ABSTRACT

FARRIS, LINDSEY. Finite Dimensional Nilpotent Leibniz Algebras with Isomorphic Maximal Subalgebras. (Under the direction of Dr. Ernest Stitzinger).

This paper classifies finite dimensional nilpotent Leibniz algebras with isomorphic maximal subalgebras by their coclass. A complete classification of the described algebras will be given for coclasses zero, one, and two. The results are field dependent.

Finite Dimensional Nilpotent Leibniz Algebras with Isomorphic Maximal Subalgebras

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BIOGRAPHY

I was born in New Castle, Pennsylvania. I began my academic career at Youngstown State University where I graduated with a bachelor's of science in Mathematics and a Criminal Justice System minor. From there, I attended the University of Pittsburgh School of Law. After graduation, I was admitted to the bar for the Commonwealth of Pennsylvania, and later, the bar for the U.S. District Court for the Western District of Pennsylvania. I went into private practice with a colleague, where we focused mostly on family law. After practicing law for about a year, I decided it was not for me. I returned to Youngstown State University where I earned my master's degree in mathematics, before coming to NC State.

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1 Introduction

In 1990 Péter Z. Hermann worked on coclass for finite p-groups with isomorphic maximal subalgebras. [2] For these groups G, we define a series of increasing normal subgroups: $Z_1(G) = Z(G)$ and $Z_{i+1}(G)$ is the subgroup such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Alternatively, it can be defined by $Z_{i+1}(G) = \{x \in G | [x,y] \in Z_i(G)\}$. If this series terminates at G, then G is nilpotent, and the upper central series of G is given by

$$\{e\} = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots \trianglelefteq Z_{c-1}(G) \trianglelefteq Z_c(G) = G$$

and we say G is of class c. If G is of order p^n, then the coclass of G is given by $cc(G) = n - c$. Hermann found that there were 3 possibilities for groups with coclass 1, and 12 possibilities for groups of coclass 2, up to isomorphism. Some of Hermann's results were later improved in ([5]).

These results were extended to Lie algebras by Karen Holmes. [4, 3] This work classified the Lie algebras of coclass 0, 1, and 2. The upper central series of a nilpotent Lie algebra L is given by

$$0 = Z_0(L) \subset Z_1(L) \subset Z_2(L) \subset \cdots \subset Z_c(L) = L$$

where $Z_i(L)$ is the largest subalgebra of L such that $[Z_i(L), L] \subseteq Z_{i-1}(L)$, for all $i \leq c$, where c is the of class L. Here, the coclass of L is given by $cc(L) = dim(L) - c$.

In this paper, we further extend the results to Leibniz algebras. All of the results contained in this paper hold over the complex numbers, but at times the results are broader. We will make note when the results are restricted to $\mathbb{C}$. Furthermore, throughout this paper, we will refer to two properties: P1 is the property that all maximal subalgebras are isomorphic; P2 refers to the property that, for any maximal subalgebra M, $dim(Z_i(M))$ depends only on i, and not M. Note that P1 implies P2. At times, we will use P2 instead of P1 as it is easier to work with. However, this does not impact the final results, as the algebras that are found can be seen to have P1. All group theory, Lie, and Leibniz results are given in Section (6).

2 Background

We begin be giving some information on Leibniz algebras. Note that Lie algebras are skew-symmetric Leibniz algebras. We begin with the formal definition.

Definition 2.1. Let A be a vector space over $\mathbb{F}$. Then A is a left Leibniz algebra if it is equipped with a bilinear map,

$$[,] : A \times A \longrightarrow A$$

which satisfies

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]. \tag{1}$$

We note that the bilinear map is often referred to as a multiplication, and (1) is called the Leibniz identity. This paper will refer to left Leibniz algebras simply as Leibniz algebras, which will be denoted as A.

Definition 2.2. Let B be a subspace of a Leibniz algebra A. Then B is subalgebra if $[B,B] \subseteq B$.

Definition 2.3. Let I be a subalgebra of A. Then I is a left ideal of A if $[A,I] \subseteq I$, and is denoted by $I \trianglelefteq_l A$. Similarly, I is a right ideal if $[I,A] \subseteq I$, denoted $I \trianglelefteq_r A$. If I is both a left and right ideal, then it is called an ideal of A**,** denoted $I \trianglelefteq A$. A proper ideal is denoted by $I \triangleleft A$.

Similar to Lie algebras, given two ideals I and J, then $I+J$ and $I \cap J$ are ideals, but $[I,J]$ need not be an ideal. For a counterexample, see ([6], Example 2.4).

Definition 2.4. Let B be a subalgebra of A. The left centralizer of B is given by $C^l(B) = \{x \in A \mid [x,b] = 0 \text{ for all } b \in B\}$, and the right centralizer is given by $C^r(B) = \{x \in A \mid [b,x] = 0 \text{ for all } b \in B\}$. Then $C(B) = C^l(B) \cap C^r(B)$ is the centralizer of B. The left center of A is given by $Z^l(A) = \{x \in A \mid [x,a] = 0 \text{ for all } a \in A\}$, and the right center is given by $Z^r(A) = \{x \in A \mid [a,x] = 0 \text{ for all } a \in A\}$. Then $Z(A) = Z^l(A) \cap Z^r(A)$ is the center of A.

Definition 2.5. Let A be a Leibniz algebra. Then A is abelian if $[x,y] = 0$ for all $x,y \in A$.

A useful proposition states that we may write elements of the Leibniz algebra as a linear combination of other elements. The following proof is adapted from ([8], Proposition 4.2). We will also see a corollary to this proposition for nilpotent algebras in the next section. We note that an element is left normed if it is expressed as $[a_1,[a_2,[\ldots,[a_{n-1},a_n]\ldots]]]$.

Proposition 2.6. *Let A be a Leibniz algebra. An element of A that is the product of n elements can be written as a linear combination of the product of the n elements, with each term left normed.*

Proof. There is nothing to prove if $n = 1$ or $n = 2$. Suppose $n = 3$, and we have the multiplication $[[a,b],c]$. By the Leibniz identity, $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$, and the result holds. By induction, we assume the result holds for $k = n-1$. Consider a product of n elements, given by $[x,y]$, where x contains i elements and y contains $n-i$ elements. If $i = 1$, the result holds since x is only one element, and y is $n-1$ elements, so using the induction hypothesis and linearity, the product can be rewritten as desired. Take $i \geq 2$. By the induction hypothesis, x can be written as $[a,t]$, where a is a single element, and t is product of $i-1$ elements which are left normed. So $[x,y] = [[a,t],y] = [a,[t,y]] - [t,[a,y]]$. Since $[t,y]$ is the product of $(i-1)+(n-i) = n-1$ elements, by induction it can be written as desired. This implies $[a,[t,y]]$ can be written as a linear combination of the product of elements as well, using the bilinearity of the given map. Similarly, $[a,y]$ has $n-i+1$ elements, and so $[a,y]$ can be written as a linear combination of the elements, all of which are left normed. Again using the bilinearity of the map, the result holds. □

Definition 2.7. Let A be a Leibniz algebra and M a vector space. We call M a module if we have two bilinear maps $[,] : A \times M \longrightarrow M$ and $[,] : M \times A \longrightarrow M$ such that

$$[a,[b,m]] = [[a,b],m] + [b,[a,m]]$$

$$[a,[m,b]] = [[a,m],b] + [m,[a,b]]$$

$$[m,[a,b]] = [[m,a],b] + [a,[m,b]]$$

for all $a,b \in A$ and $m \in M$.

We denote the associative algebra of all endomorphisms of M by $End(M)$. Let M be an A-module. Then the maps $T_a : m \longrightarrow [a,m]$ and $S_a : m \longrightarrow [m,a]$ are both endomorphisms of M. We also note that the maps from A to $End(M)$ given by $T_a : a \longrightarrow T_a$ and $S_a : a \longrightarrow S_a$ are linear. The associated representation of the A-module M is the ordered pair (T,S), where T,S are maps $T,S : A \longrightarrow End(M)$ with $T(a) = T_a$ and $S(a) = S_a$.

We denote $Leib(A) = span\{[a,a] \,|a \in A\}$. If $[a,a] = a^2 = 0$ for all $a \in A$, then A is a Lie algebra ([6], page 42). We also note that $Leib(A)$ is an ideal, and the minimal ideal such that $A/Leib(A)$ is a Lie algebra ([6], page 43).

3 Nilpotent Leibniz Algebras

In this section we begin by considering preliminary results about nilpotent Leibniz algebras. We then consider properties of the upper central series for nilpotent Leibniz algebras and the Frattini subalgebra.

Lemma 3.1. *Suppose that A is nilpotent and $I \trianglelefteq A$. Then $Z(I) \trianglelefteq A$.*

Proof. For $Z(I)$ to be an ideal of A, we need to prove that $[Z(I),A] \subseteq Z(I)$ and that $[A,Z(I)] \subseteq Z(I)$. Let $x \in Z(I)$, $y \in A$, and $z \in I$. We consider $[[x,y],z] = [x,[y,z]] - [y,[x,z]]$. Since I is an ideal, $[y,z] \in I$, and so the first term goes to 0 since $x \in Z(I)$. For the second term $[x,z] = 0$, and so this goes to 0 as well. Hence, we have that $[[x,y],z] = 0$, and so $Z^l(I) \trianglelefteq A$. Similarly, we have that $[z,[x,y]] = [[z,x],y] + [x,[z,y]] = 0$ using the same reasoning. This gives $Z^r(I) \trianglelefteq A$, and so $Z(I) \trianglelefteq A$. □

Definition 3.2. Let A be a Leibniz algebra. We say that A is nilpotent of class c if every product of $c+1$ elements is zero, and there is some product of c elements that is not zero. We will denote this by $cl(A)$.

Definition 3.3. Given a Leibniz algebra A we can define the lower central series to be

$$A = A^1 \supseteq A^2 \supseteq \cdots$$

where the A^i are ideals given by $A^{i+1} = \left[A,A^i\right]$. Note that A need not be nilpotent to define this series.

We may alternatively define nilpotent.

Corollary 3.4. *([6], Corollary 4.3) The Leibniz algebra A is nilpotent of class c if $A^{c+1} = 0$ but $A^c \neq 0$.*

Lemma 3.5. *For a nilpotent Leibniz algebra A, the $dim(Z(A)) > 0$.*

Proof. Since A is nilpotent, $A^c \neq 0$, but $A^{c+1} = 0$ for some c. Take any $0 \neq x \in A^c$. We have that $[x,A] = [A,x] = 0$ since $A^{c+1} = 0$. So $x \in Z(A)$ and $dim(Z(A)) > 0$. □

Our next Lemma will require a version of Engel's theorem for Leibniz algebras. Note that a subset S of a Leibniz algebra A is a Lie set if it is closed under multiplication and it's linear span is A.

Theorem 3.6. *([6], Theorem 4.5) Let A be a Leibniz algebra, L be a Lie subset that spans A, and M be an A-module with associated representation (T,S). Suppose that T_a is nilpotent for all $a \in L$. Then A acts nilpotently on M, and there exists an element $0 \neq m \in M$ such that $[a,m] = [m,a] = 0$ for all $a \in A$.*

More can actually be said about the operators in these circumstances.

Lemma 3.7. *([7], Lemma) Let A be a finite dimensional Leibniz algebra, and let $a \in A$. Let M be a finite dimensional A-bimodule such that T_a is nilpotent on M. Then S_a is nilpotent, and $\langle S_a, T_a \rangle$, the algebra generated by all $S_b, T_b, b \in \langle a \rangle$, is nilpotent.*

Lemma 3.8. *Suppose A is a nilpotent Leibniz algebra of class c. Then for a nontrivial ideal, $0 \neq N \trianglelefteq A$, we have $N \cap Z(A) \neq 0$.*

Proof. Suppose that A is acting on N, and consider T_a. Since A is nilpotent of class c, we have $T_a^{c+1}(n) = [a,[a,\cdots[a,n]]] = 0$ for any $a \in A$, $n \in N$. By Theorem (3.6), there exists $0 \neq m \in N$ such that $[a,m] = [m,a] = 0$ for all $a \in A$, which implies $m \in Z(A)$. Hence, $N \cap Z(A) \neq 0$. □

We may also define an upper central series for A.

Definition 3.9. Suppose A is nilpotent of class c. The upper central series is given by

$$0 = Z_0(A) \subseteq Z_1(A) \subseteq \cdots \subseteq Z_c(A) = A$$

where $Z_i(A)$ is the largest subalgebra of A such that $[Z_i(A), A] \subseteq Z_{i-1}(A)$ and $[A, Z_i(A)] \subseteq Z_{i-1}(A)$ for any $i \leq c$. Alternatively, $Z_i(A)/Z_{i-1}(A) = Z(A/Z_{i-1}(A))$. Also, $Z(A) = Z_1(A)$ since $[Z_1(A), A]$, $[A, Z_1(A)] \subseteq Z_0(A) = 0$.

Lemma 3.10. *Let $N \trianglelefteq A$ such that $dim(N) = s$ and A nilpotent. Then $N \subseteq Z_s(A)$.*

Proof. Let A act on N by left and right multiplications. Since A is nilpotent, Theorem (3.6), implies there exists $0 \neq n \in N$ such that $[a,n] = [n,a] = 0$ for all $a \in A$. Now define M_1 to be the submodule of all vectors of N that are taken to zero. Now let A act on N/M_1 with the induced multiplications. Again, we get an element $0 \neq n_2 \in N$ such that $[a,n_2] + M = [n_2,a] + M = 0 \in N/M_1$. Define M_2 to be the submodule of all vectors of N/M_1 that are taken to zero. Call this set P/M_1. Multiplications of elements in P/M_1 are taken to N/M_1, which are then taken to zero. We now have a chain of an increasing number of vectors whose multiplication by elements of A go to zero. We note that $M_1 \subseteq Z_1(A)$ and $M_2 \subseteq Z_2(A)$. We repeat this process beginning with N/P. We construct increasing chains of submodules M_i that are contained in $Z_i(A)$. We eventually get to N, which will be contained in $Z_s(A)$. □

In Lie algebra, it is a known result that if $dim(L) = n$, then $dim(Z(L)) \neq n-1$, where L is a Lie algebra ([4], Lemma 5). However, in Leibniz algebras this result does not hold since we do not require $[a,a] = 0$ for any element a. We can however make the following statement.

Lemma 3.11. *Suppose A is nilpotent, $dim(A) = n$ and $dim(Z(A)) = n-1$. Then $A = I \oplus J$, where I is the ideal with basis $\{a, a^2\}$ for some $0 \neq a \in A$ and $a^2 \in Z(A)$, and J is the ideal with the same basis elements as $Z(A)$ without a^2.*

Proof. Let $a \in A$ but $a \notin Z(A)$. Then $a^2 \neq 0$, as otherwise it would be in $Z(A)$. Then $I = span\{a, a^2\}$ with $a^2 \in Z(A)$. Take complementary subspace J of $a^2 \in Z(A)$, and the statement follows. □

Lemma 3.12. *Let $I \triangleleft A$ and $J \triangleleft A$, with I and J distinct. Suppose $dim(A) = n$ and $dim(I) = dim(J) = n-1$. Then $dim(I \cap J) = n-2$.*

Proof. Using linear algebra we have the relation

$$dim(I+J) = dim(I) + dim(J) - dim(I \cap J).$$

Since I and J are distinct, and $dim(I) = dim(J) = n-1$ with $dim(A) = n$, it must be the case that $I + J = L$. This implies

$$n = (n-1) + (n-1) - dim(I \cap J).$$

Rearranging this yields $dim(I \cap J) = n-2$. □

The next few lemmas concern the upper central series of A. Our goal is to start building up the theory necessary to relate the upper central series and the Frattini subalgebra when the property P2 holds.

Lemma 3.13. *Let M a maximal subalgebra of nilpotent A. We have that $Z_i(A) \cap M \subseteq Z_i(M)$.*

Proof. Proceed by induction on i. If $i = 1$, then $x \in Z_1(A) \cap M$ implies $[x, M] = 0 = [M, x]$, and so $x \in Z_1(M)$. Assume $Z_i(A) \cap M \subseteq Z_i(M)$. Let $x \in Z_{i+1}(A) \cap M$. Then $[x, m], [m, x] \in Z_i(A) \cap M \subseteq Z_i(M)$ by assuption, for all $m \in M$. Hence, $x \in Z_{i+1}(M)$. □

Lemma 3.14. *Let $x_1, x_2, \ldots, x_m \in A$, A nilpotent, and $y \in Z_n(A)$. Then $[y, [x_m, [\cdots [x_2, x_1]]]]$ and $[[x_m, [\cdots [x_2, x_1]]], y] \in Z_{n-m}(A)$.*

Proof. We prove this using induction. Suppose that $m = 1$. By definition, we have that $[x_1, y]$, $[y, x_1] \in Z_{n-1}(A)$. Begin with the left-normed bracket. Assume this holds for $k = m-1$, so $[y, [x_{m-1}, [\cdots [x_2, x_1]]]] \in Z_{n-(m-1)}(A)$. By the Leibniz identity, we have that

$$[y, [x_m, [\cdots [x_2, x_1]]]] = [[y, x_m], [x_{m-1}, [\cdots [x_2, x_1]]]] + [x_m, [y, [x_{m-1} [\cdots [x_2, x_1]]]]]. \tag{2}$$

Define $z = [y, x_m] \in Z_{n-1}(A)$ and $w = [y, [x_{m-1}, [\cdots [x_2, x_1]]]] \in Z_{n-(m-1)}(A)$ by the induction hypothesis. So (2) becomes

$$\begin{aligned}
[y, [x_m, [\cdots [x_2, x_1]]]] &= [z, [x_{m-1}, [\cdots [x_2, x_1]]]] + [x_{m,}, w] \\
&\subseteq Z_{n-1-(m-1)}(A) + Z_{n-(m-1)-1}(A) \\
&= Z_{n-m}(A) + Z_{n-m}(A) \\
&= Z_{n-m}(A).
\end{aligned}$$

The other side can be done similarly. ☐

Lemma 3.15. *Let A be nilpotent and $x_1,\ldots,x_{m+1}\in A$ and $x_i\in Z_n(A)$ for some $i\in 1,2,\ldots,m+1$. Then $[x_{m+1},[x_m,[\cdots[x_2,x_1]]]]\in Z_{n-m}(A)$. Further, we note that for any product (left multiplication, right multiplication, or any combination thereof) of an element $w\in Z_{n-(i-1)}(A)$, with $m+1-i$ elements of A is in $Z_{n-m}(A)$.*

Proof. If $i=m+1$ done by previous lemma. For $i\neq m+1$, define $x_i=y$. Then we have that

$$[x_{m+1},[x_m,[\cdots y,[\cdots[x_2,x_1]\cdots]]\cdots]]=[x_{m+1},[x_m,\cdots[x_{i+1},w]]]$$

where $w=[y,[\cdots[x_2,x_1]]]\in Z_{n-(i-1)}(A)$ by previous lemma. Then $[x_{m+1},[x_m,\cdots[x_{i+1},w]]]$ is the multiplication of an element of $Z_{n-(i-1)}(A)$ and $m+1-i$ elements of A, and so by definition of Z_j, we have that $[x_{m+1},[x_m,\cdots[x_{i+1},w]]]\in Z_{n-(i-1)-m-1+i}(A)=Z_{n-m}(A)$ as desired. Last, we consider $w\in Z_{n-(i-1)}(A)$. By definition, $[x,w]\in Z_{n-i}(A)$ and $[w,x]\in Z_{n-i}(A)$ for any $x\in A$. Repeating this process, the second result follows immediately. ☐

Lemma 3.16. *Let M be a maximal subalgebra of nilpotent A with $x\in Z_n(M)$ and $y_1,\ldots,y_n\in M\cup Z_n(A)$. Then $[y_n,[\cdots[y_1,x]]]=0$.*

Proof. First suppose that $y_i\in Z_n(A)$ for some $1\leq i\leq n$. By lemma (3.15), $[y_n,[\cdots[y_1,x]]]\in Z_{n-n}(A)=Z_0(A)=0$. On the other hand, suppose the $y_i\in M$ for all i. Then by the definition of $Z_n(M)$, we have $[y_n,[\cdots[y_1,x]]]=0$. ☐

Lemma 3.17. *Suppose A is nilpotent and $Z_i(A)$ is not contained in a maximal subalgebra M for some i. Then $Z_i(M)=Z_i(A)\cap M$.*

Proof. Since M is maximal, and A nilpotent, M is an ideal. Also, $Z_i(A)$ is an ideal by definition. This implies $M+Z_i(A)$ is also an ideal, and by maximality of M, $M+Z_i(A)=A$. By definition, $Z_i(A)\cap M\subseteq Z_i(M)$. Let $x\in Z_i(M)$ which is contained in M. It remains to show that $x\in Z_i(A)$. Let $a_1,\ldots,a_i\in A$ with $a_j=m_j+z_j$ for $m_j\in M$, $z_j\in Z_i(A)$, for all $1\leq j\leq i$. Now

$$[a_i,[\cdots[a_1,x]]]=\left[m_j+z_j,[\cdots[m_1+z_1,x]]\right]=\sum\left[y_j,[\cdots[y_1,x]]\right]$$

by repeatedly using bilinearity, where $y_k \in M$ or $y_k \in Z_i(A)$. By lemma (3.16), every product on the right side is 0, and so the sum is 0. Hence, $x \in Z_i(A)$. □

We can now begin discussing the Frattini subalgebra, denoted by $\phi(A)$, which is the intersection of all maximal subalgebras.

Lemma 3.18. *Suppose A is nilpotent and has P2. If $cl(A) = c$, then $Z_{c-1}(A) \subseteq \phi(A)$.*

Proof. Let M_1 be a maximal subalgebra satisfying $Z_{c-1}(A) \subseteq M_1$. This implies $Z_{c-1}(M_1) \supseteq Z_{c-1}(A)$ and so $dim(Z_{c-1}(M_1)) \geq dim(Z_{c-1}(A))$. Now suppose there exists a maximal subalgebra M_2 such that $Z_{c-1}(A) \nsubseteq M_2$. Then $Z_{c-1}(M_2) = Z_{c-1}(A) \cap M_2$ by Lemma (3.17). This implies $Z_{c-1}(M_2) \subseteq Z_{c-1}(A)$, and so $dim(Z_{c-1}(M_2)) \leq dim(Z_{c-1}(A))$. Since $dim(Z_i(M))$ depends only on i, it must be the case that $dim(Z_{c-1}(M_1)) = dim(Z_{c-1}(M_2)) = dim(Z_{c-1}(A))$. Combining this with the above, we get that $Z_{c-1}(M_1) = Z_{c-1}(M_2) = Z_{c-1}(A) \subseteq M$. This is a contradiction. Therefore, for all maximal subalgebras M, $Z_{c-1}(A) \subseteq M$. Hence, $Z_{c-1}(A) \subseteq \phi(A)$. □

Lemma 3.19. *Suppose M and N are distinct maximal subalgebras of A. Then $Z_i(M) \cap Z_j(N) \subseteq Z_{i+j-1}(A)$.*

Proof. Let $x \in Z_i(M) \cap Z_j(N) \subseteq Z_{i+j-1}(A)$. We need to show that $x \in Z_{i+j-1}(A)$. Since $A = M + N$, for any $a \in A$ we can write $a = m + n$ for some $m \in M$ and $n \in N$. Define $k = i + j - 1$, and let $a_1, \ldots, a_k \in A$ with $a_l = m_l + n_l$ for all $1 \leq l \leq k$. Consider $[a_k, [\cdots[a_1, x]]] = [m_k + n_k, [\cdots[m_1 + n_1, x]]]$. Repeatedly using bilinearity, we can simplify this to sums of multiplications of the form $[y_k, [\cdots[y_1, x]]]$ where each $y_l \in M$ or N. Since $k = i + j - 1$, there are at least i terms in M or j terms in N, and so each term goes to 0 by definition of $Z_i(M)$ and $Z_i(N)$. This implies $x \in Z_{i+j-1}(A)$. □

Before continuing, it is important to make a few notes about the Frattini subalgebra. For nilpotent Leibniz algebras, $\phi(A) = [A, A]$. To see this, we make use of the fact that the dimension of a maximal subalgebra is one less than the dimension of Leibniz algebra, where the algebra is nilpotent.

Lemma 3.20. *For a nilpotent Leibniz algebra A,* $\phi(A) = [A,A]$*. It is also the smallest ideal such that* $A/\phi(A)$ *is abelian.*

Proof. Since A is nilpotent and A/M is one dimensional, it is abelian, and so $[A,A] \subseteq M$ for all maximal subalgebra M. Hence, $[A,A] \subseteq \phi(A)$. If $x \notin [A,A]$, then there exists a maximal subalgebra that does not contain x. Hence $x \notin \phi(A)$ and $\phi(A) \subseteq [A,A]$. Therefore $\phi(A) = [A,A]$. □

Proposition 3.21. *Suppose A is nilpotent and has P2. If* $cl(A) = c$*, then* $Z_{c-1}(A) = \phi(A)$*.*

Proof. By Lemma (3.18), $Z_{c-1}(A) \subseteq \phi(A)$. By definition, $A/Z_{c-1}(A) = Z_c(A)/Z_{c-1}(A) = Z(A/Z_{c-1}(A))$ is abelian. By Lemma (3.20), $\phi(A) \subseteq Z_{c-1}(A)$, since $\phi(A)$ is the smallest subalgebra which gives an abelian quotient algebra. Hence, $Z_{c-1}(A) = \phi(A)$. □

Lemma 3.22. *Suppose* $dim(A) > 1$*. Then A is cyclic if and only if the Frattini subalgebra has codimension 1 in A.*

Proof. Since A is nilpotent, $\phi(A) = [A,A]$. Suppose A is cyclic. Then the derived algebra has codimension 1, and hence the Frattini subalgebra has codimension 1. Conversely, suppose the Frattini subalgebra is of codimension 1. Then $\phi(A)$ is the only maximal subalgebra. Let $a \in A$, such that $a \notin \phi(A)$. The algebra it generates is contained in a maximal subalgebra or is A. The former is not possible since $a \notin \phi(A)$. Hence, a generates A, and A is cyclic. □

Lemma 3.23. *Suppose A is nilpotent and* $dim(A) > 1$*. Then* $dim\left(A/A^2\right) = dim(A/\phi(A)) \geq 2$ *or A is cyclic, and* $dim\left(A/A^2\right) = 1$*.*

Proof. By Lemma (3.22), $dim\left(A/A^2\right) = 1$ if and only if A is cyclic. Otherwise, A has at least 2 maximal subalgebras and their intersection has codimension 2 in A. The Frattini subalgebra then has codimension greater than or equal to 2 in A. □

Corollary 3.24. *Suppose A is nilpotent and has P2. Then* $dim(A/Z_{c-1}(A)) \geq 2$ *or A is cyclic.*

Proof. This is an immediate consequence of Proposition (3.21) and Lemma (3.23). □

4 Coclasses 0 and 1

In this section, and the following section, we explore the coclass of A, which we will denote by $cc(A)$, and its effect on the structure of the upper central series. The coclass of A is given by $cc(A) = dim(A) - cl(A)$. For a nilpotent Leibniz algebra A, every consecutive term in the series must increase in dimension by at least one. However, when two consecutive terms increase by more than one dimension, the result is a nonzero coclass. For example, if the $cc(A) = 1$, then one of the terms of the upper central series increases in dimension by two instead of one. If $cc(A) = 2$, there are two terms in the upper central series that increase in dimension by two, or one term that increases in dimension by three. It is hard to determine where these increases in dimension occur, but we know that they occur.

Lemma 4.1. *Suppose $N \trianglelefteq A$ with A nilpotent and $dim(N) = s$. Then $cc(A/N) \leq cc(A)$.*

Proof. Lemma 3.10 implies $N \subseteq Z_s(A)$. Note that $cl(A/Z_s(A)) = cl(A) - s$, which implies $cl(A/N) \geq cl(A/Z_s(A)) = cl(A) - s$. This gives the following:

$$\begin{aligned} cc(A/N) &= dim(A/N) - cl(A/N) \\ &\leq dim(A) - dim(N) - (cl(A) - s) \\ &= dim(A) - s - cl(A) + s \\ &= dim(A) - cl(A) \\ &= cc(A). \end{aligned}$$

□

Lemma 4.2. *Let $N \trianglelefteq A$, A nilpotent, with $N \subseteq Z(A)$ and $dim(N) > 1$. Then $cc(A/N) \leq cc(A) - 1$.*

Proof. Since A is nilpotent, by definition of the upper central series $cl(A/Z(A)) = cl(A) - 1$. Since

$N \subseteq Z(A)$, $cl(A/N) \geq cl(A) - 1$. Hence,

$$\begin{aligned} cc(A/N) &= dim(A/N) - cl(A/N) \\ &\leq dim(A) - dim(N) - (cl(A) - 1) \\ &\leq dim(A) - 2 - cl(A) + 1 \\ &= cc(A) - 1. \end{aligned}$$

□

It is a known result in Lie algebra that if $dim(L) > 2$ and L has P2, then $dim(Z_2(L)) > 2$ ([4], Lemma 17). However, as we can see in the following example, this result does not hold for Leibniz algebras. The proof that the following is a Leibniz algebra can be found in ([9], Theorem 2.2). The lemma after the example provides an alternative for Leibniz algebras.

Example 4.3. Let $A = span\{x_1, x_2, x_3, x_4\}$ with nonzero multiplications given by $[x_1, x_1] = x_2$, $[x_1, x_2] = x_3$, and $[x_1, x_3] = x_4$. It is easily checked that the only maximal subalgebra is given by $\{x_2, x_3, x_4\}$ which is abelian. As it is the only maximal subalgebra, A has P2. But we have the following upper central series for A: $Z_1(A) = Z(A) = span\{x_4\}$, $Z_2(A) = span\{x_3, x_4\}$, $Z_3(A) = span\{x_2, x_3, x_4\}$, and $Z_4(A) = span\{x_1, x_2, x_3, x_4\}$. From this, we have that $dim(Z_2(A)) = 2$.

Lemma 4.4. *Suppose nilpotent A has P2. If $dim(A) \leq 2$, then A is cyclic or abelian. If $dim(A) > 2$, then A is cyclic, $dim(Leib(A)) = 1$, or $dim(Z_2(A)) > 2$.*

Proof. We may assume that $dim(A) > 2$ and that A is not cyclic. If $dim(Leib(A)) = 0$, then A is Lie and $dim(Z_2(A)) > 2$ by ([3], Lemma 6). Suppose $dim(Leib(A)) > 1$. We will show that $dim(Z_2(A)) > 2$. Suppose that $dim(Z_2(A)) = 2$. Then $dim(Z(A)) = 1$. Since A is nilpotent, $Z(A) \cap Leib(A) = Z(A)$ and $Z_2(A) \cap Leib(A) = Z_2(A)$. Now $[Leib(A), A] = 0$ always holds. Thus, for any $x \in Z_2(A)$, $x \notin Z(A)$, $[A, x] \neq 0$. The kernel, M, of R_x, is shown to be a subalgebra of A. Hence, it is an ideal since it has codimension 1 in the nilpotent A. Also, $x \in Z_2(A) \subset Leib(A)$ and $x^2 = 0$, so $x \in M$. Then $dim(Z(M)) \geq 2$ as M contains both x and $Z(A)$.

Let N be another maximal subalgebra of A such that $N \neq M$. Then N is an ideal of A. Hence, $Z(A) \cap N \neq 0$. Therefore, $Z(A) \subset N$ and $Z(A) \subset Z(N)$. Also, $dim(Z(N)) = dim(Z(M)) \geq 2$. Therefore, $Z_2(A) \cap Z(N) = Z_2(A)$ and $Z_2(A) \subset Z(N)$. So $R_x(a) = [a,x] = 0$ for all $a \in N$. Thus, $N = M$, a contradiction. □

Lemma 4.5. *Suppose $dim(A) = n$ and A is abelian. Then $cc(A) = n-1$.*

Proof. Since A is abelian, $[A,A] = 0$, which implies $cl(A) = 1$. We get that $Z(A) = A$. Hence, $cc(A) = dim(A) - cl(A) = n-1$. □

Proposition 4.6. *Suppose $cc(A) = 0$. Then A is cyclic, or $dim(A) \leq 1$.*

Proof. If A is cyclic, then $cc(A) = dim(A) - cl(A) = 0$. Suppose A is not cyclic. First, take A to be abelian. Then lemma (4.5) implies that $0 = cc(A) = dim(A) - 1$, and so $dim(A) = 1$. Now take A to not be abelian, and assume $dim(A) > 1$. Consider $\phi(A) = [A,A]$. Then by Lemma (3.23) $dim(A/\phi(A)) \geq 2$. For any $x, y \in A$, we have $[x+\phi(A), y+\phi(A)] = [x,y] + \phi(A) = \phi(A)$ since $\phi(A) = [A,A]$. This implies $[A/\phi(A), A/\phi(A)] = \phi(A)/\phi(A) = 0$, and so $A/\phi(A)$ is abelian, and thus has class 1. Then Lemma (4.1), combined with $\phi(A) \trianglelefteq A$, gives

$$\begin{aligned}
cc(A) &\geq cc(A/\phi(A)) \\
&= dim(A/\phi(A)) - cl(A/\phi(A)) \\
&\geq 2-1 \\
&= 1
\end{aligned}$$

which is a contradiction, and so $dim(A) \leq 1$. □

Theorem 4.7. *Let A be a nilpotent Leibniz algebra that satisfies P2 and is of coclass 1. Then one of the following holds:*

1.) A is a Lie algebra, and so A is abelian of dimension 2, or A is the Heisenberg Lie algebra of dimension 3

2.) $A = Z_2(A)$ and $dim(A) = 3$. If $A = span\{x,y,z\}$, then $[x,x] = z$, $[y,y] = \tau z$, $[x,y] = \lambda z$, $[y,x] = \varepsilon z$, where $\tau \neq 0$ and $(\lambda+\varepsilon)^2 - 4$ is not a square.

Proof. By Lemma (4.4), if $dim(A) > 2$, then A is cyclic, or $dim(Leib(A)) = 1$ or $dim(Z_2(A)) > 2$. We look at each case.

Case 1: If A is cyclic, then $cc(A) = 0$. If A is Lie, the result holds by ([3], Proposition 3).

Case 2: Suppose $dim(Z_2(A)) > 2$. If $dim(Z_2(A)) \geq 4$, then $cc(A) \geq 2$. Hence, $dim(Z_2(A))$ $= 3$. Then $Z_2(A) = A$ since A is not cyclic and the next to the last term in the upper central series has codimension greater than 1 in A. Therefore, $dim(Z_2(A)) = 3$ and $A = Z_2(A)$. This also implies $dim(Z(A)) = 1$. Since A is not Lie, $[A,A] = Leib(A) = Z(A)$.

Suppose $A = span\{x,y,z\}$ with non-zero squares given by one of

a.) $x^2 = z$

b.) $x^2 = z$, $y^2 = \tau z$

and $Z(A) = span\{z\}$. One maximal subalgebra is $M_1 = span\{x,z\}$ and another is $M_2 = span\{y,z\}$. Since they must be isomorphic, $\tau \neq 0$ and A satisfies (b). Therefore, the algebra satisfies the multiplication in 2 in the statement of the theorem.

M_1 is cyclic, so any maximal subalgebra is also cyclic. Hence, $M_3 = span\{\alpha x + \beta y, z\}$ with $\alpha \neq 0$ or $\beta \neq 0$ must have

$$\begin{aligned} 0 &\neq (\alpha x + \beta y)^2 \\ &= \alpha^2 [x,x] + \alpha\beta [x,y] + \alpha\beta [y,x] + \beta^2 [y,y] \\ &= \alpha^2 z + \alpha\beta\lambda z + \alpha\beta\varepsilon z + \beta^2 \tau z. \end{aligned}$$

We may take $\beta = 1$. Consider $\alpha^2 + \alpha(\lambda+\varepsilon) + \tau$. A satisfies P2 if and only if this expression is not 0 for any α, which is equivalent to $(\lambda+\varepsilon)^2 - 4\tau$ not being a square in $\mathbb{F}$.

Case 3: Suppose that $dim(Leib(A)) = 1$. Then $A/Leib(A)$ is a Lie algebra of coclass 0 or 1 and satisfies P2. If $A/Leib(A)$ has coclass 0, then $dim(A/Leib(A)) \leq 1$. If $A/Leib(A)$ has dimension 0, then $A = Leib(A)$ which is impossible. If $dim(A/Leib(A)) = 1$, then $dim(A) = 2$

and A is cyclic. Then $cc(A) = 0$, which is a contradiction.

Suppose $cc(A/Leib(A)) = 1$. Then $A/Leib(A)$ is 2-dimensional abelian or 3-dimensional Heisenberg. In the first case. $dim(A) = 3$, $dim(Z(A)) = 1$, $Z_2(A) = A$, and $Z(A) = Leib(A)$. This is the algebra considered in the last case.

Suppose $A/Leib(A)$ is Heisenberg. The next to last term in the upper central series of A has codimension greater than 1. Therefore, $dim(Z(A)) = 1$, $dim(Z_2(A)) = 2$, and $Z_3(A) = A$ is 4-dimensional. Then $Z(A) = Leib(A)$. Hence, $A = span\{w,x,y,z\}$ with $Z(A) = span\{z\}$ and $Z_2(A) = span\{y,z\}$.

The multiplication table for A is

Table 1: Multiplications in A

$[\cdot,\cdot]$	w	x	y	z
w	αz	$y + az$	bz	0
x	$-y + \hat{a}z$	βz	cz	0
y	$\hat{b}z$	$\hat{c}z$	γz	0
z	0	0	0	0

The Leibniz identity shows that $\hat{b} = -b$, $\hat{c} = -c$ and $\gamma = 0$. There must be a non-zero square, so with a change of basis, if necessary, we may assume that $\alpha \neq 0$. Comparing $M_1 = span\{w,y,z\}$ and $M_2 = span\{x,y,z\}$, we have that $\beta \neq 0$. Tables for M_1, M_2, and $M_3 = span\{mw+nx,y,z\}$ are

Table 2: $M_1 = span\{w,y,z\}$

$[\cdot,\cdot]$	w	y	z
w	αz	bz	0
y	$-bz$	0	0
z	0	0	0

Table 3: $M_2 = span\{x,y,z\}$

$[\cdot,\cdot]$	x	y	z
x	βz	cz	0
y	$-cz$	0	0
z	0	0	0

Table 4: $M_3 = span\{mw+nx, y, z\}$

$[\cdot,\cdot]$	$mw+nx$	y	z
$mw+nx$	$\left(m^2\alpha+mna+mn\hat{a}+n^2\beta\right)z$	$(mb+nc)z$	0
y	$-(mb+nc)z$	0	0
z	0	0	0

If $b=0$, then $c=0$ since the center of M_1 and M_2 have the same dimension. Then $cc(A)=2$, a contradiction. Hence, $b\neq 0\neq c$. Then $mb+nc\neq 0$. But, we can find m and n such that $mb+nc=0$. Thus, A does not satisfy P2 in this case. $\square$

5 Coclass 2

In this section, we classify nilpotent Leibniz algebras of coclass 2. Recall that P1 is used to denote the property that all maximal subalgebras of A are isomorphic, and that P2 refers to the property that $dim(Z_i(M))$ depends only on i, and not on the maximal subalgebra M. Also recall that P1 implies P2. This means $cc(A) = dim(A) - cl(A) = 2$, and so $cl(A) = dim(A) - 2$. This happens in one of two ways. The first possibility is that there are two increases of dimension two from some $Z_i(A)$ to $Z_{i+1}(A)$ for two different values of i. The other is that there is one increase of dimension 3 from some $Z_i(A)$ to $Z_{i+1}(A)$. Assume $cl(A) = c$, and that A has P1. Since $cc(A) = 2$, A is not cyclic. So by Lemma (3.22), we can immediately see that there must a increase in dimension of at least two from $Z_{c-1}(A)$ to $Z_c(A) = A$. Furthermore, Proposition (3.21) tells us that $Z_{c-1}(A) = \phi(A)$. By Lemma (3.23), $dim\left(A/A^2\right) = dim(A/\phi(a)) = dim(A/Z_{c-1}(A)) \geq 2$. If $dim(A/\phi(A)) = 3$, then there is an increase of dimension 3 from $Z_{c-1}(A)$ to $Z_c(A)$. Otherwise, we will have $dim(A/\phi(A)) = 2$, and so two increases of two dimensions. By Lemma (4.4), if $dim(Z_2(A)) > 2$, then there must be a two-dimensional increase from 0 to $Z(A)$ or from $Z(A)$ to $Z_2(A)$. This leads to several possible scenarios:

1. One increase of dimension 3 from $Z_{c-1}(A)$ to $Z_c(A) = A$
2. Two increases of dimension 2, one from $Z_{c-1}(A)$ to $Z_c(A) = A$, and the other from either 0 to $Z(A)$
3. Two increases of dimension 2, one from $Z_{c-1}(A)$ to $Z_c(A) = A$, and the other from either $Z(A)$ to $Z_2(A)$
4. Two increases of dimension 2, one from $Z_{c-1}(A)$ to $Z_c(A)$, and another from $Z_{i-1}(A)$ to $Z_i(A)$ for $i \geq 3$ (provided $cl(A) = c$ is large enough).

A known result from Lie algebra is that there is no nilpotent Lie algebra L satisfying property P1 with $Z(L) = Z(M)$ for all maximal subalgebras M and $dim(Z(L)) = 1$ ([4], Lemma 20). The following example shows that this does not hold for Leibniz algebras.

Example 5.1. Let $A = span\{x,y,z\}$ with nonzero multiplications given by $[x,x] = [y,y] = z$ over $\mathbb{R}$. Then $Z(A) = span\{z\}$ and $dim(Z(A)) = 1$. Maximal subalgebras M are of the form $M = span\{z, \alpha x + \beta y\}$, where at least one of α, β are nonzero. Since $z \in Z(A)$ and $Z(M)$, only the behavior of $(\alpha x + \beta y)^2$ needs to be considered. But

$$[\alpha x + \beta y, \alpha x + \beta y] = \alpha^2 z + \beta^2 z = \left(\alpha^2 + \beta^2\right) z \neq 0$$

over $\mathbb{R}$, which cannot be 0. Therefore, all maximal subalgebras are two dimensional cyclic, and A satisfies P1.

The following definition and lemma will be necessary for the work pertaining to $cc(A) = 2$.

Definition 5.2. If A can be written as the direct sum of at least two nontrivial ideals, then A is split. Otherwise, A is non-split.

For a nilpotent algebra A to be split, we note that the dimension of the center must be greater than 1. Suppose $A = I \oplus J$, where I, J are ideals and $I \cap J = 0$. Since A is nilpotent $Z(A)$ intersects I and J nontrivially. But since $I \cap J = 0$, $dim(Z(A)) \geq 2$.

We also need to know that if A has P2, then $A/Z_2(A)$ has P2, so that we may apply earlier theorems. This result can be seen in the following lemma, which gives a slightly stronger result.

Lemma 5.3. *Let A be a nilpotent Leibniz algebra with P1 and $cl(A) = c$. Then $A/Z_i(A)$ has P1 for all $i \leq c-1$.*

Proof. Let $M_1/Z_i(A)$ and $M_2/Z_i(A)$ be maximal subalgebras in $A/Z_i(A)$. Then M_1 and M_2 are maximal subalgebras in A and there exists an isomorphism $\sigma : M_1 \to M_2$. Since $Z_i(A)$ is σ-invariant, there is an induced automorphism from $M_1/Z_i(A)$ onto $M_2/Z_i(A)$. □

This previous Lemma will hold for any ideal contained in $\phi(A)$ that is invariant under automorphisms as the same proof will apply.

5.1 Leibniz Algebras of Coclass 2

The rest of Section (5) will focus on classifying the Leibniz algebras which are coclass 2. The final results are summarized in the theorem below.

Theorem 5.4. *The non-Lie nilpotent Leibniz algebras with P1 over $\mathbb{C}$ of coclass 2 are as follows:*

1. *If A is split, then $A = span\{x_1, x_2, x_3, x_4\}$ with multiplications $[x_1, x_1] = x_3$ and $[x_2, x_2] = x_4$*

2. *If A is non-split and $dim(A) = 4$, then $A = span\{x_1, x_2, x_3, x_4\}$, with multiplications given by one of the following:*

   (a) $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_1, x_2] = \alpha x_3, [x_2, x_2] = -x_4, \alpha \in \mathbb{C} \backslash \{-1\}$

   (b) $[x_1, x_1] = x_3, [x_1, x_2] = x_3, [x_2, x_1] = x_3 + x_4, [x_2, x_2] = x_4.$

3. *If A is non-split and $dim(A) = 6$, then $A = span\{t, u, w, x, y, z\}$, with multiplications given by $[t, u] = w = -[u, t]$, $[t, w] = x = -[w, t]$, $[u, w] = y = -[w, u]$, $[w, w] = \gamma z$, $[t, y] = dz$, $[y, t] = \hat{d}z$, $[u, x] = fz$, $[x, u] = \hat{f}z$, with the restrictions that $2\gamma = d + \hat{d} = -f - \hat{f}$, $-f = d$, and $-\hat{f} = \hat{d}$, where $\gamma, d, \hat{d}, f, \hat{f} \in \mathbb{C}$.*

Proof. All of the work is shown in the following sections. □

Since the the abelian Leibniz algebra is a Lie algebra, it is excluded from the list above. However, the proof that the abelian algebra is coclass 2 is given below.

Proposition 5.5. *Suppose $cc(A) = 2$ and that A is abelian. Then $dim(A) = 3$.*

Proof. By Lemma 4.5, $2 = cc(A) = dim(A) - 1$. This implies $dim(A) = 3$. □

The rest of the cases assume that A is not abelian. If $cc(A) = 2$, then $dim(Z_2(A))$ is between 2 and 4. If $dim(Z_2(A)) = 4$ and A is not cyclic, then $dim_{c-1}(A)$ has codimension at least two and it follows that $A = Z_2(A)$. Hence, $dim(Z_2(A)) = 2$ or 3, unless $A = Z_2(A)$, in which case $dim(A) = dim(Z_2(A)) = 4$. The following sections proceed using Lemma (4.4), which tells us that $dim(Z_2(A)) > 2$ or $dim(Leib(A)) = 1$. However, the case where $dim(A) = dim(Z_2(A)) = 4$ is considered separately from the case where $dim(Z_2(A)) = 3$. Furthermore, some of this work relies on the known classification of non-split Leibniz algebras of dimension four and five found in ([9]) and ([1]). The work determining which of these algebras have P1 can be found in Section (7).

5.2 $Dim(Z_2(A)) > 2$

Let A be a nilpotent Leibniz algebra with P1. Suppose $dim(Z_2(A)) = 3$ and $dim(Leib(A)) \neq 0$. Then there must be a increase of dimension 2 from $Z_{c-1}(A)$ to A, and the other increase of dimension two occurs below $Z_2(A)$. This implies $cc(A/Z_2(A)) = 1$, and so $A/Z_2(A)$ falls into one of the categories in Theorem (4.7) above.

Option 1: $A/Z_2(A)$ is abelian of dimension 2

Option 2: $A/Z_2(A)$ is 3 dimensional Heisenberg Lie algebra

Option 3: $A = Z_2(A) = span\{x,y,z\}$, with $[x,x] = z$, $[y,y] = \tau z$, $[x,y] = \lambda z$, $[y,x] = \varepsilon z$, where $\tau \neq 0$ and $(\lambda+\varepsilon)^2 - 4$ is not a square (A is a non-Lie Leibniz algebra)

We consider each of these cases individually. Options 1 and 3 do not result in any algebras. Option 2 gives result result 3 in (5.4).

5.2.1 Case 1: $A/Z_2(A)$ is abelian

We consider the case where $dim(Z_2(A)) = 3$, $cc(A/Z_2(A)) = 1$, $A/Z_2(A)$ is abelian, and A is nilpotent with P1. By Theorem (4.7), we have that $dim(A/Z_2(A)) = 2$, which implies $dim(A) = 5$. This implies $dim(Z_3(A)) = dim(A) = 5$, $dim(Z_2(A)) = 3$, and $dim(Z(A)) = 1$ or 2. If $dim(Z(A)) = 1$, then it is immediate that A is non-split. However, if $dim(Z(A)) = 2$, it needs to be determined if A can be split.

Lemma 5.6. *Suppose A is nilpotent and has P1, with $dim(A) = 5$, $dim(Z_2(A)) = 3$, and $dim(Z(A)) = 2$. Then A is not split.*

Proof. Suppose $A = I \oplus J$, for ideals I and J, such that $I \cap J \neq \emptyset$. Then $dim(I) = 1$ and $dim(J) = 4$ or $dim(I) = 2$ and $dim(J) = 3$.

Case 1: Suppose $dim(I) = 1$ and $dim(J) = 4$, that $A = span\{x,y,z,w,t\}$, with $Z(A) = span\{w,t\}$, and $Z_2(A) = span\{z,w,t\}$. Since A has P1, then by Lemma (3.20) and Proposition (3.21) $[A,A] = \phi(A) = Z_2(A)$. This implies there is at least one multiplication in $[A,A]$ that gives each of z, w, and t. Without loss of generality, say $I = span\{w\}$, since it must contain a center element. But then J cannot be an ideal since there is some multiplication of elements that must give w. So A cannot be split in this way.

Case 2: Suppose $dim(I) = 2$ and $dim(J) = 3$. Each ideal must contribute to the upper central series. Since $dim(Z(A)) = 2$, one center element can be in each of I and J. Similarly, each of I and J must contain an element in $Z_2(A)/Z(A)$, which implies $Z_2(A)$ is at least four dimensional, a contradiction. Hence, A cannot split in this way either, and so A is non-split. □

Proposition 5.7. *There are no nilpotent non-Lie Leibniz algebras with P1 of coclass 2 where $dim(Z_2(A)) = 3$ and $A/Z_2(A)$ is an abelian Lie algebra over $\mathbb{C}$.*

Proof. Since $dim(Z_2(A)) = 3$, we have that $cc(A/Z_2(A)) = 1$. By Theorem (4.7), we get that $A/Z_2(A)$ has dimension 2. It follows immediately that $dim(Z_3(A)) = dim(A) = 5$, $dim(Z_2(A)) = 3$, and $dim(Z(A)) = 1$ or 2. By Lemma (5.6), A is non-split regardless of the dimension of the center. Since we have a 5 dimensional non-split Leibniz algebra over $\mathbb{C}$, we may use ([1]) to determine possible algebras. By Section (7.2), the only algebras with P1 are $\mathscr{A}_{137}$, $\mathscr{A}_{138}(\alpha)$, and $\mathscr{A}_{139}$. However, examining these algebras, we see that for all of them, the center is given by $span\{x_3,x_4,x_5\}$ and the second center is given by the entire 5 dimensional algebra. Hence, these algebras are coclass 3. So there are no algebras satisfying the given conditions. □

5.2.2 Case 2: $A/Z_2(A)$ is Heisenberg

We consider the case where $dim(Z_2(A)) = 3$, $cc(A/Z_2(A)) = 1$, $A/Z_2(A)$ is the three dimensional Heisenberg Lie algebra, and A is non-Lie nilpotent with P1 over $\mathbb{C}$. As $Z_2(A)$ was dimension 3, this implies $dim(A) = 6$. So we get the upper central series where $dim(A) = 6$, $dim(Z_3(A)) = 4$, $dim(Z_2(A)) = 3$, and $dim(Z(A)) = 1$ or 2, since $cc(A) = 2$ and $Z_3(A) = \phi(A)$ cannot be codimension 1.

First, consider the case where $dim(Z(A)) = 2$. Then $A/Z(A)$ is 4 dimensional and is coclass 1. Now Theorem (4.7) covers both Lie and non-Lie Leibniz algebras of coclass 1, and there are no algebras of dimension 4. So $A/Z(A)$ gives no possibilities, and so $dim(Z(A)) \neq 2$. We now consider the case where $dim(Z(A)) = 1$. Then $A/Z(A)$ is 5 dimensional and is coclass 2. $A/Z(A)$ may be either a Lie or non-Lie Leibniz algebra. Consider the case where $A/Z(A)$ is a Lie algebra. So we consider Theorem 4 in ([3]), which has one 5 dimensional Lie algebra of coclass 2, provided $char(F) \neq 2$. If we suppose that $A = span\{t,u,w,x,y,z\}$, $Z(A) = span\{z\}$, and $Z_2(A) = span\{x,y,z\}$. We get the following possible nonzero multiplications in $A/Z(A)$:

Lie Multiplications	Multiplications in $A/Z(A)$
$[t,u] = w$	$[t,u] + Z(A) = w + Z(A)$
$[t,w] = x$	$[t,w] + Z(A) = x + Z(A)$
$[u,w] = y$	$[u,w] + Z(A) = y + Z(A)$

We note that with these multiplications, $A/Z_2(A)$ is the Heisenberg Lie algebra as required, since $x,y \in Z_2(A)$. This gives the following possible multiplication in A:

Table 5: Multiplications in A

$[\cdot,\cdot]$	t	u	w	x	y	z
t	αz	$w+az$	$x+bz$	cz	dz	0
u	$-w+\hat{a}z$	βz	$y+ez$	fz	gz	0
w	$-x+\hat{b}z$	$-y+\hat{e}z$	γz	hz	jz	0
x	$\hat{c}z$	$\hat{f}z$	$\hat{h}z$	μz	kz	0
y	$\hat{d}z$	$\hat{g}z$	$\hat{j}z$	$\hat{k}z$	σz	0
z	0	0	0	0	0	0

where $\alpha,\beta,\gamma,\mu,\sigma,a,\hat{a},b,\hat{b},c,\hat{c},d,\hat{d},e,\hat{e},f,\hat{f},g,\hat{g},h,\hat{h},j,\hat{j},k,\hat{k} \in \mathbb{C}$. We begin by considering the

Leibniz identities, which place restrictions on the constants. Some of the identities, such as

$$[t,[t,t,]] = [t,\alpha z] = 0$$

$$[[t,t],t] + [t,[t,t]] = 0$$

have 0 on both sides of the identity, and so yield no information. Similarly, identities such as

$$[t,[t,u]] = [t,w+az] = x+bz$$

$$[[t,t],u] + [t,[t,u]] = [\alpha z,u] + [t,w+az] = x+bz$$

yield no information since the result is the same on both sides. Leibniz identities of both of these types will be omitted, as well as those that give duplicate earlier results. Information gained in earlier Leibniz identities will be used in later ones for simplification.

$$[t,[u,t]] = [t,-w+\hat{a}z] = -x-bz$$

$$[[t,u],t] + [u,[t,t]] = [w+az,t] + [u,\alpha z] = -x+\hat{b}z$$

$$\Rightarrow -b = \hat{b}$$

$$[t,[w,t]] = [t,-x-bz] = -cz$$

$$[[t,w],t] + [w,[t,t]] = [x+bz,t] + [w,\alpha z] = \hat{c}z$$

$$\Rightarrow -c = \hat{c}$$

$$[t,[u,w]] = [t, y+ez] = dz$$

$$[[t,u],w] + [u,[t,w]] = [w+az, w] + [u, x+bz] = \gamma z + fz$$

$$\Rightarrow d = \gamma + f$$

$$[t,[w,u]] = [t, -y+\hat{e}z] = -dz$$

$$[[t,w],u] + [w,[t,u]] = [x+bz, u] + [w, w+az] = \hat{f}z + \gamma z$$

$$\Rightarrow -d = \hat{f} + \gamma$$

$$[t,[u,x]] = [t, fz] = 0$$

$$[[t,u],x] + [u,[t,x]] = [w+az, x] + [u, cz] = hz$$

$$\Rightarrow h = 0$$

$$[t,[x,u]] = \left[t, \hat{f}z\right] = 0$$

$$[[t,x],u] + [x,[t,u]] = [cz, u] + [x, w+az] = \hat{h}z$$

$$\Rightarrow \hat{h} = 0$$

$$[t,[u,y]] = [t,gz] = 0$$

$$[[t,u],y] + [u,[t,y]] = [w+az,y] + [u,dz] = jz$$

$$\Rightarrow j = 0$$

$$[t,[y,u]] = [t,\hat{g}z] = 0$$

$$[[t,y],u] + [y,[t,u]] = [dz,u] + [y,w+az] = \hat{j}z$$

$$\Rightarrow \hat{j} = 0$$

$$[t,[w,x]] = [t,0] = 0$$

$$[[t,w],x] + [w,[t,x]] = [x+bz,x] + [w,cz] = \mu z$$

$$\Rightarrow \mu = 0$$

$$[t,[w,y]] = [t,0] = 0$$

$$[[t,w],y] + [w,[t,y]] = [x+bz,y] + [w,dz] = kz$$

$$\Rightarrow k = 0$$

$$[t,[y,w]] = [t,0] = 0$$

$$[[t,y],w] + [y,[t,w]] = [dz,w] + [y,x+bz] = \hat{k}z$$

$$\Rightarrow \hat{k} = 0$$

$$[t,[u,u]] = [t,\beta z] = 0$$

$$[[t,u],u] + [u,[t,u]] = [w+az,u] + [u,w+az] = -y+\hat{e}z+y+ez = (e+\hat{e})z$$

$$\Rightarrow -e = \hat{e}$$

$$[u,[t,w]] = [u,x+bz] = fz$$

$$[[u,t],w] + [t,[u,w]] = [-w+\hat{a}z,w] + [t,y+ez] = -\gamma z + dz$$

$$\Rightarrow f = -\gamma + d$$

$$[u,[w,t]] = [u,-x-bz] = -fz$$

$$[[u,w],t] + [w,[u,t]] = [y+ez,t] + [w,-w+\hat{a}z] = \hat{d}z - \gamma z$$

$$\Rightarrow -f = \hat{d} - \gamma$$

$$[u,[w,u]] = [u,-y-ez] = -gz$$

$$[[u,w],u] + [w,[u,u,]] = [y+ez,u] + [w,\beta z] = \hat{g}z$$

$$\Rightarrow -g = \hat{g}$$

$$[u,[w,y]] = [u,0] = 0$$

$$[[u,w],y] + [w,[u,y]] = [y+ez,y] + [w,gz] = \sigma z$$

$$\Rightarrow \sigma = 0$$

$$[w,[t,u]] = [w,w+az] = \gamma z$$

$$[[w,t],u] + [t,[w,u]] = [-x-bz,u] + [t,-y-ez] = -\hat{f}z - dz$$

$$\Rightarrow \gamma = -\hat{f} - d$$

$$[w,[u,t]] = [w,-w+\hat{a}z] = -\gamma z$$

$$[[w,u],t] + [u,[w,t]] = [-y-ez,t] + [u,-x-bz] = -\hat{d}z - fz$$

$$\Rightarrow -\gamma = -\hat{d} - f$$

In total, there are 20 Leibniz identities given above, and 105 excluded for yielding no information, or no new information, for a total of 125 Leibniz identities considered. Using the above, we may simplify the multiplication table for A. Note that there are six results concerning γ, three unique, that will be used later.

Table 6: Updated Multiplications in A

$[\cdot,\cdot]$	t	u	w	x	y	z
t	αz	$w+az$	$x+bz$	cz	dz	0
u	$-w+\hat{a}z$	βz	$y+ez$	fz	gz	0
w	$-x-bz$	$-y-ez$	γz	0	0	0
x	$-cz$	$\hat{f}z$	0	0	0	0
y	$\hat{d}z$	$-gz$	0	0	0	0
z	0	0	0	0	0	0

This table can be further simplified. Let $w' = w+az$, $x' = x+bz$, $y' = y+ez$, and $\bar{a} = a+\hat{a}$. We can then check all of the multiplications using the new definitions.

$$[t,u] = w+az = w'$$

$$[t,w'] = x+bz = x'$$

$$[t,x'] = [t,x+bz] = cz$$

$$[t,y'] = [t,y+ez] = dz$$

$$[u,t] = -w+\hat{a}z = -w-az+\bar{a}z = -w'+\bar{a}z$$

$$[w',t] = [w+az,t] = -x'$$

$$[x',t] = [x+bz,t] = -cz$$

$$[y',t] = [y+ez,t] = \hat{d}z$$

$$[w',u] = [w+az,u] = -y'$$

$$[y',u] = [y+ez,u] = -gz$$

$$[x',u] = [x+bz,u] = \hat{f}z$$

$$[u,w'] = [u,w+az] = y'$$

$$[u,x'] = [u,x+bz] = fz$$

$$[u,y'] = [u,y+ez] = gz$$

We can then simplify the results for multiplications in A. Note that we delete the primes (')

from the new variables for ease.

Table 7: Second Updated Multiplications in A

$[\cdot,\cdot]$	t	u	w	x	y	z
t	αz	w	x	cz	dz	0
u	$-w+\bar{a}z$	βz	y	fz	gz	0
w	$-x$	$-y$	γz	0	0	0
x	$-cz$	$\hat{f}z$	0	0	0	0
y	$\hat{d}z$	$-gz$	0	0	0	0
z	0	0	0	0	0	0

We continue to work on restrictions of the constants while ensuring that A has P1. Consider maximal subalgebra $M = span\{t,w,x,y,z\}$ and $v = rx+sy \in Z(M)$. Then

$$0 = [t, rx+sy] = (rc+sd)z$$
$$0 = [rx+sy, t] = (-rc+s\hat{d})z$$

Adding the two equations together gives $0 = s\left(d+\hat{d}\right)z$. So either $s=0$ or $-d=\hat{d}$. If $s=0$, then either $r=0$ or $c=0$. If $s=0=r$, then $v=0$, and $dim(Z(M))=1$. Let $\hat{M} = span\{mt+nu,w,x,y,z\}$ and consider $\hat{r}x+\hat{s}y \in Z\left(\hat{M}\right)$. Then

$$0 = [mt+nu, \hat{r}x+\hat{s}y] = (m\hat{r}c+m\hat{s}d+n\hat{r}f+n\hat{s}g)z$$

and

$$0 = [\hat{r}x+\hat{s}y, mt+nu] = (-m\hat{r}c+n\hat{r}\hat{f}+m\hat{s}\hat{d}-n\hat{s}g)z$$

Adding these two equations together gives $0 = m\hat{s}\left(d+\hat{d}\right)+n\hat{r}\left(f+\hat{f}\right)$. Solving for $\hat{r}$ and $\hat{s}$ shows $Z(M_2)$ can have center greater than 1, which means A does not satisfy P1. Hence, we cannot have both $s=0=r$.

So either $-d=\hat{d}$ or $c=0$. Likewise, using $M_1 = span\{u,w,x,y,z\}$ and $v' = r'x+s'y \in Z(M_1)$ gives $-f=\hat{f}$ or $g=0$. There are four cases:

1.) $\hat{f} = -f$ and $\hat{d} = -d$

2.) $\hat{f} = -f$ and $c = 0$ and $g \neq 0$

3.) $\hat{d} = -d$ and $g = 0$ and $c \neq 0$

4.) $g = 0$ and $c = 0$.

Case 2 cannot hold. If it did, $M^2 = span\{x,z\} \subseteq Z(M)$ but $M_1^2 = span\{y,z\} \nsubseteq Z(M_1)$, a contradiction since $M$and M_1 must be isomorphic. For the same reason, case 3 cannot hold. From the calculations done above we have the following three identities:

$$\gamma = d - f$$

$$\gamma = -d - \hat{f}$$

$$\gamma = \hat{d} + f$$

Subtracting the third from the second gives $-d - \hat{d} = \hat{f} + f$. Adding the first to second and the first to third gives $2\gamma = d + \hat{d} = -f - \hat{f}$. Considering these equations, case 1 implies that $\gamma = 0$. If $\alpha = 0$, then M is a Lie algebra, and so M_1 must also be a Lie algebra, which means $\beta = 0$. Let $M_2 = span\{t+u, w, x, y, z\}$, which must also be Lie. This implies

$$0 = [t+u, t+u] = [t,u] + [u,t] = w - w + \bar{a}z = \bar{a}z$$

and so $\bar{a} = 0$. We now have that all multiplications in A are skew-symmetric and $Leib(A) = \{0\}$. Hence A is Lie, and is given in ([3], Theorem 4).

What remains is case 4, where $g = 0 = c$. If $\gamma = 0$, then $\hat{f} = -f$ and $\hat{d} = -d$, which was the case just considered. Suppose $\gamma \neq 0$, and take $M_3 = span\{mt + nu, w, x, y, z\}$. Consider

$$\begin{aligned}[mt+nu, mt+nu] &= m^2\alpha z + mnw - mnw + mn\bar{a}z + n^2\beta z \\ &= m^2\alpha z + mn\bar{a}z + n^2\beta z\end{aligned}$$

which is a polynomial in m. As we are over the complex numbers, for any n, we can find m to satisfy $m^2\alpha + mn\bar{a} + n^2 = 0$. If α and β are not both 0, then M_3 is not isomorphic to M or M_1. Hence $\alpha = 0 = \beta$. Then $mn\bar{a} = 0$ implies $\bar{a} = 0$ and $[t,u] = -[u,t]$. Compare the multiplication table for M and $M_4 = span\{mt+nu, w, mx+ny, y, z\}$, which must be isomorphic.

Table 8: Multiplications in M

$[\cdot,\cdot]$	t	w	x	y	z
t	0	x	0	dz	0
w	$-x$	γz	0	0	0
x	0	0	0	0	0
y	$\hat{d}z$	0	0	0	0
z	0	0	0	0	0

Table 9: Multiplications in M_4

$[\cdot,\cdot]$	$mt+nu$	w	$mx+ny$	y	z
$mt+nu$	0	$mx+ny$	$mn(d+f)z$	mdz	0
w	$-(mx+ny)$	γz	0	0	0
$mx+ny$	$mn\left(\hat{d}+\hat{f}\right)z$	0	0	0	0
y	$m\hat{d}z$	0	0	0	0
z	0	0	0	0	0

We have that $M^3 = 0$ and $M_4^3 = 0$. It can be checked that a change of a basis cannot be done to make M_4 look like M. It is necessary that $mn(d+f) = 0 = mn\left(\hat{d}+\hat{f}\right)$. Thus $d = -f$ and $\hat{d} = -\hat{f}$. We can make the table for M_4 be the same as the table for M, as follows:

Table 10: Alternative Multiplications in M_4

$[\cdot,\cdot]$	$mt+nu$	w	$mx+ny$	$\frac{1}{m}y$	z
$mt+nu$	0	$mx+ny$	0	dz	0
w	$-(mx+ny)$	γz	0	0	0
$mx+ny$	0	0	0	0	0
$\frac{1}{m}y$	$\hat{d}z$	0	0	0	0
z	0	0	0	0	0

Then the maximal subalgebras are isomorphic, A satisfies P1 and $cc(A) = 2$. We then have

Table 11: Final Multiplications in A

$[\cdot,\cdot]$	t	u	w	x	y	z
t	0	w	x	0	dz	0
u	$-w$	0	y	fz	0	0
w	$-x$	$-y$	γz	0	0	0
x	0	$\hat{f}z$	0	0	0	0
y	$\hat{d}z$	0	0	0	0	0
z	0	0	0	0	0	0

with the restrictions that $2\gamma = d + \hat{d} = -f - \hat{f}$, $-f = d$, and $-\hat{f} = \hat{d}$.

It remains to consider the case where $A/Z(A)$ is a 5 dimensional Leibniz algebra. In this case, we still know that $A/Z_2(A)$ is the 3 dimensional Heisenberg Lie algebra and $dim(Z(A)) = 1$. This implies the upper central series of $B = A/Z(A)$ is $dim(Z(B)) = 2$, $dim(Z_2(B)) = 3$, and $dim(B) = 5$. Since A has P1, B will have P1 by Lemma (5.3), so we know that $\phi(A) = [A,A] = Z_2(B)$. The only possible Leibniz algebras fitting these requirements and having P1 are $\mathscr{A}_{137}$, $\mathscr{A}_{138}(\alpha)$, and $\mathscr{A}_{139}$ in Subsection (7.2), which were coclass 3. Hence, there are no possibilities for this case.

To summarize, in this section we considered the case where $dim(A) = 6$ and $A/Z_2(A)$ is the three dimensional Heisenberg Lie algebra. The upper central series is given by $dim(A) = 6$, $dim(Z_3(A)) = 4$, $dim(Z_2(A)) = 3$, and $dim(Z(A)) = 1$. Also, $A/Z(A)$ is 5 dimensional and coclass 2. When $A/Z(A)$ is a non-Lie Leibniz algebra, no algebras are found. When $A/Z(A)$ is the 5 dimensional Lie algebra in ([3], Theorem 4), a Leibniz algebra is found. This results in the following proposition, which states result 3 in Theorem (5.4).

Proposition 5.8. *Suppose A is a nilpotent Leibniz algebra over $\mathbb{C}$ with P1 where $A/Z_2(A)$ is the three dimensional Heisenberg Lie algebra. Then $dim(A) = 6$ and $Z/(A)$ is a 5 dimensional Lie algebra. Then A is defined by the following multiplications, where $A = span\{t,u,w,x,y,z\}$ and $\gamma, d, \hat{d}, f, \hat{f} \in \mathbb{C}$: $[t,u] = w = -[u,t]$, $[t,w] = x = -[w,t]$, $[u,w] = y = -[w,u]$, $[w,w] = \gamma z$, $[t,y] = dz$, $[y,t] = \hat{d}z$, $[u,x] = fz$, $[x,u] = \hat{f}z$, with the restrictions that $2\gamma = d + \hat{d} = -f - \hat{f}$, $-f = d$, and $-\hat{f} = \hat{d}$, where $\gamma, d, \hat{d}, f, \hat{f} \in \mathbb{C}$.*

Proof. The work shown above. □

5.2.3 $A/Z_2(A)$ is a non-Lie Leibniz Algebra

The last possibility is that $A/Z_2(A)$ is a non-Lie Leibniz algebra in Theorem (4.7). However, as we are working over the complex numbers, the conditions in the proposition will not be satisfied. Hence, there are no possible algebras for this case.

5.3 $Dim(Leib(A)) = 1$

By Lemma (4.4), we know that A is cyclic, $dim(Z_2(A)) > 2$, or $dim(Leib(A)) = 1$. Above, we determined that $dim(Z_2(A)) = 2$ or 3, and $dim(Z_2(A)) = 4$ only if $A = Z_2(A)$. All possibilities for $dim(Z_2(A)) = 3$ have been considered. We now turn to assuming that $dim(Leib(A)) = 1$. Since there were no restrictions on the dimension of $Leib(A)$ in Subsection (5.2), we may assume that $dim(Z_2(A)) = 2$ or $A = Z_2(A)$ with $dim(A) = 4$.

If $dim(Z_2(A)) = 2$, this means $dim(Z(A)) = 1$, as nilpotent Leibniz algebras have a non-trivial center. We must still have at least an increase of dimension 2 from $Z_{c-1}(A)$ to $A = Z_c(A)$. If this increase is of dimension 2, there is another increase of dimension 2 from $Z_{i-1}(A)$ to $Z_i(A)$ for $i = 3, \ldots, c-1$. Otherwise, there is a increase of dimension 3 from from $Z_{c-1}(A)$ to $A = Z_c(A)$. Furthermore, since the center of a nilpotent Leibniz algebra intersects all ideals nontrivially, we have that $Leib(A) \cap Z(A) \neq 0$. Since both $Leib(A)$ and $Z(A)$ have dimension 1, it must be the case that $Leib(A) = Z(A)$. Now $A/Leib(A)$ is a Lie algebra, and since the dimension only increases by 1 from 0 to $Z(A) = Leib(A)$ in the upper central series, it must be the case that $cc(A/Leib(A)) = 2$. This means $A/Leib(A)$ is one of the algebras in the following theorem. We note that L in the following theorem is a nilpotent Lie algebra.

Theorem 5.9. *([4], Theorem 4) If $dim(L) = n$. $cc(L) = 2$, and L has P1 then L is isomorphic to one of the following algebras:*

(i) $\langle\langle a, b, c\rangle\rangle$ *where* $[a, b] = [a, c] = [b, c] = 0$

(ii) $\langle\langle x,y,z,a,b\rangle\rangle$ *where* $[x,y]=z$*,* $[x,z]=a$*,* $[y,z]=b$

(iii) $\langle\langle a,b,c,x,y,z\rangle\rangle$ *where* $[a,b]=c$*,* $[a,c]=x$*,* $[b,c]=y$*,* $[a,x]=z$*,* $[b,y]=\gamma z$ *where* $-\gamma$ *is not a perfect square*

We consider $A/Leib\,(A)$ being each of the above possibilities separately. The case where $A=Z_2\,(A)$ with $dim\,(A)=dim\,(Z_2\,(A))=4$ is handled later. None of the options (i), (ii), or (iii) result in a new Leibniz algebra.

5.3.1 $A/Leib\,(A)$ is abelian

The first possibility we consider is $A/Leib\,(A)$ being the Lie algebra generated by $\langle\langle a,b,c\rangle\rangle$ where $[a,b]=[a,c]=[b,c]=0$. In this case, $A/Leib\,(A)$ is abelian and dimension 3. This implies $dim\,(A)=4$. Suppose $A=span\,\{a,b,c,x\}$, where $Leib\,(A)=span\,\{x\}=Z\,(A)$. Based on the given Lie algebra, all of the quotient group multiplications in $A/Leib\,(A)$ are 0, and so are in the span of $Leib\,(A)$. So

$$[a,b]=\alpha_1 x \qquad [b,a]=\alpha_2 x$$
$$[a,c]=\beta_1 x \qquad [c,a]=\beta_2 x$$
$$[b,c]=\gamma_1 x \qquad [c,b]=\gamma_2 x$$

for $\alpha_i,\beta_i,\gamma_i\in\mathbb{C}$. Using the definition of $Leib\,(A)$, we also get the following:

$$[a,a]=\mu_1 x$$
$$[b,b]=\mu_2 x$$
$$[c,c]=\mu_3 x$$

for $\mu_i\in\mathbb{C}$. Since A is nilpotent, $[x,x]=0$. Since $Leib\,(A)=span\,\{x\}$, at least one of the μ_i are nonzero. By the above multiplications, it is clear that $[A,A]=span\,\{x\}$. Since A has P1, $[A,A]=Z_{c-1}\,(A)=\phi\,(A)$, and all have dimension 1. We get the following upper central series:

$$0\subseteq Z\,(A)\subseteq Z_2\,(A)=A.$$

Note that since $dim(Z(A)) = 1$, it is non-split. The only possible 4 dimensional non-split non-Lie Leibniz algebras, with $dim\left(A^2\right) = 1$, are given in Subsection (7.1), and none of which have P1 by Lemma (7.2). So there are no possible algebras for this case.

5.3.2 $A/Leib(A)$ is 5 dimensional

We now turn to the second possibility where $dim(Leib(A)) = 1$, $Leib(A) = Z(A)$, and $A/Leib(A)$ is the Lie algebra of the form $\langle\langle x,y,z,a,b\rangle\rangle$ where $[x,y] = z$, $[x,z] = a$, $[y,z] = b$. Since $dim(A/Leib(A))$ $= 5$, then $dim(A) = 6$. Say $A = span\{x,y,z,a,b,c\}$. In $A/Leib(A)$, we can list the three nontrivial multiplications as

$$[x,y] + Leib(A) = z + Leib(A)$$
$$[x,z] + Leib(A) = a + Leib(A)$$
$$[y,z] + Leib(A) = b + Leib(A)$$

with other multiplications following similarly since $A/Leib(A)$ is Lie, and so skew-symmetric. All of the other multiplications in $A/Leib(A)$ are 0, and so are in the span of $Leib(A)$. Since the multiplications in $A/Leib(A)$ are nontrivial, and x, y, and z are clearly not in $Z(A)$, it must be the case that $Leib(A) = Z(A) = span\{c\}$. Furthermore, since $Leib(A)$ is the span of squares, at least one of x^2, y^2, z^2, a^2, or b^2 must be c. Using this information, the multiplication table for A is given next.

Table 12: Multiplications in A

$[\cdot,\cdot]$	x	y	z	a	b	c
x	αc	$z+dc$	$a+ec$	fc	gc	0
y	$-z+\hat{d}c$	βc	$b+hc$	jc	kc	0
z	$-a+\hat{e}c$	$-b+\hat{h}c$	γc	mc	nc	0
a	$\hat{f}c$	$\hat{j}c$	$\hat{m}c$	μc	pc	0
b	$\hat{g}c$	$\hat{k}c$	$\hat{n}c$	$\hat{p}c$	σc	0
c	0	0	0	0	0	0

for constants $\alpha, \beta, \gamma, \mu, \sigma, d, \hat{d}, e, \hat{e}, f, \hat{f}, g, \hat{g}, h, \hat{h}, j, \hat{j}, k, \hat{k}, m, \hat{m}, n, \hat{n}, p, \hat{p} \in \mathbb{C}$.

Given the three nontrivial multiplications in $A/Leib(A)$ listed above and at least one of the squares being nonzero, as well as A having P1, we get that $dim\left(A^2\right) = dim([A,A]) = dim(Z_{c-1}(A))$ $= dim(\phi(A)) = 4$. From the beginning of this subsection, Subsection (5.3), recall that $dim(Z(A))$ $= 1$ and $dim(Z_2(A)) = 2$. So we get the upper central series

$$0 \subseteq Z(A) \subseteq Z_2(A) \subseteq Z_3(A) \subseteq Z_4(A) = A$$

with $dim(Z(A)) = 1$, $dim(Z_2(A)) = 2$, $dim(Z_3(A)) = 4$, and $dim(A) = 6$, which corresponds with $cc(A) = 2$. Looking at the table however, we can see that the $dim(Z_2(A)) = 3$. Upon exmaining the table, it can be seen that eliminating or altering constants can not be done to decrease the dimnesion of just $Z_2(A)$, given the restrictions on the multiplication in $A/Leib(A)$. In fact, although the letters have changed to match the Lie algebra in Theorem (5.9), we can see that this is the exactly the first table found in Subsection (5.2.2), in which $dim(Z_2(A)) = 3$. Hence, we must get an algebra that is isomorphic to $A = span\{t,u,w,x,y,z\}$, with multiplications given by $[t,u] = w = -[u,t]$, $[t,w] = x = -[w,t]$, = $[u,w] = y = -[w,u]$, $[w,w] = \gamma z$, $[t,y] = dz$, $[y,t] = \hat{d}z$, $[u,x] = fz$, $[x,u] = \hat{f}z$, with the restrictions that $2\gamma = d + \hat{d} = -f - \hat{f}$, $-f = d$, and $-\hat{f} = \hat{d}$, where $\gamma, d, \hat{d}, f, \hat{f} \in \mathbb{C}$. Therefore, there is no algebra which fits the criteria where $dim(Leib(A)) = 1$ and $dim(Z_2(A)) = 2$. This subsection results in no new algebras.

5.3.3 $A/Leib(A)$ is 6 dimensional

We lastly consider the possibility where $A/Leib(A)$ is the Lie algebra of the form $\langle\langle a,b,c,x,y,z\rangle\rangle$ where $[a,b] = c$, $[a,c] = x$, $[b,c] = y$, $[a,x] = z$, $[b,y] = \gamma z$ where $-\gamma$ is not a perfect square. However, we are over $\mathbb{C}$, and so $-\gamma$ will always be a perfect square. Hence, there are no possibilities in this case.

5.4 $A = Z_2(A)$ with $dim(A) = 4$

Lastly we consider the case where $A = Z_2(A)$ with $dim(A) = dim(Z_2(A)) = 4$. We still assume that A has P1, and so has P2. Since A is not cyclic it must be the case that $dim([A,A]) = dim(A^2) = dim(\phi(A)) = dim(Z) = 1$ or 2 since $Z_{c-1}(A) = Z(A)$ in this case. In this case, results 2 and 3 from Theorem (5.4) will be developed.

If $dim(Z(A)) = 1$, then A is non-split. Now suppose that $dim(Z(A)) = 2$, and that A can be split. So $A = I \oplus J$, where I and J are ideals such that $I \cap J = \{0\}$, and I and J each contain one of the center elements. Suppose first that $dim(I) = 3$ and $dim(J) = 1$. Since $dim(Z(A)) = 2$, I cannot split any further. Take maximal subalgebra M of I, then $M \oplus J$ is a maximal subalgebra of A that splits, but I is a maximal subalgebra that does not split. Thus I and $M \oplus J$ are not isomorphic, and P1 is violated. So this case is not possible. Now suppose $dim(I) = dim(J) = 2$. Then I and J are both non-split since there are not enough center elements to split I or J further. Suppose $A = span\{w,x,y,z\}$ with $Z(A) = span\{y,z\}$. Since $\phi(A) = A^2 = Z(A)$, there must exist multiplications that give elements y and z as results. Suppose $[w,w] = y$ and $[x,x] = z$. Without loss of generality, it has to be the case that $I = span\{w,y\}$ and $J = span\{x,z\}$. Since I and J are ideals, we have that $[w,x] = [x,w] = 0$. Note as well then that the Leibniz identity holds upon inspection. Note that both I and J are two cyclic ideals. All maximal subalgebras are of the form $M = span\{\alpha w + \beta x, y, z\}$ with the only non-zero product given by

$$[\alpha w + \beta x, \alpha w + \beta x] = \alpha^2 y + \beta^2 z.$$

This product is 0 if and only if $\alpha = \beta = 0$, and at least one must be non-zero. So M is the direct sum of a two-dimensional cyclic and a one dimensional algebra. Therefore, all maximal subalgebras are isomorphic and A satisfies P1 and is of coclass 2. So we get a non-Lie split Leibniz algebra $A = span\{w,x,y,z\}$ with multiplications $[w,w] = y$ and $[x,x] = z$. This gives result 1 in Theorem (5.4).

Proposition 5.10. *Suppose A is a split non-Lie nilpotent Leibniz algebra over $\mathbb{C}$ with P1 where*

$dim(A) = dim(Z_2(A)) = 4$. Then $A = span\{w,x,y,z\}$ with multiplications given by $[w,w] = y$ and $[x,x] = z$.

Proof. The work is shown above. □

The above covers A being split. If A is not of this form, it must be non-split, and we can use ([9]) to determine possible algebras. This is done in Section (7.1) below. There, we get two possible algebras that have P1:

$$\mathscr{A}_{18} : [x_1,x_1] = x_3, [x_2,x_1] = x_4, [x_1,x_2] = \alpha x_3, [x_2,x_2] = -x_4, \alpha \in \mathbb{C} \setminus \{-1\}$$

$$\mathscr{A}_{19} : [x_1,x_1] = x_3, [x_1,x_2] = x_3, [x_2,x_1] = x_3 + x_4, [x_2,x_2] = x_4.$$

It is easy to see that both of these are coclass 2. For both algebras, $Z(A) = span\{x_3,x_4\}$ and $Z_2(A)$ is the algebra itself. So $cc(A) = dim(A) - cl(A) = 4 - 2 = 2$. This gives result 2 in Theorem (5.4).

Proposition 5.11. *Suppose A is a non-split non-Lie nilpotent Leibniz algebra with P1 over $\mathbb{C}$ where $dim(A) = dim(Z_2(A)) = 4$. Then A is isomorphic to one of the following algebras:*

1. $[x_1,x_1] = x_3, [x_2,x_1] = x_4, [x_1,x_2] = \alpha x_3, [x_2,x_2] = -x_4, \alpha \in \mathbb{C} \setminus \{-1\}$
2. $[x_1,x_1] = x_3, [x_1,x_2] = x_3, [x_2,x_1] = x_3 + x_4, [x_2,x_2] = x_4.$

Proof. If A is a non-split non-Lie Leibniz algebra of dimension 4, then it must be found in ([9]). Section (7.1) of this paper determines which of the four dimensional algebras found in ([9]) have P1. The non-split non-Lie four dimensional Leibniz algebras with P1 are given in the proposition statement and can be see to be coclass two. □

6 Summary of Final Results

In this section, we list the results for Leibniz algebras, as well Lie algebras and p-groups.

Proposition 6.1. *Suppose $cc(A) = 0$ where A is a Leibniz algebra. Then A is cyclic, or $dim(A) \leq 1$.*

Theorem 6.2. *Let A be a nilpotent Leibniz algebra that satisfies P2 and is of coclass 1. Then one of the following holds:*

1. *A is a Lie algebra, and so A is abelian of dimension 2, or A is the Heisenberg Lie algebra of dimension 3*

2. *$A = Z_2(A)$ and $dim(A) = 3$. If $A = span\{x, y, z\}$ then $[x, x] = z$, $[y, y] = \tau z$, $[x, y] = \lambda z$, $[y, x] = \varepsilon z$, where $\tau \neq 0$ and $(\lambda + \varepsilon)^2 - 4$ is not a square.*

Theorem 6.3. *The non-Lie nilpotent Leibniz algebras with P1 over $\mathbb{C}$ of coclass 2 are as follows:*

1. *If A is split, then $A = span\{x_1, x_2, x_3, x_4\}$ with multiplications $[x_1, x_1] = x_3$ and $[x_2, x_2] = x_4$*

2. *If A is non-split and $dim(A) = 4$, then $A = span\{x_1, x_2, x_3, x_4\}$, with multiplications given by one of the following:*

   (a) $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_1, x_2] = \alpha x_3, [x_2, x_2] = -x_4, \alpha \in \mathbb{C} \setminus \{-1\}$

   (b) $[x_1, x_1] = x_3, [x_1, x_2] = x_3, [x_2, x_1] = x_3 + x_4, [x_2, x_2] = x_4.$

3. *If A is non-split and $dim(A) = 6$, then $A = span\{t, u, w, x, y, z\}$, with multiplications given by $[t, u] = w = -[u, t]$, $[t, w] = x = -[w, t]$, $[u, w] = y = -[w, u]$, $[w, w] = \gamma z$, $[t, y] = dz$, $[y, t] = \hat{d}z$, $[u, x] = fz$, $[x, u] = \hat{f}z$, with the restrictions that $2\gamma = d + \hat{d} = -f - \hat{f}$, $-f = d$, and $-\hat{f} = \hat{d}$, where $\gamma, d, \hat{d}, f, \hat{f} \in \mathbb{C}$.*

Proposition 6.4. *([3], Proposition 2) Let L be a nilpotent Lie algebra with P2. Then $cc(L) = 0$ implies $dim(L) \leq 1$.*

Proposition 6.5. *([3], Proposition 3) Let L be a Lie algebra with P2 $cc(L) = 1$. Then either L is two dimensional abelian or three dimensional Heisenberg.*

Theorem 6.6. *([3], Theorem 3) Let L be a Lie algebra and suppose that char* $(\mathbb{F}) \neq 2$. *If dim* $(L) = n$, $cc(L) = 2$, *and L has PI, then L is isomorphic to one of the following algebras:*

1. $\langle\langle a,b,c\rangle\rangle$ *where* $[a,b] = [a,c] = [b,c] = 0$

2. $\langle\langle x,y,z,a,b\rangle\rangle$ *where* $[x,y] = z$, $[x,z] = a$, $[y,z] = b$

3. $\langle\langle a,b,c,x,y,z\rangle\rangle$ *where* $[a,b] = c$, $[a,c] = x$, $[b,c] = y$, $[a,x] = z$, $[b,y] = \gamma z$ *where* $-\gamma$ *is not a square.*

Corollary 6.7. *([2], Corollary 1) Suppose G is a finite p-group with P1. Then G is of coclass 1 if and only if it is*

1. *elementary abelian of order* p^2, *or*

2. *nonabelian of order* p^3 *and of exponent p with* $p > 2$, *or*

3. *the quaternion group of order 8.*

Theorem 6.8. *([2], Theorem 2) Assume that G is a finite p-group with P1 and that* $cc(G) = 2$. *Then G is isomorphic to one of the groups listed below.*

1. Z_{p^3}

2. $Z_p \times Z_p \times Z_p$

3. $< a,b : a^{p^2} = b^{p^2} = 1, b^{-1}ab = a^{1+p} >$

4. $< a,b : a^9 = b^9 = [a,b]^3 = [a,b,a,a] = [a,b,b,b] = 1, [a,b,a] = b^3, [a,b,b] = a^3 >$

5. $< a,b : a^9 = b^9 = [a,b]^3 = [a,b,a,a] = [a,b,b,b] = 1, [a,b,a] = b^6, [a,b,b] = a^3 >$

6. $< a,b : a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,a] = [a,b,b,b] = 1, [a,b,a] = b^p, [a,b,b] = a^{pm} >$ *(for* $p \geq 5$ *and m the smallest quadratic non residue* $\mod p$*)*

7. $< a,b : a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,a] = [a,b,b,b] = 1, [a,b,a] = b^p, [a,b,b] = a^{pg}b^p >$ *(for* $p \geq 5$*,* $1 \leq g \leq p-1$ *and* $4g+1$ *any quadratic nonresidue* $\mod p$*, which gives* $(p-1)/2$ *groups of this type)*

8. $< a,b : a^p = b^p = [a,b]^p = [a,b,a]^p = [a,b,b]^p = [a,b,a,a] = [a,b,a,b] = [a,b,b,a] = [a,b,b,b] = 1 >$ *(for* $p \geq 5$*)*

9. $< a,b : a^{p^2} = b^{p^2} = [a,b]^p = [a,b,a,b] = [a,b,b,a] = 1, [a,b,a] = a^p, [a,b,b] = b^P >$ *(for* $p \geq 5$*)*

10. $< a,b : a^p = b^p = [a,b]^p = [a,b,a]^p = [a,b,b]^p = [a,b,a,a]^p = [a,b,a,b] = [a,b,b,a] = [a,b,a,a,a] = [a,b,a,a,b] = 1, [a,b,b,b] = [a,b,a,a]^{-m}$ *(for* $p \geq 5$ *and* m *the smallest quadratic nonresidue* $\mod p$*)*

11. $< a,b : a^9 = b^9 = [a,b]^3 = [a,b,a,a]^3 = [a,b,a,a,a] = [a,b,a,a,b] = 1, [a,b,a] = b^3, [a,b,b] = a^3, [a,b,a,a] = [a,b,b,b] >$

12. $< a,b : a^9 = b^9 = [a,b]^3 = [a,b,a,a]^3 = [a,b,a,a,a] = [a,b,a,a,b] = 1, [a,b,a]^2 \cdot [a,b,a,a] = b^3, [a,b,b] \cdot [a,b,a,a]^2 = a^3, [a,b,a,a] = [a,b,b,b] >$.

7 Determination Of Leibniz Algebras that have P1

Throughout this section, Lemma (3.20), ie that $[A,A] = \phi(A)$, is used to determine what elements are in the maximal subalgebra. Not all of the algebras of the specific dimensions are necessarily listed, only those that are relevant to the work above.

7.1 Dimension 4 Algebras Having P1

Theorem 7.1. *([9], Theorem 2.1) Let A be a four-dimensional non-split non-Lie nilpotent Leibniz algebra with dim$(A^2) = 1$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1,x_2,x_3,x_4\}$ with the nonzero products given by one of the following:*

$\mathscr{A}_1$: $[x_1,x_3] = x_4$, $[x_3,x_2] = x_4$

$\mathscr{A}_2$: $[x_1,x_3] = x_4$, $[x_2,x_2] = x_4$, $[x_2,x_3] = x_4$, $[x_3,x_1] = x_4$, $[x_3,x_2] = -x_4$

$\mathscr{A}_3$: $[x_1,x_2] = x_4$, $[x_2,x_1] = -x_4$, $[x_3,x_3] = x_4$

$\mathscr{A}_4$: $[x_1,x_2] = x_4$, $[x_2,x_1] = -x_4$, $[x_2,x_2] = x_4$, $[x_3,x_3] = x_4$

$\mathscr{A}_5$: $[x_1,x_2] = x_4$, $[x_2,x_1] = cx_4$, $[x_3,x_3] = x_4$, $x \in \mathbb{C}\setminus\{1,-1\}$

$\mathscr{A}_6$: $[x_1,x_1] = x_4$, $[x_2,x_2] = x_4$, $[x_3,x_3] = x_4$

Lemma 7.2. *None of the algebras in Theorem (7.1) have P1.*

Proof. $\mathscr{A}_1$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_4\}$, and $M_1 = Z_2(M_1)$. Now take maximal subalgebra $M_2 = span\{x_1,x_2,x_4\}$, which is abelian. Hence $\mathscr{A}_1$ does not have P2, and so does not have P1.

$\mathscr{A}_2$: Take maximal subalgebra $M_1 = span\{x_1,x_2,x_4\}$, and $M_2 = span\{2x_1 + 2x_2 + x_3 + 2x_4, 2x_1 + x_2 + 2x_3 + 2x_4, x_4\}$. Then we have the following multiplication tables:

Table 13: M_1 Multiplication Table

$[\cdot,\cdot]$	x_1	x_2	x_4
x_1	0	0	0
x_2	0	x_4	0
x_4	0	0	0

Table 14: M_1 Multiplication Table

$[\cdot,\cdot]$	$2x_1+2x_2+x_3+2x_4$	$2x_1+x_2+2x_3+2x_4$	x_4
$2x_1+2x_2+x_3+2x_4$	$12x_4$	$11x_4$	0
$2x_1+x_2+2x_3+2x_4$	$5x_4$	$9x_4$	0
x_4	0	0	0

From the table above, it can be seen that M_1 and M_2 are not isomorphic, and so $\mathscr{A}_2$ does not have P1.

$\mathscr{A}_3$: Take maximal subalgebra $M_1 = span\{x_1,x_2,x_4\}$ and maximal subalgebra $M_2 = span\{x_1,x_3,x_4\}$. Now

$$[\alpha x_1+\beta x_2+\gamma x_4, \alpha x_1+\beta x_2+\gamma x_4] = \alpha\beta x_4 - \alpha\beta x_4 = 0$$

and so $Leib(M_1) = 0$, which means M_1 is a Lie algebra. However, $Leib(M_2) = span\{x_4\}$, and so M_2 is not a Lie algebra. Therefore $\mathscr{A}_3$ does not have P1.

$\mathscr{A}_4$: Take maximal subalgebra $M_1 = span\{x_1,x_2,x_4\}$, and $M_2 = span\{x_1+2x_2+x_3+2x_4, x_1+x_2+x_3+x_4, x_4\}$. Then we have the following multiplication tables:

Table 15: M_1 Multiplication Table

$[\cdot,\cdot]$	x_1	x_2	x_4
x_1	0	x_4	0
x_2	$-x_4$	x_4	0
x_4	0	0	0

Table 16: M_2 Multiplication Table

$[\cdot,\cdot]$	$x_1+2x_2+x_3+2x_4$	$x_1+x_2+x_3+x_4$	x_4
$x_1+2x_2+x_3+2x_4$	$5x_4$	$2x_4$	0
$x_1+x_2+x_3+x_4$	$4x_4$	$2x_4$	0
x_4	0	0	0

From the table above, it can be seen that M_1 and M_2 are not isomorphic, and so $\mathscr{A}_4$ does not have

P1.

$\mathscr{A}_5$: Take maximal subalgebra $M_1 = span\{x_1,x_2,x_4\}$, with $Z(M_1) = span\{x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_2,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_1,x_3,x_4\}$, with $Z(M_2) = span\{x_1,x_4\}$ and $M_2 = Z_2(M_2) = span\{x_1,x_3,x_4\}$. Hence $\mathscr{A}_5$ does not have P2, and so does not have P1.

$\mathscr{A}_6$: Take maximal subalgebra $M_1 = span\{x_1,x_2,x_4\}$, with $Z(M_1) = span\{x_4\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_2,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_1+ix_2,x_3,x_4\}$, with $Z(M_2) = span\{x+ix_2,x_4\}$ and $M_2 = Z_2(M_2) = span\{x_1+ix_2,x_3,x_4\}$. Thus $\mathscr{A}_6$ does not have P2, and so does not have P1. ☐

Theorem 7.3. *([9], Theorem 2.3) Let A be a four-dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 2$, $dim(A^3) = 0$ *and* $dim(Leib(A)) = 1$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_8$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3 = -[x_2,x_1]$

$\mathscr{A}_9$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_4$.

Lemma 7.4. *None of the algebras in Theorem (7.3) have P1.*

Proof. $\mathscr{A}_8$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_3,x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4\}$, which is abelian. Hence $\mathscr{A}_8$ does not have P2, and so does not have P1.

$\mathscr{A}_9$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_3,x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_1+ix_2,x_3,x_4\}$, which is abelian. Therefore $\mathscr{A}_9$ does not have P2, and so does not have P1. ☐

Theorem 7.5. *([9], Theorem 2.4) Let A be a four-dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 2$ *and* $dim(Leib(A)) = 1 = dim(A^3)$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{10}$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_1,x_3] = x_4 = -[x_3,x_1]$

$\mathscr{A}_{11}$: $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_4$, $[x_1,x_3] = x_4 = -[x_3,x_1]$

$\mathscr{A}_{12}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=-x_3+x_4$, $[x_1,x_3]=x_4=-[x_3,x_1]$

$\mathscr{A}_{13}$: $[x_2,x_2]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=-x_3+x_4$, $[x_1,x_3]=x_4=-[x_3,x_1]$.

Lemma 7.6. *None of the algebras in Theorem (7.5) have P1.*

Proof. $\mathscr{A}_{10}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_4\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4\}$, which is abelian. Hence $\mathscr{A}_{10}$ does not have P2, and so does not have P1.

$\mathscr{A}_{11}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_4\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4\}$, with $Z(M_2) = span\{x_3,x_4\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4\}$. As $\mathscr{A}_{11}$ does not have P2, it does not have P1.

$\mathscr{A}_{12}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4\}$, which is abelian. Therefore $\mathscr{A}_{12}$ does not have P2, and so does not have P1.

$\mathscr{A}_{13}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4\}$, with $Z(M_1) = span\{x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4\}$, with $Z(M_2) = span\{x_3,x_4\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4\}$. Since $\mathscr{A}_{13}$ does not have P2, it does not have P1. □

Theorem 7.7. *([9], Theorem 2.5) Let A be a four-dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 2 = dim(Leib(A))$ *and* $dim(A^3) = 0$. *Then, A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4\}$ *with the nonzero products given by the following:*

$\mathscr{A}_{14}$: $[x_1,x_1]=x_3$, $[x_1,x_2]=x_4$

$\mathscr{A}_{15}$: $[x_1,x_1]=x_3$, $[x_2,x_1]=x_4$

$\mathscr{A}_{16}$: $[x_1,x_2]=x_4$, $[x_2,x_1]=x_3$, $[x_2,x_2]=-x_3$

$\mathscr{A}_{17}$: $[x_1,x_1]=x_3$, $[x_1,x_2]=x_4$, $[x_2,x_1]=\alpha x_4$, $\alpha \in \mathbb{C}\setminus\{-1,0\}$

$\mathscr{A}_{18}$: $[x_1,x_1]=x_3$, $[x_2,x_1]=x_4$, $[x_1,x_2]=\alpha x_3$, $[x_2,x_2]=-x_4$, $\alpha \in \mathbb{C}\setminus\{-1\}$

$\mathscr{A}_{19}$: $[x_1,x_1]=x_3$, $[x_1,x_2]=x_3$, $[x_2,x_1]=x_3+x_4$, $[x_2,x_2]=x_4$.

Lemma 7.8. *The only algebras in Theorem (7.7) that have P1 are* $\mathscr{A}_{18}$ *and* $\mathscr{A}_{19}$.

Proof. $\mathscr{A}_{14}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4\}$, with $Z(M_1) = span\{x_3, x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4\}$, which is abelian. Hence $\mathscr{A}_{14}$ does not have P2, and so does not have P1.

$\mathscr{A}_{15}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4\}$, with $Z(M_1) = span\{x_3, x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4\}$, which is abelian. Therefore $\mathscr{A}_{15}$ does not have P2, and so does not have P1.

$\mathscr{A}_{16}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4\}$, which is abelian. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4\}$, with $Z(M_2) = span\{x_3, x_4\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4\}$. As $\mathscr{A}_{16}$ does not have P2, it does not have P1.

$\mathscr{A}_{17}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4\}$, with $Z(M_1) = span\{x_3, x_4\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4\}$, which is abelian. Thus $\mathscr{A}_{17}$ does not have P2, and so does not have P1.

$\mathscr{A}_{18}$: All maximal subalgebras are of the form $M = span\{ax_1 + bx_2, x_3, x_4\}$. Now

$$\begin{aligned}[ax_1 + bx_2, ax_1 + bx_2] &= a^2x_3 + ab\alpha x_3 + abx_4 - b^2x_4 \\ &= \left(a^2 + ab\alpha\right)x_3 + \left(ab - b^2\right)x_4.\end{aligned}$$

Change the basis for M, and let $r = ax_1 + bx_2$ and $s = \left(a^2 + ab\alpha\right)x_3 + \left(ab - b^2\right)x_4$. Choose t to be complementary to s in $\{x_3, x_4\}$. Then all maximal subalgebras can be written as $M' = span\{r, s, t\}$ and the only multiplication is $r^2 = s$. As this holds for all maximal subalgebras, $\mathscr{A}_{18}$ has P1.

$\mathscr{A}_{19}$: All maximal subalgebras are of the form $M = span\{ax_1 + bx_2, x_3, x_4\}$. Now

$$\begin{aligned}[ax_1 + bx_2, ax_1 + bx_2] &= a^2x_3 + abx_3 + abx_3 + abx_4 + b^2x_4 \\ &= \left(a^2 + 2ab\right)x_3 + \left(ab + b^2\right)x_4.\end{aligned}$$

Change the basis for M, and let $r = ax_1 + bx_2$ and $s = \left(a^2 + 2ab\right)x_3 + \left(ab + b^2\right)x_4$. Choose t to be complementary to s in $\{x_3, x_4\}$. Then all maximal subalgebras can be written as $M' = span\{r, s, t\}$ and the only multiplication is $r^2 = s$. Since this holds for all maximal subalgebras, $\mathscr{A}_{19}$ has P1. □

7.2 Dimension 5 Algebras Having P1

The results in ([1]) give the possibilities for 5-dimensional non-split Leibniz algebras. The following theorems list the results for when $dim(A) = 5$, and $dim(Z_2(A)) = 3$, where $Z_2(A) = [A,A]$.

Theorem 7.9. *([1], Theorem 2.2) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$, $dim(A^3) = 2$, $dim(A^4) = 1$, *and* $dim(Leib(A)) = 1$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4,x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_1$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_1,x_3] = x_4 = -[x_3,x_1], [x_1,x_4] = x_5 = -[x_4,x_1]$

$\mathscr{A}_2$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $[x_1,x_4] = x_5 = -[x_4,x_1]$

$\mathscr{A}_3$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_2,x_3] = x_4 = -[x_3,x_2], [x_2,x_4] = x_5 = -[x_4,x_2]$

$\mathscr{A}_4$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_1,x_3] = x_5 = -[x_3,x_1], [x_2,x_3] = x_4 = -[x_3,x_2]$, $[x_2,x_4] = x_5 = -[x_4,x_2]$

$\mathscr{A}_5(\alpha)$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_2,x_2] = x_5, [x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = \alpha x_5 = -[x_3,x_2], [x_1,x_4] = x_5 = -[x_4,x_1]$, $\alpha \in \mathbb{C}$

$\mathscr{A}_6$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3 + x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_1,x_4] = x_5 = -[x_4,x_1]$

$\mathscr{A}_7$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3 + x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], [x_1,x_4] = x_5 = -[x_4,x_1]$

Lemma 7.10. *None of the algebras in Theorem (7.9) have P1.*

Proof. $\mathscr{A}_1$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3,x_4,x_5\}$, which is abelian. From this, it is clear that $\mathscr{A}_1$ cannot have P1, as M_1 cannot be isomorphic to M_2, as M_2 is abelian, but M_1 is not.

$\mathscr{A}_2$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 =$

$span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$, $Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Since $\mathscr{A}_2$ does not have P2, it does not have P1.

$\mathscr{A}_3$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3, x_4,x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3, x_4,x_5\}$, with $Z(M_2) = span\{x_5\}$, $Z_2(M_2) = span\{x_4,x_5\}$, and $M_2 = Z_3(M_2) = span\{x_2,x_3,x_4,x_5\}$. As $\mathscr{A}_3$ does not have P2, it does not have P1.

$\mathscr{A}_4$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_5\}$, $Z_2(M_2) = span\{x_4,x_5\}$, and $M_2 = Z_3(M_2) = span\{x_2, x_3,x_4,x_5\}$. As $\mathscr{A}_4$ does not have P2, it does not have P1.

$\mathscr{A}_5(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. If $\alpha \neq 0$, then $Z(M_2) = span\{x_4,x_5\}$ and $Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. If $\alpha = 0$, then $Z(M_2) = span\{x_3,x_{4,5}\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4, x_5\}$. In either case $\mathscr{A}_5$ does not have P2, so it does not have P1.

$\mathscr{A}_6$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, which is abelian. So $\mathscr{A}_6$ does not have P1.

$\mathscr{A}_7$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Since $\mathscr{A}_7$ does not have P2, it does not have P1. □

Theorem 7.11. *([1], Theorem 2.3) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$, $dim(A^3) = 2$, $dim(A^4) = 0$, *and* $dim(Leib(A)) = 1$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4,x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_8$: $[x_1,x_1] = x_5$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$

$\mathscr{A}_9$: $[x_1,x_1]=x_5$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_1,x_3]=x_5=-[x_3,x_1]$, $[x_2,x_3]=x_4=-[x_3,x_2]$

$\mathscr{A}_{10}$: $[x_1,x_1]=x_5$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_4=-[x_3,x_1]$, $[x_2,x_3]=x_5=-[x_3,x_2]$

$\mathscr{A}_{11}$: $[x_1,x_1]=x_5$, $[x_1,x_2]=x_3$, $[x_2,x_1]=-x_3+x_5$, $[x_1,x_3]=x_4=-[x_3,x_1]$, $[x_2,x_3]=x_5=-[x_3,x_2]$

Lemma 7.12. *None of the algebras in Theorem 7.11 have P1.*

Proof. $\mathscr{A}_8$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. In M_1, the nonzero multiplications are given by $[x_1,x_1]=x_5$, $[x_1,x_3]=x_4=-[x_3,x_1]$, so $Leib(M_1)=span\{x_5\}$, and M_1 is not a Lie algebra. In M_2, the only nonzero multiplications are given by $[x_2,x_3]=x_5=-[x_3,x_2]$, and so $Leib(M_2)=0$, which implies M_2 is Lie. Hence M_1 and M_2 are not isomorphic, and $\mathscr{A}_8$ does not have P1.

$\mathscr{A}_9$: Take $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. The nonzero multiplications in M_1 are given by: $[x_1,x_1]=x_5$, $[x_1,x_3]=x_5=-[x_3,x_1]$. So $Leib(M_1)=span\{x_5\}$, and M_1 is not Lie. The nonzero multiplications in M_2 are given by: $[x_2,x_3]=x_4=-[x_3,x_2]$. So $Leib(M_2)=0$, and hence M_2 is Lie. This implies M_1 and M_2 are not isomorphic, and $\mathscr{A}_9$ does not have P1.

$\mathscr{A}_{10}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. The nonzero multiplications in M_1 are given by: $[x_1,x_1]=x_5$, $[x_1,x_3]=x_4=-[x_3,x_1]$. We can see that $dim([M_1,M_1])=2$. The nonzero multiplications in M_2 are given by: $[x_2,x_2]=x_5$, $[x_2,x_3]=x_5=-[x_3,x_2]$. From this, we can see that $dim([M_2,M_2])=1$. Hence M_1 is not isomorphic to M_2, and $\mathscr{A}_{10}$ does not have P1.

$\mathscr{A}_{11}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. In M_1 the nonzero multiplications are given by: $[x_1,x_1]=x_5$, and $[x_1,x_3]=x_4$. So $Leib(M_1)=span\{x_5\}$, and M_1 is not Lie. In M_2, the only nonzero multiplications are given by: $[x_2,x_3]=x_5=-[x_3,x_2]$. So $Leib(M_2)=0$, and M_2 is a Lie algebra. Hence M_1 and M_2 are not isomorphic, and $\mathscr{A}_{11}$ does not have P1. □

Theorem 7.13. *([1], Theorem 2.4) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$*,* $dim(A^3) = 1$*, and* $dim(Leib(A)) = 1$*. Then A is isomorphic to a Leibniz algebra spanned by* x_1, x_2, x_3, x_4, x_5 *with the nonzero products given by one of the following:*

$\mathscr{A}_{12}$: $[x_1,x_1] = x_5$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_1,x_3] = x_4 = -[x_3,x_1]$

$\mathscr{A}_{13}$: $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$

$\mathscr{A}_{14}$: $[x_1,x_1] = x_5$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$

$\mathscr{A}_{15}$: $[x_1,x_1] = x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$.

Lemma 7.14. *None of the algebras in Theorem (7.13) have P1.*

Proof. $\mathscr{A}_{12}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_4,x_5\}$, $Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, which is abelian. So M_1 and M_2 are not isomorphic, and $\mathscr{A}_{12}$ does not have P1.

$\mathscr{A}_{13}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_4,x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_3,x_4,x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. So $\mathscr{A}_{13}$ does not have P1 as it does not have P2.

$\mathscr{A}_{14}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_4,x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_3,x_4,x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. So $\mathscr{A}_{14}$ does not have P2, and so does not have P1.

$\mathscr{A}_{15}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_4,x_5\}$ and $Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, which is abelian. Thus M_1 and M_2 are not isomorphic, and $\mathscr{A}_{15}$ does not have P1. □

Theorem 7.15. *([1], Theorem 3.5) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3 = dim(Leib(A))$*,* $dim(A^3) = 2$ *and* $dim(A^4) = 1$*. Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4,x_5\}$ *with nonzero products given by one of the following:*

$\mathscr{A}_{64}$: $[x_1,x_2]=x_3$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{65}$: $[x_1,x_2]=x_3$, $[x_2,x_1]=x_5$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{66}$: $[x_1,x_2]=x_3$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{67}$: $[x_1,x_2]=x_3$, $[x_2,x_1]=x_5$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{68}$: $[x_1,x_2]=x_3$, $[x_2,x_2]=x_4$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$, $[x_1,x_4]=x_5$

$\mathscr{A}_{69}$: $[x_1,x_2]=x_3$, $[x_2,x_1]=x_5$, $[x_2,x_2]=x_4$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$, $[x_1,x_4]=x_5$

$\mathscr{A}_{70}(\alpha)$: $[x_1,x_2]=x_3$, $[x_2,x_1]=\alpha x_5$, $[x_2,x_2]=x_4+x_5$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$, $[x_1,x_4]=x_5$, $\alpha\in\mathbb{C}$

$\mathscr{A}_{71}$: $[x_1,x_1]=x_3$, $[x_2,x_1]=x_5$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{72}$: $[x_1,x_1]=x_3$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_5$

$\mathscr{A}_{73}(\alpha)$: $[x_1,x_1]=x_3$, $[x_2,x_1]=x_4$, $[x_2,x_2]=\alpha x_5$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$, $[x_1,x_4]=x_5$, $\alpha\in\mathbb{C}$

$\mathscr{A}_{74}$: $[x_1,x_1]=x_3$, $[x_2,x_1]=x_4+x_5$, $[x_2,x_2]=2x_5$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$, $[x_1,x_4]=x_5$.

Remark. (1) If $\alpha_1,\alpha_2\in\mathbb{C}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{70}(\alpha_1)$ and $\mathscr{A}_{70}(\alpha_2)$ are not isomorphic.

(2) If $\alpha_1,\alpha_2\in\mathbb{C}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{73}(\alpha_1)$ and $\mathscr{A}_{73}(\alpha_2)$ are not isomorphic.

Lemma 7.16. *None of the algebras in Theorem (7.15) have P1.*

Proof. $\mathscr{A}_{64}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$, $Z_2(M_1)=span\{x_4,x_5\}$, $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, which is abelian. Hence, $\mathscr{A}_{64}$ does not have P1.

$\mathscr{A}_{65}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$, $Z_2(M_1)=span\{x_4,x_5\}$, and $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, which is abelian. So $\mathscr{A}_{65}$ does not have P1.

$\mathscr{A}_{66}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$, $Z_2(M_1)=span\{x_4,x_5\}$, and $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2)=span\{x_3,x_4,x_5\}$ and $M_2=Z_2(M_2)=span\{x_2,x_3,x_4,x_5\}$. So $\mathscr{A}_{66}$ does not have P2, and so does not have P1.

$\mathscr{A}_{67}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_3,x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{67}$ does not have P2, and so does not have P1.

$\mathscr{A}_{68}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{68}$ does not have P2, and so does not have P1.

$\mathscr{A}_{69}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{69}$ does not have P2, and so does not have P1.

$\mathscr{A}_{70}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{70}$ does not have P2, and so does not have P1 for any value of α.

$\mathscr{A}_{71}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, $Z_3(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_4(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, which is abelian. Hence $\mathscr{A}_{71}$ does not have P1.

$\mathscr{A}_{72}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, $Z_3(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_4(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_3,x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{72}$ does not have P2, and so does not have P1.

$\mathscr{A}_{73}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4,x_5\}$, $Z_3(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_4(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{73}$ does not have P2, and so does not have P1, for any value of α.

$\mathscr{A}_{74}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_5\}$, $Z_2(M_1) = span\{x_4, x_5\}$, $Z_3(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_4(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$ and $Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{74}$ does not have P2, and so does not have P1. □

Theorem 7.17. *([1], Theorem 3.6) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $dim(A^2) = 3$, $dim(A^3) = 2 = dim(Leib(A))$ and $A^4 = 0$. Then A is isomorphic to a Leibniz algebra spanned by x_1, x_2, x_3, x_4, x_5 with the nonzero products given by one of the following:*

$\mathscr{A}_{75}(\alpha)$: $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4$, $[x_2,x_2] = \alpha x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\{0\}$

$\mathscr{A}_{76}(\alpha)$: $[x_1,x_1] = x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4$, $[x_2,x_2] = \alpha x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{77}(\alpha)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4 + x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\{0\}$

$\mathscr{A}_{78}(\alpha)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4 + x_5$, $[x_2,x_2] = x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\{0\}$

$\mathscr{A}_{79}(\alpha)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4 + x_5$, $[x_2,x_2] = -\frac{1}{2}x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\left\{-\frac{1}{6}, 0\right\}$

$\mathscr{A}_{80}(\alpha)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_4 + x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\left\{-\frac{4}{27}, 0\right\}$

$\mathscr{A}_{81}(\alpha,\beta)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_5$, $[x_2,x_2] = x_4 + \beta x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha \in \mathbb{C}\setminus\{0\}$, $\beta \in \mathbb{C}$, $4\alpha\beta \neq 1$, $8\alpha\beta^3 - 2\beta^2 + 1 \neq 0$, $16\alpha\beta^3 \neq 1 + 6\beta^2 \pm \sqrt{4\beta^2 + 12\beta + 1}$, $-27\alpha\beta \neq 9\beta^2 + 2\beta^4 \pm 2\sqrt{\beta^2\left(3+\beta^2\right)^3}$

$\mathscr{A}_{82}(\alpha,\beta,\gamma)$: $[x_1,x_1] = \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + x_4 + \beta x_5$, $[x_2,x_2] = x_4 + \gamma x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha,\beta,\gamma \in \mathbb{C}$

$\mathscr{A}_{83}(\alpha,\beta)$: $[x_1,x_1] = x_4 + \alpha x_5$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3 + \beta x_5$, $[x_2,x_2] = x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$, $[x_2,x_3] = x_5 = -[x_3,x_2]$, $\alpha,\beta \in \mathbb{C}$.

Remarks:

1) If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{75}(\alpha_1)$ and $\mathscr{A}_{75}(\alpha_2)$ are not isomorphic.

2) If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{76}(\alpha_1)$ and $\mathscr{A}_{76}(\alpha_2)$ are not isomorphic.

3) If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{77}(\alpha_1)$ and $\mathscr{A}_{77}(\alpha_2)$ are not isomorphic.

4) If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{78}(\alpha_1)$ and $\mathscr{A}_{78}(\alpha_2)$ are not isomorphic.

5) If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \left\{-\frac{1}{6}, 0\right\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{79}(\alpha_1)$ and $\mathscr{A}_{79}(\alpha_2)$ are not isomorphic.

6) If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \left\{-\frac{4}{27}, 0\right\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{80}(\alpha_1)$ and $\mathscr{A}_{80}(\alpha_2)$ are not isomorphic.

7) Isomorphism conditions for the families $\mathscr{A}_{81}(\alpha, \beta)$, $\mathscr{A}_{82}(\alpha, \beta, \gamma)$, and $\mathscr{A}_{83}(\alpha, \beta)$ are hard to compute.

Lemma 7.18. *None of the Algebras in Theorem (7.17) have P1.*

Proof. $\mathscr{A}_{75}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. In M_1 we have that $[x_1, x_3] = x_4 = -[x_3, x_1]$. Note that $[\gamma x_1 + \beta x_3, \gamma x_1 + \beta x_3] = 0$, and the other elements are in center, so no squared element can give a multiple of x_4. In M_2, $[x_2, x_2] = \alpha x_5$, and $[x_2, x_3] = x_5 = -[x_3, x_2]$. So M_1 and M_2 are not isomorphic since you cannot get a squared element in element M_1 that gives a center element.

$\mathscr{A}_{76}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$ for all $\alpha \in \mathbb{C}$.

Consider the case where $\alpha = 0$. In M_1 we have $[x_1, x_1] = x_5$ and $[x_1, x_3] = x_4 = -[x_3, x_1]$. In M_2 we have $[x_2, x_3] = x_5 = -[x_3, x_2]$. There are no elements in M_2 that can multiply to give a second center element, so M_1 and M_2 are not isomorphic

Consider the case where $\alpha \neq 0$. In M_1 we have $[x_1, x_1] = x_5$ and $[x_1, x_3] = x_4 = -[x_3, x_1]$. In M_2 we have $[x_2, x_2] = \alpha x_5$, $[x_2, x_3] = x_5 = -[x_3, x_2]$. There are no elements in M_2 that can be multiplied to give the other center element, so the maximal subalgebras are not isomorphic.

$\mathscr{A}_{77}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$ and maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$. The nonzero multiplications in M_1 are given by: $[x_1, x_1] = \alpha x_5$, $[x_1, x_3] = x_4 = -[x_3, x_1]$. So $Leib(M_1) = span\{\alpha x_5\}$ since $\alpha \neq 0$, and M_1 is not Lie. The nonzero multiplications in M_2 are given by: $[x_2, x_3] = x_5 = -[x_3, x_2]$. Hence $Leib(M_2) = 0$, M_2 is a Lie algebra, and M_1 and M_2 are not isomorphic. Hence $\mathscr{A}_{77}(\alpha)$ does not have P1.

$\mathscr{A}_{78}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$ for all α. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. In M_1 we have $[x_1, x_1] = \alpha x_5$ and $[x_1, x_3] = x_4 = -[x_3, x_1]$ where α cannot be 0 by assumption. In M_2 we have $[x_2, x_2] = x_5$ and $[x_2, x_3] = x_5 = -[x_3, x_2]$. Hence, M_1 and M_2 are not isomorphic since there are no multiplications in M_2 that give a second center element

$\mathscr{A}_{79}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. In M_1 we have that $[x_1, x_1] = \alpha x_5$ and $[x_1, x_3] = x_4 = -[x_3, x_1]$, where α cannot be 0. In M_2 we have that $[x_2, x_2] = -\frac{1}{2}x_5$ and $[x_2, x_3] = x_5 = -[x_3, x_2]$. Therefore, M_1 and M_2 are not isomorphic since no multiplications in M_2 give the second center element.

$\mathscr{A}_{80}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$. The nonzero multiplications in M_1 are given by:$[x_1, x_1] = \alpha x_5$, $[x_1, x_3] = x_4 = -[x_3, x_1]$, and α cannot be 0. This gives $dim([M_1, M_1]) = 2$. We need to find a maximal subalgebra M_2 such that $dim([M_2, M_2]) = 1$. Since $[A, A] = \phi(A)$, and $\phi(A)$ is contained in all maximal subalgebras, $\phi(A) \in M_2$. Based on the calculations in $\mathscr{A}_{80}(\alpha)$, we can see that $\phi(A) = span\{x_3, x_4, x_5\}$. So we consider $M_2 =$

$span\{ax_1+bx_2,x_3,x_4,x_5\}$. We get the following nonzero multiplications:

$$\begin{aligned}[ax_1+bx_2,ax_1+bx_2] &= a^2\alpha x_5+b^2x_4+b^2x_5\\ &= b^2x_4+\left(\alpha a^2+b^2\right)x_5\\ [ax_1+bx_2,x_3] &= ax_4+bx_5\\ [x_3,ax_1+bx_2] &= -ax_4-bx_5.\end{aligned}$$

This gives a system of equations, and we need to coefficients to match so that $dim([M_2,M_2])=1$. We require that $a=b^2$. We now need $b=\alpha a^2+b^2=\alpha b^4+b^2$, which implies $\alpha b^4+b^2-b=0$. Using software, it can be shown that a value for b is attainable, and $\mathscr{A}_{80}(\alpha)$ does not have P1.

$\mathscr{A}_{81}(\alpha,\beta)$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$. The nonzero multiplications in M_1 are given by: $[x_1,x_1]=\alpha x_5$, $[x_1,x_3]=x_4=-[x_3,x_1]$, where $\alpha\neq 0$. We can see that $dim([M_1,M_1])=2$. We need to find a maximal subalgebra M_2 such that $dim([M_2,M_2])=1$. Since $[A,A]=\phi(A)$, and $\phi(A)$ is contained in all maximal subalgebras, $\phi(A)\in M_2$. Based on the calculations in $\mathscr{A}_{81}(\alpha,\beta)$, we can see that $\phi(A)=span\{x_3,x_4,x_5\}$. So we consider $M_2=span\{ax_1+bx_2,x_3,x_4,x_5\}$. We get the following nonzero multiplications:

$$\begin{aligned}[ax_1+bx_2,ax_1+bx_2] &= \alpha a^2x_5+abx_5+b^2x_4+\beta b^2x_5\\ &= b^2x_4+\left(\alpha a^2+ab+\beta b^2\right)x_5\\ [ax_1+bx_2,x_3] &= ax_4+bx_5\\ [x_3,ax_1+bx_2] &= -ax_4-bx_5.\end{aligned}$$

This gives a system of equations, and we need to coefficients to match so that $dim([M_2,M_2])=1$. We can see that it must be the case that $a=b^2$. We also need $b=\alpha a^2+ab+\beta b^2=\alpha b^4+b^3+\beta b^2$, which implies $\alpha b^4+b^3+\beta b^2-b=0$. Using software, it can be shown that a value for b is exists that makes $dim([M_2,M_2])=1$. Therefore, $\mathscr{A}_{81}(\alpha,\beta)$ does not have P1.

$\mathscr{A}_{82}(\alpha,\beta,\gamma)$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$. The nonzero multiplications are given by: $[x_1,x_1] = \alpha x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$. So $dim([M_1,M_1]) = 2$. We need to find a maximal subalgebra M_2 such that $dim([M_2,M_2]) = 1$. Since $[A,A] = \phi(A)$, and $\phi(A)$ is contained in all maximal subalgebras, $\phi(A) \in M_2$. Based on the calculations in $\mathscr{A}_{82}(\alpha,\beta,\gamma)$, we can see that $\phi(A) = span\{x_3,x_4,x_5\}$. So we consider $M_2 = span\{ax_1+bx_2,x_3,x_4,x_5\}$. We get the following nonzero multiplications:

$$\begin{aligned}[ax_1+bx_2,ax_1+bx_2] &= a^2\alpha x_5 + abx_4 + b^2x_4 + \gamma b^2 x_5 \\ &= \left(ab+b^2\right)x_4 + \left(\alpha a^2+\gamma b^2\right)x_5\end{aligned}$$

$$\begin{aligned}[ax_1+bx_2,x_3] &= ax_4+bx_5 \\ [x_3,ax_1+bx_2] &= -ax_4-bx_5.\end{aligned}$$

This gives a system of equations, and we need coefficients so that $dim([M_2,M_2]) = 1$. First, we get that $a = ab+b^2$, and so $a = b^2/(1-b)$. We now need $b = \alpha a^2+\gamma b^2 = \alpha b^2/(1-b)+\gamma b^2$, which implies $\alpha b^2/(1-b)+\gamma b^2 - b = 0$. Using software, we can solve for b that gives $dim([M_2,M_2])$. So $\mathscr{A}_{82}(\alpha,\beta,\gamma)$ does not have P1.

$\mathscr{A}_{83}(\alpha,\beta)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$. The nonzero multiplications are given by: $[x_1,x_1] = x_4+\alpha x_5$, $[x_1,x_3] = x_4 = -[x_3,x_1]$. So $dim([M_1,M_1]) = 2$. We need to find a maximal subalgebra M_2 such that $dim([M_2,M_2]) = 1$. Since $[A,A] = \phi(A)$, and $\phi(A)$ is contained in all maximal subalgebras, $\phi(A) \in M_2$. Based on the calculations in $\mathscr{A}_{83}(\alpha,\beta)$, we can see that $\phi(A) = span\{x_3,x_4,x_5\}$. So we consider $M_2 = span\{ax_1+bx_2,x_3,x_4,x_5\}$. We get the following nonzero multiplications:

$$\begin{aligned}
[ax_1+bx_2, ax_1+bx_2] &= a^2x_4+\alpha a^2x_5+ab\beta x_5+b^2x_5 \\
&= a^2x_4+\left(\alpha a^2+ab\beta+b^2\right)x_5 \\
[ax_1+bx_2, x_3] &= ax_4+bx_5 \\
[x_3, ax_1+bx_2] &= -ax_4-bx_5.
\end{aligned}$$

This gives a system of equations, and we need to coefficients to match so that $dim\left([M_2,M_2]\right)=1$. First, we get that $a=a^2$. This implies $a=0,1$. We now need $b=\alpha a^2+\beta ab+b^2$. If $a=0$, then $b=b^2$, and so $b=0,1$. Take $a=0$ and $b=1$, so $M_2=\{x_2,x_3,x_4,x_5\}$. Then

$$\begin{aligned}
[x_2,x_2] &= x_5 \\
[x_2,x_3] &= x_5.
\end{aligned}$$

The $dim\left([M_2,M_2]\right)=1$. So M_1 is not isomorphic M_2, and $\mathscr{A}_{83}\left(\alpha,\beta\right)$ does not have P1. □

Theorem 7.19. *([1], Theorem 3.7) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2)=3$, $dim(A^3)=1$, $dim(Z(A))=2=dim(Leib(A))$ *and* $Leib(A)\neq Z(A)$. *Then A is isomorphic to a Leibniz algebra spanned by* x_1,x_2,x_3,x_4,x_5 *with the nonzero products given by one of the following:*

- $\mathscr{A}_{84}$: $[x_1,x_2]=x_3+x_4$, $[x_2,x_1]=-x_3$, $[x_1,x_4]=x_5$
- $\mathscr{A}_{85}$: $[x_1,x_2]=x_3+x_4$, $[x_2,x_1]=-x_3$, $[x_2,x_2]=x_5$, $[x_1,x_4]=x_5$
- $\mathscr{A}_{86}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_1,x_4]=x_5$
- $\mathscr{A}_{87}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_2,x_2]=x_5$, $[x_1,x_4]=x_5$.

Lemma 7.20. *None of the algebras in Theorem (7.19) have P1.*

Proof. $\mathscr{A}_{84}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z\left(M_1\right)=span\{x_3,x_5\}$, and $M_1=Z_2\left(M_1\right)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, which is abelian. Hence $\mathscr{A}_{84}$ does not have P2, and so does not have P1.

$\mathscr{A}_{85}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, with $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Since $\mathscr{A}_{85}$ does not have P1, it does not have P2.

$\mathscr{A}_{86}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Since $\mathscr{A}_{86}$ does not have P2, it does not have P1.

$\mathscr{A}_{87}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $Z_2(M_2) = span\{x_1, x_3, x_4, x_5\}$. Since $\mathscr{A}_{87}$ does not have P2, it does not have P1. □

Theorem 7.21. *([1], Theorem 3.8) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$, $dim(A^3) = 1$, $dim(Z(A)) = 2 = dim(Leib(A))$ *and* $Leib(A) = Z(A)$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1, x_2, x_3, x_4, x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{88}$: $[x_1, x_1] = x_5$, $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{89}$: $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_2, x_2] = x_5$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{90}$: $[x_1, x_1] = x_5$, $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_2, x_2] = x_5$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{91}$: $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_5$, $[x_2, x_2] = x_4$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{92}$: $[x_1, x_1] = x_5$, $[x_1, x_2] = x_3 = -[x_2, x_1]$, $[x_2, x_2] = x_4$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{93}$: $[x_1, x_1] = x_5$, $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_5$, $[x_2, x_2] = x_4$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{94}(\alpha)$: $[x_1, x_1] = x_5$, $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4 + \alpha x_5$, $[x_2, x_2] = x_4$, $[x_1, x_3] = x_5 = -[x_3, x_1]$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{95}$: $[x_1, x_1] = x_4$, $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_5$, $[x_1, x_3] = x_5 = -[x_3, x_1]$

$\mathscr{A}_{96}$: $[x_1, x_1] = x_4$, $[x_1, x_2] = x_3 = -[x_2, x_1]$, $[x_2, x_2] = x_5$, $[x_1, x_3] = x_5 = -[x_3, x_1]$.

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}$ *such that* $\alpha_1 \neq \alpha_2$ *then* $\mathscr{A}_{94}(\alpha_1)$ *and* $\mathscr{A}_{94}(\alpha_2)$ *are isomorphic if and* *only if* $\alpha_2 = \frac{\alpha_1}{\alpha_1 - 1}$.

Lemma 7.22. *None of the algebras in Theorem (7.21) have P1.*

Proof. $\mathscr{A}_{88}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. As $\mathscr{A}_{88}$ does not have P2, it does not have P1.

$\mathscr{A}_{89}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Since $\mathscr{A}_{89}$ does not have P2, it does not have P1.

$\mathscr{A}_{90}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{90}$ does not have P2, it does not have P1.

$\mathscr{A}_{91}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. So $\mathscr{A}_{91}$ does not have P2, and so does not have P1.

$\mathscr{A}_{92}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{92}$ does not have P2, it does not have P1.

$\mathscr{A}_{93}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Since $\mathscr{A}_{93}$ does not have P2, it does not have P1.

$\mathscr{A}_{94}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{94}(\alpha)$ does not have P2, it does not have P1.

$\mathscr{A}_{95}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Hence $\mathscr{A}_{95}$ does not have P2, and so does not have P1.

$\mathscr{A}_{96}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Since $\mathscr{A}_{96}$ does not have P2, it does not have P1. □

Theorem 7.23. *([1], Theorem 3.9) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3 = dim(Leib(A))$, $dim(A^3) = 1$ *and* $dim(Z(A)) = 2$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1, x_2, x_3, x_4, x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{97}$: $[x_1, x_1] = x_3,\ [x_2, x_1] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{98}$: $[x_1, x_1] = x_3,\ [x_2, x_1] = x_4,\ [x_2, x_2] = x_5,\ [x_1, x_3] = x_5$

$\mathscr{A}_{99}$: $[x_1, x_1] = x_3,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{100}$: $[x_1, x_1] = x_3,\ [x_2, x_1] = x_5,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{101}$: $[x_1, x_1] = x_3,\ [x_2, x_1] = x_4,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{102}$: $[x_1, x_1] = x_3,\ [x_2, x_1] = x_4 + x_5,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{103}$: $[x_1, x_1] = x_3,\ [x_1, x_2] = x_4,\ [x_2, x_1] = x_5,\ [x_1, x_3] = x_5$

$\mathscr{A}_{104}(\alpha)$: $[x_1, x_1] = x_3,\ [x_1, x_2] = x_4,\ [x_2, x_1] = \alpha x_4,\ [x_1, x_3] = x_5,\ \alpha \in \mathbb{C} \setminus \{-1\}$

$\mathscr{A}_{105}(\alpha)$: $[x_1, x_1] = x_3,\ [x_1, x_2] = x_4,\ [x_2, x_1] = \alpha x_4,\ [x_2, x_2] = x_5,\ [x_1, x_3] = x_5,\ \alpha \in \mathbb{C} \setminus \{-1\}$

$\mathscr{A}_{106}$: $[x_1, x_2] = x_3,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{107}$: $[x_1, x_2] = x_3,\ [x_2, x_1] = x_5,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{108}$: $[x_1, x_2] = x_3,\ [x_2, x_1] = x_4,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{109}$: $[x_1, x_2] = x_3,\ [x_2, x_1] = x_4 + x_5,\ [x_2, x_2] = x_4,\ [x_1, x_3] = x_5$

$\mathscr{A}_{110}$: $[x_1, x_1] = x_4,\ [x_1, x_2] = x_3,\ [x_1, x_3] = x_5$

$\mathscr{A}_{111}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=x_5$, $[x_1,x_3]=x_5$

$\mathscr{A}_{112}$: $[x_1,x]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_5$

$\mathscr{A}_{113}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=x_5$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_5$

$\mathscr{A}_{114}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=x_4$, $[x_1,x_3]=x_5$

$\mathscr{A}_{115}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=x_4$, $[x_2,x_2]=x_5$, $[x_1,x_3]=x_5$

$\mathscr{A}_{116}(\alpha)$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=\alpha x_4$, $[x_2,x_2]=x_4$, $[x_1,x_3]=x_5$, $\alpha\in\mathbb{C}$

$\mathscr{A}_{117}(\alpha)$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3$, $[x_2,x_1]=\alpha x_4+x_5$, $[x_2,x_2]=x_4$, $[x_1,x_3]=x_5$, $\alpha\in\mathbb{C}$

Remark. (1) If $\alpha_1,\alpha_2\in\mathbb{C}\setminus\{-1\}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{104}(\alpha_1)$ and $\mathscr{A}_{104}(\alpha_2)$ are not isomorphic.

(2) If $\alpha_1,\alpha_2\in\mathbb{C}\setminus\{-1\}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{105}(\alpha_1)$ and $\mathscr{A}_{105}(\alpha_2)$ are not isomorphic.

(3) If $\alpha_1,\alpha_2\in\mathbb{C}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{116}(\alpha_1)$ and $\mathscr{A}_{116}(\alpha_2)$ are isomorphic if and only if $\alpha_2=-\alpha_1$.

(4) If $\alpha_1,\alpha_2\in\mathbb{C}$ such that $\alpha_1\neq\alpha_2$, then $\mathscr{A}_{117}(\alpha_1)$ and $\mathscr{A}_{117}(\alpha_2)$ are isomorphic if and only if $\alpha_2=-\alpha_1$.

Lemma 7.24. *None of the algebras in Theorem (7.23) have P1.*

Proof. $\mathscr{A}_{97}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_4,x_5\}$, and $Z_2(M_1)=span\{x_3,x_4,x_5\}$, and $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, which is abelian. Hence, $\mathscr{A}_{97}$ does not have P2, and so does not have P1.

$\mathscr{A}_{98}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_4,x_5\}$, and $Z_2(M_1)=span\{x_3,x_4,x_5\}$, and $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2)=span\{x_3,x_4,x_5\}$, and $M_2=Z_2(M_2)=span\{x_2,x_3,x_4,x_5\}$. As $\mathscr{A}_{98}$ does not have P2, it does not have P1.

$\mathscr{A}_{99}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_4,x_5\}$, and $Z_2(M_1)=span\{x_3,x_4,x_5\}$, and $M_1=Z_3(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2)=span\{x_3,x_4,x_5\}$, and $M_2=Z_2(M_2)=span\{x_2,x_3,x_4,x_5\}$. Since $\mathscr{A}_{99}$ does not have P2, it does not have P1.

$\mathscr{A}_{100}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{100}$ does not have P2, it does not have P1.

$\mathscr{A}_{101}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{101}$ does not have P2, and so does not have P1.

$\mathscr{A}_{102}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{102}$ does not have P2, it does not have P1.

$\mathscr{A}_{103}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Hence, $\mathscr{A}_{103}$ does not have P1.

$\mathscr{A}_{104}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Therefore, $\mathscr{A}_{104}(\alpha)$ does not have P1 for any value of α.

$\mathscr{A}_{105}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$, and $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{105}(\alpha)$ does not have P2, it does not have P1, for any value of α.

$\mathscr{A}_{106}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{106}$ does not have P2, and so it does not have P1.

$\mathscr{A}_{107}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. So $\mathscr{A}_{107}$ does not have P2, and thus does not have P1.

$\mathscr{A}_{108}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{108}$ does not have P2, and so it does not have P1.

$\mathscr{A}_{109}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{109}$ does not have P2, it does not have P1.

$\mathscr{A}_{110}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Therefore, $\mathscr{A}_{110}$ does not have P2, and so does not have P1.

$\mathscr{A}_{111}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. Hence $\mathscr{A}_{106}$ does not have P2, and so does not have P1.

$\mathscr{A}_{112}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{112}$ does not have P2, it does not have P1.

$\mathscr{A}_{113}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{113}$ does not have P2, and so it does not have P1.

$\mathscr{A}_{114}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$

and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, which is abelian. So $\mathscr{A}_{114}$ does not have P2, and thus does not have P1.

$\mathscr{A}_{115}$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{115}$ does not have P2, and it does not have P1.

$\mathscr{A}_{116}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Hence $\mathscr{A}_{116}(\alpha)$ does not have P2, and so it does not have P1 for any value of α.

$\mathscr{A}_{117}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_4, x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. Thus $\mathscr{A}_{117}(\alpha)$ does not have P2, and so does not have P1 for any value of α. □

Theorem 7.25. *([1], Theorem 3.10) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $dim(A^2) = 3$, $dim(A^3) = 1 = dim(Z(A))$ and $dim(Leib(A)) = 2$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathscr{A}_{118}$: $[x_1, x_2] = -x_3 + x_4$, $[x_2, x_1] = x_3$, $[x_2, x_3] = x_5 = -[x_3, x_2]$, $[x_1, x_4] = x_5$

$\mathscr{A}_{119}$: $[x_1, x_2] = -x_3 + x_4$, $[x_2, x_1] = x_3$, $[x_2, x_2] = x_5$, $[x_2, x_3] = x_5 = -[x_3, x_2]$, $[x_1, x_4] = x_5$

$\mathscr{A}_{120}(\alpha)$: $[x_1, x_2] = -x_3 = [x_2, x_1]$, $[x_2, x_2] = x_4$, $[x_2, x_3] = -\alpha x_5$, $[x_3, x_2] = (\alpha - 1)x_5$, $[x_1, x_4] = x_5$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{121}(\alpha)$: $[x_1, x_2] = -x_3 + x_4$, $[x_2, x_1] = x_3$, $[x_2, x_2] = x_4$, $[x_2, x_3] = -\alpha x_5$, $[x_3, x_2] = (\alpha - 1)x_5$, $[x_1, x_4] = x_5$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{122}$: $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_3, x_1] = x_5$, $[x_1, x_4] = x_5$

$\mathscr{A}_{123}$: $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_2, x_2] = x_5$, $[x_3, x_1] = x_5$, $[x_1, x_4] = x_5$

$\mathscr{A}_{124}$: $[x_1, x_2] = x_3$, $[x_2, x_1] = -x_3 + x_4$, $[x_3, x_1] = x_5$, $[x_2, x_3] = x_5 = -[x_3, x_2]$, $[x_1, x_4] = x_5$

$\mathscr{A}_{125}$: $[x_1,x_2]=x_3$, $[x_2,x_1]=-x_3+x_4$, $[x_2,x_2]=x_5$, $[x_3,x_1]=x_5$, $[x_2,x_3]=x_5=-[x_3,x_2]$, $[x_1,x_4]=x_5$

$\mathscr{A}_{126}(\alpha)$: $[x_1,x_2]=x_3$, $[x_2,x_1]=-x_3+x_4$, $[x_2,x_2]=x_4$, $[x_3,x_1]=x_5$, $[x_2,x_3]=\alpha x_5$, $[x_3,x_2]=(1-\alpha)x_5$, $[x_1,x_4]=x_5$, $\alpha \in \mathbb{C}\mathbb{C}/\{0\}$

$\mathscr{A}_{127}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_2,x_3]=x_5=-[x_3,x_2]$, $[x_1,x_4]=x_5$

$\mathscr{A}_{128}$: $[x_1,x_1]=x_4$, $[x_1,x_2]=x_3=-[x_2,x_1]$, $[x_2,x_2]=x_5$, $[x_2,x_3]=x_5=-[x_3,x_2]$, $[x_1,x_4]=x_5$.

Remark. (1) If $\alpha_1,\alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{120}(\alpha_1)$ and $\mathscr{A}_{120}(\alpha_2)$ are not isomorphic.

(2) If $\alpha_1,\alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{121}(\alpha_1)$ and $\mathscr{A}_{121}(\alpha_2)$ are not isomorphic.

(3) If $\alpha_1,\alpha_2 \in \mathbb{C}/\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{126}(\alpha_1)$ and $\mathscr{A}_{126}(\alpha_2)$ are not isomorphic.

Lemma 7.26. *None of the algebras in Theorem (7.25) have P1.*

Proof. $\mathscr{A}_{118}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. The nonzero multiplication in M_1 is given by: $[x_1,x_4]=x_5$. The nonzero multiplications in M_2 are given by: $[x_2,x_3]=x_5=-[x_3,x_2]$. We can see that the multiplications in M_2 are symmetric, while in M_1 they are not, and so M_1 and M_2 are not isomorphic, and $\mathscr{A}_{118}$ does not have P1.

$\mathscr{A}_{119}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_1+x_2,x_3,x_4,x_5\}$. The nonzero multiplication in M_1 is given by $[x_1,x_4]=x_5$. Note that $[ax_1+bx_4,ax_1+bx_4]=ab[x_1,x_4]=abx_5$, so $dim([M_1,M_1])=1$. The nonzero multiplications in M_2 are given by: $[x_1+x_2,x_1+x_2]=x_4+x_5$, $[x_1+x_2,x_3]=x_5$, $[x_3,x_1+x_2]=-x_5$, $[x_1+x_2,x_4]=x_5$. From this $dim([M_2,M_2])=2$. Thus M_1 and M_2 are not isomorphic, and so $\mathscr{A}_{119}$ does not have P1.

$\mathscr{A}_{120}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$ and maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$. The nonzero multiplication in M_1 given be $[x_1,x_4]=x_5$. Note that $[ax_1+bx_4,ax_1+bx_4]=abx_5$, and so $dim(Leib(M_1))=1$. The nonzero multiplications in M_2, are given by: $[x_2,x_2]=x_4$ and $[x_2,x_3]=-\alpha x_5$, $[x_3,x_2]=(\alpha-1)x_5$. Note that

$$\begin{aligned}[ax_2+bx_3,ax_2+bx_3] &= a^2x_4-\alpha abx_5+(\alpha-1)\,abx_5\\ &= a^2x_4-\alpha abx_5+\alpha abx_5-abx_5\\ &= a^2x_4-abx_5.\end{aligned}$$

From this, $dim(Leib(M_2))=2$. Therefore M_1 is not isomorphic to M_2, and $\mathscr{A}_{120}(\alpha)$ does not have P1.

$\mathscr{A}_{121}(\alpha)$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$ and take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$. In M_1, we have nonzero multiplications given by $[x_1,x_4]=x_5$. Note that $[ax_1+bx_4,ax_1+bx_4]=abx_5$. So $dim(Leib(M_1))=1$. In M_2, we have nonzero multiplications given by $[x_2,x_2]=x_4$ and $[x_2,x_3]=-\alpha x_5$, $[x_3,x_2]=(\alpha-1)x_5$. Now

$$\begin{aligned}[ax_2+bx_3,ax_2+bx_3] &= a^2x_4-\alpha abx_5+(\alpha-1)\,abx_5\\ &= a^2x_4-\alpha abx_5+\alpha abx_5-abx_5\\ &= a^2x_4-abx_5.\end{aligned}$$

From this, $dim(Leib(M_2))=2$. Therefore M_1 is not isomorphic to M_2, and $\mathscr{A}_{121}(\alpha)$ does not have P1.

$\mathscr{A}_{122}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$ and $M_1=Z_2(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, which is abelian. Hence $\mathscr{A}_{122}$ does not have P2, and so does not have P1.

$\mathscr{A}_{123}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$ and $M_1=Z_2(M_1)=span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2=span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2)=span\{x_3,x_4,x_5\}$ and $M_2=Z_2(M_2)=span\{x_2,x_3,x_4,x_5\}$. As $\mathscr{A}_{123}$ does not have P2, it does not have P1.

$\mathscr{A}_{124}$: Take maximal subalgebra $M_1=span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1)=span\{x_5\}$ and

$M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Since $\mathscr{A}_{124}$ does not have P2, it does not have P1.

$\mathscr{A}_{125}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{125}$ does not have P2, it does not have P1.

$\mathscr{A}_{126}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. As $\mathscr{A}_{126}(\alpha)$ does not have P2 for any value of $\alpha \in \mathbb{C}\setminus\{0\}$, it does not have P1.

$\mathscr{A}_{127}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_5\}$, $Z_2(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. So $\mathscr{A}_{127}$ does not have P2, it does not have P1.

$\mathscr{A}_{128}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_5\}$, $Z_2(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Hence $\mathscr{A}_{128}$ does not have P2, and so it does not have P1. □

Theorem 7.27. *([1], Theorem 3.11) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3 = dim(Leib(A))$ *and* $dim(A^3) = 1 = dim(Z(A))$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4,x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{129}(\alpha)$: $[x_1,x_1] = x_4$, $[x_1,x_2] = \alpha x_4$, $[x_2,x_1] = x_3$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{130}$: $[x_1,x_1] = x_4$, $[x_1,x_2] = -x_4$, $[x_2,x_1] = -x_3$, $[x_2,x_2] = x_3$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$

$\mathscr{A}_{131}(\alpha,\beta)$: $[x_1,x_1] = x_4$, $[x_1,x_2] = \alpha x_4$, $[x_2,x_1] = \beta x_3$, $[x_2,x_2] = x_3$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha,\beta \in \mathbb{C}$, $\alpha\beta \neq 1$

$\mathscr{A}_{132}(\alpha)$: $[x_1,x_2] = x_3$, $[x_2,x_1] = \alpha x_3$, $[x_2,x_2] = x_4$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha \in \mathbb{C}/\{-1,0\}$

$\mathscr{A}_{133}(\alpha,\beta)$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3 + \alpha x_4$, $[x_2,x_1] = \beta x_3$, $[x_2,x_2] = x_4$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}/\{-1\}$.

$\mathscr{A}_{134}(\alpha,\beta,\gamma)$: $[x_1,x_1] = \alpha x_4$, $[x_1,x_2] = x_3 + \beta x_4$, $[x_2,x_1] = \gamma x_3$, $[x_2,x_2] = x_4$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha,\beta,\gamma \in \mathbb{C}$

$\mathscr{A}_{135}(\alpha,\beta)$: $[x_1,x_1] = x_3 + \alpha x_4$, $[x_1,x_2] = x_3 + \beta x_4$, $[x_2,x_1] = -x_3 + x_4$, $[x_2,x_2] = x_4$, $[x_2,x_3] = x_5$, $[x_1,x_4] = x_5$, $\alpha,\beta \in \mathbb{C}$.

Remark.(1) If $\alpha_1,\alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{129}(\alpha_1)$ and $\mathscr{A}_{129}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.

(2) If $\alpha_1,\alpha_2 \in \mathbb{C}/\{-1,0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathscr{A}_{132}(\alpha_1)$ and $\mathscr{A}_{132}(\alpha_2)$ are not isomorphic.

(3) Isomorphism conditions for the families $\mathscr{A}_{131}(\alpha,\beta)$, $\mathscr{A}_{133}(\alpha,\beta)$, $\mathscr{A}_{134}(\alpha,\beta,\gamma)$ and $\mathscr{A}_{135}(\alpha,\beta)$ are hard to compute.

Lemma 7.28. *None of the algebras in Theorem (7.27) have P1.*

Proof. $\mathscr{A}_{129}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_5\}$, $Z_2(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_4,x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2,x_3,x_4,x_5\}$. Since $\mathscr{A}_{129}(\alpha)$ does not have P2 for any value of α, it does not have P1.

$\mathscr{A}_{130}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_5\}$, $Z_2(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_1+x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_5\}$ and $M_2 = Z_2(M_2) = span\{x_1+x_2,x_3,x_4,x_5\}$. As $\mathscr{A}_{130}$ does not have P1, it does not have P2.

$\mathscr{A}_{131}(\alpha,\beta)$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_5\}$, $Z_2(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_1+x_2,x_3,x_4,x_5\}$. Note that $[x_1+x_2,x_1+x_2] = x_4 + \alpha x_4 + \beta x_3 + x_3$. This is equal to 0 if and only if $\alpha = -1 = \beta$, which implies $\alpha\beta = 1$, a contradiction, and so

$[x_1+x_2, x_1+x_2] \neq 0$, and $[x_1+x_2, x_1+x_2] \in span\{x_3, x_4\}$. So $Z(M_2) = span\{x_5\}$, $Z_2(M_2) = span\{x_3, x_4, x_5\}$, and $M_2 = Z_3(M_2) = span\{x_1+x_2, x_3, x_4, x_5\}$. Therefore $\mathscr{A}_{131}(\alpha, \beta)$ does not have P1 for any value of α, β, and so it does not have P2.

$\mathscr{A}_{132}(\alpha)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$ and maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$. The nonzero multiplication in M_1 is given by: $[x_1, x_4] = x_5$. Note that $[ax_1+bx_4, ax_1+bx_4] = abx_5$, and so $dim(Leib(M_1)) = 1$. The nonzero multiplications in M_2 are given by: $[x_2, x_2] = x_4$, $[x_2, x_3] = x_5$. Now $[ax_2+bx_3, ax_2+bx_3] = a^2x_4+abx_5$. From this, $dim(Leib(M_2)) = 2$. So M_1 and M_2 are not isomorphic, and $\mathscr{A}_{132}(\alpha)$ does not have P1.

$\mathscr{A}_{133}(\alpha, \beta)$: Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. As $\mathscr{A}_{130}$ does not have P2 for any values of α, β, it does not have P1.

$\mathscr{A}_{134}(\alpha, \beta, \gamma)$: Assume $\alpha \neq 0$. Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. In this case, $\mathscr{A}_{134}(\alpha, \beta, \gamma)$ does not have P2, and so does not have P1. Now assume $\alpha = 0$. Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$ and maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$. The nonzero multiplication in M_1 is given by: $[x_1, x_4] = x_5$, and note that $[ax_1+bx_4, ax_1+bx_4] = abx_5$. So $dim(Leib(M_1)) = 1$. The nonzero multiplications in M_2 are given by: $[x_2, x_2] = x_4$, $[x_2, x_3] = x_5$. Now $[ax_2+bx_3, ax_2+bx_3] = a^2x_4+abx_5$. We get that $dim(Leib(M_2)) = 2$. Hence, M_1 is not isomorphic to M_2 when $\alpha = 0$. These results combined give that $\mathscr{A}_{134}(\alpha, \beta, \gamma)$ does not have P1.

$\mathscr{A}_{135}(\alpha, \beta)$: Assume $\alpha \neq 0$. Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) = span\{x_3, x_5\}$, $Z_2(M_1) = span\{x_3, x_4, x_5\}$, and $M_1 = Z_3(M_1) = span\{x_1, x_3, x_4, x_5\}$. Now take maximal subalgebra $M_2 = span\{x_2, x_3, x_4, x_5\}$, with $Z(M_2) = span\{x_4, x_5\}$ and $M_2 = Z_2(M_2) = span\{x_2, x_3, x_4, x_5\}$. In this case, $\mathscr{A}_{135}(\alpha, \beta)$ does not have P2, and so does not have P1.

Now assume $\alpha = 0$. Take maximal subalgebra $M_1 = span\{x_1, x_3, x_4, x_5\}$, with $Z(M_1) =$

$span\{x_3,x_5\}$ and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{x_1+x_2,x_3,x_4,x_5\}$, with $Z(M_2) = span\{x_5\}$, $Z_2(M_2) = span\{x_3,x_4,x_5\}$, and $M_2 = Z_3(M_2) = span\{x_1+x_2,x_3,x_4,x_5\}$. So $\mathscr{A}_{135}(\alpha,\beta)$ does not have P2, and so does not have P1, when $\alpha = 0$. Combining the above results, we get that $\mathscr{A}_{135}(\alpha,\beta)$ does not have P1. □

Theorem 7.29. *([1], Theorem 3.12) Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$, $dim(A^3) = 0$ *and* $dim(Leib(A)) = 2$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1,x_2,x_3,x_4,x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{136}$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3$, $[x_2,x_1] = -x_3+x_4+x_5$, $[x_2,x_2] = x_5$

$\mathscr{A}_{137}$: $[x_1,x_1] = x_4$, $[x_1,x_2] = x_3 = -[x_2,x_1]$, $[x_2,x_2] = x_5$.

Lemma 7.30. *From Theorem (7.29), the algebra* $\mathscr{A}_{136}$ *does not have P1, while* $\mathscr{A}_{137}$ *does have P1.*

Proof. $\mathscr{A}_{136}$: Take maximal subalgebra $M_1 = span\{x_1,x_3,x_4,x_5\}$, with $Z(M_1) = span\{x_3,x_4,x_5\}$, and $M_1 = Z_2(M_1) = span\{x_1,x_3,x_4,x_5\}$. Now take maximal subalgebra $M_2 = span\{2x_1-2x_2,x_3,x_4,x_5\}$, which is abelian since

$$[2x_1-2x_2,2x_1-2x_2] = 4x_4-4x_3-4(-x_3+x_4+x_5)+4x_5$$
$$= 0.$$

Hence $\mathscr{A}_{136}$ does not have P2, and so does not have P1.

$\mathscr{A}_{137}$: Note that all maximal subalgebras must contain $\phi(A) = [A,A] = span\{x_3,x_4,x_5\} = Z(A)$. So all maximal subalgebras are of the form $M = span\{y,x_3,x_4,x_5\}$, where $y = ax_1+bx_2+cx_3+dx_4+ex_5$, where at least one of $a,b \neq 0$, as otherwise M is not maximal. It must be the case that $Z(M) \subseteq span\{x_3,x_4,x_5\}$. Now

$$[ax_1+bx_2+cx_3+dx_4+ex_5,ax_1+bx_2+cx_3+dx_4+ex_5] = a^2x_4+b^2x_5 = 0$$

if and only if $a = b = 0$, in which case, M is not maximal, and so $y^2 \neq 0$. This implies $\mathscr{A}_{137}$ must have P2.

Consider two distinct maximal subalgebras $M_1 = span\{a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + e_1x_5, x_3, x_4, x_5\}$ and $M_2 = span\{a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 + e_2x_5, x_3, x_4, x_5\}$. We need to define a homomorphism from M_1 to M_2. The homomorphism is dependent upon which values of a_1, b_1, a_2, and b_2 are 0. However, since the homomorphism is a linear mapping, we do not need to be concerned about c_i, d_i or e_i, for $i = 1,2$.

First consider the case where $a_1, b_1, a_2, b_2 \neq 0$. Define $\phi : M_1 \to M_2$ by

$$\phi(a_1x_1 + b_1x_2) = a_2x_1 + b_2x_2$$
$$\phi(x_3) = x_3$$
$$\phi(x_4) = \frac{a_2^2}{a_1^2}x_4$$
$$\phi(x_5) = \frac{b_2^2}{b_1^2}x_5$$

Note that x_3, x_4, and x_5 are all elements of the center mapping to other center elements, so ϕ will satisfy $\phi([x_i, x]) = [\phi(x_i), \phi(x)]$, where $i = 3,4,5$, and $x \in M_1$. We need to only check the following map:

$$\begin{aligned}
\phi([a_1x_1 + b_1x_2, a_1x_1 + b_1x_2]) &= \phi\left(a_1^2x_4 + b_1^2x_5\right) \\
&= a_1^2\phi(x_4) + b_1^2\phi(x_5) \\
&= a_1^2\left(\frac{a_2^2}{a_1^2}x_4\right) + b_1^2\left(\frac{b_2^2}{b_1^2}x_5\right) \\
&= a_2^2x_4 + b_2^2x_5
\end{aligned}$$

and

$$\begin{aligned}
[\phi(a_1x_1 + b_1x_2), \phi(a_1x_1 + b_1x_2)] &= [a_2x_1 + b_2x_2, a_2x_1 + b_2x_2] \\
&= a_2^2x_4 + b_2^2x_5.
\end{aligned}$$

Hence, ϕ is a homomorphism. Since the mapping is clearly onto, and both maximal algebras have the same dimension, we have that ϕ is an isomorphism.

Next, consider the case where one of $a_1, b_1, a_2, b_2 \neq 0$. Without loss of generality, we can consider $a_1, a_2, b_2 \neq 0$, and $b_1 = 0$. This is because we can define the map from M_1 to M_2, or vice versa, and a similar map can be defined if only $a_1 = 0$. Same as before, since the homomorphism is a linear mapping, we do not need to be concerned about c_i, d_i or e_i, for $i = 1, 2$. Define $\phi : M_1 \to M_2$ by

$$\begin{aligned}
\phi(a_1x_1) &= a_2x_1 + b_2x_2 \\
\phi(x_3) &= x_3 \\
\phi(x_4) &= \frac{a_2^2}{a_1^2}x_4 + \frac{b_2^2}{a_1^2}x_5 \\
\phi(x_5) &= x_5
\end{aligned}$$

Just as before, since x_3, x_4, and x_5 are in the center, we only need to be concerned with the following calculations:

$$\begin{aligned}
\phi([a_1, x_1, a_1x_1]) &= \phi\left(a_1^2x_4\right) \\
&= a_1^2\left(\frac{a_2^2}{a_1^2}x_4 + \frac{b_2^2}{a_1^2}x_5\right) \\
&= a_2^2x_4 + b_2^2x_5
\end{aligned}$$

and

$$\begin{aligned}
[\phi(a_1x_1), \phi(a_1x_1)] &= [a_2x_1 + b_2x_2, a_2x_1 + b_2x_2] \\
&= a_2^2x_4 + b_2^2x_5.
\end{aligned}$$

Therefore, ϕ is a homomorphism. We can see that this map is onto since $\phi\left(\frac{a_1^2}{a_2^2}x_4 - \frac{b_2^2}{a_1^1}x_5\right) =$

$\frac{a_2^2}{a_1^2}\left(\frac{a_1^2}{a_2^2}x_4\right)+\frac{b_2^2}{a_1^2}x_5-\frac{b_2^2}{a_1^2}x_5=x_4$. Since M_1 and M_2 are of the same dimension, ϕ is also one-to-one, and so isomorphic.

Finally, without loss of generality, we consider the case where $a_1,b_2\neq 0$. This is sufficient since the map can defined from M_1 to M_2, or vice versa. Again, since the homomorphism is a linear mapping, we do not need to be concerned about c_i, d_i or e_i, for $i=1,2$. Define the map $\phi : M_1 \to M_2$ by

$$\begin{aligned}\phi\left(a_1x_1\right)&=b_2x_2\\ \phi\left(x_3\right)&=x_3\\ \phi\left(x_4\right)&=\frac{b_2^2}{a_1^2}x_5\\ \phi\left(x_5\right)&=x_4.\end{aligned}$$

Just as before, since x_3, x_4, and x_5 are in the center, we only need to be concerned with the following calculations:

$$\begin{aligned}\phi\left(\left[a_1x_1,a_1x_1\right]\right)&=\phi\left(a_1^2x_4\right)\\ &=a_1^2\left(\frac{b_2^2}{a_1^2}x_5\right)\\ &=b_2^2x_5\end{aligned}$$

and

$$\begin{aligned}\left[\phi\left(a_1x_1\right),\phi\left(a_1x_1\right)\right]&=\left[b_2x_2,b_2x_2\right]\\ &=b_2^2x_5.\end{aligned}$$

As above, we get this map is an isomorphism. Combining all of the above results, we get that M_1 and M_2 are isomorphic. □

Theorem 7.31. *([1], Theorem 3.13). Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $dim(A^2) = 3$, $dim(A^3) = 0$ *and* $dim(Leib(A)) = 3$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1, x_2, x_3, x_4, x_5\}$ *with the nonzero products given by one of the following:*

$\mathscr{A}_{138}(\alpha)$: $[x_1, x_1] = x_4$, $[x_1, x_2] = \alpha x_4 + x_5$, $[x_2, x_1] = x_3$, $[x_2, x_2] = x_5$, $\alpha \in \mathbb{C}$

$\mathscr{A}_{139}$: $[x_1, x_1] = x_4$, $[x_1, x_2] = x_3$, $[x_2, x_1] = x_3$, $[x_2, x_2] = x_5$

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}$ *such that* $\alpha_1 \neq \alpha_2$, *then* $\mathscr{A}_{138}(\alpha_1)$ *and* $\mathscr{A}_{138}(\alpha_2)$ *are not isomorphic.*

Lemma 7.32. *The algebras in Theorem (7.31) have P1.*

Proof. $\mathscr{A}_{138}(\alpha)$: Since $\phi(A)$ is contained in all maximal subalgebras, and $[A, A] = \phi(A)$, we know all maximal subalgebras must contain $\phi(A) = span\{x_3, x_4, x_5\}$. Take $M_1 = span\{x_1, x_3, x_4, x_5\}$ and a generic maximal subalgebra $M = span\{ax_1 + bx_2, x_3, x_4, x_5\}$. Consider the following multiplication tables:

Table 17: M_1 Multiplication Table

	M_1 multiplication table			
$[\cdot,\cdot]$	x_1	x_3	x_4	x_5
x_1	x_4	0	0	0
x_3	0	0	0	0
x_4	0	0	0	0
x_5	0	0	0	0

Table 18: M Multiplication Table

	M multiplication table			
$[\cdot,\cdot]$	$ax_1 + bx_2$	x_3	x_4	x_5
$ax_1 + bx_2$	$a^2x_4 + \alpha abx_4 + abx_5 + abx_3 + b^2x_5$	0	0	0
x_3	0	0	0	0
x_4	0	0	0	0
x_5	0	0	0	0

If we consider a linear mapping $\psi : M_1 \to M$, we would map $\psi(x_1) = ax_1 + bx_2$ and $\psi(x_4) = a^2x_4 + \alpha abx_4 + abx_5 + abx_3 + b^2x_5$, $\psi(x_3) = x_3$, and $\psi(x_5) = x_5$. Upon evaluating $\psi([\cdot,\cdot]) =$

$[\psi(\cdot), \psi(\cdot)]$, it can be seen that the tables are the same, and M_1 is isomorphic to M. Therefore, $\mathscr{A}_{138}(\alpha)$ has P1.

$\mathscr{A}_{139}$: Since $\phi(A)$ is contained in all maximal subalgebras, and $[A,A] = \phi(A)$, we know all maximal subalgebras must contain $\phi(A) = span\{x_3, x_4, x_5\}$. Take $M_1 = span\{x_1, x_3, x_4, x_5\}$ and a generic maximal subalgebra $M = span\{ax_1 + bx_2, x_3, x_4, x_5\}$. Consider the following multiplication tables:

Table 19: M_1 Multiplication Table

M_1 mult table				
$[\cdot,\cdot]$	x_1	x_3	x_4	x_5
x_1	x_4	0	0	0
x_3	0	0	0	0
x_4	0	0	0	0
x_5	0	0	0	0

Table 20: M Multiplication Table

	M mult table			
$[\cdot,\cdot]$	$ax_1 + bx_2$	x_3	x_4	x_5
$ax_1 + bx_2$	$a^2x_4 + 2abx_3 + b^2x_5$	0	0	0
x_3	0	0	0	0
x_4	0	0	0	0
x_5	0	0	0	0

If we consider a linear mapping $\psi : M_1 \to M$, which maps $\psi(x_1) = ax_1 + bx_2$ and $\psi(x_4) = a^2x_4 + 2abx_3 + b^2x_5$, $\psi(x_3) = x_3$, and $\psi(x_5) = x_5$. Upon evaluating $\psi([\cdot,\cdot]) = [\psi(\cdot), \psi(\cdot)]$, it can be seen that the tables are the same, and M_1 is isomorphic to M. Therefore, $\mathscr{A}_{139}$ has P1. $\square$

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