
#### Abstract

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We study up- and down-operators which act on a vector space whose basis consists of the elements in some given poset $P$. In particular, we characterize several associative algebras generated by subsets of these operators when $P$ is one of the following: Young's lattice, absolute order on the classical Coxeter groups, the generalized noncrossing partition lattice on finite Coxeter groups, and Bruhat order on the dihedral group. We also present several new relations among the up-operators when $P$ is Bruhat order on the symmetric group.


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# Algebras of Up- and Down-operators on Posets 

by<br>Christian Alec Smith

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## APPROVED BY:

Nathan Reading
Seth Sullivant

## DEDICATION

To my family, for all they've done.

## BIOGRAPHY

The author was born in northern California and received a BS in mathematics from University of California Davis and an MS in mathematics from North Carolina State University.

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## CHAPTER



Our work focuses on two related sets of operators called up-operators and down-operators, both of which act on vector spaces generated by the elements of a chosen poset $P$ with some fixed edge labelling. We study these operators from an algebraic perspective and they can be used to describe properties pertaining to the structure of our poset $P$. In particular, if a full list of relations among these operators is known, then we can use them to reconstruct the poset as a whole or in part. This is particularly useful when working with larger posets since one can construct and study a portion of $P$ without having to construct the entire poset. Such partial constructions can be useful in, for instance, determining topological properties for $P$. As another example, when $P$ is given by weak order on the symmetric group $\mathfrak{S}_{n}$, the study of up-operators led Fomin and Stanley [10] to define the nil-Coxeter algebra, which they then used to prove new results for Schubert polynomials and to give simpler proofs for known results about them.

In this chapter we provide the reader with some background information on this subject and we summarize our results. In Chapter 2 we discuss up- and down-operators on Young's lattice. In Chapter 3 we discuss up-operators for absolute order and the generalized noncrossing partition lattice. Finally, in Chapter 4 we discuss up-operators for Bruhat order when the underlying Coxeter group is either the dihedral group or the symmetric group.

### 1.1 Defining the Operators

Let $(P, \leq)$ be a poset with edges labeled by elements in some index set $\mathscr{I}$ and let $\mathbf{C}[P]$ be the complex linear space with basis $P$. (By edges we mean covering relations in $P$.) If $p \lessdot q \in P$, then we let $\gamma(p, q)$ denote the label assigned to the edge between $p$ and $q$. We require that the edge labelling satisfies
$\gamma\left(p, q_{1}\right) \neq \gamma\left(p, q_{2}\right)$ whenever $p \lessdot q_{1}, q_{2}$ and $q_{1} \neq q_{2}$.
We define a set of $u p$-operators $U=\left\{u_{i}: i \in \mathscr{I}\right\}$ which act on $\mathbf{C}[P]$ in the following way. If $u_{i} \in U$, then

$$
u_{i}(p)= \begin{cases}q & \text { if } p \lessdot q \text { and } \gamma(p, q)=i \\ 0 & \text { otherwise }\end{cases}
$$

for all $p \in P$, extended linearly. We similarly define a down-operator $d_{i} \in D=\left\{d_{i}: i \in \mathscr{I}\right\}$ such that $d_{i}(p)=q$ if and only if $u_{i}(q)=p$ for all $p \in P$, extended linearly.

Example 1.1.1. Consider the symmetric group $S_{3}$ and recall that for right weak order we have $\sigma_{1} \lessdot_{R} \sigma_{2}$ if and only if $\sigma_{1}^{-1} \sigma_{2}=s_{i}$ where $\sigma_{1}, \sigma_{2} \in S_{n}$ and $s_{i}$ is the transposition which switches $i$ and $i+1$. We let $\gamma\left(\sigma_{1}, \sigma_{2}\right)=s_{i}$ where $\sigma_{1}^{-1} \sigma_{2}=s_{i}$. Below is the Hasse diagram for right weak order on $S_{3}$ with the edge labels colored blue.


Our up-operators are $u_{s_{1}}, u_{s_{2}}$ and our down-operators are $d_{s_{1}}, d_{s_{2}}$. The operator $u_{s_{1}}$ acts on the elements of $S_{3}$ as follows: $u_{s_{1}}(123)=213, u_{s_{1}}(231)=321, u_{s_{1}}(132)=312$, and $u_{s_{1}}(\sigma)=0$ for all other $\sigma \in S_{3}$.

Here we have looked at the case $n=3$ but it is well known that for general $n$ the algebra generated by the up-operators for right weak order on $S_{n}$ is the nil-Coxeter algebra [10], which, as mentioned earlier, is used in the study of Schubert polynomials. The defining relations for this algebra are

$$
\begin{array}{rlr}
u_{s_{i}}^{2} & =0, & \\
u_{s_{i}} u_{s_{j}} & =u_{s_{j}} u_{s_{i}} & \text { for }|i-j| \geq 2, \\
u_{s_{i}} u_{s_{i+1}} u_{s_{i}} & =u_{s_{i+1}} u_{s_{i}} u_{s_{i+1}} . &
\end{array}
$$

Our goal is to characterize the relations among the $u_{i}$ and $d_{i}$. In other words, we wish to understand the algebras generated by certain subsets of the $u_{i}$ and $d_{i}$. Such algebras allow us to understand the structural properties of the underlying poset $P$. Studies of these types of operators have led to proofs of a Littlewood-Richardson rule and Schur positivity for certain symmetric functions [12].

We now give some general statements about up-operators on $P$. We say that an $\mathscr{I}$-word is a tuple $x=\left(\rho_{1}, \ldots, \rho_{\ell}\right)$ such that $\rho_{i} \in \mathscr{I}$ for all $1 \leq i \leq \ell$. We denote by $u_{x}$ the monomial $u_{\rho_{1}} \ldots u_{\rho_{\ell}}$. When dealing with $P$ we will often need to refer to sequences of edge labels appearing in intervals
of the poset. As such, we give the following definition which allows us to use $\mathscr{I}$-words to refer to these edge label sequences.

Definition 1.1.2. Let $x=\left(\rho_{1}, \ldots, \rho_{\ell}\right)$ be an $\mathscr{I}$-word and let $v \leq w \in P$. We say that $x$ appears in $[\nu, w](\operatorname{or}[\nu, w]$ contains $x)$ if there exists some chain $v=v_{0} \lessdot v_{1} \lessdot \cdots \lessdot v_{\ell}=w$ in $[v, w]$ such that $\gamma\left(v_{i-1}, v_{i}\right)=\rho_{i}$ for $i \in[\ell]$.

The above definition simply means that $x$ appears in $[v, w]$ if the entries of $x$ occur (in order of their indices) as edge labels of a saturated chain. The proposition and corollary below are immediate from the definition of up-operators, but we give the proof for both in the interest of being thorough. In the following we consider the right action of $u_{x}$ on $P$ rather than the left action described earlier, that is, we consider $(v) u_{x}$ rather than $u_{x}(v)$ for $v \in P$. We do this because we study the right action in Chapter 4, where the following results will primarily be of use.

Proposition 1.1.3. Let $x$ be an $\mathscr{I}$-word and $v, w \in P$ such that $v \leq w$. Then $x$ appears in $[v, w]$ if and only if $(\nu) u_{x}=w$.

Proof. Let $x=\left(\rho_{1}, \ldots, \rho_{\ell}\right)$. If $x$ appears in $[v, w]$, then there exists a unique chain $v=v_{0} \lessdot v_{1} \lessdot \cdots \lessdot v_{\ell}=$ $w$ in $[\nu, w]$ such that $\gamma\left(v_{i-1}, v_{i}\right)=\rho_{i}$ for $i \in[\ell]$. Using this chain and the definition of up-operators we see that $(\nu) u_{\rho_{1}} \ldots u_{\rho_{i}}=v_{i} \neq 0$ for all $i \in[\ell]$. From this we have $(\nu) u_{x}=w$. Now suppose that $(\nu) u_{x}=w$. By the definition of up-operators we have that $(\nu) u_{\rho_{1}} \ldots u_{\rho_{i}}=v_{i} \neq 0$ for all $i \in[\ell]$ and so [ $\nu, w$ ] contains the chain $v=v_{0} \lessdot \nu_{1} \lessdot \cdots \lessdot v_{\ell}=w$ and $\gamma\left(v_{i-1}, \nu_{i}\right)=\rho_{i}$ for $i \in[\ell]$. From this we see that $x$ appears in $[v, w]$.

The above definition tells us that if $(v) u_{x}=w \neq 0$, then $x$ must correspond to a particular saturated chain in the interval $[\nu, w]$. (An analogous statement also holds for the down-operators.) Now let $\mathscr{U}$ be the associative algebra over $\mathbf{C}$ generated by the $u_{x}$ and let $I$ be the two-sided ideal of $\mathscr{U}$ containing all elements of $\mathscr{U}$ which annihilate all of $\mathbf{C}[P]$. We have the following corollary.
 $P$.

Proof. Let $x=\left(\rho_{1}, \ldots, \rho_{\ell}\right)$ and suppose that $u_{x} \neq 0(\bmod I)$. There must then exist some $v \in P$ such that $(\nu) u_{x}=w \neq 0$. By Proposition 1.1.3 we know that $x$ appears in the interval $[v, w]$. Now suppose that $x$ appears in the interval $[v, w]$ for some $v \leq w \in P$. We know by Proposition 1.1.3 that $(v) u_{x}=w \neq 0$ and so $u_{x} \neq 0(\bmod I)$.

Finally, we note that if our poset $P$ is graded with some rank function $\operatorname{rk}(\nu)$, then the algebra $\mathscr{U} / I$ is graded by degree. This is straightforward to see since we can only have $(v) u_{x}=w \neq 0$ if the length of $x$ is equal to $\operatorname{rk}(w)-\operatorname{rk}(v)$.

### 1.2 Results for Young's Lattice

Young's lattice, denoted $\mathbf{Y}$, is the poset of integer partitions ordered by inclusion of their Young diagrams. We take the edge labelling for $\mathbf{Y}$ to be as follows: if $\lambda \lessdot_{\mathbf{Y}} \mu$, then $\gamma(\lambda, \mu)=i$ where $\mu / \lambda$ is a box in the $i$ th column. The up- and down-operators for Young's lattice can be thought of as box-adding and box-subtracting operators, respectively. Specifically, the up-operators $u_{i}$ for $i \in \mathbf{N}$ act on a partition $\lambda$ by adding a box to the $i$ th column of $\lambda$ if the result is a partition and by sending $\lambda$ to 0 otherwise. Similarly, the down-operators $d_{i}$ act on $\lambda$ by subtracting a box from the $i$ th column if the result is a partition and by sending it to 0 otherwise. These operators were introduced as Schur operators by Fomin [11] and further discussed by Fomin and Greene [12] in the context of noncommutative Schur functions. They can also be seen as refinements of the raising and lowering operators $U$ and $D$ acting on Young's lattice as defined by Stanley [24] in his study of differential posets.

In [11] and [12, Example 2.6], the authors observe that the Schur operators satisfy the relations of the local plactic monoid/algebra with relations:

$$
\begin{aligned}
u_{i} u_{j} & =u_{j} u_{i} \quad \text { for }|j-i| \geq 2, \\
u_{i} u_{i+1} u_{i} & =u_{i+1} u_{i} u_{i}, \\
u_{i+1} u_{i+1} u_{i} & =u_{i+1} u_{i} u_{i+1} .
\end{aligned}
$$

However, they remark that the full set of relations satisfied by the $u_{i}$ is unknown. We describe the complete set of relations among the $u_{i}$ and thereby give a full characterization of the associative algebra generated by the $u_{i}$ which we call the algebra of Schur operators. This characterization is given in the following theorem and will be proven in Chapter 2. (This algebra was also characterized independently using different methods by Meinel in [21], where it is called the partic algebra and studied in relation to bosonic particle configurations.)

Theorem 1.2.1. The algebra of Schur operators is defined by the relations:

$$
\begin{aligned}
u_{i} u_{j} & =u_{j} u_{i} \quad \text { for }|j-i| \geq 2, \\
u_{i} u_{i+1} u_{i} & =u_{i+1} u_{i} u_{i}, \\
u_{i+1} u_{i+1} u_{i} & =u_{i+1} u_{i} u_{i+1}, \\
u_{i+1} u_{i+2} u_{i+1} u_{i} & =u_{i+1} u_{i+2} u_{i} u_{i+1} .
\end{aligned}
$$

Interestingly, this algebra is somewhat more complicated than a more common related one, also described in [12] (see also [6]), that is generated by diagonal box-adding operators $\tilde{u}_{i}$ that add a box to the $i$ th diagonal of $\lambda$ if possible (where the diagonals are labeled $1,2, \ldots$ from bottom to top). The algebra generated by such operators was shown in [6] to be the nil-Temperley-Lieb algebra given by
the relations:

$$
\begin{array}{rlr}
\tilde{u}_{i}^{2} & =0, & \text { for }|j-i| \geq 2, \\
\tilde{u}_{i} \tilde{u}_{j} & =\tilde{u}_{j} \tilde{u}_{i} & \\
\tilde{u}_{i} \tilde{u}_{i+1} \tilde{u}_{i} & =\tilde{u}_{i+1} \tilde{u}_{i} \tilde{u}_{i+1}=0 . &
\end{array}
$$

It was also noted in [11] (using the fact that the down-operators can be thought of as transposes of the up-operators) that the $d_{i}$ satisfy:

$$
\begin{array}{rlr}
d_{j} d_{i} & =d_{i} d_{j} & \text { for }|i-j| \geq 2, \\
d_{i} d_{i+1} d_{i} & =d_{i} d_{i} d_{i+1}, & \\
d_{i+1} d_{i} d_{i+1} & =d_{i} d_{i+1} d_{i+1}, &
\end{array}
$$

and that together the $u_{i}$ and $d_{i}$ satisfy:

$$
\begin{aligned}
d_{i} u_{j} & =u_{j} d_{i} \quad \text { for } i \neq j \\
d_{1} u_{1} & =i d \\
d_{i+1} u_{i+1} & =u_{i} d_{i}
\end{aligned}
$$

Where $i d$ is the identity operator.
We give a complete description of the associative algebra generated by the $u_{i}$ and $d_{i}$, which we call the algebra of up-and down-operators for Young's lattice. The following theorem shows that quadratic relations suffice to give a presentation of this algebra.

Theorem 1.2.2. The algebra of up- and down-operators for Young's lattice is defined by the relations:

$$
\begin{aligned}
u_{i} u_{j} & =u_{j} u_{i} & & \text { for }|i-j| \geq 2, \\
d_{i} d_{j} & =d_{j} d_{i} & & \text { for }|i-j| \geq 2, \\
d_{i} u_{j} & =u_{j} d_{i} & & \text { for } i \neq j, \\
d_{1} u_{1} & =i d, & & \\
d_{i+1} u_{i+1} & =u_{i} d_{i} . & &
\end{aligned}
$$

It follows that the local plactic relations are implied by the quadratic relations in Theorem 1.2.2. In contrast, we also give a complete description of the subalgebra generated by $u_{t}$ and $d_{t}$ for a fixed $t>1$ and show that it cannot be presented using relations of bounded degree.

### 1.3 Results for the Generalized Noncrossing Partition Lattice and Absolute Order

Let $W$ be a finite Coxeter group and let $w, w^{\prime} \in W$ (in this work $W$ will always represent a finite Coxeter group). If $T$ is the set of reflections in $W$, then the absolute length of $w$, denoted $\ell_{T}(w)$, is equal to the minimal $k$ such that $w=t_{1} \cdots t_{k}$ for some $t_{1}, \ldots, t_{k} \in T$. This length function $\ell_{T}$ allows us to define a poset on the elements of $W$. We call this poset the absolute order on $W$, denoted $\operatorname{Abs}(W)$, and it is defined as follows: we have $w \leq_{T} w^{\prime}$ if and only if $\ell_{T}\left(w^{\prime}\right)=\ell_{T}(w)+\ell_{T}\left(w^{-1} w^{\prime}\right)$. Absolute order was studied by Bessis [5] in the context of dual Coxeter systems and by Armstrong [1] in his study of the generalized noncrossing partition lattice. This poset was also studied by Kallipoliti [16] in the special case when $W=B_{n}$. For further reading on $\operatorname{Abs}(W)$ see [1].

Our chosen edge labelling for $\operatorname{Abs}(W)$ is as follows: if $w \lessdot_{T} w^{\prime}$, then $\gamma\left(w, w^{\prime}\right)=w^{-1} w^{\prime}$. We will see later that we always have $w^{-1} w^{\prime} \in T$ whenever $w \lessdot_{T} w^{\prime}$. From this we see that our up-operators for $\operatorname{Abs}(W)$ are the $u_{t}$ for $t \in T$. For absolute order we again consider the associative algebra, over $\mathbf{C}$, generated by the up-operators, which we call the algebra of up-operators for $\operatorname{Abs}(W)$. We study this algebra when $W=A_{n}, B_{n}$, and $D_{n}$. In particular, we prove the three theorems below.

Theorem 1.3.1. The algebra of up-operators for $\operatorname{Abs}\left(A_{n}\right)$ is generated by the following relations:

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}, \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}, \text { and } \\
u_{t} u_{t} & \equiv 0,
\end{aligned}
$$

where $t \neq t^{\prime} \in T$.
Theorem 1.3.2. The algebra of up-operators for $\operatorname{Abs}\left(B_{n}\right)$ is generated by the following relations:

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}, \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}, \\
u_{t} u_{t} & \equiv 0, \text { and } \\
u_{\left(\left(a, a^{\prime}\right)\right)} u_{\left.\left(\left(a,-a^{\prime}\right)\right)\right)} & \equiv u_{[a]} u_{\left[a^{\prime}\right]},
\end{aligned}
$$

where $a, a^{\prime}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ with distinct absolute values and $t \neq t^{\prime} \in T$.
Theorem 1.3.3. The algebra of up-operators for $\operatorname{Abs}\left(D_{n}\right)$ is generated by the following relations:

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}} \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t} \\
u_{t} u_{t} & \equiv 0, \text { and } \\
u_{\left(\left(a_{1}, a_{2}\right)\right)} u_{\left(\left(a_{1},-a_{2}\right)\right)} u_{\left(\left(a_{3}, a_{4}\right)\right)} u_{\left(\left(a_{3},-a_{4}\right)\right)} & \equiv u_{\left(\left(a_{1}, a_{3}\right)\right)} u_{\left(\left(a_{1},-a_{3}\right)\right)} u_{\left(\left(a_{2}, a_{4}\right)\right)} u_{\left(\left(a_{2},-a_{4}\right)\right),}
\end{aligned}
$$

where $a_{1}, a_{2}, a_{3}, a_{4}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ with distinct absolute values and $t \neq t^{\prime} \in T$.
Note that in Theorem 1.3.2 the last relation uses the indices $\left(\left(a, a^{\prime}\right)\right),\left(\left(a,-a^{\prime}\right)\right),[a]$, and $\left[a^{\prime}\right]$. These are reflections in $T$ represented using a combinatorial interpretation of $B_{n}$ as a group of signed permutations. Similarly, a combinatorial interpretation for reflections in $D_{n}$ is used in the last relation of Theorem 1.3.3. Both of these combinatorial interpretations will be discussed in Chapter 3.

We also study a subposet of $\operatorname{Abs}(W)$ known as the generalized noncrossing partition lattice for $W$, which is denoted by $N C(W)$. Before defining this subposet we must first define a special class of element in $\operatorname{Abs}(W)$ called a Coxeter element. If $\pi$ is an element of the symmetric group $\mathfrak{S}_{k}$ and $s_{1}, \ldots, s_{k}$ are all of the simple reflections of $W$, then we call an element $c \in W$ a standard Coxeter element if $c=s_{\pi(1)} \cdots s_{\pi(k)}$. A Coxeter element is then a conjugate of a standard Coxeter element. We can now define $N C(W)$ as the interval $[e, c]$ in $\operatorname{Abs}(W)$ where $e$ is the identity element of $\operatorname{Abs}(W)$ and $c$ is a Coxeter element of $\operatorname{Abs}(W)$. The generalized noncrossing partition lattice has been studied by multiple people, including: Kreweras [17], who defined the lattice when $W=\mathfrak{S}_{n}$, Reiner [22], who introduced the construction for $N C\left(B_{n}\right)$, Reiner and Athanasiadis [2] who gave a combinatorial interpretation of $N C\left(D_{n}\right)$, Bessis [5] in the context of dual Coxeter systems, and Armstrong [1] in his study of $k$-divisible noncrossing partitions.

Since $N C(W)$ is a subposet of absolute order on $W$, we take the edge labeling for $N C(W)$ to be the edge labelling of $\operatorname{Abs}(W)$ restricted to $N C(W)$. From this we see that our up-operators are the $u_{t}$ for $t \in T$. We call the associative algebra generated by the $u_{t}$ the algebra of up-operators for $N C(W)$ and characterize it in the following theorem.

Theorem 1.3.4. The algebra of up-operators for $N C(W)$ is defined by the following degree 2 relations:

$$
\begin{aligned}
& u_{t} u_{t^{\prime}} \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}, \\
& u_{t} u_{t^{\prime}} \equiv u_{t t^{\prime} t} u_{t}, \text { and } \\
& u_{r} u_{r^{\prime}} \equiv 0,
\end{aligned}
$$

where $\left(t, t^{\prime}\right) \in R_{T}(w)$ for some $w \in N C(W)$ and $r=r^{\prime}$ or $r r^{\prime} \notin N C(W)$.

### 1.4 Results for Bruhat Order

Our final poset of interest is Bruhat order on a finite Coxeter group $W$. Let $S$ be the set of simple reflections for $W$ and let $w \in W$. The standard length $\ell_{S}(w)$ is the minimal $k$ such that $w=s_{i_{1}} \cdots s_{i_{k}}$ for some $s_{i_{1}}, \ldots, s_{i_{k}} \in S$. As we did with absolute order, we can use this length function to define a poset, namely Bruhat order on $W$. The definition is given below.

Definition 1.4.1. Let $W$ be a finite Coxeter group and let $T$ be the set of reflections in $W$. If $v, w \in W$ we take $v \rightarrow w$ to mean that $\ell_{S}(\nu)<\ell_{S}(w)$ and $v^{-1} w \in T$. Bruhat order on $W$ is the partial ordering where $v \leq_{\mathfrak{B}} \mathrm{W}$ if and only if there exists some sequence $\nu_{1}, \ldots, v_{k}$ such that

$$
v=v_{1} \rightarrow v_{2} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_{k}=w .
$$

We denote Bruhat order on $W$ by $\mathfrak{B}(W)$.
Our edge labelling for this poset is as follows: if $w, w^{\prime} \in W$ such that $w \lessdot_{\mathfrak{B}} w^{\prime}$, then we take $\gamma\left(w, w^{\prime}\right)=w^{-1} w^{\prime}$. It will always be the case that $w^{-1} w^{\prime} \in T$ and so just as with absolute order, the up-operators for $\mathfrak{B}(W)$ are the $u_{t}$ for $t \in T$. We call the algebra generated by the $u_{t}$ the algebra of up-operators for $\mathfrak{B}(W)$. We study this algebra when $W$ is equal to the dihedral group $I_{2}(m)$ and when $W$ is equal to the symmetric group $\mathfrak{S}_{n}$.

The structure of $\mathfrak{B}\left(I_{2}(m)\right)$ is rather simple and the proof of the next theorem relies on this simplicity.

Theorem 1.4.2. The algebra of up-operators on $\mathfrak{B}\left(I_{2}(m)\right)$ is characterized by the following relations:

$$
\begin{array}{rlr}
u_{r_{m}} u_{r_{1}} & =\sum_{i=1}^{m-1} u_{r_{i}} u_{r_{i+1}}, \\
u_{r_{1}} u_{r_{m}} & =\sum_{i=1}^{m-1} u_{r_{i+1}} u_{r_{i}}, & \\
u_{r_{1}} u_{r_{i}} & =u_{r_{j}} u_{r_{1}}, & i+j=m+2 \text { and } i, j \notin\{1,2, m\}, \\
u_{r_{m}} u_{r_{i}} & =u_{r_{j}} u_{r_{m}}, & i+j=m \text { and } i, j \notin\{1, m-1, m\}, \\
u_{r_{i}}^{2} & =0, & 1 \leq i \leq m, \\
u_{r_{i}} u_{r_{j}} & =0, & i, j \notin\{1, m\} \text { and }|i-j| \geq 2, \\
u_{r_{1}} u_{r_{i}} u_{r_{i-1}} & =0, & 3 \leq i \leq m-1, \\
u_{r_{m}} u_{r_{i}} u_{r_{i+1}} & =0, & 2 \leq i \leq m-2, \\
u_{r_{2}} u_{r_{1}} u_{r_{i}} & =0, & 1 \leq i \leq m, \\
u_{r_{m-1}} u_{r_{m}} u_{r_{i}} & =0, & 1 \leq i \leq m, \\
u_{r_{i}} u_{r_{i-1}} u_{r_{i}} & =0, & 2 \leq i \leq m, \\
u_{r_{i}} u_{r_{i+1}} u_{r_{i}} & =0, & 1 \leq i \leq m-1 .
\end{array}
$$

The up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$ were studied by Fomin and Kirillov [13] in their study of the FominKirillov algebra $\mathscr{E}_{n}$. This is the algebra over $\mathbf{C}$ generated by the formal objects $[i j]$ where $i, j \in[n]$ and $i<j$ and subject to the relations

$$
\begin{aligned}
{[i j]^{2} } & =0, & \\
{[i j][j k] } & =[j k][i k]+[i k][i j], & i<j<k, \\
{[j k[[i j]} & =[i k][j k]+[i j][i k], & i<j<k, \text { and } \\
{[i j][k l] } & =[k l][i j], & \{i, j\} \cap\{k, l\}=\emptyset .
\end{aligned}
$$

Fomin and Kirillov determined that the up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$ give an unfaithful representation of $\mathscr{E}_{n}$ where the representation map is $[i j] \mapsto u_{(i, j)}$. As a result of this representation, the $u_{(i, j)}$ inherit
the following relations:

$$
\begin{array}{rlr}
u_{(i j)}^{2} & =0, & i<j<k \\
u_{(i j)} u_{(j k)} & =u_{(j k)} u_{(i k)}+u_{(i k)} u_{(i j)}, & i<j<k, \text { and } \\
u_{(j k)} u_{(i j)} & =u_{(i k)} u_{(j k)}+u_{(i j)} u_{(i k)}, & \{i, j\} \cap\{k, l\}=\emptyset .
\end{array}
$$

Since the representation is not faithful, the $u_{(i, j)}$ satisfy other relations, two of which were stated by Fomin and Kirillov as

$$
\begin{align*}
u_{(i j)} u_{(i k)} u_{(i j)} & =0 \text { and }  \tag{1.1}\\
u_{(j k)} u_{(i k)} u_{(j k)} & =0, \tag{1.2}
\end{align*}
$$

where $i<j<k$. The full set of relations among the $u_{(i, j)}$ remains unknown but we have found several new relations using computational methods. The set of reflections $T$ for $\mathfrak{S}_{n}$ are the cycles $(i, j)$ for $1 \leq i<j \leq n$. Using these cycles in place of a generic element $t$ of $T$ we state our new relations in the proposition below.

Proposition 1.4.3. Let $i<j<k<l<m$ be integers. The following relations hold modulo $I$.

$$
\begin{align*}
u_{(j l)} u_{(i l)} u_{(i k)} & \equiv 0,  \tag{1.3}\\
u_{(j l)} u_{(j k)} u_{(i l)} u_{(j l)} & \equiv 0,  \tag{1.4}\\
u_{(k l)} u_{(j l)} u_{(i k)} u_{(i j)} & \equiv 0,  \tag{1.5}\\
u_{(k m)} u_{(j l)} u_{(i m)} u_{(l m)} & \equiv u_{(k m)} u_{(l m)} u_{(j m)} u_{(i l)},  \tag{1.6}\\
u_{(j m)} u_{(l m)} u_{(i m)} u_{(k m)} & \equiv u_{(l m)} u_{(j l)} u_{(i m)} u_{(k m)} \tag{1.7}
\end{align*}
$$

## CHAPTER

## 2

## UP- AND DOWN-OPERATORS FOR YOUNG'S LATTICE

In this chapter we discuss the up- and down-operators for Young's lattice and in particular we characterize several different algebras generated by subsets of these operators. The contents of this chapter are taken from [19] and [20] which were written by the current author in collaboration with Liu. In section 2.1 we discuss necessary background information about Young's lattice, partitions, and Knuth equivalence. In Section 2.2 we characterize the algebra generated by the up-operators for Young's lattice. In Section 2.3 we characterize the algebra generated by the down-operators for Young's lattice. Finally, in Section 2.4 we characterize the algebra generated by both the up-operators and the down-operators for Young's lattice.

### 2.1 Preliminaries

### 2.1.1 Partitions

A partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ of $|\lambda|=\sum_{i} \lambda_{i}$ is a nonincreasing sequence of nonnegative integers. (We may add or delete trailing zeroes as convenient.) To each partition, we associate a Young diagram, which is a collection of left-aligned boxes with $\lambda_{1}$ boxes in the first row, $\lambda_{2}$ boxes in the second row, and so on. We also define the conjugate partition $\lambda^{\prime}$ to be the partition whose Young diagram is obtained from that of $\lambda$ by reflecting across its main diagonal.

The set of partitions forms a partially ordered set called Young's lattice $\mathbf{Y}=(\mathbf{Y}, \subseteq)$, where $\lambda \subseteq \mu$ if and only if the Young diagram of $\lambda$ fits inside the Young diagram of $\mu$ (or equivalently, $\lambda_{i} \leq_{\mathrm{Y}} \mu_{i}$ for
all $i$ ). In this partial order, $\mu$ covers $\lambda$ if and only if $\mu / \lambda$ is a single box. Here, $\mu / \lambda$ denotes the skew Young diagram obtained by deleting those boxes in $\mu$ that are also contained in $\lambda$. In the rest of this chapter we denote $\leq_{\mathbf{Y}}$ simply as $\leq$. Our chosen labeling for $\mathbf{Y}$ is as follows: if $\lambda \lessdot \mu$, then $\gamma(\lambda, \mu)=i$ where $\mu / \lambda$ is a box in the $i$ th column.

A semistandard Young tableau (SSYT) of shape $\lambda$ is formed by filling each box of the Young diagram of $\lambda$ with a positive integer such that the numbers are weakly increasing within a row (read from left to right) and strictly increasing within a column (read from top to bottom). A standard Young tableau (SYT) is a semistandard Young tableau of shape $\lambda$ with labels $1,2, \ldots,|\lambda|$.

The reading word $\operatorname{rw}(T)$ of a tableau $T$ is the word obtained by listing the entries of the tableau by rows from bottom to top, reading each row from left to right.

Example 2.1.1. Let $\lambda=(4,3,1)$. For the semistandard Young tableau
we have $\operatorname{rw}(T)=32231114$.
The weight of a tableau $T$ is the tuple $w(T)=\left(w_{1}(T), w_{2}(T), \ldots\right)$, where $w_{i}(T)$ is the number of occurrences of $i$ in $T$. We similarly define the weight $w(x)=\left(w_{1}(x), w_{2}(x), \ldots\right)$ of any word $x$ in the alphabet $\mathbf{N}=\{1,2, \ldots\}$. (Clearly $T$ and $\operatorname{rw}(T)$ have the same weight.)

### 2.1.2 Words in the Alphabet

Let $\mathbf{N}=\{1,2, \ldots\}, \overline{\mathbf{N}}=\{\overline{1}, \overline{2}, \ldots\}$, and $\Gamma=\mathbf{N} \cup \overline{\mathbf{N}}$. We refer to elements $1,2, \ldots$ of $\mathbf{N}$ as unbarred letters and elements $\overline{1}, \overline{2}, \ldots$ of $\overline{\mathbf{N}}$ as barred letters.

Let $x=x_{1} \cdots x_{\ell}$ be a word of length $\ell$ in the alphabet $\Gamma$. The weight of $x$ is the vector $w(x)=$ $\left(w_{1}(x), w_{2}(x), \ldots\right)$ where

$$
w_{i}(x)=(\text { the number of times } i \text { appears in } x)-(\text { the number of times } \bar{i} \text { appears in } x) .
$$

We also define the $\alpha$-vector of $x$ to be $\alpha(x)=\left(\alpha_{1}(x), \alpha_{2}(x), \ldots\right)$ where

$$
\alpha_{i}(x)=\max \left\{w_{i+1}(\tilde{x})-w_{i}(\tilde{x}) \mid \tilde{x} \text { is a suffix subword of } x\right\} .
$$

Here a suffix subword $\tilde{x}$ is a word of the form $\tilde{x}=x_{j} x_{j+1} \cdots x_{\ell}$ for some $1 \leq j \leq \ell+1$. When $j=\ell+1$, $\tilde{x}$ is the empty word, in which case $w_{i+1}(\tilde{x})=w_{i}(\tilde{x})=0$, so it follows that $\alpha_{i}(x) \geq 0$ for all $i$.

Example 2.1.2. Let $x=11 \overline{332} 32 \overline{1} 21$. Then $w(x)=(2,1,-1,0, \ldots)$ and $\alpha(x)=(2,0,1,0, \ldots)$. For instance, for $\alpha_{1}(x)=2$, the maximum value of $w_{2}(\tilde{x})-w_{1}(\tilde{x})$ first occurs when $\tilde{x}=2 \overline{1} 21$.

### 2.1.3 Defining the Operators for Young's Lattice

We will write $d_{i}=u_{\bar{i}}$ for all barred letters $\bar{i}$. For any word $x=x_{1} x_{2} \ldots x_{\ell}$ in the alphabet $\Gamma$, we define $u_{x}=u_{x_{1}} u_{x_{2}} \cdots u_{x_{\ell}}$. We also use the alternate notation $\mathrm{i}=u_{i}$ for $i \in \Gamma$. To avoid potential confusion in the future, we note now that $(\mathrm{i}+\mathrm{j})$ denotes $u_{i+j}$ and not the sum $u_{i}+u_{j}$.

Let $\mathbf{C}[\mathbf{Y}]$ be the complex vector space with basis $\mathbf{Y}$. We define the up- and down-operators for $\mathbf{C}[\mathbf{Y}]$ in the following way. For $\lambda \in \mathbf{Y}$ and $i \in \mathbf{N}$, we let

$$
u_{i}(\lambda)= \begin{cases}\mu & \text { if } \mu \in \mathbf{Y} \text { and } \mu / \lambda \text { is a single box in column } i, \\ 0 & \text { otherwise },\end{cases}
$$

and

$$
d_{i}(\lambda)= \begin{cases}\mu & \text { if } \mu \in \mathbf{Y} \text { and } \lambda / \mu \text { is a single box in column } i \\ 0 & \text { otherwise }\end{cases}
$$

Example 2.1.3. Let $\lambda=(3,1)$. Then $u_{2}(\lambda)=(3,2), d_{3} u_{2}(\lambda)=(2,2)$, but $d_{1} d_{3} u_{2}(\lambda)=0$ since subtracting a box from the first column does not yield a partition.

$$
\square \backslash \xrightarrow{u_{2}} \square \xrightarrow{d_{3}} \square \xrightarrow{d_{1}} 0
$$

Note that $u_{i}(\lambda)$ is either 0 or a partition that covers $\lambda$ in $\mathbf{Y}$, so we refer to $u_{i}$ as an up-operator, and similarly we call $d_{i}$ a down-operator. These operators were introduced by Fomin [11] under the name Schur operators.

The action of $u_{x}$ on partitions is determined by the weight and $\alpha$-vector of $x$ as follows.
Proposition 2.1.4. Let $x$ be a word and $\lambda \in \mathbf{Y}$. Then

$$
u_{x}(\lambda)= \begin{cases}\left(\lambda_{1}^{\prime}+w_{1}(x), \lambda_{2}^{\prime}+w_{2}(x), \ldots\right)^{\prime} & \text { if } \lambda_{i}^{\prime}-\lambda_{i+1}^{\prime} \geq \alpha_{i}(x) \text { for all } i, \\ 0 & \text { otherwise }\end{cases}
$$

Proof. We have $u_{x}(\lambda) \neq 0$ if and only if $u_{\tilde{x}}(\lambda)$ is a partition for each suffix subword $\tilde{x}$ of $x$. Fix some $\tilde{x}$ and suppose $\mu=u_{\tilde{x}}(\lambda) \neq 0$. We then have $\mu_{i}^{\prime}=\lambda_{i}^{\prime}+w_{i}(\tilde{x})$ for all $i$. The condition for $\mu$ to be a partition is that $\mu_{i}^{\prime} \geq \mu_{i+1}^{\prime}$ for all $i$, or equivalently

$$
\lambda_{i}^{\prime}+w_{i}(\tilde{x}) \geq \lambda_{i+1}^{\prime}+w_{i+1}(\tilde{x})
$$

Rearranging this gives

$$
\lambda_{i}^{\prime}-\lambda_{i+1}^{\prime} \geq w_{i+1}(\tilde{x})-w_{i}(\tilde{x}) .
$$

By the definition of $\alpha_{i}(x)$, these inequalities hold for all suffix subwords $\tilde{x}$ if and only if $\lambda_{i}^{\prime}-\lambda_{i+1}^{\prime} \geq$ $\alpha_{i}(x)$.

The following corollary then follows from Proposition 2.1.4.

Corollary 2.1.5. Let $x$ and $y$ be words. Then $u_{x}$ and $u_{y}$ act identically on $\mathbf{Y}$ if and only if $\alpha(x)=\alpha(y)$ and $w(x)=w(y)$.

Proof. The backwards implication is immediate from Proposition 2.1.4. For the forward direction, suppose $\alpha(x) \neq \alpha(y)$. Then we may assume without loss of generality that $\alpha_{j}(x)<\alpha_{j}(y)$ for some $j$. Taking $\lambda$ such that $\lambda_{i}^{\prime}-\lambda_{i+1}^{\prime}=\alpha_{i}(x)$, we have $u_{x}(\lambda) \neq 0=u_{y}(\lambda)$, so $u_{x}$ and $u_{y}$ do not act identically. If instead $\alpha(x)=\alpha(y)$ but $w(x) \neq w(y)$, then for this same choice of $\lambda, u_{x}(\lambda) \neq u_{y}(\lambda)$ by Proposition 2.1.4.

It was noted in [11] that $u_{i}$ and $d_{i}$ are transposes with respect to the basis $\mathbf{Y}$, which we may write as $u_{i}^{T}=d_{i}$. Also in [11], various relations among the $u_{i}$ and $d_{i}$ were described, including the local plactic relations and various quadratic relations.

### 2.1.4 Knuth Equivalence and RSK

Consider words $x=x_{1} x_{2} \ldots, y=y_{1} y_{2} \ldots$ in the alphabet $\mathbf{N}=\{1,2,3, \ldots\}$. We say that $x$ and $y$ are Knuth equivalent, denoted $x \stackrel{K}{\sim} y$, if one can be obtained from the other by applying a sequence of Knuth or plactic relations of the form

$$
\begin{array}{ll}
\ldots b a c \ldots{\underset{\sim}{K}}_{\sim}^{\sim} \ldots b c a \ldots & \text { for } a<b \leq c, \\
\ldots a c b \ldots & \text { for } a \leq b<c .
\end{array}
$$

Here, the ellipses indicate that the subwords occurring before and after the swapped letters remain unchanged. (The Knuth relations define the so-called plactic monoid [18], of which the local plactic monoid is a quotient.)

The Robinson-Schensted-Knuth (RSK) algorithm gives a bijection between words $x$ and pairs of tableaux $(P, Q)$ where the insertion tableau $P$ is semistandard, the recording tableau $Q$ is standard, and $P$ and $Q$ have the same shape. (See, for instance, [23] for more information.) The exact details of the RSK algorithm will not be important for us, as we will only need the following facts.

- The insertion tableau $P$ has the same weight as $x$.
- Two words $x$ and $y$ are Knuth equivalent if and only if they have the same insertion tableau $P$.
- For any semistandard tableau $P$, the insertion tableau of $\operatorname{rw}(P)$ is $P$.

For instance, these facts imply the following proposition. (Here and elsewhere, we use $i^{k}$ to denote a subword of the form $\underbrace{i i \ldots i}_{k}$.)
Proposition 2.1.6. Let $x$ be a word with minimum letter $i$, and let $k=w_{i}(x)$ be the number of occurrences of $i$ in $x$. Then $x$ is Knuth equivalent to a word $y=\ldots i^{k} \ldots$ in which all occurrences of $i$ are consecutive.

Proof. Let $P$ be the insertion tableau of $x$, and let $y=\operatorname{rw}(P)$. Since $x$ and $y$ both insert to $P$, we have $x \stackrel{K}{\sim} y$, and $y$ has the desired form since all $i$ 's appear next to each other in the first row of $P$.

### 2.2 The Algebra of Schur Operators

In this section we are only concerned with the up-operators on $\mathbf{Y}$, that is, we only study the operators $u_{i}$ for $i \in \mathbf{N}$. As such, we will assume for the rest of this section that the entries of any word $x$ come from $\mathbf{N}$ and not from all of $\Gamma$. Let $\mathscr{U}^{\mathbf{Y}}$ be the free associative algebra over the complex field $\mathbf{C}$ generated by the elements $u_{i}$ for $i \in \mathbf{N}$. Also, let $I^{\mathbf{Y}}$ be the two-sided ideal of $\mathscr{U}^{\mathbf{Y}}$ consisting of all elements that annihilate all of $\mathbf{C}[\mathbf{Y}]$. We call $\mathscr{U}^{\mathbf{Y}} / I^{\mathbf{Y}}$ the algebra of Schur operators. To improve readability, for the rest of this section we will denote $\mathscr{U}^{\mathbf{Y}}$ and $I^{\mathbf{Y}}$ by $\mathscr{U}$ and $I$ respectively. Our overall goal is to show that $I$ is generated by the local plactic relations (2.1)-(2.3) and one additional type of relation (2.4). In other words, we wish to prove the following theorem.

Theorem 2.2.1. The algebra of Schur operators is characterized by the following relations:

$$
\begin{array}{rlr}
u_{i} u_{j} & \equiv u_{j} u_{i} & \text { for }|j-i| \geq 2, \\
u_{i} u_{i+1} u_{i} & \equiv u_{i+1} u_{i} u_{i}, & \\
u_{i+1} u_{i+1} u_{i} & \equiv u_{i+1} u_{i} u_{i+1}, \\
u_{i+1} u_{i+2} u_{i+1} u_{i} & \equiv u_{i+1} u_{i+2} u_{i} u_{i+1} . & \tag{2.4}
\end{array}
$$

Let $J^{\mathbf{Y}}$ (hereafter denoted simply as $J$ ) be the two-sided ideal generated by relations (2.1)-(2.4). To prove Theorem 2.2.1 it is sufficient to prove that $J=I$ and so this will be our main task for this section.

### 2.2.1 Properties of $I$ and Equivalences of Monomials

We first verify that $I$ is a binomial ideal (i.e., generated by elements with one or two terms), so that Corollary 2.1.5 essentially determines all of the relations in $I$.

Proposition 2.2.2. The ideal $I$ is generated by elements of the form $u_{x}-u_{y}$ for words $x$ and $y$ such that $\alpha(x)=\alpha(y)$ and $w(x)=w(y)$.

Proof. Let $I^{\prime}$ be the ideal of $\mathscr{U}$ generated by $u_{x}-u_{y}$ as described above. By Corollary 2.1.5, we have $I^{\prime} \subseteq I$. Let $R$ be any element of $I$. Then $R \equiv R^{\prime}\left(\bmod I^{\prime}\right)$ for some

$$
R^{\prime}=\sum_{k} c_{k} u_{x(k)},
$$

where for each $k, x(k)$ is a word, $0 \neq c_{k} \in \mathbf{C}$, and $u_{x(k)} \neq u_{x\left(k^{\prime}\right)}\left(\bmod I^{\prime}\right)$ for $k \neq k^{\prime}$.
Fix some weight $w$ and let $x\left(k_{1}\right), x\left(k_{2}\right), \ldots$ be those words in $R^{\prime}$ for which $w(x(k))=w$, with $\alpha\left(x\left(k_{1}\right)\right), \alpha\left(x\left(k_{2}\right)\right), \ldots$ ordered lexicographically. We construct a partition $\lambda$ such that

$$
\alpha_{i}\left(x\left(k_{1}\right)\right)=\lambda_{i}^{\prime}-\lambda_{i+1}^{\prime} \text { for all } i .
$$

Proposition 2.1.4 gives $u_{x\left(k_{1}\right)}(\lambda) \neq 0$, but $u_{x\left(k_{j}\right)}(\lambda)=0$ for all $k_{j} \neq k_{1}$ since by the lexicographic
ordering, $\alpha_{i}\left(x\left(k_{j}\right)\right)>\alpha_{i}\left(x\left(k_{1}\right)\right)=\lambda_{i}^{\prime}-\lambda_{i+1}^{\prime}$ for some $i$. This then implies that $c_{k_{1}}=0$, which is a contradiction unless $R^{\prime}=0$. Thus $R \in I^{\prime}$ and so $I=I^{\prime}$.

We therefore need only determine relations that allow us to equate $u_{x}$ for all words $x$ with a fixed $\alpha(x)$ and $w(x)$.

Another useful fact about $I$ is that it satisfies a certain shift invariance.
Corollary 2.2.3. Let $x=x_{1} \ldots x_{l}$ and $y=y_{1} \ldots y_{l}$, and define $x^{\prime}=\left(x_{1}+1\right) \ldots\left(x_{l}+1\right)$ and $y^{\prime}=\left(y_{1}+\right.$ 1)... $\left(y_{l}+1\right)$. Then

$$
u_{x} \equiv u_{y}(\bmod I) \quad \text { if and only if } \quad u_{x^{\prime}} \equiv u_{y^{\prime}}(\bmod I) .
$$

Proof. Since

$$
\begin{array}{ll}
\alpha_{i}\left(x^{\prime}\right)=\alpha_{i-1}(x), & w_{i}\left(x^{\prime}\right)=w_{i-1}(x), \\
\alpha_{i}\left(y^{\prime}\right)=\alpha_{i-1}(y), & w_{i}\left(y^{\prime}\right)=w_{i-1}(y),
\end{array}
$$

for all $i$, the result follows by Corollary 2.1.5.
We now prove the first step needed to show that $I=J$, namely, that $J \subset I$.
Proposition 2.2.4. The relations (2.1)-(2.4) hold in $\mathscr{U} / I$, or equivalently, $J \subseteq I$.
Proof. By Corollary 2.2.3, we may take $i=1$. Thus by Corollary 2.1.5, we need only check that $w(x)=$ $w(y)$ and $\alpha(x)=\alpha(y)$ for the appropriate words on both sides of the relation. This is straightforward: for instance, for relation (2.4),

$$
\begin{gathered}
w(2321)=w(2312)=(1,2,1), \\
\alpha(2321)=\alpha(2312)=(1,0) .
\end{gathered}
$$

The other relations follows similarly.
We note the following relationship with Knuth equivalence.
Lemma 2.2.5. Let $x$ and $y$ be words such that $x \stackrel{K}{\sim} y$. Then $u_{x} \equiv u_{y}(\bmod J)$.
Proof. If $x$ and $y$ are related by a Knuth move that switches $a<c$, then $u_{x} \equiv u_{y}(\bmod J)$ by (2.1) if $|a-c| \geq 2$ or by (2.2) or (2.3) if $c=a+1$.

We next demonstrate that Knuth equivalence is sufficient to describe equivalence modulo $I$ for words in two letters $i$ and $i+1$.

Proposition 2.2.6. Let $x$ and $y$ be words in $i$ and $i+1$. Then $u_{x} \equiv u_{y}(\bmod I)$ if and only if $u_{x} \equiv u_{y}$ $(\bmod J)$.

Proof. We claim that the insertion tableau of $x$ is determined by $w(x)$ and $\alpha(x)$. Indeed, since $P$ is semistandard and contains only $i$ 's and $i+1$ 's, it has at most two rows, and $i$ can only appear in the first row. Then given $w(x)=w(P)$, all that needs to be determined is the number of $i+1$ 's in the first row. Since $x \stackrel{K}{\sim} \operatorname{rw}(P)$, Corollary 2.1.5 implies that $\alpha_{i}(x)=\alpha_{i}(\operatorname{rw}(P))$. But this is clearly the number of $i+1$ 's in the first row of $P$ (since $P$ has at least as many $i$ 's in its first row as $i+$ l's in its second row).

We can now prove the proposition. The reverse direction follows from Proposition 2.2.4, so suppose $u_{x} \equiv u_{y}(\bmod I)$. By Corollary 2.1.5, we have $\alpha(x)=\alpha(y)$ and $w(x)=w(y)$. Hence $x$ and $y$ must have the same insertion tableau by the above claim, so $x \stackrel{K}{\sim} y$. By Lemma 2.2.5, it then follows that $u_{x} \equiv u_{y}(\bmod J)$.

Note that we have shown that if our words only contain two consecutive letters, then only relations (2.2) and (2.3) are needed to determine equivalence modulo $I$.

### 2.2.2 Key Lemmas

When dealing with three or more letters, we will need to utilize relations (2.1) and (2.4). The following two lemmas will show the key contexts in which these relations will be used.

Denote by $x[i, j]$ the subword of $x$ consisting only of the letters $i, i+1, \ldots, j$. For instance, if $x=1432212$, then $x[1,2]=12212$, and $x[2,4]=43222$.

Lemma 2.2.7. Let $x$ and $y$ be words in $1, \ldots$, $n$. If $x[1,2]=y[1,2]$ and $x[2, n]=y[2, n]$, then $u_{x} \equiv u_{y}$ modulo relation (2.1), that is, they are equivalent up to commutation relations.

Proof. Note that $x[1,2]$ and $x[2, n]$ must have the same number of occurrences of 2 . Then

$$
\begin{aligned}
& x[1,2]=y[1,2]=\begin{array}{lllllllll}
1^{n_{1}} & 2 & 1^{n_{2}} & 2 & 1^{n_{3}} & 2 & \ldots & 2 & 1^{n_{k}}, \\
\text {, }
\end{array} \\
& x[2, n]=y[2, n]=m^{(1)} 2 m^{(2)} \quad 2 \quad m^{(3)} \quad 2 \quad \ldots \quad 2 \quad m^{(k)} \text {, }
\end{aligned}
$$

where $m^{(j)}$ is a word in $3, \ldots, n$ for all $j=1, \ldots, k$. Then we must have that

$$
\begin{array}{rlllllllll}
x & =m_{x}^{(1)} & 2 & m_{x}^{(2)} & 2 & m_{x}^{(3)} & 2 & \ldots & 2 & m_{y}^{(k)}, \\
y & =m_{y}^{(1)} & 2 & m_{y}^{(2)} & 2 & m_{y}^{(3)} & 2 & \ldots & 2 & m_{y}^{(k)},
\end{array}
$$

where $m_{x}^{(i)}$ and $m_{y}^{(i)}$ are both words obtained by shuffling together $1^{n_{i}}$ and $m^{(i)}$. But $u_{m_{x}^{(i)}}$ and $u_{m_{y}^{(i)}}$ are both equivalent modulo relation (2.1) to $u_{1}^{n_{i}} u_{m^{(i)}}$ and hence to each other. It follows that $u_{x}$ and $u_{y}$ are also equivalent modulo relation (2.1).

The next lemma shows the key application of relation (2.4). For ease of notation, we will abbreviate $u_{1}, u_{2}, \ldots$ by $1,2, \ldots$.

Lemma 2.2.8. For any positive integer $k$, we have the relations

$$
\begin{aligned}
u_{i+1} u_{i+2}^{k} u_{i+1} u_{i} & \equiv u_{i+1} u_{i+2}^{k} u_{i} u_{i+1}(\bmod J), \\
u_{i+1} u_{i+2} u_{i}^{k} u_{i+1} & \equiv u_{i+2} u_{i+1} u_{i}^{k} u_{i+1}(\bmod J) .
\end{aligned}
$$

Proof. We may assume $i=1$. Then (using the relations indicated)

$$
\begin{align*}
23^{k} 21 & \equiv 2323^{k-1} 1  \tag{2.3}\\
& \equiv 23213^{k-1}  \tag{2.1}\\
& \equiv 23123^{k-1}  \tag{2.4}\\
& \equiv 21323^{k-1}  \tag{2.1}\\
& \equiv 213^{k} 2  \tag{2.3}\\
& \equiv 23^{k} 12 \tag{2.1}
\end{align*}
$$

Similarly,

$$
\begin{align*}
231^{k} 2 & \equiv 21^{k} 32  \tag{2.1}\\
& \equiv 1^{k-1} 2132  \tag{2.2}\\
& \equiv 1^{k-1} 2312  \tag{2.1}\\
& \equiv 1^{k-1} 2321  \tag{2.4}\\
& \equiv 1^{k-1} 3221  \tag{2.2}\\
& \equiv 1^{k-1} 3212  \tag{2.3}\\
& \equiv 31^{k-1} 212  \tag{2.1}\\
& \equiv 321^{k} 2 \tag{2.2}
\end{align*}
$$

We now have our desired result.

### 2.2.3 Equivalence of Ideals

The following proposition provides the second to last step in proving that $I=J$.
Proposition 2.2.9. For words $x$ and $y, u_{x} \equiv u_{y}(\bmod I)$ if and only if $u_{x} \equiv u_{y}(\bmod J)$.
Proof. The reverse direction is proven in Proposition 2.2.4, so we need only consider the forward direction. We will induct on $n$, the largest letter appearing in $x$ and $y$. The case $n=1$ is trivial, while the case $n=2$ follows from Proposition 2.2.6.

Assume the statement holds for words in letters $1, \ldots, n-1$ (and hence for words in any $n-1$ consecutive letters by Corollary 2.2.3). Let $x$ and $y$ be words in letters $1, \ldots, n$ such that $u_{x} \equiv u_{y}$ $(\bmod I)$, so that, by Corollary 2.1.5, $w(x)=w(y)$ and $\alpha(x)=\alpha(y)$. Since $x$ and $y$ have the same number of 2's, we can construct a word $z$ in letters $1, \ldots, n$ such that $x[2, n]=z[2, n]$ and $y[1,2]=$ $z[1,2]$. We will then show $u_{x} \equiv u_{z}(\bmod J)$ and $u_{y} \equiv u_{z}(\bmod J)$, which will imply $u_{x} \equiv u_{y}(\bmod J)$.

By assumption we have $u_{x} \equiv u_{y}(\bmod I)$, and so by Corollary 2.1.5,

$$
\begin{aligned}
& u_{x[1,2]} \equiv u_{y[1,2]}=u_{z[1,2]}(\bmod I) \\
& u_{y[2, n]} \equiv u_{x[2, n]}=u_{z[2, n]}(\bmod I)
\end{aligned}
$$

By the inductive hypothesis, we then have

$$
\begin{align*}
u_{x[1,2]} & \equiv u_{z[1,2]} \quad(\bmod J)  \tag{2.5}\\
u_{y[2, n]} & \equiv u_{z[2, n]} \quad(\bmod J) \tag{2.6}
\end{align*}
$$

We therefore need to show that if $u_{x[1,2]} \equiv u_{z[1,2]}(\bmod J)$ as in $(2.5)$ and $x[2, n]=z[2, n]$, then $u_{x} \equiv u_{z}(\bmod J)$, and similarly for $y$ and $z$ as in (2.6). It suffices to check when the two sides of (2.5) or (2.6) differ by a single application of one of the relations (2.1)-(2.4).

First suppose the relation $u_{m} \equiv u_{m^{\prime}}$ used in (2.5) involves at most one $u_{2}$. This will be the case unless we are applying (2.3) with $i=1$. Note that $m$ may not be a consecutive subword inside $x$ because there may be letters $i>2$ that occur in between the letters of $m$ in $x$. However, by Lemma 2.2.7, since there is only one occurrence of 2 in $m$, we can commute these intervening letters to the left or right to get some $u_{x^{\prime}}$ equivalent to $u_{x}$ such that $x^{\prime}$ has $m$ as a consecutive subword. Replacing $m$ with $m^{\prime}$ in $x^{\prime}$ then gives a word $z^{\prime}$ such that $z^{\prime}[1,2]=z[1,2]$ and $z^{\prime}[2, n]=z[2, n]$. Hence

$$
u_{x} \equiv u_{x^{\prime}}=\ldots u_{m} \ldots \equiv \ldots u_{m^{\prime}} \ldots=u_{z^{\prime}} \equiv u_{z}
$$

A similar argument holds if the relation $u_{m} \equiv u_{m^{\prime}}$ used in (2.6) involves at most one $u_{2}$. This will be the case unless we are applying (2.2) with $i=2$. Hence it remains to check only these remaining two cases.

Suppose in equivalence (2.5) we are applying (2.3) with $i=1$ by replacing $221 \equiv 212$. As above, 221 and 212 need not appear consecutively inside $x$ and $z$ since there may be intervening letters $i>2$. However, we may as above commute any such letters not appearing between the 2 's to the right to get words $x^{\prime}$ and $z^{\prime}$ such that

$$
\begin{aligned}
x^{\prime} & =\ldots 2 m 21 \ldots \\
z^{\prime} & =\ldots 2 m 12 \ldots
\end{aligned}
$$

where $m$ is a word in $3, \ldots, n$.
By Proposition 2.1.6, $m \stackrel{K}{\sim} m^{\prime} 3^{k} m^{\prime \prime}$ for some words $m^{\prime}$ and $m^{\prime \prime}$ in letters $4, \ldots, n$. Lemma 2.2.5
then gives $u_{m} \equiv u_{m^{\prime}} 3^{k} u_{m^{\prime \prime}}(\bmod J)$. We then have:

$$
\begin{align*}
u_{x} & \equiv \ldots 2 u_{m} 21 \ldots  \tag{Lemma2.2.7}\\
& \equiv \ldots 2 u_{m^{\prime}} 3^{k} u_{m^{\prime \prime}} 21 \ldots  \tag{Lemma2.2.5}\\
& \equiv \ldots u_{m^{\prime}} 23^{k} 21 u_{m^{\prime \prime}} \ldots  \tag{2.1}\\
& \equiv \ldots u_{m^{\prime}} 23^{k} 12 u_{m^{\prime \prime}} \ldots  \tag{2.1}\\
& \equiv \ldots 2 u_{m^{\prime}} 3^{k} u_{m^{\prime \prime}} 12 \ldots \\
& \equiv \ldots 2 u_{m} 12 \ldots \\
& \equiv u_{z}
\end{align*}
$$

$$
\equiv \ldots u_{m^{\prime}} 23^{k} 12 u_{m^{\prime \prime}} \ldots \quad(\text { Lemma 2.2.8 })
$$

(Lemma 2.2.5)
(Lemma 2.2.7).

Similarly, if in equivalence (2.6) we are applying (2.2) with $i=2$ by replacing $232 \equiv 322$, then we may commute out any l's not appearing between the 2's to get:

$$
\begin{aligned}
u_{z} & \equiv \ldots 231^{k} 2 \ldots & & (\text { Lemma 2.2.7) } \\
& \equiv \ldots 321^{k} 2 \ldots & & (\text { Lemma 2.2.8) } \\
& \equiv u_{y} & & (\text { Lemma 2.2.7) }
\end{aligned}
$$

We can now give the proof of Theorem 2.2.1.
Proof. By Proposition 2.2.2, $I$ is generated by elements of the form $u_{x}-u_{y}$. Proposition 2.2.9 then shows that the above relations generate all such elements.

Example 2.2.10. Consider the words

$$
\begin{array}{lll}
x=23443231, & x[1,2]=221, & x[2,4]=2344323 \\
y=23443132, & y[1,2]=212, & y[2,4]=2344332 .
\end{array}
$$

Using the construction described in Proposition 2.2.9, we consider the word

$$
z=23443123, \quad z[1,2]=212=y[1,2], \quad z[2,4]=2344323=x[2,4] .
$$

Note that $x[1,2]$ and $z[1,2]$ differ by a single application of (2.3) with $i=1$, but these subwords do not appear consecutively within $x$ or $z$. As in the proof of Proposition 2.2.9, we can rewrite the part of $x$ and $z$ between the 2 's using the Knuth moves $3443 \stackrel{K}{\sim} 3434 \sim \underset{\sim}{\sim} 4334$ to get the 3 's in the middle so that we can then use commutations to get a consecutive subword of the form $23^{k} 21$. We then use

Lemma 2.2.8, followed by the reverse of the previous procedure:

$$
\begin{align*}
u_{x}=23443231 & \equiv 24334231  \tag{Lemma2.2.5}\\
& \equiv 42332143  \tag{2.1}\\
& \equiv 42331243 \\
& \equiv 24334123  \tag{2.1}\\
& \equiv 23443123=u_{z}
\end{align*}
$$

(Lemma 2.2.8)
(Lemma 2.2.5).

Since $y[2,4]$ and $z[2,4]$ differ by a relation that only involves a single 2 , we need only use commutations before we can apply the appropriate relation (2.3):

$$
\begin{align*}
u_{z}=23443123 & \equiv 23441323  \tag{2.1}\\
& \equiv 23441332  \tag{2.3}\\
& \equiv 23443132=u_{y} \tag{2.1}
\end{align*}
$$

### 2.3 The Algebra of Down-Operators for Young's Lattice

Now that we have characterized the algebra generated solely by the up-operators on $\mathbf{Y}$, it is natural to also consider the algebra generated solely by the down-operators on $\mathbf{Y}$. Let $\mathscr{D}^{\mathbf{Y}}$ be the associative algebra over $\mathbf{C}$ generated by $d_{i}$ for $i \in \mathbf{N}$ (or equivalently by $u_{\bar{i}}$ for $\bar{i} \in \overline{\mathbf{N}}$ ), and let $I_{\mathscr{O}}^{\mathrm{Y}}$ be the two-sided ideal of $\mathscr{D}^{\mathbf{Y}}$ containing all elements which annihilate all of $\mathbf{C}[\mathbf{Y}]$. The algebra of down-operators for Young's lattice is then $\mathscr{D}^{\mathbf{Y}} / I_{\mathscr{D}}^{\mathrm{Y}}$. Recall that with respect to the basis $\mathbf{Y}$, we have $u_{i}^{T}=d_{i}$. Applying this transpose property to the relations in Theorem 2.2.1 gives the following characterization of $\mathscr{D}^{\mathbf{Y}} / I_{\mathscr{D}}^{\mathbf{Y}}$.

Theorem 2.3.1. The algebra of down-operators for Young's lattice is characterized by the following relations:

$$
\begin{aligned}
d_{i} d_{j} & \equiv d_{j} d_{i} \quad \text { for }|i-j| \geq 2, \\
d_{i} d_{i+1} d_{i} & \equiv d_{i} d_{i} d_{i+1}, \\
d_{i+1} d_{i} d_{i+1} & \equiv d_{i} d_{i+1} d_{i+1}, \\
d_{i} d_{i+1} d_{i+2} d_{i+1} & =d_{i+1} d_{i} d_{i+2} d_{i+1} .
\end{aligned}
$$

### 2.4 The Algebra of Up- and Down-Operators for Young's Lattice

Our final goal is to characterize the algebra generated by both the up-operators and down-operators on $\mathbf{Y}$. Let $\mathscr{U}_{\mathscr{D}}^{\mathbf{Y}}$ be the associative algebra over $\mathbf{C}$ generated by $u_{i}$ and $d_{i}$ for $i \in \mathbf{N}$ and let $I_{\mathscr{U}, \mathscr{D}}^{\mathbf{Y}}$ be the two-sided ideal of $\mathscr{U}_{\mathscr{D}}^{\mathbf{Y}}$ containing all elements which annihilate all of $\mathbf{C}[\mathbf{Y}]$. We call $\mathscr{U}_{\mathscr{D}}^{\mathbf{Y}} / I_{\mathscr{U}, \mathscr{\mathscr { P }}}^{\mathbf{Y}}$ the algebra of up-and down-operators for Young's lattice. For ease of notation we will refer to $\mathscr{U}_{\mathscr{O}}^{\mathbf{Y}}$ and $I_{\mathscr{U}, \mathscr{D}}^{\mathrm{Y}}$ as $\mathscr{U}$ and $I$ respectively. Our focus for this section will be to prove the following theorem.

Theorem 2.4.1. The algebra of up- and down-operators for Young's lattice is characterized by the following relations:

$$
\begin{align*}
u_{i} u_{j} & \equiv u_{j} u_{i} & & \text { for }|i-j| \geq 2,  \tag{2.7}\\
d_{i} d_{j} & \equiv d_{j} d_{i} & & \text { for }|i-j| \geq 2,  \tag{2.8}\\
d_{i} u_{j} & \equiv u_{j} d_{i} & & \text { for } i \neq j,  \tag{2.9}\\
d_{1} u_{1} & \equiv i d, & &  \tag{2.10}\\
d_{i+1} u_{i+1} & \equiv u_{i} d_{i} . & & \tag{2.11}
\end{align*}
$$

Let $J_{\mathscr{Q}, \mathscr{Q}}^{\mathrm{Y}}$ be the two-sided ideal generated by relations (2.7)-(2.11). To prove the above theorem it suffices to show that $I=J_{\mathscr{Q}, \mathscr{D}}^{\mathrm{Y}}$ and so proving this equivalence is our main focus. For ease of notation we denote $J_{\mathscr{U}, \mathscr{T}}^{\mathrm{Y}}$ by $J$ in the rest of this section. The following proposition gives part of our equivalence by showing that $J \subset I$.

Proposition 2.4.2. The inclusion of ideals $J \subseteq I$ holds.
Proof. It suffices to show that for each of (2.7)-(2.11), the two terms in the relation are in fact equivalent modulo $I$. We show this for relation (2.11); the other relations are similar. By Corollary 2.1.5 we need only show $\alpha(x)=\alpha(y)$ and $w(x)=w(y)$ where $x=i \bar{i}$ and $y=\overline{(i+1)}(i+1)$. Indeed, $w(x)=$ $(0,0, \ldots)=w(y)$, while $\alpha(x)=(0, \ldots, 0,1,0, \ldots)=\alpha(y)$, where the 1 occurs in the $i$ th position.

It therefore remains only to show that $I \subseteq J$. The next proposition proves that $I$ is a binomial ideal, that is, $I$ is generated by elements of the form $u_{x}-u_{y}$. The proof of this proposition is very similar to that of Proposition 2.2.2.

Proposition 2.4.3. The ideal I is a binomial ideal.
Proof. Let $I^{\prime}$ be the two-sided ideal generated by all binomials $u_{x}-u_{y}$ such that $u_{x} \equiv u_{y}(\bmod I)$, and suppose $R \in I$. Since $\mathscr{U}$ is graded by weight and $I$ is homogeneous with respect to weight, we may assume that all terms appearing in $R$ have weight $w$ for some $w=\left(w_{1}, w_{2}, \ldots\right)$. We can then find $R^{\prime} \equiv R\left(\bmod I^{\prime}\right)$ for some

$$
R^{\prime}=\sum_{i=1}^{n} c_{x^{(i)}} u_{x^{(i)}}
$$

where $x^{(i)}$ is a word in $\Gamma$ of weight $w, u_{x^{(i)}} \not \equiv u_{x^{(j)}}(\bmod I)$ whenever $i \neq j$, and $0 \neq c_{x^{(i)}} \in \mathbf{C}$ for all $i \in[n]$. In particular, by Corollary 2.1.5, the $\alpha\left(x^{(i)}\right)$ are distinct, so suppose without loss of generality that they occur in lexicographic order.

If $n \geq 1$, let $\lambda \in \mathbf{Y}$ be such that $\lambda_{k}^{\prime}-\lambda_{k+1}^{\prime}=\alpha_{k}\left(x^{(1)}\right)$ for all $k$. By Proposition 2.1.4, $u_{x^{(1)}}(\lambda) \neq 0$. For each $i>1$, by the lexicographic ordering, there exists some $s$ such that $\alpha_{s}\left(x^{(1)}\right)<\alpha_{s}\left(x^{(i)}\right)$. Then by Proposition 2.1.4, $u_{x^{(i)}}(\lambda)=0$. Thus $0=R^{\prime}(\lambda)=c_{x^{(1)}} u_{x^{(1)}}(\lambda)$, which implies $c_{x^{(1)}}=0$. This is a contradiction, so we must have $R^{\prime}=0$. Thus $I=I^{\prime}$.

Our goal for the rest of this section is to show that if $u_{x} \equiv u_{y}(\bmod I)$, then $u_{x} \equiv u_{y}(\bmod J)$. Our general strategy is as follows. Let $\left[u_{x}\right]_{I}$ be the equivalence class of $u_{x}$ modulo $I$. We will construct a representative word $[x]$ such that $u_{[x]} \in\left[u_{x}\right]_{I}$. This representative will only depend on $\alpha(x)$ and $w(x)$, so if $u_{x} \equiv u_{y}(\bmod I)$, then $[x]=[y]$. We will then show that $u_{x} \equiv u_{[x]}(\bmod J)$ and similarly for $y$, which will complete the proof.

Definition 2.4.4. For a word $x$, define

$$
\begin{aligned}
m(x) & =\max _{i \in \mathbf{N}}\left\{-\left(\alpha_{i}(x)+w_{i}(x)\right)\right\} \geq 0, \\
n(x) & =\max \{t \in \mathbf{N} \mid t \text { or } \bar{t} \text { appears in } x\} .
\end{aligned}
$$

For any $m \geq m(x), n \geq n(x)$, we let

$$
[x]_{m, n}=\left(\overline{1}^{m} \cdots \bar{n}^{m}\right)\left(n^{\beta_{n}^{m}(x)} \bar{n}^{\alpha_{n}(x)} \cdots 1_{1}^{\beta_{1}^{m}(x)} \overline{1}_{1}^{\alpha_{1}(x)}\right)
$$

where $\beta_{i}^{m}(x)=\alpha_{i}(x)+w_{i}(x)+m$.
Note that the definition of $m$ ensures that all of the exponents appearing in the definition of $[x]_{m, n}$ are nonnegative. We will often abbreviate $[x]=[x]_{m, n}$. We now show that indeed $u_{[x]} \in\left[u_{x}\right]_{I}$.

Proposition 2.4.5. For any word $x, u_{x} \equiv u_{[x]}(\bmod I)$.
Proof. Let $i \in \mathbf{N}$. Then $w_{i}([x])=-m+\beta_{i}^{m}(x)-\alpha_{i}(x)=w_{i}(x)$. We now show that $\alpha_{i}([x])=\alpha_{i}(x)$. For ease of notation, we will write $\alpha_{i}=\alpha_{i}(x), w_{i}=w_{i}(x)$, and $\beta_{i}=\beta_{i}^{m}(x)$. Since $\alpha_{i}$ relies only upon the appearances of $i, \bar{i},(i+1)$, and $\overline{(i+1)}$ in $x$, we need only consider the subword

$$
\bar{i}^{m} \overline{(i+1)}^{m}(i+1)^{\beta_{i+1}}(\overline{i+1})^{\alpha_{i+1}} i^{\beta_{i}} \bar{i}^{\alpha_{i}} .
$$

To calculate $\alpha_{i}([x])$, we need to find the maximum value of $w_{i+1}(\tilde{x})-w_{i}(\tilde{x})$ for each suffix subword $\tilde{x}$. This value only increases when adding an occurrence of $\bar{i}$ or $(i+1)$ to $\tilde{x}$. Thus we need only verify a few choices of $\tilde{x}$ :

$$
\begin{array}{ll}
\tilde{x}=\bar{i}^{\alpha_{i}}: & w_{i+1}(\tilde{x})-w_{i}(\tilde{x})=\alpha_{i}, \\
\tilde{x}=(i+1)^{\beta_{i+1}(\overline{i+1})^{\alpha_{i+1}} i^{\beta_{i}} \bar{i}^{\alpha_{i}}:} & \begin{array}{ll}
w_{i+1}(\tilde{x})-w_{i}(\tilde{x}) & =\beta_{i+1}-\alpha_{i+1}-\beta_{i}+\alpha_{i} \\
& =w_{i+1}-w_{i}, \\
\tilde{x}=[x]: &
\end{array} \\
& w_{i+1}(\tilde{x})-w_{i}(\tilde{x})=w_{i+1}-w_{i} .
\end{array}
$$

The maximum of these is just $\alpha_{i}$.
We now wish to show that $u_{x} \equiv u_{[x]_{m, n}}(\bmod J)$ for sufficiently large $m$ and $n$. To that end we will make use of the following two lemmas. As a reminder, we will use $i$ and $\bar{i}$ to represent $u_{i}$ and $d_{i}$, respectively.

Lemma 2.4.6. Let $x=\overline{1} \cdots \bar{n} n \cdots 1$ for any $n \in \mathbf{N}$. Then $u_{x} \equiv i d(\bmod J)$.
Proof. First note the equivalence

$$
\begin{equation*}
\bar{n} n(n-1) \cdots 1 \equiv(n-1)(n-2) \cdots 1 \quad(\bmod J) \tag{2.12}
\end{equation*}
$$

which holds by repeated application of (2.11) and a single use of (2.10). Then

$$
\begin{aligned}
\overline{1} \cdots \bar{n} n \cdots 1 & \equiv \overline{1} \cdots \overline{(n-1)}(n-1) \cdots 1 \\
& \equiv \overline{1} \cdots \overline{(n-2)}(n-2) \cdots 1 \\
& \vdots \\
& \equiv i d
\end{aligned}
$$

by repeated application of (2.12).
Lemma 2.4.7. The following equivalences hold modulo $J$ :

$$
\begin{align*}
u_{i} & \equiv u_{i} d_{i} u_{i}  \tag{2.13}\\
d_{i} & \equiv d_{i} u_{i} d_{i}  \tag{2.14}\\
u_{i} u_{i+1} u_{i} & \equiv u_{i+1} u_{i} u_{i}  \tag{2.15}\\
d_{i} d_{i+1} d_{i} & \equiv d_{i} d_{i} d_{i+1}  \tag{2.16}\\
u_{i+1} u_{i} u_{i+1} & \equiv u_{i+1} u_{i+1} u_{i}  \tag{2.17}\\
d_{i+1} d_{i} d_{i+1} & \equiv d_{i} d_{i+1} d_{i+1} \tag{2.18}
\end{align*}
$$

Proof. For (2.13), we have

$$
\begin{align*}
n & \equiv n \overline{1} \cdots \bar{n} n \cdots 1  \tag{Lemma2.4.6}\\
& \equiv \overline{1} \cdots \overline{(n-1)} n \bar{n} n \cdots 1  \tag{2.9}\\
& \equiv \overline{1} \cdots \overline{(n-1)} \overline{(n+1)}(n+1) n \cdots 1  \tag{2.11}\\
& \equiv \overline{(n+1)}(n+1) n \overline{1} \cdots \overline{(n-1)}(n-1) \cdots 1  \tag{2.8}\\
& \equiv \overline{(n+1)}(n+1) n  \tag{Lemma2.4.6}\\
& \equiv n \bar{n} n . \tag{2.11}
\end{align*}
$$

For (2.15) we have

$$
\begin{align*}
n(n+1) n & \equiv n(n+1) \overline{(n+1)}(n+1) n  \tag{2.13}\\
& \equiv n \overline{(n+2)}(n+2)(n+1) n  \tag{2.11}\\
& \equiv \overline{(n+2)}(n+2) n(n+1) n  \tag{2.7}\\
& \equiv(n+1) \overline{(n+1)} n(n+1) n  \tag{2.11}\\
& \equiv(n+1) \bar{n} \overline{(n+1)}(n+1) n  \tag{2.9}\\
& \equiv(n+1) n n \bar{n} n  \tag{2.11}\\
& \equiv(n+1) n n . \tag{2.13}
\end{align*}
$$

For (2.17), assume $n \geq 2$ (the case when $n=1$ is similar). Then

$$
\begin{align*}
(n+1) n(n+1) & \equiv(n+1) n \bar{n} n(n+1)  \tag{2.13}\\
& \equiv(n+1) n(n-1) \overline{(n-1)}(n+1)  \tag{2.11}\\
& \equiv(n+1) n(n+1)(n-1) \overline{(n-1)}  \tag{2.7}\\
& \equiv(n+1) n(n+1) \bar{n} n  \tag{2.11}\\
& \equiv(n+1) n \bar{n}(n+1) n  \tag{2.9}\\
& \equiv(n+1) \overline{(n+1)}(n+1)(n+1) n  \tag{2.11}\\
& \equiv(n+1)(n+1) n . \tag{2.13}
\end{align*}
$$

The proofs of (2.14), (2.16), and (2.18) are similar to the proofs of (2.13), (2.15), and (2.17), respectively.

In particular, one can observe that (2.15) and (2.17) are Knuth relations, which, together with (2.7), verify that the quadratic relations imply that $J$ contains all the relations of the local plactic monoid generated by the $u_{i}$. (See our previous discussion on Knuth relations and the local plactic monoid in Section 2.1).

We are now ready to prove the heart of our main theorem.
Proposition 2.4.8. Let $x=x_{1} \cdots x_{\ell}$ be a word. Then there exist $M, N \in \mathbf{N}$ such that $u_{x} \equiv u_{[x]_{m, n}}$ $(\bmod J)$ for all $m \geq M, n \geq N$.

Proof. As before, we will abbreviate $[x]=[x]_{m, n}$ and $[y]=[y]_{m, n}$. We proceed by induction on the length of $x$. First suppose $\ell=0$ (that is, $x$ is the empty word), and take any $m, n \geq 0$. Then we have $[x]=\overline{1}^{m} \ldots \bar{n}^{m} n^{m} \ldots 1^{m}$ and we wish to show that $u_{[x]} \equiv i d(\bmod J)$. By (2.7) and (2.17),

$$
n \cdots 1 n \equiv n(n-1) n(n-2) \cdots 1 \equiv n n(n-1) \cdots 1 .
$$

In other words, $n$ and $n \cdots 1$ commute. Therefore

$$
\begin{aligned}
(\mathrm{n} \cdots 1) \mathrm{n}^{m-1} \cdots 1^{m-1} & \equiv \mathrm{n}^{m}((\mathrm{n}-1) \cdots 1)(\mathrm{n}-1)^{m-1} \cdots 1^{m-1} \\
& \equiv \mathrm{n}^{m}(\mathrm{n}-1)^{m}((\mathrm{n}-2) \cdots 1)(\mathrm{n}-2)^{m-1} \cdots 1^{m-1} \\
& \vdots \\
& \equiv \mathrm{n}^{m} \cdots 1^{m},
\end{aligned}
$$

so $\mathrm{n}^{m} \cdots 1^{m} \equiv(\mathrm{n} \cdots 1)^{m}$. Similarly by (2.8) and (2.18), $\overline{\mathrm{I}}^{m} \cdots \overline{\mathrm{n}}^{m} \equiv(\overline{1} \cdots \overline{\mathrm{n}})^{m}$. Then applying Lemma 2.4.6 repeatedly to

$$
\overline{1}^{m} \cdots \overline{\mathrm{n}}^{m} \mathrm{n}^{m} \cdots 1^{m} \equiv(\overline{1} \cdots \overline{\mathrm{n}})^{m}(\mathrm{n} \cdots 1)^{m}
$$

gives the claim.
Now suppose the proposition statement is true for all words of length less than $\ell$. Let $x=x_{1} \cdots x_{\ell}$ and $y=x_{1} \cdots x_{\ell-1}$. By induction we know the statement holds for $y$ for some $N^{\prime}, M^{\prime} \in \mathbf{N}$. Then take $M=\max \left\{m(x), M^{\prime}\right\}$ and $N=\max \left\{n(x), N^{\prime}\right\}$ and let $m \geq M$ and $n \geq N$. By induction we have $u_{x}=u_{y} u_{x_{\ell}} \equiv u_{[y]} u_{x_{\ell}}(\bmod J)$. From this we see that it suffices to show $u_{[y]} u_{x_{\ell}} \equiv u_{[x]}(\bmod J)$. For ease of notation we let $\alpha_{i}=\alpha_{i}(y), \beta_{i}=\beta_{i}^{m}(y), w_{i}=w_{i}(y)$, and $\beta_{i}(x)=\beta_{i}^{m}(x)$ for all $i$.

We now split the argument into four cases depending on $x_{\ell}$ and $\alpha_{i}$. Note that if $x_{\ell}=t$ or $\bar{t}$ for $t \geq 1$, then $\alpha_{i}(x)=\alpha_{i}, w_{i}(x)=w_{i}$, and $\beta_{i}(x)=\beta_{i}$ for all $i \neq t, t-1$.

Case 1. Suppose $x_{\ell}=t$ and $\alpha_{t}=0$. We have $\alpha_{t-1}(x)=\alpha_{t-1}+1, w_{t-1}(x)=w_{t-1}, \alpha_{t}(x)=0$, and $w_{t}(x)=w_{t}+1$, so that $\beta_{t-1}(x)=\beta_{t-1}+1$ and $\beta_{t}(x)=\beta_{t}+1$. Then

$$
\begin{align*}
u_{[y]} u_{t} & \equiv \cdots \mathrm{t}^{\beta_{t}}(\mathrm{t}-1)^{\beta_{t-1}} \overline{(\mathrm{t}-1)}{ }^{\alpha_{t-1}} \cdots \mathrm{t} \\
& \equiv \cdots \mathrm{t}^{\beta_{t}}(\mathrm{t}-1)^{\beta_{t-1}} \overline{\mathrm{t}(\mathrm{t}-1)^{\alpha_{t-1}} \cdots}  \tag{2.7}\\
& \equiv \cdots \mathrm{t}^{\beta_{t}}(\mathrm{t}-1)^{\left.\beta_{t-1} \mathrm{t} \overline{\mathrm{t}} \overline{(\mathrm{t}-1}\right)^{\alpha_{t-1}} \cdots}  \tag{2.13}\\
& \equiv \cdots \mathrm{t}^{\beta_{t}}(\mathrm{t}-1)^{\beta_{t-1} \mathrm{t}(\mathrm{t}-1) \overline{\mathrm{t}-1)} \alpha_{t-1}^{\alpha_{t-1}} \cdots}  \tag{2.11}\\
& \equiv \cdots \mathrm{t}^{\beta_{t}+1}(\mathrm{t}-1)^{\beta_{t-1}+1} \overline{(\mathrm{t}-1)}{ }^{\alpha_{t-1}+1} \cdots  \tag{2.15}\\
& =u_{[x]} .
\end{align*}
$$

Case 2. Suppose that $x_{\ell}=t$ and $\alpha_{t} \neq 0$. We have $\alpha_{t-1}(x)=\alpha_{t-1}+1, w_{t-1}(x)=w_{t-1}, \alpha_{t}(x)=\alpha_{t}-1$, and $w_{t}(x)=w_{t}+1$, so that $\beta_{t-1}(x)=\beta_{t-1}+1$ and $\beta_{t}(x)=\beta_{t}$. Then

$$
\begin{align*}
u_{[y]} u_{t} & \equiv \cdots \mathrm{t}^{\beta_{t} \overline{\mathrm{t}}^{\alpha_{t}}}(\mathrm{t}-1)^{\beta_{t-1}} \overline{(\mathrm{t}-1)} \bar{\alpha}_{t-1}^{\alpha_{t}} \cdots \mathrm{t} \\
& \equiv \cdots \mathrm{t}^{\beta_{t} \mathrm{t}^{-\alpha_{t}-1}}(\mathrm{t}-1)^{\beta_{t-1} \overline{\mathrm{t}}(\mathrm{t}-1)^{\alpha_{t-1}} \cdots}  \tag{2.7}\\
& \equiv \cdots \mathrm{t}^{\beta_{t} \mathrm{t}^{\alpha_{t}-1}}(\mathrm{t}-1)^{\beta_{t-1}+1} \overline{\mathrm{t}-1)}{ }^{\alpha_{t-1}+1} \cdots  \tag{2.11}\\
& =u_{[x]} .
\end{align*}
$$

Case 3. Suppose that $x_{\ell}=\bar{t}$ and $\alpha_{t-1}=0$. We have $\alpha_{t-1}(x)=0, w_{t-1}(x)=w_{t-1}, \alpha_{t}(x)=\alpha_{t}+1$, and
$w_{t}(x)=w_{t}-1$, so that $\beta_{t-1}(x)=\beta_{t-1}$ and $\beta_{t}(x)=\beta_{t}$. Then

$$
\begin{align*}
u_{[y]} d_{t} & \equiv \cdots \mathrm{t}^{\beta_{t} \overline{\mathrm{t}}_{t}}(\mathrm{t}-1)^{\beta_{t-1}} \cdots \overline{\mathrm{t}} \\
& \equiv \cdots \mathrm{t}^{\beta_{t} \overline{\mathrm{t}}_{t}+1}(\mathrm{t}-1)^{\beta_{t-1}} \cdots  \tag{2.8}\\
& =u_{[x]} .
\end{align*}
$$

Case 4. Finally, suppose that $x_{\ell}=\bar{t}$ and $\alpha_{t-1} \neq 0$. We have $\alpha_{t-1}(x)=\alpha_{t-1}-1, w_{t-1}(x)=w_{t-1}$, $\alpha_{t}(x)=\alpha_{t}+1$, and $w_{t}(x)=w_{t}-1$, so that $\beta_{t-1}(x)=\beta_{t-1}-1$ and $\beta_{t}(x)=\beta_{t}$. Then

$$
\begin{align*}
& u_{[y]} d_{t} \equiv \cdots \mathrm{t}^{\beta_{t}} \overline{\mathrm{t}}^{\alpha_{t}}(\mathrm{t}-1)^{\beta_{t-1}} \overline{\mathrm{t}-1)}^{\alpha_{t-1}} \cdots \overline{\mathrm{t}} \\
& \equiv \cdots \mathrm{t}^{\beta_{t}} \overline{\mathrm{t}}^{\alpha_{t}}(\mathrm{t}-1)^{\beta_{t-1}(\overline{\mathrm{t}}-1)^{\alpha_{t-1}} \overline{\mathrm{t}} \cdots}  \tag{2.8}\\
& \equiv \cdots \mathrm{t}^{\beta_{t} \overline{\mathrm{t}}^{\alpha_{t}}}(\mathrm{t}-1)^{\beta_{t-1}-1}(\mathrm{t}-1) \overline{(\mathrm{t}-1)} \overline{\mathrm{t}} \overline{\mathrm{t}-1)}{ }^{\alpha_{t-1}-1} \ldots  \tag{2.16}\\
& \equiv \cdots \mathrm{t}^{\beta_{t}} \overline{\mathrm{t}}^{\alpha_{t}}(\mathrm{t}-1)^{\beta_{t-1}-1} \overline{\mathrm{t}} \overline{\mathrm{t}} \overline{\mathrm{t}} \overline{\mathrm{t}-1)}{ }^{\alpha_{t-1}-1} \ldots  \tag{2.11}\\
& \left.\equiv \cdots \mathrm{t}^{\beta_{t}} \overline{\mathrm{t}}^{\alpha_{t}}(\mathrm{t}-1)^{\beta_{t-1}-1} \overline{\mathrm{t}} \overline{\mathrm{t}-1}\right)^{\alpha_{t-1}-1} \cdots  \tag{2.14}\\
& \equiv \cdots \mathrm{t}^{\beta_{t} \overline{\mathrm{t}}^{\alpha_{t}+1}}(\mathrm{t}-1)^{\beta_{t-1}-1}(\overline{\mathrm{t}-1})^{\alpha_{t-1}-1} \cdots  \tag{2.9}\\
& =u_{[x]} \text {. }
\end{align*}
$$

This completes the proof.
It is now easy to complete the proof of Theorem 2.4.1.
Proof. The inclusion $J \subseteq I$ follows from Proposition 2.4.3. For the other direction, note that by Proposition 2.4.2 we need only prove that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$ for words $x$ and $y$. By Proposition 2.4.8 there exist nonnegative integers $m$ and $n$ sufficiently large such that $u_{x} \equiv u_{[x]_{m, n}}=u_{[y]_{m, n}} \equiv u_{y}(\bmod J)$.

### 2.4.1 Subalgebras

Note that the algebra of Schur operators and the algebra of down-operators for Young's lattice are both subalgebras of the algebra of up- and down-operators for Young's lattice. This of course leads to the question of whether or not the algebra of up- and down-operators for Young's lattice has other interesting subalgebras. In this subsection we define such a subalgebra and give a full characterization for it.

Fix some $1<t \in \mathbf{N}$. Let $\mathscr{B}^{\mathbf{Y}}$ be the subalgebra of $\mathscr{U}_{\mathscr{D}}^{\mathrm{Y}}$ generated by $u_{t}$ and $d_{t}$, and consider the subalgebra $\mathscr{B}^{\mathrm{Y}} / I_{\mathscr{B}}^{\mathrm{Y}}=\mathscr{B}^{\mathrm{Y}} /\left(I_{\mathscr{U}, \mathscr{D}}^{\mathrm{Y}} \cap \mathscr{B}\right) \subseteq \mathscr{U}_{\mathscr{D}}^{\mathrm{Y}} / I_{\mathscr{U}, \mathscr{D}}^{\mathrm{Y}}$. We will show that the ideal of relations $I_{\mathscr{B}}^{\mathrm{Y}}$ is generated by

$$
\begin{align*}
u_{t}^{i+1} d_{t}^{i} & \equiv u_{t}^{i+1} d_{t}^{i+1} u_{t}  \tag{2.19}\\
u_{t}^{i} d_{t}^{i+1} & \equiv d_{t} u_{t}^{i+1} d_{t}^{i+1} \tag{2.20}
\end{align*}
$$

for all $i \in \mathbf{N}$. Let $J_{\mathscr{B}}^{\mathrm{Y}}$ be the ideal generated by relations (2.19) and (2.20); we wish to show $J_{\mathscr{B}}^{\mathrm{Y}}=I_{\mathscr{B}}^{\mathrm{Y}}$. For ease of notation we will refer to $\mathscr{B}^{\mathbf{Y}}, I_{\mathscr{B}}^{\mathrm{Y}}$, and $J_{\mathscr{B}}^{\mathrm{Y}}$ as $\mathscr{B}, I_{\mathscr{B}}$, and $J_{\mathscr{B}}$ respectively.
(When $t=1$, it is straightforward to verify that the only relation between $u_{1}$ and $d_{1}$ is (2.10), namely $d_{1} u_{1} \equiv i d$, as this relation can be used to rewrite any monomial in the form $u_{1}^{i} d_{1}^{j}$, and all such monomials act independently on $\mathbf{Y}$.)

### 2.4.1.1 Peaks and valleys

One convenient way to interpret a word consisting only of the letters $t$ and $\bar{t}$ is as a graph of diagonal steps. More precisely, we construct a graph corresponding to a word $x$ in the following way. Starting at the origin in the plane we read $x$ from right to left. When we encounter a $t$ we take a diagonal step up and to the left by adding $(-1,1)$, and when we encounter a $\bar{t}$ we take a diagonal step down and to the left by adding $(-1,-1)$. One must be careful since we are reading both the word and its graph from right to left.

We call a point of the graph with maximal height a peak and a point with minimal height a valley. (Peaks and valleys need not be unique.) It is straightforward to see that if $(a, b)$ is a peak and $(c, d)$ is a valley, then $\alpha_{t-1}(x)=b$ and $\alpha_{t}(x)=-d$. Also note that if $(e, f)$ is the (leftmost) endpoint of the graph, then $w_{t}(x)=f$. Therefore by Corollary 2.1.5, the action of $x$ on $\mathbf{Y}$ is determined entirely by the heights of its peaks, valleys, and endpoint.

Example 2.4.9. The word $x=t^{2} \bar{t}^{4} t^{3}$ has the graph shown below.


This graph has a peak at $(-3,3)$ and a valley at $(-7,-1)$. Correspondingly, $\alpha_{t-1}(x)=3$ and $\alpha_{t}(x)=1$. The leftmost point of the graph is $(-9,1)$, so $w_{t}(x)=1$.

Note that relations (2.19) and (2.20) are not bounded in degree since the only condition on $i$ is that it be a nonnegative integer. This differs from the previous algebras that we examined in that the largest degree needed in those cases was 4 (as in the algebra of Schur operators). Indeed, relations of unbounded degree are required due to the following proposition.

Proposition 2.4.10. The ideal $I_{\mathscr{B}}$ cannot be generated by elements of bounded degree.
Proof. Suppose for contradiction that the largest degree appearing among the generators of $I_{\mathscr{B}}$ is $h \in \mathbf{N}$. Choose an integer $k>h$, and let $x=t^{k}$ and $y=t^{k} \bar{t}^{k} t^{k}$. Observe that $w(x)=w(y)=$ $(0, \ldots, 0, k, 0, \ldots)$ and $\alpha(x)=\alpha(y)=(0, \ldots, 0, k, 0, \ldots)$, and so $u_{x} \equiv u_{y}\left(\bmod I_{\mathscr{B}}\right)$ by Corollary 2.1.5.

Note that in the graph of $x$, there is never a peak occurring to the right of a valley. In other words, if $x=x_{1} \ldots x_{k}$, then there do not exist $i<j$ such that $\left(-i, \alpha_{t-1}(x)\right)$ and $\left(-j,-\alpha_{t}(x)\right)$ appear in the
graph of $x$. We will call an instance of a peak occurring to the right of a valley a peak/valley pair. For instance, $x$ has no peak/valley pair but $y$ does, corresponding to the suffix subwords $t^{k}$ and $\bar{t}^{k} t^{k}$, respectively.

We now show that for words $z$ satisfying $\alpha_{t-1}(z)+\alpha_{t}(z)>h$, our degree boundedness assumption implies that the existence of a peak/valley pair is invariant modulo $I_{\mathscr{B}}$. This will then lead to an immediate contradiction when applied to $x$ and $y$. Let $u_{m}-u_{m^{\prime}}$ be a generator of $I_{\mathscr{B}}$ of degree at most $h$. It suffices to show that if the word $z=m_{1} m m_{2}$ has a peak/valley pair, then so does $z^{\prime}=m_{1} m^{\prime} m_{2}$.

Since $u_{m} \equiv u_{m^{\prime}}\left(\bmod I_{\mathscr{B}}\right)$, the graphs of $m$ and $m^{\prime}$ must have their peaks, valleys, and endpoints at the same heights. Therefore $z$ has a peak or valley within $m$ if and only if $z^{\prime}$ has a peak or valley within $m^{\prime}$. If $z$ has a peak/valley pair with neither peak nor valley occurring within $m$, then $z^{\prime}$ has a peak/valley pair at the same locations. If at most one of the peak or valley occurs within $m$, say the peak, then the valley must occur within $m_{1}$, so $z^{\prime}$ will have a peak within $m^{\prime}$ and a valley within $m_{1}$ and hence a peak/valley pair. (The other case is similar.) The only remaining possibility is if both the peak and valley occur within $m$ (for they might switch order in $m^{\prime}$ ). However, since $\alpha_{t-1}(z)+\alpha_{t}(z)>h$, the difference in height between the peak and valley is more than $h$, so they cannot both appear within $m$, which has length at most $h$. This completes the proof.

### 2.4.1.2 Proof of relations

We now prove that relations (2.19) and (2.20) suffice. The proofs for the following two propositions are essentially the same as the proofs of the analogous propositions for the algebra of up- and down-operators for Young's lattice.

Proposition 2.4.11. The ideal $I_{\mathscr{B}}$ is a binomial ideal.
Proposition 2.4.12. The inclusion of ideals $J_{\mathscr{B}} \subseteq I_{\mathscr{B}}$ holds.
As in our work for the algebra of up- and down-operators for Young's lattice, our approach is to construct a standard equivalence class representative $u_{[x]}$ (modulo $I_{\mathscr{B}}$ ) for every monomial $u_{x}$ and to then show that $u_{x} \equiv u_{[x]}\left(\bmod J_{\mathscr{B}}\right)$.

Definition 2.4.13. For any word $x$ in $t$ and $\bar{t}$, define

$$
[x]=t^{w_{t}(x)+\alpha_{t}(x)} \bar{t}_{t-1}^{\alpha_{t-1}(x)+\alpha_{t}(x)} t^{\alpha_{t-1}(x)} .
$$

We say that such a word $[x]$ is the standard representative for $x$, or alternatively that it is in standard form.

Note that all the exponents appearing in $[x]$ are nonnegative: in particular, by the definition of $\alpha_{t}(x)$ we have $\alpha_{t}(x) \geq-w_{t}(x)$, and so $w_{t}(x)+\alpha_{t}(x) \geq 0$. It is straightforward to check that $w(x)=\left(0, \ldots, 0, w_{t}(x), 0, \ldots\right)=w([x])$ and $\alpha(x)=\left(0, \ldots, 0, \alpha_{t-1}(x), \alpha_{t}(x), 0, \ldots\right)=\alpha([x])$, so Corollary 2.1.5 implies that $[x]$ is the unique word in standard form such that $u_{x} \equiv u_{[x]}\left(\bmod I_{\mathscr{B}}\right)$.

Proposition 2.4.14. Let $x=x_{1} \cdots x_{\ell}$ be a word in $t$ and $\bar{t}$. We have $u_{x} \equiv u_{[x]}\left(\bmod J_{\mathscr{B}}\right)$.
Proof. We prove this by induction on the length of $x$. If $\ell=0$ or if $x=\bar{t}$, then $[x]=x$, so there is nothing to prove. If $x=t$, then $[x]=t \bar{t} t$, and $\mathrm{t} \equiv \mathrm{t} \overline{\mathrm{t}} \mathrm{by}$ (2.19) for $i=0$.

Now suppose the statement holds for all words shorter than $x$. We have that $u_{x}=u_{x_{1}} u_{y}$ where $y=x_{2} \cdots x_{\ell}$. By induction, $u_{x}=u_{x_{1}} u_{y} \equiv u_{x_{1}} u_{[y]}\left(\bmod J_{\mathscr{B}}\right)$, so we need to show $u_{x_{1}} u_{[y]} \equiv u_{[x]}$.

If $x_{1}=t$ and $w_{t}(y)<\alpha_{t-1}(y)$, then $w_{t}(x)=w_{t}(y)+1$ while $\alpha(x)=\alpha(y)$. Hence $[x]=t[y]$, so there is nothing to show. Similarly if $x_{1}=\bar{t}$ and $\alpha_{t}(y)=-w_{t}(y)$, then

$$
[x]=\bar{t}^{\alpha_{t-1}(y)+\alpha_{t}(y)+1} t^{\alpha_{t-1}(y)}=\bar{t}[y],
$$

so again there is nothing to show.
Suppose $x_{1}=t$ and $w_{t}(y)=\alpha_{t-1}(y)$. Then $w_{t}(x)=w_{t}(y)+1, \alpha_{t-1}(x)=\alpha_{t-1}(y)+1$, and $\alpha_{t}(x)=$ $\alpha_{t}(y)$. Here the graph of $x$ has a new peak at its leftmost point, so $t[y]$ is not in standard form. Applying (2.19) with $i=w_{t}(y)+\alpha_{t}(y)$ gives

$$
\begin{aligned}
u_{t} u_{[y]} & =\mathrm{t}^{w_{t}(y)+\alpha_{t}(y)+1} \overline{\mathrm{t}}^{w_{t}(y)+\alpha_{t}(y)} \mathrm{t}^{\alpha_{t-1}(y)} \\
& \equiv \mathrm{t}^{w_{t}(y)+\alpha_{t}(y)+1} \overline{\mathrm{t}}^{w_{t}(y)+\alpha_{t}(y)+1} \mathrm{t}^{\alpha_{t-1}(y)+1}=u_{[x]} .
\end{aligned}
$$

Finally, suppose $x_{1}=\bar{t}$ and $\alpha_{t}(y)>-w_{t}(y)$. We then have $w_{t}(x)=w_{t}(y)-1$ and $\alpha(x)=\alpha(y)$. Again $\bar{t}[y]$ is not in standard form since it begins with $\bar{t}$. Note that by definition $\alpha_{t-1}(y) \geq w_{t}(y)$, so $\alpha_{t-1}(y)+\alpha_{t}(y) \geq w_{t}(y)+\alpha_{t}(y)$. Therefore we can apply (2.20) with $i=w_{t}(y)+\alpha_{t}(y)-1=w_{t}(x)+\alpha_{t}(x)$ to get

$$
\begin{aligned}
d_{t} u_{[y]} & =\overline{\mathrm{t}} \mathrm{t}^{w_{t}(y)+\alpha_{t}(y) \overline{\mathrm{t}}^{\alpha_{t-1}(y)+\alpha_{t}(y)} \mathrm{t}^{\alpha_{t-1}(y)}} \\
& \equiv \mathrm{t}^{w_{t}(y)+\alpha_{t}(y)-1} \overline{\mathrm{t}}^{\alpha_{t-1}(y)+\alpha_{t}(y)} \mathrm{t}^{\alpha_{t-1}(y)}=u_{[x]} .
\end{aligned}
$$

Theorem 2.4.15. The ideals $I_{\mathscr{B}}$ and $J_{\mathscr{B}}$ are equal.
Proof. This follows from Propositions 2.4.11, 2.4.12, and 2.4.14.

### 2.4.1.3 Up- and down-operators on finite chains

Consider again the operators $u_{t}$ and $d_{t}$ for some fixed $t>1$. The action of these operators on $\mathbf{Y}$ splits up as a direct sum of the action on chains $C$, where $C$ is a set of partitions $\lambda$ that have fixed values for $\lambda_{i}^{\prime}$ for all $i \neq t$. The action is then determined entirely by $\rho=\lambda_{t-1}^{\prime}-\lambda_{t+1}^{\prime}$, the difference between the $(t-1)$ st and $(t+1)$ st columns. (Equivalently, $C$ is a chain with $\rho+1$ elements, and $u_{t}$ and $d_{t}$ act as up- and down-operators on this chain.)

Fix $\rho$, and let $I_{\mathscr{C}}$ be the two-sided ideal of $\mathscr{B}$ containing all elements which annihilate $C$, a chain with $\rho+1$ elements. We characterize the algebra $\mathscr{B} / I_{\mathscr{C}}$ by showing that $I_{\mathscr{C}}$ is generated by
the following relations:

$$
\begin{array}{rlrl}
u_{t}^{i+1} d_{t}^{i} & \equiv u_{t}^{i+1} d_{t}^{i+1} u_{t} & \text { for } 0 \leq i \leq \rho-1, \\
u_{t}^{i} d_{t}^{i+1} & \equiv d_{t} u_{t}^{i+1} d_{t}^{i+1} & & \text { for } 0 \leq i \leq \rho-1, \\
u_{t}^{\rho+1} & \equiv 0, & & \\
d_{t}^{\rho+1} & \equiv 0 . & & \tag{2.24}
\end{array}
$$

Let $J_{\mathscr{C}}$ be the ideal generated by relations (2.21)-(2.24). We will show that $J_{\mathscr{C}}=I_{\mathscr{C}}$ by exploiting the close relationship between these ideals and $I_{\mathscr{B}}$.

Theorem 2.4.16. The ideals $I_{\mathscr{C}}$ and $J_{\mathscr{C}}$ are equal.
Proof. Recall that $I_{\mathscr{B}}$ is the two-sided ideal of $\mathscr{B}$ containing all elements which annihilate $\mathbf{Y}$. Let $P$ be the two-sided ideal of $\mathscr{B}$ which is generated by relations (2.23) and (2.24). It is straightforward to see that $J_{\mathscr{C}}=I_{\mathscr{B}}+P$ (since (2.19) and (2.20) for $i \geq \rho$ are implied by (2.23) and (2.24), so we need to show that $I_{\mathscr{C}}=I_{\mathscr{B}}+P$.

The inclusion $I_{\mathscr{B}}+P \subseteq I_{\mathscr{C}}$ holds since both (2.23) and (2.24) annihilate $C$. For the reverse direction, note that by Proposition 2.4.14, $\mathscr{B} / I_{\mathscr{B}}$ has a basis consisting of the standard representatives $u_{[x]}$. A basis element $u_{[x]}$ annihilates $C$ if and only if the power of $\overline{\mathrm{t}}$ appearing in it is larger than $\rho$, which occurs if and only if it lies in $P$. The other basis elements act independently on $C$ as in the proof of Proposition 2.4.3. It follows that $I_{\mathscr{C}} \subseteq I_{\mathscr{B}}+P$.

## UP-OPERATORS FOR THE GENERALIZED NONCROSSING PARTITION LATTICE AND ABSOLUTE ORDER

In this chapter we study the up-operators for the generalized noncrossing partition lattice $N C(W)$ and for absolute order $\operatorname{Abs}(W)$. Both posets are defined with respect to a Coxeter group $W$ and so in section 3.1 we give some necessary background information on Coxeter groups followed by the formal definitions for these posets. We also state some known results for these two posets which will be of use to us later on. Then, in sections 3.2 and 3.3 we give a complete list of defining relations for the algebra generated by the up-operators for $N C(W)$ and $\operatorname{Abs}(W)$ respectively.

### 3.1 Coxeter Groups, Absolute Order, and the Generalized Noncrossing Partition Lattice

### 3.1.1 Coxeter Groups

We begin with a brief discussion on Coxeter groups. For more information on this subject see [7]. Let $S$ be a finite set. A Coxeter matrix is a matrix $m$ such that for all $s, s^{\prime} \in S$ we have

- the rows and columns of $m$ are indexed by the elements of $S$,
- $m\left(s, s^{\prime}\right) \in\{1,2, \ldots, \infty\}$,
- $m\left(s, s^{\prime}\right)=1$ if and only if $s=s^{\prime}$, and
- $m\left(s, s^{\prime}\right)=m\left(s^{\prime}, s\right)$.

Here $m\left(s, s^{\prime}\right)$ denotes the entry of $m$ in row $s$ and column $s^{\prime}$. Let $W$ be the group with generating set $S$ subject to the relations $\left(s s^{\prime}\right)^{m\left(s, s^{\prime}\right)}=1$ for all $s, s^{\prime} \in S$. We call $W$ a Coxeter group and we refer to $S$ as the set of Coxeter generators. We also refer to the pair $(W, S)$ as a Coxeter system and to $|S|$ as the rank of $(W, S)$. From here on $W$ will always represent a Coxeter group and $w$ will always be one of its elements. Furthermore, since our work focuses solely on finite Coxeter groups we will always assume that $|W|<\infty$. A reflection of $W$ is any element of the set $T=\left\{w s w^{-1}: w \in W, s \in S\right\}$. We will always use $T$ to denote the set of reflections for $W$.

One useful way to visualize a Coxeter group $W$ is to consider its Coxeter diagram (or Coxeter graph). This is the graph whose nodes are indexed by the elements of $S$ and where node $s$ and node $s^{\prime}$ are connected by an edge if $m\left(s, s^{\prime}\right) \geq 3$. If $m\left(s, s^{\prime}\right) \geq 4$, then we label the edge from $s$ to $s^{\prime}$ with $m\left(s, s^{\prime}\right)$. A list of all finite Coxeter groups and their Coxeter diagrams is given in Figure 3.5.

Example 3.1.1. Consider the symmetric group $\mathfrak{S}_{5}$. This is the Coxeter group of type $A_{4}$ (see below) with Coxeter generators $s_{i}, i=1, \ldots, 4$ where $s_{i}$ is the transposition which switches $i$ and $i+1$. The Coxeter diagram for $\mathfrak{S}_{5}$ is below.


### 3.1.2 Absolute Length and Absolute Order

Let $s_{i_{j}} \in S$. An $S$-word for $w$ is any tuple $\left(s_{i_{1}}, \ldots, s_{i_{k}}\right)$ such that $w=s_{i_{1}} \cdots s_{i_{k}}$ (we call $k$ the length of $\left(s_{i_{1}}, \ldots, s_{i_{k}}\right)$ ). The standard length of $w$, denoted $\ell_{S}(w)$, is the minimal length among all $S$-words of $w$ and furthermore, an $S$-word for $w$ is reduced if its length equals $\ell_{S}(w)$.

Both $\ell_{S}(w)$ and reduced $S$-words have been studied thoroughly, especially with respect to their importance in weak order and Bruhat order [7]. However, in this chapter we will be mostly concerned with a different type of word/tuple and a different type of length which we define as follows. Letting $t_{i_{j}} \in T$, we call a tuple $x=\left(t_{i_{1}}, \ldots, t_{i_{k}}\right)$ a $T$-word for $w$ if $w=t_{i_{1}} \cdots t_{i_{k}}$ and we say that this $T$-word has length $k$. We take the absolute length of $w$, denoted $\ell_{T}(w)$, to be the minimal length among all $T$-words for $w$ and if a $T$-word is of length $\ell_{T}(w)$, then we call it reduced. We denote by $R_{T}(w)$ the set of all reduced $T$-words for $W$. We will often call $\left(t_{i_{1}}, \ldots, t_{i_{k}}\right)$ a $T$-word rather than a $T$-word for $w$ when $w$ is either clear or unimportant. Now that we have introduced absolute length we can define one of the two posets with which this chapter is concerned.

Definition 3.1.2. Let $W$ be a Coxeter group and let $w, w^{\prime} \in W$. The absolute order on $W$, denoted $\operatorname{Abs}(W)$, is the partial order such that $w \leq_{T} w^{\prime}$ if and only if $\ell_{T}\left(w^{\prime}\right)=\ell_{T}(w)+\ell_{T}\left(w^{-1} w^{\prime}\right)$.

The earliest study of absolute order known to this author is by Hurwitz [15] who discussed it in the special case where $W=\mathfrak{S}_{n}$. It has also been studied by several others including Carter [9] in his


Figure 3.1 Absolute order on $A_{2}$.
work on conjugacy classes of Weyl groups, Bessis [5] in his work on dual Coxeter systems, Armstrong [1] in his study of the generalized noncrossing partition lattice, and Kallipoliti [16] in the special case when $W=B_{n}$. For further reading on $\operatorname{Abs}(W)$ see [1].

Figure 3.1 shows the Hasse diagram for absolute order on $A_{2}=\mathfrak{S}_{3}$. Note that $\operatorname{Abs}\left(A_{2}\right)$ has two distinct maximal elements and so in general absolute order does not have a maximum element $\hat{1}$. However, $\operatorname{Abs}(W)$ does always have a minimum element $\hat{0}$ which is equal to $e$, the identity element of $W$. Additionally, $\operatorname{Abs}(W)$ is a graded poset with the rank function $\ell_{T}(w)$. Since $\ell_{T}(w)$ is the rank function of $\operatorname{Abs}(W)$ we immediately see that if $w^{\prime}$ covers $w\left(w \lessdot_{T} w^{\prime}\right)$ then $w^{-1} w^{\prime}=t \in T$. From this we see that any saturated chain of the form

$$
e=w_{1} \lessdot_{T} \ldots \lessdot_{T} w_{k}=w
$$

can be associated with the reduced $T$-word $\left(w_{1}^{-1} w_{2}, \ldots, w_{k-1}^{-1} w_{k}\right) \in R_{T}(w)$. This correspondence demonstrates that a study of chains in $\operatorname{Abs}(W)$ requires an understanding of reduced $T$-words. The following definition will aid us in that understanding.

Definition 3.1.3. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word. We perform a Hurwitz move on $x$ when we switch some $t_{i}$ and $t_{i+1}$ and conjugate one of them in the following way:

$$
\begin{align*}
& \left(t_{1}, \ldots, t_{i}, t_{i+1}, \ldots, t_{\ell}\right) \rightarrow\left(t_{1}, \ldots, t_{i+1}, t_{i+1} t_{i} t_{i+1}, \ldots, t_{\ell}\right) \text { or }  \tag{3.1}\\
& \left(t_{1}, \ldots, t_{i}, t_{i+1}, \ldots, t_{\ell}\right) \rightarrow\left(t_{1}, \ldots, t_{i} t_{i+1} t_{i}, t_{i}, \ldots, t_{\ell}\right) . \tag{3.2}
\end{align*}
$$

We say that two $T$-words $x$ and $y$ are Hurwitz equivalent, denoted $x \stackrel{H}{\sim} y$, if one can be obtained from the other using a sequence of Hurwitz moves.

These Hurwitz moves are referred to by Bessis as dual braid moves and are in fact just a special case of the Hurwitz action (see [4]). The following lemma (which Armstrong refers to as the shifting lemma [1]) demonstrates the connection between Hurwitz moves and reduced $T$-words.

Lemma 3.1.4. Let $x$ be a reduced $T$-word for some $w \in W$. If $y$ is a $T$-word such that $x \stackrel{H}{\sim} y$, then $y$ is also a reduced $T$-word for $w$.

Proof. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ and let

$$
\begin{aligned}
& x_{1}=\left(t_{1}, \ldots, t_{i+1}, t_{i+1} t_{i} t_{i+1}, \ldots, t_{\ell}\right), \\
& x_{2}=\left(t_{1}, \ldots, t_{i} t_{i+1} t_{i}, t_{i}, \ldots, t_{\ell}\right) .
\end{aligned}
$$

Both $x_{1}$ and $x_{2}$ have length $\ell$ and the product of the entries for either $T$-word is equal to $w$.
The next proposition gives us a condition for when two elements of $W$ are comparable in $\operatorname{Abs}(W)$. This condition will prove quite useful later in this chapter.

Proposition 3.1.5. Let $w, w^{\prime} \in W$. Then $w \leq_{T} w^{\prime}$ if and only if a $T$-word for $w$ occurs as an arbitrary subword of some reduced $T$-word for $w^{\prime}$. Furthermore, if $x$ is a reduced $T$-word and $y$ is a subword of $x$, then $y$ is also reduced.

Proof. For the first statement suppose that $w \leq_{T} w^{\prime}$ and let $\left(t_{1}, \ldots, t_{\ell_{1}}\right) \in R_{T}(w)$ and $\left(r_{1}, \ldots, r_{\ell_{2}}\right) \in$ $R_{T}\left(w^{-1} w^{\prime}\right)$. It is straightforward to see that $\left(t_{1}, \ldots, t_{\ell_{1}}, r_{1}, \ldots, r_{\ell_{2}}\right) \in R_{T}\left(w^{\prime}\right)$ and so the forward implication is proven.

Now suppose that $w$ occurs as an arbitrary subword of a reduced $T$-word for $w^{\prime}$. Let $x$ be this reduced $T$-word for $w^{\prime}$ and let $y$ be the $T$-word for $w$ occurring as an arbitrary subword of $x$. Using Hurwitz moves we see that $x \stackrel{H}{\sim} x^{\prime}$ where $x^{\prime}=\left(t_{1}, \ldots, t_{\ell_{1}}, r_{1}, \ldots, r_{\ell_{2}}\right), y=\left(t_{1}, \ldots, t_{\ell_{1}}\right)$, and $z=\left(r_{1}, \ldots, r_{\ell_{2}}\right)$ is a $T$-word for $w^{-1} w^{\prime}$. It must be that both $y$ and $z$ are reduced since otherwise $x^{\prime}$ would not be reduced, contradicting Lemma 3.1.4. The fact that $y$ is reduced proves the second statement of the proposition. For the reverse implication of the first statement we see that

$$
\ell_{T}\left(w^{\prime}\right)=\ell_{1}+\ell_{2}=\ell_{T}(w)+\ell_{T}\left(w^{-1} w^{\prime}\right)
$$

and so $w \leq_{T} w^{\prime}$.
Suppose that $w \leq_{T} w^{\prime}$. The next lemma allows us to find a reduced $T$-word $x$ for $w^{\prime}$ such that $x$ either begins or ends with any reduced $T$-word for $w$.

Lemma 3.1.6. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for some $w \in W$ and let $y=\left(t_{i_{1}}, \ldots, t_{i_{k}}\right)$ be a subword of $x$. If $y^{\prime}=\left(r_{i_{1}}, \ldots, r_{i_{k}}\right)$ is any reduced $T$-word for $t_{i_{1}} \cdots t_{i_{k}}$, then there exist $T$-words $x_{1}, x_{2} \in R_{T}(w)$ such that $x_{1}=\left(r_{i_{1}}, \ldots, r_{i_{k}}, \ldots\right)$ and $x_{2}=\left(\ldots, r_{i_{1}}, \ldots, r_{i_{k}}\right)$.

Proof. We show the existence of the $T$-word $x_{1}$, the proof for $x_{2}$ is similar. We can use Hurwitz moves to find a $T$-word $x^{\prime}$ such that $x \stackrel{H}{\sim} x^{\prime}$ and $x^{\prime}=\left(t_{i_{1}}, \ldots, t_{i_{k}}, \ldots\right)$. We know that $x^{\prime}$ is reduced by Lemma 3.1.4. We can now replace $t_{i_{j}}$ with $r_{i_{j}}$ for all $j \in[k]$ in $x^{\prime}$ to get the $T$-word $x_{1}=\left(r_{i_{1}}, \ldots, r_{i_{k}}, \ldots\right)$ which must also be reduced since it has the same length as $x^{\prime}$ and is a $T$-word for $w$.

This next proposition is Lemma 2.4 in [16], and it gives us a useful isomorphism between certain intervals in $\operatorname{Abs}(W)$.

Proposition 3.1.7. Let $W$ be a Coxeter group and suppose that $\pi, \mu \in W$ such that $\pi \leq_{T} \mu$. The map $w \mapsto \pi^{-1} w$ is a poset isomorphism between the intervals $[\pi, \mu]$ and $\left[e, \pi^{-1} \mu\right]$.

Since we are interested in studying up-operators we will need a poset labeling to go along with $\operatorname{Abs}(W)$. This labeling is as follows. If $w, w^{\prime} \in \operatorname{Abs}(W)$ such that $w \lessdot_{T} w^{\prime}$, then $\gamma\left(w, w^{\prime}\right)=w^{-1} w^{\prime}$. As we noted earlier we know that $w^{-1} w^{\prime}=t \in T$. We also know that any $t \in T$ appears as an edge label. To see this simply take $w=e, w^{\prime}=t$. This tells us that our up-operators are the $u_{t}$ such that $t \in T$. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word. For ease of notation we denote by $u_{x}$ the product $u_{t_{1}} \cdots u_{t_{\ell}}$.

### 3.1.3 Coxeter Elements and the Generalized Noncrossing Partition Lattice

Let $(W, S)$ be a Coxeter system where $S=\left\{s_{1}, \ldots, s_{k}\right\}$, and let $\pi$ be any permutation in $\mathfrak{S}_{k}$. A standard Coxeter element is any $c \in W$ such that $c=s_{\pi(1)} \cdots s_{\pi(k)}$, and a Coxeter element is a conjugate of any standard Coxeter element. Coxeter elements are actually a subset of the maximal elements of $\operatorname{Abs}(W)$. In our later discussion on the combinatorics of $A_{n}, B_{n}$, and $D_{n}$ we will see that for $A_{n}$ the set of Coxeter elements and the set of maximal elements coincide, and for $B_{n}$ and $D_{n}$ the Coxeter elements are a proper subset of the maximal elements. Coxeter elements are a key object in the definition of the generalized noncrossing partition lattice and so some knowledge about them is essential to our work. We state some of the properties of Coxeter elements that will prove most useful to us. For further reading on this subject see [1]. We begin with the following lemma.

Lemma 3.1.8. Let c be a Coxeter element of $W$. The following statements are true.

- If $c^{\prime}$ is a Coxeter element of $W$, then $c$ and $c^{\prime}$ are conjugate to one another.
- If $t \in T$, then $t \leq_{T} c$.

Here we give a brief discussion about the proof of the above lemma. To prove the first statement it is sufficient to show that if $c, c^{\prime}$ are standard Coxeter elements, then $c$ and $c^{\prime}$ are conjugate to one another. A proof that $c$ and $c^{\prime}$ are conjugate is given by Humphreys [14, Proposition 3.16]. The second statement is proven by analyzing inclusions of 'moved spaces' which are vector spaces associated to each elements $w \in W$. For the full proof see [1, Lemma 2.6.2].

There is a special type of element called a parabolic Coxeter element that will be essential for our work, particularly for our understanding of reduced $T$-words. However, before defining it we must first introduce the notion of parabolic subgroups of $W$. Let $(W, S)$ be a Coxeter system and let $I \subset S$. We denote by $W_{I}$ the subgroup generated by $I$ and we call it a standard parabolic subgroup of $W$. Furthermore, for any $w \in W$ we call the group $w W_{I} w^{-1}$ a parabolic subgroup of $W$. If $c$ is a Coxeter element for some parabolic subgroup of $W$, then we call $c$ a parabolic Coxeter element of $W$.

There is actually an equivalent way to define parabolic Coxeter groups given by Bessis [5]. (The equivalence of these two definitions is proven in Corollary 4.4 of [4].) This alternate definition requires us to first define a dual Coxeter system, which is a tuple ( $W, T$ ) satisfying the following: there exists some subset $S \subset T$ such that $(W, S)$ is a Coxeter system and $T$ is the conjugate closure of
S. (Actually, in Bessis' definition a tuple for a dual Coxeter system includes a third entry $c$, which is a Coxeter element of $W$; however, this will not be important for us here.) From this definition we see that a dual Coxeter system is simply a representation of a Coxeter group which specifies its set of reflections but does not explicitly state any set of Coxeter generators. This is a useful definition because, aside from trivial cases, a Coxeter group has more than one valid set of Coxeter generators while the set of reflections is always the same.

For a dual Coxeter system let $S \subset T$ be a set of Coxeter generators for $W$ and let $I \subset S$. A parabolic subgroup of $(W, T)$ is a tuple $\left(W_{I}, T_{I}\right)$ where $W_{I}=\langle I\rangle$ and $T_{I}=T \cap W_{I}$. These $W_{I}$ coincide with the parabolic subgroups defined above and so the $\left(W_{I}, T_{I}\right)$ are just dual Coxeter systems for parabolic subgroups. A parabolic Coxeter element is then an element $w \in W$ which is a Coxeter element in some dual parabolic subgroup. The following theorem was proven by Bessis [5] and demonstrates the link between parabolic Coxeter elements and reduced $T$-words.

Theorem 3.1.9. If $w \in W$ is a parabolic Coxeter element, then $R_{T}(w)$ is transitive under the Hurwitz action.

Our study of absolute order and the generalized noncrossing partition lattice will rely heavily on reduced $T$-words and so Theorem 3.1 .9 will appear frequently in the following sections. Another useful result of Bessis [5] is stated in the following Theorem.

Theorem 3.1.10. If $w \in W$, then $w$ is a parabolic Coxeter element if and only if there exists a Coxeter element $c \in W$ such that $w \leq_{T} c$.

We now define the central object of this section.
Definition 3.1.11. Let $c$ be a Coxeter element of $W$. The interval $[e, c]$ in $\operatorname{Abs}(W)$ is called the generalized noncrossing partition lattice and it is denoted by $N C(W, c)$.

While $N C(W, c)$ is defined with respect to a Coxeter element $c$, the structure of $N C(W, c)$ is not dependent on which Coxeter element is chosen. The explanation for this is as follows. Let $c_{1}, c_{2}$ be Coxeter elements of $W$. By Lemma 3.1.8 we know that there exist some $w \in W$ such that $c_{1}=w c_{2} w^{-1}$. Consider the map $\pi$ from $\operatorname{Abs}(W)$ to itself where $\pi\left(w^{\prime}\right)=w w^{\prime} w^{-1}$. This map is a poset automorphism of $\operatorname{Abs}(W)$ and so it must be that $\pi$ is a poset isomorphism from $N C\left(W, c_{1}\right)=\left[e, c_{1}\right]$ to $N C\left(W, c_{2}\right)=\left[e, c_{2}\right]$. Since the $N C(W, c)$ are all equivalent up to isomorphism we denote the generalized noncrossing partition lattice as $N C(W)$ for any $c$.

As its name suggests, $N C(W)$ is a generalization of the noncrossing partition lattice $N C(n)$ (see section 3.1.4). In particular, $N C(n)$ is $N C(W)$ when $W=A_{n-1}$. The name of $N C(W)$ also suggests that it is a lattice and this is stated in the following theorem proven by Brady and Watt [8].

Theorem 3.1.12. The poset $N C(W)$ is a lattice.
The following lemma is due to Armstrong and will prove rather useful in our study of up-operators for $N C(W)$. The lemma's proof relies on use of the moved spaces mentioned in the proof discussion of Lemma 3.1.8.

Lemma 3.1.13. Suppose $x=\left(t_{1}, \ldots, t_{\ell}\right)$ is a reduced $T$-word for $w \in N C(W)$. Then $w=t_{1} \vee t_{2} \vee \cdots \vee t_{\ell}$.
Finally, the edge labeling for $N C(W)$ is the same as the edge labeling used for $\operatorname{Abs}(W)$. Note that by Lemma 3.1.8 we know that $t \leq c$ for all $t \in T$ and as such all $t \in T$ occur as an edge label in $N C(W)$. From this we see that our up-operators for $N C(W)$ are the $u_{t}$ for all $t \in T$.

### 3.1.4 The Classical Types

Of particular interest to us are the classical Coxeter groups of types $A_{n}, B_{n}$, and $D_{n}$ where $n$ is a positive integer (see Figure 3.5 for their Coxeter diagrams). As such, we discuss here useful combinatorial interpretations of these Coxeter groups and their poset structure under absolute order and in the generalized noncrossing partition lattice. These interpretations are taken from both [1] and [16].

### 3.1.4.1 Combinatorics of Type A

The group $A_{n}(n \geq 1)$ is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$ and so it can be viewed as the group of permutations on the integers $1, \ldots, n+1$. More explicitly, $A_{n}$ is the group of all bijections from the set $[n+1]$ to itself where the group operation is composition. The Coxeter generators of $A_{n}$ are the transpositions $s_{i}=(i, i+1)$ for $i \in[n]$ and the set of reflections is the set of all transpositions $T=\{(i, j): 1 \leq i<j \leq n\}$. We say that two cycles $x=\left(a_{1}, \ldots, a_{k}\right), y=\left(a_{k+1}, \ldots, a_{\ell}\right)$ are disjoint if there does not exist any $a_{i}$ appearing in both $x$ and $y$. Every element of $A_{n}$ can be written as a unique (up to commutation) product of disjoint cycles. If $w \in A_{n}$ and $k$ is the number of cycles in the disjoint cycle decomposition of $w$, then $\ell_{T}(w)=n+1-k$.

Let $w \in A_{n}$ and $t \in T$. If $w \lessdot_{T} w t$, then we must have

$$
\begin{equation*}
\left(a_{1}, \ldots, a_{j}\right)\left(a_{j+1}, \ldots, a_{k}\right) \xrightarrow{\left(a_{j}, a_{k}\right)}\left(a_{1}, \ldots, a_{k}\right) \tag{3.3}
\end{equation*}
$$

where $1 \leq i<j<k \leq n+1$ and $a_{1}, \ldots, a_{k}$ are elements of $[n+1$ ] with distinct values. Here the cycles to the left of the arrow are factors in the disjoint decomposition of $w$, the transposition above the arrow is $t$, and the cycle to the right of the arrow is a factor in the disjoint decomposition of $w t$. All cycles in the disjoint decomposition of both $w$ and $t$ which do not appear in (3.3) remain unchanged. We have that (3.3) fully characterizes the covering relations in $\operatorname{Abs}\left(A_{n}\right)$ and we can see from this covering relation that the maximal elements of $\operatorname{Abs}\left(A_{n}\right)$ are exactly the cycles of length $n+1$.

An element of $A_{n}$ is a Coxeter element if and only if it is equal to a single cycle of length $n$. From this we see than that the Coxeter elements of $A_{n}$ coincide exactly with the maximal elements of $N C\left(A_{n}\right)$. It is important to note that in general we can only say that the Coxeter elements of $W$ are a subset of the maximal elements of $\operatorname{Abs}(W)$. We will see that for $B_{n}$ and $D_{n}$ the Coxeter elements are a proper subset of the maximal elements.

As mentioned earlier, the generalized noncrossing partition lattice is a generalization of $N C(n)$, the noncrossing partition lattice. The following is a combinatorial representation of $N C(n)$. Label


Figure 3.2 The Hasse diagram for $N C(4) \cong N C\left(A_{3}\right)$.
the vertices of a regular $n$-gon with $1,2, \ldots, n$ counterclockwise. Let $P=\left\{P_{1}, \ldots, P_{k}\right\}$ be a partition of the numbers $1, \ldots, n$ and let $P_{i}^{\prime}$ be the convex hull of the vertices with labels coming from $P_{i}$. We say that the partition $P$ is noncrossing if for each $i \neq j \in[k]$ we have $P_{i}^{\prime} \cap P_{j}^{\prime}=\emptyset$. The lattice $N C(n)$ is the poset of noncrossing partitions under partition refinement.

In fact, we may use the above combinatorial description of $N C(n)$ for $N C\left(A_{n-1}\right)$ since $N C(n)$ is isomorphic to $N C\left(A_{n-1}\right)$. In the following we describe the isomorphism between these two posets. Recall that the structure of $N C\left(A_{n-1}\right)$ is not dependent on the specific Coxeter element chosen. As such, in defining the isomorphism from $N C\left(A_{n-1}\right)$ to $N C(n)$ we may assume that our Coxeter element is $c=(1, \ldots, n)$. Let $w \in N C\left(A_{n-1}, c\right)$ and suppose that $w=\alpha_{1} \cdots \alpha_{k}$ is the disjoint decomposition of $w$ where the $\alpha_{i}$ are cycles. Our isomorphism is the map $\pi: N C\left(A_{n-1}\right) \rightarrow N C(n)$ such that $\pi(w)=\left\{P_{1}, \ldots, P_{k}\right\}$ and $P_{i}$ is the set of elements appearing in the cycle $\alpha_{i}$. Figure 3.2 shows $N C(4) \cong N C\left(A_{3}\right)$. Each element in the poset is drawn with both its partition representation and its corresponding permutation representation. Also, for each partition $P_{1}, \ldots, P_{k}$ we have drawn the edges of the convex hull corresponding to each $P_{i}$ (if they exist). We see that for $N C\left(A_{3}\right)$ the maximum element $\hat{1}$ is $c$ and the minimum element $\hat{0}$ is the identity $e$.

### 3.1.4.2 Combinatorics of Type B

The group $B_{n}(n \geq 2)$ is the group of all bijections from the set $\{-n, \ldots,-1,1, \ldots, n\}$ (hereafter denoted by $[ \pm n])$ to itself such that if $w \in B_{n}$, then $w(i)=-w(-i)$, and where the group operation is composition. Note that $B_{n}$ is isomorphic to a subgroup of $\mathfrak{S}_{2 n}=A_{2 n-1}$ and so we can think of elements of $B_{n}$ as products of cycles. Let $a_{1}, \ldots, a_{k} \in[ \pm n]$ such that the $a_{i}$ have pairwise distinct absolute values. There are exactly two types of cycles that occur as elements in $B_{n}$. These are cycles of the form $\left(a_{1}, \ldots, a_{k}\right)\left(-a_{1}, \ldots,-a_{k}\right)$ and $\left(a_{1}, \ldots, a_{k},-a_{1}, \ldots,-a_{k}\right)$ which we refer to as paired cycles and balanced cycles respectively. We let $\left(\left(a_{1}, \ldots, a_{k}\right)\right)=\left(a_{1}, \ldots, a_{k}\right)\left(-a_{1}, \ldots,-a_{k}\right)$ and $\left[a_{1}, \ldots, a_{k}\right]=\left(a_{1}, \ldots, a_{k},-a_{1}, \ldots,-a_{k}\right)$. Every $w \in B_{n}$ can be written as a unique (up to commutation) product of disjoint paired and balanced cycles and we call this product the disjoint decomposition of $w$. We often write this disjoint decomposition as $w=b_{1} \cdots b_{k} p_{1} \cdots p_{m}$ where the $b_{i}$ are balanced cycles and the $p_{i}$ are paired cycles. If $k$ is the number of paired cycles in the disjoint cycle decomposition of $w$, then $\ell_{T}(w)=n-k$. The Coxeter generators of $B_{n}$ are the paired cycles ( $\left.(i, i+1)\right)$ for $i=1, \ldots, n-1$ and the balanced cycle [1]. Additionally, the set of reflections is given by

$$
T=\{((i, j)),((i,-j)): 1 \leq i<j \leq n\} \cup\{[i]: 1 \leq i \leq n\} .
$$

We note that the reflection $((i, \pm j))$ for $1 \leq i<j \leq n$ will also sometimes be written as $(( \pm j, i))$ or as ( $\mp j,-i)$ ).

An element $w \in \operatorname{Abs}\left(B_{n}\right)$ is maximal if and only if the disjoint decomposition of $w$ is $b_{1} \cdots b_{k}$ where the sum of the lengths of the $b_{i}$ is equal to $n$. Let $t \in T$ and suppose that $w \lessdot_{T} w t$. The covering relation $w \lessdot_{T} w t$ must look like one of the following:

$$
\begin{align*}
& \left(\left(a_{1}, \ldots, a_{k}\right)\right) \xrightarrow{\left[a_{i}\right]}\left[a_{1}, \ldots, a_{i-1}, a_{i},-a_{i+1}, \ldots,-a_{k}\right]  \tag{3.4}\\
& \left(\left(a_{1}, \ldots, a_{k}\right)\right) \stackrel{\left(\left(a_{i},-a_{j}\right)\right)}{\rightarrow}\left[a_{1}, \ldots, a_{i},-a_{j+1}, \ldots,-a_{k}\right]\left[a_{i+1}, \ldots, a_{j}\right]  \tag{3.5}\\
& {\left[a_{i}, \ldots, a_{j}\right]\left(\left(a_{j+1}, \ldots, a_{k}\right)\right) \xrightarrow{\left(\left(a_{j}, a_{k}\right)\right)}\left[a_{1}, \ldots, a_{k}\right]}  \tag{3.6}\\
& \left(\left(a_{1}, \ldots, a_{j}\right)\right)\left(\left(a_{j+1}, \ldots, a_{k}\right)\right) \xrightarrow{\left(\left(a_{j}, a_{k}\right)\right)}\left(\left(a_{1}, \ldots, a_{k}\right)\right) \tag{3.7}
\end{align*}
$$

where the given cycles are related to $w, t$, and $w t$ in the same way as in the covering relation description for $\operatorname{Abs}\left(A_{n}\right)$. Note that every covering relation other than (3.5) results in cycles of $w$ merging.

An element $w$ is a Coxeter element in $B_{n}$ if and only if the disjoint decomposition of $w$ consists of a single balanced cycle of length $n$, and it is a parabolic Coxeter element in $B_{n}$ if and only if its disjoint decomposition contains at most one balanced cycle. Again, since the structure of $N C\left(B_{n}\right)$ does not depend on our specific choice of Coxeter element we may take our Coxeter element to be $c=[1, \ldots, n]$. We now describe a combinatorial interpretation of $N C\left(B_{n}\right)$. Label the vertices of a regular $2 n$-gon with the numbers $1, \ldots, n,-1, \ldots,-n$ while moving counterclockwise, let $P=\left\{P_{1}, \ldots, P_{k}\right\}$ be a partition of the numbers $1, \ldots, n,-1, \ldots,-n$ and let $P_{i}^{\prime}$ be the convex hull of


Figure 3.3 The type $B$ noncrossing partition for $\{\{1,2,-1,-2\},\{3,4\},\{-3,-4\}\}$.
the vertices whose labels are in $P_{i}$. We say that $P$ is a type $B$ noncrossing partition if for all $i \neq j \in[k]$ we have $P_{i}^{\prime} \cap P_{j}^{\prime}=\emptyset$ and the set $P^{\prime}=P_{1}^{\prime} \cup \cdots \cup P_{k}^{\prime}$ is invariant under a 180 degree rotation in the plane. Figure 3.3 shows the type $B$ noncrossing partition for $\{\{1,2,-1,-2\},\{3,4\},\{-3,-4\}\}$. Note that in the figure we have drawn the edges of the convex hulls given by each block in the partition. Since type $B$ noncrossing partitions need to be symmetric under 180 degree rotations, they are also sometime referred to as centrally symmetric partitions. The set of type $B$ noncrossing partitions is a poset under partition refinement.

A zero block of a partition $P$ is any block containing both $i$ and $-i$ for some $i \in[n]$. Note that if $P=\left\{P_{1}, \ldots, P_{k}\right\}$ is a type $B$ noncrossing partition, then the following must hold: $P$ contains at most one zero block, and for all $i \in[k]$ such that $P_{i}$ is not the zero block, there exists some $i \neq j \in[k]$ such that $P_{j}=-P_{i}$. Here $-P_{i}$ is just the set of elements in $P_{i}$ with their signs reversed. From this we see that we can write each type $B$ noncrossing partition as $P=\left\{P_{0}, P_{1},-P_{1}, \ldots, P_{k},-P_{k}\right\}$. Using this form for $P$ we define the isomorphism $\pi$ from the poset of type $B$ noncrossing partitions to $N C\left(B_{n}, c\right)$ as follows. We let $\pi(P)=m_{0} m_{1} \cdots m_{k}$ where $m_{0}$ is the balanced cycle whose elements come from $P_{0}$ and $m_{i}=\left(\left(a_{1}, \ldots, a_{\delta}\right)\right)$ is the paired cycle such that $a_{1}, \ldots, a_{\delta}$ are the elements of $P_{i}$ for $i=1, \ldots, k$. Furthermore, the cyclic order of the elements in $m_{i}$ is chosen to be consistent with the cyclic order in our Coxeter element $c=[1, \ldots, n]$. For instance if $P=\{\{1,2,-1,-2\},\{3,4\},\{-3,-4\}\}$, then $\pi(P)=[1,2]((3,4))$. Figure 3.4 shows the Hasse diagram for $N C\left(B_{2}\right)$.

### 3.1.4.3 Combinatorics of Type D

The group $D_{n}$ is a rank 2 subgroup of $B_{n}$ which consists of all elements of $B_{n}$ whose disjoint decomposition contains an even number of balanced cycles. The Coxeter generators of $D_{n}$ are the paired cycles $((i, i+1))$ for $i=1, \ldots, n-1$ and $((1,-2))$, and

$$
T=\{((i, j)),((i,-j)): 1 \leq i<j \leq n\} .
$$

An element $w \in B_{n}$ is contained in $D_{n}$ if and only if the disjoint decomposition of $w$ contains an even number of balanced cycles. The absolute length of $w$ in $D_{n}$ is equal to its absolute length in $B_{n}$. Also, $\operatorname{Abs}\left(D_{n}\right)$ inherits all of the covering relations from $\operatorname{Abs}\left(B_{n}\right)$ except covering relation (3.4).


Figure 3.4 The Hasse diagram for $N C\left(B_{2}\right)$

An element $w \in D_{n}$ is a Coxeter element if and only if $w=\left[a_{1}, \ldots, a_{n-1}\right]\left[a_{n}\right]$ where $a_{1}, \ldots, a_{n} \in$ [ $\pm n$ ] and have distinct absolute values. As with $A_{n}$ and $B_{n}$, we can view $D_{n}$ as a poset on certain special types of partitions. The combinatorial construction for $D_{n}$ was introduced by Reiner and Athanasiadis [3]. Since it will not be important for us we do not give the construction.

### 3.2 The Algebra of Up-operators for $N C(W)$

In this section we determine the defining relations for the algebra of up-operators for $N C(W)$. Let $\mathscr{U}^{N C(W)}$ be the free associative algebra over $\mathbf{C}$ generated by the $u_{t}$ for $t \in T$. Furthermore, let $I^{N C(W)}$ be the ideal of $\mathscr{U}^{N C(W)}$ containing all elements which annihilate $\mathbf{C}[N C(W)]$. Here $\mathbf{C}[N C(W)]$ is the vector space over $\mathbf{C}$ with basis $N C(W)$. We call $\mathscr{U}^{N C(W)} / I^{N C(W)}$ the algebra of up-operators for $N C(W)$. Our goal is to prove the following theorem.

Theorem 3.2.1. The algebra of up-operators for $N C(W)$ is defined by the following degree 2 relations:

$$
\begin{align*}
& u_{t} u_{t^{\prime}} \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}},  \tag{3.8}\\
& u_{t} u_{t^{\prime}} \equiv u_{t t^{\prime} t} u_{t}, \text { and }  \tag{3.9}\\
& u_{r} u_{r^{\prime}} \equiv 0 \tag{3.10}
\end{align*}
$$

where $\left(t, t^{\prime}\right) \in R_{T}(w)$ for some $w \in N C(W)$ and $r=r^{\prime}$, or $r r^{\prime} \notin N C(W)$.


Figure 3.5 The finite Coxeter groups and their Coxeter diagrams.

Before moving on we first discuss these relations a bit. The left side of relation (3.8) is equal to $u_{x}$ where $x=\left(t, t^{\prime}\right)$ and the right side of relation (3.8) is equal to $u_{y}$ where $y=\left(t^{\prime}, t^{\prime} t t^{\prime}\right)$. Noting that $x \stackrel{H}{\sim} y$ we see that relation (3.8) is necessary to account for Hurwitz equivalence of reduced $T$-words. By a similar analysis we can see that (3.9) is also a result of Hurwitz equivalence. More specifically, relation (3.8) accounts for the Hurwitz move (3.1) and relation (3.9) accounts for the Hurwitz move (3.2). Note that if we set $r=t^{\prime}, r^{\prime}=t^{\prime} t t^{\prime}$ and substitute these into relation (3.8), then the result is actually relation (3.9) and we see that these two relations are actually equivalent. Despite this equivalence we state them separately to make clear their connection to the Hurwitz moves. Finally, relation (3.10) reflects the fact that if $\left(r, r^{\prime}\right)$ is not a reduced $T$-word for some element of $W$, then it is not possible to find $r$ and $r^{\prime}$ as consecutive labels in $N C(W)$.

The process for proving this Theorem will be similar to the process we used for the algebra of Schur operators. Specifically, it is sufficient to show that $I^{N C(W)}=J^{N C(W)}$ where $J^{N C(W)}$ is the ideal of $U^{N C(W)}$ generated by relations (3.8)-(3.10). In this section we will denote $\mathscr{U}^{N C(W)}, I^{N C(W)}, J^{N C(W)}$ by $\mathscr{U}, I, J$ respectively.

As usual, our first goal is to determine how a monomial in the $u_{t}$ acts on an element of our poset $N C(W)$. Recall that for $N C(W)$ and $\operatorname{Abs}(W)$ our operators will act on the right. We have the following proposition.

Proposition 3.2.2. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word and $w \in N C(W)$. Then

$$
(w) u_{x}= \begin{cases}w t_{1} \cdots t_{\ell} & \text { if } w t_{1} \cdots t_{\ell} \leq_{T} c \text { and } \ell_{T}\left(w t_{1} \cdots t_{\ell}\right)=\ell_{T}(w)+\ell \\ 0 & \text { otherwise }\end{cases}
$$

Proof. Suppose that $(w) u_{x}=(w) u_{t_{1}} \cdots u_{t_{\ell}} \neq 0$. Then for all $i \in[\ell]$ we have

$$
\left(w t_{1} \cdots t_{i-1}\right) u_{t_{i}} \neq 0
$$

By the definition of our up-operators and the fact that $\ell_{T}$ is the rank function for $N C(W)$ we have $\left(w t_{1} \cdots t_{i-1}\right) u_{t_{i}}=w t_{1} \cdots t_{i} \leq_{T} c$ and $\ell_{T}\left(w t_{1} \cdots t_{i}\right)=\ell_{T}\left(w t_{1} \cdots t_{i-1}\right)+1$. From this we see that when $i=\ell$ we must have $w t_{1} \cdots t_{\ell} \leq_{T} c$ and $\ell_{T}\left(w t_{1} \cdots t_{\ell}\right)=\ell_{T}(w)+\ell$.

Now suppose $(w) u_{x}=0$. Let $i \in[\ell]$ be the smallest such that $\left(w t_{1} \cdots t_{i-1}\right) u_{i}=0$. This implies that $w t_{1} \cdots t_{i}$ does not cover $w t_{1} \cdots t_{i-1}$ in $N C(W)$. If $w t_{1} \cdots t_{i} \notin N C(W)$, then we know $w t_{1} \cdots t_{i} \not \leq_{T} c$. On the other hand if $w t_{1} \cdots t_{i} \in N C(W)$, then we must have $\ell_{T}\left(w t_{1} \cdots t_{i}\right) \leq \ell_{T}\left(w t_{1} \cdots t_{i-1}\right)$. From this we see that:

$$
\begin{aligned}
\ell_{T}\left(w t_{1} \cdots t_{\ell}\right) & \leq \ell_{T}\left(w t_{1} \cdots t_{i}\right)+\ell-i \\
& \leq \ell_{T}\left(w t_{1} \cdots t_{i-1}\right)+\ell-i \\
& =\ell_{T}(w)+i-1+\ell-i \\
& =\ell_{T}(w)+\ell-1 \\
& <\ell_{T}(w)+\ell .
\end{aligned}
$$

Now that we understand how a monomial $u_{x}$ acts on an element of $N C(W)$, we can use this to understand exactly when $u_{x} \equiv 0(\bmod I)$ and when $u_{x} \equiv u_{y}(\bmod I)$. This leads to the following two corollaries.

Corollary 3.2.3. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word. Then $u_{x} \equiv 0(\bmod I)$ if and only if $x \notin R_{T}(w)$ for any $w \in N C(W)$.

Proof. If $u_{x} \equiv 0(\bmod I)$, then $(e) u_{x}=0$ where $e$ is the identity element in $N C(W)$. By Proposition 3.2.2 we know that

$$
\begin{aligned}
t_{1} \cdots t_{\ell} & \neq c, \text { or } \\
\ell_{T}\left(t_{1} \cdots t_{\ell}\right) & \neq \ell .
\end{aligned}
$$

Either statement implies that $x \notin R_{T}(w)$ for any $w \in N C(W)$.
Suppose now that $u_{x} \not \equiv 0(\bmod I)$. There must then exist some $w^{\prime} \in N C(W)$ such that $\left(w^{\prime}\right) u_{x} \neq 0$. By Proposition 3.2.2 we know that $w^{\prime} t_{1} \cdots t_{\ell} \in N C(W)$ and $\ell_{T}\left(w^{\prime} t_{1} \cdots t_{\ell}\right)=\ell^{\prime}+\ell$ where $\ell^{\prime}=\ell_{T}\left(w^{\prime}\right)$. Let $\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$ be a reduced $T$-word for $w^{\prime}$. Since $t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime} t_{1} \cdots t_{\ell}=w^{\prime} t_{1} \cdots t_{\ell}$ and $\ell_{T}\left(w^{\prime} t_{1} \cdots t_{\ell}\right)=\ell^{\prime}+\ell$ we know that $\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}, t_{1}, \ldots, t_{\ell}\right)$ is a reduced $T$-word for $w^{\prime} t_{1} \cdots t_{\ell}$. By the subword property from Proposition 3.1.5 we know that $w=t_{1} \cdots t_{\ell} \leq_{T} w^{\prime} t_{1} \cdots t_{\ell} \leq_{T} c$ and so $w=t_{1} \cdots t_{\ell} \in N C(W)$. We also know from Proposition 3.1.5 that $\ell_{T}(w)=\ell$ and so we see that $x=\left(t_{1}, \ldots, t_{\ell}\right) \in R_{T}(w)$.

Corollary 3.2.4. Let $x, y$ be $T$-words such that $u_{x}, u_{y} \not \equiv 0(\bmod I)$. Then $u_{x} \equiv u_{y}(\bmod I)$ if and only if $x, y \in R_{T}(w)$ for some $w \in N C(W)$.

Proof. Suppose that $u_{x} \equiv u_{y}(\bmod I)$ where $x=\left(t_{1}, \ldots, t_{\ell}\right), y=\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$. We assume that $u_{x}, u_{y} \not \equiv 0$ $(\bmod I)$ and so there exists some $w^{\prime} \in N C(W)$ such that $\left(w^{\prime}\right) u_{x}=\left(w^{\prime}\right) u_{y} \neq 0$. By Corollary 3.2.3 we know that $x$ and $y$ are both reduced $T$-words so it remains only to show that $x$ and $y$ are reduced $T$-words for the same $w \in N C(W)$. Indeed by Proposition 3.2.2 we have that

$$
\left(w^{\prime}\right) u_{x}=w^{\prime} t_{1} \cdots t_{\ell}=w^{\prime} t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}=\left(w^{\prime}\right) u_{y} .
$$

Multiplying the expressions above by $w^{-1}$ on the left gives us $t_{1} \cdots t_{\ell}=t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}=w$ and so $x, y \in$ $R_{T}(w)$.

Now suppose $x, y \in R_{T}(w)$ for some $w \in N C(W)$ and $u_{x}, u_{y} \not \equiv 0(\bmod I)$. We first wish to show that $\left(w^{\prime}\right) u_{x} \neq 0$ if and only if $\left(w^{\prime}\right) u_{y} \neq 0$ for any $w^{\prime} \in N C(W)$. We show only one direction since the other is identical. If $\left(w^{\prime}\right) u_{x} \neq 0$, then by Proposition 3.2.2 we have $w^{\prime} t_{1} \cdots t_{\ell} \leq_{T} c$ and $\ell_{T}\left(w^{\prime} t_{1} \cdots t_{\ell}\right)=\ell_{T}\left(w^{\prime}\right)+\ell$. From this we see that

$$
\begin{aligned}
w^{\prime} t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime} & =w^{\prime} t_{1} \cdots t_{\ell} \leq_{T} c \text { and } \\
\ell_{T}\left(w^{\prime} t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}\right) & =\ell_{T}\left(w^{\prime} t_{1} \cdots t_{\ell}\right)=\ell_{T}\left(w^{\prime}\right)+\ell
\end{aligned}
$$

and so by Proposition 3.2.2 we have $\left(w^{\prime}\right) u_{y} \neq 0$. Using Proposition 3.2.2 we see that $\left(w^{\prime}\right) u_{x}=\left(w^{\prime}\right) u_{y}$ and so we have $u_{x} \equiv u_{y}(\bmod I)$.

We can now prove that $J \subset I$, which takes us halfway to our ultimate goal of showing $J=I$.
Corollary 3.2.5. We have that $J \subset I$.
Proof. It suffices to show that modulo $I$ we have

$$
\begin{aligned}
& u_{t} u_{t^{\prime}} \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}} \\
& u_{t} u_{t^{\prime}} \equiv u_{t t^{\prime} t} u_{t}, \text { and } \\
& u_{r} u_{r^{\prime}} \equiv 0
\end{aligned}
$$

where $\left(t, t^{\prime}\right) \in R_{T}(w)$ for some $w \in N C(W)$ and $r=r^{\prime}$ or $r r^{\prime} \notin N C(W)$. If $\left(t, t^{\prime}\right) \in R_{T}(w)$ then $\left(t^{\prime}, t^{\prime} t t^{\prime}\right) \in R_{T}(w)$ since its length is 2 and $t t^{\prime}=t^{\prime}\left(t^{\prime} t t^{\prime}\right)$. By Corollary 3.2.3 we have $u_{t} u_{t^{\prime}}, u_{t^{\prime}} u_{t^{\prime} t t^{\prime}} \not \equiv$ $0(\bmod I)$ and so we can apply Corollary 3.2.4 to see that $u_{t} u_{t^{\prime}} \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}(\bmod I)$. The second equivalence can be proven similarly. For the third equivalence note that if $r=r^{\prime}$ or $r r^{\prime} \notin N C(W)$ then $u_{r} u_{r^{\prime}} \equiv 0(\bmod I)$ by Corollary 3.2.3.

As was the case with Young's lattice, we will see that $I$ is in fact a binomial ideal. The next lemma will be useful in proving this fact.

Lemma 3.2.6. Let $x=\left(t_{1}, \ldots, t_{\ell}\right), y=\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$ be $T$-words. If there exists some $w \in N C(W)$ such that $(w) u_{x}=(w) u_{y} \neq 0$, then $u_{x} \equiv u_{y}(\bmod I)$.

Proof. Suppose $(w) u_{x}=(w) u_{y} \neq 0$ for some $w \in N C(W)$. By Corollary 3.2.3 we know that $x \in$ $R_{T}\left(w^{\prime}\right)$ and $y \in R_{T}\left(w^{\prime \prime}\right)$ for some $w^{\prime}, w^{\prime \prime} \in N C(W)$. By Proposition 3.2.2 we know that

$$
(w) u_{x}=w t_{1} \cdots t_{\ell}=w t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}=(w) u_{y}
$$

Multiplying the above expressions by $w^{-1}$ on the left tells us that

$$
t_{1} \cdots t_{\ell}=t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}
$$

and so $w^{\prime}=w^{\prime \prime}$. Since $x, y \in R_{T}\left(w^{\prime}\right)$ we know by Corollary 3.2.4 that $u_{x} \equiv u_{y}(\bmod I)$.
Proposition 3.2.7. The ideal $I$ is generated by elements of the form $u_{x}-u_{y}$ where $x, y \in R_{T}(w)$ for some $w \in N C(W)$ and elements of the form $u_{z}$ where $z \notin R_{T}(w)$ for any $w \in N C(W)$.

Proof. Let $I^{\prime}$ be the ideal of $\mathscr{U}$ generated by the elements described above. By Corollaries 3.2.4 and 3.2.3 we have $I^{\prime} \subset I$. Let $R$ be any element of $I$. Then there exists $R^{\prime}$ such that $R^{\prime} \equiv R\left(\bmod I^{\prime}\right)$ and

$$
R^{\prime}=\sum_{i=1}^{m} c_{i} u_{x(i)}
$$

where $c_{i} \in \mathbf{C}$ and $u_{x(i)} \not \equiv 0(\bmod I)$ for all $i \in[m]$ and $u_{x(i)} \not \equiv u_{x(j)}(\bmod I)$ for all $i \neq j \in[m]$. For each $i \in[m]$ let $w_{i} \in W$ such that $\left(w_{i}\right) u_{x(i)}=w_{i}^{\prime} \neq 0$. By Proposition 1.1.3 we know that $x(i)$ appears in the
interval $\left[w_{i}, w_{i}^{\prime}\right.$ ]. By Proposition 3.1.7 we know that $\left[w_{i}, w_{i}^{\prime}\right]$ is isomorphic to $\left[e, w_{i}^{-1} w_{i}^{\prime}\right]$ and so $x(i)$ appears in the interval $\left[e, w_{i}^{-1} w_{i}^{\prime}\right]$. By Proposition 1.1.3 we must then have $(e) u_{x(i)}=w_{i}^{-1} w_{i}^{\prime} \neq 0$ for all $i \in[m]$. Furthermore, the $(e) u_{x(i)}$ are all distinct by Lemma 3.2.6. Since $R^{\prime} \in I$ we have $(e) R^{\prime}=0$ and since the $(e) u_{x(i)}$ are all distinct and nonzero we see that $(e) R^{\prime}=0$ only if $c_{i}=0$ for all $i \in[m]$. We then have $R^{\prime}=0 \equiv R\left(\bmod I^{\prime}\right)$ and so $I \subset I^{\prime}$.

The final portion of this section is dedicated to proving $I \subset J$. Since we now know that $I$ is a binomial ideal, it will be sufficient to show that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$ and $u_{x} \equiv 0(\bmod I)$ implies $u_{x} \equiv 0(\bmod J)$.

Proposition 3.2.8. If $x, y$ are $T$-words such that $u_{x} \equiv u_{y} \neq 0(\bmod I)$, then $u_{x} \equiv u_{y}$ modulo relations (3.8) and (3.9) and so $u_{x} \equiv u_{y}(\bmod J)$.

Proof. We know by Proposition 3.2.4 that $x, y \in R_{T}(w)$ for some $w \in N C(W)$. We also know by Theorem 3.1.10 that $w$ is a parabolic Coxeter element and so by Theorem 3.1.9 we have

$$
x \sim H_{i} .
$$

We now wish to prove that if $x \stackrel{H}{\sim} y$, then $u_{x} \equiv u_{y}(\bmod J)$. It suffices to consider two cases. The first is when $x=\left(t_{1}, \ldots, t_{i}, t_{i+1}, \ldots t_{\ell}\right)$ and $y=\left(t_{1}, \ldots, t_{i+1}, t_{i+1} t_{i} t_{i+1}, \ldots, t_{\ell}\right)$ and the second is when $x=\left(t_{1}, \ldots, t_{i}, t_{i+1}, \ldots, t_{\ell}\right)$ and $y=\left(t_{1}, \ldots, t_{i} t_{i+1} t_{i}, t_{i}, \ldots, t_{\ell}\right)$. We consider the first case; the second case is similar. We have

$$
\begin{aligned}
& u_{x}=u_{t_{1}} \cdots u_{t_{i}} u_{t_{i+1}} \cdots u_{t_{\ell}} \text { and } \\
& u_{y}=u_{t_{1}} \cdots u_{t_{i+1}} u_{t_{i+1} t_{i} t_{i+1}} \cdots u_{t_{\ell}} .
\end{aligned}
$$

Here we see that $u_{x} \equiv u_{y}$ modulo relation (3.8) and so $u_{x} \equiv u_{y}(\bmod J)$.
Now that we have shown $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$ it remains only to show that $u_{x} \equiv 0(\bmod I)$ implies $u_{x} \equiv 0(\bmod J)$. This is proven below in Proposition 3.2.10. The following lemma is needed for the proof of this proposition.

Lemma 3.2.9. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in N C(W)$, and let $t \in T$ such that $t \neq t_{i}$ and $t_{i} t \in N C(W)$ for all $i \in[\ell]$. Then $w t \in N C(W)$ and $y=\left(t_{1}, \ldots, t_{\ell}, t\right) \in R_{T}(w t)$.

The main idea of the following proof is to use the lattice structure of $N C(W)$ to show that $c t$ lies above the join of $t_{1}, \ldots, t_{\ell}$ and to then use this to prove the lemma statement.

Proof. First note that since $x$ is a reduced $T$-word for $w$ we know by Lemma 3.1.13 that $w=t_{1} \vee \cdots \vee t_{\ell}$. Now let $w^{\prime}=c t$ where $c$ is the Coxeter element (and also the maximum element) for $N C(W)$. We wish to prove two things about $w^{\prime}$ : first that $w^{\prime} \in N C(W)$ and second that $t_{i} \leq_{T} w^{\prime}$ for all $i \in[\ell]$.

We first prove that $w^{\prime} \in N C(W)$. By assumption we have $t_{i} t \leq_{T} c$ for all $i \in[\ell]$ and so by Proposition 3.1.5 we know that that $t_{i} t(i \in[\ell])$ appears as an arbitrary subword of a reduced $T$-word for $c$.

Using this, along with Lemma 3.1.6, we can construct a $T$-word of the form $\left(r_{1}, \ldots, r_{k}, t_{i}, t\right)$ which is in $R_{T}(c)$. We see that $w^{\prime}=r_{1} \cdots r_{k} t_{i}$ which means that $w^{\prime}$ appears as a subword of $\left(r_{1}, \ldots, r_{k}, t_{i}, t\right)$. By Proposition 3.1.5 we then have $w^{\prime} \leq_{T} c$ which implies $w^{\prime} \in N C(W)$. To prove that $t_{i} \leq_{T} w^{\prime}(i \in[\ell])$ we first note that $\left(r_{1}, \ldots, r_{k}, t_{i}\right) \in R_{T}\left(w^{\prime}\right)$ by Proposition 3.1.5. We then see that $t_{i}$ is a subword of $\left(r_{1}, \ldots, r_{k}, t_{i}\right)$ and so we know by Proposition 3.1.5 that $t_{i} \leq_{T} w^{\prime}$ for all $i \in[\ell]$.

We now use $w$ and $w^{\prime}$ to prove our desired result. Since $w=t_{1} \vee \cdots \vee t_{\ell}$ and $t_{i} \leq_{T} w^{\prime}(i \in[\ell])$ we have $w \leq_{T} w^{\prime}$. We then have by Proposition 3.1.5 and Lemma 3.1.6 that there exists an element in $R_{T}\left(w^{\prime}\right)$ of the form $z=\left(t_{1}, \ldots, t_{\ell}, \ldots\right)$. Also, since $t \leq_{T} c$ (by Lemma 3.1.8) there exists an element of $R_{T}(c)$ of the form $\left(r_{1}^{\prime}, \ldots, r_{k}^{\prime}, t\right)$ (by Proposition 3.1.5 and Lemma 3.1.6) where $z^{\prime}=\left(r_{1}^{\prime}, \ldots, r_{k}^{\prime}\right) \in R_{T}\left(w^{\prime}\right)$. We now substitute $z^{\prime}$ with $z$ in $\left(r_{1}^{\prime}, \ldots, r_{k}^{\prime}, t\right)$ to get $g=\left(t_{1}, \ldots, t_{\ell}, \ldots, t\right) \in R_{T}(c)$. We now use Lemma 3.1.6 to get the $T$-word $g^{\prime}=\left(t_{1}, \ldots, t_{\ell}, t, \ldots\right) \in R_{T}(c)$. Finally, we see by Proposition 3.1.5 that $w t=$ $t_{1} \cdots t_{\ell} t \leq_{T} c$ which implies that $w t \in N C(W)$, and furthermore, that $y=\left(t_{1}, \ldots, t_{\ell}, t\right) \in R_{T}(w t)$.

Proposition 3.2.10. If $x$ is a $T$-word such that $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$.
Proof. If $x=\left(t_{1}, \ldots, t_{\ell}\right)$ and $u_{x} \equiv 0(\bmod I)$, then there must exist some $\iota \in[\ell]$ such that $u_{y} \not \equiv 0$ $(\bmod I)$ for $y=\left(t_{1}, \ldots, t_{l}\right)$. From this we see that it is sufficient to consider the case where $x=$ $\left(t_{1}, \ldots, t_{\ell}\right), y=\left(t_{1}, \ldots, t_{\ell-1}\right)$ and $u_{x} \equiv 0(\bmod I), u_{y} \not \equiv 0(\bmod I)$. For any $i \in[\ell-1]$, repeated application of (3.9) on $u_{y}$ yields

$$
\begin{aligned}
u_{x} & =u_{y} u_{t_{\ell}} \\
& =u_{t_{1}} \cdots u_{t_{i}} \cdots u_{t_{\ell-1}} u_{t_{\ell}} \\
& \equiv u_{t_{1}} \cdots u_{t_{i} t_{i+1} t_{i}} u_{t_{i} t_{i+2} t_{i} \cdots u_{t_{i} t_{\ell-1} t_{i}} u_{t_{i}} u_{t_{\ell}}(\bmod J) .} .
\end{aligned}
$$

If $u_{t_{i}} u_{t_{\ell}} \equiv 0(\bmod J)$ for some $i \in[\ell-1]$, then $u_{x} \equiv 0(\bmod J)$ and we are done. Assume for contradiction that instead we have $u_{t_{i}} u_{t_{\ell}} \not \equiv 0(\bmod J)$ for all $i \in[\ell-1]$. This assumption along with Corollaries 3.2.3 and 3.2.5 imply that $\left(t_{i}, t_{\ell}\right) \in R_{T}\left(t_{i} t_{\ell}\right)$ and $t_{i} t_{\ell} \in N C(W)$ for all $i \in[\ell-1]$, which can only occur if $t_{\ell} \neq t_{i}$. Note that by Corollary 3.2.3 we have that $y \in R_{T}\left(t_{1} \cdots t_{\ell-1}\right)$. We see then that $y$ satisfied the hypothesis of Lemma 3.2.9 (where $t_{\ell}$ takes the role of $t$ ) and so we have that $t_{1} \cdots t_{\ell-1} t_{\ell} \in N C(W)$ and $x=\left(t_{1}, \ldots, t_{\ell-1}, t_{\ell}\right) \in R_{T}\left(t_{1} \cdots t_{\ell-1} t_{\ell}\right)$. Corollary 3.2.3 then implies that $u_{x} \not \equiv 0(\bmod I)$ which contradicts our assumption that $u_{x} \equiv 0(\bmod I)$.

Using all of our preceding work we are now prepared to prove the main result of this section, Theorem 3.2.1.

Proof. It suffices to show that $I=J$. By Corollary 3.2.5 we have that $J \subset I$. We know that $I$ is a binomial ideal by Proposition 3.2.7 and so to prove $I \subset J$ we need only show that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$ and $u_{x} \equiv 0(\bmod I)$ implies $u_{x} \equiv 0(\bmod J)$. These statements are proven in Proposition 3.2.8 and Proposition 3.2.10 respectively.

### 3.3 The Algebra of Up-operators for $\operatorname{Abs}(W)$ for Classical Types

Here we consider the up-operators for $\operatorname{Abs}(W)$ when $W=A_{n}, B_{n}$, and $D_{n}$. Similar to the cases for Young's lattice and $N C(W)$ we let $\mathscr{U}^{A b s(W)}$ be the associative free algebra over $\mathbf{C}$ generated by the $u_{t}$ for $t \in T$. We also let $I^{A b s(W)}$ be the two-sided ideal of $\mathscr{U}^{A b s(W)}$ containing all elements of $\mathscr{U}^{A b s(W)}$ which annihilate all elements of $\mathbf{C}[A b s(W)]$. We call $\mathscr{U}^{A b s(W)} / I^{A b s(W)}$ the algebra of up-operators for $\operatorname{Abs}(W)$. When there will be no confusion we refer to $\mathscr{U}^{A b s(W)}, I^{A b s(W)}$ as $\mathscr{U}, I$ respectively.

We begin by stating some results which hold for all possible choices of $W$ after which we will consider the cases when $W=A_{n}, B_{n}, D_{n}$ separately. For each of the classical Coxeter groups we will give a list of defining relations for the algebra of up-operators for $\operatorname{Abs}(W)$ and we will let $J^{A b s(W)}$ be the two-sided ideal of $\mathscr{U}$ generated by those relations. We will denote $J^{A b s(W)}$ by $J$ when the choice of $W$ is clear. Our task will be to show that for each of $A_{n}, B_{n}, D_{n}$ we have $I=J$.

Many of the following statements have similar counterparts in subsection 3.2 and so we will omit proofs in those cases. As usual our first task is to understand how a monomial $u_{x}$ acts on elements of $\operatorname{Abs}(W)$ and our second task is to determine when $u_{x} \equiv u_{y}(\bmod I)$ and when $u_{x} \equiv 0$ $(\bmod I)$. This leads to the following proposition and corollaries.

Proposition 3.3.1. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word and $w \in W$. Then

$$
(w) u_{x}= \begin{cases}w t_{1} \cdots t_{\ell} & \text { if } \ell_{T}\left(w t_{1} \cdots t_{\ell}\right)=\ell_{T}(w)+\ell \\ 0 & \text { otherwise }\end{cases}
$$

Corollary 3.3.2. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word. Then $u_{x} \equiv 0(\bmod I)$ if and only if $x \notin R_{T}(w)$ for any $w \in W$.

Corollary 3.3.3. Let $x, y$ be $T$-words such that $u_{x}, u_{y} \not \equiv 0(\bmod I)$. Then $u_{x} \equiv u_{y}(\bmod I)$ if and only if $x, y \in R_{T}(w)$ for some $w \in W$.

We see that once again our algebra is nice in that $I$ is a binomial ideal. This follows from the following lemma and proposition.

Lemma 3.3.4. Let $x, y$ be $T$-words. If there exists some $w \in W$ such that $(w) u_{x}=(w) u_{y} \neq 0$, then $u_{x} \equiv u_{y}(\bmod I)$.

Proposition 3.3.5. The ideal I is generated by elements of the form $u_{x}-u_{y}$ where $x, y \in R_{T}(w)$ for some $w \in W$ and elements of the form $u_{z}$ where $z \notin R_{T}(w)$ for any $w \in W$.

Finally, we prove the following proposition which will be necessary in each of the cases we study.
Proposition 3.3.6. Let $w \in A b s(W)$ and $t \in T$. If $w t$ does not cover $w$ in $A b s(W)$, then $w$ covers $w t$.
Proof. Suppose that $w t$ does not cover $w$. We first wish to determine $\ell_{T}(w t)$. Note that we must have $\ell_{T}(w t) \leq \ell_{T}(w)+1$. To see this let $\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w$. We see that $\left(t_{1}, \ldots, t_{\ell}, t\right)$ is a $T$-word (not necessarily reduced) for $w t$ and it has $\ell_{T}(w)+1$ entries. Additionally, since $w=(w t) t$
we have by similar reasoning that $\ell_{T}(w) \leq \ell_{T}(w t)+1$ and so $\ell_{T}(w t) \geq \ell_{T}(w)-1$. We know that $\ell_{T}(w t) \neq \ell_{T}(w)+1$ since otherwise the definition of absolute order would imply that $w t$ covers $w$ and we cannot have $\ell_{T}(w t)=\ell_{T}(w)$ since, due to parity, multiplication by a reflection must result in an element of $W$ with an absolute length different from that of $w$. The only possibility left is that $\ell_{T}(w t)=\ell_{T}(w)-1$. Using the definition of absolute order we see that $w t \leq_{T} w$. In fact, since $\ell_{T}$ is the rank function for $\operatorname{Abs}(W)$ it must be that $w t$ is covered by $w$.

### 3.3.1 Type $A$

In this subsection we let $W=A_{n}$. We wish to prove the following Theorem.
Theorem 3.3.7. The algebra of up-operators for $\operatorname{Abs}\left(A_{n}\right)$ is generated by the following relations:

$$
\begin{align*}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}},  \tag{3.11}\\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}, \text { and }  \tag{3.12}\\
u_{t} u_{t} & \equiv 0 \tag{3.13}
\end{align*}
$$

where $t \neq t^{\prime} \in T$.
These relations are the same as those needed for the algebra of up-operators on $N C\left(A_{n}\right)$ with the exception of the conditions on $t$ and $t^{\prime}$. The restrictions on $t, t^{\prime}$ in relations (3.8) and (3.9) ensured that $\left(t, t^{\prime}\right)$ was a reduced word for some element in $N C\left(A_{n}\right)$. For relations (3.11) and (3.12) we again want to ensure that $\left(t, t^{\prime}\right)$ is a reduced word for some element in our current poset of interest $\operatorname{Abs}\left(A_{n}\right)$. Since our poset $\operatorname{Abs}\left(A_{n}\right)$ contains $N C\left(A_{n}\right)$ there are fewer restrictions needed on $t, t^{\prime}$ for this to be satisfied. Additionally, in relation (3.10) our restrictions on $r, r^{\prime}$ ensured that ( $r, r^{\prime}$ ) was not a reduced $T$-word for some element in $N C\left(A_{n}\right)$. In $\operatorname{Abs}\left(A_{n}\right)$ the only length two $T$-words which are not reduced are those with a repeated element, hence the form of relation (3.13).

Again, to prove Theorem 3.3.7 we need to show that $I=J$. We can already prove the inclusion $J \subset I$.

Corollary 3.3.8. If $W=A_{n}$ then $J \subset I$.
Proof. We need only show that modulo $I$ we have

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}} \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}, \text { and } \\
u_{t} u_{t} & \equiv 0 .
\end{aligned}
$$

For the first relation note that $\left(t, t^{\prime}\right)$ and $\left(t^{\prime}, t^{\prime} t t^{\prime}\right)$ are both reduced $T$-words for $t t^{\prime} \in A b s(W)$. By Corollaries 3.3 .2 and 3.3 .3 we see that $u_{t} u_{t^{\prime}} \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}(\bmod I)$. The proofs for the other two relations are similar.

We now begin the task of showing that equivalence modulo $I$ implies equivalence modulo $J$. We break this task up into two parts. Corollary 3.3.9 deals with this implication when the $u_{x}$ involved are nonzero modulo $I$ while Proposition 3.3.10 considers the case when the monomials are zero modulo $I$. The corollary is a straightforward consequence of our findings in subsection 3.2.

Corollary 3.3.9. Let $W=A_{n}$ and let $x, y$ be $T$-words such that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. Then $u_{x} \equiv u_{y}$ $(\bmod J)$.

Proof. By Corollary 3.3.3 we know that $x, y \in R_{T}(w)$ for some $w \in A b s\left(A_{n}\right)$. Let $c$ be a maximal element in $\operatorname{Abs}\left(A_{n}\right)$ such that $w \leq_{T} c$. Recall that in $A_{n}$ the maximal elements in absolute order are exactly the Coxeter elements of the poset. From this we see that $w \in[e, c]=N C(W)$ where $e$ is the identity element of $A_{n}$. By Corollaries 3.2 .3 and 3.2 .4 we know that $u_{x} \equiv u_{y} \not \equiv 0\left(\bmod I^{N C\left(A_{n}\right)}\right)$ and so by Proposition 3.2.8 we know that $u_{x} \equiv u_{y}$ modulo relations (3.8) and (3.9) from $J^{N C\left(A_{n}\right)}$. Since both of these relations are contained in $J$ we have $u_{x} \equiv u_{y}(\bmod J)$.

The previous corollary considered equivalence of $u_{x}, u_{y}$ modulo $J$ when $u_{x}, u_{y} \not \equiv 0(\bmod I)$. The following proposition now deals with equivalence modulo $J$ when $u_{x}, u_{y} \equiv 0(\bmod I)$. We note here that the proof of the below proposition is not dependent on our assumption that $W=A_{n}$, that is, this same proof will work for other choices of $W$.

Proposition 3.3.10. Let $W=A_{n}$ and let $x$ be a $T$-word. If $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$.
Proof. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ and $w=t_{1} \cdots t_{\ell-1}$. It is sufficient to consider the case where $u_{x} \equiv 0(\bmod I)$ and $u_{x^{\prime}} \neq 0(\bmod I)$ where $x^{\prime}=\left(t_{1}, \ldots, t_{\ell-1}\right)$. If $e$ is the identity element in $\operatorname{Abs}\left(A_{n}\right)$ then we have

$$
(e) u_{x}=(e) u_{x^{\prime}} u_{t_{\ell}}=(w) u_{t_{\ell}}=0
$$

Since $(w) u_{t_{\ell}}=0$ we know by the definition of our up-operators that $w t_{\ell}$ does not cover $w$ and so by Proposition 3.3.6 we know that $w$ covers $w t_{\ell}$. If $y=\left(r_{1}, \ldots, r_{\ell-2}\right)$ is a reduced $T$-word for $w t_{\ell}$, then $y^{\prime}=\left(r_{1}, \ldots, r_{\ell-2}, t_{\ell}\right)$ is a reduced $T$-word for $w$. By Corollary 3.3.3 we know that $u_{x^{\prime}} \equiv u_{y^{\prime}} \not \equiv 0(\bmod I)$ and so by Corollary 3.3 .9 we have $u_{x^{\prime}} \equiv u_{y^{\prime}}(\bmod I)$. From this we have

$$
\begin{align*}
u_{x} & =u_{x^{\prime}} u_{t_{\ell}} \\
& \equiv u_{y^{\prime}} u_{t_{\ell}}  \tag{Corollary3.3.3}\\
& =\cdots u_{t_{\ell}} u_{t_{\ell}} \\
& \equiv 0 \tag{3.13}
\end{align*}
$$

We can now give the proof of Theorem 3.3.7.
Proof. By Corollary 3.3.8 we see that $J \subset I$. By Proposition 3.3 .5 we have that $I$ is binomial and so to prove $I \subset J$ we need only show that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$. This is shown in Corollary 3.3.9 and Proposition 3.3.10.

### 3.3.2 Type $B$

We now let $W=B_{n}$. We will prove the following theorem.
Theorem 3.3.11. The algebra of up operators for $\operatorname{Abs}\left(B_{n}\right)$ is generated by the following relations:

$$
\begin{align*}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}},  \tag{3.14}\\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t},  \tag{3.15}\\
u_{t} u_{t} & \equiv 0, \text { and }  \tag{3.16}\\
u_{\left(\left(a, a^{\prime}\right)\right)} u_{\left.\left(a,-a^{\prime}\right)\right)} & \equiv u_{[a]} u_{\left[a^{\prime}\right]} \tag{3.17}
\end{align*}
$$

where $a, a^{\prime}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ with distinct absolute values and $t \neq t^{\prime} \in T$.
We first briefly discuss the relations given in the theorem. Relations (3.14)-(3.16) are similar to relations (3.11)-(3.13) for $\operatorname{Abs}\left(A_{n}\right)$ with the same reasoning for the restrictions on $t$ and $t^{\prime}$. The main difference in the $B_{n}$ case is relation (3.17) which is necessary because, unlike $\operatorname{Abs}\left(A_{n}\right)$, not all elements of $\operatorname{Abs}\left(B_{n}\right)$ are parabolic Coxeter elements. As a result, for nonparabolic Coxeter elements $w \in B_{n}$ we do not necessarily have $x \stackrel{H}{\sim} y$ where $x, y \in R_{T}(w)$. As such, we do not necessarily have $u_{x} \equiv u_{y}$ modulo relations (3.14) and (3.15) and so relation (3.17) accounts for this. More specifically, we will see that if $x$ and $y$ are not Hurwitz equivalent it will be because $w$ has a pair of balanced cycles $b_{1}, b_{2}$ in its disjoint decomposition which are 'created' in different ways in $x$ and $y$. For instance, the subscripts appearing on left hand side of relation (3.17) gives us the $T$-word $\left(\left(\left(a, a^{\prime}\right)\right),\left(\left(a,-a^{\prime}\right)\right)\right)$ and the subscripts on the right give us the $T$-word ( $\left.[a],\left[a^{\prime}\right]\right)$. Both of these are reduced $T$-words for $[a]\left[a^{\prime}\right]$ but as we multiply the entries of $\left(\left(\left(a, a^{\prime}\right)\right),\left(\left(a,-a^{\prime}\right)\right)\right)$ and $\left([a],\left[a^{\prime}\right]\right)$ from left to right the balanced cycles $[a]$ and $\left[a^{\prime}\right]$ are 'created' in different ways. For $\left(\left(\left(a, a^{\prime}\right)\right),\left(\left(a,-a^{\prime}\right)\right)\right)$, multiplying from left to right, we have

$$
e^{\left(\left(a, a^{\prime}\right)\right)}\left(\left(a, a^{\prime}\right)\right) \xrightarrow{\left(\left(a,-a^{\prime}\right)\right)}[a]\left[a^{\prime}\right]
$$

and we see that $[a]$ and $\left[a^{\prime}\right]$ are created simultaneous at the step where we multiply by $\left(\left(a,-a^{\prime}\right)\right)$. On the other hand, for $\left([a],\left[a^{\prime}\right]\right)$ we have

$$
e \xrightarrow{[a]}[a] \xrightarrow{\left[a^{\prime}\right]}[a]\left[a^{\prime}\right]
$$

and so we see that $[a]$ and $\left[a^{\prime}\right]$ are not created simultaneously; rather, $[a]$ is created at the step where we multiply by [ $a$ ] while $\left[a^{\prime}\right.$ ] is created later at the step where we multiply by [ $a^{\prime}$ ]. Definitions 3.3.13 and 3.3.14 give a more formal description of what we mean when we say that two $T$-words 'create' balanced cycles differently.

Proving that $J \subset I$ is straightforward and is shown in the following corollary.
Corollary 3.3.12. Let $W=B_{n}$. Then $J \subset I$.

Proof. Similar to the proof of Corollary 3.3 .8 we need only show that modulo $I$ we have

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}, \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}, \\
u_{t} u_{t} & \equiv 0, \text { and } \\
u_{\left(\left(a, a^{\prime}\right)\right)} u_{\left.\left(\left(a,-a^{\prime}\right)\right)\right)} & \equiv u_{[a]} u_{\left[a^{\prime}\right]}
\end{aligned}
$$

Again, we will only show that one of these relations holds since the proofs for the other relations are similar. We focus on the last relation since it is the main difference between the case for $\operatorname{Abs}\left(A_{n}\right)$ and the case for $\operatorname{Abs}\left(B_{n}\right)$. The left side of the last relation is equal to $u_{x}$ where $x=\left(\left(\left(a, a^{\prime}\right)\right),\left(\left(a,-a^{\prime}\right)\right)\right)$ and the right side is equal to $u_{y}$ where $y=\left([a],\left[a^{\prime}\right]\right)$. Both $x$ and $y$ are reduced $T$-words for $[a]\left[a^{\prime}\right] \in$ $\operatorname{Abs}\left(B_{n}\right)$. Thus, by Corollaries 3.3.2 and 3.3.3 we see that $u_{\left(\left(a, a^{\prime}\right)\right)} u_{\left(\left(a,-a^{\prime}\right)\right)} \equiv u_{[a]} u_{\left[a^{\prime}\right]}$.

The remainder of this subsection will rely on the covering relations for $B_{n}$ so we encourage the reader to review the relevant information from section 3.1. Before proceeding we first introduce the following notation. If $x_{1}, \ldots, x_{k}$ are $T$-words, then we denote by $\left(x_{1}, \ldots, x_{k}\right)$ the $T$-word formed by concatenating $x_{1}, \ldots, x_{k}$.

Let $x=\left(t_{1}, \ldots, t_{\ell}\right), y=\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$ be $T$-words. To show that $I \subset J$ we focus on proving that if $u_{x} \equiv$ $u_{y}(\bmod I)$, then $u_{x} \equiv u_{y}(\bmod J)$. We will first consider the case when $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. Note that by Corollary 3.3.3 we know that $x, y \in R_{T}(w)$ for some $w \in \operatorname{Abs}\left(B_{n}\right)$ so we let $b_{1} \cdots b_{k} p_{1} \cdots p_{m}$ be the disjoint decomposition of $w$ and let $b_{i}^{\prime}$ (resp. $p_{i}^{\prime}$ ) be a reduced $T$-word for $b_{i}$ (resp. $p_{i}$ ). Our strategy is to show that $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$ where $z=\left(b_{1}^{\prime}, \ldots, b_{k}^{\prime}, p_{1}^{\prime}, \ldots, p_{m}^{\prime}\right)$.

For the most part this will be straightforward to accomplish using commutations given by relations (3.14) and (3.15). However, a complication will arise if covering relation (3.5)

$$
\left(\left(a_{1}, \ldots, a_{k}\right)\right) \xrightarrow{\left(\left(a_{i},-a_{j}\right)\right)}\left[a_{1}, \ldots, a_{i},-a_{j+1}, \ldots,-a_{k}\right]\left[a_{i+1}, \ldots, a_{j}\right]
$$

ever appears in the chain $t_{1} \lessdot_{T} t_{1} t_{2} \lessdot_{T} \cdots \lessdot_{T} t_{1} \cdots t_{\ell}$ or the chain $t_{1}^{\prime} \lessdot_{T} t_{1}^{\prime} t_{2}^{\prime} \lessdot_{T} \cdots \lessdot_{T} t_{1}^{\prime} \cdots t_{\ell^{\prime}}^{\prime}$. Specifically, if (3.5) appears, then there exists some $b_{i}, b_{j}(i \neq j)$ in the disjoint decomposition of $w$ such that $b_{i}^{\prime}, b_{j}^{\prime}$ cannot be separated using commutation alone. To deal with this complication we first show that $u_{x} \equiv$ $u_{z_{1}}, u_{y} \equiv u_{z_{2}}(\bmod J)$ where $z_{1}, z_{2}$ are $T$-words containing as a consecutive subsequence a reduced $T$-word for $b_{i} b_{j}$. We then show that $u_{z_{1}}, u_{z_{2}} \equiv u_{z}(\bmod J)$. To help create $z_{1}, z_{2}$ we introduced the following definitions: the first of which more explicitly defines these $b_{i}, b_{j}$.

Definition 3.3.13. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in \operatorname{Abs}\left(B_{n}\right)$ and let $w=$ $b_{1} \cdots b_{k} p_{1} \cdots p_{m}$ be the disjoint decomposition of $w$ into paired and balanced cycles. We will call two balanced cycles $b_{i}, b_{j}$ simultaneous with respect to $x$ if there is some $\iota \in[\ell]$ such that the disjoint decomposition of $t_{1} \cdots t_{l}$ contains balanced cycles $\bar{b}_{i}, \bar{b}_{j}$ where the elements of $\bar{b}_{i}$ are contained in $b_{i}$, and the elements of $\bar{b}_{j}$ are contained in $b_{j}$; and furthermore, the disjoint decomposition of $t_{1} \cdots t_{l-1}$ contains no such $\bar{b}_{i}$ nor $\bar{b}_{j}$.

Note that in the above definition the covering relation $t_{1} \cdots t_{l-1} \lessdot_{T} t_{1} \cdots t_{l}$ must be a covering relation of type (3.5). Also note that for any reduced $T$-word $x$ it is not possible for there to exist three balanced cycles $b_{i}, b_{j}, b_{k}$ such that $b_{i}, b_{j}$ are simultaneous with respect to $x$ and $b_{j}, b_{k}$ are also simultaneous with respect to $x$. This follows immediately from the covering relations (3.4)-(3.7).

Definition 3.3.14. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in A b s\left(B_{n}\right)$ and let $w=b_{1} \cdots b_{k} p_{1} \cdots p_{m}$ be the disjoint decomposition of $w$ into paired and balanced cycles. Let $w=m_{1} \cdots m_{\zeta}$ be a decomposition of $w$ such that if $b_{i}, b_{j}$ are simultaneous with respect to $x$, then $m_{\iota}=b_{i} b_{j}$ for some $\iota \in[\zeta]$, and for all other factors $b_{i}$ (resp. $p_{j}$ ) there exists some $\iota \in[\zeta]$ such that $m_{l}=b_{i}$ (resp. $p_{j}$ ). We call such a disjoint decomposition a simultaneous decomposition with respect to $x$.

Note that both of these definitions are dependent upon the choice of the reduced $T$-word $x$. When no confusion is possible we will say 'simultaneous' rather than simultaneous with respect to $x$ and 'simultaneous decomposition' rather than simultaneous decomposition with respect to $x$.

Example 3.3.15. Let

$$
x=(((1,2)),((2,3)),[7],((5,6)),((7,8)),((2,-3)),((3,4)))
$$

be a reduced $T$-word for $w$, where $w$ has the disjoint decomposition [1,2][3,4][7,8]((5,6)). We see that $[1,2],[3,4]$ are simultaneous with respect to $x$ since the disjoint decomposition of

$$
((1,2))((2,3))[7]((5,6))((7,8))((2,-3))
$$

(which is $[1,2][3]((5,6))[7,8])$ contains the balanced cycles $[1,2],[3]$ while the disjoint decomposition of

$$
((1,2))((2,3))[7]((5,6))((7,8))
$$

(which is $((1,2,3))((5,6))[7,8])$ contains no such balanced cycles.
We also see that $w$ has the simultaneous decomposition with respect to $x$ given by $w=m_{1} m_{2} m_{3}$ where $m_{1}=[1,2][3,4], m_{2}=[7,8]$, and $m_{3}=((5,6))$.

The following lemma constructs the $T$-words $z_{1}, z_{2}$ mentioned earlier in our discussion of the proof strategy for showing that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$.

Lemma 3.3.16. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in A b s\left(B_{n}\right)$ and let $w=$ $m_{1} \cdots m_{k}$ be a simultaneous decomposition. Then $u_{x} \equiv u_{y}(\bmod J)$ where $y=\left(m_{1}^{\prime}, \ldots, m_{k}^{\prime}\right)$ and where $m_{i}^{\prime}$ is a reduced $T$-word for $m_{i}$.

Proof. In the following, when we refer to a factor of $m_{i}$ we mean a reflection $t_{j}$ appearing as an entry in $x$ such that $t_{j} \leq_{T} m_{i}$. We can see from covering relations (3.4)-(3.7) that if $t_{i}=\left(\left(a, a^{\prime}\right)\right)($ resp. [a]) is a factor of $m_{j}$, then $a, a^{\prime}\left(\right.$ resp. $a$ ) must be contained in $m_{j}$. Since the $m_{i}$ are disjoint, we see that any factor of $m_{j_{1}}$ must be disjoint from any factor of $m_{j_{2}}$ whenever $j_{1} \neq j_{2}$. Thus, applications of relations
(3.14) and (3.15) on $u_{t} u_{t^{\prime}}$, where $t, t^{\prime}$ are factors for different $m_{i}$, simply result in commutation. We see then that

$$
u_{x} \equiv u_{m_{1}^{\prime}} \cdots u_{m_{k}^{\prime}}=u_{y} \quad(\bmod J)
$$

where $m_{i}^{\prime}$ is a $T$-word for $m_{i}$. By Proposition 3.3.12 we know that $u_{x} \equiv u_{y}(\bmod I)$ and so by Corollary 3.3.3 we have $y \in R_{T}(w)$. Since $y$ is a reduced $T$-word it must be that its subwords are also reduced $T$-words and so the $m_{i}^{\prime}$ are reduced.

Now that we have constructed $z_{1}, z_{2}$ we can begin the process of proving that $u_{z_{1}}, u_{z_{2}} \equiv u_{z}$ $(\bmod J)$. Lemmas 3.3.17, 3.3.18, and 3.3.20 will give us some tools to deal with the complications arising from simultaneous balanced cycles and Proposition 3.3 .21 shows that $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$.

Lemma 3.3.17. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in \operatorname{Abs}\left(B_{n}\right)$ such that the disjoint decomposition of $w$ is $w=b_{1} b_{2}$ where $b_{1}, b_{2}$ are simultaneous with respect to $x$. Also let $a, a^{\prime}$ be any integers appearing in $b_{1}, b_{2}$ respectively. Then there exists a $T$-word $y$ such that $u_{x} \equiv u_{y}$ $(\bmod J), y=\left(t_{1}^{\prime}, \ldots, t_{\ell}^{\prime}\right)$, and $t_{\ell}^{\prime}=\left(\left(a,-a^{\prime}\right)\right)$.

The proof for this lemma is a bit technical so before proceeding we first give a more informal description of the proof. Since $w$ is simultaneous with respect to $x$ we can see, using covering relation (3.5), that $x$ must have an entry equal to $\left(\left(a_{i},-a_{j}\right)\right)$ where $a_{i}$ is some element of $b_{1}$ and $a_{j}$ is some element of $b_{2}$. We now use relation (3.15) on $x$ to create a new $T$-word $y_{1}$ where ( $\left(a_{i},-a_{j}\right)$ ) appears in the last entry of $y_{1}$ and $u_{x} \equiv u_{y_{1}}(\bmod J)$. This $\left(\left(a_{i},-a_{j}\right)\right)$ has the same structure as the reflection $\left(\left(a,-a^{\prime}\right)\right)$, namely it is a reflection which is a paired cycle with entries from both $b_{1}$ and $b_{2}$. However, it is not necessarily the case that $\left(\left(a_{i},-a_{j}\right)\right)=\left(a,-a^{\prime}\right)$ since the values of $a_{i}$ and $a_{j}$ are determined by $x$. To account for this, we then use our relations to 'replace' $a_{i}$ and $a_{j}$ with $a$ and $\pm a^{\prime}$ respectively. We put replace in quotes because we aren't actually replacing anything, we are just finding a $T$-word equivalent to $u_{y_{1}}(\bmod J)$ that ends in $\left(\left(a, \pm a^{\prime}\right)\right)$. After applying our relations we will see that the resulting $T$-word is our desired $y$.

Proof. Let $x$ be as described in the lemma statement. Our first goal is to prove the following claim.

Claim 1 There exists a $T$-word $y_{1}$ such that $u_{x} \equiv u_{y_{1}}(\bmod J), y_{1}=\left(f_{1}, \ldots, f_{\ell}\right), f_{1} \cdots f_{\ell-1}=\left(\left(a_{1}, \ldots, a_{k}\right)\right)$, and $f_{\ell}=\left(\left(a_{i_{1}},-a_{i_{2}}\right)\right)$ for some $a_{i_{1}}, a_{i_{2}}\left(i_{1}<i_{2}\right)$ appearing in $b_{1}, b_{2}$ respectively.

The proof of claim 1 is as follows. First, suppose that $b_{1}=\left[\alpha_{1}, \ldots, \alpha_{k_{1}}\right], b_{2}=\left[\beta_{1}, \ldots, \beta_{k_{2}}\right]$. Since $b_{1}, b_{2}$ are simultaneous it must be that for some $\iota \in[\ell]$ the disjoint decomposition of $w_{1}=t_{1} \cdots t_{\iota}$ contains balanced cycles $\bar{b}_{1}, \bar{b}_{2}$ which are contained in $b_{1}, b_{2}$ respectively and $w_{2}=t_{1} \cdots t_{l-1}$ contains no such $\bar{b}_{1}, \bar{b}_{2}$. Since covering relation (3.5) is the only one which results in simultaneous balanced cycles it must be that $t_{l}=\left(\left( \pm \alpha_{i}, \pm \beta_{j}\right)\right)$ for some $i \in\left[k_{1}\right], j \in\left[k_{2}\right]$. Using relation (3.15) we see that $u_{x} \equiv u_{y_{1}}(\bmod J)$ where

$$
y_{1}=\left(t_{1}, \ldots, t_{l-1}, t_{l} t_{l+1} t_{l}, \ldots, t_{l} t_{\ell} t_{l}, t_{l}\right) .
$$

Now let

$$
w_{3}=t_{1} \cdots t_{l-1}\left(t_{l} t_{l+1} t_{l}\right) \cdots\left(t_{l} t_{\ell} t_{l}\right) .
$$

Note that $w_{3}=b_{1} b_{2}\left(\left( \pm \alpha_{i}, \pm \beta_{j}\right)\right)$ is a paired cycle and that $\pm \alpha_{i}, \pm \beta_{j}$ are contained in different cycles of this paired cycle. Letting $a_{i_{1}}= \pm \alpha_{i}$ and $-a_{i_{2}}= \pm \beta_{j}$ we see that $y_{1}$ is as described in claim 1 .

Our next goal is to prove the second claim below.

Claim 2 Let $y_{1}$ be as described in claim 1 and let $i_{3} \in[k]$ such that $a_{i_{3}}$ is contained in $b_{2}$ and $a_{i_{3}}= \pm a^{\prime}$. There exists a $T$-word $y_{2}$ such that $u_{y_{1}} \equiv u_{y_{2}}(\bmod J), y_{2}=\left(q_{1}, \ldots, q_{\ell}\right), q_{1} \cdots q_{\ell}$ is a paired cycle, and $q_{\ell}=\left(\left(a_{i_{1}},-a_{i_{3}}\right)\right)$.

The proof of claim 2 is as follows. By assumption we have that $y_{1}=\left(f_{1}, \ldots, f_{\ell}\right), f_{1} \cdots f_{\ell-1}=$ $\left(\left(a_{1}, \ldots, a_{k}\right)\right)$, and $f_{\ell}=\left(\left(a_{i_{1}},-a_{i_{2}}\right)\right)$ for some $a_{i_{1}}, a_{i_{2}}$ appearing in $b_{1}, b_{2}$ respectively. If $i_{3}=i_{2}$, then we are done, so suppose $i_{3} \neq i_{2}$. Since $a_{i_{3}}$ appears in $b_{2}$ it must be that $i_{1}<i_{3}<i_{2}$. Using this we can see that $y_{1}^{\prime}$ is a reduced $T$-word for $f_{1} \cdots f_{\ell-1}=\left(\left(a_{1}, \ldots, a_{k}\right)\right)$ where

$$
\begin{aligned}
y_{1}^{\prime}= & \left(\left(\left(a_{1}, a_{2}\right)\right), \ldots,\left(\left(a_{i_{3}-1}, a_{i_{3}}\right)\right),\left(\left(a_{i_{3}}, a_{i_{2}+1}\right)\right),\left(\left(a_{i_{2}+1}, a_{i_{2}+2}\right)\right), \ldots,\right. \\
& \left.\left(\left(a_{k-1}, a_{k}\right)\right),\left(\left(a_{i_{3}+1}, a_{i_{3}+2}\right)\right), \ldots,\left(\left(a_{i_{2}-1}, a_{i_{2}}\right)\right),\left(\left(a_{i_{3}}, a_{i_{2}}\right)\right)\right) .
\end{aligned}
$$

Note that $f_{1} \cdots f_{\ell-1} \in N C\left(B_{n}, c\right)$ where $c=\left[a_{1}, \ldots, a_{k}, \ldots, a_{n}\right]$ is a Coxeter element. Since relations (3.8) and (3.9) are contained in $J$ when $W=B_{n}$ we have by Corollary 3.2.3 and Proposition 3.2.8 that $u_{\left(f_{1}, \ldots, f_{\ell-1}\right)} \equiv u_{y_{1}^{\prime}}(\bmod J)$. We then have

$$
\begin{align*}
u_{y_{1}} & =u_{\left(f_{1}, \ldots, f_{\ell-1}\right)} u_{f_{\ell}} \\
& \equiv u_{y_{1}^{\prime}}^{\prime} u_{f_{\ell}} \\
& =\cdots u_{\left(\left(a_{i_{3}}, a_{i_{2}}\right)\right)} u_{\left(\left(a_{i_{1}},-a_{i_{2}}\right)\right)} \\
& \equiv \cdots u_{\left(\left(a_{i_{1}},-a_{\left.i_{2}\right)}\right)\right.} u_{\left(\left(a_{i_{1}},-a_{i_{3}}\right)\right)}  \tag{3.14}\\
& =u_{y_{2}} .
\end{align*}
$$

Note that

$$
\begin{aligned}
y_{2}= & \left(\left(\left(a_{1}, a_{2}\right)\right), \ldots,\left(\left(a_{i_{3}-1}, a_{i_{3}}\right)\right),\left(\left(a_{i_{3}}, a_{i_{2}+1}\right)\right),\left(\left(a_{i_{2}+1}, a_{i_{2}+2}\right)\right), \ldots,\right. \\
& \left.\left(\left(a_{k-1}, a_{k}\right)\right),\left(\left(a_{i_{3}+1}, a_{i_{3}+2}\right)\right), \ldots,\left(\left(a_{i_{2}-1}, a_{i_{2}}\right)\right),\left(\left(a_{i_{1}},-a_{i_{2}}\right)\right),\left(\left(a_{i_{1}},-a_{i_{3}}\right)\right)\right) .
\end{aligned}
$$

has the desired properties from claim 2.
Now that we have proven both of our claims we can do the following. We apply claim 2 again but this time $y_{2}$ takes the role of $y_{1}$ and $b_{1}, b_{2}$ switch roles. Doing this we obtain a $T$-word $y_{3}$ such that $u_{y_{2}} \equiv u_{y_{3}}(\bmod J), y_{3}=\left(r_{1}, \ldots, r_{\ell}\right), r_{1} \cdots r_{\ell-1}$ is a paired cycle, and $r_{\ell}=\left(\left(a_{i_{3}},-a_{i_{4}}\right)\right)=\left(\left(a_{i_{4}},-a_{i_{3}}\right)\right)$ where $a_{i_{4}}$ appears in $b_{1}$ and $a_{i_{4}}= \pm a$. We now assume that $a_{i_{4}}=a$ and consider the two possible values of $a_{i_{3}}$, either $a_{i_{3}}=a^{\prime}$ or $a_{i_{3}}=-a^{\prime}$. Note that this implicitly addresses the case when $a_{i_{4}}=-a$
since $r_{\ell}=\left(\left(a_{i_{3}},-a_{i_{4}}\right)\right)=\left(\left(a_{i_{4}},-a_{i_{3}}\right)\right)$. If $a=a_{i_{4}}$ and $a^{\prime}=a_{i_{3}}$, then $r_{\ell}=t_{\ell}^{\prime}=\left(\left(a,-a^{\prime}\right)\right)$ and we are done. Suppose instead that $a=a_{i_{4}}$ and $a^{\prime}=-a_{i_{3}}$. Note that there exists a reduced $T$-word $y_{3}^{\prime}=\left(r_{1}^{\prime}, \cdots, r_{\ell}^{\prime}\right)$ for $r_{1} \cdots r_{\ell-1}$ such that $r_{\ell}^{\prime}=\left(\left(a_{i_{3}}, a_{i_{4}}\right)\right)$ (we construct $y_{3}^{\prime}$ in the same manner as $y_{1}^{\prime}$ ). We then see that

$$
\begin{align*}
u_{y_{3}} & =u_{\left(r_{1}, \ldots, r_{\ell-1}\right)} u_{r_{\ell}} \\
& \equiv u_{y_{3}^{\prime}} u_{r_{\ell}} \\
& =\cdots u_{\left(\left(a_{3}, a_{i 4}\right)\right)} u_{\left(\left(a_{3},-a_{i 4}\right)\right)} \\
& \equiv \cdots u_{\left(\left(a_{i_{3}},-a_{\left.i_{4}\right)}\right)\right)} u_{\left(\left(a_{i_{3}}, a_{i_{4}}\right)\right)}  \tag{3.14}\\
& =u_{y_{4}}
\end{align*}
$$

where $y_{4}=\left(z_{1}, \ldots, z_{\ell}\right), z_{1} \cdots z_{\ell-1}$ is a paired cycle, and $z_{\ell}=\left(\left(a_{i_{3}}, a_{i_{4}}\right)\right)=\left(\left(a_{i_{4}}, a_{i_{3}}\right)\right)$. We now have $z_{\ell}=t_{\ell}^{\prime}=\left(\left(a,-a^{\prime}\right)\right)$ and we are done.

The following lemma gives us a way to take a reduced $T$-word $x$ (with certain conditions placed upon it) and find another reduced $T$-word $y$ such that $u_{x} \equiv u_{y}(\bmod J)$ and where $y$ is in some standard form. In fact Lemma 3.3.18 gives multiple different standard forms for $y$.

Lemma 3.3.18. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in \operatorname{Abs}\left(B_{n}\right)$ and let $w=b_{1} b_{2}$ be the disjoint decomposition of $w$ where $b_{1}, b_{2}$ are simultaneous. If $b_{1}=\left[\alpha_{1}, \ldots, \alpha_{k_{\alpha}}\right], b_{2}=\left[\beta_{1}, \ldots, \beta_{k_{\beta}}\right]$, then the following hold.
(a) We have that $u_{x} \equiv u_{y}(\bmod J)$ where $y=\left(A, B,\left(\left(\alpha_{1}, \beta_{1}\right)\right),\left(\left(\alpha_{1},-\beta_{1}\right)\right)\right)$ and

$$
\begin{aligned}
& A=\left(\left(\left(\alpha_{1},-\alpha_{2}\right)\right),\left(\left(-\alpha_{2},-\alpha_{3}\right)\right), \ldots,\left(\left(-\alpha_{k_{\alpha}-1},-\alpha_{k_{\alpha}}\right)\right)\right), \\
& B=\left(\left(\left(\beta_{1},-\beta_{2}\right)\right),\left(\left(-\beta_{2},-\beta_{3}\right)\right), \ldots,\left(\left(-\beta_{k_{\beta}-1},-\beta_{k_{\beta}}\right)\right)\right) .
\end{aligned}
$$

(b) We have that $u_{x} \equiv u_{y^{\prime}}(\bmod J)$ where $y^{\prime}=\left(\left(\left(\alpha_{1}, \beta_{1}\right)\right),\left(\left(\alpha_{1},-\beta_{1}\right)\right), A^{\prime}, B^{\prime}\right)$ and

$$
\begin{aligned}
& A^{\prime}=\left(\left(\left(\alpha_{1}, \alpha_{2}\right)\right), \ldots,\left(\left(\alpha_{k_{\alpha}-1}, \alpha_{k_{\alpha}}\right)\right)\right), \\
& B^{\prime}=\left(\left(\left(\beta_{1}, \beta_{2}\right)\right), \ldots,\left(\left(\beta_{k_{\beta}-1}, \beta_{k_{\beta}}\right)\right)\right) .
\end{aligned}
$$

Proof. We will prove (a) explicitly; the proof of (b) is similar. By Lemma 3.3.17 we know that $u_{x} \equiv u_{z}$ $(\bmod J)$ where $z=\left(t_{1}^{\prime}, \ldots, t_{\ell}^{\prime}\right), t_{1}^{\prime} \cdots t_{\ell-1}^{\prime}$ is the paired cycle $b_{1} b_{2}\left(\left(\alpha_{1},-\beta_{1}\right)\right)$, and $t_{\ell}^{\prime}=\left(\left(\alpha_{1},-\beta_{1}\right)\right)$. Note that

$$
b_{1} b_{2}\left(\left(\alpha_{1},-\beta_{1}\right)\right)=\left(\left(\alpha_{1},-\beta_{2}, \ldots,-\beta_{k_{2}}, \beta_{1},-\alpha_{2}, \ldots,-\alpha_{k_{1}}\right)\right)
$$

is contained in $N C\left(B_{n}, c\right)$ where $c=\left[\alpha_{1},-\beta_{2}, \ldots,-\beta_{k_{2}}, \beta_{1},-\alpha_{2}, \ldots,-\alpha_{k_{1}}, a_{1}, \ldots, a_{n-k_{1}-k_{2}}\right]$ is a Coxeter element such that $a_{1}, \ldots, a_{n-k_{1}-k_{2}}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ whose absolute values are pairwise distinct and which are not equal to any $\pm \alpha_{i}$ or $\pm \beta_{i}$. Note also that $z^{\prime}=\left(A, B,\left(\left(\alpha_{1}, \beta_{1}\right)\right)\right)$ is a reduced $T$-word for $b_{1} b_{2}\left(\left(\alpha_{1},-\beta_{1}\right)\right)$. Since relations (3.8) and (3.9) are contained in $J$ when $W=B_{n}$ we have
by Corollary 3.2.3 and Proposition 3.2.8 that

$$
u_{z}=u_{\left(t_{1}^{\prime}, \ldots, t_{\ell-1}^{\prime}\right)} u_{t_{\ell}^{\prime}} \equiv u_{z^{\prime}} u_{t_{\ell}^{\prime}}=u_{y}(\bmod J)
$$

The following lemma will not be necessary for our work with $\operatorname{Abs}\left(B_{n}\right)$ but it will be of use in the next subsection where we study $\operatorname{Abs}\left(D_{n}\right)$.

Lemma 3.3.19. Let $x, y$ be reduced $T$-words for some $w \in A b s\left(B_{n}\right)$ and let $w=b_{1} b_{2}$ be the disjoint decomposition of $w$ where $b_{1}, b_{2}$ are simultaneous with respect to both $x$ and $y$. Then $u_{x} \equiv u_{y}$ $(\bmod J)$.

Proof. Let $b_{1}=\left[\alpha_{1}, \ldots, \alpha_{k_{1}}\right], b_{2}=\left[\beta_{1}, \ldots, \beta_{k_{2}}\right]$. By Lemma 3.3.18 we know that $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$ where $z=\left(A, B,\left(\left(\alpha_{1}, \beta_{1}\right)\right),\left(\left(\alpha_{1},-\beta_{1}\right)\right)\right)$ and

$$
\begin{aligned}
& A=\left(\left(\left(\alpha_{1},-\alpha_{2}\right)\right),\left(\left(-\alpha_{2},-\alpha_{3}\right)\right), \ldots,\left(\left(-\alpha_{k_{\alpha}-1},-\alpha_{k_{\alpha}}\right)\right)\right), \\
& B=\left(\left(\left(\beta_{1},-\beta_{2}\right)\right),\left(\left(-\beta_{2},-\beta_{3}\right)\right), \ldots,\left(\left(-\beta_{k_{\beta}-1},-\beta_{k_{\beta}}\right)\right)\right) .
\end{aligned}
$$

Let $x, z \in R_{T}(w)$ for some $w \in B_{n}$. Recall from our discussion at the beginning of this subsection that the core problem we have in the $\operatorname{Abs}\left(B_{n}\right)$ case is finding a way to show that $u_{x} \equiv u_{z}(\bmod J)$ when $x$ and $z$ are not Hurwitz equivalent. We will see in the proof of Proposition 3.3.21 that the key to solving this problem is to first show that if the disjoint decomposition of $w$ is $b_{1} b_{2}$ and $b_{1}, b_{2}$ are simultaneous with respect to $x$, then $u_{x} \equiv u_{y}(\bmod J)$ where $y=\left(b_{1}^{\prime}, b_{2}^{\prime}\right)$ and $b_{1}^{\prime}, b_{2}^{\prime}$ are reduced $T$-words for $b_{1}, b_{2}$ respectively so that $b_{1}, b_{2}$ are not simultaneous with respect to $y$. Lemma 3.3.20 proves this equivalence by using one of the standard forms from Lemma 3.3.18 along with our relations, especially relation (3.17).

Lemma 3.3.20. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a reduced $T$-word for $w=t_{1} \cdots t_{\ell} \in A b s\left(B_{n}\right)$ such that $w=b_{1} b_{2}$ is the disjoint decomposition of $w$ where $b_{1}, b_{2}$ are simultaneous with respect to $x$. Then $u_{x} \equiv u_{y}$ $(\bmod J)$ where $y=\left(b_{1}^{\prime}, b_{2}^{\prime}\right)$ and $b_{1}^{\prime}\left(r e s p . b_{2}^{\prime}\right)$ is any reduced $T$-word for $b_{1}\left(\right.$ resp. $\left.b_{2}\right)$.

Proof. Suppose that $b_{1}=\left[\alpha_{1}, \ldots, \alpha_{k_{1}}\right]$ and $b_{2}=\left[\beta_{1}, \ldots, \beta_{k_{2}}\right]$. By Lemma 3.3.18 we have that $u_{x} \equiv u_{z_{1}}$ $(\bmod J)$ where $z_{1}=\left(A, B,\left(\left(\alpha_{1}, \beta_{1}\right)\right),\left(\left(\alpha_{1},-\beta_{1}\right)\right)\right)$ and

$$
\begin{aligned}
& A=\left(\left(\left(\alpha_{1},-\alpha_{2}\right)\right),\left(\left(-\alpha_{2},-\alpha_{3}\right)\right), \ldots,\left(\left(-\alpha_{k_{\alpha}-1},-\alpha_{k_{\alpha}}\right)\right)\right), \\
& B=\left(\left(\left(\beta_{1},-\beta_{2}\right)\right),\left(\left(-\beta_{2},-\beta_{3}\right)\right), \ldots,\left(\left(-\beta_{k_{\beta}-1},-\beta_{k_{\beta}}\right)\right)\right) .
\end{aligned}
$$

From here we see that

$$
\begin{align*}
u_{z_{1}} & =u_{A} u_{B} u_{\left(\left(\alpha_{1}, \beta_{1}\right)\right)} u_{\left(\left(\alpha_{1},-\beta_{1}\right)\right)} \\
& \equiv u_{A} u_{B} u_{\left[\alpha_{1}\right]} u_{\left[\beta_{1}\right]}  \tag{3.17}\\
& \equiv u_{A} u_{\left[\alpha_{1}\right]} u_{B} u_{\left[\beta_{1}\right]}  \tag{3.14}\\
& =u_{z_{2}}
\end{align*}
$$

where $z_{2}=\left(A,\left[\alpha_{1}\right], B,\left[\beta_{1}\right]\right)$. Note that $\left(A,\left[\alpha_{1}\right]\right),\left(B,\left[\beta_{1}\right]\right)$ are reduced $T$-words for $b_{1}, b_{2}$ respectively. Also note that $\left(A,\left[\alpha_{1}\right]\right),\left(B,\left[\beta_{1}\right]\right) \in N C(W, c)$ where

$$
c=\left[\alpha_{1}, \ldots, \alpha_{k_{1}}, \beta_{1}, \ldots, \beta_{k_{2}}, a_{1}, \ldots, a_{n-k_{1}-k_{2}}\right]
$$

is a Coxeter element such that $a_{1}, \ldots, a_{n-k_{1}-k_{2}}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ whose absolute values are pairwise distinct and which are not equal to any $\pm \alpha_{i}$ or $\pm \beta_{i}$. From this we see by Corollary 3.2.3 and Proposition 3.2.8 that

$$
u_{z_{2}} \equiv u_{\left(b_{1}^{\prime} b_{2}^{\prime}\right)}(\bmod J)
$$

where $b_{1}^{\prime}\left(\right.$ resp. $\left.b_{2}^{\prime}\right)$ is any reduced $T$-word for $b_{1}$ (resp. $b_{2}$ ).
We are now prepared to show that equivalence modulo $I$ implies equivalence modulo $J$, which is a key step in proving that $I \subset J$.

Proposition 3.3.21. Let $x, y$ be $T$-words such that $u_{x} \equiv u_{y} \equiv \equiv(\bmod I)$. Then $u_{x} \equiv u_{y}(\bmod J)$.
Proof. By Corollary 3.3.3 we know that $x, y \in R_{T}(w)$ for some $w \in \operatorname{Abs}\left(B_{n}\right)$. Let $w=b_{1} \cdots b_{k_{1}} p_{1} \cdots p_{k_{2}}$ be the disjoint decomposition of $w$ and let $b_{i}^{\prime}$ (resp. $p_{i}^{\prime}$ ) be any reduced $T$-word for $b_{i}$ (resp. $p_{i}$ ). Our goal is to show that $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$ where $z=\left(b_{1}^{\prime}, \ldots, b_{k_{1}}^{\prime}, p_{1}^{\prime}, \ldots, p_{k_{2}}^{\prime}\right)$. Let $w=m_{1} \cdots m_{\zeta}$ be a simultaneous decomposition with respect to $x$ and let $m_{i}^{\prime}$ be a reduced word for $m_{i}$. We know by Lemma 3.3.16 that $u_{x} \equiv u_{z_{1}}(\bmod J)$ where $z_{1}=\left(m_{1}^{\prime}, \ldots, m_{\zeta}^{\prime}\right)$.

First, fix some $i \in[\zeta]$ and suppose that $m_{i}=b_{j}=\left[a_{1}, \ldots, a_{k}\right]$ for some $j \in\left[k_{1}\right]$. Then $m_{i} \in$ $N C\left(B_{n}, c\right)$ where $c=\left[a_{1}, \ldots, a_{k}, \ldots, a_{n}\right]$ is a Coxeter element. Since relations (3.8) and (3.9) are contained in $J$ when $W=B_{n}$ we have by Corollary 3.2.3 and Proposition 3.2.8 that $u_{m_{i}^{\prime}} \equiv u_{b_{j}^{\prime}}$ $(\bmod J)$. Using this same process we can similarly show that if $m_{i}=p_{j}$ for some $j \in\left[k_{2}\right]$ then $u_{m_{i}^{\prime}} \equiv u_{p_{j}^{\prime}}(\bmod J)$. Now suppose $m_{i}=b_{j_{1}} b_{j_{2}}$ where $b_{j_{1}}, b_{j_{2}}$ are simultaneous with respect to $x$. By Lemma 3.3.20 we know that $u_{m_{i}^{\prime}} \equiv u_{z_{2}}(\bmod J)$ where $z_{2}=\left(b_{j_{1}}^{\prime}, b_{j_{2}}^{\prime}\right)$. From here we see that

$$
u_{x} \equiv u_{z_{1}} \equiv u_{\left(b_{1}^{\prime}, \ldots, b_{k_{1}}^{\prime}, p_{1}^{\prime}, \ldots, p_{k_{2}}^{\prime}\right)}=u_{z}(\bmod J)
$$

Using an identical process we can also show that $u_{y} \equiv u_{z}(\bmod J)$ and so $u_{x} \equiv u_{y}(\bmod J)$ as desired.

Our last task is to prove that if $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$. This is done in the following proposition.

Proposition 3.3.22. Let $x$ be a $T$-word. If $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$.
Proof. The proof is similar to the proof of Proposition 3.3.10.
Finally, we give a proof Theorem 3.3.11.
Proof. Corollary 3.3.12 gives us the inclusion $J \subset I$. For the reverse inclusion note that by Proposition 3.3.5 we know that $I$ is binomial and so it is sufficient to show that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}$ $(\bmod J)$. This follows directly from Propositions 3.3.21 and 3.3.22.

### 3.3.3 Type $D$

Here we focus on our last classical case, Coxeter groups of type $D_{n}$. In this subsection we take $W=D_{n}$. We wish to prove the following theorem. Note that any result from subsection 3.3.2 which does not rely on relation (3.17) also applies to $\operatorname{Abs}\left(D_{n}\right)$. We will make use of such results throughout this subsection.

Theorem 3.3.23. The algebra of up operators for $\operatorname{Abs}\left(D_{n}\right)$ is generated by the following relations:

$$
\begin{align*}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}}  \tag{3.18}\\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t}  \tag{3.19}\\
u_{t} u_{t} & \equiv 0, \text { and }  \tag{3.20}\\
u_{\left(\left(a_{1}, a_{2}\right)\right)} u_{\left(\left(a_{1},-a_{2}\right)\right)} u_{\left(\left(a_{3}, a_{4}\right)\right)} u_{\left(\left(a_{3},-a_{4}\right)\right)} & \equiv u_{\left(\left(a_{1}, a_{3}\right)\right)} u_{\left(\left(a_{1},-a_{3}\right)\right)} u_{\left(\left(a_{2}, a_{4}\right)\right)} u_{\left(\left(a_{2},-a_{4}\right)\right)} \tag{3.21}
\end{align*}
$$

where $a_{1}, a_{2}, a_{3}, a_{4}$ are elements of $\{ \pm 1, \ldots, \pm n\}$ with distinct absolute values and $t \neq t^{\prime} \in T$.
The relations for the up-operators on $\operatorname{Abs}\left(D_{n}\right)$ are similar to the relations for $\operatorname{Abs}\left(A_{n}\right)$ and $\operatorname{Abs}\left(B_{n}\right)$, including the conditions imposed on $t$ and $t^{\prime}$. As with the type $B$ case, the type $D$ case has a special relation needed to prove $u_{x} \equiv u_{y}(\bmod J)$ when $x$ and $y$ are not Hurwitz equivalent, namely relation (3.21). We will discuss the importance of this relation later in the subsection but for now we wish to note that relation (3.21) is of degree 4. This makes it different from the other relations we've seen in this chapter which have all been of degree 2 . An easy way to see that a degree 4 relation is needed is by looking at the Hasse diagram of the interval $[e,[1][2][3][4]]$ in $\operatorname{Abs}\left(D_{4}\right)$ (see Figure 3.6). If we let $x$ be the reduced $T$-word corresponding to the chain

$$
((1,2)) \lessdot_{T}[1][2] \lessdot_{T}[1][2]((3,4)) \lessdot_{T}[1][2][3][4]
$$

and let $y$ be the reduced $T$-word corresponding to the chain

$$
((1,3)) \lessdot_{T}[1][3] \lessdot_{T}[1][3]((2,4)) \lessdot_{T}[1][2][3][4]
$$

we see that we can have $u_{x} \equiv u_{y}(\bmod J)$ only if $J$ contains a relation of degree 4.
Once again, proving the inclusion $J \subset I$ is straightforward and is stated below.
Proposition 3.3.24. Let $W=D_{n}$. Then $J \subset I$.
Proof. We need only show that modulo $I$ we have

$$
\begin{aligned}
u_{t} u_{t^{\prime}} & \equiv u_{t^{\prime}} u_{t^{\prime} t t^{\prime}} \\
u_{t} u_{t^{\prime}} & \equiv u_{t t^{\prime} t} u_{t} \\
u_{t} u_{t} & \equiv 0, \text { and } \\
u_{\left(\left(a_{1}, a_{2}\right)\right)} u_{\left(\left(a_{1},-a_{2}\right)\right)} u_{\left(\left(a_{3}, a_{4}\right)\right)} u_{\left(\left(a_{3},-a_{4}\right)\right)} & \equiv u_{\left(\left(a_{1}, a_{3}\right)\right)} u_{\left(\left(a_{1},-a_{3}\right)\right)} u_{\left(\left(a_{2}, a_{4}\right)\right)} u_{\left(\left(a_{2},-a_{4}\right)\right)}
\end{aligned}
$$



Figure 3.6 The Hasse diagram for the interval $[e,[1][2][3][4]]$ in $\operatorname{Abs}\left(D_{4}\right)$.

We will focus on the last relation since, as mentioned earlier, it is the key difference between the $\operatorname{Abs}\left(B_{n}\right)$ case and the $\operatorname{Abs}\left(D_{n}\right)$ case. The proofs for the other relations are similar. The left side of this relation is equal to $u_{x}$ where $x=\left(\left(\left(a_{1}, a_{2}\right)\right),\left(\left(a_{1},-a_{2}\right)\right),\left(\left(a_{3}, a_{4}\right)\right),\left(\left(a_{3},-a_{4}\right)\right)\right)$ and the right side is equal to $u_{y}$ where $y=\left(\left(\left(a_{1}, a_{3}\right)\right),\left(\left(a_{1},-a_{3}\right)\right),\left(\left(a_{2}, a_{4}\right)\right),\left(\left(a_{2},-a_{4}\right)\right)\right)$. Both $x$ and $y$ are reduced $T$-words for $\left[a_{1}\right]\left[a_{2}\right]\left[a_{3}\right]\left[a_{4}\right] \in \operatorname{Abs}\left(D_{n}\right)$ and so by Corollaries 3.3.2 and 3.3.3 we see that

$$
u_{\left(\left(a_{1}, a_{2}\right)\right)} u_{\left(\left(a_{1},-a_{2}\right)\right)} u_{\left(\left(a_{3}, a_{4}\right)\right)} u_{\left(\left(a_{3},-a_{4}\right)\right)} \equiv u_{\left(\left(a_{1}, a_{3}\right)\right)} u_{\left(\left(a_{1},-a_{3}\right)\right)} u_{\left(\left(a_{2}, a_{4}\right)\right)} u_{\left.\left(\left(a_{2},-a_{4}\right)\right)\right)}(\bmod I)
$$

as desired.
We now focus on proving that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$ implies $u_{x} \equiv u_{y}(\bmod J)$. As was the case for $\operatorname{Abs}\left(B_{n}\right)$, simultaneous balanced cycles will cause a complication. However, we cannot use the strategy employed in the $\operatorname{Abs}\left(B_{n}\right)$ case since the set of reflections for $D_{n}$ does not include any balanced cycles. As such, relation (3.17) is not applicable here. Instead, our strategy will be as follows. Let $x$ be a reduced $T$-word for some $w \in A b s\left(D_{n}\right)$ and let $b_{1} \cdots b_{k_{1}} p_{1} \cdots p_{k_{2}}$ be the disjoint decomposition of $w$. Furthermore, let $\alpha_{1}, \beta_{1}, \ldots, \alpha_{h}, \beta_{h}\left(h=\frac{k_{1}}{2}\right)$ be any permutation of the indices $1, \ldots, k_{1}$. We will show that there exists a $T$-word $\left(z_{1}, z_{2}\right)$ such that $z_{1}$ is a reduced $T$-word for $b_{1} \cdots b_{k_{1}}$ where $b_{\alpha_{i}}, b_{\beta_{i}}(i \in[h])$ are simultaneous with respect to $z_{1}, z_{2}$ is a reduced $T$-word for $p_{1} \cdots p_{k_{2}}$, and $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$. The following lemma states that we can construct this $z_{1}$ in the special case where the disjoint decomposition of $w$ is $b_{1} b_{2} b_{3} b_{4}$. Note that in the lemma $w$ is never explicitly stated, but it is equal to the product $w_{1} w_{2}$ where $w_{1}=b_{1} b_{2}$ and $w_{2}=b_{3} b_{4}$. This special case will allow us to deal with the general case, which is addressed in Lemma 3.3.26.

Lemma 3.3.25. Let $x_{1}, x_{2}$ be reduced $T$-words for $w_{1}, w_{2} \in \operatorname{Abs}\left(D_{n}\right)$ respectively. Suppose that $w_{1}=$ $b_{1} b_{2}\left(\right.$ resp. $\left.w_{2}=b_{3} b_{4}\right)$ is the disjoint decomposition of $w_{1}\left(\right.$ resp. $\left.w_{2}\right)$ where $b_{1}, b_{2}\left(\right.$ resp. $\left.b_{3}, b_{4}\right)$ are simultaneous with respect to $x_{1}\left(\right.$ resp. $\left.x_{2}\right)$. Then $u_{\left(x_{1}, x_{2}\right)} \equiv u_{\left(y_{1}, y_{2}\right)}(\bmod J)$ where $y_{1} \in R_{T}\left(b_{1} b_{3}\right)$ (resp. $y_{2} \in R_{T}\left(b_{2} b_{4}\right)$ ) and $b_{1}, b_{3}$ (resp. $b_{2}, b_{4}$ ) are simultaneous with respect to $y_{1}$ (resp. $y_{2}$ ).

Proof. In the following we let

$$
\begin{aligned}
b_{1} & =\left[\alpha_{1}, \ldots, \alpha_{k_{\alpha}}\right], \\
b_{2} & =\left[\beta_{1}, \ldots, \beta_{k_{\beta}}\right], \\
b_{3} & =\left[\psi_{1}, \ldots, \psi_{k_{\psi}}\right], \text { and } \\
b_{4} & =\left[\delta_{1}, \ldots, \delta_{k_{\bar{\delta}}}\right] .
\end{aligned}
$$

We know by Lemma 3.3.18 that $u_{x_{1}} \equiv u_{z_{1}}(\bmod J)$ where $z_{1}=\left(A, B,\left(\left(\alpha_{1}, \beta_{1}\right)\right),\left(\left(\alpha_{1},-\beta_{1}\right)\right)\right)$ and

$$
\begin{aligned}
& A=\left(\left(\left(\alpha_{1},-\alpha_{2}\right)\right),\left(\left(-\alpha_{2},-\alpha_{3}\right)\right), \ldots,\left(\left(-\alpha_{k_{\alpha}-1},-\alpha_{k_{\alpha}}\right)\right)\right), \\
& B=\left(\left(\left(\beta_{1},-\beta_{2}\right)\right),\left(\left(-\beta_{2},-\beta_{3}\right)\right), \ldots,\left(\left(-\beta_{k_{\beta}-1},-\beta_{k_{\beta}}\right)\right)\right) .
\end{aligned}
$$

Similarly, we have that $u_{x_{2}} \equiv u_{z_{2}}(\bmod J)$ where $z_{2}=\left(\left(\left(\psi_{1}, \delta_{1}\right)\right),\left(\left(\psi_{1},-\delta_{1}\right)\right), C, D\right)$ and

$$
\begin{aligned}
& C=\left(\left(\left(\psi_{1}, \psi_{2}\right)\right), \ldots,\left(\left(\psi_{k_{\psi}-1}, \psi_{k_{\psi}}\right)\right)\right), \\
& D=\left(\left(\left(\delta_{1}, \delta_{2}\right)\right), \ldots,\left(\left(\delta_{k_{\delta}-1}, \delta_{k_{\delta}}\right)\right)\right)
\end{aligned}
$$

From here we see that

$$
\begin{align*}
u_{\left(x_{1}, x_{2}\right)} & \equiv u_{\left(z_{1}, z_{2}\right)}  \tag{Lemma3.3.18}\\
& =u_{A} u_{B} u_{\left(\left(\alpha_{1}, \beta_{1}\right)\right)} u_{\left(\left(\alpha_{1},-\beta_{1}\right)\right)} u_{\left(\left(\psi_{1}, \delta_{1}\right)\right)} u_{\left(\left(\psi_{1},-\delta_{1}\right)\right)} u_{C} u_{D} \\
& \equiv u_{A} u_{B} u_{\left(\left(\alpha_{1}, \psi_{1}\right)\right)} u_{\left(\left(\alpha_{1},-\psi_{1}\right)\right)} u_{\left(\left(\beta_{1}, \delta_{1}\right)\right)} u_{\left(\left(\beta_{1},-\delta_{1}\right)\right)} u_{C} u_{D}  \tag{3.21}\\
& \equiv u_{A} u_{\left(\left(\alpha_{1}, \psi_{1}\right)\right)} u_{\left(\left(\alpha_{1},-\psi_{1}\right)\right)} u_{C} u_{B} u_{\left(\left(\beta_{1}, \delta_{1}\right)\right)} u_{\left(\left(\beta_{1},-\delta_{1}\right)\right)} u_{D}  \tag{3.18}\\
& =u_{\left(y_{1}, y_{2}\right)}
\end{align*}
$$

where $y_{1}=\left(A,\left(\left(\alpha_{1}, \psi_{1}\right)\right),\left(\left(\alpha_{1},-\psi_{1}\right)\right), C\right) \in R_{T}\left(b_{1} b_{3}\right)$ and $y_{2}=\left(B,\left(\left(\beta_{1}, \delta_{1}\right)\right),\left(\left(\beta_{1},-\delta_{1}\right)\right), D\right) \in R_{T}\left(b_{2} b_{4}\right)$. Note that $b_{1}, b_{3}$ are simultaneous with respect to $y_{1}$ and $b_{2}, b_{4}$ are simultaneous with respect to $y_{2}$.

We now move on to the general case of constructing $z_{1}$ where the disjoint decomposition of $w$ is $b_{1} \cdots b_{k}$.

Lemma 3.3.26. Let $x$ be a reduced $T$-word for some $w \in \operatorname{Abs}\left(D_{n}\right)$ with the disjoint decomposition $w=b_{1} \cdots b_{k}$. Also, let $\alpha_{1}, \beta_{1} \ldots, \alpha_{h}, \beta_{h}$ be any permutation of the indices $1, \ldots, k$ where $h=\frac{k}{2}$. Then there exists some $y$ such that we have $u_{x} \equiv u_{y}(\bmod J), y=\left(y\left(\alpha_{1}, \beta_{1}\right), \ldots, y\left(\alpha_{h}, \beta_{h}\right)\right)$, and $y\left(\alpha_{i}, \beta_{i}\right)$ is a reduced $T$-word for $b_{\alpha_{i}} b_{\beta_{i}}$.

Proof. We proceed by induction on $h$. If $h=1$, then $k=2$, and the statement is true for $x=y$. Now suppose the statement is true for all $h \leq m-1$ for some integer $m>2$ and let $h=m$. Furthermore, let $w=m_{1} \cdots m_{h}$ be the simultaneous decomposition of $w$ with respect to $x$. By Lemma 3.3.16 we know that $u_{x} \equiv u_{x_{1}}(\bmod J)$ where $x_{1}=\left(m_{1}^{\prime}, \ldots, m_{h}^{\prime}\right)$ and $m_{j}^{\prime}$ is a reduced $T$-word for $m_{j}$.

There are now two cases to consider. In the first case the $b_{\alpha_{i}} b_{\beta_{i}}$ are already 'paired together' in $x_{1}$ for all $i \in[h]$ and in the second case they are not. By 'paired together' we mean that $x_{1}$ contains a consecutive subword which is a reduced $T$-word for $b_{\alpha_{i}} b_{\beta_{i}}$. This first case will be straightforward to deal with since all we have to do is use our relations to commute the $m_{j}^{\prime}$ so that the reduced $T$-words for the $b_{\alpha_{i}} b_{\beta_{i}}$ appear in the correct order. For the second case we will use our relations on $x_{1}$ in a way that will allow us to reduce to the first case.

Here we deal with the first case. Suppose that for all $i \in[h]$ there exists some $j$ such that $m_{j}=b_{\alpha_{i}} b_{\beta_{i}}$. For each $i$ we can take this corresponding $j$ and let $y\left(\alpha_{i}, \beta_{i}\right)=m_{j}^{\prime}$. As shown in the proof of Lemma 3.3.16, we can use relation (3.18) to commute any $m_{j_{1}}^{\prime}$ with any $m_{j_{2}}^{\prime}\left(j_{1} \neq j_{2}\right)$ and so modulo relation (3.18) we have $u_{x_{1}} \equiv u_{y}$ where $y=\left(y\left(\alpha_{1}, \beta_{1}\right), \ldots, y\left(\alpha_{h}, \beta_{h}\right)\right)$. This implies that $u_{x} \equiv u_{y}(\bmod J)$.

Now suppose there exists some $i \in[h]$ such that there does not exist any $j$ where $m_{j}=b_{\alpha_{i}} b_{\beta_{i}}$. There must then exist $m_{j_{1}}, m_{j_{2}}\left(j_{1} \neq j_{2}\right)$ such that $m_{j_{1}}=b_{\alpha_{i}} b_{\delta_{1}}, m_{j_{2}}=b_{\beta_{i}} b_{\delta_{2}}$ for some $\delta_{1} \neq \delta_{2} \in$ $[k]-\left\{\alpha_{i}, \beta_{i}\right\}$. We know by Lemma 3.3.25 that $u_{m_{j_{1}}^{\prime}} u_{m_{j_{2}}^{\prime}} \equiv u_{\eta_{1}} u_{\eta_{2}}(\bmod J)$ where $\eta_{1} \in R_{T}\left(b_{\alpha_{i}} b_{\beta_{i}}\right)$ and $\eta_{2} \in R_{T}\left(b_{\delta_{1}} b_{\delta_{2}}\right)$. From here we have

$$
\begin{align*}
u_{x_{1}} & \equiv \cdots u_{m_{j_{1}}^{\prime}} u_{m_{j_{2}}^{\prime}} \cdots  \tag{3.18}\\
& \equiv \cdots u_{\eta_{1}} u_{\eta_{2}} \cdots  \tag{Lemma3.3.25}\\
& \equiv u_{\eta_{1}} \cdots  \tag{3.18}\\
& =u_{\eta_{1}} u_{x_{2}} .
\end{align*}
$$

Note that $x_{2}$ is a reduced $T$-word for $w^{\prime}=w b_{\alpha_{i}}^{-1} b_{\beta_{i}}^{-1} \in A b s\left(D_{n}\right)$ where the disjoint decomposition of $w^{\prime}$ is the product of the $b_{j}$ for all $j \in[k]-\left\{\alpha_{i}, \beta_{i}\right\}$. By our inductive hypothesis there exists some $y^{\prime}$ such that $u_{x_{2}} \equiv u_{y^{\prime}}(\bmod J)$ and

$$
y^{\prime}=\left(y^{\prime}\left(\alpha_{1}, \beta_{1}\right), \ldots, y^{\prime}\left(\alpha_{i-1}, \beta_{i-1}\right), y^{\prime}\left(\alpha_{i+1}, \beta_{i+1}\right), \ldots, y^{\prime}\left(\alpha_{h}, \beta_{h}\right)\right)
$$

where $y^{\prime}\left(\alpha_{j}, \beta_{j}\right)$ is a reduced $T$-word for $b_{\alpha_{j}} b_{\beta_{j}}$. Using this we finally see that

$$
\begin{align*}
u_{\eta_{1}} u_{x_{2}} & \equiv u_{\eta_{1}} u_{y^{\prime}}  \tag{induction}\\
& \equiv u_{y} \tag{3.18}
\end{align*}
$$

where $y=\left(y\left(\alpha_{1}, \beta_{1}\right), \ldots, y\left(\alpha_{h}, \beta_{h}\right)\right)$ and $y\left(\alpha_{j}, \beta_{j}\right)$ is a reduced $T$-word for $b_{\alpha_{j}} b_{\beta_{j}}$.
Now that we have $z_{1}$ we can construct $z_{2}$. This construction is simpler than the one for $z_{1}$ and it will be done in the following proposition where we also show that $u_{x} \equiv u_{\left(z_{1}, z_{2}\right)} \equiv u_{y}(\bmod J)$.

Proposition 3.3.27. Let $x, y$ be $T$-words such that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. Then $u_{x} \equiv u_{y}(\bmod J)$.
Proof. By Corollary 3.3.3 we know that $x, y \in R_{T}(w)$ for some $w \in \operatorname{Abs}\left(D_{n}\right)$. Let $w=b_{1} \cdots b_{k_{1}} p_{1} \cdots p_{k_{2}}$ be the disjoint decomposition of $w$ and let $h=\frac{k_{1}}{2}$. Our goal is to show that $u_{x} \equiv u_{z} \equiv u_{y}(\bmod J)$ where $z$ is a $T$-word such that

$$
z=\left(z\left(\alpha_{1}, \alpha_{2}\right), \ldots, z\left(\alpha_{h-1}, \alpha_{h}\right), z(1), \ldots, z\left(k_{2}\right)\right),
$$

$z\left(\alpha_{i}, \alpha_{i+1}\right)$ is a reduced $T$-word for $b_{\alpha_{i}} b_{\alpha_{i+1}}$, and $z(i)$ is a reduced $T$-word for $p_{i}$.
We will explicitly show that $u_{x} \equiv u_{z}(\bmod J)$; the process for proving that $u_{y} \equiv u_{z}(\bmod J)$ is identical. Let $w=m_{\beta_{1}, \beta_{2}} \cdots m_{\beta_{h-1}, \beta_{h}} m_{1} \cdots m_{k_{2}}$ be the simultaneous decomposition of $w$ with respect to $x$ where $m_{\beta_{i}, \beta_{i+1}}=b_{\beta_{i}} b_{\beta_{i+1}}$ and $m_{i}=p_{i}$. By Lemma 3.3.16 we know that $u_{x} \equiv u_{x^{\prime}}(\bmod J)$ where $x^{\prime}=\left(m_{\beta_{1}, \beta_{2}}^{\prime}, \ldots, m_{\beta_{h-1}, \beta_{h}}^{\prime}, m_{1}^{\prime}, \ldots, m_{k_{2}}^{\prime}\right)$ and $m_{\beta_{i}, \beta_{i+1}}^{\prime}\left(\right.$ resp. $\left.m_{i}^{\prime}\right)$ is a reduced $T$-word for $m_{\beta_{i}, \beta_{i+1}}$ (resp. $m_{i}$ ).

First, let $x_{1}^{\prime}=\left(m_{\beta_{1}, \beta_{2}}^{\prime}, \ldots, m_{\beta_{h-1}, \beta_{h}}^{\prime}\right)$ and $z_{1}=\left(z\left(\alpha_{1}, \alpha_{2}\right), \ldots, z\left(\alpha_{h-1}, \alpha_{h}\right)\right)$. We know by Lemma 3.3.26 that $u_{x_{1}^{\prime}} \equiv u_{z_{1}^{\prime}}(\bmod J)$ where $z_{1}^{\prime}=\left(z^{\prime}\left(\alpha_{1}, \alpha_{2}\right), \ldots, z^{\prime}\left(\alpha_{h-1}, \alpha_{h}\right)\right)$ and $z^{\prime}\left(\alpha_{i}, \alpha_{i+1}\right)$ is a reduced $T$-word
$b_{\alpha_{i}} b_{\alpha_{i+1}}$. We also see by Lemma 3.3.19 that $z^{\prime}\left(\alpha_{i}, \alpha_{i+1}\right) \equiv z\left(\alpha_{i}, \alpha_{i+1}\right)(\bmod J)$. This then implies that $u_{x_{1}^{\prime}} \equiv u_{z_{1}}(\bmod J)$.

Now let $x_{2}^{\prime}=\left(m_{1}^{\prime}, \ldots, m_{k_{2}}^{\prime}\right)$ and $z_{2}=\left(z_{1}, \ldots, z_{k_{2}}\right)$. Note that for any $i \in\left[k_{2}\right]$ we have $m_{i}=p_{i}=$ $\left(\left(a_{1}, \ldots, a_{k}\right)\right) \in N C\left(D_{n}, c\right)$ where $c=\left[a_{1}, \ldots, a_{k}, \ldots, a_{n}\right]$ is a Coxeter element. Relations (3.8) and (3.9) are contained in $J$ when $W=D_{n}$ and so by Corollary 3.2.3 and Proposition 3.2.8 see that $u_{m_{i}^{\prime}} \equiv u_{z_{i}}$ $(\bmod J)$. This tells us that $u_{x_{2}^{\prime}} \equiv u_{z_{2}}(\bmod J)$. From the above we have

$$
u_{x} \equiv u_{x^{\prime}}=u_{x_{1}^{\prime}} u_{x_{2}^{\prime}} \equiv u_{z_{1}} u_{z_{2}}=u_{z}(\bmod J) .
$$

Our last step is to show that if $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$. This is done in the following proposition.

Proposition 3.3.28. Let $x$ be a $T$-word. If $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$.
Proof. The proof is similar to the proof of Proposition 3.3.10.
Now we give the proof of Theorem 3.3.23.
Proof. Corollary 3.3.24 gives us the inclusion $J \subset I$. For the reverse inclusion note that by Proposition 3.3.5 we know that $I$ is binomial and so it is sufficient to show that $u_{x} \equiv u_{y}(\bmod I)$ implies $u_{x} \equiv u_{y}$ $(\bmod J)$. This follows directly from Propositions 3.3.27 and 3.3.28.

## CHAPTER

## 4

## UP-OPERATORS FOR BRUHAT ORDER

In this chapter we discuss up-operators when our given poset is Bruhat order, which is a poset defined with respect to a Coxeter group $W$. Specifically, we consider the case where $W$ is either the dihedral group $I_{2}(m)$ or the symmetric group $\mathfrak{S}_{n}$. In section 4.1 we discuss necessary background information for Bruhat order on $I_{2}(m)$ and $\mathfrak{S}_{n}$. In section 4.2 we give a complete list of defining relations for the algebra of up-operators for Bruhat order on $I_{2}(m)$, and in section 4.3 we state several previously unknown relations among the up-operators for Bruhat order on $\mathfrak{S}_{n}$.

### 4.1 Bruhat Order

We first define Bruhat order and discuss some of its properties. The information presented in this section is taken from [7] and the reader may refer to that text for further information on the subject. As mentioned earlier, Bruhat order is defined with respect to a Coxeter group and so before proceeding the reader may wish to review the Coxeter group material covered in subsection 3.1.1.

Definition 4.1.1. Let $W$ be a finite Coxeter group and let $T$ be the set of reflections in $W$. If $v, w \in W$ we take $v \rightarrow w$ to mean that $\ell_{S}(\nu)<\ell_{S}(w)$ and $v^{-1} w \in T$. Bruhat order on $W$ is the partial order on $W$ where $v \leq_{\mathfrak{B}} w$ if and only if there exists some sequence $\nu_{1}, \ldots, v_{k}$ such that

$$
v=v_{1} \rightarrow v_{2} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_{k}=w .
$$

We denote the poset $\left(W, \leq_{\mathfrak{B}}\right)$ by $\mathfrak{B}(W)$.
Since $\mathfrak{B}(W)$ is the only poset we consider in this chapter we will write $\leq, \lessdot$ in place of $\leq_{\mathfrak{B}}, \lessdot_{\mathfrak{B}}$ respectively. We emphasize that the definition for Bruhat order relies on the standard length $\ell_{S}$ of
an element $w \in W$ rather than the absolute length $\ell_{T}$ used for absolute order. Figure 4.1 gives the Hasse diagram for Bruhat order on the Coxeter group $\mathfrak{S}_{3}$ (the permutations are given in one-line notation). The following proposition comes from Corollary 2.2.3 in [7].

Proposition 4.1.2. Let $v, w \in W$. The following are equivalent.

- $v \leq w$.
- If $x$ is a reduced $S$-word for $w$, then $x$ has a subword $x^{\prime}$ that is a reduced $S$-word for $v$.
- There exists some reduced $S$-word $x$ for $w$ such that $x$ has a subword $x^{\prime}$ that is a reduced $S$-word for $v$.

This proposition gives us a way to determine when two elements of $W$ are comparable in $\mathfrak{B}(W)$ by considering their $S$-words. For instance, consider the permutations $213,321 \in \mathfrak{S}_{3}$. The permutation 321 has ( $s_{1}, s_{2}, s_{1}$ ) as a reduced $S$-word while the permutation 213 has $\left(s_{1}\right)$ as a reduced $S$-word, where $s_{i}=(i, i+1)$. Since $\left(s_{1}\right)$ is a subword of $\left(s_{1}, s_{2}, s_{1}\right)$ we have $213 \leq 321$ by Proposition 4.1.2. Indeed, we see that $213 \leq 321$ in Figure 4.1. It is well known that $v \lessdot w$ if and only if $v^{-1} w=t$ for some $t \in T$ and $\ell_{S}(w)=\ell_{S}(\nu)+1$. From this we can also see that $\mathfrak{B}(W)$ is graded with rank function $\ell_{S}$.

As always, our main concern is the study of up-operators and as such we need some edge labeling for $\mathfrak{B}(W)$. Our edge labeling will be the natural one suggested by the nature of covering relations in Bruhat order. In particular, if $v \lessdot w$, then we take $\gamma(\nu, w)=v^{-1} w$ which is an element of $T$. Note that the set of edge labels comes from the set $T$ and so we denote a monomial $u_{t_{1}} \cdots u_{t_{\ell}}$ by $u_{x}$ where $x=\left(t_{1}, \ldots, t_{\ell}\right)$ is a $T$-word. Using the definition of up-operators and what we know about covering relations for $\mathfrak{B}(W)$ we see that

$$
(w) u_{t}= \begin{cases}w t & \text { if } \ell_{S}(w t)=\ell_{S}(w)+1, \text { and } \\ 0 & \text { otherwise }\end{cases}
$$

where $w \in W$ and $t \in T$.

### 4.1.1 Bruhat Order on the Dihedral Group

Background for the Dihedral Group: The dihedral group is the group of symmetries of a regular $m$ gon and it is isomorphic to the Coxeter group $I_{2}(m)$. The Coxeter group $I_{2}(m)$ has simple generators $s_{1}, s_{2}$ subject to $\left(s_{1} s_{2}\right)^{m}=e$, where $e$ is the identity element. The Coxeter graph of $I_{2}(m)$ is given below.


The set of reflections for $I_{2}(m)$ is $T=\left\{r_{i}: i \in[m]\right\}$ where $r_{i}=\left(s_{1} s_{2}\right)^{i-1} s_{1}$. We note that $r_{1}=s_{1}$ and $r_{m}=s_{2}$. Since our up-operators are the $u_{t}$ for $t \in T$ we see that for $I_{2}(m)$ our up-operators have the


Figure 4.1 The Hasse Diagram for $\mathfrak{B}\left(S_{3}\right)$.
form $u_{r_{i}}$ for $1 \leq i \leq m$.

Bruhat Order on the Dihedral Group: We now introduce some notation that will help us better present the structure of $\mathfrak{B}\left(I_{2}(m)\right)$. We let $a^{\delta}=\left(t_{1}, \ldots, t_{\delta}\right)$ be the $T$-word such that $t_{2 i+1}=r_{1}, t_{2 i}=r_{m}$ for all integers $i$ such that $t_{2 i}$ or $t_{2 i+1}$ appears in $a^{\delta}$. We similarly let $b^{\delta}=\left(t_{1}, \ldots, t_{\delta}\right)$ be the $T$-word such that $t_{2 i+1}=r_{m}, t_{2 i}=r_{1}$ for all integers $i$ such that $t_{2 i}$ or $t_{2 i+1}$ appears in $b^{\delta}$. In other words, $a^{\delta}$ is the $T$-word whose entries alternate between $r_{1}$ and $r_{m}$ and whose first entry is $r_{1}$, and $b^{\delta}$ is the $T$-word whose entries alternate between $r_{1}$ and $r_{m}$ and whose first entry is $r_{m}$. For instance, if $\delta=6$, then $a^{\delta}=\left(r_{1}, r_{m}, r_{1}, r_{m}, r_{1}, r_{m}\right)$ and $b^{\delta}=\left(r_{m}, r_{1}, r_{m}, r_{1}, r_{m}, r_{1}\right)$. Additionally, if $a^{\delta}=\left(t_{1}, \ldots, t_{\delta}\right)$, then we let $\bar{a}^{\delta}=\left(t_{\delta}, \ldots, t_{1}\right)$, and if $b^{\delta}=\left(t_{1}, \ldots, t_{\delta}\right)$, then we let $\bar{b}^{\delta}=\left(t_{\delta}, \ldots, t_{1}\right)$. Finally, we let $A^{\delta}$ (resp. $\bar{A}^{\delta}$ ) be the element of $I_{2}(m)$ which has $a^{\delta}$ (resp. $\bar{a}^{\delta}$ ) as a $T$-word. We define $B^{\delta}$ and $\bar{B}^{\delta}$ similarly. We note here that $A^{\delta}, \bar{A}^{\delta}, B^{\delta}$, and $\bar{B}^{\delta}$ all have length $\delta$, and $A^{m}=\bar{A}^{m}=B^{m}=\bar{B}^{m}=w_{0}$.

Now that we have the necessary notation we can discuss the structure of $\mathfrak{B}\left(I_{2}(m)\right.$ ). Figure 4.2 gives the Hasse diagram for Bruhat order on $I_{2}(m)$ along with its edge labels in blue. From the diagram we see that $\mathfrak{B}\left(I_{2}(m)\right.$ ) has exactly 1 element at rank 1 , namely the identity element $e$, and exactly 1 element at rank $m$, namely $A^{m}=B^{m}=\bar{A}^{m}=\bar{B}^{m}$. The element $e$ is the minimum element of the poset and $A^{m}=B^{m}=\bar{A}^{m}=\bar{B}^{m}=w_{0}$ is the maximum element of the poset. Furthermore, for $i \in[m]-\{1, m\}$ there are exactly two elements of rank $i$ and they are $A^{i}$ and $B^{i}$. Additionally, every element of rank $i<m$ is covered by every element of rank $i+1$. From the above we see that the structure of $\mathfrak{B}\left(I_{2}(m)\right)$ is rather simple and we will see that this simplicity is at the core of our proofs in the next section.

Recall from Definition 1.1.2 that a $T$-word $x=\left(t_{1}, \ldots, t_{\ell}\right)$ appears in $[v, w]$ (or $[v, w]$ contains $\left.x\right)$ if there exists some chain $v=v_{0} \lessdot \nu_{1} \lessdot \cdots \lessdot v_{\ell}=w$ in $[v, w]$ such that $v_{i-1}^{-1} v_{i}=t_{i}$ for $i \in[\ell]$. Remark 4.1.3 makes use of this definition.

Remark 4.1.3. Let $x$ be a $T$-word of length at least 2 such that it appears in some interval and $x \neq a^{\delta}, b^{\delta}$ for $\delta<m$. It will prove useful for us to determine in how many intervals $x$ can appear. First suppose that $x=a^{m}$ or $x=b^{m}$. It is straightforward to see that there is only one interval in


Figure 4.2 The Hasse diagram for $\mathfrak{B}\left(I_{2}(m)\right)$.
which $x$ appears, namely the interval $\left[e, w_{0}\right]$.
Now suppose that $x \neq a^{m}$ and $x \neq b^{m}$. Note that by inspection of Figure 4.2 we can see that if $i \notin\{1, m\}$, then each of the following $T$-words appears in exactly one interval:

$$
\left(r_{i}, r_{1}\right),\left(r_{1}, r_{i}\right),\left(r_{i}, r_{m}\right),\left(r_{m}, r_{i}\right),\left(r_{i}, r_{i+1}\right),\left(r_{i+1}, r_{i}\right),\left(r_{i}, r_{i-1}\right),\left(r_{i-1}, r_{i}\right) .
$$

Since $x \neq a^{m}$ and $x \neq b^{m}$ it must be that $x$ has an entry $r_{i}$ such that $i \notin\{1, m\}$. The reflection $r_{i}$ can only be preceded or succeeded by $r_{1}, r_{m}, r_{i-1}$, or $r_{i+1}$. As such, one of the length $2 T$-words listed above must appear as a consecutive subword of $x$. Since each of these length $2 T$-words appears in exactly one interval it must be that $x$ appears in exactly one interval.

Much of our work in the next section relies on understanding the intervals of $\mathfrak{B}\left(I_{2}(m)\right)$ and the $T$-words which appear in them. As such, we end this section with the following definition, which classifies all of the intervals of length at least 2 in $\mathfrak{B}\left(I_{2}(m)\right)$ in terms of $T$-words.

Definition 4.1.4. Let $v<w \in \mathfrak{B}\left(I_{2}(m)\right)$ such that $\ell_{S}(w)-\ell_{S}(v) \geq 2$, and let $\delta=\ell_{S}(w)-\ell_{S}(v)$.

- If $\left(r_{i}, a^{\delta-1}\right)$ appears in $[\nu, w]$ for some $i \in[m]-\{1, m\}$ and $v \neq e, w \neq w_{0}$, then we call $[\nu, w]$ an interval of type $A 1$.
- If $\left(r_{i}, b^{\delta-1}\right)$ appears in $[\nu, w]$ for some $i \in[m]-\{1, m\}$ and $v \neq e, w \neq w_{0}$, then we call $[\nu, w]$ an interval of type $A 2$.
- If $\left(r_{i}, a^{\delta-2}, r_{j}\right)$ appears in $[\nu, w]$ for some $i, j \in[m]-\{1, m\}$ and $v \neq e, w \neq w_{0}$, then we call $[\nu, w]$ an interval of type $B 1$.
- If $\left(r_{i}, b^{\delta-2}, r_{j}\right)$ appears in $[v, w]$ for some $i, j \in[m]-\{1, m\}$ and $v \neq e, w \neq w_{0}$, then we call [ $v, w$ ] an interval of type $B 2$.
- If $\left(r_{i}, a^{\delta-1}\right)$ appears in $[\nu, w]$ for some $i \in[m]-\{1, m\}$ and $\nu \neq e, w=w_{0}$, then we call $[\nu, w]$ an interval of type $C 1$.
- If $\left(r_{i}, b^{\delta-1}\right)$ appears in $[\nu, w]$ for some $i \in[m]-\{1, m\}$ and $v \neq e, w=w_{0}$, then we call $[v, w]$ an interval of type $C 2$.
- If $\left(\bar{a}^{\delta-1}, r_{i}\right)$ appears in $[\nu, w]$ for some $i \in[m]-\{1, m\}$ and $v=e, w \neq w_{0}$, then we call $[\nu, w]$ an interval of type $D 1$.
- If $\left(\bar{b}^{\delta-1}, r_{i}\right)$ appears in $[v, w]$ for some $i \in[m]-\{1, m\}$ and $v=e, w \neq w_{0}$, then we call $[\nu, w]$ an interval of type $D 2$.
- If $v=e$ and $w=w_{0}$, then we call $[\nu, w]$ an interval of type $E$.

This above interval classification is straightforward to see from our earlier description of $\mathfrak{B}\left(I_{2}(m)\right)$. Figures 4.3, 4.4, 4.5, and 4.6 give Hasse diagrams for examples of intervals of all types other than $E$. Interval type $E$ is just the entire poset so its Hasse diagram is given in Figure 4.2.


Figure 4.3 Examples of interval types $A 1$ (left) and $A 2$ (right).

### 4.1.2 Bruhat Order on the Symmetric Group and Known Results

Background for the Symmetric Group: We begin with some necessary notation and definitions. The symmetric group $\mathfrak{S}_{n}$ is the set of bijections from the set [ $n$ ] to itself. In other words, it is the set of permutations on the letters $1, \ldots, n$ and its group operation is function composition. Let $\pi$ be an element of $\mathfrak{S}_{n}$ and let $\pi_{i}=\pi(i)$. The one-line notation for $\pi$ is the sequence $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$ and the two-line notation for $\pi$ is the matrix

$$
\pi=\left(\begin{array}{cccc}
1 & 2 & \ldots & n \\
\pi_{1} & \pi_{2} & \ldots & \pi_{n}
\end{array}\right)
$$

To improve readability we will insert a horizontal line between the first and second row in two-line notation like so

$$
\pi=\left(\begin{array}{cccc}
1 & 2 & \ldots & n \\
\hline \pi_{1} & \pi_{2} & \ldots & \pi_{n}
\end{array}\right)
$$

Let $\pi=\pi_{1} \ldots \pi_{n}$. We say that $\left(\pi_{i}, \pi_{j}\right)$ is an inversion of $\pi$ if $i<j$ and $\pi_{i}>\pi_{j}$. It is well known that $\ell_{S}(\pi)$ is equal to the number of inversions occurring in $\pi$.

Example 4.1.5. Let $\pi$ be the element of $S_{5}$ such that $\pi(1)=4, \pi(2)=5, \pi(3)=2, \pi(4)=1$, and $\pi(5)=3$. The one-line notation for $\pi$ is $\pi=45213$ and the two-line notation is

$$
\pi=\left(\begin{array}{lllll}
1 & 2 & 3 & 4 & 5 \\
\hline 4 & 5 & 2 & 1 & 3
\end{array}\right)
$$

We see that $(4,2)$ is an inversion since $1<3$ and $\pi(1)=4>2=\pi(3)$. We also see that $(2,3)$ is not an inversion since $\pi(3)=2<3=\pi(5)$. In total, $\pi$ has 7 inversions so $\ell_{S}(\pi)=7$.


Figure 4.4 Examples of interval types $B 1$ (left) and $B 2$ (right).


Figure 4.5 Examples of interval types $C 1$ (left) and $C 2$ (right) for $m=7$.


Figure 4.6 Examples of interval types $D 1$ (left) and $D 2$ (right).

Recall from subsection 3.1.1 that $\mathfrak{S}_{n}$ is isomorphic to the Coxeter group $A_{n-1}$. We know from our earlier discussion on $A_{n-1}$ that the set of reflections for $\mathfrak{S}_{n}$ is $T=\{(i, j): i, j \in[n], i<j\}$. From this we see that our up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$ are the $u_{(i, j)}$ for $i, j \in[n]$ and $i<j$. As noted at the end of our discussion on Bruhat order, if $\pi \in \mathfrak{S}_{n}$ then

$$
(\pi) u_{(i, j)}= \begin{cases}\pi(i, j) & \text { if } \ell_{S}(\pi(i, j))=\ell_{S}(\pi)+1, \text { and } \\ 0 & \text { otherwise }\end{cases}
$$

We then see that when the action of $u_{(i, j)}$ is nonzero it amounts to multiplication on the right by $(i, j)$. In light of this we note that multiplication on the right by a transposition $(i, j)$ has the effect of switching the integers in the $i$ and $j$ position in one-line notation. For instance, if $\pi=45213$ and $t=(1,3)$ then $\pi t=\pi(1,3)=25413$.

Background for Bruhat Order on the Symmetric Group: For our purposes here, we only need one known result from the literature. The following proposition will be the main tool we use for our work with up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$. The content of this proposition is taken from Lemma 2.1.4 in [7], and it gives us a way to determine when $\pi \lessdot \sigma$.

Proposition 4.1.6. Let $\pi=\pi_{1} \ldots \pi_{n}, \sigma=\sigma_{1} \ldots \sigma_{n}$ be elements of $\mathfrak{S}_{n}$. Then $\pi \lessdot \sigma$ if and only if there exist $i<j$ such that $\sigma=\pi(i, j)$ and $\pi_{i}<\pi_{j}$ and there does not exist any $i<k<j$ such that $\pi_{i}<\pi_{k}<\pi_{j}$.

Since $(\pi) u_{(i, j)} \neq 0$ if and only if $\pi \lessdot \pi(i, j)$, Proposition 4.1.6 gives us certain inequalities among the $\pi_{i}$ which are obeyed if and only if $(\pi) u_{(i, j)} \neq 0$. We will use these imposed inequalities in the proof of our main statement in the last section.

Known Results: As mentioned previously, the up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$ were studied by Fomin and Kirillov [13] in their study of the Fomin-Kirillov algebra $\mathscr{E}_{n}$. Recall that $\mathscr{E}_{n}$ is the algebra over C generated by the formal objects $[i j]$ where $i, j \in[n]$ and $i<j$, subject to the relations

$$
\begin{aligned}
{[i j]^{2} } & =0, & \\
{[i j][j k] } & =[j k][i k]+[i k][i j], & i<j<k, \\
{[j k[[i j]} & =[i k][j k]+[i j][i k], & i<j<k, \text { and } \\
{[i j][k l] } & =[k l][i j], & \{i, j\} \cap\{k, l\}=\emptyset .
\end{aligned}
$$

Fomin and Kirillov determined that the up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$ give an unfaithful representation of $\mathscr{E}_{n}$ where the representation map is $[i j] \mapsto u_{(i, j)}$. From this representation we know that the $u_{(i, j)}$ must obey the following relations:

$$
\begin{array}{rlr}
u_{(i j)}^{2} & =0, & \\
u_{(i j)} u_{(j k)} & =u_{(j k)} u_{(i k)}+u_{(i k)} u_{(i j)}, & i<j<k, \\
u_{(j k)} u_{(i j)} & =u_{(i k)} u_{(j k)}+u_{(i j)} u_{(i k),}, & i<j<k, \text { and } \\
u_{(i j)} u_{(k l)} & =u_{(k l)} u_{(i j)}, & \{i, j\} \cap\{k, l\}=\emptyset .
\end{array}
$$

However, since the representation is not faithful, the $u_{(i, j)}$ satisfy other relations. Two of these additional relations, given by Fomin and Kirillov, are

$$
\begin{align*}
u_{(i j)} u_{(i k)} u_{(i j)} & =0 \text { and }  \tag{4.5}\\
u_{(j k)} u_{(i k)} u_{(j k)} & =0 \tag{4.6}
\end{align*}
$$

where $i<j<k$. The full set of relations among the $u_{(i, j)}$ remains unknown but we describe several more which we state in section 4.3.

### 4.2 Up-operators for Bruhat Order on the Dihedral Group

In this section we give a complete list of relations for the algebra of up-operators on $\mathfrak{B}\left(I_{2}(m)\right)$. Let $\mathscr{U}^{\mathfrak{B}\left(I_{2}(m)\right)}$ be the free associative algebra over $\mathbf{C}$ generated by the $u_{r_{i}}$ for $r_{i} \in T$. Also let $I^{\mathfrak{B}\left(I_{2}(m)\right)}$ be the two-sided ideal of $\mathscr{U}^{\mathfrak{B}\left(I_{2}(m)\right)}$ which contains all elements of $\mathscr{U}^{\mathfrak{B}\left(I_{2}(m)\right)}$ that annihilate all elements of $\mathbf{C}\left[\mathfrak{B}\left(I_{2}(m)\right)\right]$ (the vector space over $\mathbf{C}$ with basis $\mathfrak{B}\left(I_{2}(m)\right)$ ). We call $\mathscr{U}^{\mathfrak{B}\left(I_{2}(m)\right)} / I^{\mathfrak{B}\left(I_{2}(m)\right)}$ the algebra of up-operators for $\mathfrak{B}\left(I_{2}(m)\right)$. Our goal for this section is to prove the following theorem.

Theorem 4.2.1. The algebra of up-operators on $\mathfrak{B}\left(I_{2}(m)\right)$ is characterized by the following relations:

$$
\begin{array}{rlr}
u_{r_{m}} u_{r_{1}} & =\sum_{i=1}^{m-1} u_{r_{i}} u_{r_{i+1}}, \\
u_{r_{1}} u_{r_{m}} & =\sum_{i=1}^{m-1} u_{r_{i+1}} u_{r_{i},}, & \\
u_{r_{1}} u_{r_{i}} & =u_{r_{j}} u_{r_{1}}, & i+j=m+2 \text { and } i, j \notin\{1,2, m\}, \\
u_{r_{m}} u_{r_{i}} & =u_{r_{j}} u_{r_{m}}, & i+j=m \text { and } i, j \notin\{1, m-1, m\}, \\
u_{r_{i}}^{2} & =0, & 1 \leq i \leq m, \\
u_{r_{i}} u_{r_{j}} & =0, & i, j \notin\{1, m\} \text { and }|i-j| \geq 2, \\
u_{r_{1}} u_{r_{i}} u_{r_{i-1}} & =0, & 3 \leq i \leq m-1, \\
u_{r_{m}} u_{r_{i}} u_{r_{i+1}} & =0, & 2 \leq i \leq m-2, \\
u_{r_{2}} u_{r_{1}} u_{r_{i}} & =0, & 1 \leq i \leq m, \\
u_{r_{m-1}} u_{r_{m}} u_{r_{i}} & =0, & 1 \leq i \leq m, \\
u_{r_{i}} u_{r_{i-1}} u_{r_{i}} & =0, & 2 \leq i \leq m, \\
u_{r_{i}} u_{r_{i+1}} u_{r_{i}} & =0, & 1 \leq i \leq m-1 . \tag{4.18}
\end{array}
$$

Let $J^{\mathfrak{B}\left(I_{2}(m)\right)}$ be the two-sided ideal generated by relations (4.7)-(4.18). To prove Theorem 4.2.1 it is sufficient to show that $J^{\mathfrak{B}\left(L_{2}(m)\right)}=I^{\mathfrak{B}\left(I_{2}(m)\right)}$. In the remainder of this section we will denote $\mathscr{U}^{\mathfrak{B}\left(I_{2}(m)\right)}, I^{\mathfrak{B}\left(I_{2}(m)\right)}$, and $J^{\mathfrak{B}\left(I_{2}(m)\right)}$ by $\mathscr{U}, I$, and $J$ respectively. Lemma 4.2.2 gives some additional relations implied by relations (4.7)-(4.18) which will allow us to make some of our later proofs more compact.

Lemma 4.2.2. Modulo J we have

$$
\begin{align*}
u_{r_{i}} u_{r_{1}} u_{r_{2}} & \equiv 0 \text { for } 1 \leq i \leq m,  \tag{4.19}\\
u_{r_{i}} u_{r_{m}} u_{r_{m-1}} & \equiv 0 \text { for } 1 \leq i \leq m . \tag{4.20}
\end{align*}
$$

Before giving the proof we note that relations (4.19) and (4.20) are analogous to relations (4.15) and (4.16) in that both pairs ensure $u_{x} \equiv 0(\bmod J)$ if $x$ 'goes past' the top or bottom of the poset. In particular, note that $\left(r_{1}, r_{2}\right)$ (resp. $\left(r_{m}, r_{m-1}\right)$ ) appears only in the interval [ $e, r_{1} r_{2}$ ] (resp. [ $e, r_{m} r_{m-1}$ ]) and so any $T$-word of the form $\left(r_{i}, r_{1}, r_{2}\right)$ (resp. $\left(r_{i}, r_{m}, r_{m-1}\right)$ ) goes below the 'bottom' of the poset and so it cannot appear in the poset. As such, relation (4.19) (resp. (4.20)) ensures that $u_{\left(r_{i}, r_{1}, r_{2}\right)} \equiv 0$ $(\bmod J)\left(\operatorname{resp} . u_{\left(r_{i}, r_{m}, r_{m-1}\right)} \equiv 0(\bmod J)\right)$. Similarly, $\left(r_{2}, r_{1}\right)\left(\right.$ resp. $\left.\left(r_{m-1}, r_{m}\right)\right)$ appears only in the interval [ $w_{0} r_{1} r_{2}, w_{0}$ ] (resp. [ $w_{0} r_{m} r_{m-1}, w_{0}$ ]) and so a $T$-word of the form $\left(r_{2}, r_{1}, r_{i}\right)$ (resp. $\left(r_{m-1}, r_{m}, r_{i}\right)$ ) goes above the 'top' of the poset, and so relation (4.15) (resp. (4.16)) ensures that $u_{\left(r_{2}, r_{1}, r_{i}\right)} \equiv 0(\bmod J)$ (resp. $u_{\left(r_{m-1}, r_{m}, r_{i}\right)} \equiv 0(\bmod J)$ ).

Proof. We give the proof of (4.19); the proof of (4.20) is similar. If $i=1$, then $u_{1} u_{1} u_{2} \equiv 0$ modulo
relation (4.11). If $i=m$, then we have

$$
\begin{align*}
u_{r_{m}} u_{r_{1}} u_{r_{2}} & \equiv\left(u_{r_{1}} u_{r_{2}}+u_{r_{2}} u_{r_{3}}+\cdots+u_{r_{m-1}} u_{r_{m}}\right) u_{r_{2}}  \tag{4.7}\\
& =u_{r_{1}} u_{r_{2}} u_{r_{2}}+u_{r_{2}} u_{r_{3}} u_{r_{2}}+\cdots+u_{r_{m-1}} u_{r_{m}} u_{r_{2}} \\
& \equiv 0 \tag{4.11}
\end{align*}
$$

If $i=m-1$, then we have

$$
\begin{align*}
u_{r_{m-1}} u_{r_{1}} u_{r_{2}} \equiv & u_{r_{m-1}}\left(u_{r_{m}} u_{r_{1}}-u_{r_{2}} u_{r_{3}}-\cdots-u_{r_{m-1}} u_{m}\right)  \tag{4.7}\\
= & u_{r_{m-1}} u_{r_{m}} u_{r_{1}}-u_{r_{m-1}} u_{r_{2}} u_{r_{3}}-\cdots \\
& \quad-u_{r_{m-1}} u_{r_{m-2}} u_{r_{m-1}}-u_{r_{m-1}} u_{r_{m-1}} u_{m} \\
\equiv & 0 \tag{4.11}
\end{align*}
$$

If $i=2$, then $u_{r_{2}} u_{r_{1}} u_{r_{2}} \equiv 0$ modulo relation (4.18). Finally, if $i \neq 1,2, m-1, m$, then $u_{r_{i}} u_{r_{1}} u_{r_{2}} \equiv$ $u_{r_{1}} u_{r_{m-i+2}} u_{r_{2}} \equiv 0$ where the first equivalence is modulo relation (4.9) and the second equivalence is modulo relation (4.12).

The following proposition and corollary make use of Definition 1.1.2, Proposition 1.1.3, and Corollary 1.1.4 from Chapter 1 so the reader may which to review this material. The proposition and corollary give us a straightforward way to determine when $u_{x} \equiv u_{y}(\bmod I)$ given certain conditions on $x$ and $y$. Both of these statements are a direct result of the rather simple structure of $\mathfrak{B}\left(I_{2}(m)\right)$ and in particular the previously noted fact that if $x \neq a^{\delta}$ for $\delta<m$ and if $x$ is of length at least 2 then $x$ can appear in at most one interval of the poset.

Proposition 4.2.3. Let $x, y$ be $T$-words such that $x \neq y$ and $u_{x}, u_{y} \neq 0(\bmod I)$. If $u_{x} \equiv u_{y}(\bmod I)$, then $x$ and $y$ have the same length. Furthermore, $u_{x} \equiv u_{y}(\bmod I)$ if and only if $x$ and $y$ both appear in some interval $[\nu, w] \in \mathfrak{B}\left(I_{2}(m)\right)$ and $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$.

Proof. Let $x=\left(t_{1}, \ldots, t_{\ell}\right), y=\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$ and suppose that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. There must exist some $v \in I_{2}(m)$ such that $(\nu) u_{x}=(\nu) u_{y}=w \neq 0$. By Proposition 1.1.3 we then know that $x$ and $y$ both appear in the interval $[\nu, w]$. Using Definition 1.1.2 we see that if $x$ and $y$ both appear in $[v, w]$, then both of their lengths must equal $\ell_{S}(w)-\ell_{S}(\nu)$. From this we have $\ell=\ell_{S}(w)-\ell_{S}(\nu)=\ell^{\prime}$.

In the following we assume that $\ell>1$, the case where $\ell=1$ is straightforward. To prove that $x, y \notin$ $\left\{a^{\delta}, b^{\delta}\right\}$ for $\delta<m$ we consider 2 different cases. First, assume for contradiction that $x=a^{\delta}, y=b^{\delta}$ for some $1<\delta<m$. Then we have $(v) u_{x}=v A^{\delta}=v B^{\delta}=(v) u_{y}$. However, this implies that $A^{\delta}=B^{\delta}$ which is a contradiction for $\delta<m$. Second, assume again for contradiction that $x \notin\left\{a^{\delta}, b^{\delta}\right\}$ and $y \in\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$. As mentioned earlier, since $x \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $\delta<m$ it must be that $x$ appears in exactly one interval $[v, w]$. By looking at the structure of $\mathfrak{B}\left(I_{2}(m)\right)$ we can see that since $y \in\left\{a^{\delta}, b^{\delta}\right\}$ for $\delta<m$ it must appear in at least 2 intervals $[v, w]$ and $\left[v^{\prime}, w^{\prime}\right]$ where $v \neq v^{\prime}$. We then see by Proposition 1.1.3 that $\left(v^{\prime}\right) u_{y}=w^{\prime} \neq 0=\left(\nu^{\prime}\right) u_{x}$, which contradicts our assumption that $u_{x} \equiv u_{y}(\bmod I)$.

Now suppose that $x$ and $y$ both appear in some interval $[v, w]$ and $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$. By Proposition 1.1.3 we know that $(\nu) u_{x}=(\nu) u_{y}=w \neq 0$. As noted earlier, since $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$ we know that $x$ and $y$ do not appear in any interval other than $[\nu, w]$. Thus, by Proposition 1.1.3 we have $\left(\nu^{\prime}\right) u_{x}=\left(v^{\prime}\right) u_{y}=0$ for any $\nu^{\prime} \neq v$. From this we see that $u_{x} \equiv u_{y}(\bmod I)$.

Corollary 4.2.4. Let $x$ and $y$ be $T$-words such that $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$. If there exists some $v \in I_{2}(m)$ such that $(v) u_{x}=(v) u_{y}=w \neq 0$, then $u_{x} \equiv u_{y}(\bmod I)$.

Proof. This is immediate from Propositions 1.1.3 and 4.2.3.
We can now start to address the equivalence we desire, namely $J=I$. The following proposition brings us closer to proving this equivalence by showing $J \subset I$.

## Proposition 4.2.5. We have $J \subset I$.

Proof. It is sufficient to show for each of (4.7)-(4.18) that the terms on the left side of the relation are equivalent to the terms on the right side of the relation modulo $I$. We can prove this for relations (4.9)-(4.18) by considering the $T$-words appearing as indices in the relations. For relation (4.9) we can see that the $T$-word in the indices on the left is $\left(r_{1}, r_{i}\right)$ and the $T$-word in the indices on the right is $\left(r_{j}, r_{1}\right)$ where $i+j=m+2$ and $i, j \notin\{1,2, m\}$. Both of these $T$-words appear in the interval $\left[\bar{B}^{i-2}, \bar{A}^{i}\right]$. Furthermore, neither is in the set $\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$. From this we see that $u_{r_{1}} u_{r_{i}} \equiv u_{r_{j}} u_{r_{1}}(\bmod I)$ by Proposition 4.2.3. A similar argument works for relation (4.10).

For relation (4.11) we see that the $T$-word appearing in the indices on the left side is $\left(r_{i}, r_{i}\right)$ for $1 \leq i \leq m$ and there is no $T$-word for the right side. Since ( $r_{i}, r_{i}$ ) does not appear in any interval of $\mathfrak{B}\left(I_{2}(m)\right)$ we have that $u_{\left(r_{i}, r_{i}\right)} \equiv 0(\bmod I)$ by Corollary 1.1.4. A similar argument holds for relations (4.12)-(4.18).

For relations (4.7) and (4.8) we can simply check that

$$
\begin{equation*}
(\nu) u_{r_{m}} u_{r_{1}}=(v)\left(\sum_{i=1}^{m-1} u_{r_{i}} u_{r_{i+1}}\right) \tag{4.21}
\end{equation*}
$$

and

$$
\begin{equation*}
(\nu) u_{r_{1}} u_{r_{m}}=(\nu)\left(\sum_{i=1}^{m-1} u_{r_{i+1}} u_{r_{i}}\right) \tag{4.22}
\end{equation*}
$$

for all $v \in I_{2}(m)$. This is fairly straightforward to do since ( $\left.r_{m}, r_{1}\right)$ (resp. $\left(r_{1}, r_{m}\right)$ ) appears in exactly $m-1$ intervals, each of which contains exactly one chain on the right side of (4.21) (resp. 4.22).

Now that we have shown $J \subset I$ our main focus will be on proving $I \subset J$. Ultimately, we wish to show that any relation which holds modulo $I$ also holds modulo $J$. However, before addressing this more general case we first wish to prove that this holds for certain binomial relations. In other words, we wish to show, given certain conditions on $x$ and $y$, that if $u_{x} \equiv u_{y}(\bmod I)$, then $u_{x} \equiv u_{y}$
$(\bmod I)$. Proposition 4.2.6 and Corollary 4.2.8 prove this by analyzing the possible intervals which can appear in $\mathfrak{B}\left(I_{2}(m)\right)$.

Proposition 4.2.6. Let $\mathfrak{I}$ be an interval in $\mathfrak{B}\left(I_{2}(m)\right)$ and let $x, y$ be $T$-words such that $x$ and $y$ both appear in I. If $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $1<\delta<m$, then $u_{x} \equiv u_{y}(\bmod J)$.

The following proof is somewhat lengthy so we provide a concrete example of one of the cases covered in Example 4.2.7. This example uses Figure 4.7 to help the reader visualize what is happening in the process contained in the proof.

Proof. Let $\mathfrak{I}=[v, w]$ for some $v, w \in I_{2}(m)$ and let $n=\ell_{S}(w)-\ell_{S}(v)$. We denote by $\Gamma$ the set of all $T$-words appearing in $\mathfrak{I}$ which are not equal to $a^{n}$ or $b^{n}$ for $1<n<m$. Let $x$ be any element of $\Gamma$. It is sufficient to show that there exists a $T$-word $z$ appearing in $\mathfrak{I}$ such that $u_{x} \equiv u_{z}(\bmod J)$ for any possible $x$. The following is a list of our interval types and the choice of $z$ we use for each of them:

$$
\begin{aligned}
A 1: z & =\left(r_{i}, a^{n-1}\right) \text { for } i \notin\{1, m\}, \\
A 2: z & =\left(r_{i}, b^{n-1}\right) \text { for } i \notin\{1, m\}, \\
B 1: z & =\left(r_{i}, a^{n-2}, r_{j}\right) \text { for } i \neq j \text { and } i, j \notin\{1, m\}, \\
B 2: z & =\left(r_{i}, b^{n-2}, r_{j}\right) \text { for } i \neq j \text { and } i, j \notin\{1, m\}, \\
C 1: z & =\left(r_{i}, a^{n-1}\right) \text { for } i \notin\{1, m\}, \\
C 2: z & =\left(r_{i}, b^{n-1}\right) \text { for } i \notin\{1, m\}, \\
D 1: z & =\left(\bar{a}^{n-1}, r_{i}\right) \text { for } i \notin\{1, m\}, \\
D 2: z & =\left(\bar{b}^{n-1}, r_{i}\right) \text { for } i \notin\{1, m\}, \text { and } \\
E: z & =\left(r_{1}, \ldots, r_{m}\right) .
\end{aligned}
$$

Note that interval type $A 1$ is similar to type $A 2$, type $B 1$ is similar to type $B 2$, and type $C 1$ is similar to types $C 2, D 1$, and $D 2$. As such, we will only give explicit arguments for types $A 1, B 1, C 1$, and $E$. We first address the situation where $\mathfrak{I}$ is an interval of any type other than $E$. We proceed by considering each possible value of $n$. If $n=1$, then the statement is trivially true for type $A 1$ and vacuously true for types $B 1$ and $C 1$. Suppose then that $n=2$. If $\mathfrak{I}$ is of type $A 1$, then $\Gamma=\left\{\left(r_{i}, r_{1}\right),\left(r_{1}, r_{j}\right)\right\}$ where $i+j=m+2$ and $i, j \notin\{1,2, m\}$. We have $z=\left(r_{i}, r_{1}\right)$ and we see that if $x=\left(r_{i}, r_{1}\right)$, then $u_{x}=u_{z}$ and if $x=\left(r_{1}, r_{j}\right)$, then $u_{x} \equiv u_{z}$ modulo relation (4.9). If $\mathfrak{I}$ is of type $B 1$, then $\Gamma=\left\{\left(r_{i}, r_{i-1}\right)\right\}$ or $\Gamma=\left\{\left(r_{i}, r_{i+1}\right)\right\}$. In the first case we have $z=\left(r_{i}, r_{i-1}\right)$ and we can only have $x=\left(r_{i}, r_{i-1}\right)$ so $u_{x}=u_{z}$. In the second case we have $z=\left(r_{i}, r_{i+1}\right)$ and we can only have $x=\left(r_{i}, r_{i+1}\right)$ so $u_{x}=u_{z}$. Finally, if $\mathfrak{I}$ is of type $C 1$, then $\Gamma=\left\{\left(r_{1}, r_{2}\right)\right\}$. We have $z=\left(r_{1}, r_{2}\right)$ and since we can only have $x=z$ we see that $u_{x}=u_{z}$.

We now proceed by induction on $n$. For our base case we let $n=3$. (We have already done the $n=1,2$ cases but we still need to do the $n=3$ case separately since it has some unique characteristics that prevent it from being included in the inductive step.)

Type A1 Base Case. First suppose that $\Im$ is of type $A 1$. Then

$$
\Gamma=\left\{\left(r_{i}, r_{1}, r_{m}\right),\left(r_{1}, r_{m}, r_{i-2}\right),\left(r_{1}, r_{j}, r_{m}\right),\left(r_{i}, r_{i-1}, r_{i-2}\right)\right\}
$$

where $i+j=m+2$ and $i, j \notin\{1,2,3, m\}$. We have $z=\left(r_{i}, r_{1}, r_{m}\right)$. If $x=\left(r_{i}, r_{1}, r_{m}\right)$, then $u_{x}=u_{z}$. If $x=\left(r_{1}, r_{m}, r_{i-2}\right)$ we have:

$$
\begin{align*}
u_{x} & =u_{r_{1}} u_{r_{m}} u_{r_{i-2}} \\
& \equiv u_{r_{1}} u_{r_{j}} u_{r_{m}}  \tag{4.10}\\
& \equiv u_{r_{i}} u_{r_{1}} u_{r_{m}}  \tag{4.9}\\
& =u_{z} .
\end{align*}
$$

If $x=\left(r_{1}, r_{j}, r_{m}\right)$, then $u_{x} \equiv u_{z}(\bmod J)$ as shown in the work for the previous case. Finally, if $x=$ $\left(r_{i}, r_{i-1}, r_{i-2}\right)$, then

$$
\begin{align*}
u_{x}= & u_{r_{i}} u_{r_{i-1}} u_{r_{i-2}} \\
\equiv & \left(u_{r_{1}} u_{r_{m}}-u_{r_{2}} u_{r_{1}}-\cdots-u_{r_{i-1}} u_{r_{i-2}}-u_{r_{i+1}} u_{r_{i}}-\ldots\right. \\
& \left.\quad-u_{r_{m}} u_{r_{m-1}}\right) u_{r_{i-2}}  \tag{4.8}\\
= & u_{r_{1}} u_{r_{m}} u_{r_{i-2}}-u_{r_{2}} u_{r_{1}} u_{r_{i-2}}-\cdots-u_{r_{i-1}} u_{r_{i-2}} u_{r_{i-2}}-u_{r_{i+1}} u_{r_{i}} u_{r_{i-2}}-\ldots \\
& \quad-u_{r_{m}} u_{r_{m-1}} u_{r_{i-2}} \\
= & u_{r_{1}} u_{r_{m}} u_{r_{i-2}}  \tag{4.11}\\
\equiv & u_{z}
\end{align*}
$$

where the last equivalence is given by the $x=\left(r_{1}, r_{m}, r_{i-2}\right)$ case.

Type B1 Base Case. Now suppose that $\mathfrak{I}$ is of type $B 1$. Then

$$
\Gamma=\left\{\left(r_{i}, r_{1}, r_{j+1}\right),\left(r_{1}, r_{j}, r_{j+1}\right),\left(r_{i}, r_{i-1}, r_{1}\right)\right\}
$$

where $i+j=m+2$ and $i, j \notin\{1,2,3, m\}$. We have $z=\left(r_{i}, r_{1}, r_{j+1}\right)$. If $x=\left(r_{i}, r_{1}, r_{j+1}\right)$, then $u_{x}=u_{z}$. If $x=\left(r_{1}, r_{j}, r_{j+1}\right)$, then $u_{x}=u_{r_{1}} u_{r_{j}} u_{r_{j+1}} \equiv u_{r_{i}} u_{r_{1}} u_{r_{j+1}}=u_{z}$ modulo relation (4.9). Finally, if $x=$ ( $r_{i}, r_{i-1}, r_{1}$ ), then $u_{x}=u_{r_{i}} u_{r_{i-1}} u_{r_{1}} \equiv u_{r_{i}} u_{r_{1}} u_{r_{j+1}}=u_{z}$ modulo relation (4.9).

Type C1 Base Case. Now suppose that $\mathfrak{I}$ is of type $C 1$. We then have

$$
\Gamma=\left\{\left(r_{3}, r_{1}, r_{m}\right),\left(r_{1}, r_{m-1}, r_{m}\right),\left(r_{3}, r_{2}, r_{1}\right)\right\} .
$$

We see that $z=\left(r_{3}, r_{1}, r_{m}\right)$. If $x=z$, then $u_{x}=u_{z}$. If $x=\left(r_{1}, r_{m-1}, r_{m}\right)$, then $u_{x}=u_{r_{1}} u_{r_{m-1}} u_{r_{m}} \equiv$
$u_{r_{3}} u_{r_{1}} u_{r_{m}}=u_{z}$ modulo relation (4.9). Finally, if $x=\left(r_{3}, r_{2}, r_{1}\right)$, then we have

$$
\begin{align*}
u_{x} & =u_{r_{3}} u_{r_{2}} u_{r_{1}} \\
& \equiv u_{r_{3}}\left(u_{r_{1}} u_{r_{m}}-u_{r_{3}} u_{r_{2}}-\cdots-u_{r_{m}} u_{r_{m-1}}\right)  \tag{4.8}\\
& =u_{r_{3}} u_{r_{1}} u_{r_{m}}-u_{r_{3}} u_{r_{3}} u_{r_{2}}-\cdots-u_{r_{3}} u_{r_{m}} u_{r_{m-1}} \\
& \equiv u_{r_{3}} u_{r_{1}} u_{r_{m}}  \tag{4.11}\\
& =u_{z} .
\end{align*}
$$

Now that we have finished the base case $n=3$ we move on to the inductive step. Assume that the statement holds for all $n \leq k-1$ for some integer $k>4$ and let $n=k$. In the following we take $t_{1}, \ldots, t_{n}$ to be reflections such that $x=\left(t_{1}, \ldots, t_{n}\right)$.

Type B1 Induction. We first suppose that $\mathfrak{I}$ is of type $B 1$ and so we have $z=\left(r_{i}, a^{n-2}, r_{j}\right)$ where $i \neq j$ and $i, j \notin\{1, m\}$. If $x=z$, then the statement is trivially true so suppose that $x \neq z$. We consider all possibilities for $t_{n}$.

Case 1. If $t_{n} \notin\left\{r_{1}, r_{m}\right\}$, then $t_{n}=r_{j}$ and $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right)$ is a $T$-word appearing in the interval [ $v, w t_{n}$ ]. Note that $\left[v, w t_{n}\right.$ ] is a type $A 1$ interval with length $n-1$ that contains the $T$-word $z^{\prime}=$ $\left(r_{i}, a^{n-2}\right.$ ). Since $x^{\prime}$ is in a type $A 1$ interval we know that $x^{\prime} \neq a^{n-1}$ and so we can apply the induction hypothesis to get $u_{x^{\prime}} \equiv u_{z^{\prime}}(\bmod J)$. Using this we have $u_{x}=u_{x^{\prime}} u_{r_{j}} \equiv u_{z^{\prime}} u_{r_{j}}=u_{z}(\bmod J)$.

Case 2. Now suppose that $t_{n} \in\left\{r_{1}, r_{m}\right\}$ and note that $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right) \neq a^{n-1}$ since $x \in \Gamma$ and $a^{n} \notin \Gamma$. Furthermore, $x^{\prime}$ appears in the type $B 1$ interval $\left[v, w t_{n}\right]$ which is of length $n-1$ and which contains the $T$-word $z^{\prime}=\left(r_{i}, a^{n-3}, r_{j^{\prime}}\right)$ where $i \neq j^{\prime}, j^{\prime} \notin\{1, m\}$, and either $j+j^{\prime}=m$ or $j+j^{\prime}=m+2$. By induction we have $u_{x^{\prime}} \equiv u_{z^{\prime}}(\bmod J)$ and so $u_{x}=u_{x^{\prime}} u_{t_{n}} \equiv u_{z^{\prime}} u_{t_{n}}(\bmod J)$. From here we see that

$$
\begin{align*}
u_{z^{\prime}} u_{t_{n}} & =u_{r_{i}} u_{a^{n-3}} u_{r_{j^{\prime}}} u_{t_{n}} \\
& \equiv u_{r_{i}} u_{a^{n-3}} u_{t_{n}} u_{r_{j}}  \tag{4.9}\\
& =u_{z} .
\end{align*}
$$

In the above, after applying relation (4.9) or (4.10) one may wonder if the result would be something of the form $\cdots u_{t_{n}} u_{t_{n}} u_{r_{j}}$. If this occurs, then $u_{x}$ would be equivalent to 0 modulo $J$ rather than equivalent to $u_{z}$ modulo $J$. However, this is not a concern since the second to last entry of $z^{\prime}$ must equal $r_{1}$ if $t_{n}=r_{m}$ and it must equal $r_{m}$ if $t_{n}=r_{1}$. This is based on the parity of $n$.

Type A1 Induction. Now suppose that $\mathfrak{I}$ is of type $A 1$ and so $z=\left(r_{i}, a^{n-1}\right)$ for some $i \notin\{1, m\}$. If $x=z$, then $u_{x}=u_{z}$ and we are done, so suppose instead that $x \neq z$. As we did for the $B 1$ case we will consider all possibilities for $t_{n}$.

Case 1. If $t_{n} \in\left\{r_{1}, r_{m}\right\}$, then $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right)$ appears in the length $n-1$ interval $\left[v, w t_{n}\right]$. This interval is of type $A 1$ and the $T$-word $z^{\prime}=\left(r_{i}, a^{n-2}\right)$ appears in it. Since $x^{\prime}$ appears in a type $A 1$
interval it must be that $x^{\prime} \neq a^{n-1}$ and so we can apply the induction hypothesis to find $u_{x}=u_{x^{\prime}} u_{t_{n}} \equiv$ $u_{z^{\prime}} u_{t_{n}}=u_{z}(\bmod J)$. We again note that the last entry of $z^{\prime}$ is $r_{1}\left(\right.$ resp. $\left.r_{m}\right)$ if $t_{n}=r_{m}\left(\right.$ resp. $\left.t_{n}=r_{1}\right)$ so we do not have $u_{z^{\prime}} u_{t_{n}}=\cdots u_{t_{n}} u_{t_{n}} \equiv 0(\bmod J)$.

Case 2. If $t_{n} \notin\left\{r_{1}, r_{m}\right\}$, then $t_{n}=r_{j}$ for some $j \notin\{1, m\}$. We again wish to perform induction using the interval $\left[\nu, w t_{n}\right]$, which in this case is equal to $\left[v, w r_{j}\right]$. Unlike our earlier cases, here we must also consider the possible choices for $t_{n-1}$. In particular, we need to deal with the situations $t_{n-1} \in\left\{r_{1}, r_{m}\right\}$ and $t_{n-1} \notin\left\{r_{1}, r_{m}\right\}$ separately.

Case 2.1. First suppose that $t_{n-1} \in\left\{r_{1}, r_{m}\right\}$. Note that if $t_{n-1}=r_{1}$, then $t_{n}=r_{j} \neq r_{2}$ and if $t_{n-1}=r_{m}$, then $t_{n}=r_{j} \neq r_{m-1}$. We have

$$
\begin{align*}
u_{x}=\cdots u_{t_{n-1}} u_{t_{n}} & =\cdots u_{t_{n-1}} u_{r_{j}} \\
& \equiv \cdots u_{r_{j^{\prime}}} u_{t_{n-1}}  \tag{4.9}\\
& =u_{x^{\prime}}
\end{align*}
$$

where $j+j^{\prime}=m$ and $j, j^{\prime} \notin\{1, m-1, m\}$ or $j+j^{\prime}=m+2$ and $j, j^{\prime} \notin\{1,2, m\}$. We also have $x^{\prime}=$ $\left(t_{1}, \ldots, t_{n-2}, r_{j^{\prime}}, t_{n-1}\right)$. The $T$-word $x^{\prime \prime}=\left(t_{1}, \ldots, t_{n-2}, r_{j^{\prime}}\right)$ appears in the type $A 1$ interval $\left[v, w t_{n-1}\right]$. This interval is of length $n-1$ and contains the $T$-word $z^{\prime}=\left(r_{i}, a^{n-2}\right)$. Once again we note that $x^{\prime \prime} \neq a^{n-1}$ since it appears in a type $A 1$ interval and so we can apply the inductive hypothesis to find $u_{x^{\prime \prime}} \equiv u_{z^{\prime}}(\bmod J)$. From this we have $u_{x^{\prime}}=u_{x^{\prime \prime}} u_{t_{n-1}} \equiv u_{z^{\prime}} u_{t_{n-1}}=u_{z}(\bmod J)$. We again note that the last entry of $z^{\prime}$ is $r_{1}$ (resp. $r_{m}$ ) if $t_{n-1}=r_{m}$ (resp. $t_{n-1}=r_{1}$ ) and so we don't have the result $u_{z^{\prime}} u_{t_{n-1}}=\cdots u_{t_{n-1}} u_{t_{n-1}} \equiv 0(\bmod J)$ after we apply the inductive step.

Case 2.2. Finally, suppose that $t_{n-1} \notin\left\{r_{1}, r_{m}\right\}$ and so $t_{n-1}=r_{j^{\prime}}$ for some $j^{\prime} \notin\{1, m\}$. The interval $\left[\nu, w t_{n}\right]$ is of type $B 1$ and length $n-1$ and it contains the $T$-words $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right)$ and $z^{\prime}=\left(r_{i}, a^{n-3}, r_{j^{\prime}}\right)$. Since this interval is of length at least 3 we know that it contains a $T$-word $x^{\prime \prime}$ such that $x^{\prime \prime} \neq a^{n-1}$ and such that the last entry of $x^{\prime \prime}$ is in the set $\left\{r_{1}, r_{m}\right\}$. By induction we then have $u_{x}=u_{x^{\prime}} u_{t_{n}} \equiv u_{z^{\prime}} u_{t_{n}} \equiv u_{x^{\prime \prime}} u_{t_{n}}(\bmod J)$. Note that the $T$-word $\left(x^{\prime \prime}, t_{n}\right)$ is contained in $\mathfrak{I}$, its last entry is not contained in the set $\left\{r_{1}, r_{m}\right\}$, and its second to last entry is contained in the set $\left\{r_{1}, r_{m}\right\}$. These are exactly the conditions needed for Case 2.1 and so we can reduce to that case. This gives $u_{x^{\prime \prime}} u_{t_{n}} \equiv u_{z}(\bmod J)$.

Type C1 Induction. Suppose that $\mathfrak{I}$ is of type $C 1$. We then have $z=\left(r_{i}, a^{n-1}\right)$ for some $i \notin\{1,2,3, m\}$. We are done if $x=z$ so suppose that $x \neq z$. We now consider two cases based on the choice of $t_{1}$. Note that here we are interested in the possible choices for $t_{1}$ whereas for $A 1$ and $B 1$ we were interested in the possible choices for $t_{n}$.

Case 1. First suppose that $t_{1}=r_{1}$. Note that $x^{\prime}=\left(t_{2}, \ldots, t_{n}\right)$ is a $T$-word appearing in the interval [ $v t_{1}, w$ ] and that this interval is of type $C 2$ with length $n-1$. This interval also contains the $T$-word
$z^{\prime}=\left(r_{m+2-i}, b^{n-2}\right)$. By induction we have $u_{x}=u_{t_{1}} u_{x^{\prime}} \equiv u_{t_{1}} u_{z^{\prime}}(\bmod J)$. Finally, we have

$$
\begin{align*}
u_{t_{1}} u_{z^{\prime}} & =u_{t_{1}} u_{r_{m+2-i}} u_{b^{n-2}} \\
& \equiv u_{r_{i}} u_{t_{1}} u_{b^{n-2}}  \tag{4.9}\\
& =u_{z}
\end{align*}
$$

Case 2. Now suppose that $t_{1}=r_{i}$. We see that $x^{\prime}=\left(t_{2}, \ldots, t_{n}\right)$ appears in the interval $\left[v t_{i}, w\right]$ which is of type $C 1$ and length $n-1$. This interval also contains the $T$-word $z^{\prime}=\left(r_{i-1}, a^{n-2}\right)$. Note that $x^{\prime} \neq$ $a^{n-1}$ since otherwise we would have $x=\left(r_{i}, x^{\prime}\right)=\left(r_{i}, a^{n-1}\right)=z$ which contradicts our assumption $x \neq z$. Since $x^{\prime} \neq a^{n-1}$ we can use the inductive hypothesis to show that $u_{x}=u_{r_{i}} u_{x^{\prime}} \equiv u_{r_{i}} u_{z^{\prime}}$ $(\bmod J)$. From here we have

$$
\begin{align*}
u_{r_{i}} u_{z^{\prime}} & =u_{r_{i}} u_{r_{i-1}} u_{r_{1}} u_{b^{n-3}} \\
& \equiv u_{r_{i}} u_{r_{1}} u_{r_{m-i+3}} u_{b^{n-3}}  \tag{4.9}\\
& \equiv u_{r_{1}} u_{r_{m-i+2}} u_{r_{m-i+3}} u_{b^{n-3}}  \tag{4.9}\\
& =u_{x^{\prime \prime}}
\end{align*}
$$

where $x^{\prime \prime}=\left(r_{1}, r_{m-i+2}, r_{m-i+3}, b^{n-3}\right)$. Note that $x^{\prime \prime}$ appears in $\mathfrak{I}$ and it begins with $r_{1}$. These are the conditions needed for Case 1 and so we have $u_{x^{\prime \prime}} \equiv u_{z}(\bmod J)$.

Type E Intervals. All that remains is to show that the statement holds for type $E$ intervals. Here we take $z=\left(r_{1}, \ldots, r_{m}\right)$ and we let $x=\left(t_{1}, \ldots, t_{m}\right)$. If $x=z$, then we are done so let $x \neq z$. There are 2 cases to consider depending on our choice of $x$.

Case 1. Assume that $x \neq a^{m}, b^{m}$. First suppose that $t_{m}=r_{m}$. Then $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right)$ is contained in the interval $\left[e, w_{0} r_{m}\right.$ ] which is of type $D 1$. This type $D 1$ interval contains the $T$-word $z^{\prime}=\left(r_{1}, \ldots, r_{m-1}\right)$. By our previous work we then have

$$
u_{x}=u_{x^{\prime}} u_{t_{m}}=u_{x^{\prime}} u_{r_{m}} \equiv u_{z^{\prime}} u_{r_{m}}=u_{z}(\bmod J)
$$

Now suppose that $t_{m}=r_{1}$. Then $x^{\prime}=\left(t_{1}, \ldots, t_{n-1}\right)$ appears in the interval $\left[e, w_{0} r_{1}\right]$. This interval is of type $D 2$ and it contains the $T$-word $z^{\prime}=\left(r_{1}, \ldots, r_{m-2}, r_{m}\right)$. By our earlier work with type $D 2$ intervals we have $u_{x}=u_{x^{\prime}} u_{r_{1}} \equiv u_{z^{\prime}} u_{r_{1}}(\bmod J)$. Now note that $x^{\prime \prime}=\left(r_{m-2}, r_{m}, r_{1}\right)$ appears in the interval $\left[w_{0} r_{1} r_{m} r_{m-2}, w_{0}\right.$ ] which is of type $C 2$ and which contains the $T$-word $z^{\prime \prime}=\left(r_{m-2}, r_{m-1}, r_{m}\right)$. Using what we know about $C 2$ intervals we have

$$
\begin{aligned}
u_{z^{\prime}} u_{r_{1}} & =u_{r_{1}} \cdots u_{r_{m-2}} u_{r_{m}} u_{r_{1}} \\
& =u_{r_{1}} \cdots u_{r_{m-3}} u_{x^{\prime \prime}} \\
& \equiv u_{r_{1}} \cdots u_{r_{m-3}} u_{z^{\prime \prime}} \\
& =u_{z}
\end{aligned}
$$

modulo $J$.
Case 2. Now we consider the case where $x \in\left\{a^{m}, b^{m}\right\}$. If $x=b^{m}$, then we have

$$
\begin{align*}
u_{x} & =u_{r_{m}} u_{r_{1}} u_{b^{m-2}} \\
& \equiv\left(u_{r_{1}} u_{r_{2}}+\cdots+u_{r_{m-1}} u_{r_{m}}\right) u_{b^{m-2}}  \tag{4.7}\\
& =u_{r_{1}} u_{r_{2}} u_{b^{m-2}}+\cdots+u_{r_{m-1}} u_{r_{m}} u_{b^{m-2}} .
\end{align*}
$$

Note that for $i \in[m]-\{1\}$ the $T$-word $\left(r_{i}, r_{i+1}, b^{m-i-1}\right)$ appears in the type $C 2$ interval $\left[v, w_{0}\right]$ where $v=r_{1} \cdots r_{i-1}$. This interval also contains the $T$-word ( $r_{i}, r_{i+1}, \ldots, r_{m-1}, r_{m}$ ). Using our previous work for type $C 2$ intervals we have

$$
\begin{align*}
u_{r_{i}} u_{r_{i+1}} u_{b^{m-2}} & =u_{r_{i}} u_{r_{i+1}} u_{b^{m-i-1}} u_{b^{i-1}} \\
& \equiv u_{r_{i}} u_{r_{i+1}} \cdots u_{r_{m-1}} u_{r_{m}} u_{b^{i-1}}  \tag{C2intervals}\\
& \equiv 0 \tag{4.16}
\end{align*}
$$

From this we see that

$$
u_{r_{1}} u_{r_{2}} u_{b^{m-2}}+\cdots+u_{r_{m-1}} u_{r_{m}} u_{b^{m-2}} \equiv u_{r_{1}} u_{r_{2}} u_{b^{m-2}}=u_{x^{\prime}}
$$

where $x^{\prime}=\left(r_{1}, r_{2}, b^{m-2}\right)$. Since $x^{\prime} \neq a^{m}, b^{m}$ we know by Case 1 that $u_{x^{\prime}} \equiv u_{z}(\bmod J)$.
If $x=a^{m}$, then we have

$$
\begin{align*}
u_{x} & =u_{r_{1}} u_{r_{m}} u_{a^{m-2}} \\
& \equiv\left(u_{r_{2}} u_{r_{1}}+\cdots+u_{r_{m}} u_{r_{m-1}}\right) u_{a^{m-2}}  \tag{4.8}\\
& =u_{r_{2}} u_{r_{1}} u_{a^{m-2}}+\cdots+u_{r_{m}} u_{r_{m-1}} u_{a^{m-2}}
\end{align*}
$$

Now note that for $i \in[m]-\{m\}$ the $T$-word $\left(r_{i}, r_{i-1}, a^{i-1}\right)$ appears in the interval $\left[\nu, w_{0}\right]$ where $v=r_{m} \cdots r_{i+1}$. This interval is of type $C 1$ and it also contains the $T$-word $\left(r_{i}, r_{i-1}, \ldots, r_{2}, r_{1}\right)$. Given what we know about type $C 1$ intervals we have

$$
\begin{align*}
u_{r_{i}} u_{r_{i-1}} u_{a^{m-2}} & =u_{r_{i}} u_{r_{i-1}} u_{a^{i-1}} u_{a^{m-i-1}} \\
& \equiv u_{r_{i}} u_{r_{i-1}} \cdots u_{r_{2}} u_{r_{1}} u_{a^{m-i-1}}  \tag{C1intervals}\\
& \equiv 0 \tag{4.15}
\end{align*}
$$

Using this we see that

$$
u_{r_{2}} u_{r_{1}} u_{a^{m-2}}+\cdots+u_{r_{m}} u_{r_{m-1}} u_{a^{m-2}} \equiv u_{r_{m}} u_{r_{m-1}} u_{a^{m-2}}=u_{x^{\prime}}
$$

where $x^{\prime}=\left(r_{m}, r_{m-1}, a^{m-2}\right)$. Again we see that since $x^{\prime} \neq a^{m}, b^{m}$ we have by Case 1 that $u_{x^{\prime}} \equiv u_{z}$ $(\bmod J)$.

Example 4.2.7. Consider the interval $\mathfrak{I}=[\nu, w]$ in $I_{2}(7)$ where $\nu=r_{7}$ and $w=r_{1} r_{7} r_{1} r_{7} r_{1} r_{7}$. Note that $x=\left(r_{1}, r_{7}, r_{4}, r_{3}, r_{2}\right)$ is contained in this interval. Since the last two entries of $x$ are not in $\left\{r_{1}, r_{7}\right\}$ this example falls under Case 2.2 of the previous proposition. Following our proof of Proposition 4.2.6 the special $T$-word appearing in $\mathfrak{I}$ is $z=\left(r_{6}, r_{1}, r_{7}, r_{1}, r_{7}\right)$. We will show that $u_{x} \equiv u_{y}(\bmod J)$ using the process outlined in Case 2.2 and we will use Figure 4.7 to help us visualize what is happening. Any image referred to in this example will come from said figure.

The northwest image shows the Hasse diagram of our interval along with the label sequence corresponding to $x$ highlighted in red. The interval [ $v, w r_{2}$ ] is of type $B 1$ and it contains the $T$-words $x^{\prime}=\left(r_{1}, r_{7}, r_{4}, r_{3}\right), z^{\prime}=\left(r_{6}, r_{1}, r_{7}, r_{3}\right)$, and $x^{\prime \prime}=\left(r_{6}, r_{5}, r_{1}, r_{7}\right)$. By induction we have $u_{x^{\prime}} \equiv u_{z^{\prime}} \equiv u_{x^{\prime \prime}}$ $(\bmod J)$. This then gives us

$$
u_{x}=u_{x^{\prime}} u_{r_{2}} \equiv u_{z^{\prime}} u_{r_{2}} \equiv u_{x^{\prime \prime}} u_{r_{2}}
$$

The northeast image shows the label sequence corresponding to $\left(z^{\prime}, r_{2}\right)$ highlighted in red and the southwest image shows the label sequence corresponding to ( $x^{\prime \prime}, r_{2}$ ) highlighted in red. Finally, note that our current situation falls under the scope of Case 2.1 since the last entry of $\left(x^{\prime \prime}, r_{2}\right)=$ $\left(r_{6}, r_{5}, r_{1}, r_{7}, r_{2}\right)$ is not in $\left\{r_{1}, r_{7}\right\}$ while its second-to-last entry is in $\left\{r_{1}, r_{7}\right\}$. We then have $u_{x} \equiv u_{z}$ $(\bmod J)$ by Case 2.1. The southeast image shows the label sequence corresponding to $z$ highlighted in red.

Corollary 4.2.8. Let $x$ and $y$ be $T$-words such that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. Then $u_{x} \equiv u_{y}(\bmod J)$.
Proof. Suppose that $u_{x} \equiv u_{y} \not \equiv 0(\bmod I)$. We know by Proposition 4.2 .3 that $x$ and $y$ both appear in some interval $[v, w] \in \mathfrak{B}\left(I_{2}(m)\right)$ and $x, y \notin\left\{a^{\delta}, b^{\delta}\right\}$ for $\delta<m$. By Proposition 4.2.6 we know that $u_{x} \equiv u_{y}(\bmod J)$.

Now that we have shown that if $u_{x} \equiv u_{y}(\bmod I)$, then $u_{x} \equiv u_{y}(\bmod I)$ our next goal is to show that if $u_{x} \equiv 0(\bmod I)$, then $u_{x} \equiv 0(\bmod J)$. Our proof of this is lengthy, but straightforward in that it simply considers all possibilities for $x$ and then applies relations (4.7)-(4.18) to show that $u_{x} \equiv 0$ $(\bmod J)$.

Proposition 4.2.9. Let $x$ be a $T$-word such that $u_{x} \equiv 0(\bmod I)$. Then $u_{x} \equiv 0(\bmod J)$.
Proof. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ and $x^{\prime}=\left(t_{1}, \ldots, t_{\ell-1}\right)$. It is sufficient to consider the case where $u_{x} \equiv 0$ $(\bmod I)$ and $u_{x^{\prime}} \not \equiv 0(\bmod I)$. By Corollary 1.1.4 we know that $x^{\prime}$ appears in some interval of length $\ell-1$ and $x$ does not appear in any interval. Since $x^{\prime}$ appears in some interval of $\mathfrak{B}\left(I_{2}(m)\right)$ it must be that $1 \leq \ell-1 \leq m$ and so $2 \leq \ell \leq m+1$. We now proceed by considering the different values of $\ell$.

Suppose first that $\ell=2$. We can see from the description of $\mathfrak{B}\left(I_{2}\right)$ that the only $T$-words of length 2 which do not appear in some interval are $z_{1}=\left(r_{i}, r_{i}\right)$ for $i \in[m]$ and $z_{2}=\left(r_{i}, r_{j}\right)$ for $i, j \notin\{1, m\}$ and $|i-j| \geq 2$. If $x=z_{1}$, then $u_{x} \equiv 0$ modulo relation (4.11) and if $x=z_{2}$, then $u_{x} \equiv 0$ modulo relations (4.12). Next, suppose that $\ell=m+1$ and let $z=\left(r_{1}, r_{2}, \ldots, r_{m}\right)$. Note that $x^{\prime}$ and $z$ both appear in


Figure 4.7 Visualization of $u_{x} \equiv u_{y}(\bmod J)$ for $x=\left(r_{1}, r_{7}, r_{4}, r_{3}, r_{2}\right)$ and $y=\left(r_{6}, r_{1}, r_{7}, r_{1}, r_{7}\right)$ in $\mathfrak{B}\left(I_{2}(7)\right)$.
[ $e, w_{0}$ ]. Using this we have

$$
\begin{align*}
u_{x} & =u_{x^{\prime}} u_{t_{n}} \\
& \equiv u_{z} u_{t_{n}}  \tag{Proposition4.2.6}\\
& =\cdots u_{r_{m-1}} u_{r_{m}} u_{t_{n}} \\
& \equiv 0 . \tag{4.16}
\end{align*}
$$

Finally, suppose that $3 \leq \ell \leq m$. We consider different cases depending on the possible choices for $t_{\ell-1}$.

Case 1. First, consider the case where $t_{\ell-1}=r_{i}$ for $i \notin\{1, m\}$. If $t_{\ell}=r_{j}$ for $j \notin\{1, m, i+1, i-1\}$, then $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=\cdots u_{r_{i}} u_{r_{j}} \equiv 0$ modulo relation (4.12). There are now two situations to consider, either $t_{\ell} \in\left\{r_{1}, r_{i-1}\right\}$ or $t_{\ell} \in\left\{r_{m}, r_{i+1}\right\}$. Since the proofs for these two situations are similar we will only explicitly address the first one. Given that $t_{\ell-1}=r_{i} \notin\left\{r_{1}, r_{m}\right\}$ there are only two possibilities for $t_{\ell-2}$, either $t_{\ell-2} \in\left\{r_{1}, r_{i-1}\right\}$ or $t_{\ell-2} \in\left\{r_{m}, r_{i+1}\right\}$. Any other choice of $t_{\ell-2}$ would contradict our assumption that $x^{\prime}$ appears in some interval. In particular, if $t_{\ell} \in\left\{r_{1}, r_{i-1}\right\}$, it must be that $t_{\ell-2} \in\left\{r_{1}, r_{i-1}\right\}$; otherwise $x$ would appear in some interval of $\mathfrak{B}\left(I_{2}\right)$. Since $t_{\ell-2} \in\left\{r_{1}, r_{i-1}\right\}$ and $t_{\ell} \in\left\{r_{1}, r_{i-1}\right\}$ there are now 4 cases to consider depending on the choice of $t_{\ell-2}$ and $t_{\ell}$.

Case 1.1. Suppose that $t_{\ell-2}=r_{1}, t_{\ell}=r_{1}$. If $i=2$, then we have $u_{x}=\ldots u_{r_{1}} u_{r_{2}} u_{r_{1}} \equiv 0$ modulo relation (4.18). If $i \neq 2$, then we have

$$
\begin{align*}
u_{x} & =\cdots u_{r_{1}} u_{r_{i}} u_{r_{1}} \\
& \equiv \cdots u_{r_{m-i+2}} u_{r_{1}} u_{r_{1}}  \tag{4.9}\\
& \equiv 0 . \tag{4.11}
\end{align*}
$$

Case 1.2. Suppose that $t_{\ell-2}=r_{1}, t_{\ell}=r_{i-1}$. Since $u_{x}=\cdots u_{r_{1}} u_{r_{i}} u_{r_{i-1}}$ it is sufficient to show that $u_{r_{1}} u_{r_{i}} u_{r_{i-1}} \equiv 0(\bmod J)$. If $i=2$, then $u_{r_{1}} u_{r_{2}} u_{r_{1}} \equiv 0$ modulo relation (4.18). If $3 \leq i \leq m-1$, then $u_{r_{1}} u_{r_{i}} u_{r_{i-1}} \equiv 0$ modulo relation (4.13).

Case 1.3. Now suppose that $t_{\ell-2}=r_{i-1}, t_{\ell}=r_{1}$. Again, since $u_{x}=\cdots u_{r_{i-1}} u_{r_{i}} u_{r_{1}}$ it suffices to show that $u_{r_{i-1}} u_{r_{i}} u_{r_{1}} \equiv 0(\bmod J)$. If $i=2$, then $u_{r_{1}} u_{r_{2}} u_{r_{1}} \equiv 0$ modulo relation (4.18). If $i=3$, then $u_{r_{2}} u_{r_{3}} u_{r_{1}} \equiv$ $u_{r_{2}} u_{r_{1}} u_{r_{m-1}} \equiv 0$ where the first equivalence is modulo relation (4.9) and the second equivalence is modulo relation (4.15). If $i \neq 2,3$, then modulo relation (4.9) we have $u_{r_{i-1}} u_{r_{i}} u_{r_{1}} \equiv u_{r_{i-1}} u_{r_{1}} u_{r_{m-i+2}} \equiv$ $u_{r_{1}} u_{r_{m-i+3}} u_{r_{m-i+2}}$ and we can now apply case 1.2 to see that $u_{r_{1}} u_{r_{m-i+3}} u_{r_{m-i+2}} \equiv 0(\bmod J)$.
Case 1.4. Finally, suppose that $t_{\ell-2}=r_{i-1}, t_{\ell}=r_{i-1}$. We have $u_{x}=\cdots u_{r_{i-1}} u_{r_{i}} u_{r_{i-1}} \equiv 0$ modulo relation (4.17).

Case 2. Now consider the case where $t_{\ell-1}=r_{i}$ for $i \in\{1, m\}$. We will explicitly address the case where $i=1$; the case where $i=m$ is similar. Once again we have several cases to address.

Case 2.1. Suppose that $x^{\prime}=\left(t_{1}, \ldots, t_{\ell-1}\right) \neq \bar{a}^{\ell-1}$ and $\ell=3$. We consider different cases based on the choice of $t_{\ell}$. If $t_{\ell}=r_{1}$, then $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=u_{t_{\ell-2}} u_{r_{1}} u_{r_{1}} \equiv 0$ modulo relation (4.11). If $t_{\ell}=r_{m}$, then the only way $x$ does not appear in any interval is if $t_{\ell-2}=r_{2}$. From this we have $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=$ $u_{r_{2}} u_{r_{1}} u_{r_{m}} \equiv 0$ modulo relation (4.15). If $t_{\ell}=r_{2}$, then $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=u_{t_{\ell-2}} u_{r_{1}} u_{r_{2}} \equiv 0$ modulo (4.19). Finally, if $t_{\ell}=r_{j} \notin\left\{r_{1}, r_{2}, r_{m}\right\}$, then $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=u_{t_{\ell-2}} u_{r_{1}} u_{r_{j}} \equiv u_{t_{\ell-2}} u_{r_{m-j+2}} u_{r_{1}}$ modulo relations (4.9). If $\left(t_{\ell-2}, r_{m-j+2}\right)$ does not appear in any interval, then we can use our earlier work for $\ell=2$ to show that $u_{t_{\ell-2}} u_{r_{m-j+2}} \equiv 0(\bmod J)$ and so $u_{t_{\ell-2}} u_{r_{m-j+2}} u_{r_{1}} \equiv 0(\bmod J)$. If $\left(t_{\ell-2}, r_{m-j+2}\right)$ does appear in some interval, then we can use Case 1 to show $u_{t_{\ell-2}} u_{r_{m-j+2}} u_{r_{1}} \equiv 0(\bmod J)$.

Case 2.2. Suppose that $x^{\prime}=\left(t_{1}, \ldots, t_{\ell-1}\right) \neq \bar{a}^{\ell-1}$ and $3<\ell \leq m$. There are two cases to consider, either $x^{\prime}$ appears in an interval of type $C 1$ (or $C 2$ ) or it does not. First, suppose that $x^{\prime}$ appears in an interval of type $C 1$ (or $C 2$ ). Any type $C 1$ (or $C 2$ ) interval with $\ell>3$ contains a $T$-word of the form $z=\left(\ldots, r_{m-1}, r_{m}\right)$. By Proposition (4.2.6) we know that $u_{x^{\prime}} \equiv u_{z}(\bmod J)$ and so we have

$$
u_{x}=u_{x^{\prime}} u_{t_{\ell}} \equiv u_{z} u_{t_{\ell}}=\cdots u_{r_{m-1}} u_{r_{m}} u_{t_{\ell}} \equiv 0
$$

where the last equivalence is modulo relation (4.16). Now suppose that $x^{\prime}$ does not appear in an interval of type $C 1$ (or $C 2$ ). An interval of any other type which contains $x$ and has $\ell>3$ must also contain a $T$-word $z=\left(t_{1}^{\prime}, \ldots, t_{\ell-1}^{\prime}\right)$ such that $t_{\ell-1}^{\prime} \notin\left\{r_{1}, r_{m}\right\}$. By Proposition 4.2 .6 we know that $u_{x^{\prime}} \equiv u_{z}(\bmod J)$. In this case we have $u_{x}=u_{x^{\prime}} u_{t_{\ell}} \equiv u_{z} u_{t_{\ell}}(\bmod J)$ and we can now reduce to Case 1.

Case 2.3. Finally, suppose $x^{\prime}=\left(t_{1}, \ldots, t_{\ell-1}\right)=\bar{a}^{\ell-1}$. We again consider different cases based on our choice of $t_{\ell}$. Note that we cannot have $t_{\ell}=r_{m}$ since then $x=\bar{b}^{\ell}$ which appears in some interval of $\mathfrak{B}\left(I_{2}(m)\right)$. Suppose then that $t_{\ell}=r_{1}$. We have $u_{x}=u_{x^{\prime}} u_{t_{\ell}}=u_{\bar{a}^{\ell-1}} u_{t_{\ell}}=\cdots u_{r_{1}} u_{r_{1}} \equiv 0$ modulo relation (4.11). Finally, suppose that $t_{\ell}=r_{i}$ for $i \notin\{1, m\}$. Note that the $T$-word $\left(\bar{a}^{n}, r_{i}\right)$ for $1 \leq n \leq i-1$ appears in some interval of $\mathfrak{B}\left(I_{2}(m)\right)$. Since $x$ cannot appear in any interval it must be that $\ell-1>i-1$. As such, we have

$$
x=\left(x^{\prime}, r_{i}\right)=\left(\bar{a}^{\ell-1}, r_{i}\right)=\left(\bar{a}^{\ell-i}, \bar{a}^{i-1}, r_{i}\right)=\left(\bar{a}^{\ell-i}, x^{\prime \prime}\right)
$$

where $x^{\prime \prime}=\left(\bar{a}^{i-1}, r_{i}\right)$. Note that $x^{\prime \prime}$ appears in a type $D 1$ interval which also contains a $T$-word of the form $z=\left(r_{1}, r_{2}, \ldots\right)$. We then have

$$
\begin{align*}
u_{x} & =u_{\bar{a}^{\ell-i}} u_{x^{\prime \prime}} \\
& \equiv u_{\bar{a}^{\ell-i}} u_{z}  \tag{Proposition4.2.6}\\
& =u_{\bar{a}^{\ell-i}} u_{r_{1}} u_{r_{2}} \cdots \\
& \equiv 0 \tag{4.19}
\end{align*}
$$

Using the Propositions 4.2 .6 and 4.2 .9 we can finally prove our second desired inclusion $I \subset J$.
Proposition 4.2.10. We have $I \subset J$.

Proof. Let $R$ be an element of $I$. Our goal is to show that $R \equiv 0(\bmod J)$ and our first step toward this goal is to change $R$ so that it does not contain either of the terms $u_{a^{\delta}}, u_{b^{\delta}}$ for $\delta>1$. Let $x$ be a $T$-word such that the monomial $u_{x}$ appears in $R$. Suppose that $x \in\left\{a^{\delta}, b^{\delta}\right\}$ for some $\delta>1$. Then we have

$$
\begin{align*}
u_{b^{\delta}} & =u_{r_{m}} u_{r_{1}} u_{b^{\delta-2}} \\
& \equiv\left(u_{1} u_{2}+\cdots+u_{m-1} u_{m}\right) u_{b^{\delta-2}}  \tag{4.7}\\
& =u_{1} u_{2} u_{b^{\delta-2}}+\cdots+u_{m-1} u_{m} u_{b^{\delta-2}}
\end{align*}
$$

or

$$
\begin{align*}
u_{a^{\delta}} & =u_{r_{1}} u_{r_{m}} u_{a \delta-2} \\
& \equiv\left(u_{2} u_{1}+\cdots+u_{m} u_{m-1}\right) u_{a^{\delta-2}}  \tag{4.8}\\
& =u_{2} u_{1} u_{a^{\delta-2}}+\cdots+u_{m} u_{m-1} u_{a^{\delta-2}} .
\end{align*}
$$

In either case, $u_{x}$ is equivalent modulo relation (4.7) or (4.8) to a sum of monomials whose corresponding $T$-words are not equal to $a^{\delta}$ or $b^{\delta}$ for any $\delta>1$. As such, there exists an $R^{\prime}$ such that $R \equiv R^{\prime}$ modulo relations (4.7) and (4.8), and

$$
R^{\prime}=\sum_{i=1}^{k_{1}} \alpha_{i} u_{x(i)}
$$

where for all $i \in\left[k_{1}\right]$ we have that $\alpha_{i} \in \mathbf{C}$ and $x(i)$ is a $T$-word not equal to $a^{\delta}$ or $b^{\delta}$ for any $\delta>1$.
Now let $J^{\prime} \subset J$ be the left-right ideal generated by all $u_{x}-u_{y}$ such that $u_{x} \equiv u_{y}(\bmod J)$ and all $u_{x}$ such that $u_{x} \equiv 0(\bmod J)$. We now wish to use $J^{\prime}$ to take any terms of $R^{\prime}$ which are equivalent modulo $J$ and 'combine' them into one term. In other words, there exists an $R^{\prime \prime}$ such that $R^{\prime} \equiv R^{\prime \prime}$ $\left(\bmod J^{\prime}\right)$ and

$$
R^{\prime \prime}=\sum_{i=1}^{k_{2}} \beta_{i} u_{y(i)}
$$

where for all $i \in\left[k_{2}\right]$ we have $\beta_{i} \in \mathbf{C}, y(i)$ is a $T$-word, $y(i) \neq a^{\delta}, b^{\delta}$ for any $\delta>1, u_{y(i)} \not \equiv 0(\bmod I)$, and $u_{y(i)} \not \equiv u_{y(j)}(\bmod I)$ for all $i \neq j \in\left[k_{2}\right]$.

For each $u_{y(i)}\left(i \in\left[k_{2}\right]\right)$ let $w_{i} \in I_{2}(m)$ be such that $(w) u_{y(i)} \neq 0$. By Proposition 4.2 .5 we know that $J \subset I$ and so we must have $R^{\prime \prime} \equiv 0(\bmod I)$, which implies $\left(w_{i}\right) R^{\prime \prime}=0$ for all $i \in\left[k_{2}\right]$. The nonzero expressions of the form $\left(w_{i}\right) u_{y(j)}$ in $\left(w_{i}\right) R^{\prime \prime}$ are distinct and so we can only have $\left(w_{i}\right) R^{\prime \prime}=0$ for all $i \in\left[k_{2}\right]$ if $\beta_{i}=0$ for all $i \in\left[k_{2}\right]$.

We can now give the proof for Theorem 4.2.1.
Proof. It suffices to show that $I=J$. We have $J \subset I$ by Proposition 4.2.5 and we have $I \subset J$ by Proposition 4.2.10.

### 4.3 Up-operators for Bruhat Order on the Symmetric Group

Let $\mathscr{U}^{\mathfrak{B}\left(\mathfrak{G}_{n}\right)}$ be the associative algebra over $\mathbf{C}$ generated by the $u_{(i, j)}$ where $(i, j)$ is a reflection in $\mathfrak{S}_{n}$. Furthermore, let $I^{\mathfrak{B}\left(\mathfrak{S}_{n}\right)}$ be the two-sided ideal containing all elements which annihilate all elements of $\mathbf{C}\left[\mathfrak{B}\left(\mathfrak{S}_{n}\right)\right]$. We call $\mathscr{U}^{\mathfrak{B}\left(\mathfrak{S}_{n}\right)} / I^{\mathfrak{B}\left(\mathfrak{S}_{n}\right)}$ the algebra of up-operators for $\mathfrak{B}\left(\mathfrak{S}_{n}\right)$. For ease of notation we will use $\mathscr{U}$ and $I$ in place of $\mathscr{U}^{\mathfrak{B}\left(\mathfrak{S}_{n}\right)}$ and $I^{\mathfrak{B}\left(\mathfrak{S}_{n}\right)}$, respectively.

As mentioned earlier, while a complete list of relations for $\mathscr{U} / I$ remains unknown we have found several new relations which hold in the algebra. Before giving these new relations we first discuss the computational process we used to find them. The following definition will be needed for a description of this process.

Definition 4.3.1. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ be a $T$-word and let $\pi=t_{1} \cdots t_{\ell} \in \mathfrak{S}_{n}$. We call $\pi$ the $\mathfrak{S}_{n}$-degree of $u_{x}$.

This definition can now be used to state an important fact about the $u_{(i, j)}$ given in the lemma below. In the following, by 'degree' we mean the usual sense of degree, that is, the degree of $u_{x}$ is equal to the number of factors appearing in $u_{x}$. For instance, if $u_{x}=u_{(1,2)} u_{(2,3)} u_{(3,4)}$, then the degree of $u_{x}$ is 3 . Note that the degree of $u_{x}$ is always equal to the length of $x$.

Lemma 4.3.2. Let $x$ and $y$ be $T$-words and let $\pi \in \mathfrak{S}_{n}$. Suppose that $(\pi) u_{x}=(\pi) u_{y}=\sigma \neq 0$. Then $u_{x}$ and $u_{y}$ have the same degree and the same $\mathfrak{S}_{n}$-degree.

Proof. Let $x=\left(t_{1}, \ldots, t_{\ell}\right)$ and $y=\left(t_{1}^{\prime}, \ldots, t_{\ell^{\prime}}^{\prime}\right)$. Since $(\pi) u_{x}=(\pi) u_{y}=\sigma \neq 0$ we can see from the definition of up-operators and the fact that $\ell_{S}$ is the rank function for Bruhat order that we have $\ell_{S}(\sigma)=\ell_{S}(\pi)+\ell$ and $\ell_{S}(\sigma)=\ell_{S}(\pi)+\ell^{\prime}$. From this we see that $\ell=\ell^{\prime}$. We also have that $\sigma=\pi t_{1} \cdots t_{\ell}$ and $\sigma=\pi t_{1}^{\prime} \cdots t_{\ell}^{\prime}$. From this we see that $t_{1} \cdots t_{\ell}=t_{1}^{\prime} \cdots t_{\ell}^{\prime}$ and so $u_{x}$ and $u_{y}$ have the same $\mathfrak{S}_{n}$-degree.

Let $\Gamma_{m, \pi}$ be the subspace of $\mathscr{U}$ generated by all monomials in $\mathscr{U}$ whose degree is $m$ and whose $\mathfrak{S}_{n}$-degree is $\sigma$. The above lemma allows us to restrict our search for relations to the $\Gamma_{m, \pi}$ for each possible $m$ and $\pi$. To see this, suppose we have the following relation

$$
R=\sum_{i=1}^{k_{1}} c_{i} u_{x(i)} \equiv 0(\bmod I)
$$

where $c_{i} \in \mathbf{C}$ and $x(i)$ is a $T$-word for all $i \in\left[k_{1}\right]$. We may assume that $u_{x(i)} \not \equiv 0(\bmod I)$ since otherwise we can simply remove $c_{i} u_{x(i)}$ from $R$ and what remains would still be equivalent to 0 modulo $I$. Fix some positive integer $m$ and some element $\sigma$ of $\mathfrak{S}_{n}$ and let $\left\{i_{1}, \ldots, i_{k_{2}}\right\}$ be the subset of $\left[k_{1}\right]$ such that $u_{x(i)}$ has degree $m$ and $\mathfrak{S}_{n}$-degree $\sigma$. By Lemma 4.3.2 it must then be that

$$
R_{m, \pi}=\sum_{j=1}^{k_{2}} c_{i_{j}} u_{x\left(i_{j}\right)} \equiv 0(\bmod I) .
$$

From this we see that $R=\sum R_{m, \pi}$ where $R_{m, \pi} \equiv 0(\bmod I)$ and where the sum is over all pairs $(m, \pi)$ occurring (respectively) as the degree and $\mathfrak{S}_{n}$-degree of some term in $R$. Since each $R_{m, \pi} \equiv 0(\bmod I)$
is a relation involving only the elements of $\Gamma_{m, \pi}$ it is sufficient to just look for relations among the elements of $\Gamma_{m, \pi}$ to begin with.

We now know it is sufficient to look for relations among the elements of $\Gamma_{m, \pi}$ for each possible pair ( $m, \pi$ ). Using this, we can now search for relations in the following way.

1. Fix some positive integer $m$ and some $\pi \in \mathfrak{S}_{n}$.
2. Let $\Phi$ be the largest possible set of all known elements of $I$ such that every element of $\Phi$ is contained in $\Gamma_{m, \pi}$ and such that no element of $\Phi$ is linearly dependent on any other elements of $\Phi$. Note that $\Phi$ is not necessarily unique.
3. Let $\Gamma$ be the set of monomials in $\Gamma_{m, \pi}$. Also let $M_{1}$ be the matrix whose columns are indexed by the elements of $\Gamma$ and whose rows are indexed by the elements of $\Phi$. If $\phi \in \Phi$ and $u_{x} \in \Gamma$, then we put a $c(c \in \mathbf{C})$ in position $\left(\phi, u_{x}\right)$ of $M_{1}$ if $c u_{x}$ appears in $\phi$ and a 0 otherwise.
4. Let $M_{1}^{\prime}$ be the reduced echelon form of $M_{1}$ and let $\Gamma^{\prime}$ be the subset of $\Gamma$ such that $u_{x} \in \Gamma^{\prime}$ if and only if column $u_{x}$ of $M_{1}^{\prime}$ does not contain a pivot.
5. Any monomials not in $\Gamma^{\prime}$ can be eliminated from any new relation by using the previously known elements of $I$, and so we need only look for new relations among the elements of $\Gamma^{\prime}$. Let $M_{2}$ be the matrix whose columns are indexed by $\Gamma^{\prime}$ and whose rows are indexed by all elements of $\mathfrak{S}_{n}$. If $\pi \in \mathfrak{S}_{n}$ and $u_{x} \in \Gamma^{\prime}$, then we put a 1 in position $\left(\pi, u_{x}\right)$ of $M_{1}$ if $(\pi) u_{x} \neq 0$ and a 0 otherwise.
6. Let $\operatorname{ker}\left(M_{1}\right)$ be the right kernel of $M_{1}$ and let $B$ be its basis. Let $b \in B$ and let $c_{u_{x}}$ be the entry of $b$ in row $u_{x}$. Then

$$
\sum_{u_{x} \in \Gamma^{\prime}} c_{u_{x}} u_{x} \equiv 0(\bmod I)
$$

is a new relation.
All of the new relations stated in Proposition 4.3.4 were found using the above process. As a way to confirm that each of the new relations does in fact hold we provide a non-computational proof in Proposition 4.3.4. For this proof we will need the following corollary.

Corollary 4.3.3. Let $(i, j) \in T$ for $i<j$ and $\operatorname{let} \pi=\pi_{1} \ldots \pi_{n} \in \mathfrak{S}_{n}$. Then

$$
(\pi) u_{(i, j)}=\pi_{1} \ldots \pi_{i-1} \pi_{j} \pi_{i+1} \ldots \pi_{j-1} \pi_{i} \pi_{j+1} \ldots \pi_{n}
$$

if and only if $\pi_{i}<\pi_{j}$ and there does not exist some $k$ such that $i<k<j$ and $\pi_{i}<\pi_{k}<\pi_{j}$.
Proof. This follows directly from the definition of up-operators and Proposition 4.1.2.
We can now state the list of new relations we found for $\mathscr{U} / I$.

Proposition 4.3.4. Let $i<j<k<l<m$ be integers. The following relations hold modulo $I$.

$$
\begin{align*}
u_{(j l)} u_{(i l)} u_{(i k)} & \equiv 0,  \tag{4.23}\\
u_{(j l)} u_{(j k)} u_{(i l)} u_{(j l)} & \equiv 0,  \tag{4.24}\\
u_{(k l)} u_{(j l)} u_{(i k)} u_{(i j)} & \equiv 0,  \tag{4.25}\\
u_{(k m)} u_{(j l)} u_{(i m)} u_{(l m)} & \equiv u_{(k m)} u_{(l m)} u_{(j m)} u_{(i l)},  \tag{4.26}\\
u_{(j m)} u_{(l m)} u_{(i m)} u_{(k m)} & \equiv u_{(l m)} u_{(j l)} u_{(i m)} u_{(k m)} . \tag{4.27}
\end{align*}
$$

Proof. We will show that (4.26) holds; the proof for the other relations is similar. Let $\pi=\pi_{1} \ldots \pi_{n}$ be some element of $\mathfrak{S}_{n}$ and let $x=((k m),(j l),(i m),(l m))$ and let $y=((k m),(l m),(j m),(i l))$. We see that $x$ and $y$ have the same $\mathfrak{S}_{n}$-degree, namely the cycle $(i, k, m, j, l)$. It must then be that if $(\pi) u_{x},(\pi) u_{y} \neq 0$, then $(\pi) u_{x}=(\pi) u_{y}$. Thus, to prove that relation (4.26) holds we need only prove that $(\pi) u_{x} \neq 0$ if and only if $(\pi) u_{y} \neq 0$. Our strategy for proving this biconditional statement is as follows. First suppose that $(\pi) u_{x} \neq 0$. This assumption along with Corollary 4.3.3 forces certain restrictions on what $\pi$ can be. We will determine what these restrictions are and we will use them to show that $(\pi) u_{y} \neq 0$. This will prove the forward direction; for the reverse direction we will use the same approach but with the roles of $x$ and $y$ switched.

Forward Implication: First suppose that $(\pi) u_{x} \neq 0$. We must then have

$$
\begin{gather*}
(\pi) u_{(k m)} \neq 0,  \tag{4.28}\\
(\pi(k m)) u_{(j l)} \neq 0,  \tag{4.29}\\
(\pi(k m)(j l)) u_{(i m)} \neq 0, \text { and }  \tag{4.30}\\
(\pi(k m)(j l)(i m)) u_{(l m)} \neq 0 . \tag{4.31}
\end{gather*}
$$

Recall that Corollary 4.3.3 gives us certain inequalities among the $\pi_{\iota}$ when we act by an up-operator with a nonzero result. As such, each of (4.28)-(4.31) tells us something about the ordering of the $\pi_{l}$. We now determine what each of these ordering restrictions is. As we address each of (4.28)-(4.31) we will give the two-line notation for the permutation being acted upon which will provide some clarity as to what is being done. We will also list inequalities solely involving $\pi_{i}, \pi_{j}, \pi_{k}, \pi_{l}, \pi_{m}$ separately from those involving $\pi_{l}$ for any other $\iota$. We do this because the total ordering on $\pi_{i}, \pi_{j}, \pi_{k}, \pi_{l}, \pi_{m}$ will be useful for us and listing said inequalities separately will make understanding the total ordering a bit clearer.

We first consider (4.28). The two-line notation for $\pi$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{j} & \ldots & \pi_{k} & \ldots & \pi_{l} & \ldots & \pi_{m} & \ldots
\end{array}\right) .
$$

We have $(\pi) u_{(k m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{k}<\pi_{m}  \tag{4.32}\\
& \pi_{l}<\pi_{k} \text { or } \pi_{l}>\pi_{m}, \text { and }  \tag{4.33}\\
& \pi_{\alpha}<\pi_{k} \text { or } \pi_{\alpha}>\pi_{m} \text { if } k<\alpha<m \text { and } \alpha \neq l . \tag{4.34}
\end{align*}
$$

Now consider (4.29). The two-line notation for $\pi(\mathrm{km})$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{j} & \ldots & \pi_{m} & \ldots & \pi_{l} & \ldots & \pi_{k} & \ldots
\end{array}\right)
$$

We have $(\pi(k m)) u_{(j l)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{j}<\pi_{l}  \tag{4.35}\\
& \pi_{m}<\pi_{j} \text { or } \pi_{m}>\pi_{l}, \text { and }  \tag{4.36}\\
& \pi_{\alpha}<\pi_{j} \text { or } \pi_{\alpha}>\pi_{l} \text { if } j<\alpha<l \text { and } \alpha \neq k \tag{4.37}
\end{align*}
$$

Next consider (4.30). The two-line notation for $\pi(k m)(j l)$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{l} & \ldots & \pi_{m} & \ldots & \pi_{j} & \ldots & \pi_{k} & \ldots
\end{array}\right)
$$

We have $(\pi(k m)(j l)) u_{(i m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{i}<\pi_{k}  \tag{4.38}\\
& \pi_{l}<\pi_{i} \text { or } \pi_{l}>\pi_{k}  \tag{4.39}\\
& \pi_{m}<\pi_{i} \text { or } \pi_{m}>\pi_{k}  \tag{4.40}\\
& \pi_{j}<\pi_{i} \text { or } \pi_{j}>\pi_{k}, \text { and }  \tag{4.41}\\
& \pi_{\alpha}<\pi_{i} \text { or } \pi_{\alpha}>\pi_{k} \text { if } i<\alpha<m \text { and } \alpha \neq j, k, l . \tag{4.42}
\end{align*}
$$

Finally, consider (4.31). The two-line notation for $\pi(k m)(j l)(i m)$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{k} & \ldots & \pi_{l} & \ldots & \pi_{m} & \ldots & \pi_{j} & \ldots & \pi_{i} & \ldots
\end{array}\right)
$$

We have $(\pi(k m)(j l)(i m)) u_{(l m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{j}<\pi_{i} \text { and }  \tag{4.43}\\
& \pi_{\alpha}<\pi_{j} \text { or } \pi_{\alpha}>\pi_{i} \text { if } l<\alpha<m . \tag{4.44}
\end{align*}
$$

Each of (4.32)-(4.44) provides some inequality among the $\pi_{\iota}$. We will now show that given these
inequalities we must have $(\pi) u_{y} \neq 0$. In particular, we will prove that each of the following hold:

$$
\begin{gather*}
(\pi) u_{(k m)} \neq 0,  \tag{4.45}\\
(\pi(k m)) u_{(l m)} \neq 0,  \tag{4.46}\\
(\pi(k m)(l m)) u_{(j m)} \neq 0, \text { and }  \tag{4.47}\\
(\pi(k m)(l m)(j m)) u_{(i l)} \neq 0, \tag{4.48}
\end{gather*}
$$

To do this we will show that the inequalities needed for each of (4.45)-(4.48) to hold are implied by the inequalities in (4.32)-(4.44). Corollary 4.3 .3 will then imply that (4.45)-(4.48) are true. We note here that

$$
\begin{equation*}
\pi_{j}<\pi_{l}<\pi_{i}<\pi_{k}<\pi_{m} \tag{4.49}
\end{equation*}
$$

by (4.32), (4.33), (4.35), (4.36), (4.38), (4.39), and (4.43). We will again give the two-line notation for each of the permutations being acted upon.

First consider (4.45). The two-line notation for $\pi$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{j} & \ldots & \pi_{k} & \ldots & \pi_{l} & \ldots & \pi_{m} & \ldots
\end{array}\right) .
$$

We have $(\pi) u_{(k m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{k}<\pi_{m},  \tag{4.50}\\
& \pi_{l}<\pi_{k} \text { or } \pi_{l}>\pi_{m}, \text { and }  \tag{4.51}\\
& \pi_{\alpha}<\pi_{k} \text { or } \pi_{\alpha}>\pi_{m} \text { if } k<\alpha<m \text { and } \alpha \neq l . \tag{4.52}
\end{align*}
$$

We can see that (4.50) and (4.51) are both implied by (4.49) while (4.52) is implied by (4.34). From this we see that $(\pi) u_{(k m)} \neq 0$. Next consider (4.46). The two-line notation for $\pi(k m)$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{j} & \ldots & \pi_{m} & \ldots & \pi_{l} & \ldots & \pi_{k} & \ldots
\end{array}\right) .
$$

We see that $(\pi(k m)) u_{(l m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{l}<\pi_{k} \text { and }  \tag{4.53}\\
& \pi_{\alpha}<\pi_{l} \text { or } \pi_{\alpha}>\pi_{k} \text { if } l<\alpha<m . \tag{4.54}
\end{align*}
$$

We have that (4.53) is implied by (4.49) while (4.54) is implied by (4.34),(4.42),(4.44), and (4.49). From this we have $(\pi(k m)) u_{(l m)} \neq 0$. Now consider (4.47). The two-line notation for $\pi(\mathrm{km})(\mathrm{lm})$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{j} & \ldots & \pi_{m} & \ldots & \pi_{k} & \ldots & \pi_{l} & \ldots
\end{array}\right) .
$$

We see that $(\pi(k m)(l m)) u_{(j m)} \neq 0$ if and only if

$$
\begin{align*}
& \pi_{j}<\pi_{l},  \tag{4.55}\\
& \pi_{m}<\pi_{j} \text { or } \pi_{m}>\pi_{l},  \tag{4.56}\\
& \pi_{k}<\pi_{j} \text { or } \pi_{k}>\pi_{l}, \text { and }  \tag{4.57}\\
& \pi_{\alpha}<\pi_{j} \text { or } \pi_{\alpha}>\pi_{l} \text { if } j<\alpha<m \text { and } \alpha \neq k, l . \tag{4.58}
\end{align*}
$$

We see that (4.55)-(4.57) are implied by (4.49). We also have that (4.58) is implied by (4.37), (4.44), and (4.49). We see then that $(\pi(k m)(l m)) u_{(j m)} \neq 0$. Finally, consider (4.48). The two-line notation for $\pi(\mathrm{km})(\mathrm{lm})(\mathrm{jm})$ is

$$
\left(\begin{array}{ccccccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots & l & \ldots & m & \ldots \\
\hline \ldots & \pi_{i} & \ldots & \pi_{l} & \ldots & \pi_{m} & \ldots & \pi_{k} & \ldots & \pi_{j} & \ldots
\end{array}\right)
$$

We have $(\pi(k m)(l m)(j m)) u_{(i l)} \neq 0$ if and only if

$$
\begin{align*}
\pi_{i} & <\pi_{k},  \tag{4.59}\\
\pi_{l} & <\pi_{i} \text { or } \pi_{l}>\pi_{k}  \tag{4.60}\\
\pi_{m} & <\pi_{i} \text { or } \pi_{m}>\pi_{k}, \text { and }  \tag{4.61}\\
\pi_{\alpha} & <\pi_{i} \text { or } \pi_{\alpha}>\pi_{k} \text { for } i<\alpha<l \text { and } \alpha \neq j, k . \tag{4.62}
\end{align*}
$$

We see that (4.59)-(4.61) are implied by (4.49). Additionally, (4.62) is implied by (4.42). From the above we have $(\pi(\mathrm{km})(\mathrm{lm})(j m)) u_{(i l)} \neq 0$. The preceding work proves that $(\pi) u_{y} \neq 0$.

Reverse Implication: For the reverse implication we similarly need to show that each of (4.32)-(4.44) is implied by (4.50)-(4.62). We note that

$$
\begin{equation*}
\pi_{j}<\pi_{l}<\pi_{i}<\pi_{k}<\pi_{m} \tag{4.63}
\end{equation*}
$$

is implied by (4.50),(4.53),(4.55),(4.59), and (4.60). We have that (4.52) implies (4.34), (4.58) implies (4.37), (4.54) and (4.62) imply (4.42), and (4.54) and (4.58) imply (4.44). The remaining inequalities are all implied by (4.63).

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