

ABSTRACT

ALOUDAH, NOUFE. Hochschild-Serre Spectral Sequence for Lie Conformal Algebras.
(Under the direction of Bojko Bakalov).

Lie conformal algebras, originally introduced by Kac, encode an axiomatic description of the operator product expansion of chiral fields in conformal field theory. In particular, Lie conformal algebras provide a powerful tool for studying the infinite-dimensional Lie algebras and associative algebras satisfying the locality property. The most important examples of Lie conformal algebra include the Virasoro algebra Vir , the current algebra $\text{Cur } \mathfrak{g}$, and their semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$.

In this thesis, we construct the Hochschild–Serre spectral sequence for Lie conformal algebras. This construction follows a similar approach to the original work done by G. Hochschild and J-P. Serre for the case of Lie algebras. In addition, we describe the inflation-restriction exact sequence, a special case of the five-term exact sequence. As an application of this construction, we provide a different approach for calculating the cohomology of the semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with trivial coefficients. In addition, we offer explicit computations for the basic cohomology of Vir and $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in their finite conformal modules $M_{\Delta, \alpha}$ and $M_{\Delta, \alpha, U}$, respectively.

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Hochschild-Serre Spectral Sequence for Lie Conformal Algebras

by
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DEDICATION

To my dear parents and my beloved husband,
who eagerly waited for this thesis to be completed, even though they will never read it!

To my little miracle, Sattam.
Who knows? He might read it!

BIOGRAPHY

The author, Noufe Aloudah, grew up in Buraydah, Qassim, Saudi Arabia. She received her Bachelor of Science degree in Mathematics with first-class honors in 2008 and her Master of Science degree in Pure Mathematics in 2012 from Qassim University. Noufe was appointed as a teaching assistant in 2010 at Qassim University, where she was granted a full scholarship to further her graduate studies. In Fall 2016, she moved to Raleigh to join the PhD-Mathematics program at North Carolina State University. Noufe will continue her academic career as an assistant professor at Qassim University after finishing her doctoral degree.

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Chapter 1

Introduction

Conformal algebra is an axiomatic description of the operator product expansion of chiral fields in conformal field theory. It appears naturally in the context of formal distribution Lie algebras. Likewise, conformal modules over conformal algebras appear naturally in the context of conformal modules over formal distribution Lie algebras [21]. Conformal algebras were first introduced by Victor G. Kac in 1996 [20]. Moreover, conformal algebras correspond to vertex algebras in the same way as Lie algebras correspond to their associative enveloping algebras [20]. In a more general term, conformal algebras are defined as Lie algebras in a particular pseudotensor category [1].

In particular, a Lie conformal algebra is a $\mathbb{C}[\partial]$ -module \mathcal{A} endowed with a bilinear product, called a λ -bracket, $[\cdot_\lambda \cdot] : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{A}$, that satisfies the following properties

- i Conformal sesquilinearity: $[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b],$
- ii Skew-symmetry: $[a_\lambda b] = -[b_{-\lambda-\partial} a],$
- iii Jacobi identity: $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]],$

for $a, b, c \in \mathcal{A}$. It is, to some extent, a generalization of a Lie algebra and an adequate tool for the study of infinite dimensional Lie algebras satisfying the locality property [19]. Some basic examples of Lie conformal algebras are (1) the Virasoro conformal algebra Vir , a rank 1 free $\mathbb{C}[\partial]$ -module generated by an element L , with a λ -bracket defined by $[L_\lambda L] = (\partial + 2\lambda)L$, (2) the current conformal algebra associated to a finite dimensional Lie algebra \mathfrak{g} , which is defined by $\text{Cur } \mathfrak{g} = \mathbb{C}[\partial] \otimes \mathfrak{g}$ with the λ -bracket $[g_\lambda h] = [g, h]$ for $g, h \in \mathfrak{g}$.

Over the last few years, the structure theory [8], conformal module and their extensions [5, 6], and general cohomology theory [2] of finite (i.e., finitely generated as a $\mathbb{C}[\partial]$ -module) Lie conformal algebras have been well developed. The Lie conformal algebras of rank 1 and rank 2 were classified in [8, 3], respectively. Simple/semisimple Lie conformal algebras have been intensively investigated. The classification of finite simple and semisimple Lie conformal algebras was completed in [8]. It shows that a finite semisimple Lie conformal algebra is isomorphic to a direct sum of Lie conformal algebras of the following types: the Virasoro conformal algebra Vir , the current conformal algebra $\text{Cur } \mathfrak{g}$ associated with a simple finite dimensional Lie algebra \mathfrak{g} , or the semidirect product of Vir and $\text{Cur } \mathfrak{g}$.

The construction of finite nonsimple Lie conformal algebras and their structures, including central extensions, conformal derivations, and conformal modules, were also studied afterward. This includes the Schrödinger-Virasoro type and the extended Schrödinger-Virasoro type Lie conformal algebras [26], and the Lie conformal algebra $\mathcal{W}(a, b)$ [17], which are related to the Virasoro algebra. In addition, several infinite dimensional Lie conformal algebras have been developed recently such as the infinite rank Schrödinger-Virasoro type Lie conformal algebras [11], the Loop Heisenberg-Virasoro Lie conformal algebra [10], and the Lie conformal algebra of Block type [35].

The cohomology theory of conformal algebras with coefficients in an arbitrary module was developed in [2]. It describes extensions and deformations and explicitly computes cohomology of the first two types of the semisimple Lie conformal algebra. The study of the cohomology of semisimple Lie conformal algebra of the third type was done in [33]. The low dimensional cohomologies of the infinite rank general Lie conformal algebras gc_N with trivial coefficients were computed in [25]. The cohomologies of special cases of nonsimple Lie conformal algebra of type $\mathcal{W}(a, b)$ were studied in [30, 31].

Inspired by the main computational tools of Lie algebra cohomology, we construct the Hochschild-Serre spectral sequence for Lie conformal algebras. The notion of spectral sequences generally appears in homological algebra, algebraic topology, and algebraic geometry. The French mathematician Jean Leray first invented it in 1964 to compute sheaf cohomology. Henceforth, they have become powerful tools to compute the homology and cohomology of complicated spaces. In addition to the Leray spectral sequence, some well-known spectral sequences are the Serre spectral sequence of a fibration, Lyndon-Hochschild-Serre spectral sequence in group cohomology, and Adams spectral sequence in stable homotopy theory.

Cohomological spectral sequence is a collection of $\mathbb{C}[\partial]$ -module $\{E_r^{p,q}\}$ for all $r \geq 0$, together with maps $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ and $E_{r+1} =$

$\ker d_r^{p,q}/\text{im } d_r^{p-r,q+r-q}$. The term $E_r^{*,*}$ is called E_r -term and the spectral sequence collapses at the N -th term if $d_r = 0$ for all $r \gg N$.

An important observation to make about the spectral sequences is that one can proceed with computation and describing the algebraic structure of some E_r -term without knowledge of the differentials, d_r . Moreover, spectral sequence low-degree terms can be described even when a spectral sequence does not collapse. In this case, one can construct some useful exact sequence such as the five-term and seven-term exact sequence.

The Hochschild–Serre spectral sequence, named after Gerhard Hochschild and Jean-Pierre Serre, was first introduced in the context of group theory in 1953 [13]. It describes the relation between the cohomology groups of a group G , a normal subgroup N of G , and the quotient group G/N . The main result states that for any group G and a normal subgroup N of G , there is a spectral sequence of cohomological type whose E_2 -term is $H^p(G/N, H^q(N, A))$ and whose E_∞ -term is $H^{p+q}(G, A)$, where A is an arbitrary G -module. A similar result holds for Lie algebras, as shown in [14].

As mentioned previously, the objective of this thesis is to describe the Hochschild-Serre spectral sequence in the field of Lie conformal algebras. We believe this construction will provide a valuable tool to calculate the cohomology of Lie conformal algebras.

This thesis is structured as follows. In Chapter 2, we recall the basic notions of Lie conformal algebras, including their relation to the formal distribution of Lie algebras and basic examples. Then we provide a summary of the cohomology theory of Lie conformal algebras.

Chapter 3 presents the essential concepts on spectral sequences needed to describe the Hochschild-Serre spectral sequence for Lie conformal algebras.

In Chapter 4, we construct the Hochschild-Serre spectral sequence associated with Lie conformal algebras' basic complex, which is the main result of this thesis (see Theorem 4.1.1). Then we introduce the inflation-restriction exact sequence (see Theorem 4.2.2).

In Chapter 5, we conclude the thesis by giving several applications, including the computation of the basic cohomology of Vir with coefficients in the Vir -module $M_{\Delta,\alpha}$ (see Theorem 5.1.2), and the cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in the trivial module \mathbb{C}_a for $a \in \mathbb{C}$ and with coefficients in the $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ -module $M_{\Delta,\alpha,U}$ (see Theorems 5.2.1, 5.2.2, 5.4.1).

Unless otherwise specified, all vector spaces, linear maps, and tensor products are considered over the field \mathbb{C} of complex numbers, and we denote the sets of all nonnegative integers by \mathbb{Z}_+ .

Chapter 2

Lie Conformal Algebras

This chapter is an introduction of the basic notions for Lie conformal algebras that are used in this thesis. For a detailed introduction, we refer the reader to [2], [8] and [5].

2.1 Basic Definitions

Definition 2.1.1 ([21]). *A Lie conformal algebra over \mathbb{C} is a $\mathbb{C}[\partial]$ -module \mathcal{A} equipped with a bilinear product, called a λ -bracket, $[\cdot]_{\lambda} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{A}$, that satisfies the following axioms:*

- i Conformal sesquilinearity: $[\partial a_{\lambda} b] = -\lambda[a_{\lambda} b]$, $[a_{\lambda} \partial b] = (\partial + \lambda)[a_{\lambda} b]$,*
- ii Skew-symmetry: $[a_{\lambda} b] = -[b_{-\lambda-\partial} a]$,*
- iii Jacobi identity: $[a_{\lambda}[b_{\mu} c]] = [[a_{\lambda} b]_{\lambda+\mu} c] + [b_{\mu}[a_{\lambda} c]]$,*

for all $a, b, c \in \mathcal{A}$.

A **generating set** of \mathcal{A} over $\mathbb{C}[\partial]$ is a subset C of \mathcal{A} such that the smallest $\mathbb{C}[\partial]$ -module of \mathcal{A} containing C is \mathcal{A} itself. If the generating set is **finite**, then \mathcal{A} is called a finite Lie conformal algebra. Otherwise, \mathcal{A} is called **infinite**. A Lie conformal algebra \mathcal{A} is called **abelian** if $[a_{\lambda} b] = 0$ for all $a, b \in \mathcal{A}$.

Definition 2.1.2. *A subalgebra of a Lie conformal algebra \mathcal{A} is a $\mathbb{C}[\partial]$ -module \mathcal{B} such that $[a_{\lambda} b] \in \mathcal{B}[\lambda]$ for all $a, b \in \mathcal{B}$.*

Definition 2.1.3. *An ideal of a Lie conformal algebra \mathcal{A} is a subalgebra \mathcal{I} of \mathcal{A} such that $[a_{\lambda} b] \in \mathcal{I}[\lambda]$ for all $a \in \mathcal{A}$ and $b \in \mathcal{I}$.*

Definition 2.1.4 ([2]). A **conformal module** M over a Lie conformal algebra \mathcal{A} is a $\mathbb{C}[\partial]$ -module endowed with a \mathbb{C} -linear map $\mathcal{A} \otimes M \rightarrow M[\lambda], a \otimes v \mapsto a_\lambda v$, such that for any $a, b \in \mathcal{A}$ and $v \in M$ then,

$$\begin{aligned} a_\lambda(b_\mu v) - b_\mu(a_\lambda v) &= [a_\lambda b]_{\lambda+\mu} v, \\ (\partial a)_\lambda v &= -\lambda a_\lambda v, \\ a_\lambda(\partial v) &= (\partial + \lambda) a_\lambda v. \end{aligned}$$

A conformal module M is called **finite** if M is finitely generated over $\mathbb{C}[\partial]$. The **rank** of a conformal module M is its rank as a $\mathbb{C}[\partial]$ -module. An element v of a conformal module M over a Lie conformal algebra \mathcal{A} is called an **invariant** if $a_\lambda v = 0$ for all $a \in \mathcal{A}$. The set of all invariant elements of M forms a conformal \mathcal{A} -submodule of M , denoted by M^0 . An \mathcal{A} -module M is said to be **trivial** if $M^0 = M$. Moreover, a conformal module M is called **irreducible** if it has no nontrivial submodules.

An element $v \in M$ is called **torsion** if there exists a nonzero polynomial $p(\partial) \in \mathbb{C}[\partial]$ such that $p(\partial)v = 0$. A finite conformal \mathcal{A} -module M is called **torsion free** over $\mathbb{C}[\partial]$ if and only if 0 is the only torsion element of M . Moreover, a finitely-generated torsion-free \mathcal{A} -module M is free $\mathbb{C}[\partial]$ -module.

Lemma 2.1.1 ([19]). Let M be a conformal \mathcal{A} -module, and v is a torsion element of M . Then, $\mathcal{A}_\lambda v = 0$.

Definition 2.1.5 ([8]). Let M, N be modules over \mathcal{A} . A **conformal linear map** from M to N is a \mathbb{C} -linear map $f : M \rightarrow N[\lambda]$ denoted $f_\lambda : M \rightarrow N$, such that $f_\lambda \partial = (\partial + \lambda) f_\lambda$. The space of conformal linear maps is denoted by $\text{Chom}(M, N)$, and it is a $\mathbb{C}[\partial]$ and an \mathcal{A} -module with actions

$$\begin{aligned} (\partial f)_\lambda &= -\lambda f_\lambda, \\ (a_\mu f)_\lambda v &= a_\mu(f_{\lambda-\mu} v) - f_{\lambda-\mu}(a_\mu v), \end{aligned}$$

for $a \in \mathcal{A}, v \in M$, and $f \in \text{Chom}(M, N)$. The \mathcal{A} -module $\text{Chom}(M, N)$ is conformal when M and N are conformal and finite.

The space of conformal linear endomorphisms $\text{Chom}(M, M)$ is denoted by $\text{Cend } M$, and it is an associative conformal algebra. i.e., for any $f, g \in \text{Cend } M$, $v \in M$, then $(f_\lambda g)_\mu v = f_\lambda(g_{\mu-\lambda} v)$. The λ -bracket $[f_\lambda g] = f_\lambda g - g_{-\lambda-\partial} f$, defines a Lie conformal algebra structure on $\text{Cend } M$ denoted by $gc(M)$.

Definition 2.1.6. Let \mathcal{A} be a Lie conformal algebra. A **conformal derivation** of \mathcal{A} is a conformal linear map $d_\lambda : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$d_\lambda([a_\mu b]) = [(d_\lambda a)_{\lambda+\mu} b] + [a_\mu (d_\lambda b)]$$

for all $a, b \in \mathcal{A}$.

The space of all conformal derivations of \mathcal{A} is denoted by $\text{CDer}(\mathcal{A})$. For any $a \in \mathcal{A}$, then the linear map $(\text{ad } a)_\lambda : \mathcal{A} \rightarrow \mathcal{A}$ defined by $(\text{ad } a)_\lambda b = [a_\lambda b]$ for all $b \in \mathcal{A}$ is a conformal derivation of \mathcal{A} . Any conformal derivation of this type is called an **inner derivation**. The space of all inner derivations is denoted by $\text{CInn}(\mathcal{A})$.

2.2 Lie Conformal Algebras in the Context of Formal Distribution Lie Algebras

This section will review the construction of Lie conformal algebras, their mod associated to the formal distribution Lie algebras, and vice versa. This section's material primarily follows [20] and [21].

Let U be a complex vector space. A **U -valued formal distribution** in one indeterminate z is a formal power series of the form $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, where $a_n \in U$ is defined by $a_n = \text{Res}_z z^n a(z)$ such that $\text{Res}_z a(z) = a_{-1}$. The set of all such distributions form a vector space over \mathbb{C} denoted by $U[[z, z^{-1}]]$. Moreover, the derivative of $a(z)$ is defined by $\partial a(z) = \sum_{n \in \mathbb{Z}} (-n-1) a_n z^{-n-2}$.

Likewise, a U -valued formal distribution in two indeterminates z and w is a power series of the form

$$a(z, w) = \sum_{n, m \in \mathbb{Z}} a_{n, m} z^{-n-1} w^{-m-1}, \quad a_{n, m} \in U.$$

One important example of a \mathbb{C} -valued formal distributions is the *formal delta-function* $\delta(z - w)$, which is given by:

$$\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z} \right)^n.$$

Definition 2.2.1. A U -valued formal distribution $a(z, w)$ is called **local** if

$$(z - w)^N a(z, w) = 0, \text{ for } N \gg 0.$$

Theorem 2.2.1 ([20], Corollary 2.2). Every local U -valued formal distribution $a(z, w)$ is uniquely represented by the expansion

$$a(z, w) = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z - w), \quad (2.1)$$

where $c^j(w) = \text{Res}_z a(z, w)(z - w)^j$ and $\partial_w^{(j)} = \partial_w^j / j!$. This expansion is called the **operator product expansion (OPE)** of $a(z, w)$ and $c^j(w)$ are called **OPE coefficients** of $a(z, w)$.

Now let U be the Lie algebra \mathfrak{g} , and $a(z)$ and $b(w)$ be \mathfrak{g} -valued formal distributions. We will review the notion of locality of \mathfrak{g} -valued formal distributions as further explained in [20].

Definition 2.2.2. The pair $a(z)$ and $b(w)$ is called **local** if the \mathfrak{g} -valued formal distribution $[a(z), b(w)] \in \mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$ is local, i.e. if

$$(z - w)^N [a(z), b(w)] = 0, \text{ for } N \gg 0.$$

Definition 2.2.3. For each $n \in \mathbb{Z}_+$, the n -th **product** $a(w)_{(n)} b(w)$ on the space of \mathfrak{g} -valued formal distribution is given by:

$$a(w)_{(n)} b(w) = \text{Res}_z [a(z), b(w)] (z - w)^n.$$

Given two local \mathfrak{g} -valued formal distributions $a(z)$ and $b(w)$, then their OPE (2.1) is equivalent to the expansion:

$$[a(z), b(w)] = \sum_{j=0}^{N-1} (a(w)_{(j)} b(w)) \partial_w^{(j)} \delta(z - w). \quad (2.2)$$

Moreover, define the λ -bracket $[a(z)_\lambda b(w)]$ as follows:

$$[a(z)_\lambda b(w)] = \sum_{j=0}^{\infty} \lambda^{(j)} (a(w)_{(j)} b(w)), \quad \lambda^{(j)} = \lambda^j / j!. \quad (2.3)$$

Now let \mathcal{F} be a subset of $\mathfrak{g}[[z, z^{-1}]]$ consisting of pairwise local \mathfrak{g} -valued formal distributions such that the coefficients of all distributions from \mathcal{F} span \mathfrak{g} . Then the bracket of such coefficients is given by

$$[a_m, b_n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)} b)_{m+n-j}. \quad (2.4)$$

The pair $(\mathfrak{g}, \mathcal{F})$ is called a formal distribution Lie algebra. Denote by $\overline{\mathcal{F}}$ the minimal subspace of $\mathfrak{g}[[z, z^{-1}]]$ that contains \mathcal{F} such that for all $a(z), b(z) \in \overline{\mathcal{F}}$, then

$$a(z)_{(j)} b(w) \in \mathcal{F} \quad \text{and} \quad \partial(\overline{\mathcal{F}}) \subseteq \overline{\mathcal{F}}.$$

Then the pair $(\mathfrak{g}, \overline{\mathcal{F}})$ is a formal distribution Lie algebra with the bracket defined in (2.4).

Let $\text{Conf}(\mathfrak{g}, \mathcal{F}) = \mathbb{C}[\partial] \overline{\mathcal{F}}$, then the λ -bracket $[a_\lambda b] = \sum_{j=0}^{\infty} \lambda^{(j)} (a_{(j)} b)$ defines a \mathbb{C} -linear map

$$\text{Conf}(\mathfrak{g}, \mathcal{F}) \otimes \text{Conf}(\mathfrak{g}, \mathcal{F}) \longrightarrow \mathbb{C}[\lambda] \otimes_{\mathbb{C}} \text{Conf}(\mathfrak{g}, \mathcal{F})$$

The λ -bracket satisfies the conformal sesquilinearity, the skew-symmetry and the Jacobi identity axioms. Therefore, $\mathbb{C}[\partial_z] \overline{\mathcal{F}} = \text{Conf}(\mathfrak{g}, \mathcal{F})$ is a Lie conformal algebra.

The formal distribution Lie algebra associated to a Lie conformal algebra \mathcal{A} is constructed as follows. Let $\text{Lie } \mathcal{A} = \tilde{\mathcal{A}} / \tilde{\partial} \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} = \mathcal{A}[t, t^{-1}]$ with $\tilde{\partial} = \partial + \partial_t$ and the j -th product defined by:

$$(a \otimes f)_{(j)} (b \otimes g) = \sum_{s \in \mathbb{Z}_+} (a_{(j+s)} b) \otimes ((\partial_t^{(s)} f) g), \quad (2.5)$$

where $a, b \in \mathcal{A}$, $f, g \in \mathbb{C}[t, t^{-1}]$ and $j \in \mathbb{Z}_+$.

Let $a_n = a \otimes t^n$, the k -th product (2.5) is written as:

$$(a_m)_{(k)} (b_n) = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(k+j)} b)_{m+n-j}, \quad (2.6)$$

for any $m, n \in \mathbb{Z}$. Now define the Lie bracket on $\text{Lie } \mathcal{A}$ by (2.4). It follows that $\text{Lie } \mathcal{A}$ is a Lie algebra and every $a \in \mathcal{A}$ induces a formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ where $a_n \in \text{Lie } \mathcal{A}$.

Denote by \mathcal{F} the set of such distributions. Then they span $\text{Lie } \mathcal{A}$ and are pairwise local formal distributions because (2.6) is equivalent to (2.2) with $k = 0$ and $a_{(j)}b = 0$ for $j \gg 0$. Therefore, the pair $(\text{Lie } \mathcal{A}, \mathcal{F})$ is a formal distribution Lie algebra.

Observe that the derivation $-1 \otimes \partial_t$ of the 0-th product of the conformal algebra $\tilde{\mathcal{A}}$ induces a derivation T of the Lie algebra $\text{Lie } \mathcal{A}$. The derivation T is given by:

$$T(a_n) = -na_{n-1}, \quad n \in \mathbb{Z}_+.$$

Definition 2.2.4. Let \mathcal{A} be a Lie conformal algebra and let $(\text{Lie } \mathcal{A}, \mathcal{F})$ be the associated formal distribution Lie algebra. The set

$$(\text{Lie } \mathcal{A})_- = \text{span}\{a_n \mid a \in \mathcal{A}, \quad n \in \mathbb{Z}_+\},$$

is a subalgebra of $\text{Lie } \mathcal{A}$, called the **annihilation Lie algebra of \mathcal{A}** . The semidirect sum $(\text{Lie } \mathcal{A})^- = \mathbb{C}T \oplus (\text{Lie } \mathcal{A})_-$ is called the **extended annihilation Lie algebra of \mathcal{A}** .

2.3 Virasoro Conformal Algebra, Vir

Let $\mathfrak{Vect } \mathbb{C}^\times$ be the Lie algebra of regular vector fields on \mathbb{C}^\times with a basis consists of vector fields $t^n \partial_t$ where $n \in \mathbb{Z}$. The $\mathfrak{Vect } \mathbb{C}^\times$ -valued formal distribution

$$L(z) = - \sum_{n \in \mathbb{Z}} (t^n \partial_t) z^{-n-1},$$

satisfies the condition

$$[L(z), L(w)] = \partial_w L(w) \delta(z-w) + 2L(w) \partial_w \delta(z-w),$$

and hence is local with respect to itself. Then, the pair $(\mathfrak{Vect } \mathbb{C}^\times, \{L\})$ is a formal distribution Lie algebra. The corresponding Lie conformal algebra, Vir , is defined by

$$\text{Vir} = \mathbb{C}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L,$$

and is called the *Virasoro conformal algebra*. The maximal formal distribution Lie algebra is $\text{Lie}(\text{Vir}, \{L\}) \cong \mathfrak{Vect} \mathbb{C}^\times$. The corresponding annihilation algebra $(\mathfrak{Vect} \mathbb{C}^\times)_- = \mathfrak{Vect} \mathbb{C}$, the Lie algebra of regular vector fields on \mathbb{C} , and $(\mathfrak{Vect} \mathbb{C}^\times)^- \cong \mathfrak{Vect} \mathbb{C} \oplus \mathfrak{L}$, where \mathfrak{L} is the 1-dimensional Lie algebra [2],[8].

Theorem 2.3.1 ([5], Theorem 3.2). *Every irreducible finite Vir-module is $M_{\Delta, \alpha}$ where $\Delta, \alpha \in \mathbb{C}$ and $\Delta \neq 0$ such that*

$$M_{\Delta, \alpha} = \mathbb{C}[\partial]\nu, \quad L_\lambda \nu = (\partial + \alpha + \Delta\lambda)\nu.$$

2.4 Current Conformal Algebra, $\text{Cur} \mathfrak{g}$

Let $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the associated centerless current algebra to the Lie algebra \mathfrak{g} , endowed with the Lie bracket:

$$[at^m, bt^n] = [a, b]t^{m+n}, \quad a, b \in \mathfrak{g} \quad \text{and} \quad m, n \in \mathbb{Z}.$$

Let

$$a(z) = \sum_{m \in \mathbb{Z}} at^m z^{-m-1},$$

be a $\tilde{\mathfrak{g}}[[z, z^{-1}]]$ -valued formal distributions for $a \in \mathfrak{g}$. Then

$$[a(z), b(w)] = [a, b](w)\delta(z - w).$$

The family $\mathcal{F} = \{a(z) \mid a \in \mathfrak{g}\}$ consists of pairwise local formal distributions and the pair $(\tilde{\mathfrak{g}}, \mathcal{F})$ is called the current formal distribution Lie algebra. The minimal subspace

$$\overline{\mathcal{F}} = \mathbb{C}[\partial]\mathcal{F} \cong \mathbb{C}[\partial] \otimes \mathfrak{g},$$

with the λ -bracket given by:

$$[a_\lambda b] = [a, b], \quad a, b \in \mathfrak{g},$$

is a Lie conformal algebra called the *Current conformal algebra* associated to \mathfrak{g} and is denoted by $\text{Cur} \mathfrak{g}$. The Lie algebra \mathfrak{g} is identified with the subspace of $\text{Cur} \mathfrak{g}$ spanned by elements of the form $1 \otimes g$, where $g \in \mathfrak{g}$. The maximal formal distribution Lie algebra associated to $\text{Cur} \mathfrak{g}$ is $\text{Lie}(\text{Cur} \mathfrak{g}, \mathcal{F})$. The corresponding annihilation Lie algebra

is $\tilde{\mathfrak{g}}_- = \mathfrak{g}[t]$ and the extended annihilation Lie algebra is $\mathbb{C}\partial_t + \mathfrak{g}[t]$ [2, 8].

Lemma 2.4.1 ([20], Example 2.8a). *Let U be a finite dimensional module over a finite dimensional Lie algebra \mathfrak{g} . Then we have:*

1. $\tilde{U} = U[t, t^{-1}]$ is a $\tilde{\mathfrak{g}}$ -module.
2. (\tilde{U}, E) is a conformal module over the current formal distribution Lie algebra $(\tilde{\mathfrak{g}}, \mathcal{F})$, where

$$E = \{u(z) = \sum_{n \in \mathbb{Z}} (ut^n)z^{-n-1} = u\delta(z-t) \mid u \in U\}.$$

3. $M(\tilde{U}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ is a finite conformal $\text{Cur } \mathfrak{g}$ -module defined by

$$a_\lambda u = au, \quad a \in \mathfrak{g}, u \in U.$$

$M(\tilde{U})$ is irreducible iff U is a nontrivial irreducible \mathfrak{g} -module.

Theorem 2.4.1 ([5], Theorem 3.2). *Let \mathfrak{g} be a finite dimensional semisimple Lie algebra. Then every conformal finite irreducible $\text{Cur } \mathfrak{g}$ -module is $M(\tilde{U})$, where U is a finite dimensional irreducible \mathfrak{g} -module.*

2.5 The Standard Semidirect Product, $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$

Let $\text{Cur } \mathfrak{g}$ be the current conformal algebra associated to the finite dimensional Lie algebra \mathfrak{g} . Let $d^L : \text{Cur } \mathfrak{g} \rightarrow \text{Cur } \mathfrak{g}$ be a conformal linear map defined by $d_\lambda^L g = (\partial + \lambda)g$ for all $g \in \mathfrak{g}$ and L is the standard generator of Vir . Note that for any $g, h \in \mathfrak{g}$,

$$\begin{aligned} [(d_\lambda^L g)_{\lambda+\mu} h] + [g_\lambda (d_\mu^L h)] &= [(\partial + \lambda)g_{\lambda+\mu} h] + [g_\lambda (\partial + \mu)h] \\ &= (-\lambda - \mu)[g_{\lambda+\mu} h] + \lambda[g_{\lambda+\mu} h] + (\partial + \lambda)[g_\lambda h] + \mu[g_\lambda h] \\ &= -\mu[g_{\lambda+\mu} h] + (\partial + \lambda + \mu)[g_\lambda h] \\ &= -\mu[g, h] + (\partial + \lambda + \mu)[g, h] \\ &= (\partial + \lambda)[g, h] \\ &= d_\lambda^L [g_\mu h]. \end{aligned}$$

Thus, the map d^L is a conformal derivation and it satisfies $[d^L, d^L] = (\partial + 2\lambda)d^L$. The semidirect product of Vir and $\text{Cur } \mathfrak{g}$, called the standard semidirect product, is the

$\mathbb{C}[\partial]$ -module $\text{Vir} \oplus \text{Cur } \mathfrak{g}$ endowed with the λ -bracket

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [g_\lambda h] = [g, h], \quad [L_\lambda g] = (\partial + \lambda)g,$$

for $g, h \in \mathfrak{g}$. Note that $\text{Cur } \mathfrak{g}$ is an ideal of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ and for any $g, h \in \mathfrak{g}$,

$$[(L, g)_\lambda (L, h)] = ([L_\lambda L], [g, h] + [L_\lambda h] - [L_{-\partial-\lambda} g]).$$

The annihilation Lie algebra of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ is $\mathfrak{Vect } \mathbb{C} \ltimes \tilde{\mathfrak{g}}$, where $\mathfrak{Vect } \mathbb{C}$ is the Lie algebra of regular vector fields on \mathbb{C} , and $\tilde{\mathfrak{g}} = \mathfrak{g}[t]$, the positive part of the associated current algebra $\mathfrak{g}[t, t^{-1}]$. [2, 8].

Theorem 2.5.1 ([5], Theorem 3.2). *Every nontrivial finite irreducible conformal module over $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ is $M(\Delta, \alpha, U) = \mathbb{C}[\partial] \otimes U$, where U is a finite dimensional irreducible \mathfrak{g} -module, which is nontrivial if $\Delta = 0$, and*

$$L_\lambda u = (\alpha + \partial + \Delta\lambda)u, \quad g_\lambda u = g \cdot u,$$

where $\Delta, \alpha \in \mathbb{C}$, $u \in U$ and $g \in \mathfrak{g}$.

2.6 Classification of Finite Simple and Semisimple Lie Conformal Algebras

We now review the classification of finite simple and semisimple Lie conformal algebras as studied in [8].

Definition 2.6.1. *A Lie conformal algebra \mathcal{A} is called **simple** if it is not commutative and contains no nontrivial proper ideals.*

Theorem 2.6.1 ([8], Theorem 5.1). *A simple finite Lie conformal algebra is isomorphic either to a current conformal algebra $\text{Cur } \mathfrak{g}$, where \mathfrak{g} is a simple finite dimensional Lie algebra, or to the Virasoro conformal algebra, Vir .*

Definition 2.6.2. *A Lie conformal algebra is called **semisimple** if it contains no non-zero abelian ideals.*

Theorem 2.6.2 ([8], Theorem 7.1). *Any finite semisimple Lie conformal algebra is uniquely decomposed in a finite direct sum of Lie conformal algebras each of which is isomorphic to one of the following three types:*

1. Virasoro conformal algebra; Vir .
2. Current conformal algebra; $\text{Cur } \mathfrak{g}$, where \mathfrak{g} is a simple finite dimensional Lie algebra.
3. semidirect product of Vir and $\text{Cur } \mathfrak{g}$, defined by $[L_\lambda g] = (\partial + \lambda)g$, $g \in \mathfrak{g}$.

2.7 Cohomology of Lie Conformal Algebras

In this section, we will review the notations and the primary results of the Lie conformal algebra cohomology following the work of [2].

Definition 2.7.1 ([2], Definition 2.1). An n -**cochain**, ($n \in \mathbb{Z}_+$) of a conformal algebra \mathcal{A} with coefficients in a module M over it is a \mathbb{C} -linear map

$$\begin{aligned} \gamma : \mathcal{A}^{\otimes n} &\rightarrow M[\lambda_1, \dots, \lambda_n] \\ a_1 \otimes \dots \otimes a_n &\mapsto \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \end{aligned}$$

satisfying the following conditions:

i Conformal antilinearity:

$$\gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, \partial a_i, \dots, a_n) = -\lambda_i \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_i, \dots, a_n), \text{ for all } i.$$

ii Skew-symmetry:

$$\gamma_{\lambda_1, \dots, \lambda_i, \lambda_j, \dots, \lambda_n}(a_1, \dots, a_i, a_j, \dots, a_n) = -\gamma_{\lambda_1, \dots, \lambda_j, \lambda_i, \dots, \lambda_n}(a_1, \dots, a_j, a_i, \dots, a_n), \text{ for all } i, j.$$

A 0-cochain is an element of the module M , where $A^{\otimes 0} = \mathbb{C}$. If the module M is not conformal, we consider the space of formal power series $M[[\lambda_1, \dots, \lambda_n]]$ instead of the space of polynomials $M[\lambda_1, \dots, \lambda_n]$ in the Definition 2.7.1.

The **differential** d of a n -cochain γ is defined by;

$$\begin{aligned} (d\gamma)_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} a_i \lambda_i \gamma_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) \\ &+ \sum_{\substack{i, j=1 \\ i < j}}^{n+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}}([a_i \lambda_i a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{n+1}). \end{aligned} \tag{2.7}$$

If $\gamma \in M$, is a 0-cochain, then $(d\gamma)_\lambda(a) = a_\lambda \gamma$. The differential d preserves the space of cochains, and $d^2 = 0$. Hence, the space of cochains of a conformal algebra \mathcal{A} with coefficients in a module M form a complex, called the **basic complex**,

$$\tilde{C}^\bullet = \tilde{C}^\bullet(\mathcal{A}, M) = \bigoplus_{n \in \mathbb{Z}_+} \tilde{C}^n(\mathcal{A}, M).$$

The basic complex $\tilde{C}^\bullet(\mathcal{A}, M)$ is a $\mathbb{C}[\partial]$ -module with action,

$$(\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = (\partial_M + \sum_{i=1}^n \lambda_i) \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n), \quad (2.8)$$

where ∂_M is the action of ∂ on M . Moreover, the differential commutes with the action ∂ . Hence, the graded subspace $\partial \tilde{C}^\bullet(\mathcal{A}, M)$ forms a subcomplex of the complex $\tilde{C}^\bullet(\mathcal{A}, M)$. The quotient complex that is defined by,

$$C^\bullet = C^\bullet(\mathcal{A}, M) = \tilde{C}^\bullet(\mathcal{A}, M) / \partial \tilde{C}^\bullet(\mathcal{A}, M) = \bigoplus_{n \in \mathbb{Z}_+} C^n(\mathcal{A}, M). \quad (2.9)$$

is called the **reduced complex**.

Definition 2.7.2 ([2], Definition 2.2). *The **basic cohomology** $\tilde{H}^\bullet(\mathcal{A}, M)$ of a conformal algebra \mathcal{A} with coefficients in a module M is the cohomology of the basic complex $\tilde{C}^\bullet(\mathcal{A}, M)$. The **reduced cohomology** $H^\bullet(\mathcal{A}, M)$ is the cohomology of the reduced complex $C^\bullet(\mathcal{A}, M)$.*

A q -cochain $\gamma \in \tilde{C}^q(\mathcal{A}, M)$ is called a **q -cocycle** if $d\gamma = 0$, and is called a **q -coboundary** or a **trivial q -cocycle** if there is a $(q-1)$ -cochain $\phi \in \tilde{C}^{q-1}(\mathcal{A}, M)$ such that $\gamma = d\phi$. Furthermore, a q -cochain $\gamma \in \tilde{C}^q(\mathcal{A}, M)$ is called a **reduced q -cocycle** if $d\gamma = \partial\phi$ for some $\phi \in \tilde{C}^{q+1}(\mathcal{A}, M)$. Two q -cochains γ and ψ are equivalent if $\gamma - \psi$ is a q -coboundary. Denote by $\tilde{D}^q(\mathcal{A}, M)$ and by $\tilde{B}^q(\mathcal{A}, M)$ the spaces of q -cocycles and q -coboundaries, respectively. Then we have,

$$\tilde{H}^q(\mathcal{A}, M) = \tilde{D}^q(\mathcal{A}, M) / \tilde{B}^q(\mathcal{A}, M) = \{ \text{equivalent classes of } q\text{-cocycles} \}.$$

The following proposition and theorem describe the relation between the basic cohomology $\tilde{H}^\bullet(\mathcal{A}, M)$ and the reduced cohomology $H^\bullet(\mathcal{A}, M)$.

Proposition 2.7.1 ([2], Remark 2.3). *The short exact sequence of complexes $0 \rightarrow \partial \tilde{C}^\bullet \xrightarrow{\iota} \tilde{C}^\bullet \xrightarrow{\pi} C^\bullet \rightarrow 0$ where ι is the embedding and π is the natural projection gives*

the long exact sequence of cohomology groups

$$\begin{aligned}
0 \longrightarrow H^0(\partial\tilde{C}^\bullet) &\longrightarrow \tilde{H}^0(\mathcal{A}, M) \longrightarrow H^0(\mathcal{A}, M) \longrightarrow \\
&\longrightarrow H^1(\partial\tilde{C}^\bullet) \longrightarrow \tilde{H}^1(\mathcal{A}, M) \longrightarrow H^1(\mathcal{A}, M) \longrightarrow \\
&\longrightarrow H^2(\partial\tilde{C}^\bullet) \longrightarrow \tilde{H}^2(\mathcal{A}, M) \longrightarrow H^2(\mathcal{A}, M) \longrightarrow \dots
\end{aligned} \tag{2.10}$$

Theorem 2.7.1 ([2], Proposition 2.1). *The complexes \tilde{C}^\bullet and $\partial\tilde{C}^\bullet$ are isomorphic under the map $\gamma \mapsto \partial\gamma$ in degrees ≥ 1 . Therefore, $\tilde{H}^q(\mathcal{A}, M) \cong H^q(\partial\tilde{C}^\bullet)$ for all $q \geq 1$, and for all $q \geq 0$ if the module M is $\mathbb{C}[\partial]$ -free. Moreover, the sequence $0 \longrightarrow \text{Ker } \partial[0] \longrightarrow \tilde{H}^0(\mathcal{A}, M) \longrightarrow H^0(\partial\tilde{C}^\bullet) \longrightarrow 0$, where $\text{Ker } \partial[0]$ is the subcomplex $\text{Ker } \partial \subset \tilde{C}^\bullet$ concentrated in degree zero, is exact.*

In the next theorem, we review the low degree cohomology spaces of a Lie conformal algebra \mathcal{A} with coefficients in an \mathcal{A} -module M .

Theorem 2.7.2 ([2], Theorem 3.1). 1. $\tilde{H}^0(\mathcal{A}, M) = M^\mathcal{A} = \{v \in M \mid a_\lambda v = 0 \ \forall a \in \mathcal{A}\}$.

2. *The isomorphism classes of extensions $0 \rightarrow M \rightarrow E \rightarrow \mathbb{C} \rightarrow 0$ of the trivial \mathcal{A} -module \mathbb{C} by a conformal \mathcal{A} -module M correspond bijectively to $H^0(\mathcal{A}, M)$.*
3. *The isomorphism classes of $\mathbb{C}[\partial]$ -split extensions $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ of conformal modules over a conformal algebra \mathcal{A} correspond bijectively to $H^1(\mathcal{A}, \text{Chom}(N, M))$, where M and N are assumed to be finite and $\text{Chom}(N, M)$ is the \mathcal{A} -module of conformal linear maps from N to M . If, in particular, $N = \mathbb{C}$ is the trivial module, then there exist no nontrivial $\mathbb{C}[\partial]$ -split extensions.*
4. *Let C be a conformal \mathcal{A} -module, considered as a conformal algebra with respect to the zero λ -bracket. Then the equivalence classes of $\mathbb{C}[\partial]$ -split “abelian” extensions $0 \rightarrow C \rightarrow \tilde{\mathcal{A}} \rightarrow \mathcal{A} \rightarrow 0$ of the conformal algebra \mathcal{A} correspond bijectively to $H^2(\mathcal{A}, C)$.*
5. *The equivalence classes of first-order deformations of a conformal algebra \mathcal{A} , leaving the $\mathbb{C}[\partial]$ -action intact, correspond bijectively to $H^2(\mathcal{A}, \mathcal{A})$.*

For any $a \in \mathcal{A}$, the \mathcal{A} -module structure on the basic complex $\tilde{C}^\bullet(\mathcal{A}, M)$ is defined as

follows

$$\begin{aligned}
(\theta_\lambda(a)\gamma)_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) &= a_\lambda \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \\
&\quad - \sum_{i=1}^n \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_n}(a_1, \dots, [a_\lambda a_i], \dots, a_n).
\end{aligned} \tag{2.11}$$

Define the operator $\iota_\lambda(a)$ by

$$(\iota_\lambda(a)\gamma)_{\lambda_1, \dots, \lambda_{n-1}}(a_1, \dots, a_{n-1}) = \gamma_{\lambda, \lambda_1, \dots, \lambda_{n-1}}(a, a_1, \dots, a_{n-1}). \tag{2.12}$$

It follows that Cartan's identity

$$d\iota_\lambda + \iota_\lambda d = \theta_\lambda, \tag{2.13}$$

holds for Lie conformal algebras, and hence $d\theta = \theta d$. Therefore, the induced action of \mathcal{A} on the basic cohomology $\tilde{H}^\bullet(\mathcal{A}, M)$ is trivial.

Let \mathcal{A} be a conformal algebra and M is a conformal \mathcal{A} -module. Then M is a module over the annihilation Lie algebra $\mathfrak{g}_- = (\text{Lie } \mathcal{A})_-$. Let $C^\bullet(\mathfrak{g}_-, M)$ be the Chevalley–Eilenberg complex defining the cohomology of \mathfrak{g}_- with coefficients in M . Then, $C^\bullet(\mathfrak{g}_-, M)$ has the $\mathbb{C}[\partial]$ -module structure given by

$$\begin{aligned}
(\partial\gamma)(a_1 \otimes \dots \otimes a_n) &= \partial(\gamma(a_1 \otimes \dots \otimes a_n)) \\
&\quad - \sum_{i=1}^n \gamma(a_1 \otimes \dots \otimes \partial a_i \otimes \dots \otimes a_n),
\end{aligned} \tag{2.14}$$

for $\gamma \in C^n(\mathfrak{g}_-, M)$. From [2], we have the following theorem.

Theorem 2.7.3 ([2], Theorem 6.1). *The basic complex, $\tilde{C}^\bullet(\mathcal{A}, M)$ is isomorphic to the Chevalley–Eilenberg complex, $C^\bullet(\mathfrak{g}_-, M)$, and the isomorphism is compatible with the $\mathbb{C}[\partial]$ -action. Consequently, $C^\bullet(\mathcal{A}, M) \cong C^\bullet(\mathfrak{g}_-, M)/\partial C^\bullet(\mathfrak{g}_-, M)$.*

Corollary 2.7.1 ([2], Corollary 6.1). $\tilde{H}^\bullet(\mathcal{A}, M) \cong H^\bullet(\mathfrak{g}_-, M)$.

The study of the cohomology of simple conformal algebras, Vir and $\text{Cur } \mathfrak{g}$, was done in [2] as follows.

Theorem 2.7.4 ([2], Theorem 7.1 & Remark 7.2). *For the Virasoro conformal algebra, the following statements hold:*

1. For the trivial Vir -module \mathbb{C} ,

$$\dim \tilde{H}^q(\text{Vir}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0 \text{ or } 3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim H^q(\text{Vir}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 2 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

2. For the module \mathbb{C}_a , $a \neq 0$, $\tilde{H}^q(\text{Vir}, \mathbb{C}_a) \cong \tilde{H}^q(\text{Vir}, \mathbb{C})$, and $H^q(\text{Vir}, \mathbb{C}_a) = 0$, for all q .

Theorem 2.7.5 ([2], Theorem 8.1 & Remark 8.1). *For the current conformal algebra, we have:*

1. For the trivial $\text{Cur } \mathfrak{g}$ -module \mathbb{C} ,

$$\tilde{H}^\bullet(\text{Cur } \mathfrak{g}, \mathbb{C}) = H^\bullet(\mathfrak{g}, \mathbb{C}),$$

and

$$H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) = H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C}), \quad \text{for all } q.$$

2. For the module \mathbb{C}_a , $a \neq 0$, $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a) \cong \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, and $H^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a) = 0$, for all q .

Chapter 3

Spectral Sequences

The aim of this chapter is to review a few concepts on spectral sequences needed to describe the Hochschild-Serre spectral sequence for the Lie conformal algebras. The majority of this chapter's material is adapted from [24].

3.1 Definitions and Basic Properties

Definition 3.1.1. *A first quadrant cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ for all $r, p, q \geq 0$, is a collection of bigraded $\mathbb{C}[\partial]$ -modules $\{E_r^{p,q}\}$ together with a $\mathbb{C}[\partial]$ -linear map*

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1},$$

*called the **differential** such that*

$$d_r^{p-r, q+r-1} \circ d_r^{p,q} = 0,$$

and

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r),$$

where $H^{p,q}(E_r^{,*}, d_r) = \ker d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1} / \operatorname{im} d_r^{p-r, q+r-1} : E_r^{p-r, q+r-1} \longrightarrow E_r^{p,q}$.*

Observe that $E_{r+1}^{*,*}$ (not d_{r+1}) is determined by $E_r^{*,*}$ and d_r . The term $E_r^{*,*}$ is called the E_r -**term** or E_r -**page**, and the indexing can begin at any integer, regularly at 2. One can visualize the spectral sequence as a grid notebook, where each page refers to the E_r -term equipped with the differentials for all r . Here p, q refer to the position on the grid, where p is the x -coordinate and the degree q is the y -coordinate. The following

diagrams show the E_r -term for small r :

3	$E_0^{0,3}$	$E_0^{1,3}$	$E_0^{2,3}$	$E_0^{3,3}$	$E_0^{4,3}$
	\uparrow	\uparrow	\uparrow	\uparrow	\uparrow
2	$E_0^{0,2}$	$E_0^{1,2}$	$E_0^{2,2}$	$E_0^{3,2}$	$E_0^{4,2}$
	\uparrow	\uparrow	\uparrow	\uparrow	\uparrow
1	$E_0^{0,1}$	$E_0^{1,1}$	$E_0^{2,1}$	$E_0^{3,1}$	$E_0^{4,1}$
	\uparrow	\uparrow	\uparrow	\uparrow	\uparrow
0	$E_0^{0,0}$	$E_0^{1,0}$	$E_0^{2,0}$	$E_0^{3,0}$	$E_0^{4,0}$
<hr/>					
	0	1	2	3	4

Figure 3.1: $E_0^{p,q}$ -page.

3	$E_1^{0,3} \longrightarrow E_1^{1,3} \longrightarrow E_1^{2,3} \longrightarrow E_1^{3,3} \longrightarrow E_1^{4,3}$
2	$E_1^{0,2} \longrightarrow E_1^{1,2} \longrightarrow E_1^{2,2} \longrightarrow E_1^{3,2} \longrightarrow E_1^{4,2}$
1	$E_1^{0,1} \longrightarrow E_1^{1,1} \longrightarrow E_1^{2,1} \longrightarrow E_1^{3,1} \longrightarrow E_1^{4,1}$
0	$E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0} \longrightarrow E_1^{3,0} \longrightarrow E_1^{4,0}$
<hr/>	
	0 1 2 3 4

Figure 3.2: $E_1^{p,q}$ -page.

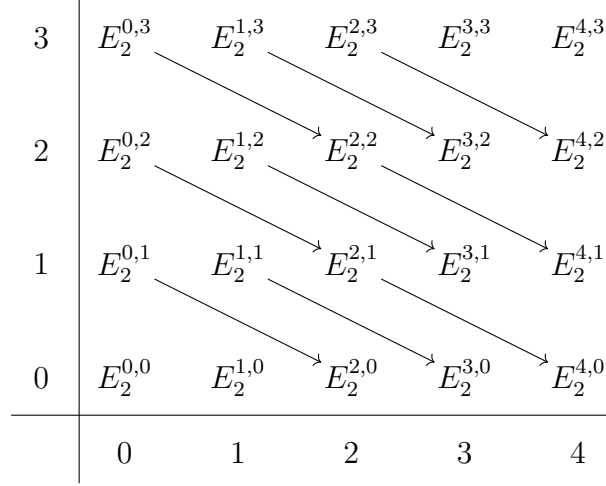


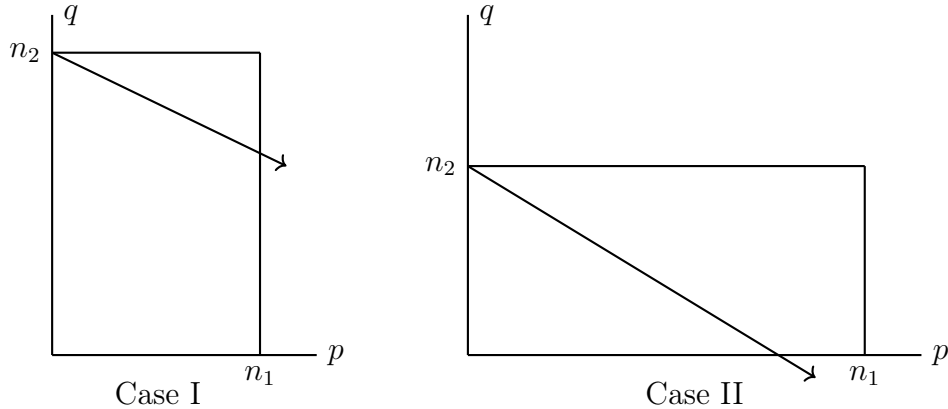
Figure 3.3: $E_2^{p,q}$ -page.

Now consider $E_r^{p,q}$ for $r > \max(p, q+1)$. Since $r > q+1$ implies $E_r^{p+r, q-r+1} = 0$, $\ker d_r = E_r^{p,q}$. Also, $r > p$ implies that $E_r^{p-r, q+r-1} = 0$, i.e., $\text{im } d_r = 0$. Hence, $E_{r+1}^{p,q} = \ker d_r = E_r^{p,q}$. Continuing the same way, we have $E_{r+k}^{p,q} = E_r^{p,q}$ for all $k \geq 0$. The $\mathbb{C}[\partial]$ -module $E_r^{p,q}$ in this case is denoted by $E_\infty^{p,q}$.

Definition 3.1.2. *The spectral sequence collapses at the N -th term if $d_r = 0$ for all $r \geq N$, and we write $E_N^{*,*} \cong E_{N+1}^{*,*} \cong \dots \cong E_\infty^{*,*}$.*

Lemma 3.1.1 ([24], Example 1.B.). *Let $\{E_r^{*,*}, d_r\}$ be a first quadrant spectral sequence. Suppose that $E_2^{p,q} = 0$ for all $p > n_1$ and $q > n_2$ where $n_1, n_2 \in \mathbb{Z}_+$. Then the spectral sequence collapses at the N^{th} -term where $N = \min(n_1 + 1, n_2 + 2)$.*

Proof. We need to show that $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ is zero for $r \geq N$. The E_r -term can be pictured in the following two cases:



Case I: Let $n_1 \leq n_2 + 1$ then $N = n_1 + 1$. Since $r \geq N$ implies that $p + r \geq p + N > n_1$, then $E_r^{p+r, q-r+1} = 0$. Hence, $d_r = 0$.

Case II: Let $n_2 + 1 < n_1$, then $N = n_2 + 2$. But, $q - r + 1 \leq q - n_2 - 1 < 0$. Thus, $E_r^{p+r, q-r+1} = 0$ and $d_r = 0$.

Therefore, the spectral sequence collapses at E_N and $E_N^{*,*} \cong E_{N+1}^{*,*} \cong \dots \cong E_\infty^{*,*}$. \square

Now, we will describe the spectral sequence as subquotient $\mathbb{C}[\partial]$ -module of E_2 . For each $r \geq 2$, write

$$\begin{aligned} Z_r &= Z_r^{p,q} = \ker d_r^{p,q}, \\ B_r &= B_r^{p,q} = \operatorname{im} d_r^{p-r, q+r-1}, \\ E_r &= E_r^{p,q}. \end{aligned}$$

Using $d \circ d = 0$, we get $B_r \subseteq Z_r \subseteq E_r$. From the definition, $E_{r+1} \cong Z_r/B_r$. Now denote $\bar{Z}_{r+1} = \ker d_{r+1}$, a $\mathbb{C}[\partial]$ -submodule of E_r . It can be written as $\bar{Z}_{r+1} = Z_{r+1}/B_r$, where Z_{r+1} is an $\mathbb{C}[\partial]$ -submodule of Z_r . Similarly, denote $\bar{B}_{r+1} = \operatorname{im} d_{r+1}$ which is isomorphic to $d(Z_{r+1})/B_r \cong B_{r+1}/B_r$, with B_{r+1} is a $\mathbb{C}[\partial]$ -submodule of Z_r . Hence we have,

$$E_{r+2} \cong \bar{Z}_{r+1}/\bar{B}_{r+1} \cong (Z_{r+1}/B_r)/(B_{r+1}/B_r) \cong Z_{r+1}/B_{r+1}.$$

which can be represented as a tower of inclusions $B_r \subset B_{r+1} \subset Z_{r+1} \subset Z_r$. Therefore, the spectral sequence can be written as an infinite sequence of $\mathbb{C}[\partial]$ -submodules of E_2 as follows:

$$B_2 \subset B_3 \subset \dots \subset B_n \subset \dots \subset Z_n \subset \dots \subset Z_3 \subset Z_2 \subset E_2. \quad (3.1)$$

with $E_{n+1} \cong Z_n/B_n$. Then we have the short exact sequence induced by d_{n+1} ,

$$0 \longrightarrow Z_{n+1}/B_n \longrightarrow Z_n/B_n \xrightarrow{d_{n+1}} B_{n+1}/B_n \longrightarrow 0.$$

which induces an isomorphism $Z_n/Z_{n+1} \cong B_{n+1}/B_n$ for all n .

Now, suppose that the spectral sequence collapses at the N^{th} -term. Denote

$$Z_\infty = Z_\infty^{p,q} = \bigcap_{r=0}^{\infty} Z_r^{p,q}, \quad B_\infty = B_\infty^{p,q} = \bigcup_{r=0}^{\infty} B_r^{p,q}$$

Then, $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$, and the tower of $\mathbb{C}[\partial]$ -submodules (3.1) becomes

$$\begin{aligned} B_2 \subset B_3 \subset \dots \subset B_{N-1} = B_N = \dots = B_\infty \\ \subset Z_\infty = \dots = Z_N = Z_{N-1} \subset \dots \subset Z_3 \subset Z_2 \subset E_2. \end{aligned}$$

3.2 Spectral Sequence of Filtered Differential Modules

Spectral sequences arise naturally in two general ways: the first is from filtered differential modules, and the second is from exact couples. These approaches are equivalent to each other. In this section, we will review the spectral sequence of a filtered complex.

A **filtration** F^* on a $\mathbb{C}[\partial]$ -module A is a family of submodules $\{F^p A\}$ for $p \in \mathbb{Z}$, such that

$$\begin{aligned} \dots \subset F^{p+1} A \subset F^p A \subset F^{p-1} A \subset \dots \subset A & \quad (\text{decreasing filtration}), \\ \text{or } \dots \subset F^{p-1} A \subset F^p A \subset F^{p+1} A \subset \dots \subset A & \quad (\text{increasing filtration}). \end{aligned}$$

The filtration is called a **bounded** filtration if there exist $p, q \in \mathbb{Z}$ such that $F^p A = 0$ and $F^q A = A$. For any filtered $\mathbb{C}[\partial]$ -module A , its **associated graded $\mathbb{C}[\partial]$ -module**, $E_0^p(A)$, is given by

$$E_0^p(A) = \begin{cases} F^p A / F^{p+1} A, & \text{when } F \text{ is decreasing,} \\ F^p A / F^{p-1} A, & \text{when } F \text{ is increasing.} \end{cases}$$

If A^* is a filtered graded $\mathbb{C}[\partial]$ -module, then one can define a filtration on each degree by $F^p A^n = F^p A^* \cap A^n$. The **associated bigraded $\mathbb{C}[\partial]$ -module** to A can be defined by:

$$E_0^{p,q}(A^*, F) = \begin{cases} F^p A^{p+q} / F^{p+1} A^{p+q}, & \text{when } F \text{ is decreasing,} \\ F^p A^{p+q} / F^{p-1} A^{p+q}, & \text{when } F \text{ is increasing.} \end{cases}$$

We now use the associated graded $\mathbb{C}[\partial]$ -module definition to describe the convergence of spectral sequences. Here and further, we will consider the case of a decreasing filtration.

Definition 3.2.1. A first quadrant cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ is said to **converge** to a graded $\mathbb{C}[\partial]$ -module H^* , often written $E_r^{p,q} \Rightarrow H^{p+q}$, if there is a bounded

filtration F on H^* such that

$$E_{\infty}^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$$

where $E_{\infty}^{*,*}$ is the **limit** term of the spectral sequence.

Now we give the definition of the filtered differential graded $\mathbb{C}[\partial]$ -module, the object needed to construct a spectral sequence.

Definition 3.2.2. A filtered differential graded $\mathbb{C}[\partial]$ -module A is a module over $\mathbb{C}[\partial]$ such that the following conditions are satisfied:

- (i) $A = \bigoplus_{n=0}^{\infty} A^n$,
- (ii) There is a $\mathbb{C}[\partial]$ -linear map, $d : A \rightarrow A$ of degree 1 satisfying $d \circ d = 0$,
- (iii) A has a decreasing filtration F such that $d : F^p A \rightarrow F^p A$.

Because the differential preserves the filtration, $d(F^p A^n) \subseteq F^p A^{n+1}$ for all p and n . Then the filtration F induces a filtration on the cohomology of A , with $F^p H(A, d)$ defined as the image of $H(F^p A, d)$ under the map induced by the inclusion $F^p A \hookrightarrow A$.

Now that the fundamental definitions are in place, we give the main theorem.

Theorem 3.2.1 ([24], Theorem 2.6). Suppose (A, d, F^*) is a filtered differential graded $\mathbb{C}[\partial]$ -module, where d has degree 1. Then there exists a spectral sequence $\{E_r^{*,*}\}$ of cohomological type such that

$$E_1^{p,q} \cong H^{p+q}(F^p A / F^{p+1} A).$$

Moreover, if the filtration is bounded, then the spectral sequence converges to $H^*(A, d)$, i.e.,

$$E_{\infty}^{p,q} \cong F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d).$$

Proof. Consider the following decreasing filtration on A :

$$\dots \subset F^{p+1} A^{p+q} \subset F^p A^{p+q} \subset F^{p-1} A^{p+q} \subset \dots$$

such that the differential preserves the filtration, i.e., $d(F^p A^{p+q}) \subset F^p A^{p+q+1}$. For all $r \geq 0$, denote:

$$\begin{aligned} Z_r^{p,q} &= \text{elements in } F^p A^{p+q} \text{ that have boundaries in } F^{p+r} A^{p+q+1} \\ &= F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1}). \\ B_r^{p,q} &= \text{elements in } F^p A^{p+q} \text{ that form the image of } d \text{ from } F^{p-r} A^{p+q-1} \\ &= F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1}). \\ Z_\infty^{p,q} &= \ker d \cap F^p A^{p+q}. \\ B_\infty^{p,q} &= \text{im } d \cap F^p A^{p+q}. \end{aligned}$$

Since the filtration is decreasing and d respects the filtration, we obtain the following tower of $\mathbb{C}[\partial]$ -submodules:

$$B_0^{p,q} \subset B_1^{p,q} \subset B_2^{p,q} \subset \dots \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset \dots \subset Z_2^{p,q} \subset Z_1^{p,q} \subset Z_0^{p,q}.$$

Note that $d(Z_r^{p-r,q+r-1}) = d(F^{p-r} A^{p+q-1} \cap d^{-1}(F^p A^{p+q})) = F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1}) = B_r^{p,q}$.

For all $0 \leq r \leq \infty$, define

$$E_r^{p,q} = Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q})$$

and let $\eta_r^{p,q} : Z_r^{p,q} \rightarrow E_r^{p,q}$ be the canonical projection with $\ker \eta_r^{p,q} = Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$. Note that $d(Z_r^{p,q}) = B_r^{p+r,q-r+1} \subset Z_r^{p+r,q-r+1}$ and,

$$\begin{aligned} d(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) &= d(Z_{r-1}^{p+1,q-1}) + d(B_{r-1}^{p,q}) \\ &\subseteq B_{r-1}^{p+r,q-r+1} + 0 \\ &\subseteq Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1}. \end{aligned}$$

Hence, the differential d as a map $d : Z_r^{p,q} \rightarrow Z_r^{p+r,q-r+1}$ induces a homomorphism $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ such that the following diagram commutes:

$$\begin{array}{ccc}
Z_r^{p,q} & \xrightarrow{d} & Z_r^{p+r,q-r+1} \\
\eta_r^{p,q} \downarrow & & \downarrow \eta_r^{p+r,q-r+1} \\
E_r^{p,q} & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}
\end{array}$$

It follows that $d_r \circ d_r = 0$, and we have

$$E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1}.$$

Now consider the following diagram:

$$\begin{array}{ccccccc}
Z_r^{p+1,q-1} + B_r^{p,q} & \hookrightarrow & Z_{r+1}^{p,q} & \hookrightarrow & Z_r^{p,q} & \xrightarrow{d} & Z_r^{p+r,q-r+1} \\
& & \downarrow \eta_r^{p,q}|_{Z_{r+1}^{p,q}} & & \downarrow \eta_r^{p,q} & & \downarrow \eta_r^{p+r,q-r+1} \\
& & \ker d_r & \xrightarrow{\quad} & E_r^{p,q} & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1} \\
& \nearrow \gamma & \downarrow & & & & \\
& & H^{p,q}(E_r^{*,*}, d_r) & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

We first prove that $\ker d_r^{p,q} = \eta_r^{p,q}(Z_{r+1}^{p,q})$. Observe that, $d_r^{p,q}(\eta_r^{p,q}(z)) = 0 \Leftrightarrow dz \in Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1}$, which is, by definition, equivalent to $z \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$. Hence, $\ker d_r^{p,q} = \eta_r^{p,q}(Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}) = \eta_r^{p,q}(Z_{r+1}^{p,q})$, because $Z_{r-1}^{p+1,q-1} \in \ker \eta_r^{p,q}$.

Now, observe that $\text{im } d_r^{p-r,q+r-1} = \eta_r^{p,q}d(Z_r^{p-r,q+r-1}) = \eta_r^{p,q}(B_r^{p,q})$, hence we have:

$$\begin{aligned}
(\eta_r^{p,q})^{-1}(\text{im } d_r^{p-r,q+r-1}) &= B_r^{p,q} + \ker \eta_r^{p,q} \\
&= B_r^{p,q} + Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \\
&= B_r^{p,q} + Z_{r-1}^{p+1,q-1},
\end{aligned}$$

and by definition,

$$\begin{aligned}
Z_{r-1}^{p+1,q-1} \cap Z_{r+1}^{p,q} &= (F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1})) \cap (F^pA^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1})) \\
&= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1}) = Z_r^{p+1,q-1}.
\end{aligned}$$

Therefore we get:

$$Z_{r+1}^{p+1,q-1} \cap (\eta_r^{p,q})^{-1} \text{im } d_r^{p-r,q+r-1} = Z_r^{p+1,q-1} + B_r^{p,q}.$$

Now we describe the isomorphism $E_{r+1} \cong H^{p,q}(E_r^{*,*}, d_r)$. Let γ , the dashed map in the diagram, be the composition of $\eta_r^{p,q}|_{Z_{r+1}^{p,q}}$ with the canonical projection $\pi : \ker d_r \rightarrow H^{p,q}(E_r^{*,*}, d_r)$. The kernel of γ is $Z_{r+1}^{p+1,q-1} \cap (\eta_r^{p,q})^{-1}(\text{im } d_r^{p-r,q+r-1})$. Because γ is surjective, by the first isomorphism theorem, we get:

$$E_{r+1}^{p,q} = Z_{r+1}^{p,q} / Z_{r+1}^{p+1,q-1} + B_r^{p,q} \xrightarrow{\cong} H^{p,q}(E_r^{*,*}, d_r).$$

Using the definition, we have $E_0^{p,q} = Z_0^{p,q} / (Z_{-1}^{p+1,q-1} + B_{-1}^{p,q})$ such that

$$Z_{-1}^{p+1,q-1} = F^{p+1}A^{p+q}, \quad B_{-1}^{p,q} = d(F^{p+1}A^{p+q-1}).$$

But d respects the filtration, so we have:

$$\begin{aligned} E_0^{p,q} &= F^p A^{p+q} \cap d^{-1}(F^p A^{p+q+1}) / F^{p+1} A^{p+q} + d(F^{p+1} A^{p+q-1}) \\ &= F^p A^{p+q} / F^{p+1} A^{p+q}. \end{aligned}$$

The differential d_0 is induced by the differential $d(F^p A^{p+q}) \subseteq F^p A^{p+q+1}$. Thus we have,

$$E_1^{p,q} \cong H^{p,q}(F^p A^{p+q} / F^{p+1} A^{p+q}).$$

Now consider $\eta_\infty^{p,q} : Z_\infty^{p,q} \rightarrow E_\infty^{p,q}$ and $\pi : \ker d \rightarrow H(A, d)$, then we obtain:

$$\begin{aligned} F^p H^{p+q}(A, d) &= \text{im } (H^{p+q}(F^p A, d) \hookrightarrow H^{p+q}(A, d)) \\ &= \pi(F^p A^{p+q} \cap \ker d) = \pi(Z_\infty^{p,q}). \end{aligned}$$

Observe that $\pi(\ker \eta_\infty^{p,q}) = \pi(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) = F^{p+1} H^{p+q}(A, d)$. Thus, π induces a map $d_\infty : E_\infty^{p,q} \rightarrow F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d)$ with

$$\begin{aligned} \ker d_\infty &= \eta_\infty^{p,q}(\pi^{-1}(F^{p+1} H^{p+q}(A, d)) \cap Z_\infty^{p,q}) \\ &= \eta_\infty^{p,q}(Z_\infty^{p+1,q-1} \cap d(A) \cap Z_\infty^{p,q}) \\ &\subset \eta_\infty^{p,q}(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) = \{0\}. \end{aligned}$$

Hence, d_∞ is an isomorphism, i.e., $E_\infty^{p,q} \cong F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d)$. \square

3.3 Applications

An important observation to make about spectral sequences is that one can proceed with computation and describing the algebraic structure of some E_r -term without knowledge of the differentials, d_r . In this section, we recall some important applications about spectral sequences that will be needed in this dissertation.

Proposition 3.3.1 ([4], Proposition 5.3, 5.3a, 5.5). **(The edge maps)** *Suppose that the filtration on H^n is convergent then the following hold:*

- (i) *If $E_\infty^{m,n-m} = 0$ for $m > p$ then $F^{p+1}H^n = 0$ and there exists a monomorphism $E_r^{p,n-p} \rightarrow E_\infty^{p,n-p} \hookrightarrow H^n(A)$.*
- (ii) *If $E_\infty^{m,n-m} = 0$ for $m < p$ then $H^n = F^p H^n$ and there exists an epimorphism $H^n(A) \rightarrow E_\infty^{p,n-p} \hookrightarrow E_r^{p,n-p}$.*
- (iii) *If $E_\infty^{m,n-m} = 0$ for $m \neq p, p+k$ where $k > 0$. Then there is a short exact sequence*

$$0 \longrightarrow E_\infty^{p+k,n-p-k} \longrightarrow H^n(A) \longrightarrow E_\infty^{p,n-p} \longrightarrow 0.$$

Example 3.3.1 ([24], Example 1.A). **(The five-term exact sequence)** *Suppose that there is a first quadrant cohomological spectral sequence such that $E_2^{p,q} \Rightarrow H^{p+q}(A)$. Then $H^0(A) = E_2^{0,0}$, and there is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(A) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(A). \quad (3.2)$$

Remark 3.3.1. The five-term exact sequence can be extended to the following **seven-term exact sequence**

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(A) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow \ker(H^2(A) \rightarrow E_2^{0,2}) \longrightarrow E_2^{1,1} \longrightarrow E_2^{3,0}. \quad (3.3)$$

Theorem 3.3.1 ([4], Theorem 5.12.). *Assume that the filtration is convergent and $E_2^{p,q} = 0$ for $p, q < 0$. Assume further that $E_2^{p,q} = 0$ for $0 < q < n$. Then*

$$E_2^{i,0} \cong H^i(A) \quad \forall \quad i < n$$

and the sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(A) \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}(A)$$

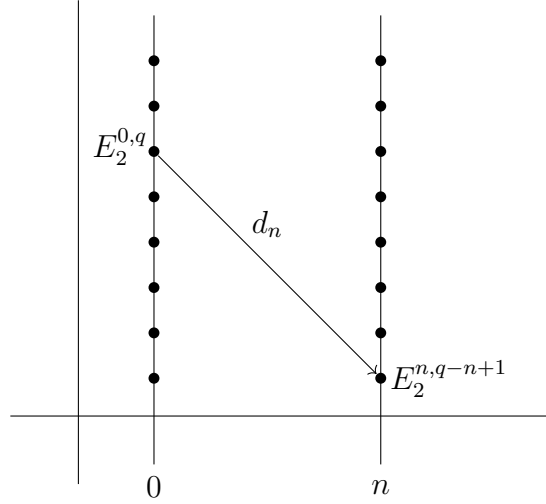
is exact.

Theorem 3.3.2 ([24], Exercise 1.3). **(The Wang sequence)** Let $\{E_r^{*,*}, d_r\}$ be a first quadrant spectral sequence of cohomological type that converges to H^* , such that $E_2^{p,q} = 0$ unless $p = 0$ or $p = n$ for some $n \geq 2$. Then there exists an exact sequence

$$\dots \longrightarrow E_2^{n,q-n} \longrightarrow H^q \longrightarrow E_2^{0,q} \xrightarrow{d_n^{0,q}} E_2^{n,q-n+1} \longrightarrow H^{q+1} \longrightarrow E_2^{0,q+1} \longrightarrow \dots$$

In particular, for $0 \leq q < n-1$, then $H^q \cong E_2^{0,q}$.

Proof. The E_2 -term has two non-trivial columns, the 0-th and n -th columns, as in the following diagram.



$E_2^{p,q}$ - page

Hence, the only possible non-trivial differential is $d_n : E_2^{0,q} \longrightarrow E_2^{n,q-n+1}$, for all $q \geq n-1$. Hence,

$$E_2^{p,q} \cong E_3^{p,q} \cong \dots \cong E_n^{p,q}$$

and,

$$E_{n+1}^{p,q} \cong E_{n+2}^{p,q} \cong \dots \cong E_{\infty}^{p,q}.$$

Now since $E_{n+1}^{p,q} \cong H(E_n^{p,q}, d_n)$, then we have

$$\begin{aligned} E_{\infty}^{0,q} &\cong \ker (d_n : E_n^{0,q} \longrightarrow E_n^{n,q-n+1}) / \operatorname{im} (d_n : E_n^{-n,q+n-1} \longrightarrow E_n^{0,q}), \\ E_{\infty}^{n,q-n+1} &\cong \ker (d_n : E_n^{n,q-n+1} \longrightarrow E_n^{2n,q-2n+2}) / \operatorname{im} (d_n : E_n^{0,q} \longrightarrow E_n^{n,q-n+1}). \end{aligned}$$

But $E_r^{*,*}$ is a first quadrant spectral sequence, so $E_n^{-n,q+n-1} \cong E_2^{-n,q+n-1} = 0$. Moreover, $E_n^{p,q} \cong E_2^{p,q}$ is trivial for $p \neq 0, n$, i.e., $E_n^{2n,q-2n+2} \cong E_2^{2n,q-2n+2} = 0$. Hence we have,

$$\begin{aligned} E_{\infty}^{0,q} &\cong \ker (d_n : E_2^{0,q} \longrightarrow E_2^{n,q-n+1}) \cong \ker d_n^{0,q}, \\ E_{\infty}^{n,q-n+1} &\cong E_2^{n,q-n+1} / \operatorname{im} (d_n : E_2^{0,q} \longrightarrow E_2^{n,q-n+1}) \cong \operatorname{coker} d_n^{0,q}. \end{aligned}$$

Thus, we have the exact sequence for each q ,

$$0 \longrightarrow E_{\infty}^{0,q} \longrightarrow E_2^{0,q} \xrightarrow{d_n} E_2^{n,q-n+1} \longrightarrow E_{\infty}^{n,q-n+1} \longrightarrow 0. \quad (3.4)$$

From Proposition 3.3.1, there exists a short exact sequence

$$0 \longrightarrow E_{\infty}^{n,q-n} \longrightarrow H^q \longrightarrow E_{\infty}^{0,q} \longrightarrow 0 \quad (3.5)$$

for each $q \geq 0$. Then we splice the exact sequences (3.4) and (3.5) together as in the following diagram.

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ \cdots & \xrightarrow{d_n} & E_2^{n,q-n} & \longrightarrow & E_{\infty}^{n,q-n} & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & H^q & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & E_{\infty}^{0,q} & \longrightarrow & E_2^{0,q} & \xrightarrow{d_n} & E_2^{n,q-n+1} \longrightarrow E_{\infty}^{n,q-n+1} \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & 0 & & & & H^{q+1} \\ & & & & & & \downarrow \\ & & & & & & 0 \longrightarrow E_{\infty}^{0,q+1} \longrightarrow E_2^{0,q+1} \xrightarrow{d_n} \cdots \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

(Note: Dashed blue arrows in the original diagram indicate the following mappings: $E_2^{n,q-n} \xrightarrow{d_n} H^q$, $H^q \xrightarrow{d_n} E_2^{0,q}$, $E_2^{n,q-n+1} \xrightarrow{d_n} H^{q+1}$, $H^{q+1} \xrightarrow{d_n} E_2^{0,q+1}$, and $E_2^{0,q+1} \xrightarrow{d_n} \cdots$.)

Hence, we obtain the desired exact sequence,

$$\cdots \longrightarrow E_2^{n,q-n} \longrightarrow H^q \longrightarrow E_2^{0,q} \xrightarrow{d_n^{0,q}} E_2^{n,q-n+1} \longrightarrow H^{q+1} \longrightarrow E_2^{0,q+1} \longrightarrow \cdots \quad (3.6)$$

Note that for $0 \leq q < n - 1$, then $E_2^{n,q-n} = 0$ and $E_2^{n,q-n+1} = 0$. Thus, $H^q \cong E_2^{0,q}$.

□

Chapter 4

Hochschild-Serre Spectral Sequence for Lie Conformal Algebras

In this chapter, we present the main result of this thesis in which we construct the Hochschild–Serre spectral sequence for Lie conformal algebras. This construction is similar to what was done in [14] for the case of Lie algebras.

4.1 The Hochschild–Serre Spectral Sequence Associated with the Basic Complex

Theorem 4.1.1. *Suppose that \mathcal{A} is a Lie conformal algebra, \mathcal{B} is an ideal of \mathcal{A} , and M is a conformal \mathcal{A} -module. Then there exists a first-quadrant cohomological spectral sequence*

$$\{E_r^{p,q}, d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}\}, \quad r \geq 0,$$

with the following properties:

- (i) $E_0^{p,q} \cong \tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M))$.
- (ii) $E_1^{p,q} \cong \tilde{H}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M))$.
- (iii) $E_2^{p,q} \cong \tilde{H}^p(\mathcal{A}/\mathcal{B}, \tilde{H}^q(\mathcal{B}, M))$, and $E_2^{p,0} \cong \tilde{H}^p(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}})$.
- (iv) The $E_\infty^{p,q}$ -page is adjoint to $\tilde{H}^*(\mathcal{A}, M)$, and the natural homomorphism $\tilde{H}^q(\mathcal{A}, M) \rightarrow \tilde{H}^q(\mathcal{B}, M)$ can be represented as the composition $\tilde{H}^q(\mathcal{A}, M) \rightarrow E_\infty^{0,q} \hookrightarrow E_1^{0,q} \cong \tilde{H}^q(\mathcal{B}, M)$.

The proof of Theorem 4.1.1 will be done by several lemmas in the following sections.

4.1.1 Spectral Sequence Associated to an Ideal

Let \mathcal{A} be a Lie conformal algebra, \mathcal{B} be an ideal of \mathcal{A} , and M be a conformal \mathcal{A} -module. Consider the filtration \tilde{F} on the basic cohomology complex associated to the conformal algebra \mathcal{A} and the \mathcal{A} -module M , $\tilde{C}^n(\mathcal{A}, M)$ defined by

$$\begin{aligned} \tilde{F}^p \tilde{C}^n(\mathcal{A}, M) &= \tilde{C}^n(\mathcal{A}, M), \quad \text{for } p \leq 0, \\ \tilde{F}^p \tilde{C}^n(\mathcal{A}, M) &= \{\gamma \in \tilde{C}^n(\mathcal{A}, M); \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = 0 \text{ if } a_1, \dots, a_{n-p+1} \in \mathcal{B}\}, \\ &\text{for } p \geq 1. \end{aligned} \quad (4.1)$$

The filtration \tilde{F} is a bounded decreasing filtration on $\tilde{C}^n(\mathcal{A}, M)$ because,

$$\begin{aligned} \tilde{C}^n(\mathcal{A}, M) &= \tilde{F}^0 \tilde{C}^n(\mathcal{A}, M) \\ &\supset \tilde{F}^1 \tilde{C}^n(\mathcal{A}, M) = \{\gamma \in \tilde{C}^n(\mathcal{A}, M); \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = 0 \text{ for } a_1, \dots, a_n \in \mathcal{B}\} \\ &\supset \tilde{F}^2 \tilde{C}^n(\mathcal{A}, M) = \{\gamma \in \tilde{C}^n(\mathcal{A}, M); \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = 0 \text{ for } a_1, \dots, a_{n-1} \in \mathcal{B}\} \\ &\supset \dots \\ &\supset \tilde{F}^n \tilde{C}^n(\mathcal{A}, M) = \{\gamma \in \tilde{C}^n(\mathcal{A}, M); \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = 0 \text{ for } a_1 \in \mathcal{B}\} \\ &\supset \tilde{F}^{n+1} \tilde{C}^n(\mathcal{A}, M) = \{0\}. \end{aligned}$$

Let $\gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \in \tilde{F}^p \tilde{C}^n(\mathcal{A}, M)$, then the differential d of γ is given by

$$\begin{aligned} (d\gamma)_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} a_i \lambda_i \gamma_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) \\ &\quad + \sum_{\substack{i,j=1 \\ i < j}}^{n+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}}([a_i \lambda_i a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{n+1}) \end{aligned}$$

Note that both terms in the right hand side vanish whenever $n - p + 2$ arguments of γ are in \mathcal{B} . This means both terms are in $\tilde{F}^p \tilde{C}^{n+1}(\mathcal{A}, M)$, and then $d\gamma \in \tilde{F}^p \tilde{C}^{n+1}(\mathcal{A}, M)$. Hence, $d\tilde{F}^p \tilde{C}^n(\mathcal{A}, M) \subset \tilde{F}^p \tilde{C}^{n+1}(\mathcal{A}, M)$.

Therefore, $(\tilde{C}^{n+1}(\mathcal{A}, M), d, \tilde{F})$ is a filtered differential graded module, and we have the following lemma.

Lemma 4.1.1. *There exists a spectral sequence $\{E_r^{p,q}, d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}\}$, $r, p, q \geq 0$,*

for $(\tilde{C}^n(\mathcal{A}, M), d, \tilde{F})$ with the following properties:

$$(i) \ E_0^{p,q} \cong \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M) / \tilde{F}^{p+1} \tilde{C}^{p+q}(\mathcal{A}, M).$$

$$(ii) \ E_{r+1}^{p,q} \cong \tilde{H}^{p+q}(E_r^{p,q}, d_r).$$

$$(iii) \ E_\infty^{p,q} \cong \tilde{F}^p \tilde{H}^{p+q}(\mathcal{A}, M) / \tilde{F}^{p+1} \tilde{H}^{p+q}(\mathcal{A}, M).$$

Proof. Follows from Theorem 3.2.1. □

Now consider a $(p+q)$ -cochain $\gamma_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}) \in \tilde{C}^{p+q}(\mathcal{A}, M)$ such that its first q -arguments are in \mathcal{B} , then

$$P(\lambda_1, \dots, \lambda_{p+q}) = \gamma_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q})$$

is a polynomial in $M[\lambda_1, \dots, \lambda_{p+q}]$. Hence γ determines a map from $\mathcal{B}^{\otimes q}$ onto $\tilde{C}^p(\mathcal{A}, M)$ given by

$$\begin{aligned} b_1 \otimes \dots \otimes b_q &\longmapsto \gamma_A^p(b_1 \otimes \dots \otimes b_q) = \gamma_{\lambda_1, \dots, \lambda_q}(b_1, \dots, b_q) \gamma_{\lambda_{q+1}, \dots, \lambda_{q+p}}(a_1, \dots, a_p) \\ &= \gamma_{\lambda_1, \dots, \lambda_q, \lambda_{q+1}, \dots, \lambda_{q+p}}(b_1, \dots, b_q, a_1, \dots, a_p). \end{aligned}$$

Consider the quotient conformal algebra $\mathcal{A}/\mathcal{B} = \{a + \mathcal{B} : a \in \mathcal{A}\}$, then the inclusion $\tilde{C}^p(\mathcal{A}/\mathcal{B}, M) \subset \tilde{C}^p(\mathcal{A}, M)$ defines a map from $\mathcal{B}^{\otimes q}$ onto $\tilde{C}^p(\mathcal{A}/\mathcal{B}, M)$ given by

$$\begin{aligned} b_1 \otimes \dots \otimes b_q &\longmapsto \bar{\gamma}_A^p(b_1 \otimes \dots \otimes b_q) = \gamma_{\lambda_1, \dots, \lambda_q}(b_1, \dots, b_q) \gamma_{\lambda_{q+1}, \dots, \lambda_{q+p}}(\bar{a}_1, \dots, \bar{a}_p) \\ &= \gamma_{\lambda_1, \dots, \lambda_q, \lambda_{q+1}, \dots, \lambda_{q+p}}(b_1, \dots, b_q, a_1, \dots, a_p) \end{aligned}$$

where $\bar{a}_1, \dots, \bar{a}_p$ are the classes of elements a_1, \dots, a_p in \mathcal{A}/\mathcal{B} . Note that the image of this map consists of all $(p+q)$ -cochains that vanish with $q+1$ arguments of a_1, \dots, a_{p+q} in \mathcal{B} . Hence, we obtain a map

$$\begin{aligned} \psi : \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M) &\longrightarrow \tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M)) \\ \gamma_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}) &\longmapsto \bar{\gamma}_A^p(b_1 \otimes \dots \otimes b_q). \end{aligned} \tag{4.2}$$

which is a $\mathbb{C}[\partial]$ -module homomorphism. Indeed, for any $\gamma, \gamma' \in \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M)$, we obtain

$$\begin{aligned} \psi(\partial\gamma + \gamma')_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}) \\ = \overline{(\partial\gamma + \gamma')_A^p}(b_1 \otimes \dots \otimes b_q) \end{aligned}$$

$$\begin{aligned}
&= (\partial\gamma + \gamma')_{\lambda_1, \dots, \lambda_q}(b_1, \dots, b_q) \gamma_{\lambda_{q+1}, \dots, \lambda_{q+p}}(\bar{a}_1, \dots, \bar{a}_p) \\
&= (\partial\gamma)_{\lambda_1, \dots, \lambda_q}(b_1, \dots, b_q) \gamma_{\lambda_{q+1}, \dots, \lambda_{q+p}}(\bar{a}_1, \dots, \bar{a}_p) + \gamma'_{\lambda_1, \dots, \lambda_q}(b_1, \dots, b_q) \gamma_{\lambda_{q+1}, \dots, \lambda_{q+p}}(\bar{a}_1, \dots, \bar{a}_p) \\
&= \partial\psi(\gamma)_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}) + \psi(\gamma')_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}).
\end{aligned}$$

Remark 4.1.1. For an ideal \mathcal{B} of a Lie conformal algebra \mathcal{A} , then \mathcal{A}/\mathcal{B} is a trivial \mathcal{B} -module since $[b_\lambda(a + \mathcal{B})] = [b_\lambda a] + \mathcal{B} = 0$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Remark 4.1.2. Note that for any $a \in \mathcal{A}$, $\tilde{C}^p(\mathcal{A}, M)$ is an \mathcal{A} -module, with the action $\theta_\lambda(a)$ given by

$$\begin{aligned}
(\theta_\lambda(a)\gamma)_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) &= a_\lambda \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \\
&\quad - \sum_{i=1}^n \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_n}(a_1, \dots, [a_\lambda a_i], \dots, a_n).
\end{aligned}$$

Thus, $\tilde{C}^p(\mathcal{A}, M)$ is a \mathcal{B} -module with the action induced by the inclusion $\mathcal{B} \subset \mathcal{A}$. Then, $\tilde{C}^p(\mathcal{A}/\mathcal{B}, M)$ has a \mathcal{B} -module structure induced by the inclusion $\tilde{C}^p(\mathcal{A}/\mathcal{B}, M) \subset \tilde{C}^p(\mathcal{A}, M)$.

Lemma 4.1.2. *The map ψ induces an isomorphism of $E_0^{p,q}$ onto $\tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M))$ for all $p, q \geq 0$.*

Proof. Let $\beta = \bar{\gamma}_A^p(b_1 \otimes \dots \otimes b_q) \in \tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M))$. Define $\gamma \in \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M)$ by

$$\gamma = \gamma_{\lambda_1, \dots, \lambda_q, \lambda_{q+1}, \dots, \lambda_{p+q}}(b_1, \dots, b_q, a_1, \dots, a_p).$$

Then, $\psi(\gamma) = \beta$. i.e., ψ is onto. Note that for any $\gamma_{\lambda_1, \dots, \lambda_{p+q}}(a_1, \dots, a_{p+q}) \in \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M)$, its image vanishes whenever q of its arguments a_1, \dots, a_{p+q} are in \mathcal{B} . Hence, $\ker \psi$ is $\tilde{F}^{p+1} \tilde{C}^{p+q}(\mathcal{A}, M)$. Therefore, ψ induces an isomorphism

$$E_0^{p,q} \cong \tilde{F}^p \tilde{C}^{p+q}(\mathcal{A}, M) / \tilde{F}^{p+1} \tilde{C}^{p+q}(\mathcal{A}, M) \longrightarrow \tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M)).$$

□

The following lemma shows that the differential commutes with the induced isomorphism in Lemma 4.1.2.

Lemma 4.1.3. *Let Ψ denote the isomorphism of $E_0^{p,q}$ onto $\tilde{C}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M))$. Then $d\Psi = \Psi d$.*

Proof. Let $\bar{\gamma} \in E_0^{p,q}$. Then $\bar{\gamma} = \gamma + \tilde{F}^{p+1}\tilde{C}^{p+q}(\mathcal{A}, M)$, where $\gamma \in \tilde{F}^p\tilde{C}^{p+q}(\mathcal{A}, M)$, and $\Psi(\bar{\gamma}) = \psi(\gamma)$. Thus, we will show that $d\psi = \psi d$.

Let $\gamma \in \tilde{F}^p\tilde{C}^{p+q}(\mathcal{A}, M)$, $b_1, \dots, b_{q+1} \in \mathcal{B}$, $a_1, \dots, a_p \in \mathcal{A}$ and $\bar{a}_1, \dots, \bar{a}_p$ be the classes of elements a_1, \dots, a_p in \mathcal{A}/\mathcal{B} . We have

$$\begin{aligned}
& (\psi d)\gamma_{\lambda_1, \dots, \lambda_{q+1}}(b_1, \dots, b_{q+1})\gamma_{\lambda_{q+2}, \dots, \lambda_{q+p+1}}(\bar{a}_1, \dots, \bar{a}_p) \\
&= d\gamma_{\lambda_1, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_{q+p+1}}(b_1, \dots, b_{q+1}, a_1, \dots, a_p) \\
&= \sum_{i=1}^{q+1} (-1)^{i+1} b_{i\lambda_i} \gamma_{\lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_{q+p+1}}(b_1, \dots, \widehat{b}_i, \dots, b_{q+1}, a_1, \dots, a_p) \\
&+ \sum_{i=1}^p (-1)^{i+1} a_{i\lambda_i} \gamma_{\lambda_1, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \widehat{\lambda}_i, \dots, \lambda_{q+p+1}}(b_1, \dots, b_{q+1}, a_1, \dots, \widehat{a}_i, \dots, a_p) \\
&+ \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_{q+p+1}}([b_{i\lambda_i} b_j], b_1, \dots, \widehat{b}_i, \dots, \widehat{b}_j, \dots, b_{q+1}, a_1, \dots, a_p) \\
&+ \sum_{\substack{i,j=1 \\ i < j}}^p (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \lambda_{q+p+1}}(b_1, \dots, b_{q+1}, [a_{i\lambda_i} a_j], a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_p) \\
&+ \sum_{i=1}^{q+1} \sum_{j=1}^p (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \widehat{\lambda}_j, \dots, \lambda_{q+p+1}}(b_1, \dots, \widehat{b}_i, \dots, b_{q+1}, [b_{i\lambda_i} a_j], a_1, \dots, \widehat{a}_j, \dots, a_p)
\end{aligned}$$

Note that γ in the second, fourth, and fifth terms of the right hand side have $q+1$ of its arguments in \mathcal{B} . Then, they all vanish, and by (4.2) and the definition of the differential we have,

$$\begin{aligned}
& (\psi d)\gamma_{\lambda_1, \dots, \lambda_{q+1}}(b_1, \dots, b_{q+1})\gamma_{\lambda_{q+2}, \dots, \lambda_{q+p+1}}(\bar{a}_1, \dots, \bar{a}_p) \\
&= \sum_{i=1}^{q+1} (-1)^{i+1} b_{i\lambda_i} \gamma_{\lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_{q+p+1}}(b_1, \dots, \widehat{b}_i, \dots, b_{q+1}, a_1, \dots, a_p) \\
&+ \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_{q+p+1}}([b_{i\lambda_i} b_j], b_1, \dots, \widehat{b}_i, \dots, \widehat{b}_j, \dots, b_{q+1}, a_1, \dots, a_p) \\
&= \sum_{i=1}^{q+1} (-1)^{i+1} b_{i\lambda_i} \psi(\gamma_{\lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_{q+1}}(b_1, \dots, \widehat{b}_i, \dots, b_{q+1})\gamma_{\lambda_{q+2}, \dots, \lambda_{q+p+1}}(\bar{a}_1, \dots, \bar{a}_p)) \\
&+ \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \psi(\gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \lambda_{q+1}}([b_{i\lambda_i} b_j], b_1, \dots, \widehat{b}_i, \dots, \widehat{b}_j, \dots, b_{q+1})\gamma_{\lambda_{q+2}, \dots, \lambda_{q+p+1}}(\bar{a}_1, \dots, \bar{a}_p))
\end{aligned}$$

$$= (d\psi)\gamma_{\lambda_1, \dots, \lambda_{q+1}}(b_1, \dots, b_{q+1})\gamma_{\lambda_{q+2}, \dots, \lambda_{q+p+1}}(\bar{a}_1, \dots, \bar{a}_p).$$

i.e., $d\psi = \psi d$ and hence, d commutes with the isomorphism Ψ . \square

Using Lemma 4.1.1, we have $E_1^{p,q} \cong \tilde{H}^{p+q}(E_0^{p,q}, d_0)$. From Lemmas, 4.1.2 and 4.1.3, we describe the first page of the Hochschild-Serre spectral sequence for Lie conformal algebras in the following theorem.

Theorem 4.1.2. *Let \mathcal{A} be a Lie conformal algebra, \mathcal{B} be an ideal of \mathcal{A} and M be an \mathcal{A} -module, then*

$$E_1^{p,q} \cong \tilde{H}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M)),$$

for all $p, q \geq 0$.

Now let $\mathfrak{g}_- = (\text{Lie } \mathcal{A}/\mathcal{B})_-$ be the annihilation Lie algebra of \mathcal{A}/\mathcal{B} . Then M is \mathfrak{g}_- -module, and we have

$$\tilde{C}^p(\mathcal{A}/\mathcal{B}, M) \cong C^p(\mathfrak{g}_-, M) \cong \text{Hom}(\Lambda^p \mathfrak{g}_-, M)$$

where $C^p(\mathfrak{g}_-, M)$ is the Chevalley-Eilenberg complex defining the cohomology of \mathfrak{g}_- with coefficients in M . In the following theorem, we give a description for the second page of the Hochschild-Serre spectral sequence for Lie conformal algebras.

Theorem 4.1.3. *Let \mathcal{A} be a Lie conformal algebra, \mathcal{B} be an ideal of \mathcal{A} and M be an \mathcal{A} -module, then*

$$E_2^{p,q} \cong \tilde{H}^p(\mathcal{A}/\mathcal{B}, \tilde{H}^q(\mathcal{B}, M)), \quad p, q \geq 0.$$

Proof. From Theorem 4.1.2, we have

$$\begin{aligned} E_1^{p,q} &\cong \tilde{H}^q(\mathcal{B}, \tilde{C}^p(\mathcal{A}/\mathcal{B}, M)) \cong \tilde{H}^q(\mathcal{B}, \text{Hom}(\Lambda^p \mathfrak{g}_-, M)) \\ &\cong (\Lambda^p \mathfrak{g}_-)^* \otimes \tilde{H}^q(\mathcal{B}, M) \\ &\cong \text{Hom}(\Lambda^p \mathfrak{g}_-, \tilde{H}^q(\mathcal{B}, M)) \\ &\cong C^p(\mathfrak{g}_-, \tilde{H}^q(\mathcal{B}, M)) \\ &\cong \tilde{C}^p(\mathcal{A}/\mathcal{B}, \tilde{H}^q(\mathcal{B}, M)). \end{aligned}$$

Hence,

$$E_2^{p,q} \cong \tilde{H}^p(\mathcal{A}/\mathcal{B}, \tilde{H}^q(\mathcal{B}, M)).$$

\square

Remark 4.1.3. The basic complex $\tilde{C}^q(\mathcal{B}, M)$ has the \mathcal{A} -module structure induced by the inclusion $\tilde{C}^q(\mathcal{B}, M) \subset \tilde{C}^q(\mathcal{A}, M)$, and the differential d commute with the action $\theta_\lambda(a)$, $a \in \mathcal{A}$. Hence, $\tilde{H}^q(\mathcal{B}, M)$ is an \mathcal{A} -module. Moreover, $\tilde{H}^q(\mathcal{B}, M)$ is a trivial \mathcal{B} -module. Thus, $\tilde{H}^q(\mathcal{B}, M)$ is an \mathcal{A}/\mathcal{B} -module.

Corollary 4.1.1. $E_2^{p,0} \cong \tilde{H}^p(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}})$ where $M^{\mathcal{B}} = \{m \in M \mid b_\lambda m = 0, \forall b \in \mathcal{B}\}$, the \mathcal{A}/\mathcal{B} -module of \mathcal{B} -invariants in M .

4.1.2 Convergence of the Spectral Sequence

Theorem 4.1.4. The $E_\infty^{p,q}$ -page is adjoint to $\tilde{H}^*(\mathcal{A}, M)$, and the natural homomorphism $\tilde{H}^q(\mathcal{A}, M) \longrightarrow \tilde{H}^q(\mathcal{B}, M)$ can be represented as the composition $\tilde{H}^q(\mathcal{A}, M) \rightarrow E_\infty^{0,q} \hookrightarrow E_1^{0,q} \cong \tilde{H}^q(\mathcal{B}, M)$.

Proof. From Lemma 4.1.1, we have

$$E_\infty^{p,q} \cong \tilde{F}^p \tilde{H}^{p+q}(\mathcal{A}, M) / \tilde{F}^{p+1} \tilde{H}^{p+q}(\mathcal{A}, M),$$

where $\tilde{F}^\bullet \tilde{H}^\bullet(\mathcal{A}, M)$ is the filtration induced by the filtration \tilde{F} on the basic complex $\tilde{C}^n(\mathcal{A}, M)$. i.e., The Hochschild-Serre spectral sequence for Lie conformal algebras converges to $\tilde{H}^\bullet(\mathcal{A}, M)$, and we write $E_r^{p,q} \Rightarrow \tilde{H}^{p+q}(\mathcal{A}, M)$.

The cochain map $\text{res} : \tilde{C}^q(\mathcal{A}, M) \longrightarrow \tilde{C}^q(\mathcal{B}, M)$ which restricts a q -cochain on \mathcal{A} to a q -cochain on \mathcal{B} induces homomorphisms $\tilde{H}^q(\mathcal{A}, M) \longrightarrow \tilde{H}^q(\mathcal{B}, M)$. If $p = 0$, then we obtain

$$E_\infty^{0,q} \cong \tilde{F}^0 \tilde{H}^q(\mathcal{A}, M) / \tilde{F}^1 \tilde{H}^q(\mathcal{A}, M) \cong \tilde{H}^q(\mathcal{A}, M) / \tilde{F}^1 \tilde{H}^q(\mathcal{A}, M).$$

So we get a surjective map $\tilde{H}^q(\mathcal{A}, M) \rightarrow E_\infty^{0,q}$. Since $E_r^{p,q}$ is a first quadrant spectral sequence, then all the differentials that are mapped into $E_r^{0,q}$ are zero. Hence we have the inclusion $E_\infty^{0,q} \hookrightarrow \dots \hookrightarrow E_{r+1}^{0,q} \hookrightarrow E_r^{0,q}$. Therefore, we have the composition $\tilde{H}^q(\mathcal{A}, M) \rightarrow E_\infty^{0,q} \hookrightarrow E_1^{0,q}$. From Theorem 4.1.2,

$$E_1^{0,q} \cong \tilde{H}^q(\mathcal{B}, \tilde{C}^0(\mathcal{A}, M)) \cong \tilde{H}^q(\mathcal{B}, M).$$

Hence, the natural homomorphism $\tilde{H}^q(\mathcal{A}, M) \longrightarrow \tilde{H}^q(\mathcal{B}, M)$ can be represented as the composition $\tilde{H}^q(\mathcal{A}, M) \rightarrow E_\infty^{0,q} \hookrightarrow E_1^{0,q} \cong \tilde{H}^q(\mathcal{B}, M)$. \square

4.2 The Inflation-Restriction Exact Sequence

Let \mathcal{A} and \mathcal{A}' be Lie conformal algebras, M and M' be \mathcal{A} -module and \mathcal{A}' -module, respectively, and φ and f be two homomorphisms $\varphi : \mathcal{A}' \rightarrow \mathcal{A}$ and $f : M \rightarrow M'$. Then φ and f are said to be **compatible pair** if $f(\varphi(a')_\lambda m) = a'_\lambda f(m)$ for $a' \in \mathcal{A}'$ and $m \in M$. From such a compatible pair of homomorphisms, we get a homomorphism $\psi : \tilde{C}^n(\mathcal{A}, M) \rightarrow \tilde{C}^n(\mathcal{A}', M')$ given by $\gamma \mapsto f \circ \gamma \circ \varphi$. Then we have the following lemma:

Lemma 4.2.1. *The maps $\psi : \tilde{C}^n(\mathcal{A}, M) \rightarrow \tilde{C}^n(\mathcal{A}', M')$ defined by $\gamma \mapsto f \circ \gamma \circ \varphi$ for $\gamma \in \tilde{C}^n(\mathcal{A}, M)$ induce maps on cohomology $\tilde{H}^n(\mathcal{A}, M) \rightarrow \tilde{H}^n(\mathcal{A}', M')$ for all $n \geq 0$.*

Proof. We need only to check that ψ is compatible with the differential d defined in (2.7). Let $\gamma \in \tilde{C}^n(\mathcal{A}, M)$. Then we have

$$\begin{aligned}
& (\psi(d\gamma))_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}) \\
&= f(d\gamma)_{\lambda_1, \dots, \lambda_{n+1}}(\varphi(a_1), \dots, \varphi(a_{n+1})) \\
&= f\left(\sum_{i=1}^{n+1} (-1)^{i+1} \varphi(a_i)_{\lambda_i} \gamma_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(\varphi(a_1), \dots, \widehat{\varphi(a_i)}, \dots, \varphi(a_{n+1}))\right. \\
&\quad \left.+ \sum_{\substack{i,j=1 \\ i < j}}^{n+1} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}}([\varphi(a_i)_{\lambda_i} \varphi(a_j)], \varphi(a_1), \dots, \widehat{\varphi(a_i)}, \dots, \widehat{\varphi(a_j)}, \dots, a_{n+1})\right) \\
&= \sum_{i=1}^{n+1} (-1)^{i+1} \varphi(a_i)_{\lambda_i} (f\gamma)_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(\varphi(a_1), \dots, \widehat{\varphi(a_i)}, \dots, \varphi(a_{n+1})) \\
&\quad + \sum_{\substack{i,j=1 \\ i < j}}^{n+1} (f\gamma)_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}}([\varphi(a_i)_{\lambda_i} \varphi(a_j)], \varphi(a_1), \dots, \widehat{\varphi(a_i)}, \dots, \widehat{\varphi(a_j)}, \dots, a_{n+1}) \\
&= d(f(\gamma))_{\lambda_1, \dots, \lambda_{n+1}}(\varphi(a_1), \dots, a_{n+1}) \\
&= d(\psi(\gamma))_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}).
\end{aligned}$$

i.e., $d\psi = \psi d$. Hence, ψ induce maps on cohomology $\tilde{H}^n(\mathcal{A}, M) \rightarrow \tilde{H}^n(\mathcal{A}', M')$ for all $n \geq 0$. \square

Consider the subalgebra \mathcal{B} of a Lie conformal algebra \mathcal{A} , and the \mathcal{A} -module M . Then the pair (ι, id) where $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ and $\text{id} : M \rightarrow M$ induces the restriction map $\text{res} : \tilde{C}^n(\mathcal{A}, M) \rightarrow \tilde{C}^n(\mathcal{B}, M)$ which by Lemma 4.2.1 gives the homomorphism on the cohomology

$$\text{res} : \tilde{H}^n(\mathcal{A}, M) \rightarrow \tilde{H}^n(\mathcal{B}, M)$$

called the **restriction**. If \mathcal{B} is an ideal of \mathcal{A} , then consider the \mathcal{B} -module $M^{\mathcal{B}} = \{m \in M \mid b_\lambda m = 0 \ \forall \ b \in \mathcal{B}\}$. The injection $\iota : M^{\mathcal{B}} \hookrightarrow M$ and the projection $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ form a compatible pair of homomorphisms. By Lemma 4.2.1, the pair (ι, π) induces a homomorphism

$$\text{inf} : \tilde{H}^n(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \longrightarrow \tilde{H}^n(\mathcal{A}, M)$$

called the **inflation** homomorphism. Note that $M^{\mathcal{B}}$ is an \mathcal{A}/\mathcal{B} -module.

Example 4.2.1. *In degree 0, the restriction homomorphism $\text{res} : M^{\mathcal{A}} \rightarrow M^{\mathcal{B}}$ and the inflation homomorphism $\text{inf} : (M^{\mathcal{B}})^{\mathcal{A}/\mathcal{B}} \rightarrow M^{\mathcal{A}}$ are the inclusion and the identity, respectively.*

Example 4.2.2. *The edge maps (Proposition 3.3.1) are the inflation $\tilde{H}^q(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \rightarrow \tilde{H}^q(\mathcal{A}, M)$ and the restriction $\tilde{H}^q(\mathcal{A}, M) \rightarrow \tilde{H}^q(\mathcal{B}, M)^{\mathcal{A}/\mathcal{B}}$.*

Now suppose that for a Lie conformal algebra \mathcal{A} there exists a Hochschild-Serre spectral sequence, $E_2^{p,q} = \tilde{H}^p(\mathcal{A}/\mathcal{B}, \tilde{H}^q(\mathcal{B}, M)) \Rightarrow \tilde{H}^{p+q}(\mathcal{A}, M)$, where $\mathcal{B} \subset \mathcal{A}$ is an ideal and M is a conformal \mathcal{A} -module. Then we have the following results which describe the inflation-restriction exact sequence that arises from the Hochschild-Serre spectral sequence.

Theorem 4.2.1. (Inflation-Restriction exact sequence) *The Hochschild-Serre spectral sequence for Lie conformal algebras yields the following exact sequence*

$$\begin{aligned} 0 \longrightarrow \tilde{H}^1(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) &\xrightarrow{\text{inf}} \tilde{H}^1(\mathcal{A}, M) \xrightarrow{\text{res}} \tilde{H}^1(\mathcal{B}, M)^{\mathcal{A}/\mathcal{B}} \longrightarrow \\ &\xrightarrow{d_2} \tilde{H}^2(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \xrightarrow{\text{inf}} \tilde{H}^2(\mathcal{A}, M) \end{aligned} \quad (4.3)$$

where $M^{\mathcal{B}}$ is the invariant submodule of M , and res and inf are the inflation and restriction maps, respectively.

Proof. Follows directly from the five-term exact sequence (see Example 3.3.1). □

Corollary 4.2.1. *Suppose that \mathcal{B} acts trivially on M and $\tilde{H}^q(\mathcal{A}/\mathcal{B}, M) = 0$ for $q = 1, 2$, then $\tilde{H}^1(\mathcal{A}, M) \cong \tilde{H}^1(\mathcal{B}, M)^{\mathcal{A}/\mathcal{B}}$.*

Proof. Follows from Theorem 4.2.1. □

Corollary 4.2.2. *Suppose $\tilde{H}^1(\mathcal{B}, M) = \tilde{H}^2(\mathcal{B}, M) = 0$, then $\tilde{H}^q(\mathcal{A}, M) \cong \tilde{H}^q(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}})$ for $q = 1, 2$.*

Proof. The seven-term exact sequence (see Remark 3.3.1), of a conformal algebra \mathcal{A} and its ideal \mathcal{B} with coefficients in an \mathcal{A} -module is given by

$$\begin{aligned} 0 \longrightarrow \tilde{H}^1(\mathcal{A}/\mathcal{B}, \tilde{H}^0(\mathcal{B}, M)) &\longrightarrow \tilde{H}^1(\mathcal{A}, M) \longrightarrow \tilde{H}^0(\mathcal{A}/\mathcal{B}, \tilde{H}^1(\mathcal{B}, M)) \longrightarrow \\ &\longrightarrow \tilde{H}^2(\mathcal{A}/\mathcal{B}, \tilde{H}^0(\mathcal{B}, M)) \longrightarrow \ker(\tilde{H}^2(\mathcal{A}, M) \rightarrow \tilde{H}^0(\mathcal{A}/\mathcal{B}, \tilde{H}^2(\mathcal{B}, M))) \longrightarrow \\ &\longrightarrow \tilde{H}^1(\mathcal{A}/\mathcal{B}, \tilde{H}^1(\mathcal{B}, M)) \longrightarrow \tilde{H}^3(\mathcal{A}/\mathcal{B}, \tilde{H}^0(\mathcal{B}, M)). \end{aligned}$$

But $\tilde{H}^q(\mathcal{B}, M) = 0$ for $q = 1, 2$, so we have

$$\begin{aligned} 0 \longrightarrow \tilde{H}^1(\mathcal{A}/\mathcal{B}, \tilde{H}^0(\mathcal{B}, M)) &\longrightarrow \tilde{H}^1(\mathcal{A}, M) \longrightarrow 0 \\ 0 \longrightarrow \tilde{H}^2(\mathcal{A}/\mathcal{B}, \tilde{H}^0(\mathcal{B}, M)) &\longrightarrow \tilde{H}^2(\mathcal{A}, M) \longrightarrow 0. \end{aligned}$$

which implies $\tilde{H}^q(\mathcal{A}, M) \cong \tilde{H}^q(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}})$ for $q = 1, 2$. □

Theorem 4.2.2. (Higher Degree Inflation-Restriction exact sequence) *Let \mathcal{B} be an ideal of a Lie conformal algebra \mathcal{A} and M an \mathcal{A} -module. Suppose that $\tilde{H}^q(\mathcal{B}, M) = 0$ for $1 \leq q < q'$. Then the inflation homomorphism induces isomorphisms*

$$\tilde{H}^q(\mathcal{A}, M) \cong \tilde{H}^q(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}})$$

for $0 \leq q < q'$, and there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \tilde{H}^{q'}(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) &\xrightarrow{\inf} \tilde{H}^{q'}(\mathcal{A}, M) \xrightarrow{\text{res}} \tilde{H}^{q'}(\mathcal{B}, M)^{\mathcal{A}/\mathcal{B}} \longrightarrow \\ &\xrightarrow{d_2} \tilde{H}^{q'+1}(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \xrightarrow{\inf} \tilde{H}^{q'+1}(\mathcal{A}, M). \end{aligned}$$

Proof. Using Theorem 3.3.1 with $n = q'$ we have

$$\tilde{H}^q(\mathcal{A}, M) \cong E_2^{q,0} = \tilde{H}^q(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \quad \forall \quad q < q'$$

and the sequence $0 \longrightarrow \tilde{H}^{q'}(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \longrightarrow \tilde{H}^{q'}(\mathcal{A}, M) \longrightarrow \tilde{H}^{q'}(\mathcal{B}, M)^{\mathcal{A}/\mathcal{B}} \xrightarrow{d_2} \tilde{H}^{q'+1}(\mathcal{A}/\mathcal{B}, M^{\mathcal{B}}) \longrightarrow \tilde{H}^{q'+1}(\mathcal{A}, M)$ is exact □

4.3 The Hochschild–Serre Spectral Sequence Associated with the Reduced Complex

Recall that for any filtration F on an $\mathbb{C}[\partial]$ –module M , and N is a submodule of M , then the induced filtrations on N and M/N , are given by $F^r N = N \cap F^r M$ and $F^r(M/N) = (F^r M + N)/N$, respectively.

Let \mathcal{A} be a Lie conformal algebra, and M be a conformal \mathcal{A} –module. Consider the induced filtration F^\bullet of \tilde{F}^\bullet on the reduced complex $C^\bullet(\mathcal{A}, M)$, then we have

$$\begin{aligned} F^p C^n(\mathcal{A}, M) &= \tilde{F}^p(\tilde{C}^n(\mathcal{A}, M)/\partial\tilde{C}^n(\mathcal{A}, M)) \\ &= (\tilde{F}^p\tilde{C}^n(\mathcal{A}, M) + \partial\tilde{C}^n(\mathcal{A}, M))/\partial\tilde{C}^n(\mathcal{A}, M), \quad \text{for all } p. \end{aligned}$$

Observe that,

$$\begin{aligned} F^0 C^n(\mathcal{A}, M) &= (\tilde{F}^0\tilde{C}^n(\mathcal{A}, M) + \partial\tilde{C}^n(\mathcal{A}, M))/\partial\tilde{C}^n(\mathcal{A}, M) \\ &\cong \tilde{F}^0\tilde{C}^n(\mathcal{A}, M)/(\tilde{F}^0\tilde{C}^n(\mathcal{A}, M) \cap \partial\tilde{C}^n(\mathcal{A}, M)) \\ &= \tilde{C}^n(\mathcal{A}, M)/(\tilde{C}^n(\mathcal{A}, M) \cap \partial\tilde{C}^n(\mathcal{A}, M)) = C^n(\mathcal{A}, M), \\ F^{n+1} C^n(\mathcal{A}, M) &= (\tilde{F}^{n+1}\tilde{C}^n(\mathcal{A}, M) + \partial\tilde{C}^n(\mathcal{A}, M))/\partial\tilde{C}^n(\mathcal{A}, M) \\ &= \partial\tilde{C}^n(\mathcal{A}, M)/\partial\tilde{C}^n(\mathcal{A}, M) = \{0\}. \end{aligned}$$

Moreover, $F^p C^n(\mathcal{A}, M) \subset F^{p-1} C^n(\mathcal{A}, M)$ for all p . Thus, F is a bounded decreasing filtration on $C^n(\mathcal{A}, M)$. Furthermore, the differential d preserves the filtration F^\bullet because

$$\begin{aligned} d(F^p C^n(\mathcal{A}, M)) &= d(\tilde{F}^p\tilde{C}^n(\mathcal{A}, M) + \partial\tilde{C}^n(\mathcal{A}, M))/\partial\tilde{C}^{n+1}(\mathcal{A}, M) \\ &= d(\tilde{F}^p\tilde{C}^n(\mathcal{A}, M)) + \partial d\tilde{C}^n(\mathcal{A}, M)/\partial\tilde{C}^{n+1}(\mathcal{A}, M) \\ &\subset \tilde{F}^p\tilde{C}^{n+1}(\mathcal{A}, M) + \partial\tilde{C}^{n+1}(\mathcal{A}, M)/\partial\tilde{C}^{n+1}(\mathcal{A}, M) \\ &= F^p C^{n+1}(\mathcal{A}, M), \end{aligned}$$

for all p . Therefore, $(C^n(\mathcal{A}, M), d, F)$ is a filtered differential graded $\mathbb{C}[\partial]$ –module. Thus, by Theorem 3.2.1, there exists a first quadrant cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ for $r \geq 0$ that satisfies the following properties:

1. $E_0^{p,q} \cong F^p C^{p+q}(\mathcal{A}, M)/F^{p+1} C^{p+q}(\mathcal{A}, M)$.
2. $E_1^{p,q} \cong H^{p+q}(E_0^{p,q}, d_0)$.

$$3. E_\infty^{p,q} \cong F^p H^{p+q}(\mathcal{A}, M) / F^{p+1} H^{p+q}(\mathcal{A}, M),$$

Now suppose that the second term of the spectral sequence associated with the reduced complex is given by $E_2^{p,q} = H^p(\mathcal{A}/\mathcal{B}, H^q(\mathcal{B}, M))$, where \mathcal{B} is an ideal of \mathcal{A} . Then by the convergence of the spectral sequence we have $E_2^{p,q} \Rightarrow H^{p+q}(\mathcal{A}, M)$. In this section we will show that the arising spectral sequence does not converge to the reduced cohomology $H^*(\mathcal{A}, M)$.

Due to [33], the dimension of the reduced cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with trivial coefficients is given by

$$\begin{aligned} \dim H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) &= \dim \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) + \dim \tilde{H}^{q+1}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \\ &= \dim H^q(\mathfrak{g}, \mathbb{C}) + \dim H^{q-3}(\mathfrak{g}, \mathbb{C}) + \dim H^{q+1}(\mathfrak{g}, \mathbb{C}) + \dim H^{q-2}(\mathfrak{g}, \mathbb{C}) \\ &= \dim H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) + \dim H^{q-3}(\text{Cur } \mathfrak{g}, \mathbb{C}), \end{aligned} \quad (4.4)$$

for all $q \geq 0$. Now suppose that the Hochschild-Serre spectral sequence with respect to the reduced complex $C^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$ is given by

$$E_2^{p,q} = H^p(\text{Vir}, H^q(\text{Cur } \mathfrak{g}, \mathbb{C})) \Rightarrow H^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \quad (4.5)$$

for all $p, q \geq 0$. Then by Theorem 2.7.5, $H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) = H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C})$ for all q , which is a trivial Vir -module (Lemma 5.2.1). Then (4.5) can be rewritten as

$$E_2^{p,q} = H^p(\text{Vir}, \mathbb{C}) \otimes H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \Rightarrow H^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}).$$

By Theorem 2.7.4, $H^p(\text{Vir}, \mathbb{C}) = 0$ unless $p = 0, 2, 3$. So, the E_2 -term is given by

$$E_2^{p,q} = \begin{cases} H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C}) & \text{if } p = 0, \\ P_2[H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C})] & \text{if } p = 2, \\ \Lambda_3[H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C})] & \text{if } p = 3, \\ 0 & \text{otherwise.} \end{cases}$$

where $q \geq 0$, $P_2 = \lambda_1^3 - \lambda_2^3$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Then the corner of the E_2 page is shown in the following figure.

6	$H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
5	$H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
4	$H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
3	$H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
2	$H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
1	0	0	0	0	0
0	\mathbb{C}	0	$P_2 \mathbb{C}$	$\Lambda_3 \mathbb{C}$	0
	0	1	2	3	4

Figure 4.1: $E_2^{p,q}$ -page for $(C^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), F^\bullet, d)$.

Suppose that any q -cochain $\bar{\gamma} \in C^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, a representative of a cohomology class $[\bar{\gamma}] \in H^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, can be extended to a q -cochain in $\bar{\gamma} \in C^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$ as the following

$$\begin{aligned} (\iota_\lambda(L)d\bar{\gamma})_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= 0, \\ \bar{\gamma}_{\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_q}(L, \dots, L, g_{s+1}, \dots, g_q) &= 0, \quad \text{for all } 2 \leq s \leq q. \end{aligned} \tag{4.6}$$

Then $\bar{\gamma} \in E_2^{0,q}$ for all $q \geq 0$. Tracking the differential d_2 in the Figure 4.1, observe that the only possible nontrivial differential is $d_2^{0,q} : H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \rightarrow P_2 H^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C})$, where $q \geq 3$. Then, for $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, a representative of a reduced q -cocycle $\bar{\gamma} \in H^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ we have

$$d_2^{0,q}(\bar{\gamma}) = d\bar{\gamma} = d\gamma \mod \partial \tilde{C}^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C}) = P_2 \beta \mod \partial \tilde{C}^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C}) = P_2 \bar{\beta}$$

for some $\beta \in \tilde{C}^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C})$, where $P_2 = \lambda_1^3 - \lambda_2^3$. On the other hand, we have

$$\begin{aligned} & d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, g_3, \dots, g_{q+1}) \\ &= (\lambda_2 - \lambda_1) \gamma_{\lambda_1 + \lambda_2, \lambda_3, \dots, \lambda_{q+1}}(L, g_3, \dots, g_{q+1}) \\ &+ \sum_{j=3}^{q+1} (-1)^{i+1} \lambda_j \gamma_{\lambda_1 + \lambda_j, \lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}}(g_j, L, g_3, \dots, \hat{g}_j, \dots, g_{q+1}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=3}^{q+1} (-1)^{i+2} \lambda_j \gamma_{\lambda_2+\lambda_j, \lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}}(g_j, L, g_3, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& + \sum_{\substack{i,j=3 \\ i < j}}^{q+1} (-1)^{i+j} \gamma_{\lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}}([g_i \lambda_i g_j], L, L, g_3, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& = (\lambda_2 - \lambda_1) \gamma_{\lambda_1+\lambda_2, \lambda_3, \dots, \lambda_{q+1}}(L, g_3, \dots, g_{q+1}) \\
& + \sum_{j=3}^{q+1} \lambda_j \gamma_{\lambda_2, \dots, \lambda_1+\lambda_j, \dots, \lambda_{q+1}}(L, g_3, \dots, g_j, \dots, g_{q+1}) \\
& - \sum_{j=3}^{q+1} \lambda_j \gamma_{\lambda_1, \dots, \lambda_2+\lambda_j, \dots, \lambda_{q+1}}(L, g_3, \dots, g_j, \dots, g_{q+1}). \tag{4.7}
\end{aligned}$$

where the last equality holds due to (4.6) and the skew symmetry of γ . Since $H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) = H^q(\mathfrak{g}, \mathbb{C}) \oplus H^{q+1}(\mathfrak{g}, \mathbb{C})$, $\bar{\gamma}$ does not depend on $\lambda_3, \dots, \lambda_{q+1}$. Then we can set $\gamma = c'_0$ for some $c'_0 \in \mathbb{C}$ when evaluated on the elements of \mathfrak{g} . On the other hand, we have $d\bar{\gamma} = P_2 \bar{\beta} \in P_2 H^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C})$ for some $\beta \in \tilde{C}^{q-1}(\text{Cur } \mathfrak{g}, \mathbb{C})$. Then $\bar{\beta}$ is a constant and $d\bar{\gamma}$ is a skew symmetric polynomial of degree 3. It follows that γ is a quadratic homogeneous polynomial in one variable. Then we can rewrite (4.7) as

$$\begin{aligned}
& d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, g_3, \dots, g_{q+1}) \\
& = c'_0(\lambda_2 - \lambda_1)(\lambda_1 + \lambda_2)^2 + c'_0(\lambda_3 + \dots + \lambda_{q+1})\lambda_2^2 \\
& - c'_0(\lambda_3 + \dots + \lambda_{q+1})\lambda_1^2 \\
& = c'_0(\lambda_2^2 - \lambda_1^2)(\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_{q+1})
\end{aligned} \tag{4.8}$$

Since $\partial \tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) = (\sum_{i=1}^q \lambda_i) \partial \tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$, $d\gamma = 0$ and thus, $d_2^{0,q}$ vanishes for all $q \geq 0$. Therefore, $E_3^{p,q} \cong E_2^{p,q}$ for all $p, q \geq 0$. The corner of the E_3 -page is shown in the following diagram.

6	$H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^6(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
5	$H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^5(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
4	$H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^4(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
3	$H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^3(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
2	$H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	0	$P_2 H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	$\Lambda_3 H^2(\text{Cur } \mathfrak{g}, \mathbb{C})$	0
1	0	0	0	0	0
0	\mathbb{C}	0	$P_2 \mathbb{C}$	$\Lambda_3 \mathbb{C}$	0
	0	1	2	3	4

Figure 4.2: $E_3^{p,q}$ -page for $(C^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), F^\bullet, d)$.

Now assume that any q -cochain $\bar{\gamma} \in C^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, a representative of a cohomology class $[\bar{\gamma}] \in H^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, can be extended to a q -cochain in $\bar{\gamma} \in C^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$ as the following

$$\begin{aligned}
(\iota_\lambda(L)d\bar{\gamma})_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= 0, \\
\bar{\gamma}_{\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_q}(L, \dots, L, g_{s+1}, \dots, g_q) &= 0, \quad \text{for all } 3 \leq s \leq q.
\end{aligned} \tag{4.9}$$

Then $\bar{\gamma} \in E_3^{0,q}$ for all $q \geq 0$. From Figure 4.2, the only possible nontrivial differential is $d_3^{0,q} : H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \rightarrow H^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C})$ for $q \geq 2$. Observe that for any $\bar{\gamma} \in C^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, the differential $d_3^{0,q}$ can be written as follows

$$d_3^{0,q}\bar{\gamma} = d\bar{\gamma} = d\gamma \mod \partial\tilde{C}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) = \Lambda_3\beta \mod \partial\tilde{C}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) = \Lambda_3\bar{\beta}$$

for some $\beta \in \tilde{C}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C})$, where $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. By the definition of the differential, we have

$$\begin{aligned}
d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, L, g_3, \dots, g_{q+1}) \\
&= (\lambda_2 - \lambda_1)\gamma_{\lambda_1+\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1}) \\
&+ (\lambda_1 - \lambda_3)\gamma_{\lambda_1+\lambda_3, \lambda_2, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1}) \\
&+ (\lambda_3 - \lambda_2)\gamma_{\lambda_2+\lambda_3, \lambda_1, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{i-1}, \lambda_1 + \lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_j, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_1, \lambda_3, \lambda_4, \dots, \lambda_{i-1}, \lambda_2 + \lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_j, \dots, g_{q+1}) \\
& - \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_1, \lambda_2, \lambda_4, \dots, \lambda_{i-1}, \lambda_3 + \lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_j, \dots, g_{q+1}). \tag{4.10}
\end{aligned}$$

Since $d\bar{\gamma} = \Lambda_3 \bar{\beta} \in \Lambda_3 H^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ where $\beta \in \tilde{C}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C})$, then γ is a quadratic skew symmetric homogeneous polynomial in two variables that is a constant on \mathfrak{g} . So we can rewrite (4.10) as

$$\begin{aligned}
& d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, L, g_3, \dots, g_{q+1}) \\
& = (\lambda_2 - \lambda_1)[(\lambda_1 + \lambda_2)^2 - \lambda_3^2] + (\lambda_1 - \lambda_3)[(\lambda_1 + \lambda_3)^2 - \lambda_2^2] \\
& + (\lambda_3 - \lambda_2)[(\lambda_2 + \lambda_3)^2 - \lambda_1^2] - \sum_{j=4}^{q+1} \lambda_j [\lambda_2^2 - \lambda_3^2] \\
& + \sum_{j=4}^{q+1} \lambda_j [\lambda_1^2 - \lambda_3^2] - \sum_{j=4}^{q+1} \lambda_j [\lambda_1^2 - \lambda_2^2] = 0. \tag{4.11}
\end{aligned}$$

So we have $d_3^{0,q}(\bar{\gamma}) = d\bar{\gamma} = 0$ for all $q \geq 0$. It follows that $E_4^{p,q} \cong E_3^{p,q}$ for all $p, q \geq 0$. Moreover, the differential d_r for all $r \geq 4$ is zero, and hence $E_\infty^{p,q} \cong E_4^{p,q}$ for all $p, q \geq 0$. Thus we have,

$$\begin{aligned}
H^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) & \cong \bigoplus_{p+q=n} H^p(\text{Vir}, \mathbb{C}) \otimes H^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \\
& \cong H^0(\text{Vir}, \mathbb{C}) \otimes H^n(\text{Cur } \mathfrak{g}, \mathbb{C}) \oplus H^2(\text{Vir}, \mathbb{C}) \otimes H^{n-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) \\
& \oplus H^3(\text{Vir}, \mathbb{C}) \otimes H^{n-3}(\text{Cur } \mathfrak{g}, \mathbb{C}) \\
& \cong H^n(\text{Cur } \mathfrak{g}, \mathbb{C}) \oplus P_2 H^{n-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 H^{n-3}(\text{Cur } \mathfrak{g}, \mathbb{C}).
\end{aligned}$$

where $P_2 = \lambda_1^3 - \lambda_2^3$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. i.e., $\dim H^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) = \dim H^n(\text{Cur } \mathfrak{g}, \mathbb{C}) + \dim H^{n-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) + \dim H^{n-3}(\text{Cur } \mathfrak{g}, \mathbb{C})$, which contradicts (4.4).

Chapter 5

Applications

According to [8], every finite semisimple Lie conformal algebra is uniquely decomposed in a finite direct sum of Lie conformal algebras each of which is isomorphic to a *Virasoro conformal algebra*, Vir , a *current conformal algebra*, $\text{Cur } \mathfrak{g}$ where \mathfrak{g} is a semisimple finite dimensional Lie algebra, or a *semidirect product of Vir and Cur g*, defined by $[L_\lambda g] = (\partial + \lambda)g$, $g \in \mathfrak{g}$. The cohomology of the first two types was extensively studied in [2], and the cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ was done in [33].

In this chapter, we will start by computing the basic cohomology of Vir with coefficient in a Vir -module $M_{\Delta, \alpha}$. Then we will use the Lie conformal algebra's Hochschild-Serre spectral sequence to compute the cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with trivial coefficients. In addition, we give explicit computations for the basic cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in the module $M_{\Delta, \alpha, U}$.

5.1 Cohomology of Vir with coefficients in $M_{\Delta, \alpha}$

Recall (Theorem 2.3.1) that every irreducible finite Vir -module is $M_{\Delta, \alpha}$ where $\Delta, \alpha \in \mathbb{C}$ and $\Delta \neq 0$ such that

$$M_{\Delta, \alpha} = \mathbb{C}[\partial]\nu, \quad L_\lambda \nu = (\partial + \alpha + \Delta\lambda)\nu.$$

The following theorem describes the reduced cohomology of Vir with coefficients in $M_{\Delta, \alpha}$ which was studied in [2].

Theorem 5.1.1 ([2], Theorem 7.2). 1. $H^\bullet(\text{Vir}, M_{\Delta, \alpha}) = 0$ if $\alpha \neq 0$.

2. $H^q(\text{Vir}, M_{\Delta, 0}) \cong H^q(\mathfrak{Vect}_0 \mathbb{C}, U_{\Delta-1}) \oplus H^{q-1}(\mathfrak{Vect}_0 \mathbb{C}, U_{\Delta-1})$ for any q .

3. $\dim H^q(\text{Vir}, M_{\Delta,0}) = \dim H^q(\mathfrak{Vect}\mathbb{C}, \mathbb{C}[t, t^{-1}](dt)^{1-\Delta})$. Explicitly:

$$\dim H^q(\text{Vir}, M_{1-(3r^2 \pm r)/2,0}) = \begin{cases} 2 & \text{for } q = r + 1 \\ 1 & \text{for } q = r, r + 2 \\ 0 & \text{otherwise} \end{cases}$$

and $H^q(\text{Vir}, M_{\Delta,0}) = 0$ if $\Delta \neq 1 - (3r^2 \pm r)/2$ for any $r \in \mathbb{Z}_+$.

In this section, we will compute the basic cohomology of Vir with coefficients in the Vir -module $M_{\Delta,\alpha}$ where $\Delta, \alpha \in \mathbb{C}$.

Theorem 5.1.2. *For the Virasoro conformal algebra Vir , for any $\alpha \in \mathbb{C}$,*

1. $\dim \tilde{H}^q(\text{Vir}, M_{0,\alpha}) = 0$ for $q > 3$
2. $\dim \tilde{H}^q(\text{Vir}, M_{1,\alpha}) = 0$ for $q \neq 1, 2$.
3. $\dim \tilde{H}^q(\text{Vir}, M_{(3r-r^2)/2,\alpha}) = 0$ for $q > r$ where $r \in \mathbb{Z}_+$ and $r \geq 4$.

Proof. We first identify the basic cohomology complex $\tilde{C}^\bullet(\text{Vir}, M_{\Delta,\alpha})$. Any n -cochain $\gamma \in \tilde{C}^n(\text{Vir}, M_{\Delta,\alpha})$ is determined by its value on $L^{\otimes n}$:

$$P(\lambda_1, \dots, \lambda_n) = \gamma_{\lambda_1, \dots, \lambda_n}(L, \dots, L),$$

where $P(\lambda_1, \dots, \lambda_n)$ is a skew-symmetric polynomial in n variables with values in $M_{\Delta,\alpha}$. The differential is given by the following formula:

$$\begin{aligned} (dP)(\lambda_1, \dots, \lambda_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} (\partial_M + \alpha + \Delta \lambda_i) P(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}) \\ &\quad + \sum_{\substack{i,j=1 \\ i < j}}^{n+1} (-1)^{i+j} (\lambda_i - \lambda_j) P(\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}). \end{aligned}$$

Following [2], consider the homotopy operator

$$\begin{aligned} \kappa : \tilde{C}^q(\text{Vir}, M_{\Delta,\alpha}) &\longrightarrow \tilde{C}^{q-1}(\text{Vir}, M_{\Delta,\alpha}) \\ \kappa \gamma_{\lambda_1, \dots, \lambda_{q-1}}(L, \dots, L) &= \frac{\partial}{\partial \lambda} \iota_\lambda(L) \gamma_{\lambda_1, \dots, \lambda_{q-1}}(L, \dots, L)|_{\lambda=0}. \end{aligned}$$

Then by Cartan's identity we have

$$\begin{aligned}
& (\kappa d + d\kappa)\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L) \\
&= \frac{\partial}{\partial \lambda}(\iota_\lambda(L)d + d\iota_\lambda(L))\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L)|_{\lambda=0} \\
&= \frac{\partial}{\partial \lambda}\theta_\lambda(L)\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L)|_{\lambda=0} \\
&= \frac{\partial}{\partial \lambda}((\partial + \alpha + \Delta\lambda)\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L)|_{\lambda=0} + \sum_{i=1}^q(\lambda_i - \lambda)\gamma_{\lambda_1, \dots, \lambda_i + \lambda, \dots, \lambda_q}(L, \dots, L)|_{\lambda=0}) \\
&= (\deg_\lambda \gamma - q + \Delta)\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L).
\end{aligned}$$

where $\deg_\lambda \gamma$ is the total degree of γ in $\lambda_1, \dots, \lambda_q$. Then, for a q -cochain γ , it contributes to the cohomology of $\tilde{C}^q(\text{Vir}, M_{\Delta, \alpha})$ only if its degree as a polynomial is equal to $q - \Delta$. Thus, Δ must be an integer, otherwise $\tilde{H}^q(\text{Vir}, M_{\Delta, \alpha}) = 0$ for all q . Since $\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L)$ as a polynomial in $\lambda_1, \dots, \lambda_q$ is a skew-symmetric, then it is divisible by $\prod_{i < j}(\lambda_i - \lambda_j)$ whose polynomial degree is $q(q-1)/2$.

Consider the quadratic inequality $q(q-1)/2 \leq q - \Delta$, whose discriminant is $9 - 8\Delta$. Then $q \in I = [(3 - \sqrt{9 - 8\Delta})/2, (3 + \sqrt{9 - 8\Delta})/2]$. Let r be a nonnegative integer such that $r \in I$, then we have $\Delta = (3r - r^2)/2$. It follows that the only integral solutions for the inequality $q(q-1)/2 \leq q - \Delta$ are

$$q = \begin{cases} 0, 1, 2, 3 & \text{if } \Delta = 0, \\ 1, 2 & \text{if } \Delta = 1, \\ 0, 1, \dots, r & \text{if } \Delta = (3r - r^2)/2. \end{cases}$$

where $r \in \mathbb{Z}_+$ such that $r \geq 4$. Therefore, for $\Delta = 0$, we have $\tilde{H}^q(\text{Vir}, M_{0, \alpha}) = 0$ for all $q > 3$, and for $\Delta = 1$ we obtain $\tilde{H}^q(\text{Vir}, M_{1, \alpha}) = 0$ for all $q \neq 1, 2$. If $\Delta = (3r - r^2)/2$ for $r \in \mathbb{Z}_+$ such that $r \geq 4$, then we have $\tilde{H}^q(\text{Vir}, M_{\Delta, \alpha}) = 0$ for all $q > r$. \square

As shown in Proposition 2.7.1, the short exact sequences of complexes $0 \longrightarrow \partial\tilde{C}^\bullet \longrightarrow \tilde{C}^\bullet \longrightarrow C^\bullet \longrightarrow 0$ gives the following long exact sequence of cohomology groups:

$$\begin{aligned}
0 &\longrightarrow H^0(\partial\tilde{C}^\bullet) \xrightarrow{\iota_0} \tilde{H}^0(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{\pi_0} H^0(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{w_0} \\
&\longrightarrow H^1(\partial\tilde{C}^\bullet) \xrightarrow{\iota_1} \tilde{H}^1(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{\pi_1} H^1(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{w_1} \\
&\longrightarrow H^2(\partial\tilde{C}^\bullet) \xrightarrow{\iota_2} \tilde{H}^2(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{\pi_1} H^2(\text{Vir}, M_{\Delta, \alpha}) \xrightarrow{w_2} \dots
\end{aligned} \tag{5.1}$$

where ι_i and π_i are the homomorphisms induced by the embedding $\partial\tilde{C}^\bullet \hookrightarrow \tilde{C}^\bullet$ and the canonical projection $\pi : \tilde{C}^\bullet \rightarrow \tilde{C}^\bullet/\partial\tilde{C}^\bullet$, respectively, and w_i is the i -th connecting homomorphism. Then, we have the following lemma.

Lemma 5.1.1. *The map $\iota_q : H^q(\partial\tilde{C}^\bullet) \rightarrow \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha})$ is zero for all $q \geq 0$.*

Proof. Let $\gamma \in \tilde{C}^q(\text{Vir}, M_{\Delta,\alpha})$ be a representative of cohomology class $[\gamma] \in \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha})$ then by Theorem 5.1.2, $\gamma_{\lambda_1, \dots, \lambda_q}(L, \dots, L) = P(\lambda_1, \dots, \lambda_q)$ where $P(\lambda_1, \dots, \lambda_q)$ is a skew symmetric polynomial in $\lambda_1, \dots, \lambda_q$ of total degree, $\deg_\lambda P = q - \Delta$.

Consider the q -cocycle $[\partial\gamma] \in H^q(\partial\tilde{C}^\bullet)$ for nonzero $[\gamma] \in \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha})$, then $\iota_q([\partial\gamma]) = [\partial\gamma] \in \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha})$. So, $\partial\gamma$ can be identified with a skew symmetric polynomial ∂P of degree $q - \Delta$. From the definition we have

$$\partial \cdot \gamma_{\lambda_1, \dots, \lambda_q} = \partial \cdot P(\lambda_1, \dots, \lambda_q) = (\partial_M + \sum_{i=1}^q \lambda_i) P(\lambda_1, \dots, \lambda_q).$$

So we get $\deg_\lambda \partial\gamma = \deg_\lambda P + 1 = q - \Delta + 1$, a contradiction. Then, $\partial\gamma$ must be zero in $\tilde{H}^q(\text{Vir}, M_{\Delta,\alpha})$, i.e., the image of ι_q is zero for all $q \geq 0$. \square

Theorem 5.1.3. *For all $q \geq 0$,*

$$\dim H^q(\text{Vir}, M_{\Delta,\alpha}) = \dim \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha}) + \dim \tilde{H}^{q+1}(\text{Vir}, M_{\Delta,\alpha}).$$

Proof. By Lemma 5.1.1, we have $\text{im } \iota_q = 0$ for all $q \geq 0$. Thus, $\ker \pi_q = 0$, and $\text{im } w_q = H^{q+1}(\partial\tilde{C}^\bullet)$. Then the long exact sequence (5.1) gives the following short exact sequence for each $q \geq 0$:

$$0 \longrightarrow \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha}) \longrightarrow H^q(\text{Vir}, M_{\Delta,\alpha}) \longrightarrow \tilde{H}^{q+1}(\partial\tilde{C}^\bullet) \longrightarrow 0$$

Thus, for all $q \geq 0$ we have $\dim H^q(\text{Vir}, M_{\Delta,\alpha}) = \dim \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha}) + \dim \tilde{H}^{q+1}(\partial\tilde{C}^\bullet) = \dim \tilde{H}^q(\text{Vir}, M_{\Delta,\alpha}) + \dim \tilde{H}^{q+1}(\text{Vir}, M_{\Delta,\alpha})$ by Theorem 2.7.1. \square

Theorem 5.1.4. $\tilde{H}^\bullet(\text{Vir}, M_{\Delta,\alpha}) = 0$ for $\alpha \neq 0$.

Proof. Follows from Theorem 5.1.1 and Theorem 5.1.3. \square

Theorem 5.1.5. *For the Virasoro conformal algebra,*

$$\dim \tilde{H}^q(\text{Vir}, M_{0,0}) = \begin{cases} 1 & \text{if } q = 2 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\tilde{H}^2(\text{Vir}, M_{0,0}) \cong \mathbb{C}(\lambda_1^2 - \lambda_2^2)\nu$, and $\tilde{H}^3(\text{Vir}, M_{0,0}) \cong \mathbb{C}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\nu$.

Proof. By Theorem 5.1.2, we have $r = 0$ or 3 and $\dim \tilde{H}^q(\text{Vir}, M_{0,0}) = 0$ for all $q \geq 4$. Using Theorem 5.1.1, we obtain

$$\dim H^q(\text{Vir}, M_{0,0}) = \begin{cases} 2 & \text{if } q = 2, \\ 1 & \text{if } q = 1 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Theorem 5.1.3, $\dim \tilde{H}^q(\text{Vir}, M_{0,0}) = 1$ for $q = 2$ or 3 , and is zero otherwise. To find a basis for $\tilde{H}^q(\text{Vir}, M_{0,0})$ and $q = 2, 3$, recall that the only skew symmetric homogeneous polynomials of degree $q - \Delta$ contribute to the cohomology of $\tilde{C}^\bullet(\text{Vir}, M_{0,0})$. For $q = 2$, the only skew symmetric homogeneous polynomial of degree 2 in two variables is $\lambda_1^2 - \lambda_2^2$. Setting $\gamma_{\lambda,\mu}(L, L) = m(\lambda^2 - \mu^2)$ where $m = p(\partial)\nu \in M_{0,0}$, we get

$$\begin{aligned} & d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) \\ &= L_{\lambda_1} \gamma_{\lambda_2, \lambda_3}(L, L) - L_{\lambda_2} \gamma_{\lambda_1, \lambda_3}(L, L) + L_{\lambda_3} \gamma_{\lambda_1, \lambda_2}(L, L) - \gamma_{\lambda_1 + \lambda_2, \lambda_3}([L_{\lambda_1} L], L) \\ &+ \gamma_{\lambda_1 + \lambda_3, \lambda_2}([L_{\lambda_1} L], L) - \gamma_{\lambda_2 + \lambda_3, \lambda_1}([L_{\lambda_2} L], L) \\ &= L_{\lambda_1} (p(\partial)(\lambda_2^2 - \lambda_3^2)\nu) - L_{\lambda_2} (p(\partial)(\lambda_1^2 - \lambda_3^2)\nu) + L_{\lambda_3} (p(\partial)(\lambda_1^2 - \lambda_2^2)\nu) \\ &- (\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + (\lambda_1 - \lambda_3) \gamma_{\lambda_1 + \lambda_3, \lambda_2}(L, L) - (\lambda_2 - \lambda_3) \gamma_{\lambda_2 + \lambda_3, \lambda_1}(L, L) \\ &= p(\partial + \lambda_1)(\partial_M(\lambda_2^2 - \lambda_3^2)\nu) - p(\partial + \lambda_2)(\partial_M(\lambda_1^2 - \lambda_3^2)\nu) + p(\partial + \lambda_3)(\partial_M(\lambda_1^2 - \lambda_2^2)\nu) \\ &- p(\partial)(\lambda_1 - \lambda_2)((\lambda_1 + \lambda_2)^2 - \lambda_3^2)\nu + p(\partial)(\lambda_1 - \lambda_3)((\lambda_1 + \lambda_3)^2 - \lambda_2^2)\nu \\ &- p(\partial)(\lambda_2 - \lambda_3)((\lambda_2 + \lambda_3)^2 - \lambda_1^2)\nu \\ &= p(\partial + \lambda_1)(\partial - \lambda_2 - \lambda_3)(\lambda_2^2 - \lambda_3^2)\nu - p(\partial + \lambda_2)(\partial - \lambda_1 - \lambda_3)(\lambda_1^2 - \lambda_3^2)\nu \\ &+ p(\partial + \lambda_3)(\partial - \lambda_1 - \lambda_2)(\lambda_1^2 - \lambda_2^2)\nu. \end{aligned} \tag{5.2}$$

Setting $p(\partial) = \sum_{i=0}^n u_i \partial^i$ with $u_n \neq 0$, then plugging this into (5.2) with $\lambda_3 = 0$ yields

$$\begin{aligned} & (\partial \lambda_2^2 - \lambda_2^3) \sum_{i=0}^n u_i (\partial + \lambda_1)^i - (\partial \lambda_1^2 - \lambda_1^3) \sum_{i=0}^n u_i (\partial + \lambda_2)^i \\ & + (\partial - \lambda_1 - \lambda_2)(\lambda_1^2 - \lambda_2^2) \sum_{i=0}^n u_i \partial^i = 0. \end{aligned} \quad (5.3)$$

Comparing the coefficients of $\lambda_2^3 \partial^{n-1}$ in (5.3) gives $nu_n \lambda_1 = 0$, i.e., $n = 0$. Hence, $d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) = 0$ only if $p(\partial)$ is a constant, i.e., an element of \mathbb{C} . Thus, $\lambda_1^2 - \lambda_2^2$, up to a constant factor ($\nu \in M_{0,0}$), is a 2-cocycle of $\tilde{C}^2(\text{Vir}, M_{0,0})$.

Observe that $\lambda_1^2 - \lambda_2^2$ represents a nontrivial class in the cohomology. Indeed, assume that there exists a nonzero $\phi \in \tilde{C}^1(\text{Vir}, M_{0,0})$ such that $d\phi_{\lambda_1, \lambda_2}(L, L) = \lambda_1^2 - \lambda_2^2$. Since ϕ is identified with a polynomial in one variable λ , then we can set $\phi_\lambda(L) = \lambda f(\lambda)$ for some polynomial $f(\lambda) = \sum_{i=0}^n a_i(\lambda) \partial^i \in M_{0,0}[\lambda]$. So we have

$$\begin{aligned} \lambda_1^2 - \lambda_2^2 &= d\phi_{\lambda_1, \lambda_2}(L, L) \\ &= L_{\lambda_1}(\lambda_2 f(\lambda_2)) - L_{\lambda_2}(\lambda_1 f(\lambda_1)) - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)f(\lambda_1 + \lambda_2) \\ &= \left(\sum_{i=0}^n a_i(\lambda_2)(\partial + \lambda_1)^i \right) (\partial - \lambda_2) \lambda_2 - \left(\sum_{i=0}^n a_i(\lambda_1)(\partial + \lambda_2)^i \right) (\partial - \lambda_1) \lambda_1 \\ &\quad - (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right). \end{aligned} \quad (5.4)$$

Letting $\lambda_2 = 0$ in (5.4) we get

$$\begin{aligned} -\lambda_1^2 &= \lambda_1(\partial - \lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) + \lambda_1^2 \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) \\ &= (\lambda_1(\partial - \lambda_1) + \lambda_1^2) \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) = \lambda_1 \sum_{i=0}^n a_i(\lambda_1) \partial^{i+1}. \end{aligned} \quad (5.5)$$

Note that the left hand side of (5.5) does not depend on ∂ , hence all coefficients of nonzero powers of ∂ must be 0. So we have $a_i(\lambda) \lambda = 0$, which implies that $a_i(\lambda) = 0$ for all $0 \leq i \leq n$. i.e., $f(\lambda) = 0$, a contradiction. Therefore, $\lambda_1^2 - \lambda_2^2$ is not a 2-coboundary and $\tilde{H}^2(\text{Vir}, M_{0,0}) \cong \mathbb{C}(\lambda_1^2 - \lambda_2^2)\nu$.

For $q = 3$, the only skew symmetric homogeneous polynomial of degree 3 in 3 variables is $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Let $\gamma_{\lambda, \mu, \beta}(L, L, L) = m \Lambda_3^{\lambda, \mu, \beta}$ where $m = p(\partial)\nu$ a

nonzero element of $M_{0,0}$ and $\Lambda_3^{\lambda,\mu,\beta} = (\lambda - \mu)(\lambda - \beta)(\mu - \beta)$. Then we have

$$\begin{aligned}
& d\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, L, L) \\
&= L_{\lambda_1}\gamma_{\lambda_2,\lambda_3,\lambda_4}(L, L, L) - L_{\lambda_2}\gamma_{\lambda_1,\lambda_3,\lambda_4}(L, L, L) + L_{\lambda_3}\gamma_{\lambda_1,\lambda_2,\lambda_4}(L, L, L) \\
&\quad - L_{\lambda_4}\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, L, L) - (\lambda_1 - \lambda_2)\gamma_{\lambda_1+\lambda_2,\lambda_3,\lambda_4}(L, L, L) + (\lambda_1 - \lambda_3)\gamma_{\lambda_1+\lambda_3,\lambda_2,\lambda_4}(L, L, L) \\
&\quad - (\lambda_1 - \lambda_4)\gamma_{\lambda_1+\lambda_4,\lambda_2,\lambda_3}(L, L, L) - (\lambda_2 - \lambda_3)\gamma_{\lambda_2+\lambda_3,\lambda_1,\lambda_4}(L, L, L) \\
&\quad + (\lambda_2 - \lambda_4)\gamma_{\lambda_2+\lambda_4,\lambda_1,\lambda_3}(L, L, L) - (\lambda_3 - \lambda_4)\gamma_{\lambda_3+\lambda_4,\lambda_1,\lambda_2}(L, L, L) \\
&= p(\partial + \lambda_1)(\partial - \lambda_2 - \lambda_3 - \lambda_4)\Lambda_3^{\lambda_2,\lambda_3,\lambda_4}\nu - p(\partial + \lambda_2)(\partial - \lambda_1 - \lambda_3 - \lambda_4)\Lambda_3^{\lambda_1,\lambda_3,\lambda_4}\nu \\
&\quad + p(\partial + \lambda_3)(\partial - \lambda_1 - \lambda_2 - \lambda_4)\Lambda_3^{\lambda_1,\lambda_2,\lambda_4}\nu - p(\partial + \lambda_4)(\partial - \lambda_1 - \lambda_2 - \lambda_3)\Lambda_3^{\lambda_1,\lambda_2,\lambda_3}\nu \\
&\quad - p(\partial)(\lambda_1 - \lambda_2)\Lambda_3^{\lambda_1+\lambda_2,\lambda_3,\lambda_4}\nu + p(\partial)(\lambda_1 - \lambda_3)\Lambda_3^{\lambda_1+\lambda_3,\lambda_2,\lambda_4}\nu - p(\partial)(\lambda_1 - \lambda_4)\Lambda_3^{\lambda_1+\lambda_4,\lambda_2,\lambda_3}\nu \\
&\quad - p(\partial)(\lambda_2 - \lambda_3)\Lambda_3^{\lambda_2+\lambda_3,\lambda_1,\lambda_4}\nu + p(\partial)(\lambda_2 - \lambda_4)\Lambda_3^{\lambda_2+\lambda_4,\lambda_1,\lambda_3}\nu - p(\partial)(\lambda_3 - \lambda_4)\Lambda_3^{\lambda_3+\lambda_4,\lambda_1,\lambda_2}\nu.
\end{aligned}$$

Letting $p(\partial) = \sum_{i=0}^n u_i \partial^i$ with $u_n \neq 0$, then $d\gamma_{\lambda_1,\lambda_2,\lambda_3,0}(L, L, L, L) = 0$ becomes

$$\begin{aligned}
& \lambda_2\lambda_3(\partial - \lambda_2 - \lambda_3)(\lambda_2 - \lambda_3) \sum_{i=0}^n u_i(\partial + \lambda_1)^i - \lambda_1\lambda_3(\partial - \lambda_1 - \lambda_3)(\lambda_1 - \lambda_3) \sum_{i=0}^n u_i(\partial + \lambda_2)^i \\
& \quad + \lambda_1\lambda_2(\partial - \lambda_1 - \lambda_2)(\lambda_1 - \lambda_2) \sum_{i=0}^n u_i(\partial + \lambda_3)^i + (\lambda_1^2(\lambda_2 - \lambda_3)(\lambda_1 - \partial) \\
& \quad + \lambda_2^2(\lambda_1 - \lambda_3)(\lambda_2 + \partial) - \lambda_3^2(\lambda_1 - \lambda_2)(\lambda_3 + \partial)) \left(\sum_{i=0}^n u_i \partial^i \right) = 0.
\end{aligned}$$

Comparing the coefficients if $\lambda_1\lambda_2^3\lambda_3\partial^{n-1}$ implies that $nu_n = 0$, hence $n = 0$. It follows that $d\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, L, L) = 0$ only if $p(\partial)$ is a constant. Thus, Λ_3 , up to a constant factor ($\nu \in M_{0,0}$), is a 3-cocycle.

Now assume that there exists a 2-cochain $\phi \in \tilde{C}^2(\text{Vir}, M_{0,0})$ such that $d\phi = \Lambda_3^{\lambda_1,\lambda_2,\lambda_3}$. Because ϕ can be identified with a skew symmetric polynomial in two variables, then we can write $\phi_{\lambda,\mu}(L, L) = (\lambda - \mu)f(\lambda, \mu)$ for some nonzero symmetric polynomial $f(\lambda, \mu) \in M_{0,0}[\lambda, \mu]$. So we have

$$\begin{aligned}
\Lambda_3^{\lambda_1,\lambda_2,\lambda_3} &= d\phi_{\lambda_1,\lambda_2,\lambda_3}(L, L, L) \\
&= L_{\lambda_1}((\lambda_2 - \lambda_3)f(\lambda_2, \lambda_3)) - L_{\lambda_2}((\lambda_1 - \lambda_3)f(\lambda_1, \lambda_3)) + L_{\lambda_3}((\lambda_1 - \lambda_2)f(\lambda_1, \lambda_2)) \\
&\quad - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3)f(\lambda_1 + \lambda_2, \lambda_3) + (\lambda_1 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2)f(\lambda_1 + \lambda_3, \lambda_2) \\
&\quad + (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3)f(\lambda_1, \lambda_2 + \lambda_3). \tag{5.6}
\end{aligned}$$

Setting $f(\lambda, \mu) = \sum_{i=0}^n a_i(\lambda, \mu) \partial^i$ in (5.6). Then, for $\lambda_3 = 0$, (5.6) becomes

$$\begin{aligned}
\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) &= \partial_M(\lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - \partial_M(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) \\
&+ \partial_M(\lambda_1 - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^i \right) - (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) \\
&+ (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^i \right) \\
&= (\partial - \lambda_2)(\lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - (\partial - \lambda_1)(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) \\
&+ (\partial - \lambda_1 - \lambda_2)(\lambda_1 - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^i \right) - (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) \\
&+ (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^i \right) \\
&= (\partial - \lambda_2)(\lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - (\partial - \lambda_1)(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) \\
&+ \partial(\lambda_1 - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^i \right) - (\lambda_1^2 - \lambda_2^2) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) \\
&= \lambda_1 \left((\lambda_1 - \partial) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) + \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} - \lambda_1 \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) \right) \\
&+ \lambda_2 \left((\partial - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} + \lambda_2 \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) \right).
\end{aligned} \tag{5.7}$$

Equating similar terms in both sides of (5.7) we obtain

$$\begin{aligned}
(\lambda_1 - \partial) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) + \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} - \lambda_1 \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) &= \lambda_2 \lambda_1 - \lambda_2^2, \\
(\partial - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} + \lambda_2 \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) &= \lambda_1^2 - \lambda_1 \lambda_2,
\end{aligned} \tag{5.8}$$

which implies that

$$\begin{aligned}
& (\lambda_1 - \partial) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) + (\partial - \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) \\
& + (\lambda_2 - \lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right) = \lambda_1^2 - \lambda_2^2.
\end{aligned} \tag{5.9}$$

Now letting $\lambda_2 = 0$ in (5.9) yields

$$\partial \left(\sum_{i=0}^n a_i(0) (\partial + \lambda_1)^i \right) - \sum_{i=0}^n a_i(\lambda_1) \partial^{i+1} = \lambda_1^2. \tag{5.10}$$

The right hand side of (5.10) does not depend on ∂ , then all the coefficients of nonzero powers of ∂ must be zero. So we have $a_n(\lambda_1) = a_n(0)$ and $a_i(\lambda_1) = a_i(0) + (i+1)a_{i+1}(0)\lambda_1$ for all $i = 0, \dots, n-1$. Plugging this into (5.8) gives

$$\begin{aligned}
& (\lambda_1 - \partial) \left(\sum_{i=0}^n (a_i(0) + (i+1)a_{i+1}(0)\lambda_1) (\partial + \lambda_2)^i \right) + \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} \\
& - \lambda_1 \left(\sum_{i=0}^n (a_i(0) + (i+1)a_{i+1}(0)(\lambda_1 + \lambda_2)) \partial^i \right) = \lambda_2 \lambda_1 - \lambda_2^2.
\end{aligned} \tag{5.11}$$

Comparing the coefficients of $\lambda_1 \partial^{i+1}$ for all $i = 0, \dots, n-1$ in (5.11), we get $(i+1)a_{i+1}(0) = 0$. It follows that $a_i(0) = 0$ for all $i = 1, \dots, n$. Hence, $a_i(\lambda_1) = 0$ for $i = 1, \dots, n$ and $a_0(\lambda_1) = a_0(0)$. Plugging this into (5.8) yields

$$-a_0(0)\partial + \sum_{i=0}^n a_i(\lambda_1, \lambda_2) \partial^{i+1} = \lambda_2 \lambda_1 - \lambda_2^2$$

It follows that $f(\lambda_1, \lambda_2) = a_0(\lambda_1, \lambda_2) = a_0(0)$, a constant. Set $f(\lambda_1, \lambda_2) = c\nu$ for some $c \in \mathbb{C}$. Then $d\phi_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) = 0$, a contradiction. Therefore, Λ_3 represents a nontrivial class in the cohomology and we have $\tilde{H}^3(\text{Vir}, M_{0,0}) \cong \mathbb{C}\Lambda_3\nu$. \square

Theorem 5.1.6. *For the Virasoro conformal algebra,*

$$\dim \tilde{H}^q(\text{Vir}, M_{1,0}) = \begin{cases} 1 & \text{if } q = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\tilde{H}^1(\text{Vir}, M_{1,0}) \cong \mathbb{C}\nu$ and $\tilde{H}^2(\text{Vir}, M_{1,0}) \cong \mathbb{C}(\lambda_1 - \lambda_2)\nu$.

Proof. Using Theorem 5.1.1, we get $\dim H^1(\text{Vir}, M_{1,0}) = 2$, $\dim H^q(\text{Vir}, M_{1,0}) = 1$ for $q = 0, 2$ and is zero otherwise. Then by Theorem 5.1.2 and Theorem 5.1.3, $\dim \tilde{H}^q(\text{Vir}, M_{1,0}) = 1$ for $q = 1, 2$ and zero otherwise. Now, we construct a basis for $\tilde{H}^q(\text{Vir}, M_{1,0})$ where $q = 1, 2$.

For $q = 1$, then only polynomials of degree 0 contribute to the cohomology of $\tilde{C}^1(\text{Vir}, M_{1,0})$. For any $\gamma \in \tilde{C}^1(\text{Vir}, M_{1,0})$ such that $\deg_\lambda \gamma = 0$ then we can write $\gamma_\lambda(L) = p(\partial)\nu$ for some $p(\partial) \in \mathbb{C}[\partial]$. So we have

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2}(L, L) &= L_{\lambda_1}\gamma_{\lambda_2}(L) - L_{\lambda_2}\gamma_{\lambda_1}(L) - \gamma_{\lambda_1+\lambda_2}([L_{\lambda_1}L]) \\ &= p(\partial + \lambda_1)(\partial_M + \lambda_1)\nu - p(\partial + \lambda_2)(\partial_M + \lambda_2)\nu - (\lambda_1 - \lambda_2)p(\partial)\nu \\ &= p(\partial + \lambda_1)(\partial + \lambda_1)\nu - p(\partial + \lambda_2)(\partial + \lambda_2)\nu - (\lambda_1 - \lambda_2)p(\partial)\nu. \end{aligned}$$

Let $p(\partial) = \sum_{i=0}^n u_i \partial^i$ such that $u_n \neq 0$. Then $d\gamma_{\lambda_1, \lambda_2}(L, L) = 0$ yields

$$\sum_{i=0}^n u_i (\partial^i + \lambda_1)^i \nu - \sum_{i=0}^n u_i (\partial^i + \lambda_2)^i \nu - (\lambda_1 - \lambda_2) \sum_{i=0}^n u_i (\partial^i)^i \nu = 0. \quad (5.12)$$

Setting $\lambda_2 = 0$ in (5.12) and comparing the coefficients of ∂^n gives $nu_n\lambda = 0$, which implies $n = 0$. i.e., $p(\partial)$ must be a constant. Then, $\gamma = c\nu$ for some $c \in \mathbb{C}$ is a 1-cocycle. Now suppose that there exists $\phi \in \tilde{C}^0(\text{Vir}, M_{1,0})$ such that $d\phi = \gamma$ and $\phi = p(\partial)\nu$, then we have $d\phi = L_\lambda(p(\partial)\nu) = p(\partial + \lambda)(\partial_M + \lambda)\nu$. i.e., $\deg_\lambda(d\phi) = 1$. Thus, γ is not 1-coboundary and we have $\tilde{H}^1(\text{Vir}, M_{1,0}) \cong \mathbb{C}\nu$.

For $q = 2$, let $\gamma_{\lambda_1, \lambda_2}(L, L)$ be a 2-cocycle, then it must be a skew-symmetric homogeneous polynomial of degree 1 in two variables, so it is $\lambda_1 - \lambda_2$. Let $\gamma_{\lambda, \mu}(L, L) = m(\lambda - \mu)$ for some nonzero $m = p(\partial)\nu \in M_{1,0}$. Then we have

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) &= p(\partial + \lambda_1)(\partial_M + \lambda_1)(\nu(\lambda_2 - \lambda_3)) - p(\partial + \lambda_2)(\partial_M + \lambda_2)(\nu(\lambda_1 - \lambda_3)) \\ &\quad + p(\partial + \lambda_3)(\partial_M + \lambda_3)(\nu(\lambda_1 - \lambda_2)) - (\lambda_1 - \lambda_2)(p(\partial)\nu(\lambda_1 + \lambda_2 - \lambda_3)) \\ &\quad + (\lambda_1 - \lambda_3)(p(\partial)\nu(\lambda_1 + \lambda_3 - \lambda_2)) - (\lambda_2 - \lambda_3)(p(\partial)\nu(\lambda_2 + \lambda_3 - \lambda_1)) \\ &= p(\partial + \lambda_1)(\partial_M + \lambda_1)(\nu(\lambda_2 - \lambda_3)) - p(\partial + \lambda_2)(\partial_M + \lambda_2)(\nu(\lambda_1 - \lambda_3)) \\ &\quad + p(\partial + \lambda_3)(\partial_M + \lambda_3)(\nu(\lambda_1 - \lambda_2)) \\ &= p(\partial + \lambda_1)(\partial - \lambda_2 - \lambda_3 + \lambda_1)(\nu(\lambda_2 - \lambda_3)) - p(\partial + \lambda_2)(\partial - \lambda_1 - \lambda_3 + \lambda_2)(\nu(\lambda_1 - \lambda_3)) \\ &\quad + p(\partial - \lambda_1 - \lambda_2 + \lambda_3)(\partial_M + \lambda_3)(\nu(\lambda_1 - \lambda_2)). \end{aligned}$$

Let $p(\partial) = \sum_{i=0}^n u_i \partial^i$ such that $u_n \neq 0$. Then $d\gamma_{\lambda_1, \lambda_2, 0}(L, L, L) = 0$ implies that

$$\begin{aligned} \lambda_2(\partial - \lambda_2 + \lambda_1) \left(\sum_{i=0}^n u_i (\partial + \lambda_1)^i \right) - \lambda_1(\partial - \lambda_1 + \lambda_2) \left(\sum_{i=0}^n u_i (\partial + \lambda_2)^i \right) \\ + (\lambda_1 - \lambda_2)(\partial - \lambda_1 - \lambda_2) \left(\sum_{i=0}^n u_i \partial^i \right) = 0. \end{aligned}$$

Comparing the coefficients of $\lambda_2 \lambda_1^{i+1}$ gives $u_i = 0$ for $i = 1, \dots, n$. i.e., $d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) = 0$ only if m is independent of ∂ . Thus, $\lambda_1 - \lambda_2$, up to a constant factor $\nu \in M_{1,0}$, is a 2-cocycle. Moreover, if $\phi \in \tilde{C}^1(\text{Vir}, M_{1,0})$ such that $d\phi_{\lambda_1, \lambda_2}(L, L) = \lambda_1 - \lambda_2$, then

$$\begin{aligned} \lambda_1 - \lambda_2 &= d\phi_{\lambda_1, \lambda_2}(L, L) \\ &= (\partial_M + \lambda_1)\phi_{\lambda_2}(L) - (\partial_M + \lambda_2)\phi_{\lambda_1}(L) - (\lambda_1 - \lambda_2)\phi_{\lambda_1 + \lambda_2}(L) \\ &= (\partial + \lambda_1 - \lambda_2)\phi_{\lambda_2}(L) - (\partial - \lambda_1 + \lambda_2)\phi_{\lambda_1}(L) - (\lambda_1 - \lambda_2)\phi_{\lambda_1 + \lambda_2}(L). \end{aligned} \quad (5.13)$$

Since ϕ is identified with a polynomial in one variable, then it can be written as $\phi_\lambda(L) = \lambda f(\lambda)$ for a nonzero polynomial $f(\lambda) \in M_{1,0}[\lambda]$. Setting $f(\lambda) = \sum_{i=0}^n a_i(\lambda) \partial^i$ such that $a_n(\lambda) \neq 0$, then (5.13) becomes

$$\begin{aligned} \lambda_1 - \lambda_2 &= (\partial + \lambda_1 - \lambda_2)(\lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) \\ &\quad - (\partial - \lambda_1 + \lambda_2)(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \left(\sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \right). \end{aligned} \quad (5.14)$$

Letting $\lambda_2 = 0$ in (5.14), we get

$$\lambda_1 = -(\partial - \lambda_1)(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) - \lambda_1^2 \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) = -\lambda_1 \sum_{i=0}^m a_i(\lambda_1) \partial^{i+1}. \quad (5.15)$$

The left hand side of (5.15) does not depend on ∂ . Then all coefficients of nonzero powers of ∂ must be zero. So, $a_i(\lambda) \lambda_1 = 0$ for all $i = 0, \dots, n$. It follows that $\phi_\lambda(L) = 0$, a contradiction. Thus, $\lambda_1 - \lambda_2$ is not a 2-coboundary and $\tilde{H}^2(\text{Vir}, M_{1,0}) = \mathbb{C}(\lambda_1 - \lambda_2)\nu$. \square

Theorem 5.1.7. *For $q = 0, 1$, $\tilde{H}^q(\text{Vir}, M_{(3r-r^2)/2,0}) = \{0\}$ where $r \in \mathbb{Z}_+$ and $r \geq 4$.*

Proof. By Theorem 2.7.2, part (1) we have $\tilde{H}^0(\text{Vir}, M_{\Delta,0}) = M_{\Delta,0}^{\text{Vir}} = \{m \in M_{\Delta,0} | L_\lambda m = 0\}$.

So, for $m = \sum_{i=0}^n u_i \partial^i \in M_{\Delta,0}$ such that $L_\lambda m = 0$, then we have

$$\begin{aligned} L_\lambda m &= (\partial + \Delta\lambda) \left(\sum_{i=0}^n u_i (\partial + \lambda)^i \right) \\ &= \sum_{i=0}^n u_i (\partial + \lambda)^i \partial + \Delta\lambda \sum_{i=0}^n u_i (\partial + \lambda)^i = 0. \end{aligned} \tag{5.16}$$

Comparing the coefficients of ∂^{n+1} and ∂^n in (5.16), we get $u_n = 0$ and $(\Delta + n)u_n \lambda + u_{n-1} = 0$, respectively, which yields $u_{n-1} = 0$. Proceeding in a similar way, we obtain

$$((\Delta + k)u_k \lambda + u_{k-1}) \partial^k = 0, \quad \forall 0 \leq k \leq n.$$

It follows that $u_i = 0$ for all i , so we have $m = 0$, i.e., $\tilde{H}^0(\text{Vir}, M_{(3r-r^2)/2,0}) = \{0\}$.

For $q = 1$, then any 1-cocycle $\gamma \in \tilde{C}^1(\text{Vir}, M_{\Delta,0})$ can be identified with a skew symmetric polynomial of degree $1 - \Delta$ in one variable. So we can write $\gamma_\lambda(L) = \lambda f(\lambda)$ for some nonzero symmetric polynomial $f(\lambda) \in M_{\Delta,0}[\lambda]$ of degree of degree $n = -\Delta$. Letting $f(\lambda) = \sum_{i=0}^n a_i(\lambda) \partial^i$, then we have

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2}(L, L) &= L_{\lambda_1} \gamma_{\lambda_2}(L) - L_{\lambda_2} \gamma_{\lambda_1}(L) - \gamma_{\lambda_1 + \lambda_2}([L_{\lambda_1} L]) \\ &= L_{\lambda_1} \left(\lambda_2 \sum_{i=0}^n a_i(\lambda_2) \partial^i \right) - L_{\lambda_2} \left(\lambda_1 \sum_{i=0}^n a_i(\lambda_1) \partial^i \right) - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \\ &= (\partial_M + \Delta\lambda_1)(\lambda_2) \left(\sum_{i=0}^n a_i(\lambda_2) (\partial + \lambda_1)^i \right) - (\partial_M + \Delta\lambda_2)(\lambda_1) \left(\sum_{i=0}^n a_i(\lambda_1) (\partial + \lambda_2)^i \right) \\ &\quad - (\lambda_1^2 - \lambda_2^2) \sum_{i=0}^n a_i(\lambda_1 + \lambda_2) \partial^i \end{aligned}$$

where $\Delta = (3r - r^2)/2$ for $r \in \mathbb{Z}_+$ and $r \geq 4$. Letting $\lambda_2 = 0$, then $d\gamma = 0$ implies

$$(\partial_M \lambda_1 + \lambda_1^2) \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right) = ((\partial - \lambda_1)\lambda_1 + \lambda_1^2) f(\lambda_1) = \partial \lambda_1 f(\lambda_1) = 0.$$

So $f(\lambda_1) = 0$, a contradiction. Thus γ must be zero and therefore $\tilde{H}^1(\text{Vir}, M_{\Delta,0}) = \{0\}$. □

5.2 Cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with Trivial Coefficients

In this section we will compute, using the Hochschild-Serre spectral sequence for Lie conformal algebra, the cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with trivial coefficients \mathbb{C} , where \mathfrak{g} is a finite dimensional semisimple Lie algebra with $\partial v = 0$ and $a_\lambda v = 0$ for $v \in \mathbb{C}$ and $a \in \text{Vir} \ltimes \text{Cur } \mathfrak{g}$. The computation of $\tilde{H}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$ was done in [33] using the Hochschild-Serre spectral sequence for $\mathfrak{Vect } \mathbb{C} \ltimes \mathfrak{g}[t]$, that is the Lie annihilation algebra of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$.

5.2.1 The Basic Cohomology

Theorem 5.2.1. *For the standard semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$, then*

$$\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^n(\mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 H^{n-3}(\mathfrak{g}, \mathbb{C}), \quad \text{for all } n \geq 0,$$

where we set $H^k(\mathfrak{g}, \mathbb{C}) = 0$ for all $k < 0$.

The proof of Theorem 5.2.1 will be done by the following lemmas.

Lemma 5.2.1. *$\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ is a trivial Vir -module.*

Proof. Let $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, be a representative of a cohomology class $[\gamma]$ in $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, then by (2.8) we have

$$\begin{aligned} (\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= (\partial_{\mathbb{C}} + \sum_{i=1}^q \lambda_i) \gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) \\ &= \left(\sum_{i=1}^q \lambda_i \right) \gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q). \end{aligned}$$

So $\partial \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) = \left(\sum_{i=1}^q \lambda_i \right) \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$. Using Theorem 2.7.5, we have $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C})$. Thus, γ is independent of the choice of λ_i 's. i.e., γ is constant on \mathfrak{g} . Then we can set $\lambda_1 = \dots = \lambda_q = 0$, and hence we get,

$$(\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) = 0,$$

for all $[\gamma] \in \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$. By Lemma 2.1.1, we conclude that $\text{Vir}_\lambda \gamma = 0$ for all $[\gamma] \in \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$. i.e., $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ is a trivial Vir -module. □

Lemma 5.2.2. *The Hochschild-Serre spectral sequence for $\mathcal{A} = \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, $\mathcal{B} = \text{Cur } \mathfrak{g}$ and $M = \mathbb{C}$ is given by*

$$E_2^{p,q} = \begin{cases} H^q(\mathfrak{g}, \mathbb{C}), & \text{if } p = 0, \\ \Lambda_3 H^q(\mathfrak{g}, \mathbb{C}), & \text{if } p = 3, \\ 0, & \text{otherwise.} \end{cases}$$

where $q \geq 0$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Moreover,

(i) $d_2^{p,q} = 0$ for all $p, q \geq 0$.

(ii) $E_3^{p,q} \cong E_2^{p,q}$ for all $p, q \geq 0$.

(iii) The spectral sequence collapses at the fourth page.

Proof. Using Theorem 4.1.1, the Hochschild-Serre spectral sequence for $\mathcal{A} = \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, $\mathcal{B} = \text{Cur } \mathfrak{g}$ and $M = \mathbb{C}$ is given by

$$E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), \quad p, q \geq 0.$$

From Theorem 2.7.4 and Theorem 2.7.5 we have $\tilde{H}^p(\text{Vir}, \mathbb{C}) = 0$ for all $p \neq 0, 3$ and $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C})$ for all q . Hence, the $E_2^{p,q}$ terms vanish unless p is zero or three, and we get

$$\begin{aligned} E_2^{0,q} &\cong \tilde{H}^0(\text{Vir}, \mathbb{C}) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C}), \\ E_2^{3,q} &\cong \tilde{H}^3(\text{Vir}, \mathbb{C}) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong \Lambda_3 \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong \Lambda_3 H^q(\mathfrak{g}, \mathbb{C}). \end{aligned}$$

where $q \geq 0$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. It follows that the only nonzero columns of the E_2 page are the zeroth and the third columns. For a finite dimensional semisimple complex Lie algebra \mathfrak{g} , $H^i(\mathfrak{g}, \mathbb{C}) = 0$, where $i = 1, 2, 4$ ([7], (18.3)). Thus, the corner of the E_2 page is shown in Figure 5.1.

6	$H^6(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^6(\mathfrak{g}, \mathbb{C})$	0	0
5	$H^5(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^5(\mathfrak{g}, \mathbb{C})$	0	0
4	0	0	0	0	0	0
3	$H^3(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^3(\mathfrak{g}, \mathbb{C})$	0	0
2	0	0	0	0	0	0
1	0	0	0	0	0	0
0	\mathbb{C}	0	0	$\Lambda_3 \mathbb{C}$	0	0
	0	1	2	3	4	5

Figure 5.1: $E_2^{p,q}$ -page for $(\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), \tilde{F}^\bullet, d)$.

Observe that the differential $d_2^{p,q}$ shifts two points to the right and one point down, i.e., $d_2^{p,q} : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ and it is zero for all p, q . Hence, $E_3^{p,q} \cong \ker d_2^{p,q} \cong E_2^{p,q}$ for all p, q . Since $H^q(\mathfrak{g}, \mathbb{C}) = 0$ for all $q > \dim \mathfrak{g} = n \geq 2$ ([7]), then $E_2^{p,q} = 0$ for all $p > 3$ and $q > n$. By Lemma 3.1.1, we have $N = \min(4, n+2) = 4$. Thus, the Hochschild-Serre spectral sequence collapses at $E_4^{p,q}$.

□

In the following lemma, we will describe the isomorphism $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \xrightarrow{\cong} E_3^{0,q}$.

Lemma 5.2.3. *Let $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, be a representative of the cohomology class $[\gamma]$ in $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, that satisfies the following conditions:*

$$\begin{aligned}
 (\iota_\lambda(L)d\gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= 0, \\
 \gamma_{\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_q}(L, \dots, L, g_{s+1}, \dots, g_q) &= 0, \quad \text{for all } 3 \leq s \leq q.
 \end{aligned} \tag{5.17}$$

Then $\gamma \in E_3^{0,q}$ for all $q \geq 0$.

Proof. Let $\mathcal{A} = \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, and $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ be a q -cocycle that satisfies the conditions in (5.17). First, from the proof of Theorem 3.2.1, we have

$$E_3^{0,q} = Z_3^{0,q} / B_2^{0,q}$$

$$\begin{aligned}
&= \frac{\{\gamma \in \tilde{C}^q(\mathcal{A}, \mathbb{C}) \mid d\gamma \in F^3 \tilde{C}^{q+1}(\mathcal{A}, \mathbb{C})\}}{d(\tilde{C}^{q-1}(\mathcal{A}, \mathbb{C}))} \\
&= \frac{\{\gamma \in \tilde{C}^q(\mathcal{A}, \mathbb{C}) \mid d\gamma_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) = 0 \text{ whenever } a_1, \dots, a_{q-1} \in \text{Cur } \mathfrak{g}\}}{d(\tilde{C}^{q-1}(\mathcal{A}, \mathbb{C}))}.
\end{aligned}$$

Now observe that

$$d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(g_1, \dots, g_{q+1}) = 0, \quad (5.18)$$

because γ is a q -cocycle. Moreover, since $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ is a trivial Vir-module, then by Cartan's identity (2.13), the first condition of (5.17) yields

$$(d\iota_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) = 0 \implies d\gamma_{\lambda, \lambda_1, \dots, \lambda_q}(L, g_1, \dots, g_q) = 0. \quad (5.19)$$

Now, let $\phi_{\lambda_1, \dots, \lambda_{q-1}}(g_1, \dots, g_{q-1}) = \iota_{\lambda'}(L)\gamma_{\lambda_1, \dots, \lambda_{q-1}}(g_1, \dots, g_{q-1})$, then from (5.19) we get

$$\begin{aligned}
&(d\iota_\lambda(L)\phi)_{\lambda_1, \dots, \lambda_{q-1}}(g_1, \dots, g_{q-1}) = 0 \\
\implies &(d\iota_\lambda(L)(\iota_{\lambda'}(L)\gamma))_{\lambda_1, \dots, \lambda_{q-1}}(g_1, \dots, g_{q-1}) = 0 \\
\implies &(d\iota_\lambda(L)\gamma)_{\lambda', \lambda_1, \dots, \lambda_{q-1}}(L, g_1, \dots, g_{q-1}) = 0 \\
\implies &d\gamma_{\lambda, \lambda', \lambda_1, \dots, \lambda_{q-1}}(L, L, g_1, \dots, g_{q-1}) = 0.
\end{aligned} \quad (5.20)$$

It follows that $d\gamma$ vanishes whenever evaluated at at least $q-1$ elements of $\text{Cur } \mathfrak{g}$. So $d\gamma \in \tilde{F}^3 \tilde{C}^{q+1}(\mathcal{A}, \mathbb{C})$, where the filtration \tilde{F} is as defined in (4.1). Thus, $\gamma \in Z_3^{0,q}$. Note that $\gamma \notin B_2^{0,1} = d(\tilde{C}^{q-1}(\mathcal{A}, \mathbb{C}))$ since if there exists a $(q-1)$ -cochain ϕ such that $\gamma = d\phi$, then γ is a q -coboundary. Hence, $\gamma \in E_3^{0,q}$ for all $q \geq 0$. \square

Using Lemma 5.2.2, the E_3 page is pictured in the following diagram.

6	$H^6(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^6(\mathfrak{g}, \mathbb{C})$	0	0	0
5	$H^5(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^5(\mathfrak{g}, \mathbb{C})$	0	0	0
4	0	0	0	0	0	0	0
3	$H^3(\mathfrak{g}, \mathbb{C})$	0	0	$\Lambda_3 H^3(\mathfrak{g}, \mathbb{C})$	0	0	0
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	\mathbb{C}	0	0	$\Lambda_3 \mathbb{C}$	0	0	0
	0	1	2	3	4	5	6

Figure 5.2: $E_3^{p,q}$ -page for $(\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), \tilde{F}^\bullet, d)$.

Note that the differential $d_3^{p,q}$ shifts three points to the right and two points down, i.e., $d_3^{p,q} : E_3^{p,q} \rightarrow E_3^{p+3,q-2}$ and it is zero for all p, q except when $p = 0$ and $q = 5, 7, 8, \dots$. In the following lemma we will show that $d_3^{0,q} = 0$ for all $q \geq 0$.

Lemma 5.2.4. *The differential $d_3^{0,q}$ vanishes for all q .*

Proof. The differential $d_3^{0,q} : E_3^{0,q} \rightarrow E_3^{3,q-2}$ is induced by the differential d of the basic complex $\tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$. Since $E_3^{0,q} \cong \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$ and $E_3^{3,q-2} \cong \Lambda_3 \tilde{H}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C})$, then for each $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, a representative of the cohomology class $[\gamma]$ in $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$, such that γ satisfies the conditions (5.17), we have

$$d_3^{0,q}([\gamma]) = [d\gamma] = \Lambda_3[\beta], \quad \beta \in \tilde{C}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C}). \quad (5.21)$$

The differential $d\gamma \in \Lambda_3 \tilde{H}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C})$ is given by:

$$\begin{aligned}
& d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, L, g_4, \dots, g_{q+1}) \\
&= -\gamma_{\lambda_1+\lambda_2, \lambda_3, \dots, \lambda_{q+1}}([L_{\lambda_1}L], L, g_4, \dots, g_{q+1}) \\
&\quad + \gamma_{\lambda_1+\lambda_3, \lambda_2, \lambda_4, \dots, \lambda_{q+1}}([L_{\lambda_1}L], L, g_4, \dots, g_{q+1}) \\
&\quad - \gamma_{\lambda_2+\lambda_3, \lambda_1, \lambda_4, \dots, \lambda_{q+1}}([L_{\lambda_2}L], L, g_4, \dots, g_{q+1}) \\
&\quad + \sum_{j=4}^{q+1} (-1)^{j+1} \gamma_{\lambda_1+\lambda_j, \lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}}([L_{\lambda_1}g_j], L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1})
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=4}^{q+1} (-1)^{j+2} \gamma_{\lambda_2+\lambda_j, \lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} ([L\lambda_2 g_j], L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} (-1)^{j+3} \gamma_{\lambda_3+\lambda_j, \lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} ([L\lambda_3 g_j], L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& + \sum_{\substack{i,j=4 \\ i < j}}^{q+1} (-1)^{j+i} \gamma_{\lambda_i+\lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} ([g_i \lambda_i g_j], L, L, L, g_4, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& = (\lambda_2 - \lambda_1) \gamma_{\lambda_1+\lambda_2, \lambda_3, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& + (\lambda_1 - \lambda_3) \gamma_{\lambda_1+\lambda_3, \lambda_2, \lambda_4, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& + (\lambda_3 - \lambda_2) \gamma_{\lambda_2+\lambda_3, \lambda_1, \lambda_4, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} (-1)^j \lambda_j \gamma_{\lambda_1+\lambda_j, \lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} (g_j, L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} (-1)^{j+1} \lambda_j \gamma_{\lambda_2+\lambda_j, \lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} (g_j, L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} (-1)^j \lambda_j \gamma_{\lambda_3+\lambda_j, \lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_{q+1}} (g_j, L, L, g_4, \dots, \hat{g}_j, \dots, g_{q+1}) \\
& = (\lambda_2 - \lambda_1) \gamma_{\lambda_1+\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& + (\lambda_1 - \lambda_3) \gamma_{\lambda_1+\lambda_3, \lambda_2, \lambda_4, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& + (\lambda_3 - \lambda_2) \gamma_{\lambda_2+\lambda_3, \lambda_1, \lambda_4, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_{q+1}) \\
& - \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{i-1}, \lambda_1+\lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_j, \dots, g_{q+1}) \\
& + \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_1, \lambda_3, \lambda_4, \dots, \lambda_{i-1}, \lambda_2+\lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_j, \dots, g_{q+1}) \\
& - \sum_{j=4}^{q+1} \lambda_j \gamma_{\lambda_1, \lambda_2, \lambda_4, \dots, \lambda_{i-1}, \lambda_3+\lambda_j, \lambda_{i+1}, \dots, \lambda_{q+1}} (L, L, g_4, \dots, g_j, \dots, g_{q+1}).
\end{aligned}$$

By Theorem 2.7.5, we have $\widetilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C})$. Thus, γ is independent of the choice of $\lambda_4, \dots, \lambda_{q+1}$, i.e., γ is constant on the elements of \mathfrak{g} . Then we can set $\lambda_4, \dots, \lambda_{q+1}$ to be

all zero. Hence we have

$$\begin{aligned}
d\gamma_{\lambda_1, \dots, \lambda_{q+1}}(L, L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0} \\
= (\lambda_2 - \lambda_1)\gamma_{\lambda_1+\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0} \\
+ (\lambda_1 - \lambda_3)\gamma_{\lambda_1+\lambda_3, \lambda_2, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0} \\
+ (\lambda_3 - \lambda_2)\gamma_{\lambda_2+\lambda_3, \lambda_1, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0}.
\end{aligned} \tag{5.22}$$

From (5.21), the differential $d\gamma$ is written as a product of $\Lambda_3\beta$ where $\beta \in \widetilde{H}^{q-2}(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^{q-2}(\mathfrak{g}, \mathbb{C})$. It follows that $\gamma_{\lambda, \mu, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0}$ must be a homogeneous quadratic polynomial in λ and μ where $\lambda = \lambda_i + \lambda_j$, $\mu = \lambda_k$ and $i, j, k = 1, 2, 3$. From the definition of cochains, γ is a skew symmetric polynomial. Since the only quadratic skew symmetric polynomial in two variables is $\lambda^2 - \mu^2$, then we obtain

$$\gamma_{\lambda, \mu, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0} = c_0(\lambda^2 - \mu^2),$$

for some $c_0 \in \mathbb{C}$. Plugging this in (5.22), we get

$$\begin{aligned}
\gamma_{\lambda, \mu, \lambda_4, \dots, \lambda_{q+1}}(L, L, g_4, \dots, g_{q+1})|_{\lambda_4=0, \dots, \lambda_{q+1}=0} &= c_0(\lambda_2 - \lambda_1)[(\lambda_1 + \lambda_2)^2 - \lambda_3^2] \\
&+ c_0(\lambda_1 - \lambda_3)[(\lambda_1 + \lambda_3)^2 - \lambda_2^2] \\
&+ c_0(\lambda_3 - \lambda_2)[(\lambda_2 + \lambda_3)^2 - \lambda_1^2] = 0.
\end{aligned}$$

Therefore, $d\gamma = 0$ for all $q \geq 0$, and consequently, $d_3^{0,q} = 0$ for all $q \geq 0$. □

Proof of Theorem 5.2.1.

Proof. In Lemma 5.2.2, we proved that the Hochschild-Serre spectral sequence collapses at the fourth page, i.e., $E_4^{p,q} \cong E_\infty^{p,q}$ for all p, q . Since $d_3^{p,q}$ is zero for all p, q , then $\ker d_3^{p,q} = E_3^{p,q}$. But $E_3^{p,q} \cong E_2^{p,q}$ and $E_4^{p,q} \cong E_3^{p,q}$ for all p, q . That implies $E_4^{p,q} \cong E_2^{p,q}$ for all p, q . Therefore,

$$\begin{aligned}
\widetilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) &\cong \bigoplus_{p+q=n} \widetilde{H}^p(\text{Vir}, \mathbb{C}) \otimes \widetilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \\
&\cong \widetilde{H}^0(\text{Vir}, \mathbb{C}) \otimes \widetilde{H}^n(\text{Cur } \mathfrak{g}, \mathbb{C}) \oplus \widetilde{H}^3(\text{Vir}, \mathbb{C}) \otimes \widetilde{H}^{n-3}(\text{Cur } \mathfrak{g}, \mathbb{C}) \\
&\cong \mathbb{C} \otimes H^n(\mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 \mathbb{C} \otimes H^{n-3}(\mathfrak{g}, \mathbb{C}) \\
&\cong H^n(\mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 H^{n-3}(\mathfrak{g}, \mathbb{C}),
\end{aligned}$$

where $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$ and $H^k(\mathfrak{g}, \mathbb{C}) = \{0\}$ for all $k < 0$. □

Remark 5.2.1. Let γ be a q -cocycle in $\tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$, then γ can be written as $\gamma = \phi + \Lambda_3 \alpha$ where $\phi \in H^q(\mathfrak{g}, \mathbb{C})$ and $\alpha \in H^{q-3}(\mathfrak{g}, \mathbb{C})$. Since each k -cocycle of $H^k(\mathfrak{g}, \mathbb{C})$ is independent of the choice of λ'_i s, i.e., ϕ and α are of degree 0. Then we conclude that γ is either a constant or a skew symmetric polynomial of degree 3 in $\lambda_1, \lambda_2, \lambda_3$.

Corollary 5.2.1 ([33], Theorem 1.1 (i)). *For the standard semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$\dim \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) = \dim H^q(\mathfrak{g}, \mathbb{C}) + \dim H^{q-3}(\mathfrak{g}, \mathbb{C})$$

where $\dim H^k(\mathfrak{g}, \mathbb{C}) = 0$ for $k < 0$.

5.2.2 The Reduced Cohomology

To compute the reduced cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with trivial coefficients, we prove the following theorem. The purpose of the theorem is generalize the approach used in [2] to describe the reduced cohomology of the current conformal algebra $\text{Cur } \mathfrak{g}$ with trivial coefficients. It is also used to compute the reduced cohomology of the Heisenberg-Virasoro conformal algebra \mathcal{HV} in [30] and of the $\mathcal{W}(2, 2)$ conformal algebra in [31].

Theorem 5.2.2. *Let \mathcal{A} be a Lie conformal algebra. Suppose that any q -cocycle $\gamma \in \tilde{C}^q(\mathcal{A}, \mathbb{C})$ can be written as a sum of cocycles $\gamma_1, \dots, \gamma_n$, in which each cocycle γ_i is of λ -degree k_{q_i} for some nonnegative integer k_{q_i} such that $k_{q_i} - k_{q_j} \neq 1$ for all $i, j = 1, \dots, n$ where λ -degree refers to the total degree in $\lambda_1, \dots, \lambda_q$. Then*

$$H^q(\mathcal{A}, \mathbb{C}) \cong \tilde{H}^q(\mathcal{A}, \mathbb{C}) \oplus \tilde{H}^{q+1}(\mathcal{A}, \mathbb{C}), \quad q \geq 0,$$

where \mathbb{C} is the trivial \mathcal{A} -module.

Proof. Let $\gamma \in \tilde{C}^q(\mathcal{A}, \mathbb{C})$, be representative of the cohomology class $[\gamma]$ in $\tilde{H}^q(\mathcal{A}, \mathbb{C})$, such that γ is written as a direct sum of cocycle $\gamma_1, \dots, \gamma_n$, in which each cocycle γ_i is a homogeneous polynomial of λ -degree k_{q_i} for some nonnegative integers k_{q_i} such that $k_{q_i} - k_{q_j} \neq 1$ for all $i, j = 1, \dots, n$, where λ -degree refers to the total degree of γ in $\lambda_1, \dots, \lambda_q$.

Consider the long exact sequence of cohomology groups,

$$\begin{aligned} \dots &\longrightarrow H^q(\partial \tilde{C}^\bullet) \xrightarrow{\iota_q} \tilde{H}^q(\mathcal{A}, \mathbb{C}) \xrightarrow{\pi_q} H^q(\mathcal{A}, \mathbb{C}) \xrightarrow{\omega_q} \\ &\longrightarrow H^{q+1}(\partial \tilde{C}^\bullet) \xrightarrow{\iota_{q+1}} \tilde{H}^{q+1}(\mathcal{A}, \mathbb{C}) \xrightarrow{\pi_{q+1}} H^{q+1}(\mathcal{A}, \mathbb{C}) \xrightarrow{\omega_{q+1}} \dots \end{aligned} \quad (5.23)$$

where ι_n, π_n are the maps induced by the embedding $\iota : \partial\tilde{C}^\bullet \rightarrow \tilde{C}^\bullet$ and the natural projection $\pi : \tilde{C}^q \rightarrow C^\bullet$, respectively, and ω_n is the n -th connecting homomorphism [see Proposition 2.7.1].

For $q = 0$, then $\tilde{C}^0(\mathcal{A}, \mathbb{C}) = \mathbb{C}$, and $\partial\tilde{C}^0 = \partial\mathbb{C} = 0$. Hence, $H^0(\partial\tilde{C}^\bullet) = 0$, and $\iota_0 : H^0(\partial\tilde{C}^\bullet) \rightarrow \tilde{H}^0(\mathcal{A}, \mathbb{C})$ is the zero map.

For $q \geq 1$, let ϕ be a cocycle in $H^q(\partial\tilde{C}^\bullet)$ such that $\phi = \partial\gamma$ for a nonzero cocycle $\gamma \in \tilde{H}^q(\mathcal{A}, \mathbb{C})$. Then $\iota_q([\phi]) = [\iota \circ \phi] = [\phi] \in \tilde{H}^q(\mathcal{A}, \mathbb{C})$, which implies that $\phi = \partial\gamma$ can be written as $\partial\gamma = \partial\gamma_1 + \dots + \partial\gamma_q$, where each of γ_i is a homogeneous of λ -degree k_{q_i} . Since the action of ∂ on \mathbb{C} is trivial, then

$$\begin{aligned} (\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) &= (\partial\gamma_1)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) + \dots + (\partial\gamma_q)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) \\ &= \left(\sum_{i=1}^q \lambda_i \right) \gamma_{1\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) + \dots + \left(\sum_{i=1}^q \lambda_i \right) \gamma_{q\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q). \end{aligned}$$

It follows that each $\partial\gamma_i$ is of λ -degree $k_{q_i} + 1$. Since $k_{q_i} - k_{q_j} \neq 1$ for all $i, j = 1, \dots, n$, then $k_{q_i} + 1 \neq k_{q_j}$ for any $i, j = 1, \dots, n$. Hence, ϕ must be a zero cocycle in $\tilde{H}^q(\mathcal{A}, \mathbb{C})$. i.e., the image of ι_q is zero for all $q \geq 1$. The exactness of (5.23) shows that

$$\begin{aligned} \ker \pi_q &= \text{im } \iota_q = 0 \\ \text{im } \omega_q &= \ker \iota_{q+1} = H^{q+1}(\partial\tilde{C}^\bullet), \quad q \geq 1. \end{aligned}$$

Thus, the long exact sequence (5.23) splits into the following short exact sequences

$$0 \longrightarrow \tilde{H}^q(\mathcal{A}, \mathbb{C}) \xrightarrow{\pi_q} H^q(\mathcal{A}, \mathbb{C}) \xrightarrow{\omega_q} H^{q+1}(\partial\tilde{C}^\bullet) \longrightarrow 0.$$

Therefore,

$$H^q(\mathcal{A}, \mathbb{C}) \cong \tilde{H}^q(\mathcal{A}, \mathbb{C}) \oplus \tilde{H}^{q+1}(\mathcal{A}, \mathbb{C}), \quad q \geq 0.$$

□

Corollary 5.2.2. *For the standard semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \cong \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \oplus \tilde{H}^{q+1}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), \quad q \geq 0.$$

Proof. Remark 5.2.1 shows that each q -cocycle $\gamma \in \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C})$ can be written as a sum of two cocycles γ_1 and γ_2 such that $\deg_\lambda \gamma_1 = 0$ and $\deg_\lambda \gamma_2 = 0$ or 3 . By Theorem

5.2.2, we get

$$H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \cong \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \oplus \tilde{H}^{q+1}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}), \quad q \geq 0.$$

□

Corollary 5.2.3 ([33], Theorem 1.1 (ii)). *For the standard semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$\begin{aligned} \dim H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) &= \dim H^q(\mathfrak{g}, \mathbb{C}) + \dim H^{q-3}(\mathfrak{g}, \mathbb{C}) + \dim H^{q+1}(\mathfrak{g}, \mathbb{C}) \\ &\quad + \dim H^{q-2}(\mathfrak{g}, \mathbb{C}). \end{aligned}$$

for all $q \geq 0$, and $\dim H^q(\mathfrak{g}, \mathbb{C}) = 0$ for $q < 0$.

5.2.3 Central Extensions

Let \mathcal{A} be a Lie conformal algebra and C be an abelian conformal algebra. A short exact sequence of Lie conformal algebras

$$0 \longrightarrow C \longrightarrow \hat{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow 0$$

is called an **extension of \mathcal{A} by C** . This extension is called **central** if $[C_\lambda \hat{\mathcal{A}}] = 0$ and $\partial C = 0$. Furthermore, a central extension

$$0 \longrightarrow \hat{C} \longrightarrow \hat{\mathcal{A}} \xrightarrow{\hat{\pi}} \mathcal{A} \longrightarrow 0$$

is called **universal** if for every central extension

$$0 \longrightarrow \tilde{C} \longrightarrow \tilde{\mathcal{A}} \xrightarrow{\tilde{\pi}} \mathcal{A} \longrightarrow 0$$

there exists a unique homomorphism $\phi : \hat{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ such that $\hat{\pi} = \tilde{\pi} \circ \phi$.

The λ -bracket on $\hat{\mathcal{A}}$ is given by:

$$[a_\lambda b]^\wedge = [a_\lambda b] + \alpha_\lambda(a, b),$$

where $[a_\lambda b]$ is the λ -bracket on \mathcal{A} and $\alpha_\lambda(\cdot, \cdot) : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}[\lambda] \otimes_{\mathbb{C}} C$ is a \mathbb{C} -bilinear map.

From the axioms of Lie conformal algebras, $\alpha_\lambda(\cdot, \cdot)$ satisfies the following properties:

$$\begin{aligned}\alpha_\lambda(\partial a, b) &= -\lambda \alpha_\lambda(a, b), & \alpha_\lambda(a, \partial b) &= (\lambda + \partial) \alpha_\lambda(a, b), \\ \alpha_\lambda(a, b) &= -\alpha_{-\lambda-\partial}(b, a), \\ \alpha_\lambda(a, [b_\mu c]) &= \alpha_{\lambda+\mu}([a_\lambda b], c) + \alpha_\mu(b, [a_\lambda c]),\end{aligned}$$

for all $a, b, c \in \mathcal{A}$. The map α_λ is called a 2-cocycle of \mathcal{A} , and is a 2-coboundary or trivial 2-cocycle if there exists a $\mathbb{C}[\partial]$ -module homomorphism $f : \mathcal{A} \rightarrow C$ such that

$$\alpha_\lambda(a, b) = f([a_\lambda b]).$$

Two 2-cocycles α_λ and α'_λ are equivalent if $\alpha'_\lambda - \alpha_\lambda$ is a 2-coboundary, i.e.,

$$\alpha'_\lambda(a, b) = \alpha_\lambda(a, b) + f([a_\lambda b]).$$

The trivial 2-cocycle defines a trivial extension, and the equivalent 2-cocycles $\alpha'_\lambda(a, b)$ and $\alpha_\lambda(a, b)$ define isomorphic extensions. By Theorem 2.7.2, part (4), the central extensions of \mathcal{A} by C are parameterized by $H^2(\mathcal{A}, C)$.

Let $\widehat{\text{Vir} \ltimes \text{Cur } \mathfrak{g}}$ be a central extension of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$. Then, $\widehat{\text{Vir} \ltimes \text{Cur } \mathfrak{g}} = \text{Vir} \ltimes \text{Cur } \mathfrak{g} \oplus \mathbb{C}\mathfrak{c}$ and the λ -bracket on $\widehat{\text{Vir} \ltimes \text{Cur } \mathfrak{g}}$ is given by

$$\begin{aligned}[L_\lambda L]^\wedge &= (\partial + 2\lambda)L + \alpha_\lambda(L, L)\mathfrak{c}, \\ [L_\lambda g]^\wedge &= (\partial + \lambda)g + \alpha_\lambda(L, g)\mathfrak{c}, \\ [g_\lambda h]^\wedge &= [g, h] + \alpha_\lambda(g, h)\mathfrak{c}.\end{aligned}\tag{5.24}$$

where $g, h \in \mathfrak{g} \subset \text{Cur } \mathfrak{g}$, L is the standard generator of Vir , and $\alpha_\lambda(a, b)$ is in $\mathbb{C}[\lambda]$ for $a, b \in \{L\} \cup \mathfrak{g}$. Moreover, we have $\partial\mathfrak{c} = 0$ and $[a_\lambda \mathfrak{c}]^\wedge = 0$.

Using Theorem 5.2.2 and Corollary 5.2.1, we obtain

$$\begin{aligned}\dim H^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) &= \dim \widetilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) + \dim \widetilde{H}^3(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}) \\ &= \dim H^2(\mathfrak{g}, \mathbb{C}) + \dim H^{-1}(\mathfrak{g}, \mathbb{C}) + \dim H^3(\mathfrak{g}, \mathbb{C}) \\ &\quad + \dim H^0(\mathfrak{g}, \mathbb{C}) = 2.\end{aligned}$$

It follows that there are, up to isomorphism, only two nontrivial central extensions of the semidirect product Lie conformal algebra, $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$.

According to [8], the universal central extension of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ is a combination of the universal central extension of Vir and $\text{Cur } \mathfrak{g}$ where \mathfrak{g} is a finite dimensional semisimple Lie algebra. The universal central extension of Vir has one-dimensional center and the corresponding 2-cocycle is given by $\alpha_\lambda(L, L) = a\lambda^3$, $a \in \mathbb{C}$. Furthermore, the universal central extension of $\text{Cur } \mathfrak{g}$ is identified with the space \mathcal{B} of all invariant bilinear forms on \mathfrak{g} and the corresponding 2-cocycle is $\alpha_\lambda^f(g, h) = f(g, h)$ where $f \in \mathcal{B}$ and $g, h \in \mathfrak{g}$. In this section, we will compute explicitly the universal central extension of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$.

First we compute $\alpha_\lambda(L, L)$ by using the Jacobi identity for (L, L, L) . Here we have

$$\begin{aligned}
& [L_\lambda[L_\mu L]]^\wedge = [[L_\lambda L]_{\lambda+\mu} L]^\wedge + [L_\mu[L_\lambda L]]^\wedge \\
\implies & \alpha_\lambda(L, [L_\mu L]) = \alpha_{\lambda+\mu}([L_\lambda L], L) + \alpha_\mu(L, [L_\lambda L]) \\
\implies & \alpha_\lambda(L, (\partial + 2\mu)L) = \alpha_{\lambda+\mu}((\partial + 2\lambda)L, L) + \alpha_\mu(L, (\partial + 2\lambda)L) \\
\implies & (\lambda + 2\mu)\alpha_\lambda(L, L) = (\lambda - \mu)\alpha_{\lambda+\mu}(L, L) + (\mu + 2\lambda)\alpha_\mu(L, L).
\end{aligned} \tag{5.25}$$

Now set $\alpha_\lambda(L, L) = \sum_{i=0}^n a_i \lambda^i$ such that $a_i \in \mathbb{C}$ for all i and $a_n \neq 0$. Then we rewrite (5.25) as the following

$$(\lambda + 2\mu) \sum_{i=0}^n a_i \lambda^i = (\lambda - \mu) \sum_{i=0}^n a_i (\lambda + \mu)^i + (\mu + 2\lambda) \sum_{i=0}^n a_i \mu^i.$$

Comparing the coefficients of λ^n in both sides we obtain

$$2a_n \mu \lambda^n = (na_n \mu - a_n \mu) \lambda^n \implies 2a_n \mu = (n - 1)a_n \mu \implies n = 3.$$

Thus, we have $\alpha_\lambda(L, L) = a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{C}$. By the skew symmetry of α_λ we have

$$\begin{aligned}
\alpha_\lambda(L, L)\mathfrak{c} &= (a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3)\mathfrak{c} \\
&= -(a_0 + a_1(-\lambda - \partial) + a_2(-\lambda - \partial)^2 + a_3(-\lambda - \partial)^3)\mathfrak{c} \\
&= -\alpha_{-\lambda-\partial}(L, L)\mathfrak{c}.
\end{aligned}$$

Comparing the coefficients of similar terms, we obtain $a_0 = a_2 = 0$ and hence,

$$\alpha_\lambda(L, L) = a_1 \lambda + a_3 \lambda^3, \quad a_1, a_3 \in \mathbb{C}. \tag{5.26}$$

Now we compute $\alpha_\lambda(L, g)$ by writing the Jacobi identity for (L, L, g) , so we have

$$(\lambda + \mu)\alpha_\lambda(L, g) = (\lambda - \mu)\alpha_{\lambda+\mu}(L, g) + (\mu + \lambda)\alpha_\mu(L, g). \quad (5.27)$$

Similarly, we set $\alpha_\lambda(L, g) = \sum_{i=0}^n b_i \lambda^i$; $b_i \in \mathbb{C}$, $b_n \neq 0$ in (5.27) and then compare the coefficients of λ^n . We obtain

$$b_n \mu \lambda^n = (n b_n \mu - b_n \mu) \lambda^n \implies b_n \mu = (n - 1) b_n \mu \implies n = 2.$$

Hence, $\alpha_\lambda(L, g) = b_0 + b_1 \lambda + b_2 \lambda^2$ for some $b_0, b_1, b_2 \in \mathbb{C}$. By the skew symmetry of α_λ we have

$$\begin{aligned} \alpha_\lambda(L, g)\mathfrak{c} &= (b_0 + b_1 \lambda + b_2 \lambda^2)\mathfrak{c} \\ &= -(b_0 + b_1(-\lambda - \partial) + b_2(-\lambda - \partial)^2)\mathfrak{c} = -\alpha_{-\lambda-\partial}(L, g)\mathfrak{c}. \end{aligned}$$

which implies that $b_0 = b_2 = 0$. Therefore,

$$\alpha_\lambda(L, g) = b_1 \lambda, \quad b_1 \in \mathbb{C}. \quad (5.28)$$

Finally, we compute $\alpha_\lambda(g, h)$ for $g, h \in \mathfrak{g}$, we apply the Jacobi identity to (L, g, h) . Here we get

$$\alpha_\lambda(L, [g, h]) = -\mu\alpha_{\lambda+\mu}(g, h) + (\lambda + \mu)\alpha_\mu(g, h). \quad (5.29)$$

Let $\alpha_\mu(g, h) = \sum_{i=0}^n c_i \mu^i$ and $\alpha_{\lambda+\mu}(g, h) = \sum_{i=0}^n c_i (\lambda + \mu)^i$ where $c_i \in \mathbb{C}$ for all i and $c_n \neq 0$. By (5.28), we have $\alpha_\lambda(L, [g, h]) = b_1 \lambda$. Plugging this into (5.29) and comparing the coefficients of similar terms in both sides, we obtain

$$b_1 \lambda = c_0 \lambda \implies b_1 = c_0,$$

$$\text{and} \quad -n c_n \lambda \mu^n + c_n \lambda \mu^n = 0 \implies n = 1.$$

Hence, $\alpha_\lambda(g, h) = c_0 + c_1 \lambda$ for some $c_0, c_1 \in \mathbb{C}$ such that $c_0 = b_1$. Using the skew-symmetry of α_λ we have

$$\alpha_\lambda(g, h)\mathfrak{c} = (c_0 + c_1 \lambda)\mathfrak{c} = -(c_0 + c_1(-\lambda - \partial))\mathfrak{c} = -\alpha_{-\lambda-\partial}(h, g)\mathfrak{c}.$$

Which implies that $c_0 = 0$, thus $b_1 = 0$, and therefore we have

$$\alpha_\lambda(L, g) = 0, \quad \text{and} \quad \alpha_\lambda(g, h) = c_1 \lambda, \quad c_1 \in \mathbb{C}. \quad (5.30)$$

Theorem 5.2.3 ([8], Remark 8.4). 1. For nonzero $a, b \in \mathbb{C}$, there exists a unique nontrivial central extension of the semidirect product Lie conformal algebra, $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ by a Lie conformal algebra $\mathbb{C}\mathfrak{c}$ where \mathfrak{c} is a central element and the λ -brackets given by:

$$\begin{aligned} [L_\lambda L]^\wedge &= (\partial + 2\lambda)L + a\lambda^3\mathfrak{c}, & [L_\lambda g]^\wedge &= (\partial + \lambda)g, \\ [g_\lambda h]^\wedge &= [g, h] + b\lambda\mathfrak{c}. \end{aligned} \quad (5.31)$$

2. There exists a unique nontrivial central extension of the semidirect product Lie conformal algebra, $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ by a Lie conformal algebra $\mathbb{C}\mathfrak{c} \oplus \mathbb{C}\mathfrak{c}'$ where \mathfrak{c} and \mathfrak{c}' are central elements and the λ -brackets defined by:

$$\begin{aligned} [L_\lambda L]^\wedge &= (\partial + 2\lambda)L + \lambda^3\mathfrak{c}, & [L_\lambda g]^\wedge &= (\partial + \lambda)g, \\ [g_\lambda h]^\wedge &= [g, h] + \lambda\mathfrak{c}'. \end{aligned} \quad (5.32)$$

Proof. (1) Plugging (5.26) and (5.30) into (5.24) and replacing L by $L - \frac{1}{2}a\mathfrak{c}$ with $a_3 = a$ and $c_1 = b$ for nonzero $a, b \in \mathbb{C}$, we obtain (5.31) as desired.

(2) Follows from part (1). □

5.3 Cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with Coefficients in \mathbb{C}_a

Consider the $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ -module \mathbb{C}_a , that is the one-dimensional vector space \mathbb{C} such that $\partial v = av$ and $u_\lambda v = 0$ for $u \in \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, $v \in \mathbb{C}$ and a is a nonzero complex number. Note that \mathfrak{g} is a finite dimensional complex semisimple Lie algebra.

5.3.1 The Basic Cohomology

Theorem 5.3.1. For all $n \geq 0$, we have

$$\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) \cong H^n(\mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 H^{n-3}(\mathfrak{g}, \mathbb{C}).$$

Proof. The second term of the Hochschild-Serre spectral sequence for $\mathcal{A} = \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, $\mathcal{B} = \text{Cur } \mathfrak{g}$ and $M = \mathbb{C}_a$ is

$$E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a)) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a).$$

By Theorem 2.7.5, we have $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a) \cong \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C})$ for all $q \geq 0$. Let $\gamma \in \tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a)$ be a representative of a cohomology class $[\gamma] \in \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a)$. Then we have

$$(\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) = \left(a + \sum_{i=1}^q \lambda_i\right) \gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q).$$

By the isomorphism $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \cong H^q(\mathfrak{g}, \mathbb{C})$, γ is constant on the element of \mathfrak{g} , hence $\partial\gamma = a\gamma$. Using Lemma 2.1.1, we conclude that $\tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a)$ is a trivial Vir -module. Thus,

$$\begin{aligned} E_2^{p,q} &\cong \tilde{H}^p(\text{Vir}, \mathbb{C}_a) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}_a) \\ &\cong \tilde{H}^p(\text{Vir}, \mathbb{C}) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a). \end{aligned}$$

From Lemmas 5.2.2 and 5.2.4, we know that $d_r^{p,q} = 0$ for all p, q and $r \geq 2$. Hence, $E_\infty^{p,q} \cong E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \mathbb{C}) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C})$. So we obtain

$$\tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) \cong \bigoplus_{p+q=n} \tilde{H}^p(\text{Vir}, \mathbb{C}) \otimes \tilde{H}^q(\text{Cur } \mathfrak{g}, \mathbb{C}).$$

From Theorem 5.2.1, we conclude that

$$\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) \cong H^n(\mathfrak{g}, \mathbb{C}) \oplus \Lambda_3 H^{n-3}(\mathfrak{g}, \mathbb{C}).$$

□

Corollary 5.3.1 ([33], Theorem 1.1 (i)). *For the standard semidirect product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$\dim \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) = \dim H^q(\mathfrak{g}, \mathbb{C}) + \dim H^{q-3}(\mathfrak{g}, \mathbb{C})$$

where $a \neq 0$ and $\dim H^k(\mathfrak{g}, \mathbb{C}) = 0$ for $k < 0$.

5.3.2 The Reduced Cohomology

Theorem 5.3.2 ([33], Theorem 1.1 (ii)). *If $a \neq 0$, then $\dim H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) = 0$ for all $q \geq 0$.*

Proof. For $a \neq 0$, define an operator $\kappa : \tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) \rightarrow \tilde{C}^{q-1}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a)$ as

follows:

$$\kappa\gamma_{\lambda_1, \dots, \lambda_{q-1}}(a_1, \dots, a_{q-1}) = \iota_\lambda(L)\gamma_{\lambda_1, \dots, \lambda_{q-1}}(a_1, \dots, a_{q-1})|_{\lambda=0},$$

for $a_1, \dots, a_{q-1} \in \text{Vir} \ltimes \text{Cur } \mathfrak{g}$, where ι is the operator defined in (2.12). Note that $\partial\tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) = (a + \sum_{i=1}^q \lambda_i)\tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a)$. Using (2.13) and (2.11), we have

$$\begin{aligned} ((d\kappa + \kappa d)\gamma)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q) &= ((d\iota_\lambda(L) + \iota_\lambda(L)d)\gamma)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q)|_{\lambda=0} \\ &= (\theta_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q)|_{\lambda=0} \\ &= L_\lambda\gamma_{\lambda_1, \dots, \lambda_q}(a_1, \dots, a_q)|_{\lambda=0} \\ &\quad - \sum_{i=1}^q \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(a_1, \dots, [L_\lambda a_i], \dots, a_q)|_{\lambda=0} \\ &= - \sum_{i=1}^q \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(a_1, \dots, [L_\lambda a_i], \dots, a_q)|_{\lambda=0}. \end{aligned} \tag{5.33}$$

Now suppose that γ is a q -cochain of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ evaluated at k copies of L and $q - k$ elements of \mathfrak{g} , g_{k+1}, \dots, g_q . Since $[L_\lambda a_i] = (\partial + 2\lambda)L$ when $a_i = L$ and $[L_\lambda g_i] = (\partial + \lambda)g_i$ when $a_i = g_i$ for some $g_i \in \mathfrak{g}$, then (5.33) can be rewritten as

$$\begin{aligned} ((d\kappa + \kappa d)\gamma)_{\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_q}(L, \dots, L, g_{k+1}, \dots, g_q) &= \sum_{i=1}^k (\lambda_i - \lambda)\gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_q}(L, \dots, L, g_{k+1}, \dots, g_q)|_{\lambda=0} \\ &\quad + \sum_{i=1}^k \lambda_i \gamma_{\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda + \lambda_i, \dots, \lambda_q}(L, \dots, L, g_{k+1}, \dots, g_q)|_{\lambda=0} \\ &= (\partial - a)\gamma_{\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_q}(L, \dots, L, g_{k+1}, \dots, g_q) \\ &= -a\gamma_{\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_q}(L, \dots, L, g_{k+1}, \dots, g_q) \mod \partial\tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a). \end{aligned} \tag{5.34}$$

Now assume that γ is a reduced q -cocycle, i.e., $\gamma \in \tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a)$ such that $d\gamma \in \partial\tilde{C}^{q+1}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a)$. Then, there exists a $(q+1)$ -cochain ϕ such that $d\gamma = \partial\phi = (a + \sum_{i=1}^{q+1} \lambda_i)\phi$. Thus,

$$\kappa d\gamma = \kappa \partial\phi = (a + \sum_{i=1}^q \lambda_i)\kappa\phi \in \partial\tilde{C}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a).$$

By (5.34), we have $d\kappa\gamma \equiv -a\gamma \implies \gamma \equiv d(a^{-1}\kappa\gamma)$, i.e., γ is a reduced q -coboundary. Therefore, $H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, \mathbb{C}_a) = 0$ for all $q \geq 0$. \square

5.4 Cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with Coefficients in $M_{\alpha,\Delta,U}$

Recall that, (Theorem 2.5.1), every finite irreducible conformal module over $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ is given by $M_{\Delta,\alpha,U} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$, where U is a finite dimensional irreducible \mathfrak{g} -module, and

$$L_\lambda u = (\alpha + \partial + \Delta\lambda)u, \quad g_\lambda u = g \cdot u,$$

for $\Delta, \alpha \in \mathbb{C}$, $u \in U$ and $g \in \mathfrak{g}$. The \mathfrak{g} -module U is nontrivial if $\Delta = 0$. In this section we will compute the basic cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in the module $M_{\alpha,\Delta,U}$ for some $\Delta, \alpha \in \mathbb{C}$, and the reduced cohomology when $\alpha \neq 0$.

5.4.1 The Basic Cohomology

The second term of the Hochschild-Serre spectral sequence associated to the basic complex of the semi direct product $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in $M_{\Delta,\alpha,U}$ is given by

$$E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})$$

for all $p, q \geq 0$. To describe the cohomology space $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U}))$ we first introduce the following lemma.

Lemma 5.4.1. *For $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})$, we have*

$$\begin{aligned} (\theta_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= (\partial + \alpha + \Delta\lambda)\gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) \\ &\quad + \sum_{i=1}^q (\partial - \partial_M - \lambda - \sum_{\substack{j=1 \\ j \neq i}}^q \lambda_j) \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q). \end{aligned}$$

Proof. Let $\gamma \in \tilde{C}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})$ then by (2.11), we get

$$\begin{aligned} &(\theta_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) \\ &= L_\lambda \gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) - \sum_{i=1}^q \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, [L_\lambda g_i], \dots, g_q) \end{aligned}$$

$$= (\partial + \alpha + \Delta\lambda)\gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) + \sum_{i=1}^q \lambda_i \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q). \quad (5.35)$$

Using (2.8), the $\mathbb{C}[\partial]$ -module structure on $\tilde{C}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})$ is given by

$$(\partial \cdot \gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) = (\partial_M + \sum_{i=1}^q \lambda_i) \gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q).$$

Then we can rewrite $\lambda_i \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q)$ as

$$\lambda_i \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q) = (\partial - \partial_M - \lambda - \sum_{\substack{j=1 \\ j \neq i}}^q \lambda_j) \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q).$$

Plugging this into (5.35) we obtain

$$\begin{aligned} (\theta_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) &= (\partial + \alpha + \Delta\lambda)\gamma_{\lambda_1, \dots, \lambda_q}(g_1, \dots, g_q) \\ &\quad + \sum_{i=1}^q (\partial - \partial_M - \lambda - \sum_{\substack{j=1 \\ j \neq i}}^q \lambda_j) \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_q}(g_1, \dots, g_q). \end{aligned}$$

□

Theorem 5.4.1. *For the Virasoro conformal algebra,*

1. $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0, \alpha, U})) = 0$ for $p > 3$.
2. $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2, \alpha, U})) = 0$ for $p \neq 1, 2$.
3. $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{3r-r^2, \alpha, U})) = 0$ for $p > r$ where $r \in \mathbb{Z}_+$ and $r \geq 4$.
4. $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) = 0$ if $\Delta \neq 3r - r^2$ for any $r \in \mathbb{Z}_+$.

Proof. As in the proof of Theorem 5.1.2, we first describe the basic cohomology complex $\tilde{C}^\bullet(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$. For any n -cochain $\gamma \in \tilde{C}^n(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$, then it is determined by its value on $L^{\otimes n}$:

$$P(\lambda_1, \dots, \lambda_n) = \gamma_{\lambda_1, \dots, \lambda_n}(L, \dots, L),$$

where $P(\lambda_1, \dots, \lambda_n)$ is a skew-symmetric polynomial in n variables with values in $\tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})$. The differential is given by the following formula:

$$\begin{aligned} (dP)(\lambda_1, \dots, \lambda_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} (\partial + \alpha + \Delta \lambda_i) P(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}) \\ &\quad + \sum_{i=1}^{n+1} \left(\sum_{j=1}^{n+1} \lambda_j P(\lambda_1, \dots, \lambda_i + \lambda_j, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}) \right) \\ &\quad + \sum_{\substack{i,j=1 \\ i < j}}^{n+1} (-1)^{i+j} (\lambda_i - \lambda_j) P(\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}). \end{aligned}$$

Following [2], consider the homotopy operator

$$\begin{aligned} \kappa : \tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) &\longrightarrow \tilde{C}^{p-1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \\ \kappa \gamma_{\lambda_1, \dots, \lambda_{p-1}}(L, \dots, L) &= \frac{\partial}{\partial \lambda} \iota_\lambda(L) \gamma_{\lambda_1, \dots, \lambda_{p-1}}(L, \dots, L)|_{\lambda=0}. \end{aligned}$$

Then we have

$$\begin{aligned} &(\kappa d + d\kappa) \gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L) \\ &= \frac{\partial}{\partial \lambda} (L_\lambda \gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} - \sum_{i=1}^p \gamma_{\lambda_1, \dots, \lambda_i + \lambda, \dots, \lambda_p}(L, \dots, [L_\lambda L], \dots, L)|_{\lambda=0}) \\ &= \frac{\partial}{\partial \lambda} ((\partial + \alpha + \Delta \lambda) \gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} \\ &\quad + \sum_{i=1}^p (\partial - \partial_M - \lambda - \sum_{\substack{j=1 \\ j \neq i}}^p \lambda_j) \gamma_{\lambda_1, \dots, \lambda_i + \lambda, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} \\ &\quad + \sum_{i=1}^p (\lambda_i - \lambda) \gamma_{\lambda_1, \dots, \lambda_i + \lambda, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0}) \\ &= (2(\deg_\lambda \gamma - p) + \Delta) \gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L), \end{aligned}$$

where $\deg_\lambda \gamma$ is the total degree of γ in $\lambda_1, \dots, \lambda_p$. Thus, a p -cocycle γ contributes to the cohomology of $\tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$ if its degree as a polynomial is equal to $(2p - \Delta)/2$. These polynomials are skew symmetric, hence divisible by $\Lambda_p = \prod_{i \leq j} (\lambda_i - \lambda_j)$, whose polynomial degree is $p(p-1)/2$. Consider the quadratic inequality $p(p-1)/2 \leq (2p - \Delta)/2$. Then we have $p \in [(3 - \sqrt{9 - 4\Delta})/2, (3 + \sqrt{9 - 4\Delta})/2]$. A straightforward computation shows that $\Delta = 3r - r^2$ for any $r \in \mathbb{Z}_+$, otherwise $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) = \{0\}$. Thus,

the only integral solutions for the inequality $p(p-1)/2 \leq (2p-\Delta)/2$ are the following

$$p = \begin{cases} 0, 1, 2, 3 & \text{if } \Delta = 0, \\ 1, 2 & \text{if } \Delta = 2, \\ 0, 1, \dots, r & \text{if } \Delta = (3r - r^2), \end{cases}$$

where $r \in \mathbb{Z}_+$ such that $r \neq 0, 1, 2, 3$. It follows that

- i. $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,\alpha,U})) = \{0\} \quad \forall \quad p > 3.$
- ii. $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,\alpha,U})) = \{0\} \quad \forall \quad p \neq 1, 2.$
- iii. $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{3r-r^2,\alpha,U})) = \{0\} \quad \forall \quad p > r, \quad r \in \mathbb{Z}_+ \text{ and } r \geq 4.$
- iv. $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})) = \{0\}$ if $\Delta \neq 3r - r^2$ for any $r \in \mathbb{Z}_+.$

□

As shown in Proposition 2.7.1, we obtain the following long exact sequence of cohomology groups:

$$\begin{aligned} 0 &\longrightarrow H^0(\partial\tilde{C}^\bullet) \xrightarrow{\iota_0} \tilde{H}^0(\text{Vir}, N) \xrightarrow{\pi_0} H^0(\text{Vir}, N) \xrightarrow{w_0} \\ &\longrightarrow H^1(\partial\tilde{C}^\bullet) \xrightarrow{\iota_1} \tilde{H}^1(\text{Vir}, N) \xrightarrow{\pi_1} H^1(\text{Vir}, N) \xrightarrow{w_1} \\ &\longrightarrow H^2(\partial\tilde{C}^\bullet) \xrightarrow{\iota_2} \tilde{H}^2(\text{Vir}, N) \xrightarrow{\pi_2} H^2(\text{Vir}, N) \xrightarrow{w_2} \dots \end{aligned} \quad (5.36)$$

where $N = \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})$, and ι_i and π_i are the homomorphisms induced by the embedding $\partial\tilde{C}^\bullet \xrightarrow{\iota} \tilde{C}^\bullet$ and the canonical projection $\pi : \tilde{C}^\bullet \longrightarrow \tilde{C}^\bullet/\partial\tilde{C}^\bullet$, respectively, and w_i is the i -th connecting homomorphism. Then, we have the following theorem.

Theorem 5.4.2. *For the Virasoro conformal algebra Vir,*

$$\begin{aligned} \dim H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})) &= \dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})) \\ &\quad + \dim \tilde{H}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U})) \end{aligned}$$

for all $p \geq 0$.

Proof. Let $\partial\gamma \in H^p(\partial\tilde{C}^\bullet)$, for a nonzero $\gamma \in \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U}))$ then $\iota_p([\partial\gamma]) = [\partial\gamma] \in \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U}))$. Using Theorem 5.4.1, $\partial\gamma$ can be identified with a

skew symmetric polynomial of degree $(2p - \Delta)/2$ for all $p \geq 0$. On the other hand, by definition, we have $\deg_\lambda \partial\gamma = \deg_\lambda \gamma + 1 = (2p - \Delta)/2 + 1$. Thus, $\partial\gamma = 0$, and the image of ι_p is zero for all $p \geq 0$. By the exactness of (5.36), $\text{im } \iota_p = \ker \pi_p = 0$ and $\text{im } w_p = \ker \iota_{p+1} = H^{p+1}(\partial\tilde{C}^\bullet) \cong \tilde{H}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$ for all $p \geq 0$. Therefore, we get the following short exact sequence

$$\begin{aligned} 0 \longrightarrow \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) &\longrightarrow H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \longrightarrow \\ &\longrightarrow \tilde{H}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \longrightarrow 0 \end{aligned}$$

which implies that for all $p \geq 0$,

$$\begin{aligned} \dim H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) &= \dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \\ &\quad + \dim \tilde{H}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})). \end{aligned}$$

□

Theorem 5.4.3. *If $\alpha \neq 0$, then $\tilde{H}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}) = 0$.*

Now we introduce the following lemma, which is essential to prove Theorem 5.4.3.

Lemma 5.4.2. *$H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) = 0$ for all $p \geq 0$ and $\alpha \neq 0$.*

Proof. Define the homotopy operator

$$\begin{aligned} \kappa : C^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) &\longrightarrow C^{p-1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \\ \kappa\gamma_{\lambda_1, \dots, \lambda_{p-1}}(L, \dots, L) &= \iota_\lambda(L)\gamma_{\lambda_1, \dots, \lambda_{p-1}}(L, \dots, L)|_{\lambda=0}. \end{aligned}$$

Then we have,

$$\begin{aligned} ((d\kappa + \kappa d)\gamma)_{\lambda_1, \dots, \lambda_p}(L, \dots, L) &= (\theta_\lambda(L)\gamma)_{\lambda_1, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} \\ &= (\partial_M + \alpha + \Delta\lambda)\gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} + \sum_{i=0}^p \lambda_i \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} \\ &\quad - \sum_{i=1}^p (\lambda - \lambda_i) \gamma_{\lambda_1, \dots, \lambda + \lambda_i, \dots, \lambda_p}(L, \dots, L)|_{\lambda=0} \\ &= (\partial_M + \alpha + 2 \sum_{i=1}^p \lambda_i) \gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L) \end{aligned}$$

$$\begin{aligned}
&= (\partial_M + \alpha + 2(\partial - \partial_M))\gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L) \\
&= (\alpha - \partial_M)\gamma_{\lambda_1, \dots, \lambda_p}(L, \dots, L) \mod \partial \tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})). \quad (5.37)
\end{aligned}$$

Assume that γ is a reduced p -cocycle, then $\gamma \in \tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$ such that $d\gamma \in \partial \tilde{C}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))$, i.e., there exists a $(p+1)$ -cochain ϕ such that $d\gamma = \partial\phi = (\partial_M + \sum_{i=1}^p \lambda_i)\phi$. Hence,

$$\kappa d\gamma = \kappa \partial\alpha = (\partial_M + \sum_{i=1}^p \lambda_i)\kappa\phi \in \partial \tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})).$$

Now from (5.37), for $\alpha \neq 0$, we have $d\kappa\gamma \equiv (\alpha - \partial_M)\gamma \implies \gamma \equiv d((\alpha - \partial_M)^{-1}\kappa\gamma)$, i.e., γ is a reduced p -coboundary. Therefore, $H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) = 0$ for all $q \geq 0$ and $\alpha \neq 0$. \square

Proof of Theorem 5.4.3.

Proof. Using Theorem 5.4.2 and Lemma 5.4.2, we have

$$\begin{aligned}
0 = \dim H^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) &= \dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) \\
&\quad + \dim \tilde{H}^{p+1}(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}))
\end{aligned}$$

which implies that $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U})) = \{0\}$ for all p . Hence, $E_2^{p,q} = \{0\}$ for all p, q , and then $d_r^{p,q} = 0$ for all p, q and $r \geq 2$. Therefore, the Hochschild-Serre spectral sequence collapses at the second page and we obtain $\tilde{H}^p(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}) = 0$ for all $p \geq 0$ and $\alpha \neq 0$. \square

Theorem 5.4.4. *For the semidirect product Lie conformal algebra $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$\dim \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U}) = \dim \tilde{H}^{q-2}(\text{Cur } \mathfrak{g}, M_{0,0,U}) + \dim \tilde{H}^{q-3}(\text{Cur } \mathfrak{g}, M_{0,0,U})$$

for all $q \geq 0$. In particular,

$$\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U}) \cong P_2 \tilde{H}^{n-2}(\text{Cur } \mathfrak{g}, M_{0,0,U}) \oplus \Lambda_3 \tilde{H}^{n-3}(\text{Cur } \mathfrak{g}, M_{0,0,U}),$$

where $P_2 = \lambda_1^2 - \lambda_2^2$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$.

Proof. The second term of the Hochschild-Serre spectral sequence associated to the basic complex of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in $M_{0,0,U}$ is given by

$$E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U})$$

for all $p, q \geq 0$. From Theorem 5.4.1, we have $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) = 0$ for all $p > 3$ and only p -cochains that is identified with skew symmetric polynomials of degree p contribute to the cohomology of $\tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U}))$ where $p = 0, 1, 2$ and 3 .

For $p = 0$, and for any nonzero $\gamma \in \tilde{C}^0(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) = \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$ such that $d\gamma = 0$ we have

$$\begin{aligned} 0 = d\gamma_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) &= L_\lambda \gamma_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) \\ &= \partial_M \gamma_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) + \sum_{i=1}^q \mu_i \gamma_{\mu_1, \dots, \lambda + \mu_i, \dots, \mu_q}(g_1, \dots, g_q). \end{aligned}$$

Set $\gamma = \sum_{i=0}^n a_i(\mu_1, \dots, \mu_q) \partial^i$ such that $a_n(\mu_1, \dots, \mu_q) \neq 0$. Letting $\lambda = 0$ in (5.38) yields

$$\begin{aligned} 0 = d\gamma_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) &= \partial_M \left(\sum_{i=0}^n a_i(\mu_1, \dots, \mu_q) \partial^i \right) + \sum_{i=1}^q \mu_i \left(\sum_{i=0}^n a_i(\mu_1, \dots, \mu_q) \partial^i \right) \\ &= \left(\partial_M + \sum_{i=1}^q \mu_i \right) \sum_{i=0}^n a_i(\mu_1, \dots, \mu_q) \partial^i \\ &= \partial \left(\sum_{i=0}^n a_i(\mu_1, \dots, \mu_q) \partial^i \right). \end{aligned}$$

It follows that $a_i(\lambda_1, \dots, \lambda_q) = 0$ for all i , i.e., $\gamma = 0$, a contradiction. Thus, any 0-cocycle γ must be zero and so we get $\tilde{H}^0(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) = \{0\}$.

For $p = 1$, let $\gamma \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U}))$ such that $d\gamma = 0$, then λ which is the only polynomial of degree 1 in one variable. Letting $\gamma_\lambda(L) = \gamma' \lambda$ where γ' is a nonzero q -cocycle in $\tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$, we have

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2}(L, L) &= \partial_M \gamma_{\lambda_2}(L) + \lambda_2 \gamma_{\lambda_1 + \lambda_2}(L) - \partial_M \gamma_{\lambda_1}(L) - \lambda_1 \gamma_{\lambda_1 + \lambda_2}(L) - (\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2}(L) \\ &= (\partial - \lambda_2)(\gamma' \lambda_2) + \gamma' \lambda_2(\lambda_1 + \lambda_2) - (\partial - \lambda_1)(\gamma' \lambda_1) - \gamma' \lambda_1(\lambda_1 + \lambda_2) \\ &\quad - \gamma'(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \\ &= \gamma'(\lambda_2 - \lambda_1)(\partial + \lambda_1 + \lambda_2). \end{aligned}$$

i.e., $d\gamma_\lambda(L) = 0$ implies that $\gamma_\lambda(L) = 0$. Hence, $\tilde{H}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) = \{0\}$.

If $p = 2$, then the only skew symmetric homogeneous polynomial of degree 2 in two variables is $\lambda_1^2 - \lambda_2^2$. Here we have

$$\begin{aligned}
d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) &= \partial_M \gamma_{\lambda_2, \lambda_3}(L, L) + \lambda_2 \gamma_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + \lambda_3 \gamma_{\lambda_2, \lambda_1 + \lambda_3}(L, L) - \partial_M \gamma_{\lambda_1, \lambda_3}(L, L) \\
&- \lambda_1 \gamma_{\lambda_1 + \lambda_2, \lambda_3}(L, L) - \lambda_3 \gamma_{\lambda_1, \lambda_2 + \lambda_3}(L, L) + \partial_M \gamma_{\lambda_1, \lambda_2}(L, L) + \lambda_1 \gamma_{\lambda_1 + \lambda_3, \lambda_2}(L, L) \\
&+ \lambda_2 \gamma_{\lambda_1, \lambda_2 + \lambda_3}(L, L) - (\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + (\lambda_1 - \lambda_3) \gamma_{\lambda_1 + \lambda_3, \lambda_2}(L, L) \\
&- (\lambda_2 - \lambda_3) \gamma_{\lambda_2 + \lambda_3, \lambda_1}(L, L) \\
&= \partial_M (\lambda_1^2 - \lambda_2^2) + \lambda_2 ((\lambda_1 + \lambda_2)^2 - \lambda_3^2) + \lambda_3 (\lambda_2^2 - (\lambda_1 + \lambda_3)^2) - \partial_M (\lambda_1^2 - \lambda_3^2) \\
&- \lambda_1 ((\lambda_1 + \lambda_2)^2 - \lambda_3^2) - \lambda_3 (\lambda_1^2 - (\lambda_2 + \lambda_3)^2) + \partial_M (\lambda_1^2 - \lambda_2^2) + \lambda_1 ((\lambda_1 + \lambda_3)^2 - \lambda_2^2) \\
&+ \lambda_2 (\lambda_1^2 - (\lambda_2 + \lambda_3)^2) - (\lambda_1 - \lambda_2) ((\lambda_1 + \lambda_2)^2 - \lambda_3^2) + (\lambda_1 - \lambda_3) ((\lambda_1 + \lambda_3)^2 - \lambda_2^2) \\
&- (\lambda_2 - \lambda_3) ((\lambda_2 + \lambda_3)^2 - \lambda_1^2) = 0.
\end{aligned}$$

It follows that $\lambda_1^2 - \lambda_2^2$ is 2-cocycle. Now let $\phi \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U}))$ such that $d\phi = \lambda_1^2 - \lambda_2^2$ then we have

$$\begin{aligned}
\lambda_1^2 - \lambda_2^2 &= d\phi_{\lambda_1, \lambda_2}(L, L) \\
&= \partial_M \phi_{\lambda_2}(L) + \lambda_2 \phi_{\lambda_1 + \lambda_2}(L) - \partial_M \phi_{\lambda_1}(L) - \lambda_1 \phi_{\lambda_1 + \lambda_2}(L) - (\lambda_1 - \lambda_2) \phi_{\lambda_1 + \lambda_2}(L) \\
&= (\partial - \lambda_2) \phi_{\lambda_2}(L) + \lambda_2 \phi_{\lambda_1 + \lambda_2}(L) - (\partial - \lambda_1) \phi_{\lambda_1}(L) - \lambda_1 \phi_{\lambda_1 + \lambda_2}(L) \\
&- (\lambda_1 - \lambda_2) \phi_{\lambda_1 + \lambda_2}(L).
\end{aligned} \tag{5.38}$$

Since ϕ is a 1-cochain that it is identified with a polynomial in one variable λ , then we can rewrite $\phi_\lambda(L) = \lambda f(\lambda)$ for some nonzero polynomial $f(\lambda) \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$. Plugging this into (5.38) gives

$$\begin{aligned}
\lambda_1^2 - \lambda_2^2 &= (\partial - \lambda_2) \lambda_2 f(\lambda_2) + \lambda_2 (\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2) - (\partial - \lambda_1) \lambda_1 f(\lambda_1) \\
&- \lambda_1 (\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2) (\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2) \\
&= (\partial - \lambda_2) \lambda_2 f(\lambda_2) - (\partial - \lambda_1) \lambda_1 f(\lambda_1) + 2(\lambda_2 - \lambda_1) (\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2).
\end{aligned} \tag{5.39}$$

Letting $\lambda_2 = 0$ in (5.39) implies $\lambda_1^2 = -(\partial + \lambda_1) \lambda_1 f(\lambda_1)$. Set $f(\lambda_1) = \sum_{i=0}^n a_i(\lambda_1) \partial^i$ such

that $a_n(\lambda_1) \neq 0$. Assume that $n > 1$, then we have

$$\lambda_1^2 = -(\partial + \lambda_1)\lambda_1 \left(\sum_{i=0}^n a_i(\lambda_1) \partial^i \right). \quad (5.40)$$

Equating the coefficients of ∂^{n+1} we get $a_n(\lambda_1) = 0$, a contradiction. Then $f(\lambda_1) = a_0(\lambda_1) + a_1(\lambda_1)\partial$. Plugging this into (5.40) we get

$$\lambda_1^2 = -(\partial + \lambda_1)\lambda_1 (a_0(\lambda_1) + a_1(\lambda_1)\partial).$$

Equating similar terms of both sides yields $a_1(\lambda_1) = a_0(\lambda_1) = 0$, hence $f(\lambda_1) = 0$. That implies $\phi = 0$, a contradiction. Hence, $\lambda_1^2 - \lambda_2^2$ represents a nontrivial class in the cohomology, and we have $\tilde{H}^2(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) = (\lambda_1^2 - \lambda_2^2) \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$.

For $p = 3$, the only skew symmetric homogeneous polynomial of degree 3 in three variables is $\Lambda_3^{\lambda_1, \lambda_2, \lambda_3} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$, and it is a 3-cocycle because

$$\begin{aligned} & d\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, L) \\ &= \partial_M \gamma_{\lambda_2, \lambda_3, \lambda_4}(L, L, L) + \lambda_2 \gamma_{\lambda_1 + \lambda_2, \lambda_3, \lambda_4}(L, L, L) + \lambda_3 \gamma_{\lambda_2, \lambda_1 + \lambda_3, \lambda_4}(L, L, L) \\ &+ \lambda_4 \gamma_{\lambda_2, \lambda_3, \lambda_1 + \lambda_4}(L, L, L) - \partial_M \gamma_{\lambda_1, \lambda_3, \lambda_4}(L, L, L) - \lambda_1 \gamma_{\lambda_1 + \lambda_2, \lambda_3, \lambda_4}(L, L, L) \\ &- \lambda_3 \gamma_{\lambda_1, \lambda_2 + \lambda_3, \lambda_4}(L, L, L) - \lambda_4 \gamma_{\lambda_1, \lambda_3, \lambda_2 + \lambda_4}(L, L, L) + \partial_M \gamma_{\lambda_1, \lambda_2, \lambda_4}(L, L, L) \\ &+ \lambda_1 \gamma_{\lambda_1 + \lambda_3, \lambda_2, \lambda_4}(L, L, L) + \lambda_2 \gamma_{\lambda_1, \lambda_2 + \lambda_3, \lambda_4}(L, L, L) + \lambda_4 \gamma_{\lambda_1, \lambda_2, \lambda_3 + \lambda_4}(L, L, L) \\ &- \partial_M \gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) - \lambda_1 \gamma_{\lambda_1 + \lambda_4, \lambda_2, \lambda_3}(L, L, L) - \lambda_2 \gamma_{\lambda_1, \lambda_2 + \lambda_4, \lambda_3}(L, L, L) \\ &- \lambda_3 \gamma_{\lambda_1, \lambda_2, \lambda_3 + \lambda_4}(L, L, L) - (\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2, \lambda_3, \lambda_4}(L, L, L) \\ &+ (\lambda_1 - \lambda_3) \gamma_{\lambda_1 + \lambda_3, \lambda_2, \lambda_4}(L, L, L) - (\lambda_1 - \lambda_4) \gamma_{\lambda_1 + \lambda_4, \lambda_2, \lambda_3}(L, L, L) \\ &- (\lambda_2 - \lambda_3) \gamma_{\lambda_2 + \lambda_3, \lambda_1, \lambda_4}(L, L, L) + (\lambda_2 - \lambda_4) \gamma_{\lambda_2 + \lambda_4, \lambda_1, \lambda_3}(L, L, L) \\ &- (\lambda_3 - \lambda_4) \gamma_{\lambda_3 + \lambda_4, \lambda_1, \lambda_2}(L, L, L) \\ &= \partial_M \Lambda_3^{\lambda_2, \lambda_3, \lambda_4} + \lambda_2 \Lambda_3^{\lambda_1 + \lambda_2, \lambda_3, \lambda_4} + \lambda_3 \Lambda_3^{\lambda_2, \lambda_1 + \lambda_3, \lambda_4} + \lambda_4 \Lambda_3^{\lambda_2, \lambda_3, \lambda_1 + \lambda_4} \\ &- \partial_M \Lambda_3^{\lambda_1, \lambda_3, \lambda_4} - \lambda_1 \Lambda_3^{\lambda_1 + \lambda_2, \lambda_3, \lambda_4} - \lambda_3 \Lambda_3^{\lambda_1, \lambda_2 + \lambda_3, \lambda_4} - \lambda_4 \Lambda_3^{\lambda_1, \lambda_3, \lambda_2 + \lambda_4} \\ &+ \partial_M \Lambda_3^{\lambda_1, \lambda_2, \lambda_4} + \lambda_1 \Lambda_3^{\lambda_1 + \lambda_3, \lambda_2, \lambda_4} + \lambda_2 \Lambda_3^{\lambda_1, \lambda_2 + \lambda_3, \lambda_4} + \lambda_4 \Lambda_3^{\lambda_1, \lambda_2, \lambda_3 + \lambda_4} \\ &- \partial_M \Lambda_3^{\lambda_1, \lambda_2, \lambda_3} - \lambda_1 \Lambda_3^{\lambda_1 + \lambda_4, \lambda_2, \lambda_3} - \lambda_2 \Lambda_3^{\lambda_1, \lambda_2 + \lambda_4, \lambda_3} - \lambda_3 \Lambda_3^{\lambda_1, \lambda_2, \lambda_3 + \lambda_4} \\ &- (\lambda_1 - \lambda_2) \Lambda_3^{\lambda_1 + \lambda_2, \lambda_3, \lambda_4} + (\lambda_1 - \lambda_3) \Lambda_3^{\lambda_1 + \lambda_3, \lambda_2, \lambda_4} - (\lambda_1 - \lambda_4) \Lambda_3^{\lambda_1 + \lambda_4, \lambda_2, \lambda_3} \\ &- (\lambda_2 - \lambda_3) \Lambda_3^{\lambda_2 + \lambda_3, \lambda_1, \lambda_4} + (\lambda_2 - \lambda_4) \Lambda_3^{\lambda_2 + \lambda_4, \lambda_1, \lambda_3} - (\lambda_3 - \lambda_4) \Lambda_3^{\lambda_3 + \lambda_4, \lambda_1, \lambda_2} = 0, \end{aligned}$$

where $\Lambda_3^{\lambda_i, \lambda_j, \lambda_k} = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)$ for $i, j, k \in \{1, 2, 3, 4\}$. Now suppose that there exists a 2-cochain ϕ such that $d\phi = \Lambda_3^{\lambda_1, \lambda_2, \lambda_3}$. Then we have

$$\begin{aligned}
(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) &= d\phi_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) \\
&= \partial_M \phi_{\lambda_2, \lambda_3}(L, L) + \lambda_2 \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + \lambda_3 \phi_{\lambda_2, \lambda_1 + \lambda_3}(L, L) - \partial_M \phi_{\lambda_1, \lambda_3}(L, L) \\
&\quad - \lambda_1 \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) - \lambda_3 \phi_{\lambda_1, \lambda_2 + \lambda_3}(L, L) + \partial_M \phi_{\lambda_1, \lambda_2}(L, L) + \lambda_1 \phi_{\lambda_1 + \lambda_3, \lambda_2}(L, L) \\
&\quad + \lambda_2 \phi_{\lambda_1, \lambda_2 + \lambda_3}(L, L) - (\lambda_1 - \lambda_2) \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + (\lambda_1 - \lambda_3) \phi_{\lambda_1 + \lambda_3, \lambda_2}(L, L) \\
&\quad - (\lambda_2 - \lambda_3) \phi_{\lambda_2 + \lambda_3, \lambda_1}(L, L) \\
&= (\partial - \lambda_2 - \lambda_3) \phi_{\lambda_2, \lambda_3}(L, L) + \lambda_2 \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + \lambda_3 \phi_{\lambda_2, \lambda_1 + \lambda_3}(L, L) \\
&\quad - (\partial - \lambda_1 - \lambda_3) \phi_{\lambda_1, \lambda_3}(L, L) - \lambda_1 \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) - \lambda_3 \phi_{\lambda_1, \lambda_2 + \lambda_3}(L, L) \\
&\quad + (\partial - \lambda_1 - \lambda_2) \phi_{\lambda_1, \lambda_2}(L, L) + \lambda_1 \phi_{\lambda_1 + \lambda_3, \lambda_2}(L, L) + \lambda_2 \phi_{\lambda_1, \lambda_2 + \lambda_3}(L, L) \\
&\quad - (\lambda_1 - \lambda_2) \phi_{\lambda_1 + \lambda_2, \lambda_3}(L, L) + (\lambda_1 - \lambda_3) \phi_{\lambda_1 + \lambda_3, \lambda_2}(L, L) \\
&\quad - (\lambda_2 - \lambda_3) \phi_{\lambda_2 + \lambda_3, \lambda_1}(L, L). \tag{5.41}
\end{aligned}$$

Since $\phi_{\lambda, \mu}(L, L)$ is a skew symmetric polynomial then it can be written as $(\lambda - \mu)f(\lambda, \mu)$ for some symmetric polynomial $f(\lambda, \mu) \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})[\lambda, \mu]$. Plugging this into (5.41) yields

$$\begin{aligned}
&(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \\
&= (\partial - \lambda_2 - \lambda_3)(\lambda_2 - \lambda_3)f(\lambda_2, \lambda_3) + \lambda_2(\lambda_1 + \lambda_2 - \lambda_3)f(\lambda_1 + \lambda_2, \lambda_3) \\
&\quad + \lambda_3(\lambda_2 - \lambda_1 - \lambda_3)f(\lambda_2, \lambda_1 + \lambda_3) - (\partial - \lambda_1 - \lambda_3)(\lambda_1 - \lambda_3)f(\lambda_1, \lambda_3) \\
&\quad - \lambda_1(\lambda_1 + \lambda_2 - \lambda_3)f(\lambda_1 + \lambda_2, \lambda_3) - \lambda_3(\lambda_1 - \lambda_2 - \lambda_3)f(\lambda_1, \lambda_2 + \lambda_3) \\
&\quad + (\partial - \lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)f(\lambda_1, \lambda_2) + \lambda_1(\lambda_1 + \lambda_3 - \lambda_2)f(\lambda_1 + \lambda_3, \lambda_2) \\
&\quad + \lambda_2(\lambda_1 - \lambda_2 - \lambda_3)f(\lambda_1, \lambda_2 + \lambda_3) - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3)f(\lambda_1 + \lambda_2, \lambda_3) \\
&\quad + (\lambda_1 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2)f(\lambda_1 + \lambda_3, \lambda_2) - (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1)f(\lambda_2 + \lambda_3, \lambda_1) \\
&= (\partial - \lambda_2 - \lambda_3)(\lambda_2 - \lambda_3)f(\lambda_2, \lambda_3) - (\partial - \lambda_1 - \lambda_3)(\lambda_1 - \lambda_3)f(\lambda_1, \lambda_3) \\
&\quad + (\partial - \lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)f(\lambda_1, \lambda_2) - 2(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3)f(\lambda_1 + \lambda_2, \lambda_3) \\
&\quad + 2(\lambda_1 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2)f(\lambda_1 + \lambda_3, \lambda_2) - 2(\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1)f(\lambda_2 + \lambda_3, \lambda_1). \tag{5.42}
\end{aligned}$$

Letting $\lambda_3 = 0$ in (5.42), we obtain

$$\begin{aligned} \lambda_1 \lambda_2 (\lambda_1 - \lambda_2) &= \lambda_1 ((\partial + \lambda_1) f(\lambda_1, \lambda_2) - (\partial - \lambda_1) f(\lambda_1) - 2\lambda_1 f(\lambda_1 + \lambda_2)), \\ &+ \lambda_2 ((\partial - \lambda_2) f(\lambda_2) - (\partial + \lambda_2) f(\lambda_1, \lambda_2) + 2\lambda_2 f(\lambda_1 + \lambda_2)). \end{aligned} \quad (5.43)$$

Equating similar terms of both sides, we have

$$\begin{aligned} \lambda_2 (\lambda_1 - \lambda_2) &= (\partial + \lambda_1) f(\lambda_1, \lambda_2) - (\partial - \lambda_1) f(\lambda_1) - 2\lambda_1 f(\lambda_1 + \lambda_2) \\ \lambda_1 (\lambda_1 - \lambda_2) &= (\partial - \lambda_2) f(\lambda_2) - (\partial + \lambda_2) f(\lambda_1, \lambda_2) + 2\lambda_2 f(\lambda_1 + \lambda_2) \\ \implies \lambda_1^2 - \lambda_2^2 &= (\partial - \lambda_2) f(\lambda_2) - (\partial - \lambda_1) f(\lambda_1) - 2(\lambda_1 - \lambda_2) f(\lambda_1 + \lambda_2) \\ &+ (\lambda_1 - \lambda_2) f(\lambda_1, \lambda_2). \end{aligned} \quad (5.44)$$

Letting $\lambda_2 = 0$ in the last equation of (5.44) gives $\lambda_1^2 = \partial(f(0) - f(\lambda_1))$. Observe that the left hand side does not depend on ∂ , so we have $f(0) - f(\lambda_1) = 0$. Plugging this into the first equation of (5.44) we get

$$\begin{aligned} \lambda_2 (\lambda_1 - \lambda_2) &= (\partial + \lambda_1) f(\lambda_1, \lambda_2) - (\partial - \lambda_1) f(0) - 2\lambda_1 f(0) \\ &= (\partial + \lambda_1) (f(\lambda_1, \lambda_2) - f(0)). \end{aligned} \quad (5.45)$$

Set $f(\lambda) = \sum_{i=0}^n a_i(\lambda) \partial^i$ such that $a_n(\lambda) \neq 0$, and $f(\lambda, \mu) = \sum_{i=0}^n a_i(\lambda, \mu) \partial^i$ such that $a_i(\lambda, \mu) \neq 0$. Plugging this into (5.45) and comparing the coefficients of similar terms we get $a_i(\lambda_1, \lambda_2) = a_i(0)$ for all i . Hence, $f(\lambda, \mu) = \sum_{i=0}^n a_i(0) \partial^i$. Using this in (5.42) yields $\Lambda_3 = 0$, a contradiction. It follow that Λ_3 is not a 3-coboundary and thus we have $\tilde{H}^3(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) \cong \Lambda_3 \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$.

Therefore, the second term of the Hochschild-Serre spectral sequence is given by

$$E_2^{p,q} = \begin{cases} P_2 \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U}) & \text{if } p = 2 \\ \Lambda_3 \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U}) & \text{if } p = 3 \\ 0 & \text{otherwise,} \end{cases}$$

where $P_2 = \lambda_1^2 - \lambda_2^2$ and $\Lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Then the corner of the E_2 page for the Hochschild-Serre spectral sequence associated to $\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U})$ is shown in the following figure.

5	0	0	$P_2 N_{0,0,U}^5$	$\Lambda_3 N_{0,0,U}^5$	0	
4	0	0	$P_2 N_{0,0,U}^4$	$\Lambda_3 N_{0,0,U}^4$	0	\rightarrow
3	0	0	$P_2 N_{0,0,U}^3$	$\Lambda_3 N_{0,0,U}^3$	0	\rightarrow
2	0	0	$P_2 N_{0,0,U}^2$	$\Lambda_3 N_{0,0,U}^2$	0	\rightarrow
1	0	0	$P_2 N_{0,0,U}^1$	$\Lambda_3 N_{0,0,U}^1$	0	\rightarrow
0	0	0	$P_2 N_{0,0,U}^0$	$\Lambda_3 N_{0,0,U}^0$	0	\rightarrow
	0	1	2	3	4	

Figure 5.3: $E_2^{p,q}$ -page for $(\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U}), \tilde{F}^\bullet, d)$.

Here $N_{0,0,U}^q = \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})$. Since the differential $d_r^{p,q}$ with $r \geq 2$ shifts r points to the right and $q - r + 1$ points down, we conclude that $d_r^{p,q} = 0$ for all $p, q \geq 0$ and $r \geq 2$. It follows that the Hochschild-Serre spectral sequence collapses at the second page, i.e., $E_\infty^{p,q} \cong E_2^{p,q}$ for all $p, q \geq 0$. Therefore,

$$\begin{aligned}
\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{0,0,U}) &\cong \bigoplus_{p+q=n} \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{0,0,U})) \\
&\cong \tilde{H}^2(\text{Vir}, \tilde{H}^{n-2}(\text{Cur } \mathfrak{g}, M_{0,0,U})) \oplus \tilde{H}^3(\text{Vir}, \tilde{H}^{n-3}(\text{Cur } \mathfrak{g}, M_{0,0,U})) \\
&\cong P_2 \tilde{H}^{n-2}(\text{Cur } \mathfrak{g}, M_{0,0,U}) \oplus \Lambda_3 \tilde{H}^{n-3}(\text{Cur } \mathfrak{g}, M_{0,0,U})
\end{aligned}$$

for all $n \geq 2$ and $\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}) = \{0\}$ for $n \leq 1$. □

Theorem 5.4.5. *For the semidirect product Lie conformal algebra $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$,*

$$\dim \tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}) = \dim \tilde{H}^{q-1}(\text{Cur } \mathfrak{g}, M_{2,0,U}) + \dim \tilde{H}^{q-2}(\text{Cur } \mathfrak{g}, M_{2,0,U})$$

for all $q \geq 0$. In particular,

$$\tilde{H}^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}) \cong \tilde{H}^{q-1}(\text{Cur } \mathfrak{g}, M_{2,0,U}) \oplus (\lambda_1 - \lambda_2) \tilde{H}^{q-2}(\text{Cur } \mathfrak{g}, M_{2,0,U}).$$

Proof. As mentioned previously in this section, we have

$$E_2^{p,q} \cong \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})) \Rightarrow \tilde{H}^{p+q}(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U})$$

for all $p, q \geq 0$. We begin our proof by computing the cohomology $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$. Using Theorem 5.4.1, we know that $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})) = 0$ unless $p = 1, 2$. Moreover, only skew symmetric polynomials of degree $p - 1$ contribute to the cohomology of $\tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$.

For $p = 1$, then any 1-cocycle is a polynomial of degree zero, i.e., an element of $\tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})$. Now, let $\gamma \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$ such that $\gamma_{\lambda_1}(L) = \phi$ for some nonzero $\phi \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})$. Then γ is a 1-cocycle because

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2}(L, L) &= (\partial_M + 2\lambda_1)\gamma_{\lambda_2}(L) + \lambda_2\gamma_{\lambda_1+\lambda_2}(L) - (\partial_M + 2\lambda_2)\gamma_{\lambda_1}(L) \\ &\quad - \lambda_1\gamma_{\lambda_1+\lambda_2}(L) - (\lambda_1 - \lambda_2)\gamma_{\lambda_1+\lambda_2}(L) \\ &= (\partial_M + 2\lambda_1)\gamma_{\lambda_2}(L) - (\partial_M + 2\lambda_2)\gamma_{\lambda_1}(L) - 2(\lambda_1 - \lambda_2)\gamma_{\lambda_1+\lambda_2}(L) \\ &= (\partial_M + 2\lambda_1)\phi - (\partial_M + 2\lambda_2)\phi - 2(\lambda_1 - \lambda_2)\phi = 0. \end{aligned} \tag{5.46}$$

Now suppose that there exists a 0-cochain $\gamma' \in \tilde{C}^0(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$ such that $d\gamma'_\lambda(L) = L_\lambda\gamma' = \gamma$ for each $\gamma \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$, a representative of a 1-cocycle in $\tilde{H}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$. Since $\deg_\lambda(d\gamma') = 1$, and $\deg_\lambda(\gamma) = 0$, then γ represents a nontrivial class in the cohomology and we have $\tilde{H}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})) \cong \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})$.

For $p = 2$, the only skew symmetric homogeneous polynomial of degree 1 in two variable is $\lambda_1 - \lambda_2$, which is a 2-cocycle.

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) &= (\partial_M + 2\lambda_1)\gamma_{\lambda_2, \lambda_3}(L, L) + \lambda_2\gamma_{\lambda_1+\lambda_2, \lambda_3}(L, L) + \lambda_3\gamma_{\lambda_2, \lambda_1+\lambda_3}(L, L) \\ &\quad - (\partial_M + 2\lambda_2)\gamma_{\lambda_1, \lambda_3}(L, L) - \lambda_1\gamma_{\lambda_1+\lambda_2, \lambda_3}(L, L) - \lambda_3\gamma_{\lambda_1, \lambda_2+\lambda_3}(L, L) \\ &\quad + (\partial_M + 2\lambda_3)\gamma_{\lambda_1, \lambda_2}(L, L) + \lambda_1\gamma_{\lambda_1+\lambda_3, \lambda_2}(L, L) + \lambda_2\gamma_{\lambda_1, \lambda_2+\lambda_3}(L, L) \\ &\quad - (\lambda_1 - \lambda_2)\gamma_{\lambda_1+\lambda_2, \lambda_3}(L, L) + (\lambda_1 - \lambda_3)\gamma_{\lambda_1+\lambda_3, \lambda_2}(L, L) - (\lambda_2 - \lambda_3)\gamma_{\lambda_2+\lambda_3, \lambda_1}(L, L) \\ &= (\partial_M + 2\lambda_1)(\lambda_2 - \lambda_3) + \lambda_2(\lambda_1 + \lambda_2 - \lambda_3) + \lambda_3(\lambda_2 - \lambda_1 - \lambda_3) \\ &\quad - (\partial_M + 2\lambda_2)(\lambda_1 - \lambda_3) - \lambda_1(\lambda_1 + \lambda_2 - \lambda_3) - \lambda_3(\lambda_1 - \lambda_2 - \lambda_3) \\ &\quad + (\partial_M + 2\lambda_3)(\lambda_1 - \lambda_2) + \lambda_1(\lambda_1 + \lambda_3 - \lambda_2) + \lambda_2(\lambda_1 - \lambda_2 - \lambda_3) \\ &\quad - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3) + (\lambda_1 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2) - (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1) = 0. \end{aligned}$$

Observe that $\lambda_1 - \lambda_2$ represents a nontrivial cohomology class because if there exists $\phi \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U}))$ such that $d\phi_{\lambda_1, \lambda_2}(L, L) = \lambda_1 - \lambda_2$, then we have

$$\begin{aligned}
\lambda_1 - \lambda_2 &= d\phi_{\lambda_1, \lambda_2}(L, L) \\
&= (\partial_M + 2\lambda_1)\phi_{\lambda_2}(L) + \lambda_2\phi_{\lambda_1+\lambda_2}(L) - (\partial_M + 2\lambda_2)\phi_{\lambda_1}(L) \\
&\quad - \lambda_1\phi_{\lambda_1+\lambda_2}(L) - (\lambda_1 - \lambda_2)\phi_{\lambda_1+\lambda_2}(L) \\
&= (\partial - \lambda_2 + 2\lambda_1)\phi_{\lambda_2}(L) - (\partial - \lambda_1 + 2\lambda_2)\phi_{\lambda_1}(L) \\
&\quad - 2(\lambda_1 - \lambda_2)\phi_{\lambda_1+\lambda_2}(L).
\end{aligned} \tag{5.47}$$

Since $\phi_\lambda(L)$ is a 1-cochain that is identified with a polynomial in one variable λ , then it can be written as $\phi_\lambda(L) = \lambda f(\lambda)$ for a nonzero polynomial $f(\lambda) \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})[\lambda]$. Then (5.47) becomes

$$\begin{aligned}
\lambda_1 - \lambda_2 &= (\partial - \lambda_2 + 2\lambda_1)(\lambda_2 f(\lambda_2)) - (\partial - \lambda_1 + 2\lambda_2)(\lambda_1 f(\lambda_1)) \\
&\quad - 2(\lambda_1^2 - \lambda_2^2)f(\lambda_1 + \lambda_2).
\end{aligned} \tag{5.48}$$

Set $f(\lambda) = \sum_{i=0}^n a_i(\lambda) \partial^i$ such that $a_n(\lambda) \neq 0$. Plugging this into (5.48) with $\lambda_2 = 0$ we obtain

$$\lambda_1 = -(\partial\lambda_1 + \lambda_1^2)f(\lambda_1) = -(\partial\lambda_1 + \lambda_1^2) \sum_{i=0}^n a_i(\lambda_1) \partial^i. \tag{5.49}$$

Since the left hand side of (5.49) does not depends on ∂ , then all the coefficients of nonzero powers of ∂ are zero. So we have,

$$\begin{aligned}
a_n(\lambda_1)\lambda_1\partial^{n+1} &= 0, & -a_0(\lambda_1)\lambda_1^2 &= \lambda_1, \\
(a_{i+1}(\lambda_1)\lambda_1^2 + a_i(\lambda_1)\lambda_1)\partial^{i+1} &= 0 \quad \forall i \geq 0.
\end{aligned}$$

It follows that $a_i(\lambda_1) = 0$ for all i . Thus, $\phi_\lambda(L) = 0$, a contradiction. Therefore, $\tilde{H}^2(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})) \cong (\lambda_1 - \lambda_2)\tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})$.

Then the corner of the E_2 page for the Hochschild-Serre spectral sequence associated to $\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U})$ is shown in the following figure.

5	0	$\tilde{H}^5(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^5(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
4	0	$\tilde{H}^4(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^4(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
3	0	$\tilde{H}^3(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^3(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
2	0	$\tilde{H}^2(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^2(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
1	0	$\tilde{H}^1(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^1(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
0	0	$\tilde{H}^0(\text{Cur } \mathfrak{g}, M_{2,0,U})$	$\Lambda_1 \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{2,0,U})$	0
	0	1	2	3

Figure 5.4: $E_2^{p,q}$ -page for $(\tilde{C}^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}), \tilde{F}^\bullet, d)$.

Since the differential $d_r^{p,q}$ with $r \geq 2$ shifts r points to the right and $1 - r$ points down, then from Figure 5.4 we conclude that $d_r^{p,q} = 0$ for all $p, q \geq 0$ and $r \geq 2$. It follows that the Hochschild-Serre spectral sequence collapses at the second page, i.e., $E_\infty^{p,q} \cong E_2^{p,q}$ for all $p, q \geq 0$. Therefore,

$$\begin{aligned}
\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}) &\cong \bigoplus_{p+q=n} \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{2,0,U})) \\
&\cong \tilde{H}^1(\text{Vir}, \tilde{H}^{n-1}(\text{Cur } \mathfrak{g}, M_{2,0,U})) \oplus \tilde{H}^2(\text{Vir}, \tilde{H}^{n-2}(\text{Cur } \mathfrak{g}, M_{2,0,U})) \\
&\cong \tilde{H}^{n-1}(\text{Cur } \mathfrak{g}, M_{2,0,U}) \oplus \Lambda_1 \tilde{H}^{n-2}(\text{Cur } \mathfrak{g}, M_{2,0,U}),
\end{aligned}$$

for all $n \geq 1$ and $\tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{2,0,U}) = \{0\}$ for $n \leq 0$. □

Lemma 5.4.3. *Let $E_r^{*,*}$ be the Hochschild-Serre spectral sequence associated to the basic complex of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in the module $M_{\Delta,0,U}$ where $\Delta = 3r - r^2$ for $r \in \mathbb{Z}_+$ and $r \geq 4$. Then we have*

1. $\dim \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) = 0$ for $p = 0, 1$.
2. $E_2^{p,q} = \tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))$ for all $2 \leq p \leq r$ and $q \geq 0$, and $E_2^{p,q} = 0$ otherwise.
3. If $\dim \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}) = 0$ for all $q > s$ then $E_\infty^{p,q} = E_n^{p,q}$ where $n = \min(r+1, s+2)$. Otherwise, $E_\infty^{p,q} = E_{r+1}^{p,q}$.

Proof. 1. From Theorem 5.4.1, only homogeneous skew symmetric polynomial of degree $(2p - \Delta)/2$ contribute to the cohomology of $\tilde{C}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))$. For $p = 0$, let $\gamma \in \tilde{C}^0(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))$ be a 0-cocycle. Then γ can be identified with a nonzero q -cochain $\gamma' \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})$ and $d\gamma = L_\lambda \gamma' = 0$. Then we have

$$0 = L_\lambda \gamma'_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) = (\partial_M + \Delta \lambda) \gamma'_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) + \sum_{i=1}^q \mu_i \gamma'_{\mu_1, \dots, \mu_i + \lambda, \dots, \mu_q}(g_1, \dots, g_q)$$

Setting $\lambda = 0$ yields

$$0 = (\partial_M + \sum_{i=1}^q \mu_i) \gamma'_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q) = \partial \gamma'_{\mu_1, \dots, \mu_q}(g_1, \dots, g_q).$$

It follows that $\tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})$ is a trivial Vir -module (Lemma 2.1.1), contradiction. Then γ' must be zero thus $\tilde{H}^0(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) = \{0\}$.

If $p = 1$, let $\gamma \in \tilde{C}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))$ be a 1-cocycle. Then γ is a polynomial of degree $n = (2 - \Delta)/2$ in one variable, so we can write $\gamma_\lambda(L) = \lambda f(\lambda)$ for nonzero polynomial $f(\lambda) \in \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})[\lambda]$. By the definition of the differential we have

$$\begin{aligned} d\gamma_{\lambda_1, \lambda_2}(L, L) &= (\partial_M + \Delta \lambda_1) \gamma_{\lambda_2}(L) + \lambda_2 \gamma_{\lambda_1 + \lambda_2}(L) - (\partial_M + \Delta \lambda_2) \gamma_{\lambda_1}(L) - \lambda_1 \gamma_{\lambda_1 + \lambda_2}(L) \\ &\quad - (\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2}(L) \\ &= (\partial - \lambda_2 + \Delta \lambda_1) \lambda_2 f(\lambda_2) + 2\lambda_2(\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2) - (\partial - \lambda_1 + \Delta \lambda_2) \lambda_1 f(\lambda_1) \\ &\quad - 2\lambda_1(\lambda_1 + \lambda_2) f(\lambda_1 + \lambda_2). \end{aligned}$$

Letting $\lambda_2 = 0$ yields $(\partial \lambda_1 + \lambda_1^2) f(\lambda_1) = 0$, so $f(\lambda_1) = 0$, a contradiction. Then, γ must be zero and so we have $\tilde{H}^1(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))$.

2. Follows directly from Theorem 5.4.1 and part (1).

3. If $\dim \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}) = 0$ for all $q > s$, then by Lemma 3.1.1 and part (2), the Hochschild-Serre spectral sequence collapses at the n^{th} -page where $n = \min(r + 1, s + 2)$. Otherwise, by part (2), $E_2^{p,q} = 0$ for all $p \geq r + 1$ and thus $E_s^{p,q} = 0$ for all $p \geq r + 1$. Since the differential d_r shifts r points to the right and $1 - r$ points down. Then $d_{r+1} = 0$ for all $r > 4$. Hence, the Hochschild-Serre spectral sequence collapses at the $(r + 1)^{\text{th}}$ -page. \square

Theorem 5.4.6. *If $\Delta = 3r - r^2$ such that $r \in \mathbb{Z}_+$ and $r \geq 4$ then*

1. $\dim \tilde{H}^n(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) = 0$ for $n = 0, 1$.
2. $\dim \tilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) = \dim \tilde{H}^2(\text{Vir}, M_{\Delta,0,U}^{\text{Cur } \mathfrak{g}})$.

Proof. Let $\Delta = 3r - r^2$ for $r \in \mathbb{Z}_+$ and $r \geq 4$. From Example, 3.3.1, we have $\tilde{H}^0 = E_\infty^{0,0} = E_2^{0,0}$. By Lemma 5.4.3, $E_2^{0,0} = 0$. Thus, $\dim \tilde{H}^0(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) = 0$. For $n = 1, 2$, the seven-term exact sequence (3.3) looks as follows:

$$\begin{aligned}
0 \longrightarrow \tilde{H}^1(\text{Vir}, \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) &\longrightarrow \tilde{H}^1(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow \\
&\longrightarrow \tilde{H}^0(\text{Vir}, \tilde{H}^1(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) \longrightarrow \tilde{H}^2(\text{Vir}, \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) \longrightarrow \\
&\longrightarrow \ker(\tilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \rightarrow \tilde{H}^0(\text{Vir}, \tilde{H}^2(\text{Cur } \mathfrak{g}, M_{\Delta,0,U}))) \longrightarrow \\
&\longrightarrow \tilde{H}^1(\text{Vir}, \tilde{H}^1(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) \longrightarrow \tilde{H}^3(\text{Vir}, \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})).
\end{aligned} \tag{5.50}$$

From Lemma 5.4.3, $\tilde{H}^p(\text{Vir}, \tilde{H}^q(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) = 0$ for $p = 0$ and 1 , then the long exact sequence (5.50) can be rewritten as

$$\begin{aligned}
0 \longrightarrow \tilde{H}^1(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) &\longrightarrow 0 \longrightarrow \tilde{H}^2(\text{Vir}, \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) \longrightarrow \\
&\longrightarrow \ker(\tilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \rightarrow 0) \longrightarrow 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{H}^1(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) &= \{0\}, \\
\tilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) &\cong \tilde{H}^2(\text{Vir}, \tilde{H}^0(\text{Cur } \mathfrak{g}, M_{\Delta,0,U})) \cong \tilde{H}^2(\text{Vir}, M_{\Delta,0,U}^{\text{Cur } \mathfrak{g}}).
\end{aligned}$$

□

5.4.2 The Reduced Cohomology

According to [33], $H^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U}) = 0$ if $\alpha \neq 0$. In this section we will give another proof for this result using Theorem 5.4.3 and show that the zeroth reduced cohomology of $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in $M_{\Delta,\alpha,U}$ vanishes for all $\Delta, \alpha \in \mathbb{C}$ and $\Delta \neq 2$.

Theorem 5.4.7 ([33], Theorem 1.1 (iii)). $H^\bullet(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,\alpha,U}) = 0$ if $\alpha \neq 0$.

Proof. Using Theorem 5.4.3, the long exact sequence of cohomology groups associated with (2.10) for the semi direct product Lie conformal algebra $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients

in the irreducible module $M_{\Delta, \alpha, U}$ where $\alpha \neq 0$ is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(\partial\widetilde{C}^\bullet) & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^0(\mathrm{Vir} \ltimes \mathrm{Cur} \, \mathfrak{g}, M_{\Delta, \alpha, U}) \longrightarrow \\ & & & & & & \\ & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^1(\mathrm{Vir} \ltimes \mathrm{Cur} \, \mathfrak{g}, M_{\Delta, \alpha, U}) \longrightarrow \\ & & & & & & \\ & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^2(\mathrm{Vir} \ltimes \mathrm{Cur} \, \mathfrak{g}, M_{\Delta, \alpha, U}) \longrightarrow \dots \end{array}$$

Hence, $H^q(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta, \alpha, U}) = \{0\}$ for all $q \geq 0$.

Theorem 5.4.8. $H^0(\mathrm{Vir} \ltimes \mathrm{Cur} \mathfrak{g}, M_{\Delta,0,U}) = \{0\}$ for all $\Delta \in \mathbb{C}$ and $\Delta \neq 2$.

Proof. The long exact sequence of cohomology groups associated with (2.10) for the semidirect product Lie conformal algebra $\text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with coefficients in the irreducible module $M_{\Delta,0,U}$ looks as follows

$$\begin{aligned}
0 \longrightarrow H^0(\partial \tilde{C}^\bullet) &\longrightarrow \tilde{H}^0(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow H^0(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow \\
&\longrightarrow H^1(\partial \tilde{C}^\bullet) \longrightarrow \tilde{H}^1(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow H^1(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow \\
&\longrightarrow H^2(\partial \tilde{C}^\bullet) \longrightarrow \tilde{H}^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow H^2(\text{Vir} \ltimes \text{Cur } \mathfrak{g}, M_{\Delta,0,U}) \longrightarrow \cdots
\end{aligned} \tag{5.51}$$

From Theorem 5.4.4 and Theorem 5.4.6, $\dim \widetilde{H}^n(\mathrm{Vir} \ltimes \mathrm{Cur} \mathfrak{g}, M_{\Delta,0,U}) = 0$ for $n = 0, 1$ and $\Delta \neq 2$. Plugging this into (5.51) we get $\dim H^0(\mathrm{Vir} \ltimes \mathrm{Cur} \mathfrak{g}, M_{\Delta,0,U}) = 0$ for all $\Delta \in \mathbb{C}$ and $\Delta \neq 2$. \square

Remark 5.4.1. Since $H^2(\mathrm{Vir}, M_{\alpha, \Delta, U}) = 0$ for $\alpha \neq 0$, we conclude that there exist no nontrivial $\mathbb{C}[\partial]$ -split extension of $\mathrm{Vir} \ltimes \mathrm{Cur} \mathfrak{g}$ by $M_{\alpha, \Delta, U}$.

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