ABSTRACT

KIRK, SAMANTHA L. Toroidal Lie Algebras and Their Vertex Operator Representations. (Under the direction of Bojko Bakalov.)

Toroidal Lie algebras were first introduced in 1990 by Robert Moody, Senapathi Eswara Rao, and Takeo Yokonuma in their paper *Toroidal Lie algebras and vertex representations*. These algebras can be described as a generalization of a special class of infinite-dimensional Lie algebras known as affine Kac–Moody algebras. Many influential mathematicians have studied the properties of toroidal Lie algebras and have shown toroidal Lie algebras can be applied to several branches of mathematics and physics.

The representation theory of infinite-dimensional Lie algebras is more complex than the representation theory of finite-dimensional Lie algebras. In 1986, Richard Borcherds discovered vertex algebras while trying to construct representations of infinite-dimensional Lie algebras. Since then, several mathematicians have used vertex algebras as a tool in building representations of affine Kac–Moody algebras and toroidal Lie algebras.

Just like twisted affine Kac–Moody algebras, we can use a finite order automorphism of a simple finite-dimensional Lie algebra as a foundation to build a twisted toroidal Lie algebra. Representations of certain twisted affine Kac–Moody algebras and toroidal Lie algebras have been explored in works such as [14, 33, 8, 17] . The study of twisted affine Kac–Moody algebras led to the development of twisted vertex operators and twisted modules over vertex algebras.

The purpose of this thesis is to construct representations of twisted toroidal Lie algebras using twisted modules over vertex algebras. We begin by reviewing some basic concepts related to finite-dimensional Lie algebras and affine Kac–Moody algebras. Next, we review the theory of vertex algebras, lattice vertex algebras, affine vertex algebras, and twisted modules over vertex algebras. In Chapter 4, we discuss toroidal Lie algebras and present a construction of representations of twisted toroidal Lie algebras using twisted modules over vertex algebras. Later, we discuss areas of future research involving toroidal Lie algebras including a new class of twisted toroidal Lie algebras and their potential representations.

Toroidal Lie Algebras and Their Vertex Operator Representations

by
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DEDICATION

In memory of my brother Johnathan.

BIOGRAPHY

Samantha Kirk was born on June 24, 1987 in Washington, North Carolina. She spent her childhood and early adult years in eastern North Carolina. In 2015, she moved to Raleigh, NC to attend North Carolina State University to study mathematics. She was awarded a Doctor of Philosophy in Mathematics from North Carolina State University in the summer of 2021. At the time of this writing, she is preparing to move to South Hadley, MA to teach at Mount Holyoke College as part of a postdoctoral appointment.

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INTRODUCTION

1.1 Introduction

A toroidal Lie algebra is a special type of infinite-dimensional Lie algebra that was first introduced in 1990 by the mathematicians Robert Moody, Senapathi Eswara Rao, and Takeo Yokonuma in their paper *Toroidal Lie algebras and vertex representations* [47]. Since then, many influential mathematicians have studied the properties of toroidal Lie algebras and their representations. Toroidal Lie algebras have many applications to mathematics and physics including combinatorics, non-linear partial differential equations, quantum algebras, and conformal field theory.

To describe a toroidal Lie algebra, we start by taking a simple finite-dimensional Lie algebra $\mathfrak{g}$ with a Lie bracket $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ and tensor $\mathfrak{g}$ with the Laurent polynomials in one variable $\mathbb{C}[t,t^{-1}]$ to form an infinite dimensional Lie algebra $\mathscr{L}(\mathfrak{g})$ called a **loop algebra**,

$$\mathscr{L}(\mathfrak{g})=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}].$$

The loop algebra can be thought of as the collection of all $\mathfrak{g}$-valued polynomial maps on the circle. An **affine Kac–Moody algebra** $\hat{\mathscr{L}}(\mathfrak{g})$ is created when we form a central extension of $\mathscr{L}(\mathfrak{g})$ by “adding" an element K to the center along with a derivation d,

$$\hat{\mathscr{L}}(\mathfrak{g})=\hat{\mathscr{L}}(\mathfrak{g})\oplus\mathbb{C}K\oplus\mathbb{C}d.$$

The process of constructing a central extension of the loop algebra $\hat{\mathscr{L}}(\mathfrak{g})$ is called the affinization of the Lie algebra $\mathfrak{g}$.

A **toroidal Lie algebra** can be described as a generalization of the affinization process. We use a simple finite-dimensional Lie algebra $\mathfrak{g}$ and tensor it with the Laurent polynomials, but this time in several variables, $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_r^{\pm 1}]$ to form a large loop algebra

$$\mathscr{L}_{r+1}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_r^{\pm 1}].$$

The loop algebra $\mathscr{L}_{r+1}(\mathfrak{g})$ is the collection of all $\mathfrak{g}$-valued polynomial maps on the $(r+1)$-dimensional torus. We from a central extension of $\mathscr{L}_{r+1}(\mathfrak{g})$ by adding a certain ideal $\mathscr{K}$ to the center along with an algebra $\mathscr{D}_+$ of derivations (however, in the case when $r \geq 1$, both the ideal $\mathscr{K}$ and the algebra $\mathscr{D}_+$ are infinite-dimensional)

$$\mathscr{L}_{r+1}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_r^{\pm 1}] \oplus \mathscr{K} \oplus \mathscr{D}_+.$$

The construction of representations of infinite-dimensional Lie algebras is more complex than the construction of representations of finite-dimensional Lie algebras. In [47], Moody, Rao, and Yokonuma used vertex operators to form representations of toroidal Lie algebras when $r = 1$. As expected, their results were similar in spirit to vertex operator representations of affine Kac–Moody algebras as a framework.

Vertex algebras were discovered by Richard Borcherds in 1986 while studying the representation theory of infinite-dimensional Lie algebras. Once he developed a working definition of vertex algebras, many mathematicians were able to use vertex algebras and vertex operators as tools in constructing representations of affine Kac–Moody algebras [14, 22] and the use of tensor products of vertex algebras to create representations of toroidal Lie algebras [8].

We can "twist" an affine Lie algebra $\hat{\mathscr{L}}(\mathfrak{g})$ by working with a finite order automorphism of the foundation Lie algebra $\mathfrak{g}$ and lifting the automorphism to $\hat{\mathscr{L}}(\mathfrak{g})$. Investigations into the representations of twisted affine Lie algebras naturally let to the development of twisted vertex operators [35, 44] and twisted modules over a vertex algebra [5, 16, 19, 41]. In a similar fashion, twisted toroidal Lie algebras were constructed and the representations are currently under investigation. Representations involving certain finite-order automorphism σ, such as when σ is the Coxeter element or σ is a diagram automorphism, have been studied in works such as [12, 17, 18, 31, 32, 48, 49, 25, 2, 15, 30].

The goal of this thesis is to contribute to the representation theory of twisted toroidal Lie algebras and to expand its connections with vertex algebras. We will explicitly show how twisted modules over a tensor product of an affine and a lattice vertex algebra can be used to construct representations of twisted toroidal Lie algebras. We will also discuss future projects involving representations of twisted toroidal Lie algebras that use automorphisms of infinite order or automorphisms of affine Kac–Moody algebras in their construction.

This thesis in divided into 5 chapters. In Chapter 2, we will review the necessary background material on Lie algebras. This includes simple finitie-dimensional Lie algebras, root system decompositions, Cartan matrices, and Dynkin diagrams. We will discuss affine Kac–Moody algebras and how to use automorphisms of simple Lie algebras to construct twisted affine Kac–Moody algebras.

In Chapter 3, we shift our attention to vertex algebras. We will review affine vertex algebras, lattice vertex algebras, homomorphisms of vertex algebras and modules of vertex algebras. We will also use automorphisms of vertex algebras to construct twisted modules over vertex algebras. We will discuss vertex operator representations of affine Kac–Moody Lie algebras and their ties with twisted modules over both affine vertex algebras and lattice vertex algebras.

In Chapter 4, we will construct vertex operator representations of twisted toroidal Lie algebras. We will start by defining toroidal Lie algebras and twisted toroidal Lie algebras. Next, we will discuss tensor products of vertex algebras and use the representations of affine Kac–Moody algebras presented in Chapter 3 as the framework for building representations of twisted toroidal Lie algebras. We will compute n-th products related to a certain lattice vertex algebra we call V_J and then show how the tensor product of an affine vertex algebra and V_J lead to our representations.

In Chapter 5, we provide concluding remarks and discuss some potential future projects. One of these projects is the construction of representations of twisted affine Kac–Moody algebras $\hat{\mathscr{L}}(\mathfrak{g})$ where $\mathfrak{g}$ is not simply laced. Another potential project involves the construction of representations of twisted toroidal Lie algebras using a class of finite-order and infinite-order automorphisms that were not considered in Chapter 4.

Unless otherwise specified, all vector spaces, linear maps and tensor products will be over the field $\mathbb{C}$ of complex numbers. Parts of this thesis have been published in [6].

CHAPTER

2

LIE ALGEBRAS

In this chapter, we will review some of the basic concepts in the theory of Lie algebras. These concepts include root systems, Cartan matrices, Dynkin diagrams, derivations, and automorphisms of Lie algebras. We will also define untwisted and twisted affine Kac–Moody Lie algebras in this chapter. For more information on these topics, we refer the reader to [29, 33].

2.1 Definition of a Lie algebra and other basic notions

Definition 2.1.1. A complex **Lie algebra** is a vector space $\mathfrak{g}$ over $\mathbb{C}$ equipped with a bilinear product $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, known as the Lie bracket, that satisfies the properties

$$[x,y]=-[y,x] \qquad \text{(skew-symmetry)} \tag{2.1}$$

$$\big[x,[y,z]\big]=\big[[x,y],z\big]+\big[y,[x,z]\big] \qquad \text{(Jacobi identity)} \tag{2.2}$$

for all $x,y,z\in\mathfrak{g}$.

Example 2.1.2. A Lie algebra can be constructed from the set of all $n\times n$ matrices with complex entries. If we let A and B be any two $n\times n$ matrices and AB the product of the matrices, then we obtain a Lie algebra when we define the Lie bracket as $[A,B]=AB-BA$. This Lie algebra is known as the **general linear Lie algebra** and is denoted by $\mathfrak{gl}(n,\mathbb{C})$.

Example 2.1.3. In a fashion similar to $\mathfrak{gl}(n,\mathbb{C})$, we can construct a Lie algebra using products of linear operators on a vector space V with dimension n. If we let $x,y\in\operatorname{End}(V)$ (the set of all linear operators on V) then we can define the Lie bracket as $[x, y] = xy - yx$ where xy is the product of the operators x and y. This Lie algebra is denoted by $\mathfrak{gl}(V)$.

Example 2.1.4. Let V be a vector space with $\dim(V) = 2n+1$. Let $\mathscr{B} = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z\}$ be an ordered basis for V. We can construct a Lie algebra by defining a Lie bracket on the basis vectors of V as follows:

$$[x_i, y_j] = \delta_{i,j} z, \qquad [\mathfrak{g}, z] = 0,$$

where $\delta_{i,j}$ is the Kronecker delta function. This Lie algebra is known as the **Heisenberg Lie algebra**.

Definition 2.1.5. A **subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace where for any $x, y \in \mathfrak{h}$ we have $[x, y] \in \mathfrak{h}$. If $\mathfrak{h}$ also possesses the property $[x, z] \in \mathfrak{h}$ for any $x \in \mathfrak{h}$ and $z \in \mathfrak{g}$, then we call $\mathfrak{h}$ an **ideal**.

It is important to notice that the closure of the Lie bracket in the definition of a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ allows us to consider $\mathfrak{h}$ as a Lie algebra in its own right.

Example 2.1.6. The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ has many subalgebras including the set of all $n \times n$ matrices with trace 0, denoted $\mathfrak{sl}(n, \mathbb{C})$, the set of all upper triangular matrices, denoted $\mathfrak{t}(n, \mathbb{C})$, and the set of all strictly upper triangular matrices, denoted $\mathfrak{n}(n, \mathbb{C})$. The subalgebra $\mathfrak{n}(n, \mathbb{C})$ is an ideal of $\mathfrak{t}(n, \mathbb{C})$.

Definition 2.1.7. If a Lie algebra $\mathfrak{g}$ is nonabelian and contains no nontrivial proper ideals, then we call $\mathfrak{g}$ **simple**.

Example 2.1.8. It is easy to see that $\mathfrak{gl}(n, \mathbb{C})$ is not simple since $\mathfrak{sl}(n, \mathbb{C})$ is an ideal of $\mathfrak{gl}(n, \mathbb{C})$. However, $\mathfrak{sl}(n, \mathbb{C})$ is an example of a simple Lie algebra.

Definition 2.1.9. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over $\mathbb{C}$. A Lie algebra **homomorphism** is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ that preserves the Lie bracket. That is, $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}_1$. A Lie algebra **isomorphism** is a homomorphism that is both one-to-one and onto.

Example 2.1.10. If we let $\mathfrak{g}_1 = \mathfrak{t}(n, \mathbb{C})$ with basis $\{E_{ij} | 1 \le i < j \le n\}$ and $\mathfrak{g}_2$ be the set of lower triangular matrices with basis $\{E_{ij} | 1 \le j < i \le n\}$ then we can define a homomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ by $\phi(E_{ij}) = -E_{ji}$. This map is clearly one-to-one. Since the dimensions of both spaces is $n(n-1)/2$, the map is also onto which makes ϕ an isomorphism.

Example 2.1.11. Let $\mathfrak{g}_1 = \mathfrak{gl}(V)$ for any vector space V of dimension n and $\mathfrak{g}_2 = \mathfrak{gl}(n, \mathbb{C})$ where $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ takes a linear operator to its matrix relative to a fixed ordered basis of V. The map ϕ is an isomorphism.

The theory of simple finite-dimensional Lie algebras is highly developed and it is well-known that every simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is isomorphic to one of the **classical Lie algebras** $A_\ell, B_\ell, C_\ell, D_\ell$, or isomorphic to one of the **exceptional Lie algebras** E_6, E_7, E_8, F_4, G_2. We will review the classical Lie algebras in the next few examples.

Example 2.1.12. The Lie algebra A_ℓ for $\ell \geq 1$ is the Lie algebra of all $\ell+1 \times \ell+1$ matrices with trace 0 that is also denoted by $\mathfrak{sl}(\ell+1,\mathbb{C})$. That is,

$$A_\ell := \{A \in \mathfrak{gl}(\ell+1,\mathbb{C}) \,|\, \mathrm{tr}(A) = 0\}.$$

A basis for A_ℓ is the set $\{E_{ij},\, E_{kk} - E_{k+1,k+1} \,|\, 1 \leq i, j \leq \ell+1,\, i \neq j,\, 1 \leq k \leq \ell\}$ and $\dim(A_\ell) = (\ell+1)^2 - 1$. This Lie algebra is called the **special linear Lie algebra**.

Example 2.1.13. The Lie algebra B_ℓ is the Lie algebra of all matrices $A \in \mathfrak{gl}(2\ell+1,\mathbb{C})$ that are skew-symmetric. That is,

$$B_\ell := \{A \in \mathfrak{gl}(2\ell+1,\mathbb{C}) \,|\, A + A^T = 0\}.$$

This Lie algebra is called the **odd-dimensional orthogonal Lie algebra** and denoted $\mathfrak{so}(2\ell+1,\mathbb{C})$. The dimension of B_ℓ is $2\ell^2+\ell$. Similarly, the Lie algebra D_ℓ is the Lie algebra of all matrices $A \in \mathfrak{gl}(2\ell,\mathbb{C})$ that are skew-symmetric.That is,

$$D_\ell := \{A \in \mathfrak{gl}(2\ell,\mathbb{C}) \,|\, A + A^T = 0\}.$$

This Lie algebra is known as the **even-dimensional orthogonal Lie algebra** and denoted $\mathfrak{so}(2\ell,\mathbb{C})$. The dimension of D_ℓ is $2\ell^2 - \ell$.

Example 2.1.14. Let $A \in \mathfrak{gl}(2\ell,\mathbb{C})$ and J_ℓ be the block matrix

$$J_\ell = \begin{pmatrix} 0_\ell & I_\ell \\ -I_\ell & 0_\ell \end{pmatrix}$$

where I_ℓ is the $\ell \times \ell$ identity matrix and 0_ℓ is the $\ell \times \ell$ zero matrix. The **symplectic Lie algebra** C_ℓ is defined as

$$C_\ell := \{A \in \mathfrak{gl}(2\ell,\mathbb{C}) \,|\, J_\ell A + A^T J_\ell = 0\}.$$

The Lie algebra C_ℓ is also denoted by $\mathfrak{sp}(2\ell,\mathbb{C})$. The dimension of C_ℓ is $2\ell^2+\ell$.

2.2 Representations and derivations of Lie algebras

In this thesis, we will construct representations and derivations of certain infinite-dimensional Lie algebras. We will review the concepts of representations and derivations in this section.

Definition 2.2.1. A **representation** of a complex Lie algebra $\mathfrak{g}$ on a vector space V over $\mathbb{C}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Example 2.2.2. Consider the Lie algebra $A_1 = \mathfrak{sl}(2,\mathbb{C})$ consisting of all 2×2 traceless matrices. Let V be the vector space of all polynomials in t of degree n or less. That is, $V = \mathbb{C}[t]_{\deg \leq n}$. Recall that $A_1 = \mathrm{span}_{\mathbb{C}}\{E_{11} - E_{22},\, E_{12},\, E_{21}\}$. Let ϕ be the map that sends

$$E_{11} - E_{22} \mapsto n - 2t\frac{d}{dt},\;\; E_{12} \mapsto \frac{d}{dt},\;\; E_{21} \mapsto nt - t^2\frac{d}{dt}.$$

The map $\phi : A_1 \to \mathfrak{gl}(V)$ is a representation.

Example 2.2.3. Since every Lie algebra $\mathfrak{g}$ is a vector space, we can define representations of $\mathfrak{g}$ on itself. One important example of such a representation is the **adjoint representation** $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where for each $x, y \in \mathfrak{g}$ we have

$$\mathrm{ad}(x) = \mathrm{ad}_x, \qquad \mathrm{ad}_x(y) = [x, y].$$

Let $\mathfrak{g}$ be a Lie algebra, V a vector space, and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Since $\phi(x)$ for all $x \in \mathfrak{g}$ is a linear operator, then every representation gives rise to an **action** of $\mathfrak{g}$ on V defined as $x \cdot v = \phi(x)v \in V$ for all $x \in \mathfrak{g}$, $v \in V$. This leads to the notion of a $\mathfrak{g}$-module.

Definition 2.2.4. A $\mathfrak{g}$**-module** is a vector space V equipped with a bilinear product $\cdot : \mathfrak{g} \times V \to V$ that satisfies $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$. A **submodule** U of V is a subspace such that $x \cdot u \in U$ for all $x \in \mathfrak{g}$ and $u \in U$. We call a $\mathfrak{g}$-module V **irreducible** if its only submodules are $\{0\}$ and V.

In Example 2.2.2, $V = \mathbb{C}[t]_{\deg \leq n}$ is an irreducible A_1-module under ϕ. In Example 2.2.3, $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action. If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module under the adjoint action. If $\mathfrak{g}$ is not simple, then the $\mathfrak{g}$-submodules under the adjoint action correspond to the ideals of $\mathfrak{g}$.

Definition 2.2.5. A **derivation** D of a Lie algebra $\mathfrak{g}$ is a linear operator $D : \mathfrak{g} \to \mathfrak{g}$ that satisfies

$$D[x, y] = [Dx, y] + [x, Dy]$$

for all $x, y \in \mathfrak{g}$. The set of all derivations of a Lie algebra $\mathfrak{g}$ is denoted $\mathrm{Der}(\mathfrak{g})$.

Example 2.2.6. Let $\mathfrak{g}$ be a Lie algebra and let ad_x for $x \in \mathfrak{g}$ be the linear operator defined by the adjoint representation $\mathrm{ad}_x(y) = [x, y]$ for any $y \in \mathfrak{g}$. The Jacobi identity can be rewritten as

$$\mathrm{ad}_x\left([y, z]\right) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

Therefore, the operator ad_x for all $x \in \mathfrak{g}$ is a derivation of $\mathfrak{g}$ called an **inner derivation.**

2.3 The root space decomposition of a simple Lie algebra

In this section, we will review the root space decomposition of a simple finite-dimensional Lie algebra. We will define concepts such as diagonalizable operators, toral subalgebras, and root systems associated to simple finite-dimensional Lie algebras.

Definition 2.3.1. Let V be a finite-dimensional vector space over $\mathbb{C}$ and let T be a linear operator on V. If all the roots of the minimal polynomial of T are distinct, then T is called **diagonalizable**. Equivalently, if there exists a basis $\mathscr{B}$ for V such that the matrix of T relative to $\mathscr{B}$ is diagonal, then T is a diagonalizable operator.

Definition 2.3.2. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ consisting only of elements $x \in \mathfrak{g}$ where ad_x is a diagonalizable operator. Then the subalgebra $\mathfrak{h}$ is known as a **toral** subalgebra.

Remark 2.3.3. Every toral subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is abelian.

Definition 2.3.4. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$ (that is, a toral subalgebra that is not a proper subalgebra of any other toral subalgebra in $\mathfrak{g}$). Then we call $\mathfrak{h}$ a **Cartan subalgebra** of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. The set $\{\mathrm{ad}_x \mid x \in \mathfrak{h}\}$ is a class of commuting diagonalizable linear operators of $\mathfrak{g}$ and thus simultaneously diagonalizable. That means $\mathfrak{g}$ can be written as a direct sum of subspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \text{where} \qquad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{h}\}.$$

All nonzero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{g}_\alpha \neq 0$ are called the **roots** of $\mathfrak{g}$ relative to $\mathfrak{h}$ and the subspaces $\mathfrak{g}_\alpha$ are called the **root spaces** of $\mathfrak{g}$. The set of all such roots is called the **root system** of $\mathfrak{g}$ and denoted by Δ. That is,

$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}.$$

Remark 2.3.5. The subspace $\mathfrak{g}_0$ is exactly the Cartan subalgebra $\mathfrak{h}$.

Definition 2.3.6. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}$. The **root space decomposition** of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

In the next proposition, we will give more information about the root system Δ.

Proposition 2.3.7. For any simple finite-dimensional Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$,

(a) The set of roots Δ is finite, does not contain 0, and spans the dual space $\mathfrak{h}^*$.

(b) For any root $\alpha \in \Delta$, the only multiples of α in Δ are $\pm\alpha$.

(c) For each $\alpha \in \Delta$, the subspaces $\mathfrak{g}_\alpha$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ are one dimensional.

(d) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(e) If $\alpha, \beta \in \Delta$ but $\alpha + \beta \notin \Delta \cup \{0\}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$.

Definition 2.3.8. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with root system Δ and let Π be a subset of Δ. We call Π a **base** if the roots in Π form a basis for $\mathfrak{h}^*$ and each $\alpha \in \Delta$ can be written as $\alpha = \sum_{\alpha \in \Delta} k(\alpha)\alpha$ where $k : \Delta \to \mathbb{Z}$ and $k(\alpha)$ are all nonegative or all nonpositive.

The set Π is also known as the set of **simple roots**. We can fix an ordering of the simple roots Π and label them $\alpha_1, \alpha_2, \cdots, \alpha_\ell$. If $\dim(\mathfrak{h}) = \ell$, then we say $\mathfrak{g}$ has **rank** ℓ.

Example 2.3.9. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is a vector space consisting of all $n \times n$ traceless matrices with the basis

$$\left\{E_{ij},\, E_{kk} - E_{k+1,k+1} \,|\, 1 \leq i,j \leq n,\ i \neq j,\ \text{and } 1 \leq k < n\right\}.$$

If we set $h_k = E_{kk} - E_{k+1,k+1}$, then the vector space $\mathfrak{h} = \operatorname{span}_{\mathbb{C}}\{h_k \,|\, 1 \leq k < n\}$ forms a Cartan subalgebra of $\mathfrak{g}$. Let $h \in \mathfrak{h}$ and write

$$h = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

where $\sum_{i=1}^{n} a_i = 0$. Let $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ be the linear functional such that $\epsilon_i(h) = a_i$. Then $\epsilon_i \in \mathfrak{h}^*$ and we have

$$[h, E_{ij}] = (\epsilon_i - \epsilon_j)(h)E_{ij}.$$

If we let $\alpha_{ij} = \epsilon_i - \epsilon_j$, then the root system Δ related to $\mathfrak{h}$ is the set

$$\Delta = \{\alpha_{ij} \,|1 \leq i,j \leq n,\ \text{and } i \neq j\},$$

and if we let $\alpha_i = \alpha_{i,i+1}$ then the set of simple roots Π is given by

$$\Pi = \{\alpha_i \,|1 \leq i \leq n-1\}.$$

The root spaces are given by $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$ and the root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\substack{1 \leq i,j \leq n \\ i \neq j}} \mathbb{C}E_{ij} \right).$$

2.4 The Killing form and Cartan matrices of simple Lie algebras

In this section, we will use the Killing form to build a Cartan matrix associated to a simple finite-dimensional Lie algebra.

Definition 2.4.1. Let $\mathfrak{g}$ be any Lie algebra. The **Killing form** of $\mathfrak{g}$ is defined as the bilinear form

$$\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}, \qquad \kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x, \operatorname{ad}_y) \quad \text{for all } x, y \in \mathfrak{g}.$$

The Killing form satisfies the properties (for $x, y, z \in \mathfrak{g}$):

$$\begin{aligned} \kappa(x, y) &= \kappa(y, x) && \text{(symmetry)}, \\ \kappa\big([x, y], z\big) &= \kappa\big(x, [y, z]\big) && \text{(invariance)}. \end{aligned}$$

For any simple Lie algebra $\mathfrak{g}$ of rank ℓ, the restriction of the killing form to a Cartan subalgebra $\mathfrak{h}$ is nondegenerate, which means we can identify $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$. That is, for every $\alpha \in \Delta$ we can find a *unique* $h_\alpha \in \mathfrak{h}$ that satisfies $\alpha(h) = \kappa(h_\alpha, h)$. We can also define a nondegenerate bilinear form $(\cdot|\cdot) : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ as follows (let $\alpha \in \mathfrak{h}^*, h \in \mathfrak{h}$):

$$(\alpha|h) = \kappa(h_\alpha, h).$$

Since the elements $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ form a base for Δ, then the elements $\{h_{\alpha_i}\}_{i=1}^{\ell}$ form a basis for $\mathfrak{h}$. It is more advantageous to work with certain scalar multiples of the elements h_i. We will define $\alpha^\vee = \frac{2}{(\alpha|\alpha)} h_\alpha$ for $\alpha \in \Delta$ and let $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_\ell^\vee\}$. The set $\Pi^\vee$ is often referred to as the set of **simple coroots**.

Definition 2.4.2. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank ℓ with a Cartan subalgebra $\mathfrak{h}$, the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ related to $\mathfrak{h}$, and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_\ell^\vee\}$ the set of simple coroots. We can define a $\ell \times \ell$ matrix $C = \big(c_{ij}\big)_{1 \le i, j \le \ell}$ where $c_{ij} = (\alpha_i | \alpha_j^\vee)$. The matrix C is called the **Cartan matrix** associated to $\mathfrak{g}$.

Suppose $\mathfrak{g}$ is a simple finite-dimensional Lie algebra of rank ℓ. Then there is exactly one Cartan matrix C associated to $\mathfrak{g}$ and C satisfies the following properties ($1 \le i, j \le \ell,\ i \ne j$):

(a) $c_{ii} = 2$,

(b) c_{ij} are nonpositive integers for $i \ne j$,

(c) $c_{ij} = 0 \iff c_{ji} = 0$,

(d) $\det(C) \ne 0$.

We list the Cartan matrices associated to the classical Lie algebras A_ℓ, B_ℓ, C_ℓ, and D_ℓ in the following examples.

Example 2.4.3. For the Lie algebra A_ℓ, the Cartan matrix is

$$A_\ell : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Example 2.4.4. For the Lie algebra B_ℓ, the Cartan matrix is

$$B_\ell : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

Notice that the 2×2 block in the bottom right-hand corner of the Cartan matrix for B_ℓ is different from the 2×2 block in the bottom right-hand corner of the Cartan matrix for A_ℓ.

Example 2.4.5. For the Lie algebra C_ℓ, the Cartan matrix is

$$C_\ell : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Notice that the Cartan matrix for C_ℓ is the transpose of the Cartan matrix for B_ℓ.

Example 2.4.6. For the Lie algebra D_ℓ, the Cartan matrix is

$$D_\ell : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Notice that the Cartan matrices for A_ℓ and D_ℓ are symmetric.

2.5 Dynkin diagrams of simple Lie algebras

Definition 2.5.1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $C = \left(c_{ij}\right)_{1 \le i,j \le \ell}$ be its associated Cartan matrix. We can construct a graph based off the matrix C known as the **Dynkin diagram**

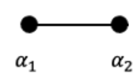
of $\mathfrak{g}$ and denoted $D(\mathfrak{g})$. In this graph, we draw ℓ nodes to represent the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_\ell$. If $|c_{ij}| \geq |c_{ji}|$, then the nodes α_i and α_j are connected by $|c_{ij}|$ lines. If $|c_{ij}| > |c_{ji}|$, then we include an arrow pointing toward the node α_i.

Example 2.5.2. The Cartan matrix associated to the simple Lie algebra of type A_2, then

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the Dynkin diagram $D(A_2)$ is given by the figure below.

Figure 2.1 The Dynkin Diagram of A_2.

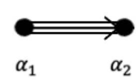


Example 2.5.3. The Cartan matrix associated to the simple Lie algebra of type G_2, then

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

and the Dynkin diagram $D(G_2)$ is given by the figure below.

Figure 2.2 The Dynkin Diagram of G_2.



In the following figures, we give the Dynkin diagrams for some of the classical and exceptional Lie algebras.

Figure 2.3 The Dynkin diagrams associated to $A_\ell - D_\ell$.

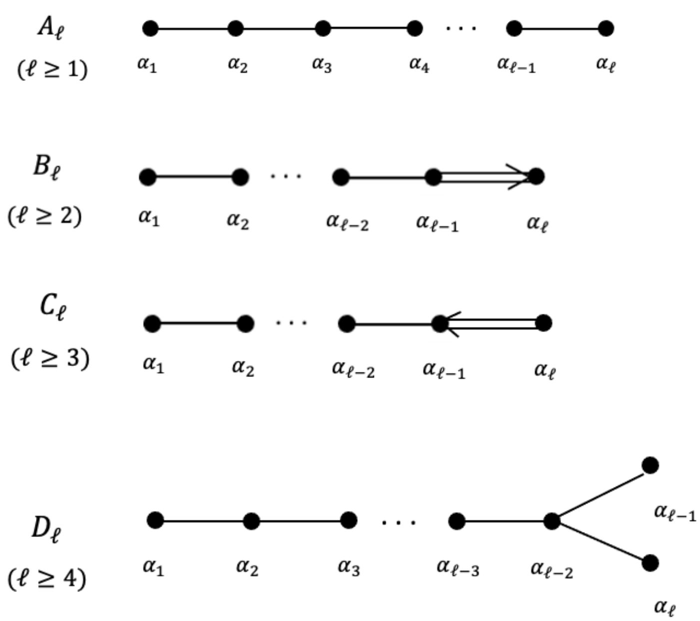
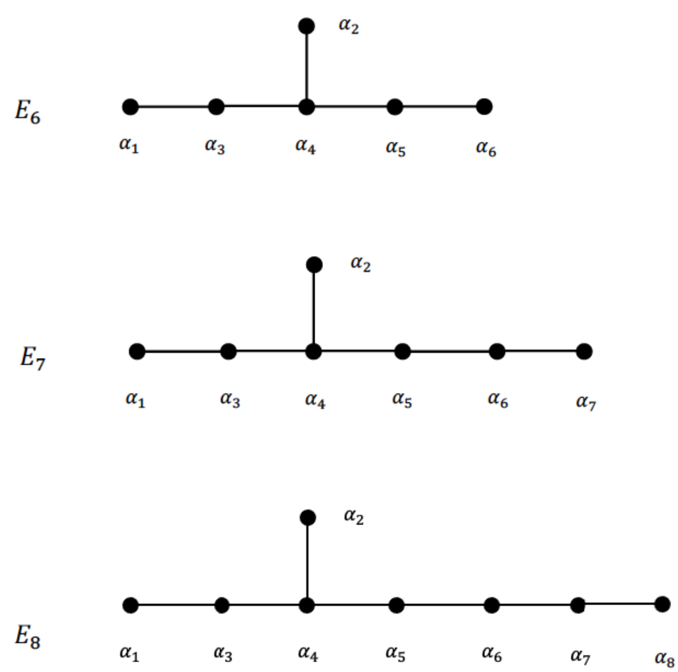


Figure 2.4 The Dynkin diagrams of the exceptional Lie algebras $E_6, E_7,$ and E_8.



Notice that the Dynkin diagrams assoiciated to the Lie algebras A_ℓ, D_ℓ, E_6, E_7 and E_8 do not have any multiple edges between nodes. We call the Dynkin diagrams of these simple finite-dimensional Lie algebras **simply laced**.

2.6 Lattices and automorphisms of Lie algebras

A large portion of this thesis deals with lattices and automorphisms of Lie algebras. We provide a review of these concepts in this section.

Definition 2.6.1. An integral **lattice** of rank ℓ is a free abelian group Q that is generated by ℓ elements and equipped with a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot|\cdot): Q \times Q \to \mathbb{Z}$.

Example 2.6.2. Suppose $\Pi = \{\alpha_i | i = 1, 2, \cdots, \ell\}$ is the set of simple roots for a simple finite-dimensional Lie algebra $\mathfrak{g}$ and let $C = (c_{ij})_{i,j=1}^{\ell}$ be its associated Cartan matrix. We can construct a **root lattice** by defining Q in the following way:

$$Q := \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$$

and we can use the Killing form to define the bilinear form $(\cdot|\cdot): Q \times Q \to \mathbb{Z}$. That is,

$$(\alpha_i|\alpha_j) = \kappa\left(\alpha_i^{\vee}, h_{\alpha_j}\right).$$

In this thesis, we will focus on root lattices defined using the simple roots of simply laced Lie algebras. Suppose Q is a root lattice of a simple Lie algebra $\mathfrak{g}$ of type A_ℓ, D_ℓ, or E_ℓ. We can extract the root system Δ from Q by considering the following set:

$$\Delta = \{\alpha \in Q | (\alpha|\alpha) = 2\}.$$

Also notice that the dual space $\mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} Q$ and (since we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$) we can write $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. We can define a function $\varepsilon : Q \times Q \to \{\pm 1\}$ where $\varepsilon(\alpha_i, \alpha_j) = -1$ when $i = j$ or $i < j$ and the nodes α_i and α_j are connected in the Dynkin diagram. If the nodes α_i and α_j are connected and $i > j$ then we set $\varepsilon(\alpha_i, \alpha_j) = 1$. If the nodes α_i and α_j are not connected then $\varepsilon(\alpha_i, \alpha_j) = 1$. We extend this map to $Q \times Q$ by the following bimultiplicitivity properties (for any $\alpha, \beta, \gamma \in Q$):

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma),$$
$$\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma).$$

Definition 2.6.3. We can assign a one-dimensional vector space $\mathbb{C}E_\alpha$ to every $\alpha \in \Delta$ and build the Lie algebra $\mathfrak{g}$ (of type A_ℓ, D_ℓ or E_ℓ) associated to Δ via the **Frenkel-Kac construction** [33],

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C}E_\alpha \right), \tag{2.3}$$

where the Lie brackets are defined as

$$
\begin{aligned}
&[h,h']=0 && \text{if } h,h'\in\mathfrak{h},\\
&[h,E_\alpha]=(h|\alpha)E_\alpha && \text{if } h\in\mathfrak{h} \text{ and } \alpha\in\Delta,\\
&[E_\alpha,E_{-\alpha}]=-\alpha && \text{if } \alpha\in\Delta,\\
&[E_\alpha,E_\beta]=0 && \text{if } \alpha,\beta\in\Delta \text{ and } \alpha+\beta\notin\Delta\cup\{0\},\\
&[E_\alpha,E_\beta]=\varepsilon(\alpha,\beta)E_{\alpha+\beta} && \text{if } \alpha,\beta\in\Delta \text{ and } \alpha+\beta\in\Delta.
\end{aligned}
$$

Definition 2.6.4. An **automorphism** ϕ of a Lie algebra $\mathfrak{g}$ is a Lie algebra isomorphism from $\mathfrak{g}$ to itself. That is, $\phi:\mathfrak{g}\to\mathfrak{g}$ is linear, onto, one-to-one, and preserves the Lie bracket. Similarly, an **automorphism** ϕ of an integral lattice Q is a linear, onto, and one-to-one map $\phi:Q\to Q$ such that $(\alpha|\beta)=(\phi\alpha|\phi\beta)$ for all $\alpha,\beta\in Q$.

Definition 2.6.5. An automorphism ϕ of a Lie algebra $\mathfrak{g}$ is called an **inner** automorphism if ϕ is generated by products of $\exp(\mathrm{ad}_x)$ where $x\in\mathfrak{g}$ and $(\mathrm{ad}_x)^n=0$ for some positive integer n. Any automorphism of a Lie algebra that is not an inner automorphism is known as an an **outer** automorphism.

Definition 2.6.6. Let ϕ be an automorphism of a root lattice Q from a simple finite-dimensional Lie algebra $\mathfrak{g}$. A **diagram automorphism** $\mu:Q\to Q$ is an automorphism that preserves the Dynkin diagram associated to $\mathfrak{g}$.

Example 2.6.7. Let Q be the root lattice associated to the Lie algebra A_ℓ and recall the Dynkin diagram associated to A_ℓ given in Figure 2.3. The only nontrivial diagram automorphism associated to A_ℓ is the map that sends

$$\alpha_i\mapsto\alpha_{\ell+1-i}$$

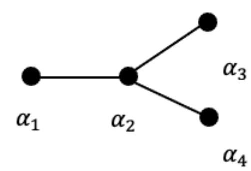
for all $1\le i\le\ell$.

Example 2.6.8. Let Q be the root lattice for the simple finite-dimensional Lie algebra D_ℓ and recall the Dynkin diagram associated to D_ℓ given in Figure 2.4. For $\ell\neq 4$, there is one nontrivial diagram automorphisms associated to D_ℓ that sends

$$\alpha_{\ell-1}\mapsto\alpha_\ell,\qquad \alpha_i\mapsto\alpha_i \text{ for } i\neq\ell-1,\ell.$$

For $\ell=4$, the Dynkin diagram takes the shape in the figure below.

Figure 2.5 The Dynkin Diagram of D_4.



and the set of diagram automorphisms associated to D_4 consists of all possible permutations of the roots α_1, α_3, and α_4.

Let Q be a root lattice corresponding to a simple finite-dimensional Lie algebra $\mathfrak{g}$ that is simply laced. Recall that we can use Q to reconstruct the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha \right).$$

If μ is a diagram automorphism of Q, then μ induces an automorphism $\overline{\mu}$ of $\mathfrak{g}$ defined by

$$\overline{\mu}(\alpha) = \mu(\alpha), \quad \overline{\mu}(E_\alpha) = E_{\mu(\alpha)}.$$

The induced automorphism $\overline{\mu}$ is called a **diagram automorphism** of $\mathfrak{g}$.

2.7 Affine Kac–Moody algebras

In this section, we review affine Kac–Moody algebras and their root systems. More details can be found in [33].

Definition 2.7.1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of type X_ℓ with Lie bracket $[\cdot,\cdot]$. An **affine Kac-Moody algebra** of type $X_\ell^{(1)}$ is the Lie algebra

$$\hat{\mathscr{L}}(\mathfrak{g}) = \left(\mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right]\right) \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where the Lie bracket $[\cdot,\cdot]$ is extended to $\hat{\mathscr{L}}(\mathfrak{g})$ in the following way:

$$\begin{aligned} [a \otimes t^m, b \otimes t^n] &= [a,b] \otimes t^{m+n} + m\delta_{m,-n}(a|b)K, \\ [K, \hat{\mathscr{L}}(\mathfrak{g})] &= 0, \\ [d, a \otimes t^m] &= m a \otimes t^m, \end{aligned} \tag{2.4}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here $(\cdot|\cdot)$ is a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ (which can be any scalar multiple of the Killing form). We will denote by $\hat{\mathscr{L}}'(\mathfrak{g})$ the subalgebra of $\hat{\mathscr{L}}(\mathfrak{g})$ given by

$$\hat{\mathscr{L}}'(\mathfrak{g}) = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K.$$

A Cartan subalgebra $\mathfrak{h}$ of $\hat{\mathscr{L}}(\mathfrak{g})$ can be constructed by using the Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathfrak{g}$ with the addition of the central element K and the derivation d. That is,

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Similarly, the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ (which has the same dimension as $\mathfrak{h}$) can be constructed using the dual space $\mathring{\mathfrak{h}}^*$ of $\mathfrak{g}$ with the addition of two extra elements to its basis, which we will call α_0 and Λ_0,

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\alpha_0 \oplus \mathbb{C}\Lambda_0.$$

If $\mathring{\Pi} = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ is the set of simple roots for $\mathfrak{g}$, then a base for the simple roots of $\hat{\mathscr{L}}(\mathfrak{g})$ is $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$. We can also build a set of coroots $\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ for $\hat{\mathscr{L}}(\mathfrak{g})$. Just like in the simple finite-dimensional case, we can also construct a nondegenerate bilinear form $(\cdot|\cdot) : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ as follows (let $\alpha \in \mathfrak{h}^*, h \in \mathfrak{h}$):

$$(\alpha|h) = \alpha(h).$$

This bilinear form leads to the creation of a Cartan matrix C for $\hat{\mathscr{L}}(\mathfrak{g})$

$$C = \left(c_{ij}\right)_{0 \le i,j \le \ell}$$

where $c_{ij} = (\alpha_i|\alpha_j^\vee)$. Since $\mathring{\Pi} \subset \Pi$, the Cartan matrix for $\hat{\mathscr{L}}(\mathfrak{g})$ will have the Cartan matrix for $\mathfrak{g}$ as a submatrix.

Example 2.7.2. The Cartan matrix for the affine Kac–Moody algebra of type $A_1^{(1)}$ is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and the Cartan matrix for the affine Kac–Moody algebra of type $A_\ell^{(1)}$ for $\ell \ge 2$ is

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}.$$

Similar to the simple finite-dimensional case, we can construct a Dynkin diagram based off the Cartan matrix C of $\hat{\mathscr{L}}(\mathfrak{g})$. We provide the Dynkin diagram of $A_\ell^{(1)}$ for $\ell \geq 1$ as an example in the figures below. For more information, see [33].

Figure 2.6 The Dynkin diagram of $A_1^{(1)}$.

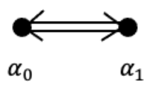
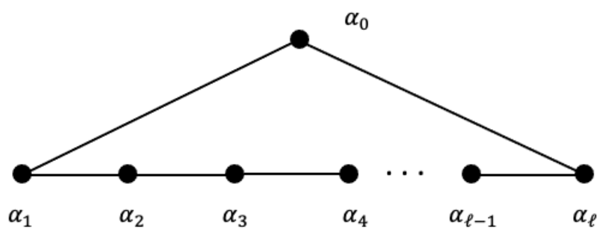


Figure 2.7 The Dynkin diagram of $A_\ell^{(1)}$ for $\ell \geq 2$.



Let C be the Cartan matrix associated to an affine Kac–Moody algebra $\hat{\mathscr{L}}(\mathfrak{g})$. There exists a *unique* vector $\hat{\delta} = (a_0, a_1, \ldots, a_\ell)^T$ where the a_i's are positive relatively prime integers and $C\hat{\delta} = 0$. Let $\delta = \sum_{i=0}^{\ell} a_i \alpha_i$ where $\{\alpha_i | i = 0, 1, \ldots, \ell\}$ is the set of simple roots. We call δ the **null root** and δ is orthogonal to all roots

$$(\delta|\alpha_i) = 0 \ \text{ for } 0 \leq i \leq \ell, \qquad (\delta|\Lambda_0) = 1.$$

Using δ we can re-express the dual space as

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0.$$

Let $\mathring{\Delta}$ be the set of roots for $\mathfrak{g}$. The root system Δ of $\hat{\mathscr{L}}(\mathfrak{g})$ consists of the union of **real roots** (denoted Δ^{re}) and **imaginary roots** (denoted Δ^{im}) given below

$$\Delta = \Delta^{\text{re}} \cup \Delta^{\text{im}} \text{ where } \Delta^{\text{re}} = \{\alpha + n\delta | \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\} \text{ and } \Delta^{\text{im}} = \{n\delta | n \in \mathbb{Z}\}.$$

2.8 Twisted affine Kac–Moody algebras

We briefly review automorphisms of finite order and twisted affine Kac–Moody algebras in this section. Unless specified otherwise, $\mathfrak{g}$ will be a simple finite-dimensional Lie algebra of type X_ℓ.

Suppose σ is an automorphism of the Lie algebra $\mathfrak{g}$ with finite order N. Then the minimal polynomial for σ is $p(\sigma) = \sigma^N - I$ which is a product of distinct linear factors. That means σ is diagonalizable and the set of eigenvalues of σ is $\{e^{\frac{2\pi i j}{N}} \mid j \in \mathbb{Z}/N\mathbb{Z}\}$ where $i=\sqrt{-1}$. The Lie algebra $\mathfrak{g}$ can be written as a direct sum of eigenspaces

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/N\mathbb{Z}} \mathfrak{g}_j \quad \text{where} \quad \mathfrak{g}_j = \{a \in \mathfrak{g} \mid \sigma a = e^{2\pi \mathrm{i} j/N} a\}.$$

We can use the automorphism σ and the associated eigenspaces $\mathfrak{g}_j$ to construct what is known as a *twisted* affine Kac–Moody Lie algebra.

Definition 2.8.1. Let σ be an automorphism of the Lie algebra $\mathfrak{g}$ of finite order N. We can define a subalgebra $\hat{\mathscr{L}}(\mathfrak{g},\sigma)$ of $\hat{\mathscr{L}}(\mathfrak{g})$ by

$$\hat{\mathscr{L}}(\mathfrak{g},\sigma) = \bigoplus_{j\in\mathbb{Z}} \mathscr{L}(\mathfrak{g},\sigma)_j \oplus \mathbb{C}K \oplus \mathbb{C}d, \tag{2.5}$$

where

$$\mathscr{L}(\mathfrak{g},\sigma)_j = \mathfrak{g}_j \otimes \mathbb{C}t^j, \qquad j \in \mathbb{Z}. \tag{2.6}$$

When $\mathfrak{g}$ is simply laced (of type $X = A, D, E$) and σ is a diagram automorphism of order $N=2$ or 3, then the Lie algebra $\hat{\mathscr{L}}(\mathfrak{g},\sigma)$ is known as the **twisted affine Kac–Moody algebra** of type $X_\ell^{(N)}$. We will denote by $\hat{\mathscr{L}}'(\mathfrak{g},\sigma)$ the subalgebra of $\hat{\mathscr{L}}(\mathfrak{g},\sigma)$ given by

$$\hat{\mathscr{L}}'(\mathfrak{g},\sigma) = \bigoplus_{j\in\mathbb{Z}} \mathscr{L}(\mathfrak{g},\sigma)_j \oplus \mathbb{C}K.$$

Proposition 2.8.2 ([33])**.** Let σ be an automorphism of $\mathfrak{g}$ of finite order N. Then there exists an associated diagram automorphism μ of $\mathfrak{g}$ (where the order of μ is $1,2$, or 3, depending on $\mathfrak{g}$) and an inner automorphism φ of $\mathfrak{g}$ such that $\sigma = \mu\varphi$.

When σ is not a diagram automorphism of $\mathfrak{g}$, then by Proposition 2.8.2 we have $\sigma = \mu\varphi$ where μ is a diagram automorphism and φ is an inner automorphism. Let r be the order of μ. This results in the Lie algebra $\hat{\mathscr{L}}(\mathfrak{g},\sigma)$ being isomorphic to the twisted affine Kac–Moody algebra $\hat{\mathscr{L}}(\mathfrak{g},\mu)$ of type $X_\ell^{(r)}$ when $X = A, D$, or E. For more information, see Theorem 8.5 in [33].

Remark 2.8.3. For an automorphism σ of $\mathfrak{g}$ of finite order N, notice that σ^{-1} also has order N. We can consider the subalgebra $\hat{\mathscr{L}}(\mathfrak{g},\sigma^{-1})$ of $\hat{\mathscr{L}}(\mathfrak{g})$ where

$$\hat{\mathscr{L}}(\mathfrak{g},\sigma^{-1}) = \bigoplus_{j\in\mathbb{Z}} \mathscr{L}(\mathfrak{g},\sigma^{-1})_j \oplus \mathbb{C}K \oplus \mathbb{C}d, \tag{2.7}$$

where

$$\begin{aligned} \mathscr{L}(\mathfrak{g},\sigma^{-1})_j &= \mathfrak{g}_j \otimes \mathbb{C}t^j, \qquad j \in \mathbb{Z}, \\ \mathfrak{g}_j &= \{a \in \mathfrak{g} \mid \sigma a = e^{-2\pi \mathrm{i} j/N} a\}. \end{aligned} \tag{2.8}$$

The Lie algebras $\hat{\mathscr{L}}(\mathfrak{g},\sigma^{-1})$ and $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$ will be important when we construct vertex operator representations of affine Kac–Moody algebras and toroidal Lie algebras.

CHAPTER

3

VERTEX ALGEBRAS AND TWISTED MODULES OVER VERTEX ALGEBRAS

In this chapter, we will review some basic concepts from the theory of vertex algebras. These concepts include automorphisms of vertex algebras, subalgebras, and twisted modules over vertex algebras. We will also construct representations of affine Kac–Moody algebras from twisted modules over affine vertex algebras and lattice vertex algebras. For more information on these topics, we refer the reader to [34].

3.1 Vertex algebras and examples

Let V be a vector space and $\mathbb{Z}^{\geq 1}$ be the set of positive integers. Recall that $V[z,z^{-1}]$ is the set of finite sums

$$V[z,z^{-1}]=\left\{\sum_{n\in\mathbb{Z}} v_n z^{n-1} \mid v_n\in V \quad \text{all but finitely many} \quad v_n=0\right\},$$

and $V[z]$ is the subset of $V[z,z^{-1}]$ consisting of finite sums with nonnegative powers of z

$$V[z]=\left\{\sum_{n\in\mathbb{Z}^{\geq 1}} v_n z^{n-1} \mid v_n\in V \quad \text{all but finitely many} \quad v_n=0\right\}.$$

In this thesis, we will be concerned with **formal power series** which are sums that are allowed to be infinite. We will denote $V[[z,z^{-1}]]$ as the set

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^{n-1} \mid v_n \in V \right\},$$

and $V[[z]]$ as the subset of $V[[z,z^{-1}]]$ containing infinite sums with nonnegative powers of z

$$V[[z]] = \left\{ \sum_{n\in\mathbb{Z}^{\geq 1}} v_n z^{n-1} \mid v_n \in V \right\}.$$

Definition 3.1.1. A **vertex algebra** [20, 34, 42] is a vector space V with a distinguished vector $\mathbf{1} \in V$ (called the *vacuum vector*), equipped with bilinear *n-th products* for $n \in \mathbb{Z}$:

$$V \otimes V \to V, \qquad a \otimes b \mapsto a_{(n)}b, \tag{3.1}$$

subject to the following axioms. First, for every fixed $a, b \in V$, we have

$$a_{(n)}b = 0, \quad \text{ for } n \gg 0, \tag{3.2}$$

where $n \gg 0$ stands for sufficiently large n. This means that there exists some positive integer N such that for all integers n such that$n > N$ we have $a_{(n)}b = 0$. Second, the vacuum vector $\mathbf{1}$ plays the role of a unit in the sense that

$$a_{(-1)}\mathbf{1} = \mathbf{1}_{(-1)}a = a, \qquad \mathbf{1}_{(n)}a = 0 \ \text{ for } \ n \in \mathbb{Z},\ n \neq -1, \qquad a_{(n)}\mathbf{1} = 0 \ \text{ for } \ n \geq 0. \tag{3.3}$$

Lastly, the **Borcherds identity** must be satisfied for all $a, b, c \in V$ and $k, m, n \in \mathbb{Z}$:

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{m}{j} (a_{(k+j)}b)_{(m+n-j)}c = \sum_{i=0}^{\infty} \binom{k}{i} (-1)^i \, a_{(m+k-i)}(b_{(n+i)}c) \\ - \sum_{i=0}^{\infty} \binom{k}{i} (-1)^{k+i} \, b_{(n+k-i)}(a_{(m+i)}c). \end{aligned} \tag{3.4}$$

As a consequence of Axiom (3.2), all the sums in the Borcherds identity are finite. If we set $k = 0$ in the Borcherds identity, we obtain the **commutator formula**

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}. \tag{3.5}$$

Definition 3.1.2. The n-th products $a_{(n)}b$ for $a, b \in V$ and $n \in \mathbb{Z}$ can be viewed as a sequence of linear operators $a_{(n)} \in \mathrm{End}(V)$ acting on $b \in V$. We call the operators $a_{(n)}$ the **modes** of a. We can organize the modes into formal power series

$$Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}, \qquad a\in V, \tag{3.6}$$

called **fields** or vertex operators. The linear map $Y: V\to(\mathrm{End}\,V)[[z,z^{-1}]]$ is known as the **state-field correspondence**. Observe that $Y(\mathbf{1},z)=I$ is the identity operator.

Definition 3.1.3. We can define an operator $T\in\mathrm{End}(V)$ by $Ta=a_{(-2)}\mathbf{1}$. The operator T is called the **translation operator** and T satisfies the properties

$$[T,Y(a,z)]=Y(Ta,z)=\partial_z Y(a,z), \tag{3.7}$$

or equivalently,

$$[T,a_{(n)}]=(Ta)_{(n)}=-na_{(n-1)}. \tag{3.8}$$

Another important consequence of the Borcherds identity is the (-1)**-st product identity** where for $a,b\in V$ we have

$$\begin{aligned} Y(a_{(-1)}b,z) &= :Y(a,z)Y(b,z): \\ &= \sum_{n<0} a_{(n)}z^{-n-1}Y(b,z)+\sum_{n\geq 0} Y(b,z)a_{(n)}z^{-n-1}. \end{aligned} \tag{3.9}$$

The double colons in Equation (3.9) denote the **normally-ordered product**. We can consider the normally ordered product of more than two fields by applying the normally ordered product from right to left. For example $(a,b,c\in V)$,

$$\begin{aligned} :Y(a,z)Y(b,z)Y(c,z): &= :Y(a,z)(Y(b,z)Y(c,z)): \\ &= :Y(a,z)\left(\sum_{n<0} b_{(n)}z^{-n-1}Y(c,z)+\sum_{n\geq 0} Y(c,z)b_{(n)}z^{-n-1}\right): \\ &= \sum_{j<0} a_{(j)}z^{-j-1}\left(\sum_{n<0} b_{(n)}z^{-n-1}Y(c,z)+\sum_{n\geq 0} Y(c,z)b_{(n)}z^{-n-1}\right) \\ &\qquad +\left(\sum_{n<0} b_{(n)}z^{-n-1}Y(c,z)+\sum_{n\geq 0} Y(c,z)b_{(n)}z^{-n-1}\right)\sum_{j\geq 0} a_{(j)}z^{-j-1}. \end{aligned} \tag{3.10}$$

We can combine Equations (3.7)–(3.9), we get

$$Y(a_{(-1-m)}b,z)=:(\partial_z^{(m)}Y(a,z))Y(b,z):, \quad m\geq 0, \ \text{ and } \ \partial_z^{(m)}=\frac{1}{m!}\partial_z^m. \tag{3.11}$$

Finally, the Borcherds identity also provides us with a property known as **locality** where for $a,b\in V$ we have

$$(z-w)^N[Y(a,z),Y(b,w)]=0$$

for sufficiently large N, depending on a, b (for example, we can take N to be such that $a_{(n)}b = 0$ for $n \geq N$).

In the two following subsections we will discuss two important types of vertex algebras that will be used later in this thesis: affine vertex algebras and lactice vertex algebras.

3.1.1 Affine vertex algebras

Definition 3.1.4. Suppose $\mathfrak{g}$ is a simple finite-dimensional Lie algebra equipped with a nondegenerate symmetric invariant bilinear form $(\cdot|\cdot)$. Consider the Lie algebra $\hat{\mathscr{L}}'(\mathfrak{g})$ with the brackets given by the first brackets in Equation (2.4). For a fixed $k \in \mathbb{C}$ (called the **level**), consider the (generalized) Verma module for $\hat{\mathfrak{g}}$:

$$V_k(\mathfrak{g}) = \mathrm{Ind}_{(\mathfrak{g}\otimes\mathbb{C}[t])\oplus\mathbb{C}K}^{\hat{\mathscr{L}}'(\mathfrak{g})}\mathbb{C},$$

where $\mathfrak{g}\otimes\mathbb{C}[t]$ acts as 0 on $\mathbb{C}$ and K acts as multiplication by k. The module $V_k(\mathfrak{g})$ has the structure of a vertex algebra [24], called the **universal affine vertex algebra** at level k. The $\hat{\mathfrak{g}}$-module $V_k(\mathfrak{g})$ has a unique irreducible quotient $V^k(\mathfrak{g})$, which is also a vertex algebra [24], known as the **simple affine vertex algebra** at level k.

Let us review the vertex algebra structure of $V = V_k(\mathfrak{g})$; the same applies to $V = V^k(\mathfrak{g})$ as well. The vacuum vector $\mathbf{1}$ is the highest-weight vector of the $\hat{\mathfrak{g}}$-module V. For $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, let $a_{(n)}$ act as $a \otimes t^n$ on V. We embed $\mathfrak{g}$ in V so that we identify $a \in \mathfrak{g}$ with $a_{(-1)}\mathbf{1} \in V$. The fields

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}, \qquad a \in \mathfrak{g},$$

are known as **currents**. All other fields in V are obtained by applying repeatedly Formula (3.11) as shown below (see [20, 34, 42]):

$$Y(a_{1(-m_1)}a_{2(-m_2)}\cdots a_{s(-m_s)}\mathbf{1},z) = :\left(\partial_z^{(m_1-1)}Y(a_1,z)\right)\left(\partial_z^{(m_2-1)}Y(a_2,z)\right)\cdots\left(\partial_z^{(m_s-1)}Y(a_s,z)\right):$$

for $a_i \in \mathfrak{g}$, $m_i \in \mathbb{N}$, and $i \in \{1,2,\ldots,s\}$. For $a,b \in \mathfrak{g} \subset V$, their modes satisfy the commutation relations of the Lie algebra $\hat{\mathfrak{g}}$:

$$[a_{(m)}, b_{(n)}] = [a,b]_{(m+n)} + m\delta_{m,-n}(a|b)k. \tag{3.12}$$

By the commutator formula (3.5), this is equivalent to the j-th products

$$a_{(0)}b = [a,b], \qquad a_{(1)}b = (a|b)k\mathbf{1}, \qquad a_{(j)}b = 0 \ (j \geq 2). \tag{3.13}$$

3.1.2 Lattice vertex algebras

Let Q be an integral lattice of rank ℓ with a symmetric bilinear form $(\cdot|\cdot)\colon Q\times Q \to \mathbb{Z}$. We will assume that Q is even, which means that $|\alpha|^2 = (\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$. Let $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} Q$ and extend the form

$(\cdot|\cdot)$ to $\mathfrak{h}$ using bilinearity. We can construct a Heisenberg Lie algebra $\hat{\mathfrak{h}}$ defined as

$$\hat{\mathfrak{h}} = (\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K,$$

with the Lie brackets ($h, h' \in \mathfrak{h}$, $m, n \in \mathbb{Z}$):

$$[h \otimes t^m, h' \otimes t^n] = m\delta_{m,-n}(h|h')K, \qquad [\hat{\mathfrak{h}}, K] = 0. \tag{3.14}$$

We will use the notation $h_{(m)} = h \otimes t^m$.

Definition 3.1.5. The bosonic **Fock space** B is the induced module

$$B = \mathrm{Ind}_{\mathfrak{h}[t] \oplus \mathbb{C}K}^{\hat{\mathfrak{h}}} \mathbb{C} \simeq S(t^{-1}\mathfrak{h}[t^{-1}]),$$

where K acts as I and $\mathfrak{h}[t]$ acts trivially on $\mathbb{C}$. The Fock space B has the structure of a vertex algebra with a vacuum vector $\mathbf{1}$ the highest-weight vector and the state-field correspondence Y defined as follows. For $h \in \mathfrak{h}$, we identify h with $h_{(-1)}\mathbf{1} \in B$ and let

$$Y(h, z) = \sum_{m \in \mathbb{Z}} h_{(m)} z^{-m-1}, \qquad h \in \mathfrak{h}, \tag{3.15}$$

be the free boson fields. All other fields in B are obtained by applying repeatedly Formula (3.11). That is,

$$Y(h_{1(-m_1)} h_{2(-m_2)} \cdots h_{s(-m_s)}\mathbf{1}, z) = :\left(\partial_z^{(m_1-1)} Y(h_1, z)\right)\left(\partial_z^{(m_2-1)} Y(h_2, z)\right) \cdots \left(\partial_z^{(m_s-1)} Y(h_s, z)\right):,$$

for $h_i \in \mathfrak{h}$, $m_i \in \mathbb{N}$, and $i \in \{1, 2, \ldots, s\}$. The Lie algebra $\hat{\mathfrak{h}}$ has a unique highest-weight representation on the Fock space [20, 34, 42].

We define a bimultiplicative function $\varepsilon \colon Q \times Q \to \{\pm 1\}$ such that

$$\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2/2}, \qquad \alpha \in Q, \tag{3.16}$$

and satisfies

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}, \qquad \alpha, \beta \in Q.$$

We use ε to define the twisted group algebra $\mathbb{C}_\varepsilon[Q]$ with basis $\{e^\alpha\}_{\alpha \in Q}$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}, \qquad e^0 = 1. \tag{3.17}$$

We can construct the tensor product V_Q defined as

$$V_Q := B \otimes \mathbb{C}_\varepsilon[Q],$$

and the representation of $\hat{\mathfrak{h}}$ can be extended to V_Q by the action

$$h_{(m)}(u\otimes e^{\beta})=(h_{(m)}u+\delta_{m,0}(h|\beta)u)\otimes e^{\beta},$$

for $h\in\mathfrak{h}$, $m\in\mathbb{Z}$, $u\in B$ and $\beta\in Q$. In particular, note that e^{β} is a highest-weight vector for the Heisenberg Lie algebra:

$$h_{(m)}e^{\beta}=\delta_{m,0}(h|\beta)e^{\beta},\qquad m\geq 0,\ \ h\in\mathfrak{h},\ \ \beta\in Q. \tag{3.18}$$

We can also represent the algebra $\mathbb{C}_{\varepsilon}[Q]$ on V_Q by

$$e^{\alpha}(u\otimes e^{\beta})=\varepsilon(\alpha,\beta)(u\otimes e^{\alpha+\beta}),$$

for $u\in B$ and $\alpha,\beta\in Q$.

For simplicity of notation, we will write e^{α} for $\mathbf{1}\otimes e^{\alpha}\in V_Q$ and h for $h_{(-1)}\mathbf{1}\otimes e^{0}\in V_Q$, where $\alpha\in Q$ and $h\in\mathfrak{h}$.

Definition 3.1.6. The space V_Q has the structure of a vertex algebra called the **lattice vertex algebra**, with a vacuum vector $\mathbf{1}\otimes e^{0}$ and a state-field correspondence generated by the free boson fields (3.15) and vertex operators

$$Y(e^{\alpha},z)=e^{\alpha}z^{\alpha_{(0)}}e^{\left(\sum_{n=1}^{\infty}\alpha_{(-n)}\frac{z^{n}}{n}\right)}e^{\left(\sum_{n=1}^{\infty}\alpha_{(n)}\frac{z^{-n}}{-n}\right)}. \tag{3.19}$$

In this formula, $z^{\alpha_{(0)}}$ acts on V_Q by

$$z^{\alpha_{(0)}}(u\otimes e^{\beta})=z^{(\alpha|\beta)}(u\otimes e^{\beta}),\qquad u\in B,\ \alpha,\beta\in Q.$$

For future use, we need to compute the action of the translation operator T on V_Q. Taking advantage of Formulas (3.7) and (3.9) we get

$$Y(Te^{\alpha},z)=\partial_z Y(e^{\alpha},z)=:Y(\alpha,z)Y(e^{\alpha},z):=Y\left(\alpha_{(-1)}e^{\alpha},z\right),$$

which gives us

$$Te^{\alpha}=\alpha_{(-1)}e^{\alpha},\qquad \alpha\in Q. \tag{3.20}$$

3.2 Subalgebras, homomorphisms, and modules over vertex algebras

Definition 3.2.1. A **subalgebra** of a vertex algebra V is a subspace U of V that contains the vacuum vector $\mathbf{1}$ and is closed under n-th products

$$a_{(n)}u\in U\quad\text{ for all }\quad a,u\in U.$$

Example 3.2.2. Consider any simple finite dimensional Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$.

Recall from Section 3.1 that we can construct the affine vertex algebra $V_k(\mathfrak{g})$ where we identify $a \in \mathfrak{g}$ with $a_{(-1)}\mathbf{1} \in V_k(\mathfrak{g})$. If we let U be the subalgebra generated by all $h \in \mathfrak{h}$, then U is closed under all n-th products and U is a subalgebra of the vertex algebra $V_k(\mathfrak{g})$.

Definition 3.2.3. Let V_1 and V_2 be two vertex algebras. A vertex algebra **homomorphism** is a linear map $\phi : V_1 \to V_2$ that preserves the n-th products

$$\phi\left(a_{(n)}b\right) = \phi(a)_{(n)}\phi(b) \quad \text{for all} \quad a, b \in V_1, \quad n \in \mathbb{Z}.$$

A vertex algebra **isomorphism** is a homomorphism that is both one-to-one and onto. A vertex algebra **automorphism** is an isomorphism from V_1 to itself.

Definition 3.2.4. Let V be a vertex algebra. An (*untwisted*) V**-module** is a vector space M endowed with a linear map $Y^M \colon V \to (\mathrm{End} M)[[z, z^{-1}]]$,

$$Y^M(a, z) = \sum_{m \in \mathbb{Z}} a^M_{(m)} z^{-m-1}, \qquad a \in V, \tag{3.21}$$

where for every $a \in V$ and $v \in M$, we have

$$a^M_{(m)} v = 0, \quad \text{for } m \gg 0. \tag{3.22}$$

Next, $Y^M(\mathbf{1}, z) = I$ and the Borcherds identity is satisfied any $a, b \in V$, $c \in M$, $k, m, n \in \mathbb{Z}$:

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{m}{j} (a_{(k+j)}b)^M_{(m+n-j)} c &= \sum_{i=0}^{\infty} \binom{k}{i} (-1)^i \, a^M_{(m+k-i)} (b^M_{(n+i)} c) \\ &- \sum_{i=0}^{\infty} \binom{k}{i} (-1)^{k+i} \, b^M_{(n+k-i)} (a^M_{(m+i)} c). \end{aligned} \tag{3.23}$$

A **submodule** U of M is a subspace such that $a^M_{(n)} u \in U$ for all $a \in V$, $n \in \mathbb{Z}$, and $u \in U$. We call a M **irreducible** if its only submodules are $\{0\}$ and M.

Example 3.2.5. Every vertex algebra V is a V-module.

Example 3.2.6. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and recall the universal affine vertex algebra $V_k(\mathfrak{g})$. Every $V_k(\mathfrak{g})$-module is the same as a $\hat{\mathscr{L}}'(\mathfrak{g})$-module M with the property that $a_{(n)} v = 0$ for $a \in \mathfrak{g}$, $v \in M$ and $n \gg 0$. If M is an irreducible $\hat{\mathscr{L}}'(\mathfrak{g})$-module then M is an irreducible $V_k(\mathfrak{g})$-module. (see [24, 37, 42]).

3.3 Twisted modules over vertex algebras

We devote this section to the discussion of twisted modules over vertex algebras. The reader should compare the definition of twisted modules over vertex algebras to that of untwisted modules over vertex algebras in Section 3.2.

3.3.1 Definition of twisted modules over vertex algebras

Definition 3.3.1. Let V be a vertex algebra and σ be an automorphism of V of order N. We can write V as a direct sum of subspaces

$$V = \bigoplus_{j\in\mathbb{Z}/N\mathbb{Z}} V_j, \qquad \text{where } V_j = \{a \in V \mid \sigma a = e^{-2\pi \mathrm{i} j/N} a\}.$$

A σ**-twisted** V**-module** is a vector space M endowed with a linear map $Y^M\colon V \to (\mathrm{End} M)[[z^{1/N}, z^{-1/N}]]$ where

$$Y^M(a,z) = \sum_{m\in\frac{1}{N}\mathbb{Z}} a^M_{(m)} z^{-m-1}, \qquad a \in V, \tag{3.24}$$

subject to the following axioms. First, for every $a \in V$, $v \in M$, we have

$$a^M_{(m)} v = 0, \quad \text{for } m \gg 0. \tag{3.25}$$

Next, $Y^M(\mathbf{1}, z) = I$ and

$$Y^M(\sigma a, z) = Y^M(a, e^{2\pi \mathrm{i}} z), \tag{3.26}$$

where the meaning of the right-hand side is that we replace z^{-m-1} with $e^{-2\pi\mathrm{i}(m+1)} z^{-m-1}$ in each summand of Equation (3.24) [16, 19]. Explicitly, Equation (3.26) means that if a is an eigenvector of σ, then in Equation (3.26) we only have terms with $m \in \frac{1}{N}\mathbb{Z}$ such that $\sigma a = e^{-2\pi\mathrm{i} m} a$. Finally, the σ**-twisted Borcherds identity** must be satisfied for any $a, b \in V$, $c \in M$, $k \in \mathbb{Z}$, $m, n \in \frac{1}{N}\mathbb{Z}$:

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{m}{j} (a_{(k+j)} b)^M_{(m+n-j)} c = \sum_{i=0}^{\infty} \binom{k}{i} (-1)^i a^M_{(m+k-i)} (b^M_{(n+i)} c) \\ - \sum_{i=0}^{\infty} \binom{k}{i} (-1)^{k+i} b^M_{(n+k-i)} (a^M_{(m+i)} c), \end{aligned} \tag{3.27}$$

provided that $\sigma a = e^{-2\pi\mathrm{i} m} a$.

If we can set $k = 0$ in the σ-twisted Borcherds identity, then we obtain the σ**-twisted commutator formula** for $a, b \in V$ and $m, n \in \frac{1}{N}\mathbb{Z}$ such that $\sigma a = e^{-2\pi\mathrm{i} m} a$:

$$[a^M_{(m)}, b^M_{(n)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)} b)^M_{(m+n-j)}. \tag{3.28}$$

Notice that the j-th product $a_{(j)} b$ in formula (3.28) corresponds to the j-th product defined in the lattice vertex algebra V and not the j-th product corresponding to the σ-twisted module over V. If σ is the identity operator, then M becomes an untwisted module over V.

The translation covariance properties in Equations (3.7) and (3.8) continue to hold for twisted modules. However, Formula (3.11) does not hold for twisted modules. It is replaced by **Bakalov's**

Formula [37]:

$$\frac{1}{k!}\partial_z^k\left((z-w)^N Y^M(a,z)Y^M(b,w)\right)c\Big|_{z=w} = Y(a_{(N-1-k)}b,w)c, \tag{3.29}$$

for all $a,b\in V,\ c\in M,\ k\geq 0$ and sufficiently large N.

Example 3.3.2. Suppose $\sigma\in \mathrm{Aut}(Q)$ where $\sigma^N = I$ and extend σ to $\mathfrak{h}=(\mathfrak{h}\otimes\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}K$ by linearity. We can lift σ to an automorphism of the Heisenberg Lie algebra $\hat{\mathfrak{h}}$ by

$$\sigma(h_{(m)}) = (\sigma h)_{(m)},\quad \sigma(K)=K,$$

and to an automorphism of the Fock space B so that $\sigma\mathbf{1}=\mathbf{1}$. Since the cocycles $\varepsilon(\alpha,\beta)$ and $\varepsilon(\sigma\alpha,\sigma\beta)$ are equivalent, there is a map $\eta\colon Q\to\{\pm 1\}$ such that

$$\eta(\alpha)\eta(\beta)\varepsilon(\alpha,\beta)=\eta(\alpha+\beta)\varepsilon(\sigma\alpha,\sigma\beta),\qquad \alpha,\beta\in Q.$$

We can choose the map η so that $\eta(\alpha)=1$ if $\sigma\alpha=\alpha$. (see [4]). We can also lift σ to an automorphism of $V_Q = B\otimes\mathbb{C}_\varepsilon[Q]$ by

$$\sigma(h_{(-m)}\mathbf{1}\otimes e^\alpha)=(\sigma h)_{(-m)}\mathbf{1}\otimes\eta(\alpha)e^{\sigma\alpha}$$

for $h\in\mathfrak{h}$. Notice that the order of the lift $\sigma\in\mathrm{Aut}(V_Q)$ is N or $2N$. The irreducible σ-twisted V_Q-modules were classified in [5].(see also [16, 41]).

Example 3.3.3. 2.8, As in Section 2.8, 2.8, let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $\sigma\in\mathrm{Aut}(\mathfrak{g})$ such that $\sigma^N=I$. Recall that

$$\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}/N\mathbb{Z}}\mathfrak{g}_j\quad\text{where}\quad \mathfrak{g}_j=\{a\in\mathfrak{g}\,|\,\sigma a = e^{2\pi \mathrm{i} j/N}a\}.$$

From Proposition 2.8.2, we can write $\sigma=\mu\varphi$ where φ is an inner automorphism and μ is a diagram automorphism. We extend σ uniquely to an automorphism of the universal affine vertex algebra $V_k(\mathfrak{g})$ by

$$\sigma(\mathbf{1})=\mathbf{1},\quad \sigma(a_{(m)})=(\sigma a)_{(m)},\qquad a\in\mathfrak{g},\ m\in\mathbb{Z}. \tag{3.30}$$

3.3.2 Vertex operator representations of affine Kac–Moody algebras

Recall from Section 2.8 that if σ is an automorphism of finite order of $\mathfrak{g}$ then we can construct the twisted affine Lie algebras $\hat{\mathscr{L}}'(\mathfrak{g},\sigma)$ and $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$. The following proposition connects $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$ to a σ-twisted $V_k(\mathfrak{g})$-module M.

Proposition 3.3.4 (cf. [37])**.** For any σ-twisted $V_k(\mathfrak{g})$-module M, the Lie algebra spanned by the modes of the fields $Y^M(a,z)$ for $a\in\mathfrak{g}$ form a representation of the twisted affine Kac–Moody algebra $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$ on M of level k.

Proof. We will define a Lie algebra homomorphism from $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$ to the modes of the fields $Y^M(a,z)$. By linearity, since σ is diagonalizable, we can assume that $a\in\mathfrak{g}$ is an eigenvector of σ.

Recall that the set of eigenvalues for σ is $\left\{e^{\frac{2\pi i j}{N}} \,|\, j \in \mathbb{Z}\right\}$. Suppose that $\sigma a = e^{2\pi \mathrm{i} j/N} a$ for some $j \in \mathbb{Z}$. Then $\sigma^{-1} a = e^{-2\pi \mathrm{i} j/N} a$. Recall that $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1})$ is given by

$$\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1}) = \bigoplus_{j \in \mathbb{Z}} \mathscr{L}(\mathfrak{g}, \sigma^{-1})_j \oplus \mathbb{C}K,$$

where

$$\mathscr{L}(\mathfrak{g}, \sigma^{-1})_j = \mathfrak{g}_j \otimes \mathbb{C}t^j, \qquad j \in \mathbb{Z},$$
$$\mathfrak{g}_j = \{a \in \mathfrak{g} \,|\, \sigma a = e^{-2\pi \mathrm{i} j/N} a\}.$$

Recall from Equation (3.24) that the the modes of the fields $Y^M(a, z)$ for $a \in \mathfrak{g}$ take the form $a^M_{(m)}$ for $m \in \frac{1}{N}\mathbb{Z}$. We define a linear map from $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1})$ to the span of these modes, by sending $a \otimes t^j$ to $a^M_{(j/N)}$. We also send K to kI as a linear operator on M.

To check that this is a Lie algebra homomorphism, we compute the commutator of modes, which is given by the σ-twisted commutator formula (3.28). Using the brackets in (3.13), we obtain

$$[a^M_{(m)}, b^M_{(n)}] = [a, b]^M_{(m+n)} + m\delta_{m,-n}(a|b)k$$

for $a, b \in \mathfrak{g}$ and $m, n \in \frac{1}{N}\mathbb{Z}$ such that $\sigma a = e^{-2\pi \mathrm{i} m} a$ and $\sigma b = e^{-2\pi \mathrm{i} n} b$. As this coincides with the Lie bracket in (2.4) in $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1})$ with $K = kI$, the claim follows. □

Remark 3.3.5. When σ is an inner automorphism of $\mathfrak{g}$, then $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1}) \simeq \hat{\mathscr{L}}'(\mathfrak{g})$ is an untwisted affine Lie algebra.

Suppose now $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ where $X = A, D, E$ (simply laced). Let Δ be its root system and $Q = \mathbb{Z}\Delta$ its root lattice. Let σ be an automorphism of the lattice Q of finite order N. Recall that we can use Q to construct the lattice vertex algebra V_Q, and in Example 3.3.2 we lifted σ to an automorphism of the lattice vertex algebra V_Q.

Corollary 3.3.6 (cf. [36, 37])**.** For any σ-twisted V_Q-module M, the modes of the free bosons $Y^M(h, z)$ for $h \in \mathfrak{h}$ and the modes of the vertex operators $Y^M(e^\alpha, z)$ for $\alpha \in \Delta$ span a representation of the twisted affine Kac–Moody algebra $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1})$ on M.

Proof. By the Frenkel–Kac construction [22], the lattice vertex algebra V_Q is isomorphic to the simple affine vertex algebra $V^1(\mathfrak{g})$ (see [34, Theorem 5.6 (c)]). The map σ induces an automorphism of $\mathfrak{g}$ and hence of $V^1(\mathfrak{g})$. Recall that $V^1(\mathfrak{g})$ is a quotient of the universal affine vertex algebra $V_1(\mathfrak{g})$. Thus any σ-twisted $V^1(\mathfrak{g})$-module M is also a σ-twisted module for $V_1(\mathfrak{g})$. The claim then follows from Proposition 3.3.4. □

CHAPTER

4

VERTEX OPERATOR REPRESENTATIONS OF TOROIDAL LIE ALGEBRAS

In this chapter, we construct representations of twisted toroidal Lie algebras using twisted modules over vertex algebras. We start by defining toroidal Lie algebras and twisted toroidal Lie algebras. Next, we build on the information presented in Chapter 3 to construct representations of twisted toroidal Lie algebras.

4.1 Toroidal Lie Algebras

We now provide the definitions of toroidal Lie algebras and twisted toroidal Lie algebras. The reader should notice a connection between these definitions and that of twisted and untwisted affine Kac–Moody algebras presented in Sections 2.7 and 2.8.

4.1.1 Untwisted toroidal Lie algebras

Let $\mathfrak{g}$ be a fixed simple finite-dimensional Lie algebra equipped with a Lie bracket $[\cdot,\cdot]$ and a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot)$. Let r be a fixed integer. We will use variables $t_0, t_1, \dots, t_r$ and multi-index notation

$$\boldsymbol{t} = (t_1, \dots, t_r), \quad \boldsymbol{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r, \quad \boldsymbol{t}^{\boldsymbol{m}} = t_1^{m_1} \cdots t_r^{m_r}.$$

Definition 4.1.1. Consider the loop algebra $\mathscr{L}_{r+1}(\mathfrak{g})$ in $r+1$ variables

$$\mathscr{L}_{r+1}(\mathfrak{g}) = \mathfrak{g} \otimes \mathscr{O}, \qquad \mathscr{O} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \dots, t_r^{\pm 1}],$$

where the Lie bracket $[\cdot,\cdot]$ is extended to $\hat{\mathscr{L}}(\mathfrak{g})$ in the following way ($a, b \in \mathfrak{g}$, $m_0, n_0 \in \mathbb{Z}$, $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^r$):

$$[a \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, b \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = [a, b] \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}.$$

Next, we create a central extension $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ of $\mathscr{L}_{r+1}(\mathfrak{g})$:

$$\hat{\mathscr{L}}'_{r+1}(\mathfrak{g}) = \mathscr{L}_{r+1}(\mathfrak{g}) \oplus \mathscr{K},$$

where

$$\mathscr{K} = \Big(\bigoplus_{i=0}^{r} \mathbb{C} K_i \otimes \mathscr{O}\Big) \Big/ \operatorname{span}_{\mathbb{C}} \Big\{ \sum_{i=0}^{r} m_i K_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}} \;\Big|\; m_i \in \mathbb{Z} \Big\}. \tag{4.1}$$

The Lie brackets in $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ are given by:

$$[a \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, b \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = [a, b] \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} + (a|b) \sum_{i=0}^{r} m_i K_i \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}, \tag{4.2}$$

$$[\mathscr{K}, \hat{\mathscr{L}}'_{r+1}(\mathfrak{g})] = 0. \tag{4.3}$$

By Kassel's Theorem, the central extension $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ of $\mathscr{L}_{r+1}(\mathfrak{g})$ is universal [40] (setting $K_i = t_i^{-1} d t_i$ allows us to identify $\mathscr{K}$ with $\Omega^1 / d\mathscr{O}^1$). The Lie algebra $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ is known as the **toroidal Lie algebra** [47].

Remark 4.1.2. Notice that when $r = 0$, we have

$$\hat{\mathscr{L}}_1(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t_0^{\pm 1}] \oplus \mathscr{K},$$

where $\mathscr{K} = \Big(\mathbb{C} K_0 \otimes \mathbb{C}[t_0^{\pm 1}]\Big) \Big/ \operatorname{span}_{\mathbb{C}} \Big\{ \sum_{i=0}^{r} m_0 K_0 \otimes t_0^{m_0} \;\Big|\; m_0 \in \mathbb{Z} \Big\}$. We can see that $\mathscr{K} \simeq \mathbb{C} K_0$ and $\hat{\mathscr{L}}'_1(\mathfrak{g})$ is the affine algebra $\hat{\mathscr{L}}'(\mathfrak{g})$ defined in Section 2.7.

It will be convenient to slightly modify the definition of $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ as follows. For a given complex number k (called the **level**), we replace the bracket (4.2) with

$$[a \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, b \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = [a, b] \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} + k(a|b) \sum_{i=0}^{r} m_i K_i \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}. \tag{4.4}$$

The resulting Lie algebra $\mathscr{L}_{r+1}(\mathfrak{g}) \oplus \mathscr{K}$ with bracket (4.4) and (4.3) will be denoted as $\hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g})$ and called the **toroidal Lie algebra of level** k. Notice that, for $k \neq 0$, formulas (4.2) and (4.4) are equivalent after rescaling the bilinear form $(\cdot|\cdot)$ or rescaling the central elements $K_0, \dots, K_r$.

Now we will add derivations to our toroidal Lie algebra, as in [12]. We let $\mathscr{D}$ be the Lie algebra of derivations of $\mathscr{O}$ given by

$$\mathscr{D} = \Big\{ \sum_{i=0}^{r} d_i \otimes f_i \,\Big|\, f_i \in \mathscr{O} \Big\}, \qquad d_i = t_i \frac{\partial}{\partial t_i},$$

and $\mathscr{D}_+$ be the subalgebra of $\mathscr{D}$ given by

$$\mathscr{D}_+ = \Big\{ \sum_{i=1}^{r} d_i \otimes f_i \,\Big|\, f_i \in \mathscr{O} \Big\}.$$

Then the elements of $\mathscr{D}_+$ extend uniquely to derivations of the Lie algebra $\hat{\mathscr{L}}'_{r+1}(\mathfrak{g})$ by ($a \in \mathfrak{g}$, $1 \le i, j \le r$):

$$(d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}})(a \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}) = n_i a \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}},$$
$$(d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}})(K_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}) = n_i K_j \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} + \delta_{i,j} \sum_{l=0}^{r} m_l K_l \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}.$$

The Lie algebra we will consider in this thesis, which we will refer to again as a **toroidal Lie algebra of level** k, is

$$\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g}) = \hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g}) \oplus \mathscr{D}_+,$$

with the Lie brackets given by (4.4), (4.3), and ($a \in \mathfrak{g}$, $m_l, n_l \in \mathbb{Z}$, $1 \le i, j \le r$):

$$[d_i \otimes {t_0}^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, a \otimes {t_0}^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = n_i a \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}, \tag{4.5}$$

$$[d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, K_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = n_i K_j \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} + \delta_{i,j} \sum_{l=0}^{r} m_l K_l \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}, \tag{4.6}$$

$$\begin{aligned}[d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, d_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = n_i d_j \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} &- m_j d_i \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} \\ &- n_i m_j \sum_{l=0}^{r} m_l K_l \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}.\end{aligned} \tag{4.7}$$

The last term in (4.7) corresponds to a $\mathscr{K}$-valued 2-cocycle on $\mathscr{D}_+$. In fact, we can define the bracket in (4.7) more generally as

$$\begin{aligned}[d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, d_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}] = n_i d_j \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} &- m_j d_i \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}} \\ &+ \tau\big(d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, d_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}\big)\end{aligned}$$

where τ is any linear combination of the two cocycles τ_1 and τ_2 defined by

$$\tau_1\big(d_i \otimes t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}}, d_j \otimes t_0^{n_0} \boldsymbol{t}^{\boldsymbol{n}}\big) = -m_j n_i \sum_{l=0}^{r} m_l K_l \otimes t_0^{m_0+n_0} \boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}$$

and

$$\tau_2\big(d_i \otimes t_0^{m_0}\boldsymbol{t}^{\boldsymbol{m}}, d_j \otimes t_0^{n_0}\boldsymbol{t}^{\boldsymbol{n}}\big) = -m_i n_j \sum_{l=0}^{r} m_l K_l \otimes t_0^{m_0+n_0}\boldsymbol{t}^{\boldsymbol{m}+\boldsymbol{n}}.$$

In our definition of a toroidal Lie algebra of level k, we will only consider $\tau = \tau_1 + 0\tau_2$. For more information on derivations of toroidal Lie algebras and 2-cocycles, see [8].

Notice that, for $k \neq 0$, if we rescale the generators $K_0, \dots, K_r$ in order to replace (4.4) with (4.2), the 2-cocycle gets rescaled by $1/k$.

4.1.2 Twisted toroidal Lie algebras

As before, fix a level $k \in \mathbb{C}$, and let σ be an automorphism of order N of a simple finite-dimensional Lie algebra $\mathfrak{g}$. As in (3.29), we denote by $\mathfrak{g}_j$ ($j \in \mathbb{Z}$) the eigenspace of σ with eigenvalue $e^{2\pi \mathrm{i} j/N}$. The nondegenerate symmetric invariant bilinear form $(\cdot|\cdot)$ we associated with $\mathfrak{g}$ is σ-invariant. That is, $(a|b) = (\sigma a|\sigma b)$ for all $a, b \in \mathfrak{g}$. If we choose $a \in \mathfrak{g}_m$ and $b \in \mathfrak{g}_n$, then the σ-invariance of $(\cdot|\cdot)$ gives us

$$\begin{aligned}(a|b) &= (\sigma a|\sigma b)\\ &= \big(e^{2\pi \mathrm{i} m/N} a | e^{2\pi \mathrm{i} n/N} b\big)\\ &= e^{2\pi \mathrm{i}(m+n)/N}(a|b)\end{aligned}$$

which implies $e^{2\pi \mathrm{i}(m+n)/N} = 1$ or $(a|b) = 0$. Therefore, we have

$$(a|b) = 0, \qquad a \in \mathfrak{g}_m,\ \ b \in \mathfrak{g}_n,\ \ m+n \not\equiv 0 \mod N. \tag{4.8}$$

Consider the subalgebra $\mathscr{L}_{r+1}(\mathfrak{g},\sigma)$ of the loop algebra $\mathscr{L}_{r+1}(\mathfrak{g})$ given by

$$\mathscr{L}_{r+1}(\mathfrak{g},\sigma) = \bigoplus_{m_0 \in \mathbb{Z}} \mathscr{L}_{r+1}(\mathfrak{g},\sigma)_{m_0}$$

where

$$\mathscr{L}_{r+1}(\mathfrak{g},\sigma)_{m_0} = \mathfrak{g}_{m_0} \otimes \operatorname{span}_{\mathbb{C}}\big\{ t_0^{m_0}\boldsymbol{t}^{\boldsymbol{m}} \,\big|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r \big\}.$$

Let

$$\mathscr{K}' = \frac{\operatorname{span}_{\mathbb{C}}\big\{K_i \otimes t_0^{Nm_0}\boldsymbol{t}^{\boldsymbol{m}} \,\big|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r,\ i = 0,\dots,r\big\}}{\operatorname{span}_{\mathbb{C}}\big\{\big(Nm_0K_0 + \sum_{i=1}^r m_i K_i\big) \otimes t_0^{Nm_0}\boldsymbol{t}^{\boldsymbol{m}} \,\big|\, m_i \in \mathbb{Z}\big\}}.$$

We can identify $\mathscr{K}'$ as the subspace of $\mathscr{K}$ given by the image of

$$\operatorname{span}_{\mathbb{C}}\big\{K_i \otimes t_0^{Nm_0}\boldsymbol{t}^{\boldsymbol{m}} \,\big|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r,\ i = 0,\dots,r\big\}$$

under the quotient map $\bigoplus_{i=0}^r \mathbb{C}K_i \otimes \mathscr{O} \to \mathscr{K}$ (cf. (4.1)). Then the central extension

$$\hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g},\sigma) = \mathscr{L}_{r+1}(\mathfrak{g},\sigma) \oplus \mathscr{K}' \subset \hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g})$$

is a subalgebra of $\hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g})$, thanks to (4.4), (4.8).

Definition 4.1.3. When σ is a diagram automorphism of $\mathfrak{g}$, $\hat{\mathscr{L}}'_{r+1,1}(\mathfrak{g},\sigma)$ is known as the **twisted toroidal Lie algebra** [25]. We will continue to use that terminology for an arbitrary finite-order automorphism σ of $\mathfrak{g}$.

As in Section 4.1.1, we can add to $\hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g},\sigma)$ a subalgebra of the Lie algebra of derivations $\mathscr{D}$ of $\mathscr{O}$. We define

$$\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma) = \hat{\mathscr{L}}'_{r+1,k}(\mathfrak{g},\sigma) \oplus \mathscr{D}'_{+} \tag{4.9}$$

where

$$\mathscr{D}'_{+} = \operatorname{span}_{\mathbb{C}}\left\{ d_i \otimes t_0^{N m_0} \boldsymbol{t}^{\boldsymbol{m}} \,\middle|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r,\ i = 1, \dots, r \right\}.$$

It is easy to see from (4.5)–(4.7) that $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma)$ is a subalgebra of the toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g})$. We will also call $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma)$ the **twisted toroidal Lie algebra of level** k.

4.2 Tensor products of vertex algebras and twisted modules

Our construction of vertex operator representations of toroidal Lie algebras will involve twisted modules over a tensor product of vertex algebras. We review tensor products of vertex algebras and twisted modules over tensor products in this section.

Definition 4.2.1. The **tensor product** $V_1 \otimes V_2$ of two vertex algebras V_1 and V_2 is a vertex algebra [21] with a vacuum vector $\mathbf{1} \otimes \mathbf{1}$ and a state-field correspondence given by ($a \in V_1$, $b \in V_2$):

$$Y(a \otimes b, z) = Y(a,z) \otimes Y(b,z) = \sum_{k,m \in \mathbb{Z}} a_{(k)} \otimes b_{(m)} z^{-k-m-2}. \tag{4.10}$$

In terms of modes, we have

$$(a \otimes b)_{(n)} = \sum_{k \in \mathbb{Z}} a_{(k)} \otimes b_{(n-k-1)}. \tag{4.11}$$

Recall that in a vertex algebra we can define a translation operator $T \in \operatorname{End}(V_1 \otimes V_2)$ such that $T(a \otimes b) = (a \otimes b)_{(-2)}(\mathbf{1} \otimes \mathbf{1})$. By Equation (4.11),

$$\begin{aligned}
(a \otimes b)_{(-2)} \mathbf{1} \otimes \mathbf{1} &= \sum_{k \in \mathbb{Z}} a_{(k)} \otimes b_{(-2-k-1)} (\mathbf{1} \otimes \mathbf{1}) \\
&= \sum_{k \in \mathbb{Z}} a_{(k)} \mathbf{1} \otimes b_{(n+2-1)} \mathbf{1}
\end{aligned}$$

and from the vacuum axiom in Equation (3.3),

$$\begin{aligned}
(a \otimes b)_{(-2)} \mathbf{1} \otimes \mathbf{1} &= a_{(-2)} \mathbf{1} \otimes b_{(-1)} \mathbf{1} + a_{(-1)} \mathbf{1} \otimes b_{(-2)} \mathbf{1} \\
&= a_{(-2)} \mathbf{1} \otimes b + a \otimes b_{(-2)} \mathbf{1} \\
&= (T_1 \otimes I_2 + I_1 \otimes T_2)(a \otimes b)
\end{aligned}$$

where T_i is the translation operator and I_i is the identity operator in V_i for $i = 1, 2$. Therefore the translation operator T in $V_1 \otimes V_2$ is

$$T = T_1 \otimes I_2 + I_1 \otimes T_2.$$

Example 4.2.2. Let V_1 and V_2 be vertex algebras with finite-order automorphisms σ_1 and σ_2, respectively. Suppose M_i $(i = 1, 2)$ are σ_i-twisted V_i-modules. Then $\sigma_1 \otimes \sigma_2$ is an automorphism of the tensor product vertex algebra $V_1 \otimes V_2$ and the tensor product $M_1 \otimes M_2$ is a $\sigma_1 \otimes \sigma_2$-twisted module over $V_1 \otimes V_2$ with

$$Y^{M_1 \otimes M_2}(a \otimes b, z) = Y^{M_1}(a, z) \otimes Y^{M_2}(b, z), \qquad a \in V_1,\ b \in V_2, \tag{4.12}$$

see [4, 21].

Remark 4.2.3. Let Q_1 and Q_2 be two integral lattices of rank ℓ_1 and ℓ_2, respectively. Let $\hat{Q}$ be the orthogonal direct sum of Q_1 and Q_2 of rank $\ell_1 + \ell_2$. That is, let $\hat{Q} = Q_1 \oplus Q_2$ such that $(a|b) = 0$ for any $a \in Q_1$ and $b \in Q_2$. Then the lattice vertex algebra $V_{\hat{Q}} \simeq V_{Q_1} \otimes V_{Q_2}$ (see [23] and [42]).

4.3 Twisted modules over $V_k(\mathfrak{g}) \otimes V_J$ and twisted toroidal Lie algebras

We will now explore the relationship between twisted toroidal Lie algebras in $r+1$ variables and the tensor product of the universal affine vertex algebra $V_k(\mathfrak{g})$ with the lattice vertex algebra corresponding to r copies of a certain rank-2 lattice. The level $k \in \mathbb{C}$ will be fixed through the end of the section.

As before, consider an automorphism σ of finite order N of a simple finite-dimensional Lie algebra $\mathfrak{g}$, and the twisted toroidal Lie algebra (of level k) $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g}, \sigma)$ defined by (4.9). For $i = 1, \dots, r$, let J_i be the lattice given by

$$J_i = \mathbb{Z}\delta^i \oplus \mathbb{Z}\Lambda_0^i, \qquad (\delta^i|\Lambda_0^i) = 1, \quad (\delta^i|\delta^i) = (\Lambda_0^i|\Lambda_0^i) = 0. \tag{4.13}$$

We define a bimultiplicative function $\varepsilon \colon J_i \times J_i \to \{\pm 1\}$ satisfying Equation (3.16) by $\varepsilon(\delta^i, \Lambda_0^i) = -1$ and $\varepsilon = 1$ for all other pairs of generators. Note that J_i is an even integral lattice. Then we can form the lattice vertex algebra V_{J_i} as in Subsection 3.1.2.

Introduce the orthogonal direct sum

$$J = J_1 \oplus \cdots \oplus J_r, \tag{4.14}$$

and extend ε to $J \times J$ by $\varepsilon(\delta^i, \Lambda_0^j) = -1$ for $i = j$ and $\varepsilon = 1$ for all other pairs of generators. Then the lattice vertex algebra V_J is isomorphic to the tensor product:

$$V_J \simeq V_{J_1} \otimes \cdots \otimes V_{J_r}. \tag{4.15}$$

As preparation for our main theorem, we need to calculate some n-th products in V_J. We will use the notation

$$\boldsymbol{p}\boldsymbol{\delta} = \sum_{i=1}^{r} p_i \delta^i, \qquad \boldsymbol{p} = (p_1, \dots, p_r) \in \mathbb{Z}^r.$$

Lemma 4.3.1. The lattice vertex algebra V_J has the following n-th products for $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{Z}^r$ and $i, j = 1, \dots, r$:

$$\begin{aligned}
e^{\boldsymbol{p}\boldsymbol{\delta}}_{(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}} &= e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
e^{\boldsymbol{p}\boldsymbol{\delta}}_{(-2)} e^{\boldsymbol{q}\boldsymbol{\delta}} &= (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(u_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)} \left(v_{(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= 0 \ \text{ for } \ n \geq 0, \ \ u, v \in \{\mathbf{1}, \delta^1, \dots, \delta^r\}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(0)} e^{\boldsymbol{q}\boldsymbol{\delta}} &= q_i \, e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)} e^{\boldsymbol{q}\boldsymbol{\delta}} &= 0 \ \text{ for } \ n \geq 1, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(0)} \left(\delta^j_{(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= q_i \delta^j_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} + \delta_{i,j} (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(1)} \left(\delta^j_{(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= \delta_{i,j} \, e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)} \left(\delta^j_{(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= 0 \ \text{ for } \ n \geq 2, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(0)} \left(\Lambda^j_{0(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= \left(-p_j \Lambda^i_{0(-1)} + q_i \Lambda^j_{0(-1)} - q_i p_j (\boldsymbol{p}\boldsymbol{\delta})_{(-1)}\right) e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(1)} \left(\Lambda^j_{0(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= -q_i p_j \, e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}, \\
\left(\Lambda^i_{0(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)} \left(\Lambda^j_{0(-1)} e^{\boldsymbol{q}\boldsymbol{\delta}}\right) &= 0 \ \text{ for } \ n \geq 2.
\end{aligned}$$

Proof. The n-th product $\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)} \left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)$ is the coefficient of z^{-n-1} in $Y\left(e^{\boldsymbol{p}\boldsymbol{\delta}}, z\right) e^{\boldsymbol{q}\boldsymbol{\delta}}$. We use Equation (3.19) to obtain:

$$Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}} = e^{\boldsymbol{p}\boldsymbol{\delta}} z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}} \exp\left(\sum_{m>0} (\boldsymbol{p}\boldsymbol{\delta})_{(-m)} \frac{z^m}{m} \right) \exp\left(\sum_{m>0} (\boldsymbol{p}\boldsymbol{\delta})_{(m)} \frac{z^{-m}}{-m} \right) e^{\boldsymbol{q}\boldsymbol{\delta}}.$$

For $\mathfrak{h} = \mathbb{C} \otimes J$, the vector $e^{\boldsymbol{q}\boldsymbol{\delta}}$ commutes with the exponentials in $Y\left(e^{\boldsymbol{p}\boldsymbol{\delta}}, z\right)$ due to the action (3.18) and $\exp\left(\sum_{m>0} (\boldsymbol{p}\boldsymbol{\delta})_{(m)} \frac{z^{-m}}{-m}\right) \mathbf{1} = 1$. From this we are able to simplify $Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}}$ to the following:

$$Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}} = e^{\boldsymbol{p}\boldsymbol{\delta}} z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}} \exp\left(\sum_{m>0} (\boldsymbol{p}\boldsymbol{\delta})_{(-m)} \frac{z^m}{m} \right) e^{\boldsymbol{q}\boldsymbol{\delta}}.$$

By definition of the operator $z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}}$, we have $z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}} e^{\boldsymbol{q}\boldsymbol{\delta}} = z^{(\boldsymbol{p}\boldsymbol{\delta}|\boldsymbol{q}\boldsymbol{\delta})} e^{\boldsymbol{q}\boldsymbol{\delta}}$. Since $(\boldsymbol{p}\boldsymbol{\delta}|\boldsymbol{q}\boldsymbol{\delta}) = 0$, we obtain $z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}} e^{\boldsymbol{q}\boldsymbol{\delta}} = e^{\boldsymbol{q}\boldsymbol{\delta}}$. From (3.17), the product $e^{\boldsymbol{p}\boldsymbol{\delta}} e^{\boldsymbol{q}\boldsymbol{\delta}}$ can be expressed as $\varepsilon(\boldsymbol{p}\boldsymbol{\delta}, \boldsymbol{q}\boldsymbol{\delta}) e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} = e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}$. Moreover, by the definition of the map ε, we get $\varepsilon(\delta^i, \delta^j) = 1$, which results in $\varepsilon(\boldsymbol{p}\boldsymbol{\delta}, \boldsymbol{q}\boldsymbol{\delta}) = 1$. This leads to

$$Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}} = \exp\left(\sum_{m>0} (\boldsymbol{p}\boldsymbol{\delta})_{(-m)} \frac{z^m}{m} \right) e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.$$

Notice that this sum can be expanded as follows:

$$Y(e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}} = \exp\left(1+(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}\frac{z^1}{1}+\ \dots\ \right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} \tag{4.16}$$

where the dots refer to higher powers of z. To evaluate $\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}\left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)$ for $n=-1$, we look for the coefficient in front of $z^{-(-1)-1}=z^0$ in $Y(e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}$, which gives us

$$\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(-1)}\left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)=e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.$$

For $\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}\left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)$ when $n=-2$, we look for the coefficient of $z^{-(-2)-1}=z^1$ in $Y(e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}$, and we find

$$\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(-2)}\left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)=(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.$$

Notice that there are no negative powers of z in Equation 4.16, so $\left(e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}\left(e^{\boldsymbol{q}\boldsymbol{\delta}}\right)=0,$ for $n\geq 0$.

The remaining n-th products are found in a similar fashion. We can find $\left(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}\left(v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\right)$ for $n\geq 0$ and $u,v\in\{\mathbf{1},\delta^1,\dots,\delta^r\}$ when we consider the coefficient of z^{-n-1} in $Y\left(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z\right)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}$ as shown

$$\begin{aligned}
Y\left(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z\right)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}} &= \left(\sum_{l\in\mathbb{Z}}u_{(l)}z^{-l-1}\right)e^{\boldsymbol{p}\boldsymbol{\delta}}z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(-m)}\frac{z^m}{m}\right)\\
&\qquad\times\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(m)}\frac{z^{-m}}{-m}\right)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\\
&= v_{(-1)}\left(\sum_{l\in\mathbb{Z}_{>0}}u_{(-l)}z^{l-1}\right)z^0\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&= v_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\left(1+u_{(-1)}z^0+u_{(-2)}z^1+\ \dots\ \right)\left(1+(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}\frac{z^1}{1}+\ \dots\ \right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.
\end{aligned}$$

This sum has no negative powers of z. Therefore $\left(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}\left(v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\right)=0$ for $n\geq 0$.

Next, the n-th product $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(n)}e^{\boldsymbol{q}\boldsymbol{\delta}}$ is the coefficient of z^{-n-1} in $Y\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z\right)e^{\boldsymbol{q}\boldsymbol{\delta}}$ as shown below,

$$\begin{aligned}
&Y(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}\\
&\quad=:Y(\Lambda^i_0,z)Y(e^{\boldsymbol{p}\boldsymbol{\delta}},z):e^{\boldsymbol{q}\boldsymbol{\delta}}\\
&\quad=\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)e^{\boldsymbol{p}\boldsymbol{\delta}}z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(m)}\frac{z^{-m}}{-m}\right)e^{\boldsymbol{q}\boldsymbol{\delta}}\\
&\qquad+e^{\boldsymbol{p}\boldsymbol{\delta}}z^{(\boldsymbol{p}\boldsymbol{\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p}\boldsymbol{\delta})_{(m)}\frac{z^{-m}}{-m}\right)\left(\sum_{l\geq 0}\Lambda^i_{0(l)}z^{-l-1}\right)e^{\boldsymbol{q}\boldsymbol{\delta}},
\end{aligned}$$

which we can simplify to the following:

$$
\begin{aligned}
&Y(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}, z)e^{\boldsymbol{q\delta}}\\
&\qquad=\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad\qquad+e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)(\Lambda^i_0|\boldsymbol{q\delta})e^{\boldsymbol{q\delta}}z^{-1}\\
&\qquad=\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}+q_iz^{-1}z^0\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad=\left(\Lambda^i_{0(-1)}z^0+\Lambda^i_{0(-2)}z^1+\dots\right)\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\dots\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad\qquad+q_iz^{-1}\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\dots\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.
\end{aligned}
$$

For $n=0$, the coefficient of z^{-1} is $q_ie^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}$. Thus, $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(0)}e^{\boldsymbol{q\delta}}=q_ie^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}$. Since there are no negative powers of z, then for $n\geq 0$, we have $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(n)}e^{\boldsymbol{q\delta}}=0$.

Now we will consider the n-th product $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(n)}\left(\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}\right)$. This is the coefficient of z^{-n-1} in $Y\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}},z\right)\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}$ as shown:

$$
\begin{aligned}
&Y(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}, z)\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}\\
&\qquad=:Y(\Lambda^i_0,z)Y(e^{\boldsymbol{p\delta}},z):\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}\\
&\qquad=\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}\\
&\qquad\qquad+e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\left(\sum_{l\geq 0}\Lambda^i_{0(l)}z^{-l-1}\right)\delta^j{}_{(-1)}e^{\boldsymbol{q\delta}}\\
&\qquad=\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)\delta^j{}_{(-1)}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad\qquad+e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\left((\Lambda^i_0|\boldsymbol{q\delta})\delta^j{}_{(-1)}z^{-1}+(\Lambda^i_0|\delta^j)z^{-2}\right)e^{\boldsymbol{q\delta}}\\
&\qquad=\delta^j{}_{(-1)}\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad\qquad+\left(q_i\delta^j{}_{(-1)}z^{-1}+\delta_{i,j}z^{-2}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad=\delta^j{}_{(-1)}\left(\Lambda^i_{0(-1)}z^0+\Lambda^i_{0(-2)}z^1+\dots\right)\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\dots\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\qquad\qquad+\left(q_i\delta^j{}_{(-1)}z^{-1}+\delta_{i,j}z^{-2}\right)\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\dots\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.
\end{aligned}
$$

For $n=0$, the coefficient of z^{-1} is $\left(q_i\delta^j_{0(-1)}+\delta_{i,j}(\boldsymbol{p\delta})_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}$. For $n=1$, the coefficient of z^{-2} is $\delta_{i,j}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}$. Also, there are no powers of z less than -2. Thus,

$$\begin{aligned}
\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(0)}\left(\delta^j_{(-1)}e^{\boldsymbol{q\delta}}\right) &= q_i\delta^j_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}+\delta_{i,j}(\boldsymbol{p\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}},\\
\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(1)}\left(\delta^j_{(-1)}e^{\boldsymbol{q\delta}}\right) &= \delta_{i,j}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}},\\
\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(n)}\left(\delta^j_{(-1)}e^{\boldsymbol{q\delta}}\right) &= 0 \quad \text{for} \quad n\geq 2,
\end{aligned}$$

Finally, the n-th product $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(n)}\left(\Lambda_0{}^j{}_{(-1)}e^{\boldsymbol{q\delta}}\right)$ is the coefficient of z^{-n-1} in $Y\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}},z\right)\Lambda_0{}^j{}_{(-1)}e^{\boldsymbol{q\delta}}$ as shown:

$$\begin{aligned}
&Y(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}},z)\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\\
&\quad =:Y(\Lambda^i_0,z)Y(e^{\boldsymbol{p\delta}},z):\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\\
&\quad =\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\\
&\quad +e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\left(\sum_{l\geq 0}\Lambda^i_{0(l)}z^{-l-1}\right)\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\\
&\quad =\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\left(\Lambda^j_{0(-1)}-\left(\boldsymbol{p\delta}|\Lambda^j_0\right)z^{-1}\right)e^{\boldsymbol{q\delta}}\\
&\quad +e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(m)}\frac{z^{-m}}{-m}\right)\left(\Lambda^i_0|\boldsymbol{q\delta}\right)\Lambda^j_{0(-1)}z^{-1}e^{\boldsymbol{q\delta}}\\
&\quad =\left(\Lambda^j_{0(-1)}-p_jz^{-1}\right)\left(\sum_{l<0}\Lambda^i_{0(l)}z^{-l-1}\right)\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\quad +e^{\boldsymbol{p\delta}}z^{(\boldsymbol{p\delta})_{(0)}}\exp\left(\sum_{m>0}(\boldsymbol{p\delta})_{(-m)}\frac{z^m}{m}\right)\left(q_i\Lambda^j_{0(-1)}z^{-1}-q_i\left(\boldsymbol{p\delta}|\Lambda^j_0\right)z^{-2}\right)e^{\boldsymbol{q\delta}}.
\end{aligned}$$

From here, we obtain

$$\begin{aligned}
&Y(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}},z)\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\\
&\quad =\left(\Lambda^j_{0(-1)}-p_jz^{-1}\right)\left(\Lambda^i_{0(-1)}z^0+\Lambda^i_{0(-2)}z^1+\ \dots\ \right)\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\ \dots\ \right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\\
&\quad +\left(q_i\Lambda^j_{0(-1)}z^{-1}-q_ip_jz^{-2}\right)\left(1+(\boldsymbol{p\delta})_{(-1)}\frac{z^1}{1}+(\boldsymbol{p\delta})_{(-2)}\frac{z^2}{2}+\ \dots\ \right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.
\end{aligned}$$

To find $\left(\Lambda^i_{0(-1)}e^{\boldsymbol{p\delta}}\right)_{(n)}\left(\Lambda^j_{0(-1)}e^{\boldsymbol{q\delta}}\right)$ for $n\geq 0$, we extract the terms with negative powers of z from the above expression

$$\left(-p_j\Lambda^i_{0(-1)}z^{-1}+q_i\Lambda^j_{0(-1)}z^{-1}-q_ip_jz^{-2}-q_ip_j(\boldsymbol{p\delta})_{(-1)}z^{-1}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}.$$

This gives us

$$\begin{aligned}
&\big(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)_{(0)}\big(\Lambda^j_{0(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)=\big(-p_j\Lambda^i_{0(-1)}+q_i\Lambda^j_{0(-1)}-q_ip_j(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}\big)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}},\\
&\big(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)_{(1)}\big(\Lambda^j_{0(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)=-q_ip_j\,e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}},\\
&\big(\Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)_{(n)}\big(\Lambda^j_{0(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)=0\quad\text{for}\ \ n\geq 2.
\end{aligned}$$

This completes the proof of the lemma. □

Recall the universal affine vertex algebra $V_k(\mathfrak{g})$ of level k, defined in Subsection 3.1.1. We will now consider the tensor product of vertex algebras $V_k(\mathfrak{g})\otimes V_J$ (see Section 4.2). We extend σ from a finite order automorphsim of $\mathfrak{g}$ to a finite order automorphism σ of $V_k(\mathfrak{g})$ also of order N as described in Formula (3.30). Next, we extend σ to an automorphism of the vertex algebra $V_k(\mathfrak{g})\otimes V_J$, again of order N, by letting

$$\sigma(a\otimes b)=\sigma a\otimes b. \tag{4.17}$$

Let M be a σ-twisted $V_k(\mathfrak{g})$-module, and M' be a V_J-module (untwisted). Then $\mathscr{M}=M\otimes M'$ is a σ-twisted $V_k(\mathfrak{g})\otimes V_J$-module with a state-field correspondence $Y^{\mathscr{M}}$ given by (4.12). Now we can formulate our main theorem, which uses twisted $V_k(\mathfrak{g})\otimes V_J$-modules to create representations of the twisted toroidal Lie algebra.

Theorem 4.3.2. Let σ be an automorphism of order N of a simple finite-dimensional Lie algebra $\mathfrak{g}$, and let $\mathscr{M}=M\otimes M'$ be a σ-twisted $V_k(\mathfrak{g})\otimes V_J$-module. Then the Lie algebra of modes of the fields

$$\begin{aligned}
&Y^{\mathscr{M}}(a\otimes e^{\boldsymbol{p}\boldsymbol{\delta}},z), &&Y^{\mathscr{M}}(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}},z),\\
&Y^{\mathscr{M}}(\mathbf{1}\otimes \delta^i_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z), &&Y^{\mathscr{M}}(\mathbf{1}\otimes \Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z),
\end{aligned}$$

where $a\in\mathfrak{g}$ and $\boldsymbol{p}\in\mathbb{Z}^r$, form a representation of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma^{-1})$ of level k on $\mathscr{M}$. Explicitly, we have a Lie algebra homomorphism

$$\begin{aligned}
a\otimes t_0^m\boldsymbol{t}^{\boldsymbol{p}}&\mapsto\big(a\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(\frac{m}{N})},\\
K_0\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}&\mapsto\frac{1}{N}\big(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m-1)},\\
K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}&\mapsto\big(\mathbf{1}\otimes \delta^i_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\\
d_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}&\mapsto\big(\mathbf{1}\otimes \Lambda^i_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},
\end{aligned}\tag{4.18}$$

for $\boldsymbol{p}\in\mathbb{Z}^r$, $1\leq i\leq r$, and $a\in\mathfrak{g}$, $m\in\mathbb{Z}$ such that $\sigma a=e^{-2\pi\mathrm{i}m/N}a$.

Proof. Recall that $\mathscr{L}_{r+1}(\mathfrak{g},\sigma^{-1})_m$ for $m\in\mathbb{Z}$ is spanned by all elements of the form $a\otimes t_0^m\boldsymbol{t}^{\boldsymbol{p}}$ where $\boldsymbol{p}\in\mathbb{Z}^r$ and $a\in\mathfrak{g}$ with $\sigma^{-1}a=e^{2\pi\mathrm{i}m/N}a$. The latter condition is equivalent to $\sigma a=e^{-2\pi\mathrm{i}m/N}a$. Hence,

$\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma^{-1})$ is spanned by all elements in the left-hand side of (4.18), subject to the relations:

$$N m K_0 \otimes t_0^{Nm} \boldsymbol{t}^{\boldsymbol{p}} + \sum_{i=1}^{r} p_i K_i \otimes t_0^{Nm} \boldsymbol{t}^{\boldsymbol{p}} = 0.$$

For the map ϕ to be well defined, we need to check that

$$N m \frac{1}{N} \big(\mathbf{1} \otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m-1)} + \sum_{i=1}^{r} p_i \big(\mathbf{1} \otimes \delta^i_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)} = 0,$$

or equivalently

$$m\big(\mathbf{1} \otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m-1)} + \big(\mathbf{1} \otimes (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)} = 0.$$

This follows from the translation covariance property (3.8):

$$-m\big(\mathbf{1} \otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m-1)} = \big(T(\mathbf{1} \otimes e^{\boldsymbol{p}\boldsymbol{\delta}})\big)^{\mathscr{M}}_{(m)} = \big(\mathbf{1} \otimes T e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)} = \big(\mathbf{1} \otimes (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)}$$

where we used (3.20) and that T acts as $T_1 \otimes I_2 + I_1 \otimes T_2$ on a tensor product of vertex algebras.

To show that ϕ is a homomorphism, we need to check that the defining Lie brackets (4.3)–(4.7) of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma^{-1})$ match with the commutators of modes in the right-hand side of (4.18). The latter are determined by the commutator formula (3.28). We can apply (3.28) because $\sigma a = e^{-2\pi \mathrm{i} m/N} a$ implies $\sigma(a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}}) = e^{-2\pi \mathrm{i} m/N}(a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}})$, since σ acts as the identity on V_J. Similarly, if $b \in \mathfrak{g}$ and $n \in \mathbb{Z}$ satisfy $\sigma b = e^{-2\pi \mathrm{i} n/N} b$, then $\sigma(b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}}) = e^{-2\pi \mathrm{i} n/N}(b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ for $\boldsymbol{q} \in \mathbb{Z}^r$. Hence,

$$\Big[(a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}})^{\mathscr{M}}_{(\frac{m}{N})}, (b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}})^{\mathscr{M}}_{(\frac{n}{N})}\Big] = \sum_{l=0}^{\infty} \binom{\frac{m}{N}}{l} \big((a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)} (b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}})\big)^{\mathscr{M}}_{(\frac{m}{N} + \frac{n}{N} - l)}. \tag{4.19}$$

The l-th product $(a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ is the coefficient of z^{-l-1} in the expression

$$Y(a \otimes e^{\boldsymbol{p}\boldsymbol{\delta}}, z)(b \otimes e^{\boldsymbol{q}\boldsymbol{\delta}}) = Y(a,z)b \otimes Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}}.$$

We are only interested in the negative powers of z. By the products (3.13) in the affine vertex algebra $V_k(\mathfrak{g})$, we have

$$Y(a,z)b = (a|b)k\mathbf{1} z^{-2} + [a,b] z^{-1} + \ \ldots \tag{4.20}$$

where $\ldots$ denote terms with higher powers of z. The products in the lattice vertex algebra V_J were computed in Lemma 4.3.1. In particular,

$$Y(e^{\boldsymbol{p}\boldsymbol{\delta}}, z) e^{\boldsymbol{q}\boldsymbol{\delta}} = e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} + (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} z + \ \ldots. \tag{4.21}$$

Putting (4.20) and (4.21) together, we get

$$\begin{aligned}Y(a\otimes e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}) &= \big([a,b]z^{-1}\otimes + (a|b)k\mathbf{1}z^{-2}\big)\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}} + (a|b)k\mathbf{1}z^{-2}\otimes(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z + \dots\\ &= [a,b]\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-1} + (a|b)k\mathbf{1}\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-2} + (a|b)k\mathbf{1}\otimes(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-1} + \dots.\end{aligned}$$

Using the map (4.18), the commutator formula (4.19), and the computation above, we obtain the bracket

$$\begin{aligned}&\big[\phi\big(a\otimes t_0^m\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}\big)\big]\\ &\quad= \Big[\big(a\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{m}{N})},\big(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{n}{N})}\Big]\\ &\quad= \big([a,b]\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{m}{N}+\frac{n}{N})} + (a|b)k\big(\mathbf{1}\otimes(\boldsymbol{p}\boldsymbol{\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{m}{N}+\frac{n}{N})}\\ &\qquad + \frac{m}{N}(a|b)k\big(\mathbf{1}\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{m}{N}+\frac{n}{N}-1)}\\ &\quad= \phi\big([a,b]\otimes t_0^{m+n}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}}\big) + (a|b)k\sum_{i=1}^{r}p_i\phi\big(K_i\otimes t_0^{m+n}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}}\big)\\ &\qquad + m(a|b)k\phi\big(K_0\otimes t_0^{m+n}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}}\big).\end{aligned}$$

By (4.4), the last expression is exactly $\phi([a\otimes t_0^m\boldsymbol{t}^{\boldsymbol{p}}, b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}])$. Next, for $u\in\{\mathbf{1},\delta^i\}$ we have

$$\Big[\big(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(m)},\big(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{n}{N})}\Big] = \sum_{l=0}^{\infty}\binom{m}{l}\big((\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})\big)^{\mathcal{M}}_{(m+\frac{n}{N}-l)}$$

and the l-th product $(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ is the coefficient of z^{-l-1} in the expression

$$\begin{aligned}Y(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}) &= Y(\mathbf{1},z)b\otimes Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}\\ &= bz^0\otimes Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}\end{aligned}$$

and from Lemma 4.3.1, $Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}$ has no negative powers of z. Hence, the expression $Y(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ will have no negative powers of z and

$$\Big[\big(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(m)},\big(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{n}{N})}\Big] = 0.$$

From (4.18) and the computation above, we see that

$$\begin{aligned}\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}\big)\big] &= \Big[\frac{1}{N}\big(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(m)},\big(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{n}{N})}\Big] = 0,\\ \big[\phi\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}\big)\big] &= \Big[\big(\mathbf{1}\otimes\delta^i_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(m)},\big(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathcal{M}}_{(\frac{n}{N})}\Big] = 0,\end{aligned}$$

and since $\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}\big)\big] = \big[\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(b\otimes t_0^n\boldsymbol{t}^{\boldsymbol{q}}\big)\big] = 0$ in the twisted toroidal Lie algebra, then we get the following:

$$\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(b\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\phi\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(b\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\big)\big],$$
$$\big[\phi\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(b\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\phi\big[\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(b\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\big)\big].$$

Next, for the bracket $\Big[\big(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\big(\mathbf{1}\otimes v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]$ where $u\in\{\mathbf{1},\delta^i\}$ and $v\in\{\mathbf{1},\delta^j\}$ we have

$$\Big[\big(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\big(\mathbf{1}\otimes v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]=\sum_{l=0}^{\infty}\binom{m}{l}\big((\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(\mathbf{1}\otimes v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}})\big)^{\mathscr{M}}_{(m+n-l)},$$

and the l-th product $(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}})$ is the coefficient of z^{-l-1} in

$$\begin{aligned}Y(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}})&=Y(\mathbf{1},z)\otimes Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\\&=\mathbf{1}z^0\otimes Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}.\end{aligned}$$

The expression $Y(u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z)v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}$ has no negative powers of z. Hence,

$$\Big[\big(\mathbf{1}\otimes u_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\big(\mathbf{1}\otimes v_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]=0.$$

From (4.18), notice that

$$\big[\phi\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\Big[\big(\mathbf{1}\otimes\delta^i_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\big(\mathbf{1}\otimes\delta^j_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]=0,$$
$$\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\Big[\frac{1}{N}\big(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\big(\mathbf{1}\otimes\delta^j_{(-1)}e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]=0,$$
$$\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\Big[\frac{1}{N}\big(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(m)},\frac{1}{N}\big(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\big)^{\mathscr{M}}_{(n)}\Big]=0,$$

and since in the twisted toroidal Lie algebra we have

$$\begin{aligned}&\big[\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=0,\\&\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=0,\\&\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=0,\end{aligned}$$

then

$$\big[\phi\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\phi\big[\big(K_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big],$$
$$\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\phi\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_j\otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\big)\big],$$
$$\big[\phi\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\phi\big(K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\big)\big]=\phi\big[\big(K_0\otimes t_0^{N(m+1)}\boldsymbol{t}^{\boldsymbol{p}}\big),\big(K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\big)\big].$$

For the bracket $\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(\frac{n}{N})}^{\mathcal{M}}\right]$, we have

$$\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(\frac{n}{N})}^{\mathcal{M}}\right]=\sum_{l=0}^{\infty}\binom{m}{l}\left((\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})\right)_{(m+\frac{n}{N}-l)}^{\mathcal{M}}$$

and the l-th product $(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ is the coefficient of z^{-l-1} in the expression

$$\begin{aligned}Y(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})&=Y(\mathbf{1},z)b\otimes Y(\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}\\&=bz^{0}\otimes Y(\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}.\end{aligned}$$

The negative powers of $Y(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ must come from $Y(\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}$. Taking advantage of Lemma 4.3.1, the only negative power of $Y(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ with non-zero coefficients is $b\otimes q_{i}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-1}$, which is when $l=0$. Thus

$$\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(\frac{n}{N})}^{\mathcal{M}}\right]=\left(b\otimes q_{i}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)_{(m+\frac{n}{N})}^{\mathcal{M}}.$$

From Equation (4.18) and the equation above we get

$$\begin{aligned}&\left[\phi\left(d_{i}\otimes t_{0}^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\phi\left(b\otimes t_{0}^{n}\boldsymbol{t}^{\boldsymbol{q}}\right)\right]\\&\qquad=\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(b\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(\frac{n}{N})}^{\mathcal{M}}\right]\\&\qquad=\left(b\otimes q_{i}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)_{(m+\frac{n}{N})}^{\mathcal{M}},\end{aligned}$$

and from the bracket (4.5) in the twisted toroidal Lie algebra and from (4.18) we get

$$\phi\left(\left[d_{i}\otimes t_{0}^{Nm}\boldsymbol{t}^{\boldsymbol{p}},b\otimes t_{0}^{n}\boldsymbol{t}^{\boldsymbol{q}}\right]\right)=q_{i}\phi\left(b\otimes t_{0}^{Nm+n}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}}\right)=q_{i}\left(b\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)_{(\frac{mN+n}{N})},$$

which gives us

$$\left[\phi\left(d_{i}\otimes t_{0}^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\phi\left(b\otimes t_{0}^{n}\boldsymbol{t}^{\boldsymbol{q}}\right)\right]=\phi\left[\left(d_{i}\otimes t_{0}^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\left(b\otimes t_{0}^{n}\boldsymbol{t}^{\boldsymbol{q}}\right)\right].$$

For the bracket $\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(n)}^{\mathcal{M}}\right]$, we have

$$\left[\left(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}}\right)_{(m)}^{\mathcal{M}},\left(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}}\right)_{(n)}^{\mathcal{M}}\right]=\sum_{l=0}^{\infty}\binom{m}{l}\left((\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})\right)_{(m+n-l)}^{\mathcal{M}}$$

and the l-th product $(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}})_{(l)}(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ is the coefficient of z^{-l-1} in the expression

$$\begin{aligned}Y(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})&=Y(\mathbf{1},z)\mathbf{1}\otimes Y(\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}\\&=\mathbf{1}z^{0}\otimes Y(\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)e^{\boldsymbol{q}\boldsymbol{\delta}}.\end{aligned}$$

Similar to the last example, the only negative power of $Y(\mathbf{1}\otimes\Lambda_{0}{}_{(-1)}^{i}e^{\boldsymbol{p}\boldsymbol{\delta}},z)(\mathbf{1}\otimes e^{\boldsymbol{q}\boldsymbol{\delta}})$ is $\mathbf{1}\otimes q_{i}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-1}$.

Thus

$$\left[\left(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta}\right)^{\mathcal{M}}_{(m)},\left(\mathbf{1}\otimes e^{\boldsymbol{q}\delta}\right)^{\mathcal{M}}_{(n)}\right]=\left(\mathbf{1}\otimes q_i\, e^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)^{\mathcal{M}}_{(m+n)}.$$

From (4.18) we get

$$\begin{aligned}&\left[\phi\left(d_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\phi\left(K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\right)\right]\\&\qquad=\left[\left(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta}\right)^{\mathcal{M}}_{(m)},\frac{1}{N}\left(\mathbf{1}\otimes e^{\boldsymbol{q}\delta}\right)^{\mathcal{M}}_{(n)}\right]\\&\qquad=\frac{1}{N}\left(\mathbf{1}\otimes q_i\, e^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)^{\mathcal{M}}_{(m+n)}\end{aligned}$$

and from the bracket (4.6) in the twisted toroidal Lie algebra and from (4.18) we get

$$\phi\left(\left[d_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}},K_0\otimes t_0^{N(n+1)}\boldsymbol{t}^{\boldsymbol{q}}\right]\right)=\phi\left(q_iK_0\otimes t_0^{N(m+n+1)}\boldsymbol{t}^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)=\frac{q_i}{N}\left(\mathbf{1}\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)_{(m+n)}.$$

Therefore,

$$\left[\phi\left(d_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\phi\left(K_0\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\right)\right]=\phi\left[\left(d_i\otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right),\left(K_0\otimes t_0^{n}\boldsymbol{t}^{\boldsymbol{q}}\right)\right].$$

For the bracket $\left[\left(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta}\right)^{\mathcal{M}}_{(m)},\left(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta}\right)^{\mathcal{M}}_{(n)}\right]$, we have

$$\left[\left(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta}\right)^{\mathcal{M}}_{(m)},\left(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta}\right)^{\mathcal{M}}_{(n)}\right]=\sum_{l=0}^{\infty}\binom{m}{l}\left((\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta})_{(l)}(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta})\right)^{\mathcal{M}}_{(m+n-l)}$$

and the l-th product $(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta})_{(l)}(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta})$ is the coefficient of z^{-l-1} in the expression

$$\begin{aligned}Y(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta},z)(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta})&=Y(\mathbf{1},z)\mathbf{1}\otimes Y(\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta},z)\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta}\\&=\mathbf{1}z^0\otimes Y(\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta},z)\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta}.\end{aligned}$$

From Lemma 4.3.1, the negative powers of $Y(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta},z)(\mathbf{1}\otimes\delta^{j}{}_{(-1)}e^{\boldsymbol{q}\delta})$ are

$$\mathbf{1}\otimes\left(\left(q_i\delta^{j}_{(-1)}+\delta_{i,j}(\boldsymbol{p}\delta)_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\delta}z^{-1}+\delta_{i,j}e^{(\boldsymbol{p}+\boldsymbol{q})\delta}z^{-2}\right).$$

Thus

$$\begin{aligned}&\left[\left(\mathbf{1}\otimes\Lambda_{0}{}^{i}_{(-1)}e^{\boldsymbol{p}\delta}\right)^{\mathcal{M}}_{(m)},\left(\mathbf{1}\otimes\delta^{i}_{(-1)}e^{\boldsymbol{q}\delta}\right)^{\mathcal{M}}_{(n)}\right]=\\&\qquad\left(\mathbf{1}\otimes\left(q_i\delta^{j}_{(-1)}+\delta_{i,j}(\boldsymbol{p}\delta)_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)^{\mathcal{M}}_{(m+n)}+m\left(\mathbf{1}\otimes\delta_{i,j}e^{(\boldsymbol{p}+\boldsymbol{q})\delta}\right)^{\mathcal{M}}_{(m+n-1)}.\end{aligned}$$

From (4.18) we have

$$\begin{aligned}
&\left[\phi\left(d_i \otimes t_0^{Nm}\boldsymbol{t^p}\right),\phi\left(K_j \otimes t_0^{n}\boldsymbol{t^q}\right)\right]\\
&\qquad\qquad\qquad = \left[\left(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}}\right)^{\mathscr{M}}_{(m)}, \left(\mathbf{1}\otimes \delta^{j}_{(-1)} e^{\boldsymbol{p\delta}}\right)^{\mathscr{M}}_{(n)}\right]\\
&\qquad\qquad\qquad = \left(\mathbf{1}\otimes\left(q_i\delta^{j}_{(-1)}+\delta_{i,j}(\boldsymbol{p\delta})_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n)} + m\left(\mathbf{1}\otimes\delta_{i,j}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n-1)}
\end{aligned}$$

and from the bracket (4.6) in the twisted toroidal Lie algebra and from (4.18) we get

$$\begin{aligned}
&\phi\left(\left[d_i\otimes t_0^{Nm}\boldsymbol{t^p}, K_j\otimes t_0^{Nn}\boldsymbol{t^q}\right]\right)\\
&= \phi\left(q_iK_j\otimes t_0^{N(m+n)}\boldsymbol{t^{p+q}}+\delta_{i,j}\left(\sum_{l=1}^{r}p_lK_l\otimes t_0^{(m+n)N}\boldsymbol{t^{p+q}}+NmK_0\otimes t_0^{(m+n)N}\boldsymbol{t^{p+q}}\right)\right)\\
&= q_i\left(\mathbf{1}\otimes\delta^{j}_{(-1)}e^{(\boldsymbol{q}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n)}+\delta_{i,j}\left(\left(\mathbf{1}\otimes(\boldsymbol{p\delta})_{(-1)}e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n)}+\frac{Nm}{N}\left(\mathbf{1}\otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{\left(\frac{(m+n)N}{N}-1\right)}\right).
\end{aligned}$$

Therefore we have

$$\left[\phi\left(d_i\otimes t_0^{Nm}\boldsymbol{t^p}\right),\phi\left(K_j\otimes t_0^{n}\boldsymbol{t^q}\right)\right]=\phi\left[\left(d_i\otimes t_0^{Nm}\boldsymbol{t^p}\right),\left(K_j\otimes t_0^{n}\boldsymbol{t^q}\right)\right].$$

For the final bracket $\left[\left(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}}\right)^{\mathscr{M}}_{(m)}, \left(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}\right)^{\mathscr{M}}_{(n)}\right]$, we have

$$\left[\left(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}}\right)^{\mathscr{M}}_{(m)}, \left(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}\right)^{\mathscr{M}}_{(n)}\right]=\sum_{l=0}^{\infty}\binom{m}{l}\left((\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}})_{(l)}(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}})\right)^{\mathscr{M}}_{(m+n-l)}$$

and the l-th product $(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}})_{(l)}(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}})$ is the coefficient of z^{-l-1} in the expression

$$\begin{aligned}
Y(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}},z)(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}) &= Y(\mathbf{1},z)\mathbf{1}\otimes Y(\Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}},z)\Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}\\
&= \mathbf{1}z^0\otimes Y(\Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}},z)\Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}.
\end{aligned}$$

From Lemma 4.3.1, the negative powers of $Y(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}},z)(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}})$ are

$$\mathbf{1}\otimes\left(\left(-p_j\Lambda^{i}_{0(-1)}+q_i\Lambda^{j}_{0(-1)}-q_ip_j(\boldsymbol{p\delta})_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-1}-q_ip_je^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}z^{-2}\right).$$

Thus

$$\begin{aligned}
&\left[\left(\mathbf{1}\otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p\delta}}\right)^{\mathscr{M}}_{(m)}, \left(\mathbf{1}\otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{q\delta}}\right)^{\mathscr{M}}_{(n)}\right]=\\
&\qquad\left(\mathbf{1}\otimes\left(-p_j\Lambda^{i}_{0(-1)}+q_i\Lambda^{j}_{0(-1)}-q_ip_j(\boldsymbol{p\delta})_{(-1)}\right)e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n)}-m\left(\mathbf{1}\otimes q_ip_je^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathscr{M}}_{(m+n-1)}.
\end{aligned}$$

From (4.18) and the calculation above we have

$$\begin{aligned}
&\left[\phi\left(d_i \otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right), \phi\left(d_j \otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\right)\right] \\
&\qquad = \left[\left(\mathbf{1} \otimes \Lambda_{0}{}^{i}_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(m)}, \left(\mathbf{1} \otimes \Lambda_{0}{}^{j}_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(n)}\right] \\
&\qquad = \left(\mathbf{1} \otimes \left(-p_j \Lambda^i_{0(-1)} + q_i \Lambda^j_{0(-1)} - q_i p_j (\boldsymbol{p}\boldsymbol{\delta})_{(-1)}\right) e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(m+n)} - m\left(\mathbf{1} \otimes q_i p_j e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(m+n-1)}
\end{aligned}$$

and from the bracket (4.7) in the twisted toroidal Lie algebra and from (4.18) we get

$$\begin{aligned}
&\phi\left(\left[d_i \otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}, d_j \otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\right]\right) \\
&= \phi\left(q_i d_j \otimes t_0^{N(m+n)}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}} - p_j d_i \otimes t_0^{N(m+n)}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}} - q_i p_j\left(\sum_{l=1}^{r} p_l K_l \otimes t_0^{N(m+n)}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}} + NmK_0 \otimes t_0^{N(m+n)}\boldsymbol{t}^{\boldsymbol{p}+\boldsymbol{q}}\right)\right) \\
&= q_i\left(\mathbf{1} \otimes \Lambda_{0}{}^{j}_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(m+n)} - p_j\left(\mathbf{1} \otimes \Lambda_{0}{}^{i}_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{(m+n)} - q_i p_j\left(\mathbf{1} \otimes (\boldsymbol{p}\boldsymbol{\delta})_{(-1)} e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{\left(\frac{N(m+n)}{N}\right)} \\
&\qquad - q_i p_j \frac{mN}{N}\left(\mathbf{1} \otimes e^{(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{\delta}}\right)^{\mathcal{M}}_{\left(\frac{N(m+n)}{N}-1\right)}.
\end{aligned}$$

Therefore,

$$\left[\phi\left(d_i \otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right), \phi\left(d_j \otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\right)\right] = \phi\left[\left(d_i \otimes t_0^{Nm}\boldsymbol{t}^{\boldsymbol{p}}\right), \left(d_j \otimes t_0^{Nn}\boldsymbol{t}^{\boldsymbol{q}}\right)\right].$$

This completes the proof of the theorem. □

Now let $\mathfrak{g}$ be a simple Lie algebra of type X_ℓ where $X = A, D, E$ (simply laced), with a root system Δ and a root lattice $Q = \mathbb{Z}\Delta$. Let σ be an automorphism of the lattice Q of finite order N. Consider the orthogonal direct sum of lattices

$$L = Q \oplus J$$

where, as before, J is defined by (4.13) and (4.14). We extend σ to an automorphism of L, acting as the identity on J, and we lift it to an automorphism of the lattice vertex algebra

$$V_L \simeq V_Q \otimes V_J.$$

Finally, let

$$\mathfrak{H} = \mathbb{C} \otimes_{\mathbb{Z}} L = \mathfrak{h} \oplus \operatorname{span}\{\delta^i, \Lambda_0^i\}_{1\le i\le r}$$

where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ is the Cartan subalgebra of $\mathfrak{g}$. With the above notation, using the Frenkel–Kac construction [22], we can reformulate Theorem 4.3.2.

Corollary 4.3.3. For any σ-twisted V_L-module $\mathcal{M}$, the Lie algebra of modes of the fields ($\alpha \in \Delta \cup \{0\}$, $h \in \mathfrak{H}$, $\boldsymbol{p} \in \mathbb{Z}^r$):

$$Y^{\mathcal{M}}(e^{\alpha+\boldsymbol{p}\boldsymbol{\delta}}, z), \quad Y^{\mathcal{M}}(h_{(-1)} e^{\boldsymbol{p}\boldsymbol{\delta}}, z) \qquad (\alpha \in \Delta \cup \{0\},\ h \in \mathfrak{H},\ \boldsymbol{p} \in \mathbb{Z}^r)$$

form a representation of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,1}(\mathfrak{g}, \sigma^{-1})$ of level 1 on $\mathcal{M}$.

Proof. Since $V_L \simeq V_Q \otimes V_J$ and $V_Q \simeq V^1(\mathfrak{g})$ by the Frenkel–Kac construction (see [22]), then $V_L \simeq V^1(\mathfrak{g}) \otimes V_J$. By Corollary 3.3.6, any σ-twisted V_Q-module M is also a σ-twisted module for $V_1(\mathfrak{g})$. Then $\mathscr{M} = M \otimes M'$ will be a σ-twisted module for V_L and the claim follows from Theorem 3.3.4. $\square$

In the special case when σ is a Coxeter element in the Weyl group of $\mathfrak{g}$, the above corollary recovers Billig's construction from [12]. Observe that, for $r = 1$, the lattice $Q \oplus \mathbb{Z}\delta$ is the root lattice of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ of type $X_\ell^{(1)}$, while $\mathfrak{H} = \mathfrak{h} \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ is the Cartan subalgebra of $\hat{\mathfrak{g}}$. Moreover, the set $\{\alpha + p\delta \mid \alpha \in \Delta \cup \{0\}, \, p \in \mathbb{Z}\}$ is the union of $\{0\}$ and the root system of $\hat{\mathfrak{g}}$. In the paper [13], Billig and Lau used an approach that is similar to our construction and produced certain irreducible modules over twisted toroidal Lie algebras.

CHAPTER

5

CONCLUSION AND FUTURE DIRECTIONS

5.1 Conclusion

In this thesis, we were able to construct vertex operator representations of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma^{-1})$ of level k where $\mathfrak{g}$ was a simple finite-dimensional Lie algebra and σ was an automorphism of $\mathfrak{g}$ of finite order. We saw that σ could be lifted to the affine vertex algebra $V_k(\mathfrak{g})$ and then to the tensor product $V_k(\mathfrak{g})\otimes V_J$ where J corresponded to r copies of a certain rank-2 lattice. Next, we took a σ-twisted $V_k(\mathfrak{g})$-module called M and tensored it with an untwisted V_J-module we called M' to form the σ-twisted $V_k(\mathfrak{g})\otimes V_J$-module we called $\mathscr{M}=M\otimes M'$. We showed that the modes of the vertex operators

$$\begin{aligned} &Y^{\mathscr{M}}(a\otimes e^{\boldsymbol{p}\boldsymbol{\delta}},z), &&Y^{\mathscr{M}}(\mathbf{1}\otimes e^{\boldsymbol{p}\boldsymbol{\delta}},z),\\ &Y^{\mathscr{M}}(\mathbf{1}\otimes \delta^{i}_{(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z), &&Y^{\mathscr{M}}(\mathbf{1}\otimes \Lambda^{i}_{0(-1)}e^{\boldsymbol{p}\boldsymbol{\delta}},z),\end{aligned}$$

for $a\in\mathfrak{g}$ and $\boldsymbol{p}\in\mathbb{Z}^r$ formed a representation of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g},\sigma^{-1})$ of level k on $\mathscr{M}$.

We saw that when $\mathfrak{g}$ was a simple finite-dimensional Lie algebra of type X_ℓ where $X=A,D$, or E, then the lattice vertex algebra V_Q (where Q is the root system for $\mathfrak{g}$) was isomorphic to the affine vertex algebra $V_1(\mathfrak{g})$. We noticed that the vertex algebras V_L and $V_Q\otimes V_J$ were isomorphic when $L=Q\oplus J$ and that $\mathscr{M}=M\otimes M'$ (where M is a twisted V_Q-module and M' an untwisted V_J-module) forms a twisted module over V_L. Finally, we showed that the modes of the vertex operators

$$Y^{\mathscr{M}}(e^{\alpha+\boldsymbol{p}\delta},z), \quad Y^{\mathscr{M}}(h_{(-1)}e^{\boldsymbol{p}\delta},z) \qquad (\alpha\in\Delta\cup\{0\},\ h\in\mathfrak{H},\ \boldsymbol{p}\in\mathbb{Z}^r).$$

formed a representation of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,1}(\mathfrak{g},\sigma^{-1})$ on $\mathscr{M}$.

Many questions naturally arise from the work presented in this thesis. These questions include, can we consider a different class of automorphisms in constructing twisted toroidal Lie algebras and their representations? Is it possible to build irreducible representations of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,1}(\mathfrak{g},\sigma^{-1})$ from σ-twisted modules over a tensor product of lattice vertex algebras when $\mathfrak{g}$ is not simply laced? Questions such as these are described in more detail in the following sections of this chapter.

5.2 Irreducible Representations of $\hat{\mathscr{L}}_{r+1,1}(\mathfrak{g},\sigma^{-1})$ when $\mathfrak{g}$ is not simply laced

In Example 3.3.2, we demonstrated that an automorphism σ of a root lattice Q associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$ can be lifted to the lattice vertex algebra V_Q. In Corollary 3.3.6, we saw that when $\mathfrak{g}$ is simply laced, and M is a σ-twisted V_Q module, then the modes of the σ-twisted free bosons $Y^M(h,z)$ for $h\in\mathfrak{h}$ and the modes of the vertex operators $Y^M(e^\alpha,z)$ for $\alpha\in\Delta$ correspond to a representation of the affine Kac–Moody Lie algebra $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$. One question that naturally arises: is it possible to construct a representation of an affine Kac–Moody Lie algebra $\hat{\mathscr{L}}'(\mathfrak{g},\sigma^{-1})$ using modes from a lattice vertex algebra if $\mathfrak{g}$ is not simply laced?

The Frenkel–Kac construction which states V_Q is isomorphic to the simple affine vertex algebra of level 1, denoted $V^1(\mathfrak{g})$, only holds when Q is a root lattice associated to a simply laced Lie algebra. However, it is well known that every non-simply laced Lie algebra can be embedded in a “larger" simply laced Lie algebra. This can be done by working with a diagram automorphism μ of the simply laced Lie algebra $\mathfrak{g}$. Depending on the choice of the simply laced Lie algebra, the order of μ is $N=2$ or 3. Recall from Section 2.8, we can decompose $\mathfrak{g}$ into a direct sum of eigenspaces

$$\mathfrak{g}=\bigoplus_{j=0}^{N-1}\mathfrak{g}_j$$

where $\mathfrak{g}_0$ is the set of fixed points of μ and $\mathfrak{g}_1$ is the eigenspace of μ with eigenvalue -1 if $N=2$ (or $\mathfrak{g}_j$ is the eigenspace with eigenvalue $e^{\frac{2\pi ij}{3}}$ if $N=3$). The eigenspace $\mathfrak{g}_0$ corresponds to a non-simply laced Lie algebra. Embedding a non-simply laced Lie algebra into a σ-twisted simply laced Lie algebra $\mathfrak{g}$ for any finite order automorphism is much more involved since σ changes the internal structure of the Lie algebra $\mathfrak{g}$. Finding such an embedding for any σ of finite order is an open problem. A good starting point to finding such embeddings is to consider the special case where σ is a finite order inner automorphism of a non-simply laced Lie algebra $\mathfrak{g}_0$ and then lift σ to the larger simply laced Lie algebra $\mathfrak{g}$. The lifted map produces an automorphism of $\mathfrak{g}$ that commutes with the diagram automorphism, which makes it easier to identify the embedding.

Once an embedding is identified, the next issues is to find the irreducible modules for the affine Kac–Moody algebra $\hat{\mathscr{L}}'(\mathfrak{g}_0, \sigma^{-1})$ associated to the non-simply laced Lie algebra. The irreducible modules for the larger affine Kac–Moody Lie algebra $\hat{\mathscr{L}}'(\mathfrak{g}, \sigma^{-1})$ may be reducible when we restrict to $\hat{\mathscr{L}}'(\mathfrak{g}_0, \sigma^{-1})$. Once the irreducible modules are found, we should be able to generalize the results to construct irreducible representations of the twisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1,1}(\mathfrak{g}_0, \sigma^{-1})$. Irreducible level-one representations for affine Kac–Moody algebras associated to non-simply laced Lie algebras where σ is a diagram automorphism or the Coxeter element have been explored in papers such as [11, 26, 45, 46, 50].

5.3 New types of twisted toroidal Lie algebras and their representations

Recall from Section 4.1.2 that we build a twisted toroidal Lie algebra by starting with an automorphism σ with finite order N of a simple finite-dimensional Lie algebra $\mathfrak{g}$ and using it to create a twisted subalgebra $\mathscr{L}_{r+1}(\mathfrak{g}, \sigma)$ of the loop algebra $\mathscr{L}_{r+1}(\mathfrak{g})$ given by

$$\mathscr{L}_{r+1}(\mathfrak{g}, \sigma) = \bigoplus_{m_0 \in \mathbb{Z}} \mathscr{L}_{r+1}(\mathfrak{g}, \sigma)_{m_0},$$

where

$$\mathscr{L}_{r+1}(\mathfrak{g}, \sigma)_{m_0} = \mathfrak{g}_{m_0} \otimes \operatorname{span}_{\mathbb{C}} \left\{ t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}} \,\middle|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r \right\}$$

and that the twisted toroidal Lie algebra was the central extension of this twisted loop algebra (with derivations) given by

$$\hat{\mathscr{L}}_{r+1,k}(\mathfrak{g}, \sigma) = \mathscr{L}_{r+1}(\mathfrak{g}, \sigma) \oplus \mathscr{K}' \oplus \mathscr{D}'_{+}.$$

What if instead of starting with an automorphism σ of $\mathfrak{g}$ that we started with an automorphism σ of finite order N of the affine Kac–Moody algebra $\hat{\mathscr{L}}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$? That is, what if we defined a new type of twisted toroidal Lie algebra in the following way: let $\overline{\sigma}$ be the induced automorphism on the loop algebra $\mathscr{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ (instead of using $\mathscr{L}(\mathfrak{g})$ we will call this space $\overline{\mathfrak{g}}$ to simplify notation). The new twisted toroidal Lie algebra will be the central extension (with derivations) of the space

$$\mathscr{L}_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma}) = \bigoplus_{m_0 \in \mathbb{Z}} \mathscr{L}_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma})_{m_0},$$

where

$$\mathscr{L}_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma})_{m_0} = \overline{\mathfrak{g}}_{m_0} \otimes \operatorname{span}_{\mathbb{C}} \left\{ t_0^{m_0} \boldsymbol{t}^{\boldsymbol{m}} \,\middle|\, m_0 \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^r \right\}$$
$$\overline{\mathfrak{g}}_{m_0} = \{ x \in \overline{\mathfrak{g}} \mid \overline{\mu}(x) = e^{2\pi i m_0/N} \}.$$

The new **twisted toroidal Lie algebra** is the space

$$\hat{\mathscr{L}}_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma}) = \mathscr{L}_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma}) \oplus \mathscr{K}^* \oplus \mathscr{D}^*_{+},$$

which is a subalgebra of the untwisted toroidal Lie algebra $\hat{\mathscr{L}}_{r+1}(\overline{\mathfrak{g}}) \subset \hat{\mathscr{L}}_{r+2}(\mathfrak{g})$ for some $\mathscr{K}^* \subset \mathscr{K}$ and $\mathscr{D}_+^* \subset \mathscr{D}_+$. Notice that the twisted toroidal Lie algebra will contain $r+2$ variables since $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. We can think of this new type of twisted toroidal Lie algebra as an extension of our original definition of a twisted toroidal Lie algebra. In fact, if σ is an automorphism of $\mathfrak{g}$, then we can lift σ to $\hat{\mathscr{L}}(\mathfrak{g})$ so that σ acts as the identity on K and d.

So many questions can be asked here including, what exactly are the sets $\mathscr{K}^*$ and $\mathscr{D}^*$ for any fixed σ of finite order N? Does Proposition 2.8.2 hold for not just a finite-dimensional Lie algebra but also for an affine Kac–Moody algebra? How do we construct vertex operator representations of these twisted toroidal Lie algebras using twisted modules over vertex algebras?

In the paper [15], this type of twisted toroidal Lie algebra was explored when σ was a diagram automorphism of the affine Kac–Moody algebra $\hat{\mathscr{L}}'(\mathfrak{g})$ without derivation. The authors described the action of the induced automorphism $\overline{\sigma}$ on $\overline{\mathfrak{g}}$, identified the subalgebra $\mathscr{K}^*$ needed to construct a universal central extension, and calculated the Lie brackets of the generators of $\hat{\mathscr{L}}'_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma})$. The authors also presented $\hat{\mathscr{L}}'_{r+1}(\overline{\mathfrak{g}}, \overline{\sigma})$ in a similar way to the construction of the untwisted toroidal Lie algebras by Moody, Rao, and Yokonuma in the paper [47]. It is conceivable that the results from [15] could be applied for any σ of finite order N.

To construct vertex operator representations of these toroidal Lie algebras, we could start with an automorphism σ of order N of $\hat{\mathscr{L}}(\mathfrak{g})$ of type $X_\ell^{(1)}$ where $X = A, D$, or E. The automorphism σ would induce an automorphism on the extended root lattice $\hat{Q} = Q \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$ of $\hat{\mathscr{L}}(\mathfrak{g})$ where Q is the root lattice associated to the finite-dimensional simple Lie algebra of type X_ℓ. We could then lift σ to the lattice vertex algebra $V_{\hat{Q}}$ and then to $V_L = V_{\hat{Q}} \otimes V_J$ where J is r-copies of the rank-2 lattice described in Equation (4.13). We could then study the σ-twisted $V_{\hat{Q}} \otimes V_J$-modules $\mathscr{M} = M \otimes M'$ where M is a σ-twisted $V_{\hat{Q}}$ module and M' is an untwisted V_J-module. The tricky part is the identification of modes from the $V_{\hat{Q}}$-module M. More research is needed in this area, however it seems that since $V_{\hat{Q}} \simeq V_Q \otimes V_{\mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0}$ then perhaps a lift can be constructed from the induced automorphism $\overline{\sigma}$ on the root lattice Q to σ on $\hat{Q}$ in a way that is similar in principle to the lift from $\overline{\mathfrak{g}}$ to $\hat{\mathscr{L}}'_1(\overline{\mathfrak{g}}, \overline{\sigma})$ used in [15].

Finally, suppose instead of working with a finite order automorphism of the affine Kac–Moody algebra $\hat{\mathscr{L}}(\mathfrak{g})$ we used certain infinite-order automorphisms of $\hat{\mathscr{L}}(\mathfrak{g})$. Specifically, any automorphism from the Weyl group of $\hat{\mathscr{L}}(\mathfrak{g})$. Within the last 10 years, the notion of logarithmic vertex operators and σ-twisted logarithmic modules for an infinite order automorphism σ of a vertex algebra have been developed (see [1, 3, 27, 28]). Bakalov and Sullivan expanded the work in this area to include σ-twisted logarithmic modules of lattice vertex algebras in [7]. It seems likely that the key to finding vertex operator representations of twisted toroidal Lie algebras corresponding to infinite order automorphisms from the Weyl group of an affine Kac–Moody Lie algebra may lie in the understanding of logarithmic vertex operators from a tensor product of two lattice vertex algebras.

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