \forall x \in [x^\diamond, x^*],$$

which yields (3.61). □

From the formula of $\beta(\cdot)$ in (3.58), one can actually show that β is locally Lipschitz and

$$0 \;\leq\; \beta'(t) \;\leq\; \frac{\rho(t)}{t} \qquad a.e.\ t \in [x, x^*]. \tag{3.65}$$

Notice that

$$\lim_{t \to 0+} \frac{\rho(t)}{t} = \rho'(0) < +\infty$$

and

$$\int_{x^*-\delta}^{x^*} \rho(t)dt \;<\; +\infty \quad \text{for some } \delta > 0 \qquad \Longrightarrow \qquad \int_0^{x^*} \frac{\rho(t)}{t} dt \;<\; +\infty$$

In this case, we have

$$\sup_{t\in[0,x^*[} \beta(t) \ \le\ \beta^* := \int_0^{x^*} \frac{\rho(t)}{t} dt + \frac{\lambda+r}{\theta_{\min}} + \frac{\sigma^2}{2}. \tag{3.66}$$

As a consequence of Proposition 3.3.1, we can deduce the following statements about the optimal feedback strategies.

Corollary 3.3.2. *Under the same assumptions in Theorem 3.2.1, it holds that*

(i) Suppose that $\int_{x^*-\delta}^{x^*} \rho(t)\,dt < +\infty$ *for some* $\delta > 0$ *and, letting* β^* *be as in* (3.66)*, assume that*

$$\beta^* < Br \cdot \min\left\{ \frac{1}{c'(0)}, \frac{1}{L'(0)x^*} \right\}.$$

Then there exists some $\bar{x} \in]0, x^*[$ *sufficiently close to* x^* *such that*

$$u^*(x) \ >\ 0 \qquad \text{and} \qquad v^*(x) \ >\ 0 \qquad \forall x \in [\bar{x}, x^*[.$$

(ii) If (3.60) holds then there exists $\hat{x} \in]0, x^*[$ *sufficiently close to* x^* *such that*

$$u^*(x) \ =\ v^*(x) \ =\ 0 \qquad \forall x \in [\hat{x}, x^*[.$$

Proof. **(i).** For every given $x_2 \in]0, x^*[$, (3.59) and (3.66) imply that

$$V(x^*) - V(x_2) \ \ge\ V(x^*) - V_1(x_2) \ \ge\ B \cdot \frac{r\ln(x_1/x_2)}{r\ln(x_1/x_2) + \beta^*} \quad \forall x_1 \in [x_2, x^*[.$$

In particular, we have

$$V(x^*) - V(x_2) \ \ge\ \sup_{x_1\in[x_2,x^*[} \left[B \cdot \frac{r\ln(x_1/x_2)}{r\ln(x_1/x_2) + \beta^*} \right] \ =\ B \cdot \frac{r\ln(x^*/x_2)}{r\ln(x^*/x_2) + \beta^*}.$$

By mean value theorem, there exists $x_c \in [x_2, x^*]$ such that

$$V'(x_c) \cdot x_c \ \ge\ B \cdot \frac{r\ln(x^*/x_2)}{r\ln(x^*/x_2) + \beta^*} \cdot \frac{x_2}{x^* - x_2}. \tag{3.67}$$

On the other hand, from the first equation of (3.17) and Lemma 3.1.2, it holds

$$\frac{\sigma^2}{2}[x^2V''(x) + xV'(x)] \ \ge\ -cxV'(x) - \rho(x)B \qquad \forall x \in]0, x^*[,$$

with constant $c := \frac{r+\lambda}{\theta_{\min}} + \frac{\sigma^2}{2}$. Set $Z(x) = xV'(x)$, $c_1 = \frac{4c}{\sigma^2 x^*}$ and $c_2 = \frac{4B}{\sigma^2 x^*}$, we have

$$Z'(x) \ \geq \ -c_1 Z(x) - c_2 \rho(x) \qquad \forall x \in]x^*/2, x^*[.$$

Solving the differential inequality yields

$$Z(x) \ \geq \ e^{c_1(x_0 - x)} \cdot Z(x_0) - c_2 \int_{x_0}^{x} \rho(s) ds\,, \qquad \forall \ \frac{x^*}{2} < x_0 \leq x < x^*.$$

In particular, recalling (3.67), we have

$$Z(x) \ \geq \ Be^{c_1(x_2 - x^*)} \frac{r \ln(x^*/x_2)}{r \ln(x^*/x_2) + \beta^*} \cdot \frac{x_2}{x^* - x_2} - c_2 \int_{x_2}^{x^*} \rho(s) ds \ =: \ I(x_2),$$

for all $x^*/2 < x_2 < x_c \leq x < x^*$. Since

$$\beta^* < Br \cdot \min\left\{\frac{1}{c'(0)}, \frac{1}{L'(0)x^*}\right\} \qquad \text{and} \qquad \int_0^{x^*} \rho(t)\, dt < +\infty\,,$$

it holds that

$$\lim_{x_2 \to x^*} I(x_2) \ = \ \frac{Br}{\beta^*} \ > \ \max\left\{c'(0), L'(0)x^*\right\}.$$

Thus, there exists $x_2 \in [x^*/2, x^*[$ sufficiently close to x^* such that

$$Z(x) \ \geq \ I(x_2) \ > \ c'(0) \qquad \text{and} \qquad V'(x) \ \geq \ \frac{I(x_2)}{x} \ > \ L'(0) \ \geq \ L'(0)p(x),$$

for all $x \in [x_c, x^*]$ and, recalling (3.12)-(3.13), this yields (i) for $\bar{x} = x_c$.

(ii). Assuming that (3.60) holds, we recall $x^\diamond$ in Proposition 3.3.1. By the mean value theorem, for every $x_1 \in [x^\diamond, x^*[$, there exists $x_c \in]x_1, x^*[$ such that

$$\begin{aligned} V'(x_c) \cdot x_c \ &= \ \frac{V(x^*) - V(x_1)}{x^* - x_1} \cdot x_c \ \leq \ \frac{B - V_2(x_1)}{x^* - x_1} \cdot x^* \\ &= \ \frac{B}{\ln\left(\frac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \cdot \ln\left(\frac{x^*}{x_1}\right). \end{aligned} \tag{3.68}$$

From the first differential equation of (3.17) and Lemma 3.1.2, we can estimate

$$\frac{\sigma^2}{2} \cdot [x^2 V''(x) + xV'(x)] \ \leq \ rB + (\lambda + \mu + v_{\max})\, xV'(x) \qquad \forall x \in]x^*/2, x^*[.$$

Recalling that $Z(x) = xV'(x)$, we have

$$Z'(x) \ \leq \ c_3 Z(x) + c_4 \qquad \forall x \in [x^*/2, x^*],$$

with

$$c_3 := \frac{4rB}{\sigma^2 x^*} \qquad \text{and} \qquad c_4 := \frac{2\,(\lambda + \mu + v_{\max})}{\sigma^2 x^*}\,.$$

Thus, by applying Gronwall's inequality, we have

$$Z(x) \ \leq \ e^{c_3\cdot(x-x_c)} \cdot Z(x_c) + \frac{c_4}{c_3} \cdot \left(e^{c_3\cdot(x-x_c)} - 1\right)\,,$$

and (3.68) implies that

$$Z(x) \ \leq \ \frac{B}{\ln\left(\dfrac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \cdot e^{c_3\cdot(x^*-x_1)} \cdot \ln\left(\frac{x^*}{x_1}\right) + \frac{c_4}{c_3} \cdot \left(e^{c_3\cdot(x^*-x_1)} - 1\right) \ =: \ J(x_1),$$

for all $x \in [x_c, x^*[$. Since $\lim_{x_1 \to x^*-} J(x_1) = 0$, there exists $x_1 \in]x^\diamond, x^*[$ such that

$$Z(x) \ \leq \ J(x_1) \ \leq \ c'(0)\,,$$

and therefore

$$V'(x) \ \leq \ \frac{J(x_1)}{x} \ \leq \ \theta_{\min} L'(0) \ \leq \ L'(0) \cdot p(x),$$

for all $x \in [x_c, x^*[$. Recalling (3.12)-(3.13) and setting $\hat{x} = x_c$ yields (ii). □

Chapter 4

Fine regularity of the Burgers-Poisson equation

The aim of this chapter is to show that weak entropy solutions to the Burgers-Poisson equation are in $SBV(\mathbb{R}^+ \times \mathbb{R})$. The Burgers-Poisson equation is given by the balance law obtained from Burgers' equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G * u]_x\,, \tag{4.1}$$

where, $G(x) = -\frac{1}{2}e^{-|x|}$ is the Poisson kernel such that

$$[G * f](x) \ = \ \int_{-\infty}^{+\infty} G(x-y) \cdot f(y)\, dy$$

solves Poisson's equation

$$\varphi_{xx} - \varphi \ = \ f\,. \tag{4.2}$$

Equation (4.1) was first derived in [80] as a simplified model of shallow-water waves and admits conservation of both momentum and energy. In the literature, it has also been referred to as the Fornberg-Whitham equation ([55],[54],[57],[56],[78]). Well-posedness analysis in [44] established the local existence and uniqueness of solutions of the associated Cauchy problem of (4.1) and initial data

$$u(x,0) \ = \ u_0(x), \qquad x \in \mathbb{R}\,. \tag{4.3}$$

for sufficiently regular u_0. Additionally, their analysis of traveling waves showed that the equation features wave breaking in finite time. More generally, it has been demonstrated that (4.1) does not admit a global smooth solution ([62]). Hence, further analysis has focused on the concept of weak entropy solutions, in the sense of Kruzkov.

Based on the vanishing viscosity approach, the existence result for a global weak solution was provided for $u_0 \in BV(\mathbb{R})$ in [44]. However, in the more general case with initial data in $\mathbf{L}^1(\mathbb{R})$, this approach cannot be applied. Moreover, there are no uniqueness or continuity results for global weak entropy solutions of (4.1) established in [44]. More recently, the existence and continuity results for global weak entropy solutions of (4.1) were established for $\mathbf{L}^1(\mathbb{R})$ initial data in [53]. The entropy weak solutions are constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers equation and the Lipschitz continuity of solutions to the Poisson equation, approximating solutions satisfy an Oleinik-type inequality for any positive time. As a consequence, the sequence of approximating solutions is precompact and converges in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, using an energy estimate, they show that the characteristics are Hölder continuous, which is used to achieve the continuity property of the solutions. The Oleinik-type inequality gives that the solution $u(t,\cdot)$ is in $BV_{\mathrm{loc}}(\mathbb{R})$ for every $t > 0$. In particular, this implies that the Radon measure $Du(t,\cdot)$ is divided into three mutually singular measures (see Section 2.5)

$$Du(t,\cdot) \ = \ D^a u(t,\cdot) + D^j u(t,\cdot) + D^c u(t,\cdot)$$

where $D^a u(t,\cdot)$ is the absolutely continuous measure with respect to the Lebesgue measure, $D^j u(t,\cdot)$ is the *jump part* which is a countable sum of weighted Dirac measures, and $D^c u(t,\cdot)$ is the non-atomic singular part of the measure called the *Cantor part*. For a given $w \in BV_{\mathrm{loc}}(\mathbb{R})$, the Cantor part of Dw does not vanish in general. A typical example of $D^c w$ is the derivative of the Cantor-Vitali ternary function. If $D^c w$ vanishes, then we say the function w is locally in the space of special functions of bounded variation, denoted by $SBV_{\mathrm{loc}}(\mathbb{R})$. The space of SBV functions was first introduced in [37]. Motivated by results on SBV regularity for hyperbolic conservation laws ([4, 73, 11, 63]), we show the following.

Theorem 4.0.1. *Let $u : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$ be the unique locally BV-weak entropy solution of (4.1) with initial data $u_0 \in \mathbf{L}^1(\mathbb{R})$. Then there exists a countable set $\mathcal{T} \subset \mathbb{R}^+$ such that*

$$u(t,\cdot) \ \in \ SBV_{\mathrm{loc}}(\mathbb{R}) \qquad \forall t \in \mathbb{R}^+ \setminus \mathcal{T}.$$

As a consequence, the slicing theory of BV functions and the chain rule of Vol'pert [5] implies that the weak entropy solution u is in $SBV_{\mathrm{loc}}([0,+\infty[\times\mathbb{R})$. This is the first result showing that a SBV-regularization effect holds for solutions to a scalar conservation law with a nonlocal source term.

4.1 A nonlocal dispersive balance law

We recall results of the Burgers-Poisson equation, which has also been called the Fornberg-Whitham equation. Introduced by Whitham in [80], as a dispersive model for the center of shallow water wave model captures the balance between linear dispersion and nonlinear effects, while showing indications of wave breaking (see also Whitman's models in [79]). The wave height is described as a function of space and time $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. Throughout this chapter, we will examine the existence, uniqueness and regularity results of (4.1) subject to initial condition (4.3). The equation (4.1) emerged in [44], as a rescaled formulation of the Whitham equation (see also [50]), from a system of equations. Note that by setting $\phi := -G * u$, we obtain the system of partial differential equations

$$\begin{cases} u_t + uu_x \ = \ \phi_x \\ \phi_{xx} \ = \ \phi + u \,. \end{cases}$$

This formulation was called the *Burgers-Poisson system* in [44], while the analog that we will focus on (4.1) is called the *Burgers-Poisson equation* (as in [53]).

Remark 4.1.1. By applying the operator $1 - \partial_x^2$ to (4.1), we obtain the third order PDE

$$u_t - u_{xxt} + uu_x + u_x \ = \ 3u_x u_{xx} + uu_{xxx} \,. \tag{4.4}$$

Compare the terms in (4.4) to those of the Camassa-Holm equation [21]:

$$u_t - u_{xxt} + 3uu_x + 2\kappa u_x \ = \ 2u_x u_{xx} + uu_{xxx} \,, \tag{4.5}$$

where $\kappa \geq 0$ is related to the critical shallow water wave speed. Using the bi-Hamiltonian property of (4.5), Camassa and Holm derived an infinite sequence of conservation laws and thus that the associated flows are completely integrable. The equation (4.5) is often considered with $\kappa = 0$, in which case it has peaked soliton solutions. Since we will look at the system through the lens of scalar balance laws, we will use the form (4.1).

Remark 4.1.2. The Burgers-Poisson equation (4.1) becomes the Benjamin-Ono equation

([10],[69]) when $G * u$ is replaced by $-2H(u_x)$, where H is the Hilbert transform:

$$H(u)(y) \;=\; \text{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(x)}{x-y}\, dx\,.$$

The Benjamin-Ono equation is used to study long internal gravity waves in stratified fluids of great depth. It is another completely integrable hamiltonian system, possessing multi-soliton solutions.

Let us consider the conserved quantities for solutions u (of sufficient regularity and integrability properties). Integrating (4.1) with respect to x yields

$$\int \partial_t u(x,t)\, dx \;=\; \int \partial_x \left[-\frac{u^2}{2} + [G * u(t,\cdot)] \right] dx \;=\; 0$$

thus we have that

$$\int_{\mathbb{R}} u(t,x)\, dx \;=\; \int_{\mathbb{R}} u_0(x)\, dx, \qquad \forall t \geq 0\,.$$

Another conserved quantity is that of spatial $\mathbf{L}^2$-norm

Theorem 4.1.3 (Conservation of energy)**.** *Let u a distributional solution of* (4.1)*. Then it holds that*

$$\frac{d}{dt} \int_{\mathbb{R}} u(t,x)^2\, dx \;=\; 0 \qquad \forall t \geq 0\,.$$

Proof. Consider the Burgers-Poisson system:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x \;=\; \phi_x \\ \phi_{xx} - \phi = u\,. \end{cases}$$

Multiplying the first equation by u, we derive a second conservation law

$$\begin{aligned} 0 \;&=\; \left(\frac{1}{2}u^2\right)_t + \left(\frac{u^3}{3}\right)_x - \phi_x u \\ &=\; \left(\frac{1}{2}u^2\right)_t + \left(\frac{u^3}{3}\right)_x - \phi_x\left(\phi - \phi_x x\right) \\ &=\; \left(\frac{1}{2}u^2\right)_t + \left(\frac{u^3}{3} + \frac{\phi^2}{2} - \frac{\phi_x^2}{2}\right)_x . \end{aligned}$$

Assuming that either u is periodic in x or that u and its derivatives vanish sufficiently rapidly at infinity, we recover that

$$\int \frac{1}{2}u_0^2\,dx \;=\; \int \frac{1}{2}u(t,x)^2\,dx, \qquad \forall t > 0\,.$$

□

However, as shown in [58], the Burgers-Poisson equation is known to belong to the class of dispersive wave equations that are not completely integrable.

4.1.1 Classical Solutions

As in Section 2.6, the well-posedness analysis of (4.1) began in [65] by searching for functions $u : \Omega \subset \mathbb{R} \times [0, T[\mapsto \mathbb{R}$ which are classical solutions in the sense that u possesses first-order partial derivatives, the convolution $[G * u(t, \cdot)]$ is defined on $\mathbb{R}$ for every $t > 0$, and (4.1) holds point-wise for every $(x, t) \in \Omega$. The preliminary on classical smooth solutions was established in [65] for initial data in H^∞. The result is established through a strategy of successive approximation, where each approximation is found by solving a linear hyperbolic equation. By using properties of the characteristics, they show convergence of the scheme leading to the following result.

Theorem 4.1.4. *Let $u_0 \in H^\infty(\mathbb{R})$. Then there is some $T > 0$ such that the Cauchy problem* (4.1)-(4.3) *has a unique solution $u \in C^\infty([0, T], H^\infty(\mathbb{R}))$.*

The next results for classical solutions came in the paper [44]. Relying on a contraction argument for the map $v \mapsto u$, where u solves

$$u_t + uu_x \;=\; [G * v]_x$$

they show the following.

Theorem 4.1.5. *Assume $u_0 \in H^k(\mathbb{R})$ with $k > \frac{3}{2}$. Then, there exists a time $T > 0$, depending on $\|u_0\|_{H^{k+1}}$ such that the Cauchy problem* (4.1)-(4.3) *has a unique solution u with regularity*

$$u \in \mathbf{L}^\infty(]0, T[; H^k(\mathbb{R})) \cap C([0, T]; H^{k-1}(\mathbb{R}))$$

It holds that the solution regularity is better than the given space $\mathbf{L}^\infty(]0, T[; H^k(\mathbb{R})) \cap C([0, T]; H^{k-1}(\mathbb{R}))$ and that the solution is a strong solution. By the equation (4.1), it follows that $D_t u = -uD_x u + [G * u]_x \in \mathbf{L}^\infty([0, T], H^k(\mathbb{R}))$. Hence, the map $t \mapsto u(t, \cdot)$ is Lipschitz continuous with values in $H^k(\mathbb{R})$.

For the periodic case, a similar result, but with spatial H^{s+1} regularity for some $s \in \mathbb{R}$ such that $s > 1/2$ and solution $u \in C([0,T], H^{s+1}(\mathbb{T}))$ was established in [55]. The method of the proof uses a Galerkin aprroximation argument and derives energy estimates yielding convergence in the appropriate function spaces. Again, by utilizing the balance law (4.1), it holds that $u \in C^1([0,T], H^s(\mathbb{T}))$ and we have a strong solution. Moreover, they note the continuous dependence of u on the initial data u_0.

4.1.2 Wave Breaking

As discussed in Section 2.6, solutions to conservation laws may not be continuous for all time. In this subsection, we review related analysis of the Burgers-Poisson equation and focus on the prediction of wave breaking. We say that *wave breaking* occurs for u for some time $T > 0$ [29, Definition 6.1], if the wave $u(T,\cdot)$ remains bounded while its slope becomes unbounded, i.e.

$$\sup_{t\in[0,T[} \|u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \;<\; +\infty \quad \text{and} \quad \limsup_{t\uparrow T} \|D_x u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \;=\; +\infty\,.$$

In particular, we are interested in identifying if, for initial data $u_0(\cdot) \in C^1(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$, wave breaking occurs for some time $T > 0$. The traditional method of analysis is to follow solutions along the characteristics, as long as they exist in the classical sense. The first indication that wave breaking occurs was given in [74] and followed up in [65], where a sketch of arguments detailing a quantitative asymmetry condition in terms of the minimum and maximum slopes of the initial wave profile. The first rigorous proof of a wave breaking result was given in [28] which we state here. Availability of more general well-posedness results allows for less regular initial data than the original theorem [28, Theorem 3.2].

Theorem 4.1.6. *Let $u_0 \in H^3(\mathbb{R})$ such that u_0 satisfies*

$$\inf_{x\in\mathbb{R}} u_0'(x) + \sup_{x\in\mathbb{R}} u_0'(x) \;\leq\; -1\,.$$

Then wave breaking occurs in the solution to the Cauchy problem (4.1), (4.3) *with initial data u_0.*

In [62], they show a similar wave breaking result in the case of periodic initial data. Again, it holds that the nonlocal source term cannot prevent the nonlinear breaking of smooth solutions when the slope of the initial data is sufficiently large. More precisely, they show that:

Theorem 4.1.7. *Consider the solutions u of the system* (4.1), (4.3) *subject to smooth periodic initial data $u_0(x)$ such that $u_0(x+1) = u_0(x)$. Let $m = -\inf D_x u_0(x)$ and suppose that*

$$\frac{m^3}{4 + 2m\left(\frac{1}{2} + \frac{e^{0.5}+e^{-0.5}}{e^{0.5}-e^{-0.5}}\right)} > 4\,\|u_0\|_{\mathbf{L}^\infty(\mathbb{R})} .$$

Then the smooth solutions u break down, and wave breaking occurs in finite time, before $T = 2/m$.

A similar result is provided in [53] in the non-periodic case. They give the following precise statements on when wave breaking *will not* occur as well as a sufficient condition for wave breaking in relatively short time.

Theorem 4.1.8. *Let $u(t,x)$ be the solution of* (4.1),(4.3) *with $u_0(\cdot) \in C^1(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$. Denote by*

$$m = \inf_{x\in\mathbb{R}} u_0'(x) \qquad \text{and} \qquad M = \|u_0\|_{\mathbf{L}^\infty(\mathbb{R})} .$$

Then the following statements hold

(i) $D_x u(t,x(t))$ remains bounded for all $t \in [0, T^*[$ where

$$T^* = \ln\left(1 + \frac{1}{\sqrt{|m| + 3M}}\left(\frac{\pi}{2} - \arctan\left[\frac{|m|}{\sqrt{2M}}\right]\right)\right)$$

(ii) If

$$m < -1 - \sqrt{\frac{5}{2}M + 1} \leq 0,$$

then $D_x u(t,x(t))$ becomes unbounded within the time interval $[0, T[$ where

$$T = \frac{2}{|2m + 1 + 2\sqrt{|m| + \frac{5}{2}M + \frac{1}{4}}} .$$

Moreover, in [53], they show the following negative wave breaking result. In particular, they show that if the $\mathbf{L}^\infty$ norm of the slope of the initial data is small, then the corresponding entropy solution will remain smooth for a long time.

Theorem 4.1.9. *Let $u(t,x)$ be the solution of* (4.1), (4.3) *with $u_0 \in C^1(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ and let $m = \|u_0'\|_{\mathbf{L}^\infty(\mathbb{R})}$. Then $\|D_x u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}$ remains bounded for all $t \in [0, \ln(1 + \frac{1}{m})[$.*

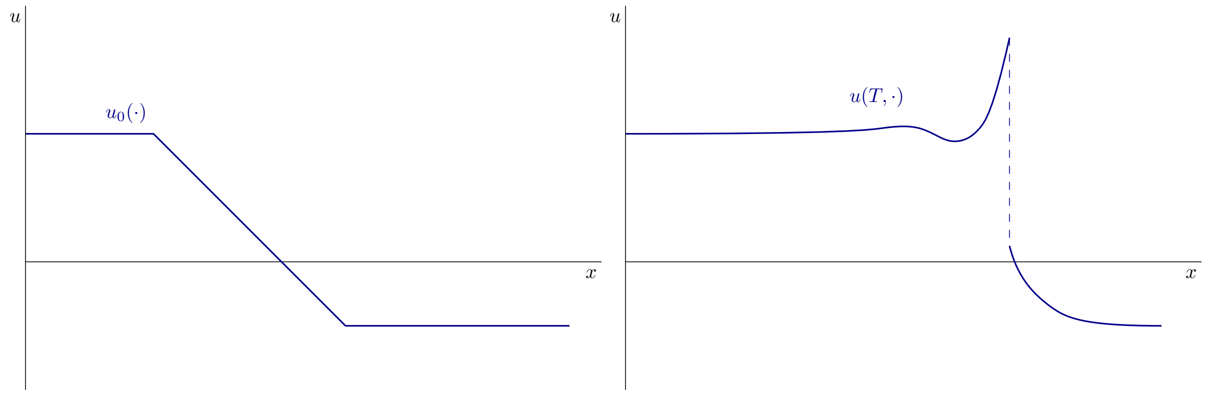

Figure 4.1 Numerical simulations (see, e.g. [44, Figure 4.2]) demonstrate the wave breaking behavior of solutions to the Burgers-Poisson equation at time T with continuous initial data.

4.1.3 Entropy admissible solutions and Oleinik inequality

As shown in the previous section, the classical solutions to (4.1) break down in finite time even with smooth initial data. Instead, we extend the notion of an entropy weak solution in the Kruzkov sense to (4.1).

Definition 4.1.10. A function $u \in \mathbf{L}^1_{\mathrm{loc}}([0,\infty[\times\mathbb{R}) \cap \mathbf{L}^\infty_{\mathrm{loc}}(]0,\infty[,\mathbf{L}^\infty(\mathbb{R}))$ is an *entropy weak solution* of (4.1) if u satisfies the following properties:

(i) the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R})$, i.e.,

$$\|u(t,\cdot)-u(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; L\cdot|t-s| \qquad \forall 0\leq s\leq t$$

for some constant $L>0$.

(ii) For any $k\in\mathbb{R}$ and any non-negative test function $\phi\in C^1_c(]0,\infty[\times\mathbb{R},\mathbb{R})$ one has

$$\int\int\Big[|u-k|\phi_t+\operatorname{sign}(u-k)\left(\frac{u^2}{2}-\frac{k^2}{2}\right)\phi_x+\operatorname{sign}(u-k)[G_x*u(t,\cdot)](x)\phi\Big]\,dx\,dt \;\geq\; 0\,.$$

A standard way of showing existence of entropy solutions is through the method of vanishing viscosity. The first indication that this method may produce a convergent scheme towards an entropy solution for is given in [65]. The vanishing viscosity approach was later employed in [44] to produce a rigorous statement on the existence of weak entropy solutions for initial data with spatially BV regularity. In particular, they consider the regularized equation

$$u_t+uu_x \;=\; [G*u]_x+\varepsilon u_{xx}$$

with $\varepsilon > 0$ and derive a convergent scheme as $\varepsilon \to 0$ to show the following [44, Theorem 4.2]

Theorem 4.1.11. *Assume that $u_0 \in BV(\mathbb{R})$. Then there exists a global weak solution to the Cauchy problem* (4.1), (4.3)

$$u \in \mathbf{L}^\infty_{loc}([0,\infty[; BV(\mathbb{R}))$$

satisfying the entropy condition

$$(u^2)_t + \left(\frac{2}{3}u^3 + \phi^2 - \phi_x^2\right)_x \le 0$$

in the distributional sense.

The vanishing viscosity method used in [44] for initial data in BV can not be applied to the more general case where $u_0 \in \mathbf{L}^1(\mathbb{R})$. Instead, in [53], the entropy weak solutions are constructed by a flux-splitting method. The proof of follows the idea in [16], where the authors provide an ee existence result for the Burgers-Hilbert equation. Relying on the decay properties of the semigroup generated by Burgers equation and the Lipschitz continuity of solutions to the Poisson equation (see Propositions 4.1.13 and 4.1.14), approximating solutions satisfy an Oleinik-type inequality for any positive time. As a consequence, the sequence of approximating solutions is precompact and converges in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$. Moreover, using an energy estimate, they show that the characteristics are Hölder continuous, which is used to achieve the continuity property of the solutions. Their main results are presented here [53, Theorem 1.2].

Theorem 4.1.12. *The Cauchy problem* (4.1)-(4.2) *with initial data $u_0 := u(0,\cdot) \in \mathbf{L}^1(\mathbb{R})$ admits a unique solution $u(t,x)$ such that for all $t > 0$ the following hold:*

(i) the $\mathbf{L}^1$-norm is bounded by

$$\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\le\; e^t \cdot \|u_0\|_{\mathbf{L}^1(\mathbb{R})}\,; \tag{4.6}$$

(ii) the solution satisfies the following Oleinik-type inequality

$$u(t,y) - u(t,x) \;\le\; \frac{K_t}{t} \cdot (y-x) \qquad \forall\, y > x \tag{4.7}$$

with

$$K_t \;=\; 1 + 2t + 2t^2 + 4t^2 e^t \cdot \|u_0\|_{\mathbf{L}^1(\mathbb{R})}\,;$$

(iii) Given $v(t,x)$, the solution of (4.1) *with initial data in $v_0 \in \mathbf{L}^1(\mathbb{R})$, it holds that*

$$\|u(t,\cdot) - v(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; e^t \,\|u_0 - v_0\|_{\mathbf{L}^1(\mathbb{R})} \;;$$

(iv) the $\mathbf{L}^\infty$-norm is bounded by

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; \sqrt{\frac{2K_t}{t}\,\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}} \;\leq\; \sqrt{\frac{2K_t e^t}{t}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\,. \tag{4.8}$$

We recall some properties of the approximating solutions, used in the proof Theorem 4.1.12. Let S^B denote the semigroup generated by Burgers equation, i.e., $t \mapsto S^B_t(u_0)$ denotes the Kruzkov entropy solution to

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; 0, \qquad u(0,x) = u_0(x) \in \mathbf{L}^1(\mathbb{R}).$$

In a refinement to the typical one-sided jump estimate in Theorem 2.6.16, if the initial data is one-sided Lipschitz, we recover a stronger inequality for the solution.

Proposition 4.1.13 (Positive decay of Burgers semigroup)**.** *[53, Lemma 2.1]: If u_0 is such that*

$$u_0(x_2) - u_0(x_1) \leq K \cdot (x_2 - x_1), \qquad\qquad \forall x_1 < x_2 \tag{4.9}$$

then

$$S^B_t(u_0)(x_2) - S^B_t(u_0)(x_1) \;\leq\; \frac{K}{1+Kt}\cdot(x_2 - x_1), \qquad\qquad \forall x_1 < x_2 \tag{4.10}$$

Proof. Since u satisfies the Lax entropy condition, it holds that

$$u(t,x-) \;\geq\; u(t,x+), \qquad \forall t > 0\,.$$

Hence, it is sufficient to prove (4.10) for any point of continuity x_i of $S^B_t(u_0)$. Let $\xi_{x_i}(\cdot)$ be the characteristic through the point (t,x_i). The characteristic equations imply that

$$x_i \;=\; \xi_{x_i}(0) + t u_0(\xi_{x_i}(0)) \qquad \text{and} \qquad S^B_t(u_0)(x_i) \;=\; u_0(\xi_{x_i}(0))\,.$$

Let $x_1 < x_2$ be such points of continuity. From the assumption (4.9), it holds that

$$\begin{aligned} x_2 - x_1 \; &= \xi_{x_2}(0) - \xi_{x_1}(0) + t\cdot\left(u_0(\xi_{x_2}(0)) - u_0(\xi_{x_1}(0))\right) \\ &\leq (1+Kt)\cdot\left(\xi_{x_2}(0) - \xi_{x_1}(0)\right)\,, \end{aligned}$$

therefore we have

$$\begin{aligned} S_t^B(u_0)(x_2) - S_t^B(u_0)(x_1) &= \frac{1}{t}\left(x_2 - x_1 - (\xi_{x_2}(0) - \xi_{x_1}(0))\right) \\ &\leq \frac{K}{1+Kt} \cdot (x_2 - x_1)\ , \end{aligned}$$

which completes the proof.

□

Another significant result is that solutions to the Poisson equation retain (one-sided) Lipschitz continuity. The following is crucial to the analysis provided in [53] and summarized in Theorem 4.1.12.

Proposition 4.1.14 (Lipschitz continuity of solutions to Poisson equation)**.** *[53, Lemma 2.2]: Let u_0 is such that*

$$u_0(x_2) - u_0(x_1) \ \leq\ K \cdot (x_2 - x_1), \qquad\qquad \forall x_1 < x_2\,. \tag{4.11}$$

Then

$$\left|[G * u_0]_x(x_2) - [G * u_0]_x(x_1)\right| \ \leq\ \left(\sqrt{2K\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + \|u_0\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot |x_2 - x_1|.$$

Proof. Since $x_1 < x_2$ we can estimate from above

$$\begin{aligned} \left|[G * u_0(\cdot)]_x(x_2) - [G * u_0(\cdot)]_x(x_1)\right| \ &\leq\ \frac{1}{2}\cdot\int_{-\infty}^{x_1} |u_0(z)|\cdot\left|e^{z-x_2} - e^{z-x_1}\right|\ dz \\ +\frac{1}{2}\cdot\int_{x_1}^{x_2} |u_0(z)|\cdot\left|e^{z-x_2} + e^{x_1-z}\right|\ dz &+ \frac{1}{2}\cdot\int_{x_2}^{+\infty} |u_0(z)|\cdot\left|e^{x_1-z} - e^{x_2-z}\right|\ dz \end{aligned}$$

By Hölder's inequality, it holds that

$$\begin{aligned} \frac{1}{2}\cdot\int_{-\infty}^{x_1} |u_0(z)|\cdot\left|e^{z-x_2} - e^{z-x_1}\right|\ dz + \frac{1}{2}\cdot\int_{x_2}^{+\infty} |u_0(z)|\cdot\left|e^{x_1-z} - e^{x_2-z}\right|\ dz \\ \leq\ \left(1 - e^{x_1-x_2}\right)\|u_0\|_{\mathbf{L}^1(\mathbb{R})}\,. \end{aligned}$$

By a comparison principal argument, it holds that $1 - e^{-x} \leq x$ for all $x > 0$, hence, we estimate that

$$\left|[G * u_0(\cdot)]_x(x_2) - [G * u_0(\cdot)]_x(x_1)\right| \ \leq\ \left(\|u_0\|_{\mathbf{L}^1(\mathbb{R})} + \|u_0\|_{\mathbf{L}^\infty(\mathbb{R})}\right)\cdot\left|x_2 - x_1\right|. \tag{4.12}$$

On the other hand, note that $u_0 \in BV(\mathbb{R})$ and since u_0 satisfies (4.11), we have that

$Du_0 \leq K$ in the sense of measures. Hence, Proposition 2.5.12 yields

$$\|u_0\|_{\mathbf{L}^\infty} \leq \sqrt{2K \|u_0\|_{\mathbf{L}^1}}$$

which when applied to (4.12), completes the proof. □

The previous two Propositions yield that an intermediate Oleinik-type inequality holds for the approximating solutions used in the proof of Theorem 4.1.12. A similar Oleinik-type inequality (4.7) also holds for the entropy weak solutions, which in turn implies that u retains a BV regularity for all positive time.

Proposition 4.1.15 (BV-regularity of solutions)**.** *Let u be the entropy weak solution of* (4.1) *then $u(t,\cdot) \in BV_{loc}(\mathbb{R})$ for all $t > 0$ with total variation bounded by*

$$|Du(t,\cdot)|(I) \;\leq\; \frac{2K_t}{t} \cdot \mathcal{L}(I) + 2\,\|u(t,\cdot)\|_{\mathbf{L}^\infty(I)} \tag{4.13}$$

for any compact interval $I \subset \mathbb{R}$.

Proof. For any $t > 0$, define the function $g_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) \;=\; \frac{K_t}{t} \cdot x - u(t,x)\,.$$

By the Oleinik inequality (4.7), $g_t(\cdot)$ is monotone increasing on $\mathbb{R}$ and bounded on any compact subset of $\mathbb{R}$. In particular, this implies that for any $[a,b] \subset \mathbb{R}$, the variation of g_t is bounded by

$$|Dg_t|([a,b]) \;=\; g_t(b) - g_t(a) \;\leq\; \frac{K_t}{t} \cdot (b-a) + 2\,\|u(t,\cdot)\|_{\mathbf{L}^\infty([a,b])}$$

Therefore $g \in BV_{loc}(\mathbb{R})$ and $u(t,\cdot) \in BV_{loc}(\mathbb{R})$ as a sum of two BV_{loc} functions with total variation bounded by

$$|Du(t,\cdot)|(I) \;\leq\; |Dg_t|(I) + \frac{K_t}{t} \cdot \mathcal{L}(I) \qquad \forall I \subset\subset \mathbb{R}\,,$$

implying (4.13). □

Remark 4.1.16. By taking the limit as $y \to x$, in (4.7), it is clear that for all $t > 0$, $u(t,\cdot)$ satisfies

$$u(t,x-) \;\geq\; u(t,x+) \qquad \forall x \in \mathbb{R}. \tag{4.14}$$

4.1.4 Properties of generalized characteristics

We recall the definition and theory of generalized characteristic curves associated with (4.1). For a more in-depth theory of generalized characteristics, we direct the readers to [36].

Definition 4.1.17. For any $(t,x) \in]0,+\infty[\times\mathbb{R}$, an absolutely continuous curve $\xi_{(t,x)}(\cdot)$ is called a *backward characteristic curve* starting from (t,x) if it is a solution of the differential inclusion

$$\dot{\xi}_{(t,x)}(s) \;\in\; \left[u\left(s,\xi_{(x,t)}(s)+\right),\, u\left(s,\xi_{(t,x)}(s)-\right)\right] \qquad a.e.\ s\in[0,t] \tag{4.15}$$

with $\xi_{(t,x)}(t) = x$. If $s \in [t,+\infty[$ in (4.15) then ξ is called a *forward characteristic curve*, denoted by $\xi^{(t,x)}(\cdot)$. The characteristic curve ξ is called *genuine* if $u(t,\xi(t)-) = u(t,\xi(t)+)$ for almost every t.

The existence of backward (forward) characteristics was studied by Fillipov. As in [36] and [73], the speed of the characteristic curves are determined and genuine characteristics are essentially classical characteristics:

Proposition 4.1.18. *Let $\xi : [a,b] \to \mathbb{R}$ be a characteristic curve for the Burgers-Poisson equation (4.1), associated with an entropy weak solution u. Then for almost every time $t \in [a,b]$, it holds that*

$$\dot{\xi}(t) \;=\; \begin{cases} u(t,\xi(t)) & \text{if} \quad u(t,\xi(t)+) \;=\; u(t,\xi(t)-)\,, \\[2ex] \dfrac{u(t,\xi(t)+)+u(t,\xi(t)-)}{2} & \text{if} \quad u(t,\xi(t)+) \;<\; u(t,\xi(t)-)\,. \end{cases} \tag{4.16}$$

In addition, if ξ is genuine on $[a,b]$, then there exists $v(t) \in C^1([a,b])$ such that

$$u(t,\xi(t)-) \;=\; v(t) \;=\; u(t,\xi(t)+) \qquad \forall t \in]a,b[\tag{4.17}$$

and $(\xi(\cdot),v(\cdot))$ solve the system of ODEs

$$\begin{cases} \dot{\xi}(t) \;=\; v(t) \\[1ex] \dot{v}(t) \;=\; [G * u(t,\cdot)]_x(\xi(t)) \end{cases} \qquad \forall t \in]a,b[\,. \tag{4.18}$$

Proof. We first prove (4.16), following the proof of Theorem 10.2.3 of [36]. Wherever $u(t,\cdot)$ is continuous, the differential inclusion (4.15) reduces to a single point and we recover that

$$\dot{\xi}(t) = u(t, \xi(t))$$

Applying the measure equality (4.1) to arbitrary subarcs of the graph of ξ, and using Theorem 1.7.8 of [36], yields

$$\begin{aligned}
\dot{\xi}(t)\,[u(t,\xi(t)+) - u(t,\xi(t)-)] &= \frac{[u(t,\xi(t)+)]^2}{2} - \frac{[u(t,\xi(t)-)]^2}{2} \\
&\quad - ([G * u(t,\cdot)]\,(\xi(t)+) - [G * u(t,\cdot)]\,(\xi(t)-)) \\
&= \frac{[u(t,\xi(t)+)]^2}{2} - \frac{[u(t,\xi(t)-)]^2}{2},
\end{aligned}$$

almost everywhere on $[a,b]$. Consequently, for almost all $t \in [a,b]$ with $u(t,\xi(t)+) < u(t,\xi(t)-)$, we have that $\dot{\xi}(t) = s$ where s is the speed of a shock. To prove the rest, we follow the idea in Theorem 11.9.1 of [36]. Let ξ be genuine on $[a,b]$ and let

$$I = \{t \in (a,b) \,:\, u(t,\xi(t)+) = u(t,\xi(t)-)\}\,.$$

For any $t \in I$, set $v(t) = u(t,\xi(t)\pm)$, we have

$$\dot{\xi}(t) = v(t) \qquad \text{a.e. on } (a,b) \tag{4.19}$$

Fix r and s such that $a \le r < s \le b$. For some $\varepsilon > 0$, we integrate (4.1) over the set

$$\{(x,t) \,:\, r < t < s,\ \xi(t) - \varepsilon < x < \xi(t)\}$$

and apply Leibniz's rule to get

$$\begin{aligned}
&\int_{\xi(s)-\varepsilon}^{\xi(s)} u(s,x)\,dx - \int_{\xi(r)-\varepsilon}^{\xi(r)} u(r,x)\,dx + \int_r^s \int_{\xi(t)-\varepsilon}^{\xi(t)} [G * u(t,\cdot)]_x(x)\,dx\,dt \\
&\quad = \int_r^s \frac{u^2(t,\xi(t)-\varepsilon)}{2} - \frac{u^2(t,\xi(t))}{2} - \dot{\xi}(t)\,[u(t,\xi(t)-\varepsilon) - u(t,\xi(t))]\;dt \\
&\quad = \int_r^s \frac{u^2(t,\xi(t)-\varepsilon)}{2} - \frac{v^2(t)}{2} - v(t)\,[u(t,\xi(t)-\varepsilon) - v(t)]\;dt \\
&\quad = \int_r^s \frac{1}{2}\,[u(t,\xi(t)-\varepsilon) - v(t)]^2\;dt \;\ge\; 0\,.
\end{aligned}$$

Multiplying by $1/\varepsilon$ and letting $\varepsilon \downarrow 0$ yields

$$u(s,\xi(s)-) \;\ge\; u(r,\xi(r)-) - \int_r^s [G * u(t,\cdot)]_x(\xi(t))\,dt\,. \tag{4.20}$$

Applying the same procedure as above to the set

$$\{(x,t)\ :\ r<t<s,\ \xi(t)<x<\xi(t)+\varepsilon\}\ ,$$

yields

$$u(s,\xi(s)+)\ \le\ u(r,\xi(r)+)-\int_r^s [G*u(t,\cdot)]_x(\xi(t))\,dt\,. \tag{4.21}$$

For any $t\in]a,b[$, we apply (4.20) and (4.21), first for $r=t$ and $s\in I\cap]t,b[$, then for $s=t$ and $r\in I\cap]a,t[$. This yields that $u(t,\xi(t)-)=u(t,\xi(t)+)$. Therefore $I=]a,b[$ and (4.17) holds. Hence, for any r and s in (a,b), (4.20) and (4.21) yield

$$v(s)\ =\ v(r)-\int_r^s [G*u(t,\cdot)]_x(\xi(t))\,dt\,,$$

which, in conjunction with (4.19) implies that $(\xi(\cdot),v(\cdot))$ are C^1 functions on $[a,b]$ which satisfy the system (4.18).

□

As defined in [35], backward characteristics $\xi_{(t,x)}(\cdot)$ are confined between a maximal and minimal backward characteristics, $\big($denoted by $\xi_{(t,x+)}(\cdot)$ and $\xi_{(t,x-)}(\cdot)\ \big)$. Relying on the above proposition and (4.14), we can obtain properties of generalized characteristics, associated with entropy weak solutions of the Burgers-Poisson equation, including the non-crossing property of two genuine characteristics.

Proposition 4.1.19. *Let u be an entropy weak solution to* (4.1)*. Then for any $(t,x)\in\,]0+\infty[\times\mathbb{R}$, the following holds:*

- *(i) The maximal and minimal backward characteristics $\xi_{(t,x\pm)}$ are genuine and thus the function $u\left(\tau,\xi_{(t,x\pm)}(\tau)\right)$ solves (4.18) for $\tau\in]0,t[$ with initial data $u(t,\xi_{(t,x\pm)}(t))$.*
- *(ii) [Non-crossing of genuine characteristics] Two genuine characteristics may intersect only at their endpoints.*
- *(iii) If $u(t,\cdot)$ is discontinuous at a point x, then there is a unique forward characteristic $\xi^{(t,x)}$ which passes though (t,x) and*

$$u\left(\tau,\xi^{(t,x)}(\tau)-\right)\ >\ u\left(\tau,\xi^{(t,x)}(\tau)+\right)\qquad\forall\tau\ge t\,.$$

Let us introduce some notation used throughout the rest of this chapter. Given u, an entropy weak solution to (4.1), we denote by $\mathcal{J}(t)=\{x\in\mathbb{R}:u(t,x-)>u(t,x+)\}$, the jump set of $u(t,\cdot)$ for any $t > 0$. For any $x \in \mathcal{J}(t)$, the base of the backward characteristic cone starting from (t,x) at time $s \in [0,t[$ is

$$I_{(t,x)}(s) \ := \ ]\xi_{(t,x-)}(s), \xi_{(t,x+)}(s)[. \tag{4.22}$$

For any $T > 0$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, we denote the open interval

$$A^T_{[z_1,z_2]}(s) \ := \ ]\xi_{(T,z_1)}(s), \xi_{(T,z_2)}(s)[\,. \tag{4.23}$$

By the non-crossing property, the set

$$\mathcal{A}^T_{[z_1,z_2]} \ := \ \left\{(s,x) \in [0,T] \times \mathbb{R} \ : \ x \in A^T_{[z_1,z_2]}(s)\right\} \tag{4.24}$$

confines all backward characteristics starting from (T,x) with $x \in]z_1, z_2[$. For any $0 < s < \tau \leq T$, we denote by

$$I^{\tau,T}_{[z_1,z_2]}(s) \ = \ \bigcup_{x \in A^T_{[z_1,z_2]}(\tau) \bigcap \mathcal{J}(\tau)} I_{(\tau,x)}(s). \tag{4.25}$$

Due to the no-crossing property of two genuine backward characteristics and the

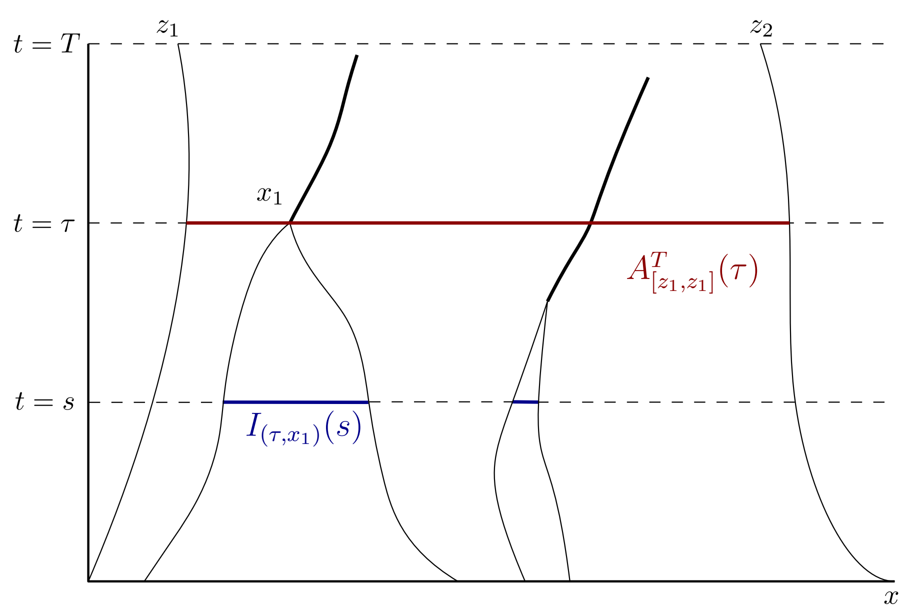

Figure 4.2 Characteristic curves in the xt-plane. Shocks (discontinuities) are denoted by bold black lines. The set $I^{\tau,T}_{[z_1,z_2]}(s)$ is the union of the blue intervals. By the non-crossing property, the intervals are disjoint.

uniqueness of forward characteristics in Proposition 4.1.19, the following holds:

Corollary 4.1.20. *Given $T > 0$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, the map $\tau \mapsto I^{\tau,T}_{[z_1,z_2]}(s)$ is increasing in the interval $]s, T]$ in the following sense*

$$I^{\tau_1,T}_{[z_1,z_2]}(s) \subseteq I^{\tau_2,T}_{[z_1,z_2]}(s) \qquad \forall 0 \leq s < \tau_1 \leq \tau_2 \leq T. \tag{4.26}$$

Moreover, for any $x \in A^T_{[z_1,z_2]}(\tau_1) \setminus I^{\tau_2,T}_{[z_1,z_2]}(\tau_1)$ with $0 < \tau_1 < \tau_2 < t$, the unique forward characteristic $\xi^{(\tau_1,x)}$ passing through (τ_1, x) is genuine in $[\tau_1, \tau_2]$.

Proof. Let $x \in \mathcal{J}(\tau_1) \cap A^T_{[z_1,z_2]}(\tau_1)$ and let $\chi(\cdot)$ be the unique forward characteristic emanating from (τ_1, x). By property (iii) of Proposition 4.1.19, for a fixed $\tau_2 \in [\tau_1, T]$ we have that $\chi(\tau_2) \in \mathcal{J}(\tau_2)$ and by the non-crossing property, $\chi(\tau_2) \in A^T_{[z_1,z_2]}(\tau_2)$. Since the backward characteristics that form the base of a characteristic cone are genuine, the non-crossing property implies that

$$I_{(\tau_1,x)}(s) \subseteq I_{(\tau_2,\chi(\tau_2))}(s) \subset A^T_{[z_1,z_2]}(s) \qquad \forall s \in [0, \tau_1]$$

yielding (4.26). The latter statement follows directly. $\square$

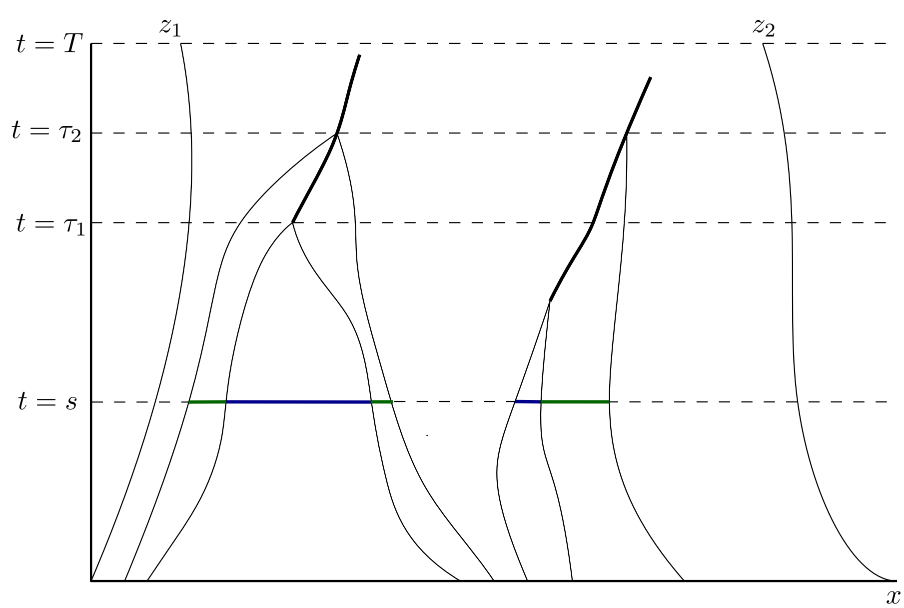

Figure 4.3 The union of intervals $I^{\tau_1,T}_{[z_1,z_2]}(s)$ is denoted by the solid blue lines. Due to the geometric properties of shock-free characteristics, the set $I^{\tau_2,T}_{[z_1,z_2]}(s)$ contains the previous intervals and potentially increases in size (denoted by the solid green lines).

4.2 SBV regularity of the Burgers-Poisson equation

In this section, we seek to prove Theorem 4.0.1 and show that the entropy weak solution u of (4.1) is in $SBV_{\text{loc}}(\mathbb{R})$ for but countably many time. First, we will recount some recent history on the connection between the space SBV and the regularity of hyperbolic conservation laws. It has been shown that admissible solutions of the general conservation law (2.15) do not preserve the SBV regularity of their initial condition (see [4],[38]). More precisely,

Remark 4.2.1. There exists a bounded Lipschitz initial data u_0 such that, if $\bar{u}$ denotes the unique entropy solution of (2.15)-(2.16) then $\bar{u}(1, \cdot)$ is a "Cantor-type" function which belongs to $\mathbf{L}^\infty \cap (BV \setminus SBV)$.

On the other hand, entropy solutions may be SBV functions when we consider them as a function of two variables. Indeed, an SBV function of two variables can have the Cantor ternary function as a trace on a given line. Consider an example of (2.15), where f is a linear function. For some constant a we have

$$u_t + au_x \;=\; 0\,. \tag{4.27}$$

The only distributional solution of (4.27) with the initial condition $u_0(\cdot)$ is $u(t,x) = u_0(x - at)$. Therefore, if u_0 is in the space $(BV \setminus SBV)$, then the solution u is also not in SBV. In order to get a SBV-regularization effect, we need a "sufficient nonlinearity" assumption on the flux function f.

The key conjecture, contributed to Bressan [38], is that: if u is an entropy solution of (2.15) with a convex flux function and, for a certain positive time τ, $u(\tau, \cdot)$ is not in SBV, then at future times $\tau + \varepsilon$ the "Cantor part" of $u(\tau, \cdot)$ gets transformed into jump singularities. From this idea, $u(t, \cdot)$ has been shown in a number of cases to be almost always in SBV. The first result was provided in [4]:

Theorem 4.2.2. *Assume $f \in C^2(\mathbb{R})$ and $f'' > 0$. Let u be a bounded entropy solution of* (2.15) *in a domain $\Omega \subset \mathbb{R} \times \mathbb{R}$. Then there exists a set $S \subset \mathbb{R}$ at most countable such that the following holds for all $\tau \notin S$*

$$u(\tau, \cdot) \in SBV(I) \qquad \textit{for every open interval} \quad I \subset\subset \Omega \cap \{t = \tau\}$$

Moreover, $u \in SBV(\Gamma)$ for every domain $\Gamma \subset\subset \Omega$.

In addition, the theorem was extended in [73] in two directions:

(i) the assumption $f'' > 0$ can be replaced by the discreteness of the set $\{f'' = 0\}$. In that case, they assume that the solution is BV since there are bounded entropy solutions with unbounded variation.

(ii) They allow for sufficiently regular source terms and flux functions depending on (x, t) and the value of the function $u(t, x)$, i.e. the result was extended to balance laws of the type

$$u_t + [f(u, x, t)]_x = g(u, x, t)\,.$$

Finally, the Theorem 4.2.2 has been extended to the case of hyperbolic systems in [[11, Theorem 1.1], which we present below. The initial steps in this direction were taken in [34] for self-similar solutions.

Theorem 4.2.3. *Consider a system of conservation laws in 1 space dimension*

$$u_t + [f(u)]_x = 0$$

coupled with initial data $u(0, \cdot) = u_0(\cdot)$. Assume that the system is strictly hyperbolic (i.e., the matrix $Df(u)$ has real and distinct eigenvalues for every u) and assume that each characteristic field is genuinely nonlinear (see Definition 5.2 of [13]). If the BV norm of u_0 is sufficiently small, then the same conclusions of Theorem 4.2.2 hold for the unique semigroup solution of the corresponding Cauchy problem.

The common theme of the proofs on these recent results of SBV regularity involve an appropriate geometric functional which has monotonicity properties and jumps at time $t > 0$ if $u(t, \cdot)$ does not belong to SBV (see e.g. in [4]). More precisely, let $\mathcal{J}(t)$ be the set of jump discontinuities of $u(t, \cdot)$. For each $x_j \in \mathcal{J}(t)$, there are minimal and maximal backward characteristics $\xi_j^-(s)$ and $\xi_j^+(s)$ emanating from (t, x_j) which define a nonempty interval $I_j(s) :=]\xi_j^-(s), \xi_j^+(s)[$ for any $s < t$. In this case, the functional $F_s(t)$ defined as the sum of the measures of $I_j(s)$ is monotonic and bounded. Relying on a careful study of generalized characteristics, we show that if the measure $Du(t, \cdot)$ has a non-vanishing Cantor part then the function F_s jumps up at time t which implies that the Cantor part is only present at countably many t.

Due to the nonlocal source, the details are more complicated than the above picture. In particular, the source term guarantees that $u(t, \cdot)$ does not possess compact support for any $t > 0$. Thus, we approach the domain by first looking at compact sets and then "glue" the sections together to recover the entire domain.

4.2.1 Proof of Theorem 4.0.1

Throughout this section, let $u : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ be the unique locally BV-weak entropy solution of (4.1) for some initial data $u_0 \in \mathbf{L}^1(\mathbb{R})$. The section aims to prove Theorem 4.0.1. For simplicity, denote the jump and Cantor parts of $Du(t,\cdot)$ by

$$\nu_t \;=\; D^j u(t,\cdot) \qquad \text{and} \qquad \mu_t \;=\; D^c u(t,\cdot) \qquad \text{for any} \;\; t \in]0,+\infty[$$

which, by (4.7), are both non-positive. We will show that $\mu_t(\mathbb{R}) < 0$ for at most countable positive times $t > 0$. In order to do so, let us first establish some basic bounds on backward characteristics.

Lemma 4.2.4. *For any given $0 < t_0 < t$ and $x_1 \le x_2$, let $\xi_i(\cdot)$ be a genuine backward characteristic starting from (t, x_i) and denote the solution along the characteristics by*

$$v_i(s) \;=\; u(s,\xi_i(s)) \quad \forall\, s \in [0,t], \;\; i \in \{1,2\}.$$

Then the followings hold:

$$|v_2(s) - v_1(s)| + |\xi_2(s) - \xi_1(s)| \;\le\; c_t(s)\cdot\big(|v_2(t) - v_1(t)| + |\xi_2(t) - \xi_1(t)|\big) \tag{4.28}$$

for all $s \in [0,t]$ and

$$\xi_2(t_0) - \xi_1(t_0) \;\ge\; \frac{x_2 - x_1 + (v_1(t_0) - v_2(t_0))\cdot(t - t_0)}{\Gamma_{[t_0,t]}} \tag{4.29}$$

with

$$\begin{cases} c_t(s) = \exp\left\{2\cdot\left(\sqrt{2K_t e^t\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + \left(e^t\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})} + 1\right)\cdot\sqrt{t}\right)\cdot\left(\sqrt{t} - \sqrt{s}\right)\right\}, \\[2ex] \Gamma_{[t_0,t]} = 1 + \left(\sqrt{\dfrac{2K_t e^t}{t_0}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + e^t\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot\dfrac{e^{K_t}t}{t_0}\cdot(t - t_0)^2. \end{cases} \tag{4.30}$$

Proof. **1.** Let us first proof (4.28). From Proposition 4.1.18, it holds that

$$\begin{cases} \dot{\xi}_i(s) &=\; v_i(s) \\[1ex] \dot{v}_i(s) &=\; [G * u(s,\cdot)]_x(\xi_i(s)) \end{cases} \qquad \forall s \in]0,t[, \;\; i \in \{1,2\}. \tag{4.31}$$

In particular, this implies that, for all $s \in]0,t[$,

$$\frac{d}{ds}\left|\xi_2(s)-\xi_1(s)\right| \;\geq\; -\left|v_2(s)-v_1(s)\right|$$

and

$$\frac{d}{ds}\left|v_2(s)-v_1(s)\right| \;\geq\; -\Big|[G*u(s,\cdot)]_x(\xi_2(s))-[G*u(s,\cdot)]_x(\xi_1(s))\Big|.$$

Since for all $s\in]0,t]$ it holds that $\xi_2(s)\geq\xi_1(s)$ and $u(s,\cdot)$ satisfies (4.7), we can apply Proposition 4.1.14 yielding

$$\begin{aligned}&\Big|[G*u(s,\cdot)]_x(\xi_2(s))-[G*u(s,\cdot)]_x(\xi_1(s))\Big|\\ &\qquad\leq\;\left(\sqrt{2K_s\left\|u(s,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})}}+\left\|u(s,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot\left|\xi_2(s)-\xi_1(s)\right|.\end{aligned} \tag{4.32}$$

Hence, (4.6) and (4.8) imply that

$$\begin{aligned}&\Big|[G*u(s,\cdot)]_x(\xi_2(s))-[G*u(s,\cdot)]_x(\xi_1(s))\Big|\\ &\qquad\leq\;\left(\sqrt{\frac{2K_te^t}{s}\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}}+e^t\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot\left|\xi_2(s)-\xi_1(s)\right|.\end{aligned} \tag{4.33}$$

Setting $M_t=\sqrt{2K_te^t\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}}+(e^t\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}+1)\cdot\sqrt{t}$, we have

$$\frac{d}{ds}\Big(\left|\xi_2(s)-\xi_1(s)\right|+\left|v_2(s)-v_1(s)\right|\Big)\;\geq\;-\frac{M_t}{\sqrt{s}}\cdot\Big(\left|\xi_2(s)-\xi_1(s)\right|+\left|v_2(s)-v_1(s)\right|\Big),$$

for all $s\in]0,t]$, and Grönwall's inequality yields (4.28).

2. In order to prove (4.29), we first apply (4.7) to (4.31) to get

$$\dot\xi_2(s)-\dot\xi_1(s)\;=\;u(s,\xi_2(s))-u(s,\xi_1(s))\;\leq\;\frac{K_t}{s}\cdot(\xi_2(s)-\xi_1(s)),$$

and this implies

$$\xi_2(s)-\xi_1(s)\;\leq\;\frac{e^{K_t}s}{t_0}\cdot(\xi_2(t_0)-\xi_1(t_0))\;\leq\;\frac{e^{K_t}t}{t_0}\cdot(\xi_2(t_0)-\xi_1(t_0))\quad\forall s\in[t_0,t]. \tag{4.34}$$

Therefore, from (4.31) and (4.33), it holds for $s \in [t_0, t]$ that

$$\begin{aligned} v_2(s) - v_1(s) &= v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} [G * u(\tau, \cdot)]_x(\xi_2(\tau)) - [G * u(\tau, \cdot)]_x(\xi_1(\tau)) \, d\tau \\ &\leq v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} \left(\sqrt{\frac{2K_t e^t}{t_0}} \|u_0\|_{\mathbf{L}^1(\mathbb{R})} + e^t \|u_0\|_{\mathbf{L}^1(\mathbb{R})} \right) \cdot \big(\xi_2(\tau) - \xi_1(\tau)\big) \, d\tau \\ &\leq v_2(t_0) - v_1(t_0) + \gamma_{[t_0,t]} \cdot (\xi_2(t_0) - \xi_1(t_0)) \end{aligned}$$

with

$$\gamma_{[t_0,t]} \;=\; \left(\sqrt{\frac{2K_t e^t}{t_0}} \|u_0\|_{\mathbf{L}^1(\mathbb{R})} + e^t \|u_0\|_{\mathbf{L}^1(\mathbb{R})} \right) \cdot \frac{e^{K_t} t}{t_0} \cdot (t - t_0).$$

Integrating the first equation in (4.31) over $[t_0, t]$, we get

$$\begin{aligned} \xi_2(t) - \xi_1(t) &= \xi_2(t_0) - \xi_1(t_0) + \int_{t_0}^{t} v_2(\tau) - v_1(\tau) \, d\tau \\ &\leq \; (v_2(t_0) - v_1(t_0)) \cdot (t - t_0) + \Big(1 + \gamma_{[t_0,t]} \cdot (t - t_0)\Big) \cdot (\xi_2(t_0) - \xi_1(t_0)) \end{aligned}$$

and this yields (4.29). □

As a consequence, we obtain the following two corollaries. Given $x \in \mathcal{J}(t)$, the jump set of $u(t, \cdot)$, the first corollary provides an upper bound on the base of the characteristic cone emanating from (t, x) at any time $s \in [0, t[$.

Corollary 4.2.5. *For any $(t, x) \in]0, +\infty[\times \mathcal{J}(t)$, it holds that*

$$\left| I_{(t,x)}(s) \right| \;\leq\; -c_t(s) \cdot \nu_t(\{x\}) \quad \forall s \in [0, t[. \tag{4.35}$$

Proof. Since $x \in \mathcal{J}(t)$, the inequality (4.14) implies that

$$\nu_t(\{x\}) \;=\; u(t, x+) - u(t, x-) \;<\; 0.$$

Thus, recalling (4.28), we obtain

$$\left| \xi_{(t,x+)}(s) - \xi_{(t,x-)}(s) \right| \;\leq\; c_t(s) \cdot |u(t, x+) - u(t, x-)|$$

and this yields (4.35). □

In the following corollary, we show that two distinct characteristics are separated for

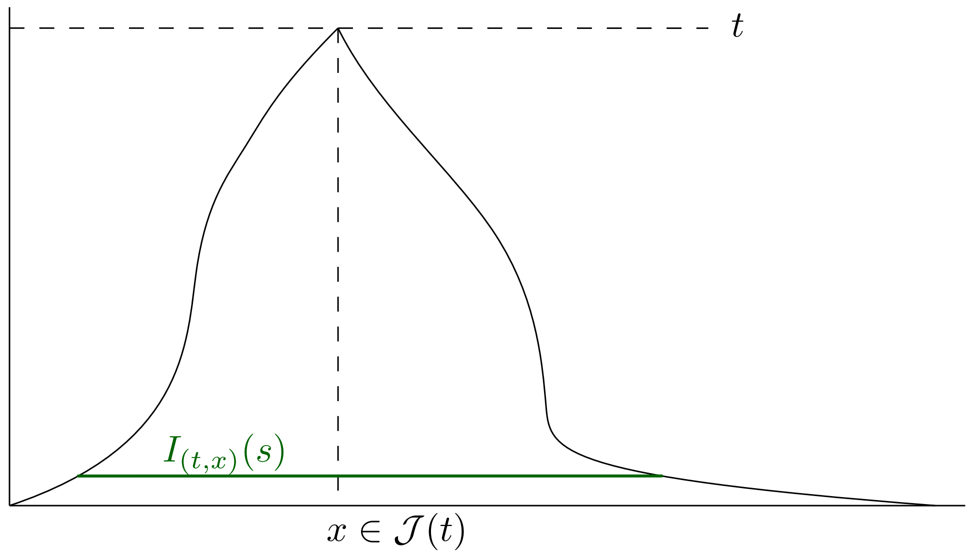

Figure 4.4 The interval between the minimal and maximal backwards characteristics emanating from a point (t, x) at time s is bounded by the size of the shock at the point (t, x)

all positive time; moreover, the distance between them is proportional to the difference in the values of the solution along the characteristics.

Corollary 4.2.6. *Given $x_1 < x_2$ and $0 < \sigma < t \le T$, let $\xi_i(\cdot)$ be a genuine backward characteristic starting from (t, x_i) and*

$$v_i(s) \;=\; u(s, \xi_i(s)) \quad \forall s \in [0, t[, \;\; i \in \{1, 2\}.$$

Then it holds that

$$\xi_2(\sigma/2) - \xi_1(\sigma/2) \;\ge\; \kappa_{[\sigma,T]} \cdot (v_1(t) - v_2(t)) \tag{4.36}$$

where, with $\Gamma_{[\sigma/2,T]}$ as defined in (4.30),

$$\kappa_{[\sigma,T]} \;=\; \frac{\sigma}{2}\left[\Gamma_{[\sigma/2,T]} + \left(\sqrt{\frac{4K_T e^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + e^T\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot e^{K_T} T \cdot (T - \sigma/2)\right]^{-1}.$$

Proof. Integrating the second equation in (4.18) over $[\sigma/2, t]$ yields

$$\begin{aligned} v_1(t) - v_2(t) \;&= v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} [G * u(\tau,\cdot)]_x(\xi_1(\tau)) - [G * u(\tau,\cdot)]_x(\xi_2(\tau))\, d\tau \\ &\le v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} \Big|[G * u(\tau,\cdot)]_x(\xi_2(\tau)) - [G * u(\tau,\cdot)]_x(\xi_1(\tau))\Big|\, d\tau \end{aligned}$$

and by (4.33)-(4.34) it holds that

$$v_1(t) - v_2(t) \leq v_1(\sigma/2) - v_2(\sigma/2) + \left(\sqrt{\frac{4K_T e^T}{\sigma}\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + e^T \|u_0\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot \frac{2e^{K_T}T}{\sigma} \cdot (T - \sigma/2) \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)) . \tag{4.37}$$

On the other hand, by (4.29) from Lemma 4.2.4, we have that

$$v_1(\sigma/2) - v_2(\sigma/2) \leq \frac{\Gamma_{[\sigma/2,t]}}{t - \sigma/2} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)) \leq \frac{2\Gamma_{[\sigma/2,T]}}{\sigma} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)).$$

which, when applied to (4.37), implies (4.36). □

We now seek to show that, for a certain positive time s, if $u(s, \cdot)$ is not in SBV, then at future times $s + \varepsilon$ the Cantor part of $u(s, \cdot)$ gets transformed into jump singularities. Let us recall the notation defined in (4.22)-(4.25). Following the main idea in [4, 73], for any $s \in]0, T[$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, we consider the set of points $E^T_{[z_1,z_2]}(s)$ in $A^T_{[z_1,z_2]}(s)$ where the Cantor part of $Du(s, \cdot)$ prevails, i.e.,

$$E^T_{[z_1,z_2]}(s) = \left\{ x \in A^T_{[z_1,z_2]}(s) : \lim_{\eta \to 0+} \frac{\eta + |Du(s, \cdot) - \mu_s|([x - \eta, x + \eta])}{-\mu_s([x - \eta, x + \eta])} = 0 \right\} . \tag{4.38}$$

Besicovitch differentiation theorem (Theorem 2.4.6) gives that the measure μ_s is concentrated on $E^T_{[z_1,z_2]}(s)$, i.e.

$$\mu_s \left(A^T_{[z_1,z_2]}(s) \setminus E^T_{[z_1,z_2]}(s) \right) = 0 ,$$

and

$$\lim_{\eta \to 0^+} \frac{u^-(s, x - \eta) - u^+(s, x + \eta)}{-\mu_s([x - \eta, x + \eta])} = 1 \qquad \forall x \in E^T_{[z_1,z_2]}(s). \tag{4.39}$$

Moreover, due to the continuity of μ_s, we have the following characterization of points in $E^T_{[z_1,z_2]}(s)$.

Proposition 4.2.7. *Given $s > 0$, for μ_s-a.e. x in $E^T_{[z_1,z_2]}(s)$, it holds*

$$\lim_{\eta \to 0} \frac{u(s, x + \eta) - u(s, x)}{\eta} = -\infty. \tag{4.40}$$

Proof. Fix $s > 0$, $m \in \mathbb{N}$ and define the set

$$K_m = \left\{ x \in E^T_{[z_1,z_2]}(s) \ : \ \lim_{\eta \to 0+} \frac{-\mu_s([x-\eta, x])}{\eta} \le m \right\} .$$

By the continuity of μ_s, there exists $\eta_0 > 0$ such that for any $\eta \in]0, \eta_0[$, there exists $\delta_\eta > 0$ such that

$$-\mu_s(]x-\eta, x+\delta_\eta[) \le 2m \cdot \eta \le 2m \cdot (\eta + \delta_\eta) \qquad \forall x \in K_m .$$

Let $\varepsilon > 0$. Then there exists $\eta_\varepsilon > 0$, $\delta_\varepsilon > 0$, and K_ε open such that

$$K_m \subset \bigcup_{x \in K_m}]x - \eta_\varepsilon, x + \delta_\varepsilon[\subset K_\varepsilon \quad \text{and} \quad \mathcal{L}(K_\varepsilon) < \varepsilon . \tag{4.41}$$

Furthermore, by the definition of K_m and the continuity of μ_s, η_ε and δ_ε can be chosen sufficiently small such that

$$-\mu_s(]x-\eta_\varepsilon, x+\delta_\varepsilon[) \le 2m \cdot (\eta_\varepsilon + \delta_\varepsilon) \qquad \forall x \in K_m . \tag{4.42}$$

It is clear that

$$-\mu_s(K_m) \le -\mu_s \left(\bigcup_{x \in K_m}]x - \eta_\varepsilon, x + \delta_\varepsilon[\right) \le -\sum_{x \in K_m} \mu_s \left( ]x - \eta_\varepsilon, x + \delta_\varepsilon[\right) ,$$

and by (4.42), it holds that

$$-\mu_s(K_m) \le 2m \sum_{x \in K_m} \mathcal{L} \left( ]x - \eta_\varepsilon, x + \delta_\varepsilon[\right) . \tag{4.43}$$

For each $x \in K_m$, denote the open balls

$$B_x := B\left(\frac{2x - \eta_\varepsilon + \delta_\varepsilon}{2}, \frac{\eta_\varepsilon + \delta_\varepsilon}{2} \right), \qquad B_{5x} := B\left(\frac{2x - \eta_\varepsilon + \delta_\varepsilon}{2}, 5 \cdot \frac{\eta_\varepsilon + \delta_\varepsilon}{2} \right) .$$

By the Vitali Covering Lemma, there exists a countable subcollection $K'_m \subset K_m$ such that

$$\bigcup_{x \in K_m}]x - \eta_\varepsilon, x + \delta_\varepsilon[= \bigcup_{x \in K_m} B_x \subset \bigcup_{x \in K'_m} B_{5x} ,$$

where B_x are mutually disjoint for every $x \in K'_m$. Hence,

$$2m \sum_{x \in K_m} \mathcal{L}\left(]x - \eta_\varepsilon, x + \delta_\varepsilon[\right) \;\le\; 2m \sum_{x \in K'_m} \mathcal{L}\left(B_{5x}\right) \;=\; 10m \sum_{x \in K'_m} \mathcal{L}\left(B_x\right) \;=\; 10m\mathcal{L}\left(\bigcup_{x \in K'_m} B_x\right).$$

Now, relying on (4.41), we have that

$$-\mu_s(K_m) \;=\; 10m\mathcal{L}\left(\bigcup_{x \in K'_m} B_x\right) \;\le\; 10m\mathcal{L}\left(K_\varepsilon\right) \;\le\; 10m \cdot \varepsilon\,.$$

Letting $\varepsilon \to 0$, we recover that $\mu_s(K_m) = 0$, and therefore it also holds that

$$\mu_s\left(\bigcup_{m \in \mathbb{N}} K_m\right) \;=\; 0\,.$$

The statement (4.40) follows from (4.39). $\square$

Lemma 4.2.8. *Let $0 < s < t \le T$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$ be fixed. Then, it holds for μ_s-a.e. $x \in A^T_{[z_1,z_2]}(s)$ that*

$$]x - \eta_x, x + \eta_x[\;\subset\; I^{t,T}_{[z_1,z_2]}(s) \qquad \text{for some } \eta_x > 0.$$

Proof. Since $I^{t,T}_{[z_1,z_2]}(s)$ is open, it is sufficient to prove that every point $x \in E^T_{[z_1,z_2]}(s) \setminus \mathcal{J}(s)$ satisfying (4.40) is in $I^{t,T}_{[z_1,z_2]}(s)$. Moreover, by the uniqueness property of forward characteristics in Proposition 4.1.19, it holds that $\partial\left(I^{t,T}_{[z_1,z_2]}(s)\right) = \partial\left(\overline{I^{t,T}_{[z_1,z_2]}(s)}\right)$. Assume by a contradiction that either

$$x \in A^T_{[z_1,z_2]}(s) \setminus \overline{I^{t,T}_{[z_1,z_2]}(s)} \qquad \text{or} \qquad x \in \partial(\overline{I^{t,T}_{[z_1,z_2]}(s)}).$$

1. If $x \in A^T_{[z_1,z_2]}(s) \setminus \overline{I^{t,T}_{[z_1,z_2]}(s)}$ then

$$]x - \eta_0, x + \eta_0[\bigcap \overline{I^{t,T}_{[z_1,z_2]}(s)} \;=\; \emptyset \qquad \text{for some } \eta_0 > 0. \tag{4.44}$$

Given any $\eta \in [0, \eta_0[$, let $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ be the unique forward characteristics emanating from $x - \eta$ and $x + \eta$ at time s. From Corollary 4.1.20, both $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ are genuine in $[s, t]$ and

$$\xi^\eta_2(\tau) - \xi^\eta_1(\tau) \;\ge\; 0 \qquad \forall \tau \in [s, t] \tag{4.45}$$

Thus, (4.29) in Lemma 4.2.4 implies

$$
\begin{aligned}
2\eta &= \xi_2^\eta(s) - \xi_1^\eta(s) \geq \frac{\xi_2^\eta(t) - \xi_1^\eta(t) + (u(s,x-\eta) - u(s,x+\eta)) \cdot (t-s)}{\Gamma_{[s,t]}} \\
&\geq -\frac{(u(s,x+\eta) - u(s,x-\eta)) \cdot (t-s)}{\Gamma_{[s,t]}}
\end{aligned}
$$

which yields a contradiction to (4.40) when η is sufficiently small.

2. Suppose that $x \in \partial(\overline{I^{t,T}_{[z_1,z_2]}(s)})$. In this case, $\xi_{(s,x)}(\cdot)$ is either a minimal or maximal backward characteristic in $[s,t]$. Moreover, for every $\eta > 0$ there exists $x_\eta \in]x - \eta, x[\bigcup]x, x+\eta[$ such that $x_\eta \notin \overline{I^{t,T}_{[z_1,z_2]}(s)}$ and the unique forward characteristics $\xi^{(s,x_\eta)}(\cdot)$ emanating from x_η at time s is genuine and does not cross $\xi_{(s,x)}(\cdot)$ in the time interval $[s,t]$. With the same computation in the previous step, we get

$$
\frac{u(s,x_\eta) - u(s,x)}{x_\eta - x} \geq -\frac{\Gamma_{[s,t]}}{t-s}
$$

and this also yields a contradiction to (4.40) when η is sufficiently small. $\square$

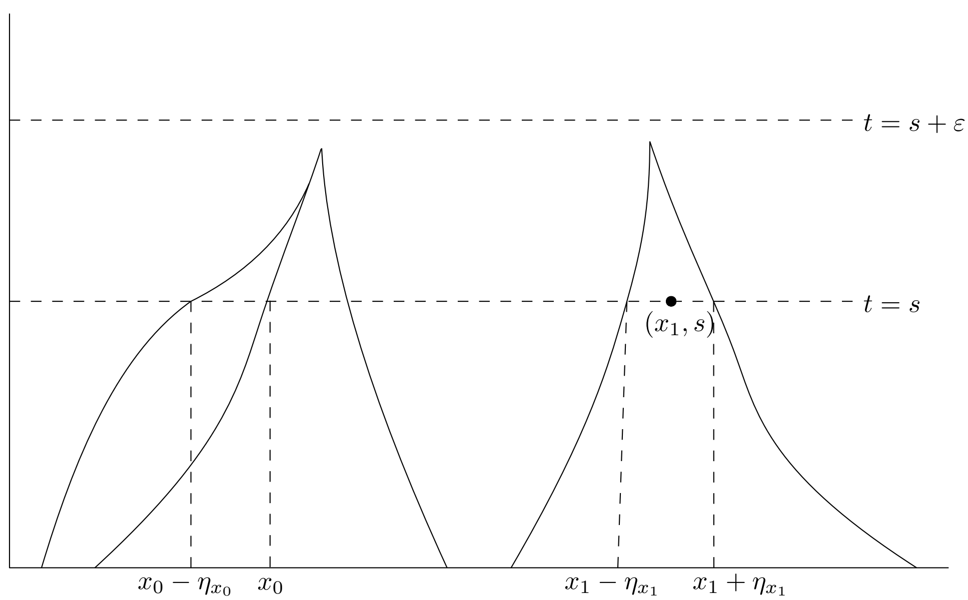

Figure 4.5 In the proof of Lemma 4.2.8, if a Cantor part is present at time s, two cases may occur. In case 1, at the point (x_1, s) there exist forward characteristics near x_1 that cross before time $s + \varepsilon$. In case 2, the forward characteristic from the point (x_0, s) is not a minimal characteristic if there is a Cantor part at x_0, as assumed.

We are now ready to prove the main theorem.
Proof of Theorem 4.0.1. The proof is divided into two steps:

Step 1. Fix $T > 0$ and $z_1, z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$ with $z_1 < z_2$ and, recalling (4.23)-(4.25) let

$$\mathcal{A} \;=\; \mathcal{A}^{T}_{[z_1,z_2]}, \qquad A_t \;=\; A^{T}_{[z_1,z_2]}(t) \qquad \text{and} \qquad I^{t}(s) \;=\; I^{t,T}_{[z_1,z_2]}(s)$$

for all $0 < s < t \leq T$. We claim that the set

$$\mathcal{T}_{[z_1,z_2]} \;:=\; \{t \in]0,T] \;:\; \mu_t\left(A_t\right) \text{ does not vanish}\}$$

is at most countable.

(i). Fix $\sigma \in]0,T[$. By Proposition 4.1.18 and (4.8), it holds that

$$|A_t| \;\leq\; |z_2 - z_1| + 2\sqrt{\frac{2K_T e^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\cdot T \qquad \forall t \in [\sigma, T],$$

and the Oleinik-type inequality (4.7) yields

$$|Du(t,\cdot)|\big(A_t\big) \;\leq\; M^{T}_{\sigma} \qquad \forall t \in [\sigma, T]$$

with

$$M^{T}_{\sigma} \;=\; 2\sqrt{\frac{2K_T e^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + \frac{2K_T}{\sigma}\cdot\left(|z_2 - z_1| + 2\sqrt{\frac{2K_T e^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\cdot T\right).$$

Let the geometric functional $F_\sigma : [\sigma, T] \to [0, +\infty[$ be defined by

$$F_\sigma(t) \;=\; \left|I^{t}(\sigma/2)\right| \;=\; \sum_{x \in \mathcal{J}(t) \bigcap A_t} \left|I_{(t,x)}\left(\sigma/2\right)\right| \qquad \forall t \in [\sigma, T]\,,$$

where the second equality follows by the non-crossing property. By Corollaries 4.1.20 and 4.2.5, the map $t \mapsto F_\sigma(t)$ is non-decreasing in $[\sigma, T]$ and uniformly bounded:

$$\sup_{t \in [\sigma,T]} F_\sigma(t) \;\leq\; c_T(\sigma/2)\cdot \sup_{t \in [\sigma,T]} \left(|\nu_t|(A_t)\right) \;\leq\; c_T(\sigma/2)\cdot M^{T}_{\sigma} \tag{4.46}$$

where $c_T(\sigma/2)$ is defined in (4.30).

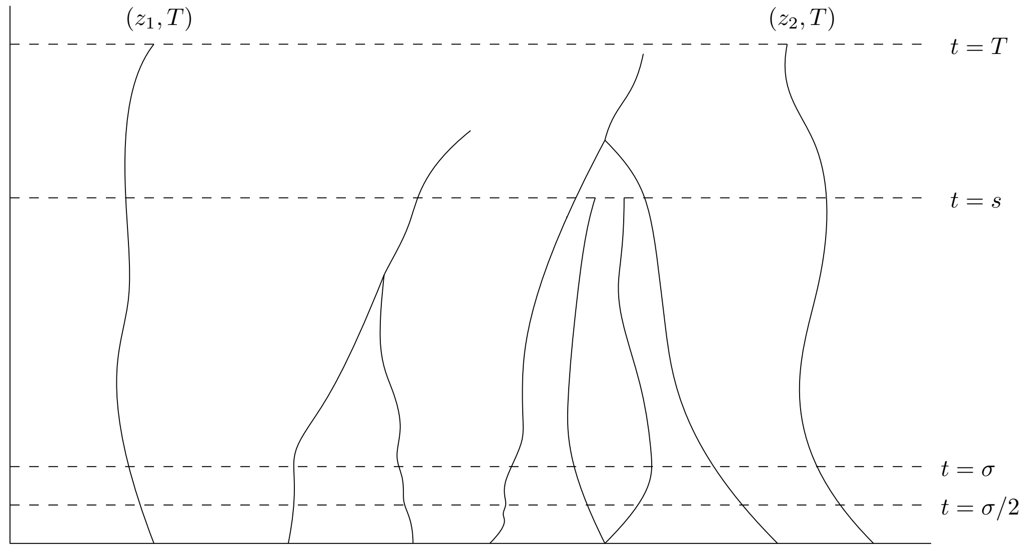

Figure 4.6 $F_\sigma(s)$ will have a jump discontinuity if a Cantor part is present at time s. Using Corollary 4.2.6, the size of the jump is estimated by the distance between the backward characteristics defined in Lemma 4.2.8.

(ii). Assume that a Cantor part is present in $\mathcal{A}$ at time $t \in]\sigma, T[$, i.e.,

$$\mu_t\left(A_t\right) \;\leq\; -\alpha \qquad \text{for some } \alpha > 0\,. \tag{4.47}$$

By (4.38) the Cantor part is concentrated on $E_t := E^T_{[z_1,z_2]}(t)$. We will show that

$$F_\sigma(t+) - F_\sigma(t) \;\geq\; \frac{\kappa_{[\sigma,T]}}{2} \cdot \alpha \tag{4.48}$$

where $\kappa_{[\sigma,T]}$ is defined in Corollary 4.2.6. It is sufficient to prove that

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \;=\; \left|I^{t+\varepsilon}(\sigma/2) \setminus I^t(\sigma/2)\right| \;\geq\; \frac{\kappa_{[\sigma,T]}}{2} \cdot \alpha$$

for any given $\varepsilon \in]0, T-t[$. By Lemma 4.2.8, for μ_t-a.e. $x \in E_t$ there exists $\eta_x > 0$ such that

$$]x - \eta_x, x + \eta_x[\;\subset\; I^{t+\varepsilon}(t). \tag{4.49}$$

On the other hand, given $x \in E_t$ we denote the interval

$$J^{\sigma/2}_{x,\eta} \;=\; \left]\xi_{(t,x-\eta)}\left(\sigma/2\right), \xi_{(t,x+\eta)}\left(\sigma/2\right)\right[,$$

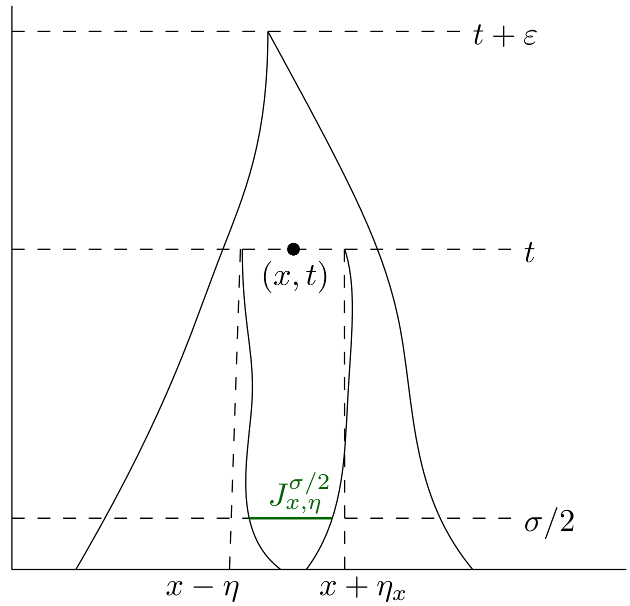

Figure 4.7 The interval $J^{\sigma/2}_{x,\eta}$ is contained in $I^{t+\varepsilon}(\sigma/2)$ and its magnitude is bounded below by the difference in solutions along the characteristics at time t.

for any $\eta \in \{\gamma > 0 \,:\, x \pm \gamma \notin \mathcal{J}(t)\}$. Corollaries 4.2.5 and 4.2.6 imply that

$$\begin{aligned} \left|J^{\sigma/2}_{x,\eta} \setminus I^t(\sigma/2)\right| &= \xi_{(t,x+\eta)}(\sigma/2) - \xi_{(t,x-\eta)}(\sigma/2) - \left|J^{\sigma/2}_{x,\eta} \cap I^t(\sigma/2)\right| \\ &\geq \kappa_{[\sigma,T]} \cdot (u(t,x-\eta) - u(t,x+\eta)) + c_T(\sigma/2)\nu_t(]x-\eta,x+\eta[)\,. \end{aligned}$$

Therefore, by (4.39) and the definition of E_t, there exists $\eta_0 > 0$ such that

$$\left|J^{\sigma/2}_{x,\eta} \setminus I^t(\sigma/2)\right| \;\geq\; -\frac{\kappa_{[\sigma,T]}}{2}\mu_t(]x-\eta,x+\eta[) \qquad \forall \eta \in]0,\eta_0]\,. \tag{4.50}$$

By the Besicovitch covering theorem (Theorem 2.4.5), we can cover μ_t-a.e. E_t with pairwise disjoint intervals $[x_j-\eta_j, x_j+\eta_j]$ where η_j is chosen such that both (4.49)and (4.50) hold. Proposition 4.1.19 (ii) implies that the intervals $J^{\sigma/2}_{x_j,\eta_j}$ are also pairwise disjoint and by (4.49) we have that $J^{\sigma/2}_{x_j,\eta_j}$ is contained in $A_{\sigma/2}$. Therefore, it holds that

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \;=\; \left|I^{t+\varepsilon}(\sigma/2) \setminus I^t(\sigma/2)\right| \;\geq\; \sum_j \left|J^{\sigma/2}_{x_j,\eta_j} \setminus I^t(\sigma/2)\right|\,.$$

Applying (4.50) and then (4.47) to the above inequality yields

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \;\geq\; -\frac{\kappa_{[\sigma,T]}}{2}\sum_j \mu_t\left([x_j-\eta_j, x_j+\eta_j]\right) \;\geq\; -\frac{\kappa_{[\sigma,T]}}{2}\mu_t\left(E_t\right) \;\geq\; \frac{\kappa_{[\sigma,T]}}{2}\alpha\,,$$

and therefore (4.48) holds.

(iii). By the monotonicity of F_σ and (4.46), F_σ has at most countable many discontinuities on $[\sigma, T]$. Thus, for any given $\sigma \in]0, T[$, (4.47)-(4.48) imply that the set

$$\bigcup_{n\in\mathbb{N}} \{t \in [\sigma, T] \ : \ \mu_t(A_t) \le -2^{-n}\} \ = \ \{t \in [\sigma, T] \ : \ \mu_t(A_t) < 0\}$$

is at most countable and therefore,

$$\bigcup_{n\in\mathbb{N}} \{t \in [2^{-n}, T] \ : \ \mu_t(A_t) < 0\} \ = \ \mathcal{T}_{[z_1,z_2]} \ \text{ is countable.}$$

Step 2. To complete the proof, it is sufficient to show that for any given $T > 0$, there exists an at most countable subset $\mathcal{T}_T$ of $[0, T]$ such that

$$u(t,\cdot) \ \in \ SBV_{\text{loc}}(\mathbb{R}) \qquad \forall t \in [0, T] \setminus \mathcal{T}_T\,. \tag{4.51}$$

For any $k \in \mathbb{Z}$, we pick a point $\bar{z}_k \in]k, k+1[\setminus \mathcal{J}(T)$. Let $\xi_k(\cdot)$ be the unique genuine backward characteristic starting at point $(T, \bar{z}_k)$ for every $k \in \mathbb{Z}$ and define

$$\mathcal{A}_k^T \ = \ \mathcal{A}^T_{[\bar{z}_k,\bar{z}_{k+1}]} \bigcup \{(\xi_k(t), t) : t \in [0, T]\} \qquad \text{and} \qquad A_k^T(t) \ = \ A^T_{[\bar{z}_k,\bar{z}_{k+1}]}(t) \bigcup \{\xi_k(t)\}\,.$$

Due to the no-crossing property of two genuine backward characteristics (Proposition 4.1.19 (ii)), it holds that

$$\bigcup_{k\in\mathbb{Z}} \mathcal{A}_k^T \ = \ [0, T] \times \mathbb{R} \qquad \text{and} \qquad \bigcup_{k\in\mathbb{Z}} A_k^T(t) \ = \ \mathbb{R} \qquad \forall t \in [0, T].$$

From Step 1, it holds that, for every $k \in \mathbb{Z}$, the set

$$\Big\{t \in [0, T] \ : \ \mu_t(A_k^T(t)) \ \neq \ 0\Big\} \ \text{ is countable.}$$

Hence,

$$\mathcal{T}_T \ = \ \Big\{t \in [0, T] \ : \ \mu_t(A_k^T(t)) \ \neq \ 0 \ \text{ for some } k \in \mathbb{Z}\Big\} \ \text{ is also countable,}$$

and this yields (4.51). □

To conclude this section, we show that Theorem 4.0.1 implies that $u \in SBV_{\text{loc}}$ as a function of two variables.

Corollary 4.2.9. *Let $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be the unique locally BV-weak entropy solution of (4.1) with initial data $u_0 \in \mathbf{L}^1(\mathbb{R})$. Then $u \in SBV_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$.*

Proof. Using the slicing theory of BV functions, we can recover the Cantor part of the 2-dimensional measure $D_x u$ from the 1-dimensional measure $D^c u(t, \cdot)$. In particular, for any $\Omega \subset\subset \mathbb{R}^+ \times \mathbb{R}$, by Theorem 4.0.1 and Theorem 2.5.9 it holds that

$$(D_x^c u)(\Omega) \;=\; \int_{\mathbb{R}^+} D^c u(t, \cdot)(\Omega_x(t))\, dt \;=\; 0 \qquad \text{where} \quad \Omega_x(t) = \{x \in \mathbb{R} \,:\, (t, x) \in \Omega\} \ ,$$

i.e., the measure $D_x u$ has no Cantor part. Since for all $t > 0$, Proposition 4.1.14 implies that $x \mapsto [G * u(t, \cdot)]_x(x)$ is continuous, we can apply the chain rule of Vol'pert [5, Theorem 3.96], to the Burgers-Poisson equation (4.1), yielding that the Cantor part of $D_t u$ vanishes as well. Therefore $u \in SBV_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$. □

BIBLIOGRAPHY

[1] M. Aguiar, M. Amador, E. Farhi, and G. Gopinath, Crisis and Commitment: Inflation Credibility and the Vulnerability to Sovereign Debt Crises, *National Bureau of Economic Research Working Paper Series*, No. 19516 (2013).

[2] M. Aguiar and G. Gopinath, Defaultable debt, interest rates and the current account, *Journal of International Economics*, Elsevier, vol. 69(1),(2006) 64–83.

[3] H. Amann, Invariant sets and existence theorems for semilinear parabolic and elliptic systems, *Journal of Mathematical Analysis and Applications*, 65(2):432–467, 1978.

[4] L. Ambrosio and C. De Lellis, A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations, *Journal of Hyperbolic Differential Equations* **1** (2004), 813–826.

[5] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, 2000.

[6] F. Ancona, O. Glass, and K. T. Nguyen, Lower compactness estimates for scalar balance laws, *Comm. Pure Appl. Math.* **65** (2012), 1303–1329.

[7] C. Arellano, Default Risk and Income Fluctuations in Emerging Economies, *American Economic Review*, American Economic Association, vol. 98(3), (2008) 690–712.

[8] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, Boston, (1997).

[9] R. Bellman, Dynamic Programming, *Princeton Univ. Press*, Princeton, NH, (1957).

[10] B. Benjamin, Internal waves of permanent form in fluids of great depth, *Journal of Fluid Mechanics*, **29** (03): 559, (1967).

[11] S. Bianchini and L. Caravenna, SBV regularity for genuinely nonlinear, strictly hyperbolic systems of conservation laws in one space dimension, *Comm. Math. Phys.* 313 (2012), 1–33.

[12] L. Bocola, and A. Dovis, Self-Fulfilling Debt Crises: A Quantitative Analysis, *American Economic Review*, 109 (2019), 4343–77.

[13] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-dimensional Cauchy Problem*, Oxford Lecture Series in Mathematics and its Applications, 2000.

[14] A. Bressan and Y. Jiang, The vanishing viscosity limit for a system of H-J equations related to a debt management problem, *Discrete and Continuous Dynamical Systems - Series S*, 11(5):793–824, 2018.

[15] A. Bressan, A. Marigonda, K.T. Nguyen, and M. Palladino, A Stochastic Model of Optimal Debt Management and Bankruptcy, *SIAM Journal on Financial Mathematics*, 8(1):841–873, 2017.

[16] A. Bressan and K.T. Nguyen, Global Existence of Weak Solutions for the Burgers-Hilbert Equation, *SIAM Journal on Mathematical Analysis*, **46** (2014), 2884–2904

[17] A. Bressan and K.T. Nguyen, An equilibrium model of debt and bankruptcy, *ESAIM: COCV*, 22(4):953–982, 2016.

[18] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York, New York, Springer, (2011).

[19] D. Brigo and F. Mercurio, Interest Rate Models: Theory and Practice, 2nd ed. *Springer-Verlag*, Berlin, (2006).

[20] G.A. Calvo, Servicing the Public Debt: The Role of Expectations, *The American Economic Review*, 78 (1988), 647–61 .

[21] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (11), pp. 1661–1664 (1993).

[22] A. Camous, and R. Cooper, Monetary Policy and Debt Fragility, *National Bureau of Economic Research Working Paper Series*, No. 20650 (2014).

[23] P. Cannarsa and C. Sinestrari, Semiconcave functions, Hamilton-Jacobi equations, and optimal control. *Progress in Nonlinear Differential Equations and Their Applications*, vol. 58 (2004).

[24] R. Capuani, S. Gilmore, and K.T. Nguyen A model of debt with bankruptcy risk and currency devaluation *Minimax Theory and its Application*, 5:251-274, 2020.

[25] I. Capuzzo-Dolcetta and P.L. Lions. Viscosity solutions of Hamilton-Jacobi-Bellman equations and state constraints, *Trans. A.M.S.*, **318** (1990).

[26] D.A. Carlson, A.B. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control : Deterministic and Stochastic Systems*, Berlin: Berlin ; New York : Springer-Verlag, (1991)

[27] H.L. Cole and T.J. Kehoe, Self-Fulfilling Debt Crises, *The Review of Economic Studies*, 67 (2000), 91–116.

[28] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations. *Acta Mathematica*, **181**(2) 229-243 (1998).

[29] A. Constantin, Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis, *CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 81. Society for Industrial and Applied Mathematics, pp. 241–73, (2011).

[30] M. Copelovitch, J. Frieden, and S. Walter, The Political Economy of the Euro Crisis, *Comparative Political Studies*. 49 (7): 811–840 (2016).

[31] G. Corsetti and L. Dedola, The Mystery of the Printing Press: Monetary Policy and Self-Fulfilling Debt Crises, *Journal of the European Economic Association*, 14 (2016), 1329–71 .

[32] M.G. Crandall and P.L. Lions, Conditions d'unicité pour les solutions generalises des equations d'Hamilton-Jacobi du premier ordre. *C.R. Acad. Sc. Paris Sér. I Math.*, 292:478-502, (1981).

[33] M.G. Crandall and P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277:1-42, 1983.

[34] C.M. Dafermos, Wave fans are special. *Acta Math. Appl. Sin. Engl. Ser. 24*, 3 (2008), 369–374.

[35] C.M. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws, *Indiana. U. Math. J.* **26** (1977), 1097–1119.

[36] C.M. Dafermos, Hyperbolic conservation laws in continuum physics, *Grundlehren der matematischen Wissenshaften* **325** Second edition (2005).

[37] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del Calcolo delle Variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl.* **82** (1988), 199–210.

[38] C. De Lellis, Hyperbolic equations and SBV functions, *Journées équations aux deŕivées partielles* , no. 6, 10 (2010)

[39] J. Eaton, and M. Gersovitz, Debt with Potential Repudiation: Theoretical and Empirical Analysis, *The Review of Economic Studies*, 48 (1981), 289–309

[40] L. C. Evans, Partial differential equations. *American Mathematical Society*, 2010.

[41] L. C. Evans, Classical solutions of the Hamilton-Jacobi-Bellman equation for uniformly elliptic operators, *Trans. A.M.S.*, **275** 245-255, (1983).

[42] L. C. Evans, R. F. Gariepy, *Measure Thoery and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Ann Arbor, (1992).

[43] H. Federer, *Geometric Measure Theory*, Springer-Verlang, Berlin, 1969.

[44] K. Fellner and C. Schmeiser, Burgers–Poisson: A nonlinear dispersive model equation, *SIAM Journal Applied Mathematics* **64** (2004), 1509–1525.

[45] W. H. Fleming, Optimal control of Markov processes, *Proc. Internat. Congress of Math.*, Warzawa, 71-84 (1983).

[46] W. H. Fleming, and R. W. Rishel, *Deterministic and stochastic optimal control*, Applications of mathematics. Springer-Verlag, 1975.

[47] W.H. Fleming, and H.M. Soner *Controlled Markov Processes and Viscosity Solutions* Stochastic Modeling and Applied Probability. Springer-Verlang, 2006.

[48] A. F. Filippov, Differential equation with discontinuous right-hand side, *Mat. Sb.* **51** (1960), 99–128. (English translation: *Amer. Math. Soc. Transl. Ser.* **42** (1964), 199–231)

[49] Financial Crisis Inquiry Commission, *The Financial Crisis Inquiry Report: Final Report of the National Commission on the Causes of the Financial and Economic Crisis in the United States*. Washington, D. C.: Government Printing Office, (2011).

[50] B. Fornberg and G.B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, *Philosophical Transactions of the Royal Society of London*. Series A, Mathematical and Physical Sciences, (1978), 289373–404.

[51] J. Frieden and S. Walter, Understanding the Political Economy of the Eurozone Crisis, *Annual Review of Political Science*. 20 (1): 371–390, (2017).

[52] S. Gilmore, and K. T. Nguyen, SBV Regularity for Burgers-Poisson Equation, *Journal of Mathematical Analysis and Applications*, 500 (2021).

[53] K. Grunert and K. T. Nguyen, On the Burgers-Poisson Equation, *Journal of Differential Equations* **23** (2016), no. 4, 23–46.

[54] S.V. Haziot, Wave Breaking for the Fornberg–Whitham Equation, *Journal of Differential Equations*, 263 (2017), 8178–85

[55] J. Holmes, Well-posedness of the Fornberg–Whitham equation on the circle, *Journal of Differential Equations* **260** (12), (2016).

[56] J. Holmes, and R.C. Thompson, Well-Posedness and Continuity Properties of the Fornberg–Whitham Equation in Besov Spaces, *Journal of Differential Equations*, 263 (2017), 4355–81

[57] G. Hörmann, Wave Breaking of Periodic Solutions to the Fornberg-Whitham Equation, *Discrete & Continuous Dynamical Systems*, 38, 1605–13 (2017)

[58] R. Ivanov, On the Integrability of a Class of Nonlinear Dispersive Wave Equations, *Journal of Nonlinear Mathematical Physics*, 12:4, 462-468, (2005) DOI: 10.2991/jnmp.2005.12.4.2

[59] P. Lax, Hyperbolic systems of conservation laws. *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[60] P. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves* Philadelphia: : SIAM, (1973)

[61] H.E. Leland, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *The Journal of Finance*, 49 (1994), 1213–5.

[62] H. Liu, Wave Breaking in a Class of Nonlocal Dispersive Wave Equations, *Journal of Nonlinear Mathematical Physics* **13** (2006), 441–466.

[63] E. Marconi, Regularity estimates for scalar conservation laws in one space dimension, *Journal of Hyperbolic Differential Equations*, **15** , no. 4, 623–691 (2018).

[64] A. Marigonda, and K.T. Nguyen, *A Debt Management Problem with Currency Devaluation*. 49 pages, submitted (https://arxiv.org/abs/1805.05043).

[65] P. I. Naumkin and I. A. Shishmarev, Nonlinear Nonlocal Equations in the Theory of Waves. Translations of Mathematical Monographs, vol. 133 American Mathematical Society, Providence, RI (1994).

[66] G. Nuño, and C. Thomas, *Monetary Policy and Sovereign Debt Vulnerability*. Ssrn, 2015.

[67] O. Oleinik, Discontinuous solutions of non-linear differential equations, *Amer. Math. Soc. Transl*, **26** (1963), 95–172.

[68] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Hochschultext / Universitext. Springer, 2003.

[69] H. Ono, Algebraic solitary waves in stratified fluids, *Journal of the Physical Society of Japan*, **39** (4): 1082–1091 (1975).

[70] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishechenko, The Mathematical Theory of Optimal Processes, Interscience, New York, (1962).

[71] F. Ramsey, A Mathematical Theory of Saving, *Economic Journal*, **38**, pp.543-549, (1928).

[72] R. Reis, The Mystique Surrounding the Central Bank's Balance Sheet, Applied to the European Crisis, *The American Economic Review*, 103 (2013), 135–40.

[73] R. Robyr, SBV regularity of entropy solutions for a class of genuinely nonlinear scalar balance laws with non-convex flux function, *J. Hyper. Differential Equations* **05** (2008) .

[74] R.L. Selger, A note on the breaking of waves. *Proc. R. Soc. Lond. Ser. A* **303**, 493-496 (1968)

[75] S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Number v. 11 in Springer Finance Textbooks Springer, 2004.

[76] S. Sundaresan, A Review of Merton's Model of the Firm's Capital Structure with Its Wide Application, *Annual Review of Financial Economics*, 5 (2013), 21–41.

[77] W. Walter, On the strong maximum principle for parabolic differential equations, *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 29, 1986.

[78] L. Wei, Wave Breaking Analysis for the Fornberg–Whitham Equation, *Journal of Differential Equations*, 265 (2018), 2886–96.

[79] G. B. Whitham, *Varational methods and applications to water waves*, Proc. R. Soc. Lond. Ser. A 299 (1967) 6-25.

[80] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.