ABSTRACT

DINKINS, ERICA CHRISTINE SWAIN. $C_n^{(1)}$ Geometric Crystal Corresponding to Dynkin Node $i = n$ and its Ultra-Discretization for $n = 2, 3, 4$. (Under the direction of Kailash Misra).

Kac-Moody algebras were introduced by Victor Kac and Robert Moody in 1967 as a generalization of finite dimensional semisimple Lie algebras. There are 3 types of Kac-Moody algebras: finite, affine, and indefinite. Affine Lie algebras have been completely classified, and one family of affine Lie algebras is $C_n^{(1)}$. In 1985, Vladimir Drinfeld and Michio Jimbo introduced quantum groups, which are deformations of the universal enveloping algebras of Lie algebras. In 1990 Masaki Kashiwara introduced the notion of a crystal base, a parametrization of a basis of a certain module of a quantum group which is a powerful combinatorial tool. A particular type of crystal called a perfect crystal was introduced by Seok-Jin Kang, Kashiwara, Kailash Misra, Tetsuji Miwa, Toshiki Nakashima, and Atsushi Nakayashiki in 1992. Perfect crystals naturally arise as the crystals associated with Kirillov-Reshetikhin modules of quantum affine algebras. A family of perfect crystals $\{B^l\}_{l\geq 1}$ is called a coherent family of perfect crystals if it has a projective limit B^∞.

In 2000 Arkady Berenstein and David Kazhdan introduced the notion of a geometric crystal, which is composed of a variety along with $\mathbb{C}^\times$ actions e_i and rational functions γ_i, ε_i which satisfy certain relations. Geometric crystals are a geometric analogue of Kashiwara crystals. The ultradiscretization (or tropicalization) functor $\mathcal{UD}$ maps the category of positive geometric crystals to the category of Kashiwara crystals. In 2008, Kashiwara, Nakashima, and Okado made a conjecture that for each affine Lie algebra $\mathfrak{g}$ there exists a unique variety X endowed with a positive geometric crystal structure corresponding to each nonzero Dynkin node and the ultra-discretization of X is isomorphic to the limit of a coherent family of perfect crystals associated with the Langlands dual $\mathfrak{g}^L$. This conjecture has been proven to be true for Dynkin index $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ by Kashiwara, Nakashima and Okado, $\mathfrak{g} = D_4^{(3)}$ by Igarashi, Misra and Nakashima, and $\mathfrak{g} = G_2^{(1)}$ by Nakashima. For Dynkin index $i > 1$, the conjecture has been proven for $A_n^{(1)}$ by Misra and Nakashima. For Dynkin index $i = n$ this conjecture has been proven for $D_5^{(1)}$ by Igarashi, Misra, and Pongprasert and $D_6^{(1)}$ [27, 28] by Misra and Prongprasert. In this thesis, we will present the proof of Kashiwara, Nakashima, and Okado's conjecture for Dynkin index $i = n$, $\mathfrak{g} = C_n^{(1)}$, and $n = 2, 3, 4$.

$C_n^{(1)}$ Geometric Crystal Corresponding to Dynkin Node $i = n$ and its Ultra-Discretization for $n = 2, 3, 4$

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BIOGRAPHY

Erica Swain Dinkins was born in Houston, Texas in 1996. She graduated from Houston Christian High School in 2014. She later earned a Bachelor of Science degree in Mathematics from Wheaton College in 2017. She attended North Carolina State University beginning in 2017 and earned her Master of Science in Mathematics in 2019, before completing her Doctor of Philosophy in Mathematics with a minor in Operations Research in 2022. She will move to Boston to start a preceptor position at Harvard University in the fall.

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INTRODUCTION

In the 1870s Sophus Lie discovered what later became known as Lie algebras. Lie was studying local group actions on a space, and discovered that by studying infinitesimal actions, he could reconstruct the entire action. This approach led to the discovery of Lie algebras which are formed from these infinitesimal actions. Lie algebras are linear, so they are easier to work with while still giving information about the associated Lie group. Both Lie algebras and Lie groups study the symmetries of a space, which can reveal helpful information about the space's geometry, which was Lie's original purpose. In the 1890s Wilhelm Killing classified all simple Lie algebras, and this was later made rigorous by Elie Cartan in the following years [10]. He used a maximal nilpotent subalgebra in the work of classification, and this was later named the Cartan subalgebra. The simple Lie algebras over $\mathbb{C}$ can be classified into 4 classical types A_n, B_n, C_n, D_n for $n \geq 1$ along with the exceptional cases E_6, E_7, E_8, F_4, G_2 [10]. The classical types correspond to the Lie groups $\mathrm{SU}(n+1)$, $\mathrm{SO}(2n+1)$, $\mathrm{SP}(2n)$, and $\mathrm{SO}(2n)$[3]. Claude Chevalley and Jean-Pierre Serre showed that any semisimple Lie algebra can be expressed in terms of generators and relations. Eugene Dynkin associated each semisimple Lie algebra with a graph known as a Dynkin diagram [10].

The study of representations of semisimple Lie algebras began in the 1890s onward, and Cartan determined the finite dimensional irreducible representations of simple Lie algebras [3]. These can be characterized based on their highest weights, which are integral eigenvalues of the Cartan subalgebra. Herman Weyl discovered a number of important results in the representation theory of Lie algebras during the 1920s and 1930s, including his theorem of complete reducibility that shows it is often sufficient to study irreducible representations [1]. Representation theory became an important field because it studies how symmetries act on spaces, which has important applications in many areas, especially in physics.

Simultaneously, Robert Moody and Victor Kac discovered a generalization of finite dimensional semisimple Lie algebras in 1967, now known as Kac-Moody algebras [29, 14]. A Kac-Moody algebra $\mathfrak{g}$ along with index set I is constructed using a generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$ where $a_{ii}=0$, $a_{ij}\leq 0$ and $a_{ij}=0$ if and only if $a_{ji}=0$ [14]. These algebras are defined using the generalized Cartan matrix and some additional data called the Cartan datum [14]. They can be classified into 3 types: finite, affine and indefinite. In this paper we will focus on Kac-Moody algebras of affine type, denoted $\mathfrak{g}$. Affine algebras have a one dimensional center and they have been completely classified [14]. Affine algebras also have a particular index set, $I=\{0,\ldots,n\}$, and there is a Dynkin node in their Dynkin diagram corresponding to each element in the index set I. In this paper we will focus on the affine Lie algebra $C_n^{(1)}$. The representation theory of Kac-Moody algebras developed with many similar results as the theory for finite dimensional semisimple Lie algebras. In chapter 2 we will review many of the basic definitions and facts related to Lie algebras, Kac Moody algebras, and their representations.

In 1985, Vladimir Drinfeld and Michio Jimbo discovered quantum groups, which are deformations of the universal enveloping algebra of Kac-Moody algebras [4, 13]. These arise naturally in certain lattice models in quantum physics. In 1990, Lusztig presented a basis for quantum group modules called the canonical base [24]. In the same year, Kashiwara introduced the notion of a crystal base, which is a parametrization of the basis of a quantum group module [18]. Later, Kashiwara also developed a basis for quantum group modules that is equivalent to Lusztig's canonical base, called a global base [17]. For a dominant weight λ of level $\lambda(\mathbf{c})$ where $\mathbf{c}$ is the canonical central element, a crystal base is a pair $(\mathcal{L}(\lambda),\mathcal{B}(\lambda))$ associated with the $U_q(\mathfrak{g})$ module $V^q(\lambda)$ [18], and $\mathcal{B}(\lambda)$ is called a crystal. Crystals are powerful combinatorial tools that simplify the computation of tensor product decomposition and the computations of the dimension of weight spaces. Therefore, they are important in the study of representation theory of quantum groups. In Chapter 3 we recall many of the basic definitions and facts related to quantum groups and crystal bases.

In order to give an explicit description of crystals corresponding to quantum affine algebras, perfect crystals were introduced in [16]. They arise when computing the path realization of crystals of integrable highest weight $U_q(\mathfrak{g})$ modules. The path realization is a semi-infinite tensor product, where the components are elements of a perfect crystal B^l. They also arise naturally as the crystals associated with Kirillov-Reshetikhin modules (KR-modules) of the quantum affine algebras $U_q(\mathfrak{g})$ [8, 7, 22]. A family of perfect crystals $\{B^l\}_{l\geq 1}$ is called a coherent family of perfect crystals if it has a limit B^∞. We will review the basic facts and definitions related to perfect crystals and their limits in Chapter 4.

Geometric crystals were introduced by Arkady Berenstein and David Kazhdan in 1999. They are a geometric analogue of Kashiwara crystals. They were first defined over reductive groups in [2] and later extended to Kac Moody groups in [30]. Geometric crystals consist of a variety V along with $\mathbb{C}^\times$-actions $e_i : \mathbb{C} \times V \longrightarrow V$, and rational functions $\gamma_i, \varepsilon_i : V \longrightarrow \mathbb{C}$ $(i \in I)$ that satisfy certain relations. A positive geometric crystal is a geometric crystal with a positive structure. Roughly speaking, this means that the rational functions are ratios of polynomials with positive coefficients. In Chapter 5 we will present the definition of a geometric crystal and important related facts.

Since geometric crystals are an analogue of Kashiwara crystals, a connection between them might be expected. The ultradiscretization (or tropicalization) functor $\mathcal{UD}$ maps the category of positive geometric crystals to the category of Kashiwara crystals. $\mathcal{UD}$ sends rational functions to piecewise linear functions via:

$$x \times y \longrightarrow x + y,\ \frac{x}{y} \longrightarrow x - y,\ x + y \longrightarrow \max\{x, y\}$$

In 2008, Kashiwara, Nakashima, and Okado made a conjecture [20] that

1. For any Dynkin node $i \in I/\{0\}$, there exists a unique variety X endowed with a positive $\mathfrak{g}$-geometric crystal structure.

2. The ultra-discretization of X is isomorphic to the limit of a coherent family of perfect crystals associated with the Langlands dual $\mathfrak{g}^L$.

This conjecture has been proven to be true in a number of cases. It has been proven for Dynkin index $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ [20], $\mathfrak{g} = D_4^{(3)}$ [12], $\mathfrak{g} = G_2^{(1)}$ [32]. For Dynkin index $i > 1$, the conjecture has been proven for $A_n^{(1)}$ [25, 26]. For Dynkin index $i = n$ this conjecture has been proven for $D_5^{(1)}$ [11] and $D_6^{(1)}$ [27, 28].

In this thesis we will prove Kashiwara, Nakashima, and Okado's conjecture for Dynkin index $i = n$, $\mathfrak{g} = C_n^{(1)}$, and $n = 2, 3, 4$. To do this we construct the positive geometric crystals associated with the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_n$. These subalgebras are are both isomorphic to C_n, and are formed using the modified index set $I/\{0\}$ and $I/\{n\}$ respectively. Using these geometric crystals, we form the positive geometric crystal for the affine algebra $\mathfrak{g}$. Then we reparametrize the perfect crystals associated with the Langlands dual of $\mathfrak{g}$, $D_{n+1}^{(2)}$, and compute the limit of the coherent family of perfect crystals. We then apply the ultra-discretization functor to the geometric crystal we constructed. Finally we construct the crystal isomorphism between the limit of the coherent family of perfect crystals and the ultra discretization of the geometric crystal. This proves the conjecture in [20] for Dynkin index $i = n$, $\mathfrak{g} = C_n^{(1)}$, and $n = 2, 3, 4$.

CHAPTER

2

LIE ALGEBRAS AND KAC-MOODY ALGEBRAS

In this chapter we will review the basic definitions and facts related to Lie algebras and their representations, as well as Kac-Moody algebras and their representations. For a more detailed exposition of these facts, the reader can refer to [1, 10, 14].

2.1 Lie Algebras

In this section, we will recall the basic definitions and facts related to Lie algebras as presented in [1],[10].

Definition 2.1.1. *A Lie algebra over a field $\mathbb{F}$ is a vector space L equipped with a bilinear product $[\cdot,\cdot] : L \times L \to L$ called the Lie bracket such that*

- $[x, x] = 0$
- $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

for all $x, y, z \in L$. This second condition is called the Jacobi identity.

A common introductory example of a Lie algebra is the special linear algebra, $\mathfrak{sl}(2,\mathbb{F})$, the vector space of 2×2 matrices with trace 0. The Lie bracket is the commutator bracket, $[x,y]=xy-yx$. A useful basis of $\mathfrak{sl}(2,\mathbb{F})$ is the following:

$$e=\begin{bmatrix}0&1\\0&0\end{bmatrix}\ f=\begin{bmatrix}0&0\\1&0\end{bmatrix}\ h=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

If we look at the Lie bracket of these elements, we get the following:

$$[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$$

This can be extended to any dimension n.

Another example of a Lie algebra is $\mathfrak{gl}(V)$, the set of endomorphisms of V, called the general linear algebra. By picking a basis, this is equivalent to all $n\times n$ matrices if $\dim(V)=n$.

Now we will give a number of useful definitions.

Definition 2.1.2. *K is a subalgebra of a Lie algebra L if K is a subspace of L as vector spaces and if for all $x,y\in K$, $[x,y]\in K$.*

Definition 2.1.3. *Let L and L' be Lie algebras. A map $\phi: L\to L'$ is a Lie algebra homomorphism if ϕ is a vector space homomorphism and $\phi([x,y])=[\phi(x),\phi(y)]$ for all $x,y\in L$.*

Definition 2.1.4. *A Lie algebra L is abelian if $[x,y]=0$ for all $x,y\in L$.*

From the definition of a subalgebra, we can see that $\mathfrak{sl}(n)\subset\mathfrak{gl}(V)$ where $\dim(V)=n$. Another subalgebra of $\mathfrak{gl}(V)$ is the symplectic algebra.

Example 2.1.5. *Assume $\dim(V)=2n$. Let f be a nondegenerate skew symmetric bilinear form on V. It can be shown that V must have even dimension for f to be nondegenerate. By skew symmetric, we mean that $f(v,w)=-f(w,v)$. Then the symplectic algebra $\mathfrak{sp}(2n,\mathbb{F})$ (or C_n) is the algebra of all endomorphisms on V satisfying $f(x(v),w)=-f(v,x(w))$. Choosing a basis $e_1,...,e_{2n}$, this means that any matrix in $\mathfrak{sp}(2n,\mathbb{F})$ must have the block form*

$$\begin{bmatrix}A&B\\C&A^T\end{bmatrix}$$

where B,C are both symmetric. From this information we can get a basis for $\mathfrak{sp}(2n,\mathbb{F})$, and it can be shown that $\dim(\mathfrak{sp}(2n,\mathbb{F}))=2n^2+n$.

This key example will be studied further in the following chapters.

We now give some additional definitions.

Definition 2.1.6. *An ideal of a Lie algebra L is a subspace I of L such that $[x, y] \in I$ for $x \in I$, $y \in L$.*

Definition 2.1.7. *A Lie algebra L is simple if it is nonabelian, and has no nontrivial proper ideals.*

Definition 2.1.8. *Consider the derived series of L, defined inductively by $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. Then $L^{(0)} \supset L^{(1)} \supset L^{(2)} \supset \dots$. L is solvable if $L^{(n)} = \{0\}$ for some $n \geq 1$.*

Definition 2.1.9. *L is called semisimple if it can be written as a direct sum of simple ideals.*

Now one of the downsides of a Lie algebra is that it is not necessarily an associative algebra. In order to extend a Lie algebra L to an associative algebra, we associate with each Lie algebra its universal enveloping algebra $U(L)$, which we define below.

Definition 2.1.10. *A universal enveloping algebra associated to L is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is an $\mathbb{F}$-associative algebra with identity and $i : L \to \mathfrak{U}$ is a map satisfying*

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

for any $x, y \in L$. Additionally, for any $\mathbb{F}$ associative algebra with identity $\mathfrak{V}$ and map $j : L \to \mathfrak{V}$ satisfying the above condition, there exists a unique homomorphism $\phi : \mathfrak{U} \to \mathfrak{V}$ such that $\phi \circ i = j$.

Universal enveloping algebras are unique up to isomorphism. They can be constructed as follows:

$$\mathfrak{U}(L) = \mathfrak{T}(L)/J$$

where $\mathfrak{T}(L)$ is the tensor algebra of L and J is the two sided ideal generated by

$$\{x \otimes y - y \otimes x - [x, y]\}$$

for all $x, y \in L$. The important results related to the universal enveloping algebra come from the Poincare-Birkoff-Witt (PBW) Theorem.

Theorem 2.1.11 ([10]). **(PBW Theorem)**

1. *The map* $i : L \to \mathfrak{U}(L)$ *is injective.*

2. *Let* $(x_1, x_2, ...)$ *be any ordered basis for* L*. Then the elements* $x_{i(1)}x_{i(2)}...x_{i(m)} = \pi(x_{i(1)} \otimes ... \otimes x_{i(m)})$, $m \in \mathbb{Z}^+$, $i(1) \leq i(2) \leq ... \leq i(m)$ *along with the identity form a basis of* $\mathfrak{U}(L)$.

This in particular proves that any Lie algebra is isomorphic to a subalgebra of an associative algebra. Now we move on to studying results about representations of Lie algebras.

2.2 Representations of Lie Algebras

In this section we will recall the definition of a representation of a Lie algebra, provide an example and recall relevant statements related to these representations as presented in [1], [10].

Definition 2.2.1. *A representation of a Lie algebra* L *on a vector space* V *is a Lie algebra homomorphism* $\rho : L \to \mathfrak{gl}(V)$*. This induces an action of* L *on* V, $xv = \rho(x)v \in V$ *for* $x \in L$, $v \in V$.

An equivalent notion to consider is an L-module.

Definition 2.2.2. *An* L*-module is a vector space* V *equipped with a bilinear product* $L \times V \to V$ *such that*

$$[x, y]v = x(yv) - y(xv)$$

for $x, y \in L$ *and* $v \in V$.

We now recall a few important definitions regarding L-modules.

Definition 2.2.3. *For two* L*-modules* V *and* W*, a module homomorphism is a linear map* $\varphi : V \to W$ *which commutes with the* L *action, i.e.* $\varphi(xv) = x\varphi(v)$.

Definition 2.2.4. *A submodule of an* L*-module* V *is a subspace* W *which is closed under the action of* L.

Definition 2.2.5. *An* L*-module* V *is irreducible if its only submodules are* $\{0\}$ *and* V.

Definition 2.2.6. *An* L*-module* V *is completely reducible if it is a direct sum of irreducible submodules.*

We now give a few basic examples of representations of Lie algberas.

Example 2.2.7. *For each Lie algebra, there is the adjoint representation* $ad: L \to \mathfrak{gl}(L)$ *where* $ad_x(y) = [x, y]$ *for each* $x \in L$.

Example 2.2.8. *For* $L = \mathfrak{gl}(V)$*, there is a vector representation where* $\rho : L \to \mathfrak{gl}(V)$ *is the identity, and the action of* L *on* V *is matrix multiplication.*

In the following chapters, we will be studying L modules corresponding to the vector representation. Before we extend these ideas to Kac-Moody algebras and representations, we provide a final important result due to Weyl, Weyl's Complete Reducibility Theorem. This theorem is crucial to the study of representations because it shows that it is often sufficient to study irreducible L-modules.

Theorem 2.2.9 ([10])**.** *(Weyl's Complete Reducibility Theorem) Let* L *be a finite dimensional semisimple Lie algebra over a field of characteristic zero. Then every finite dimensional* L*-module is completely reducible.*

2.3 Kac-Moody Algebras

Victor Kac and Robert Moody both developed a generalization of Lie algebras, called Kac-Moody algebras. In this section, following the work of Victor Kac in [14] we will recall the basic definitions and facts related to Kac-Moody algebras. The first ingredient needed to define a Kac-Moody algebra is a generalized Cartan matrix.

Let I be a finite index set.

Definition 2.3.1. *A square matrix* $A = (a_{ij})_{i,j \in I}$ *with entries in* $\mathbb{Z}$ *is called a generalized Cartan matrix if it satisfies*

1. $a_{ii} = 2$ *for all* $i \in I$
2. $a_{ij} \leq 0$ *if* $i \neq j$
3. $a_{ij} = 0$ *if and only if* $a_{ji} = 0$

There are a few important properties we might be interested in related to generalized Cartan matrices. First, if there exists a diagonal matrix $D = \text{diag}(s_i)$ with all $s_i \in \mathbb{Z}_{\geq 0}$ such that DA is symmetric, then A is called symmetrizable. Second, A is indecomposable if for every nonempty pair of subsets I_1, I_2 such that $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and

$j \in I_2$ such that $a_{ij} \neq 0$. The Kac-Moody algebra is uniquely defined by the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. We now define each of these components:

- A is the generalized Cartan matrix as defined above.
- $P^\vee$ is a free abelian group with basis $\{h_i | i \in I\} \cup \{d_s | s = 1, ..., |I| - \text{rank}(A)\}$. $P^\vee$ is referred to as the coweight lattice. We also define $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} P^\vee$ to be the $\mathbb{F}$-linear space spanned by $P^\vee$, called the Cartan subalgebra.
- $P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbb{Z}\}$ called the weight lattice.
- $\Pi = \{\alpha_i | i \in I\}$ is the set of simple roots
- $\Pi^\vee = \{\alpha_i^\vee | i \in I\}$ is the set of simple co roots

We require that Π and $\Pi^\vee$ are linearly independent sets and $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. Now we have all of the components to define a Kac-Moody algebra:

Definition 2.3.2. *The Kac-Moody algebra $\mathfrak{g}$ over $\mathbb{F}$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements $e_i, f_i (i \in I)$ and $h \in P^\vee$ subject to the following relations:*

1. $[h, h'] = 0$ *for* $h, h' \in P^\vee$
2. $[e_i, f_i] = \delta_{ij} h_i$
3. $[h, e_i] = \alpha_i(h) e_i$ *for* $h \in P^\vee$
4. $[h, f_i] = -\alpha_i(h) f_i$ *for* $h \in P^\vee$
5. $(ad(e_i))^{1-a_{ij}} e_j = 0$ *for* $i \neq j$
6. $(ad(f_i))^{1-a_{ij}} f_j = 0$ *for* $i \neq j$

The first 4 relations are referred to as the Chevalley relations, and the last 2 are the Serre relations.

Let $Q = \sum_{i=1}^{n} \mathbb{Z} \alpha_i$, which is called the root lattice. An element $\alpha \in Q$ is called a root if $\alpha \neq 0$ and the dimension of the weight space of α, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h}\}$, is nonzero. Let Δ be the set of all roots, called the root system, and Δ_+ is the set of positive roots. Additionally, the fundamental weights are Λ_i such that $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let W be the Weyl group, generated by simple reflections $s_i(\lambda) = \lambda - \lambda(\check{\alpha}_i) \alpha_i$. A root α is real if for some

$w \in W$, $w(\alpha)$ is equal to a simple root. If a root is not real, we call it imaginary. The set of real roots is denoted Δ^{re}.

We now recall some of the main properties of Kac-Moody algebras:

Theorem 2.3.3 ([14])**.** *A Kac-Moody algebra* $\mathfrak{g}$ *has the following decompositions:*

1. *We have a triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_+$ *is generated by* e_i *and* $\mathfrak{g}_-$ *is generated by* f_i

2. *We have a root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_\alpha = 0$ *or dim* $\mathfrak{g}_\alpha < \infty$ *for all* $\alpha \in Q$.

We can classify Kac-Moody algebras based on their generalized Cartan matrices.

Theorem 2.3.4 ([14])**.** *Let A be an indecomposable generalized Cartan matrix. Then one and only one of the following three possibilities hold for A and* A^t*:*

1. *Finite type: det*$(A) \neq 0$*; there exists a vector* $u > 0$ *such that* $Au > 0$*;* $Av \geq 0$ *implies that* $v > 0$ *or* $v = 0$

2. *Affine type: corank*$(A) = 1$*; there exists a vector* $u > 0$ *such that* $Au = 0$*;* $Av \geq 0$ *implies* $Av = 0$.

3. *Indefinite type: There exists* $u > 0$ *such that* $Au < 0$*;* $Av \geq 0$ *and* $v \geq 0$ *imply* $v = 0$.

Associated with each generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ is an oriented graph called a Dynkin diagram. The diagram consists of vertices indexed by I and edges with arrows defined as follows: If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$ then the vertices i and j are connected with $|a_{ij}|$ edges equipped with an arrow pointing toward i if $|a_{ij}| > 1$.

We are interested in Kac-Moody algebras of affine type for the rest of the paper, called affine Lie algebras. These have been completely classified into the following types: $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $A_{2n}^{(2)}$, $A_{2n+1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$.

We know that if A is of affine type, there exists $u = (a_0, a_1, ..., a_n)$ such that $Au^T = 0$. This means that the null space of A is 1 dimensional. The element $\delta = a_0\alpha_0 + a_1\alpha_1 + ... + a_n\alpha_n$ is called the null root. The canonical central element is $c = \sum_{i=0}^{n} a_i^\vee \alpha_i^\vee$ where $a_i^\vee$ come from a corresponding vector $v = (a_0^\vee, a_1^\vee, ..., a_n^\vee)$ such that $A^T v^T = 0$ and $\gcd(a_0^\vee, a_1^\vee, ..., a_n^\vee) = 1$.

We are particularly interested in the affine Lie algebra $C_n^{(1)}$. The corresponding generalized Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -2 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & ... & 0 & -1 & 2 & -2 \\ 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

$C_n^{(1)}$ has simple roots $\alpha_0, \alpha_1, ..., \alpha_n$, null root $\delta = \alpha_0 + 2\alpha_1 ... + 2\alpha_{n-1} + \alpha_n$ and central element $c = h_0 + h_1 + ... + h_{n-1} + h_n$. It has Dynkin diagram

$$\circ \Rightarrow \circ — \cdots — \circ \Leftarrow \circ$$

A Dynkin diagram uniquely determines the corresponding generalized Cartan matrix, so we can classify Kac-Moody algebras by their Dynkin diagrams.

We end this section by recalling the extension of Kac-Moody algebras to a universal enveloping algebra.

Theorem 2.3.5 ([14])**.** *The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra over $\mathbb{F}$ with unity generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ subject to the following relations:*

1. $hh' = h'h$ *for* $h.h' \in \mathfrak{h}$
2. $e_i f_j - f_j e_i = \delta_{ij} h_i$ *for* $i, j \in I$
3. $he_i - e_i h = \alpha(h) e_i$ *for* $h \in \mathfrak{h}$, $i \in I$
4. $hf_i - f_i h = \alpha(h) f_i$ *for* $h \in \mathfrak{h}$, $i \in I$
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$

The universal enveloping algebra also has a triangular decomposition and a root space decomposition just like the Kac-Moody Algebra. Now we turn to the study of representations of Kac-Moody algebras.

2.4 Representations of Kac-Moody Algebras

Now we will explore the theory of weight modules and Verma modules, as presented in [14],[9].

Definition 2.4.1. *A $\mathfrak{g}$-module V is called a weight module if it admits a weight space decomposition*

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \text{ where } V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$$

A vector $v \in V_\mu$ is called a weight vector of weight μ

We then can define the category $\mathcal{O}$, which is the category of weight modules over $\mathfrak{g}$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, ..., \lambda_s \in \mathfrak{h}^*$ such that $\text{wt}(V) \subset D(\lambda_1) \cup ... \cup D(\lambda_s)$ where $D(\lambda) = \{\mu \in \mathfrak{h}^* | \mu \leq \lambda\}$ and $\text{wt}(V)$ is the set of all weights in V. The statement $\mu \leq \lambda$ is equivalent to $\lambda = \mu + \sum_{0 \leq i \leq n} m_i \alpha_i$ where $m_i \in \mathbb{Z}^{\geq 0}$. With this definition of weight modules, we can explore the most interesting example of weight modules in $\mathcal{O}$, highest weight modules.

Definition 2.4.2. *A weight module V is a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$, called a highest weight vector such that*

$$\begin{aligned} &e_i v_\lambda = 0 \text{ for all } i \in I \\ &h v_\lambda = \lambda(h) v_\lambda \text{ for all } h \in \mathfrak{h} \\ &V = U(\mathfrak{g}) v_\lambda \end{aligned}$$

Now we define a specific kind of highest weight module, Verma modules.

Definition 2.4.3. *Fix $\lambda \in \mathfrak{h}^*$ and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by all e_i and $h - \lambda(h)1$. Then the Verma module is $M(\lambda) = U(\mathfrak{g})/J(\lambda)$.*

Verma modules have the following properties:

Theorem 2.4.4 ([14])**.** *Let $M(\lambda)$ be a Verma module. Then:*

1. *$M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$*

2. *$M(\lambda)$ has a unique maximal submodule, $N(\lambda)$. Then the irreducible highest weight module is $V(\lambda) = M(\lambda)/N(\lambda)$.*

3. *Every irreducible* $\mathfrak{g}$*-module in the category* $\mathcal{O}$ *is isomorphic to* $V(\lambda)$ *for some* $\lambda \in \mathfrak{h}^*$

We now define the category of modules we will be studying. But first, we need a few small definition:

Definition 2.4.5. $x \in \mathfrak{g}$ *is locally nilpotent on an* $\mathfrak{g}$*-module* V *if for any* $v \in V$ *there exists a positive integer* N *such that* $x^N v = 0$.

Definition 2.4.6. *A module* V *is integrable if all* e_i *and* f_i *are locally nilpotent on* V.

The category $\mathcal{O}_{\text{int}}$ is the category of integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$. We now give some important results about $\mathcal{O}_{\text{int}}$. These results allow us to focus our study on irreducible highest weight modules, greatly simplifying the study of modules in $\mathcal{O}_{\text{int}}$. This theorem provides the same flexibility in studying Kac-Moody representations as Weyl's Complete Reducibility Theorem gave in the context of studying Lie algebra representations.

Theorem 2.4.7 ([14])**.** *Let* $\mathfrak{g}$ *be a Kac-Moody algebra associated with a Cartan datum* $(A, \Pi, \Pi^\vee, P, P^\vee)$. *Then every* $\mathfrak{g}$*-module in the category* $\mathcal{O}_{int}$ *is isomorphic to a direct sum of irreducible highest weight modules* $V(\lambda)$ *with* $\lambda \in P^+$*, the weight lattice restricted to positive weights. Additionally, the tensor product of a finite number of* $\mathfrak{g}$*-modules in the category* $\mathcal{O}_{int}$ *is completely reducible.*

CHAPTER

3

QUANTUM GROUPS AND CRYSTAL BASES

In this chapter, we introduce quantum groups, which are quantum deformations of the universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$. We will recall the definition and present some important facts related to quantum groups, as presented in [9]. After defining quantum groups, we will discuss the representation theory related to quantum groups. Finally, we will recall the notion of crystal bases, which are roughly the limit of a representation as q goes to 0. Quantum groups and crystal bases were partially motivated by lattice models in quantum physics. The additional parameter q corresponds to temperature within the model. Then when $q \to 0$, this corresponds to the temperature going to zero Kelvin. Physically, this means we might expect some simplification in this case because at zero Kelvin particles are much more stable.

3.1 Quantum Groups

Before we define quantum groups, we need to define some additional notation.

Let $n \in \mathbb{Z}$ and let q be an indeterminate. Then

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

This is called a q-integer. Let $[n]_q! = [n]_q[n-1]_q...[1]_q$ and $[0]_q! = 1$. Then for $m \geq n \geq 0$ we define an analogue of binomial coefficients:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

called the q-binomial coefficient. Now let A be a symmetrizable generalized Cartan matrix and $D = \text{diag}(s_i | i \in I)$ be its symmetrizing matrix. Also let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A. Now we define the quantum group:

Definition 3.1.1 ([9])**.** *The quantum group $U_q(\mathfrak{g})$ associated with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{F}(q)$ with identity generated by e_i, f_i, and q^h for $i \in I$ and $h \in P^\vee$ with the following defining relations:*

1. $q^0 = 1$, $q^h \cdot q^{h'} = q^{h+h'}$ *for* $h, h' \in P^\vee$
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ *for* $h \in P^\vee$
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ *for* $h \in P^\vee$
4. $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ *for* $i, j \in I$
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$

where $q_i = q^{s_i}$ *and* $K_i = q^{s_i h_i}$.

From the definition, we can see that the relations are similar to those imposed to get a Kac-Moody algebra, with the first four corresponding to the Chevalley relations, and the last two corresponding to the Serre relations. Because all the relations above are homogeneous, there is a root space decomposition for a quantum group,

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (U_q)_\alpha$$

where $(U_q)_\alpha = \{u \in U_q(\mathfrak{g}) | q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in P^\vee\}$ and $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the root lattice.

An additional property of interest for quantum groups is that a quantum group is also a Hopf algebra. A Hopf algebra is an associative algebra which is also a coalgebra. It provides additional structure that gives a module structure on the tensor products of its modules and the duals of its modules.

Theorem 3.1.2 ([9])**.** *The quantum group $U_q(\mathfrak{g})$ has a Hopf algebra structure with the comultiplication Δ, counit ε and antipode S defined by:*

1. $\Delta(q^h) = q^h \otimes q^h$
2. $\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \qquad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$
3. $\varepsilon(q^h) = 1, \qquad \varepsilon(e_i) = \varepsilon(f_i) = 0$
4. $S(q^h) = q^{-h}, \qquad S(e_i) = -e_i K_i\,, \qquad S(f_i) = -K_i^{-1} f_i$

One final important property is that the quantum group admits a triangular decomposition

$$U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+$$

where U_q^- is the subalgebra generated by $\{f_i\}$, U_q^+ is the subalgebra generated by $\{e_i\}$ and U_q^0 is the subalgebra generated by $\{q^h\}$.

3.2 Representation Theory of Quantum Groups

There are many similarities between the representation theory of quantum groups and the representation theory of Kac-Moody algebras. Thus we will explore similar concepts and facts in this section as seen in Chapter 2.

A $U_q(\mathfrak{g})$-module V^q is a weight module if it admits a weight space decomposition $V^q = \bigoplus_{\mu \in P} V_\mu^q$, where $V_\mu^q = \{v \in V^q | q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$. We define highest weight vectors and highest weights in parallel ways to the theory of Kac-Moody algebra representations.

Definition 3.2.1. *A weight module V^q is called a highest weight module with highest weight $\lambda \in P$ if there exists a nonzero $v_\lambda \in V^q$ such that*

1. $e_i v_\lambda = 0$ *for all* $i \in I$

2. $q^h v = q^{\lambda(h)} v$ *for all* $h \in P^\vee$

3. $V^q = U_q(\mathfrak{g}) v_\lambda$

The vector v_λ *is called the highest weight vector.*

Now let $J^q(\lambda)$ be the ideal of $U_q(\mathfrak{g})$ generated by e_i for all $i \in I$ and $q^h - q^{\lambda(h)}1$ for all $h \in P^\vee$. Then the Verma module $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$. It can be shown that this module has a unique maximal submodule, $N^q(\lambda)$, and by taking $M^q(\lambda)/N^q(\lambda)$, we obtain an irreducible highest weight module with highest weight λ, which we denote $V^q(\lambda)$.

We now define the category $\mathcal{O}^q_{\text{int}}$:

Definition 3.2.2 ([9])**.** *The category* $\mathcal{O}^q_{int}$ *consists of* $U_q(\mathfrak{g})$*-modules* V^q *satisfying the following conditions:*

1. V^q *has a weight space decomposition* $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ *and* $\dim V^q_\lambda < \infty$ *for all* $\lambda \in P$
2. *There exist a finite number of elements* $\lambda_1, \ldots, \lambda_s \in P$ *such that* $wt(V^q) \subset D(\lambda_1) \cup \ldots \cup D(\lambda_s)$ *where* $D(\lambda_i) = \{\mu \in P | \mu \le \lambda_i\}$
3. *All* e_i *and* f_i $(i \in I)$ *are locally nilpotent on* V^q

This category is closed under taking direct sums and finite tensor products.

We now present a few useful facts:

Theorem 3.2.3 ([9])**.** *If* $V^q(\lambda)$ *is the irreducible highest weight* $U_q(\mathfrak{g})$*-module with highest weight* $\lambda \in P$*, then* $V^q(\lambda)$ *belongs in the category* $\mathcal{O}^q_{int}$ *if and only if* $\lambda \in P^+$*.*

Let $U_q(\mathfrak{g}_{(i)})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Then $U_q(\mathfrak{g}_{(i)}) \cong U_{q_i}(\mathfrak{sl}_2)$, and we have a similar result as we had previously for $\mathfrak{g}$-modules.

Theorem 3.2.4 ([9])**.** *Let* V^q *be a* $U_q(\mathfrak{g})$*- module in* $\mathcal{O}^q_{int}$*. Then for each* $i \in I$*,* V^q *decomposes into a direct sum of* $U_q(\mathfrak{h})$ *invariant finite dimensional, irreducible* $U_q(\mathfrak{g}_{(i)})$*- submodules.*

Finally, as we might hope to be true, we can completely reduce any integrable $U_q(\mathfrak{g})$ module into a direct sum of irreducible, highest weight modules.

Theorem 3.2.5 ([9])**.** *Let* $U_q(\mathfrak{g})$ *be the quantum group associated with the Cartan datum* $(A, \Pi, \Pi^\vee, P, P^\vee)$*. Then every* $U_q(\mathfrak{g})$*-module in the category* $\mathcal{O}^q_{int}$ *is isomorphic to a direct sum of irreducible highest weight modules* $V^q(\lambda)$*. Additionally, the tensor product of a finite number of* $U_q(\mathfrak{g})$*-modules in the category* $\mathcal{O}^q_{int}$ *is completely reducible.*

This result allows us to focus on the characterization of irreducible highest weight modules in order to study $U_q(\mathfrak{g})$-modules in $\mathcal{O}^q_{\text{int}}$.

3.3 Crystal Bases

Crystal bases were introduced by Kashiwara in 1990. They are advantageous because they provide a good description of the character of a module and they also behave simply with respect to the tensor product. They retain the information we might be interested in studying while simplifying computations. In this section we will define crystal bases and crystals, discuss the tensor product as well as the definition of crystal morphisms, following the presentation in [9].

In this section, we limit our discussion of $U_q(\mathfrak{g})$ modules to those in the category $\mathcal{O}^q_{\text{int}}$. We first need the following theorem:

Theorem 3.3.1 ([9]). *Let $M = \bigoplus_{\lambda\in P} M_\lambda$ be a $U_q(\mathfrak{g})$ module in the category O^q_{int}. For each $i \in I$, every weight vector $u \in M_\lambda$ where $\lambda \in wt(M)$ may be written in the form*

$$u = u_0 + f_i u_1 + ... + f_i^{(N)} u_N$$

where $N \in \mathbb{Z}_{\geq 0}$ and $u_k \in M_{\lambda+k\alpha_i} \cap ker(e_i)$ for all $k = 0, 1, ..., N$. Each u_k is uniquely determined by u and $u_k \neq 0$ only if $\lambda(h_i) + k \geq 0$.

This allows us to define Kashiwara operators.

Definition 3.3.2. *The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ on M defined by*

$$\tilde{e}_i = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \tilde{f}_i = \sum_{k=1}^{N} f_i^{(k+1)} u_k$$

One important property of the Kashiwara operators is that they commute with $U_q(\mathfrak{g})$-module homomorphisms.

Before we give the definition of a crystal base, we will first discuss a helpful motivating example following its presentation in [9], $U_q(\mathfrak{sl}(2))$ acting on the two dimensional representation $V = \mathbb{F}(q)v_+ \oplus \mathbb{F}(q)v_-$. $U_q(\mathfrak{sl}(2))$ is generated by e, f, and $K^{\pm 1}$. The nonzero actions on the module are as follows:

$$e v_- = v_+, \;\; f v_+ = v_-, \;\; K v_+ = q v_+, \;\; K v_- = q^{-1} v_-$$

Then an obvious basis for the tensor product $V \otimes V$ might be $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$, and $v_- \otimes v_-$, but this is not an actual basis for $V \otimes V$.

The actual basis is $v_+ \otimes v_+$, $q v_+ \otimes v_- + v_- \otimes v_+$, $v_+ \otimes v_- - q v_- \otimes v_+$, and $v_- \otimes v_-$.

We can observe that when $q = 0$, the actual basis corresponds with our ideal basis. Thus, we can expect that the crystal base, which corresponds with $q = 0$, will often give a simplified basis. Note that the resulting crystal base might not always be an actual basis.

Now we define a crystal lattice, which generates M as a vector space and crystal bases, which is a lattice along with a basis. Before this, we need to define a ring $\mathbb{A}_0 = \{g/h | g, h \in \mathbb{F}[q] | h(0) \neq 0\}$ to prevent an undefined coefficient as $q \to 0$.

Definition 3.3.3 ([9])**.** *Let M be a $U_q(\mathfrak{g})$-module. A free $\mathbb{A}_0$-submodule $\mathcal{L}$ of M is called a crystal lattice if*

1. *$\mathcal{L}$ generates M as a vector space over $\mathbb{F}(q)$*
2. *$\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for all $\lambda \in P$*
3. *$\tilde{e}_i \mathcal{L} \subset \mathcal{L}$, $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$.*

Definition 3.3.4 ([9])**.** *A crystal basis of a $U_q(\mathfrak{g})$-module M is a pair $(\mathcal{L}, \mathcal{B})$ such that*

1. *$\mathcal{L}$ is a crystal lattice of M*
2. *$\mathcal{B}$ is an $\mathbb{F}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{F} \otimes_{\mathbb{A}_0} \mathcal{L}$*
3. *$\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda$ where $\mathcal{B}_\lambda = \mathcal{B} \cap \mathcal{L}_\lambda / q\mathcal{L}_\lambda$*
4. *$\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, for all $i \in I$*
5. *for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$*

Suppose our $U_q(\mathfrak{g})$ module is an irreducible highest weight module $V(\lambda)$ with highest weight $\lambda \in P^+$ and highest weight vector v_λ.

Theorem 3.3.5 ([9])**.** *Let $\mathcal{L}(\lambda)$ be the free $\mathbb{A}_0$-submodule spanned by vectors of the form $\tilde{f}_{i_1} \ldots \tilde{f}_{i_r} v_\lambda$ and $\mathcal{B}(\lambda) = \{\tilde{f}_{i_1} \ldots \tilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda)\} / \{0\}$. Then $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal basis for $V(\lambda)$.*

As previously mentioned, one of the advantages of the crystal base is that there is a simplified way to compute the tensor product. This simplified version allows us to find the basis of the tensor product and more easily decompose the tensor product of 2 modules into a direct sum of irreducible modules.

Theorem 3.3.6. *(Tensor Product Rule)[[9]] Let M_j be a $U_q(\mathfrak{g})$-module and let $(\mathcal{L}_j, \mathcal{B}_j)$ be a crystal basis of M_j for $j = 1, 2$. Set $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathbb{A}_0} \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Then $(\mathcal{L}, \mathcal{B})$ is a crystal basis of $M_1 \otimes_{\mathbb{F}(q)} M_2$, where the actions of Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{B}$ $(i \in I)$ are given by*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

Therefore,

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle)$$

$$\varphi_i(b_1 \otimes b_2) = max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle)$$

This allows us to easily compute the decomposition of the tensor product into irreducible modules.

If we define the maps $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ as

$$\varepsilon_i(b) = \max\{k \geq 0 | \tilde{e}_i^k b \in \mathcal{B}\}, \quad \varphi_i(b) = \max\{k \geq 0 | \tilde{f}_i^k b \in \mathcal{B}\}$$

then we can define crystals abstractly apart from a specific module.

Definition 3.3.7. *Let I be a finite index set and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A crystal associated with the Cartan datum is a set $\mathcal{B}$ together with the maps $wt : \mathcal{B} \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ satisfying the following properties:*

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle$ *for all* $i \in I$
2. $wt(\tilde{e}_i b) = wt\, b + \alpha_i$ *if* $\tilde{e}_i b \in \mathcal{B}$
3. $wt(\tilde{f}_i b) = wt\, b - \alpha_i$ *if* $\tilde{f}_i b \in \mathcal{B}$
4. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ *if* $\tilde{e}_i b \in \mathcal{B}$
5. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ *if* $\tilde{f}_i b \in \mathcal{B}$
6. *for any* $b, b' \in \mathcal{B}$ *and* $i \in I$, *we have* $\tilde{f}_i b = b'$ *if and only if* $\tilde{e}_i b' = b$

7. *if* $\varphi_i(b) = -\infty$ *for* $b \in \mathcal{B}$, *then* $\tilde{e}_i b = \tilde{f}_i b = 0$

Each crystal has a colored graph structure where we connect two elements $b, b' \in \mathcal{B}$ with an i-arrow from $b \to b'$ if $f_i b = b'$ for some $i \in I$. Now we give an example of a crystal and its graph. All of the crystal elements are semistandard Young tableau.

Example: Consider the $U_q(\mathfrak{sl}(3))$ module $V(\Lambda_1 + \Lambda_2)$. This has the following crystal graph:

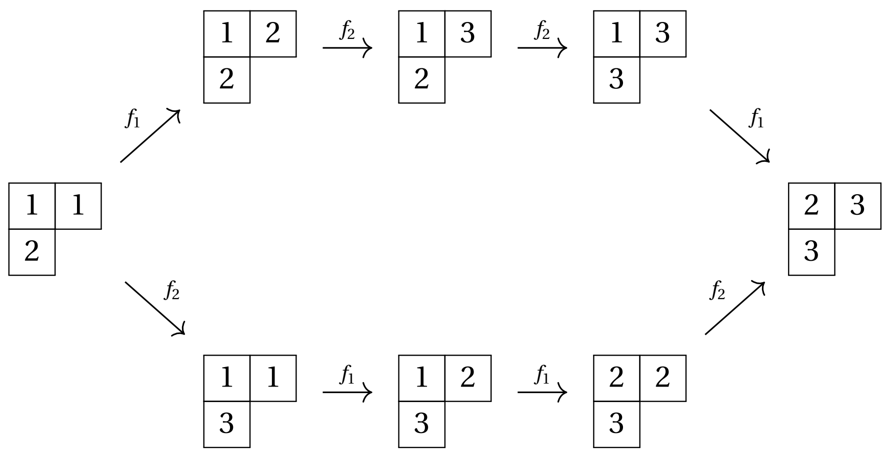

Figure 3.1: Crystal Graph of the $U_q(\mathfrak{sl}(3))$ module $V(\Lambda_1 + \Lambda_2)$

Finally, we define morphisms between crystals.

Definition 3.3.8. *Let* $\mathcal{B}_1, \mathcal{B}_2$ *be crystals associated with the Cartan datum* $(A, \Pi, \Pi^\vee, P, P^\vee)$. *A crystal morphism* $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ *is a map* $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ *such that*

1. $\Psi(0) = 0$

2. *if* $b \in \mathcal{B}_1$ *and* $\Psi(b) \in \mathcal{B}_2$, *then* $wt(\Psi(b)) = wt(b)$, $\varepsilon(\Psi(b)) = \varepsilon(b)$ *and* $\varphi(\Psi(b)) = \varphi(b)$ *for all* $i \in I$

3. *if* $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$ *and* $\tilde{f}_i(b) = b'$, *then* $\tilde{f}_i \Psi(b) = \Psi(b')$ *and* $\Psi(b) = \tilde{e}_i \Psi(b')$ *for all* $i \in I$.

CHAPTER

4

PERFECT CRYSTALS

4.1 Quantum Affine Algebras

Let $\mathfrak{g}$ be a symmetrizable affine Kac-Moody algebra generated by e_i, f_i, and $\mathfrak{h}$ where $i \in I = \{0, \ldots, n\}$, and $\mathfrak{h}$ is the Cartan subalgebra over $\mathbb{C}$. Each affine algebra can also be characterized by its Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ where $A = (a_{ij})_{i,j=0}^n$ is an affine generalized Cartan matrix, Π is the set of roots, $\Pi^\vee$ the set of coroots, P the weight lattice, and $P^\vee$ the coweight lattice. Let $\mathbf{c}$ be the canonical central element and δ be the null root. The Cartan subalgebra can be expressed as $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$. Recall that $\Lambda_0, \ldots, \Lambda_n \in P$ are the fundamental weights. The roots, coroots, and fundamental weights must satisfy $\alpha_j(\alpha_i^\vee) = a_{ij}$ and $\Lambda_j(\alpha_i^\vee) = \delta_{ij}$.

Definition 4.1.1. *The algebra $U_q(\mathfrak{g})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ where A is an affine generalized Cartan matrix is called a quantum affine algebra. It is generated by e_i, f_i, q^h and satisfies all of the quantum group relations listed in Definition 3.1.1.*

Notice that $U_q(\mathfrak{g})$ is also a Hopf algebra. We can write the weight lattice as

$$P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbb{Z}\}$$

The set of affine dominate integral weights is

$$P^+ = \{\lambda \in \mathfrak{h}^* | \lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0}\}$$

The level of a weight $\lambda \in P^+$ is a nonnegative integer $\lambda(\mathbf{c})$, where $\mathbf{c}$ is the canonical central element of $\mathfrak{g}$.

Let $\bar{P} = \bigoplus \mathbb{Z}\Lambda_i$ and $\bar{P}^\vee = \bigoplus \mathbb{Z}\check{\alpha}_i$.

Definition 4.1.2. *The algebra $U_q'(\mathfrak{g})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ is also called a quantum affine algebra. It is generated by $e_i, f_i, K_i^{\pm 1}$ (for the definition of K_i, refer to Section 3.1).*

Notice that $U_q'(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$.

4.2 Perfect Crystals

4.2.1 Definitions and Background

Let $\mathcal{B}$ be a classical crystal, a finite crystal associated with the Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$. It is also called a $U_q'(\mathfrak{g})$ crystal. Then

$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i \quad \varphi(b) = \sum \varphi_i(b)\Lambda_i$$

Therefore $\mathrm{wt}(b) = \varphi(b) - \varepsilon(b)$. Also let $\bar{P}_l^+ = \{\lambda \in P | \langle c, \lambda \rangle = l\}$. With this notation, we can present the definition of a perfect crystal, as presented in [9]

Definition 4.2.1. *For a positive integer $l > 0$, $\mathcal{B}$ is a perfect crystal of level l if it satisfies the following conditions:*

1. *there exists a finite dimensional $U_q'(\mathfrak{g})$ module with a crystal basis whose crystal graph is isomorphic to $\mathcal{B}$*
2. *$\mathcal{B} \otimes \mathcal{B}$ is connected.*
3. *there exists a classical weight $\lambda_0 \in \bar{P}$ such that*

$$wt(\mathcal{B}) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\geq 0}\alpha_i$$

where $\#(\mathcal{B}_{\lambda_0}) = 1$

4. *for any* $b \in \mathcal{B}$, *we have* $\langle c, \varepsilon(b)\rangle \geq l$

5. *for each* $\lambda \in \bar{P}_l^+$, *there exist unique vectors* $b^\lambda \in \mathcal{B}$, $b_\lambda \in \mathcal{B}$ *such that* $\varepsilon(b^\lambda) = \lambda$ *and* $\varphi(b_\lambda) = \lambda$

We also define $\mathcal{B}^{\min} = \{b \in \mathcal{B} | \langle c, \varepsilon(b)\rangle = l\}$. Additionally, we can observe from the definitions that $\varepsilon, \varphi : \mathcal{B}^{\min} \to \bar{P}_l^+$ are both bijective.

This brings us to an important theorem:

Theorem 4.2.2 ([9]). *Let* $\mathcal{B}$ *be a perfect crystal of level* $l > 0$. *For any classical dominant integral weight* $\lambda \in \bar{P}_l^+$ *there exists a crystal isomorphism*

$$\Psi : \mathcal{B}(\lambda) \longrightarrow \mathcal{B}(\mu) \otimes \mathcal{B}$$

given by $u_\lambda \mapsto u_\mu \otimes b_\lambda$ *where* u_λ *and* u_μ *are highest weight vectors of* $\mathcal{B}(\lambda)$ *and* $\mathcal{B}(\mu)$ *respectively and* b_λ *is the unique vector as described in Definition 4.2.1.*

This theorem is crucial in the work related to path realizations as well as the specific crystal we are interested in, the limit of a coherent family of perfect crystals. We call $\{\mathcal{B}_l\}_{l \geq 1}$ a family of perfect crystals. Also, we define $J := \{(l, b) | l > 0, b \in \mathcal{B}_l^{\min}\}$. Now we define the limit of a family of perfect crystals:

Definition 4.2.3 ([15]). *A crystal* $\mathcal{B}_\infty$ *with element* b_∞ *is called a limit of a family of perfect crystals* $\{\mathcal{B}_l\}_{l \geq 1}$ *if*

1. $wt(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$

2. *For any* $(l, b) \in J$, *there exists an embedding of crystals:*

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes \mathcal{B}_l \otimes T_{-\varphi(b)} \hookrightarrow \mathcal{B}_\infty$$

$$t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty$$

   where $T_\lambda := \{t_\lambda\}$ *for* $\lambda \in P$ *and* $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$ *and* $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = \infty$ *and* $wt(t_\lambda) = \lambda$

3. $\mathcal{B} = \bigcup_{(l,b) \in J} Im_{f_{(l,b)}}$

If a limit exists, then $\{\mathcal{B}_l\}_{l \geq 1}$ is called a coherent family of perfect crystals.

We give one final theorem:

Theorem 4.2.4 ([9])**.** *Let $\mathcal{B}(\infty)$ be a crystal of $U_q^-(\mathfrak{g})$. Then there exists the following isomorphism:*

$$\mathcal{B}(\infty)\otimes\mathcal{B}_\infty \xrightarrow{\sim} \mathcal{B}(\infty)$$

In the rest of the chapter, we will construct the coherent families of perfect crystals for the Langlands dual of $C_n^{(1)}$, $n=2,3,4$, and construct their respective limits. Note that in this case, we can obtain the generalized Cartan matrix of the Langlands dual of $C_n^{(1)}$ by taking the transpose of its Cartan matrix. By doing this we see that the Langlands dual of $C_n^{(1)}$ is $D_{n+1}^{(2)}$. We will prove that our constructions are indeed perfect crystals and that the family of perfect crystals has a limit.

$C_2^{(1)}$ Perfect Crystal

The Langlands dual of $C_2^{(1)}$ is $D_3^{(2)}$. We can reparametrize the level l $U_q(D_3^{(2)})$ perfect crystal $B(l\Lambda_2)$ that was presented in [17] and its limit $B^{2,\infty}$ as follows:

$$B^{2,l}=\left\{(b_{ij})_{1\le i\le 2, i\le j\le i+2}\;\middle|\;\begin{array}{l}(b_{ij})\in\mathbb{Z}^{\ge 0}, \sum_{j=i}^{i+2} b_{ij}=l, 1\le i\le 2\\ \sum_{j=1}^{t} b_{1j}\ge\sum_{j=2}^{t+1} b_{2j}, 1\le t\le 3\\ b_{11}=b_{22}+b_{23}, b_{24}=b_{12}+b_{13}\end{array}\right\}$$

$$B^{2,\infty}=\left\{(b_{ij})_{1\le i\le 2, i\le j\le i+2}\;\middle|\;\begin{array}{l}(b_{ij})\in\mathbb{Z}, \sum_{j=i}^{i+2} b_{ij}=0, 1\le i\le 2\\ b_{11}=b_{22}+b_{23}, b_{24}=b_{12}+b_{13}\end{array}\right\}$$

Associated with these crystals are the following actions and maps, $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$ for $i=0,1,2$. The conditions in parentheses correspond only to the conditions required for $B^{2,l}$, not $B^{2,\infty}$. For $0\le k\le 2$, $\tilde{e}_k(b)=(b'_{ij})$, where

$$\begin{cases} k=0 \quad b'_{11}=b_{11}-1, b'_{24}=b_{24}+1 \begin{cases} b'_{23}=b_{23}-1, b'_{13}=b_{13}+1 & b_{23}>b_{12}, (b_{11}, b_{23}>0)\\ b'_{22}=b_{22}-1, b'_{12}=b_{12}+1 & b_{23}\le b_{12}, (b_{11}, b_{22}>0)\end{cases}\\ k=1 \quad b'_{11}=b_{11}+1, b'_{12}=b_{12}-1, b'_{23}=b_{23}+1, b'_{24}=b_{24}-1 \ (b_{24}, b_{12}>0)\\ k=2 \quad \begin{cases} b'_{13}=b_{13}-1, b'_{12}=b_{12}+1 & b_{12}\le b_{23}, (b_{13}>0)\\ b'_{23}=b_{23}-1, b'_{22}=b_{22}+1 & b_{12}>b_{23}, (b_{23}>0)\end{cases}\end{cases}$$

and $b'_{ij}=b_{ij}$ otherwise.

For $0 \le k \le 2$, $\tilde{f}_k(b) = (b'_{ij})$ where

$$\begin{cases} k=0 & b'_{11} = b_{11}+1, b'_{24} = b_{24}-1 \text{ and} \begin{cases} b'_{23} = b_{23}+1, b'_{13} = b_{13}-1 & b_{23} \ge b_{12}, (b_{23} > 0) \\ b'_{22} = b_{22}+1, b'_{12} = b_{12}-1 & b_{23} < b_{12}, (b_{22} > 0) \end{cases} \\ k=1 & b'_{11} = b_{11}-1, b'_{12} = b_{12}+1, b'_{23} = b_{23}-1, b'_{24} = b_{24}+1 \ (b_{23}, b_{11} > 0) \\ k=2 & \begin{cases} b'_{13} = b_{13}+1, b'_{12} = b_{12}-1 & b_{12} > b_{23}, (b_{12} > 0) \\ b'_{23} = b_{23}+1, b'_{22} = b_{22}-1 & b_{12} \le b_{23}, (b_{22} > 0) \end{cases} \end{cases}$$

and $b_{ij} = b'_{ij}$ otherwise. We now give the maps ε_i, φ_i for $i = 0, 1, 2$.

$$\varepsilon_1(b) = b_{12},\ \varphi_1(b) = b_{23}$$

$$\varepsilon_2(b) = b_{13} + \max(b_{23} - b_{12}, 0)$$

$$\varphi_2(b) = b_{22} + \max(b_{12} - b_{23}, 0))$$

$$\varepsilon_0(b) = \begin{cases} l - b_{24} - \min(b_{12}, b_{23}) & b \in B^{2,l} \\ -b_{24} - \min(b_{12}, b_{23}) & b \in B^{2,\infty} \end{cases}$$

$$\varphi_0(b) = \begin{cases} l - b_{11} - \min(b_{12}, b_{23}) & b \in B^{2,l} \\ -b_{11} - \min(b_{12}, b_{23}) & b \in B^{2,\infty} \end{cases}$$

One can observe that $(B^{2,l})_{\min} = \{b \in B^{2,l} | b_{12} = b_{23}\}$. For $\lambda \in \bar{P}$, we consider the crystal $T_\lambda = \{t_\lambda\}$ with $\tilde{e}_k(t_\lambda) = \tilde{f}_k(t_\lambda) = 0$,
$\varepsilon_k(t_\lambda) = \varphi_k(t_\lambda) = -\infty$, and $\mathrm{wt}\,(t_\lambda) = \lambda$ for $k = 0, 1, 2$. For $\lambda, \mu \in \bar{P}$, $T_\lambda \otimes B^{2,l} \otimes T_\mu$ is a crystal given by:

$\tilde{e}_k(t_\lambda \otimes b \otimes t_\lambda) = t_\lambda \otimes \tilde{e}_k(b) \otimes t_\lambda$
$\varepsilon_k(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_k(b) - \langle \check{\alpha}_k, \lambda \rangle$
$\mathrm{wt}(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + \mathrm{wt}\,(b)$

$\tilde{f}_k(t_\lambda \otimes b \otimes t_\lambda) = t_\lambda \otimes \tilde{f}_k(b) \otimes t_\lambda$
$\varphi_k(t_\lambda \otimes b \otimes t_\mu) = \varphi_k(b) + \langle \check{\alpha}_k, \mu \rangle$

Theorem 4.2.5. *$B^{2,l}$ is a coherent family of perfect crystals and the crystal $B^{2,\infty}$ is its limit with the vector $b_\infty = (0)_{2\times 4}$.*

Proof. To prove this theorem, we must show the 3 condition from Definition 4.2.3 hold.
1. For each $k = 0, 1, 2$, $\varepsilon_k(b_\infty) = \varphi_k(b_\infty) = 0$, so $\varphi(b_\infty) = \varepsilon(b_\infty) = \mathrm{wt}\,(b_\infty) = 0$
2. We need to check the embedding. To do this, we parametrize elements $b^0 = (b^0_{ij}) \in (B^{2,l})_{\min}$ with $a_k \in \mathbb{Z}_{\ge 0}$ s.t. $a_0 + 2a_1 + a_2 = l$ and $\varphi(b^0) = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2$ and $\varepsilon(b^0) = a_2\Lambda_0 + a_1\Lambda_1 + a_0\Lambda_2$.

Then we have the following:
$b^0_{11} = a_1 + a_2$, $b^0_{12} = a_1$, $b^0_{13} = a_0$, $b^0_{22} = a_2$, $b^0_{23} = a_1$, and $b^0_{24} = a_0 + a_1$. For $b \in B^{2,l}$, we define

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes B^{2,l} \otimes T_{-\varphi(b)} \to B^{2,\infty}$$

by $f_{(l,b)}(t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}) = b' = (b_{ij})'$ where $b'_{ij} = b_{ij} - b^0_{ij}$ We can see easily that
$\varepsilon_0(b') = \varepsilon_0(b) - \varepsilon_0(b^0) = \varepsilon_0(b) - a_2 = \varepsilon_0(b) - \langle \check{\alpha}_0, \varepsilon_0(b^0)\rangle$
$\varepsilon_1(b') = \varepsilon_1(b) - \varepsilon_1(b^0) = \varepsilon_1(b) - 2a_1 = \varepsilon_1(b) - \langle \check{\alpha}_1, \varepsilon_1(b^0)\rangle$
$\varepsilon_2(b') = \varepsilon_2(b) - \varepsilon_2(b^0) = \varepsilon_2(b) - a_0 = \varepsilon_2(b) - \langle \check{\alpha}_2, \varepsilon_2(b^0)\rangle$
$\varphi_0(b') = \varphi_0(b) - \varphi_0(b^0) = \varphi_0(b) - a_0 = \varphi_0(b) + \langle \check{\alpha}_0, -\varphi_0(b^0)\rangle$
$\varphi_1(b') = \varphi_1(b) - \varphi_1(b^0) = \varphi_1(b) - 2a_1 = \varphi_1(b) + \langle \check{\alpha}_1, -\varphi_1(b^0)\rangle$
$\varphi_2(b') = \varphi_2(b) - \varphi_2(b^0) = \varphi_2(b) - a_2 = \varphi_2(b) + \langle \check{\alpha}_2, -\varphi_2(b^0)\rangle$
Additionally, $\mathrm{wt}\,(b') = \sum_k (\varphi(b') - \varepsilon(b'))\Lambda_k = \mathrm{wt}\,(b) + \varepsilon(b^0) - \varphi(b^0)$.

It can be easily shown that the conditions for the actions $\tilde{e}_k(b')$ and $\tilde{f}_k(b')$ only depend on the conditions for the actions $\tilde{e}_k(b)$ and $\tilde{f}_k(b)$. So $\tilde{e}_k(b') = \tilde{e}_k(b) - b^0$. Then we can see that $f_{(l,b^0)}(\tilde{e}_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})) = f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes \tilde{e}_k(b) \otimes t_{-\varphi(b^0)}) = \tilde{e}_k(b) - b^0 = \tilde{e}_k(b') = \tilde{e}_k(f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}))$. Similarly, we see the same is true for $\tilde{f}_k$ for $k = 0,1,2$. Finally, clearly $f_{(l,b^0)}$ is injective with $f_{(l,b^0)}((t_{\varepsilon(b^0)} \otimes b^0 \otimes t_{-\varphi(b^0)}) = b^\infty$. This proves 2.

Now we prove 3. First, we see that $\sum_{j=i}^{i+2} b'_{ij} = \sum_{j=i}^{i+2} b_{ij} - \sum_{j=i}^{i+2} b^0_{ij} = l - l = 0$ for $i = 1,2$. Now we check that the other relations of a perfect crystal hold.

$$b'_{11} = b_{11} - b^0_{11} = b_{22} + b_{23} - a_1 - a_2 = b_{22} - b^0_{22} + b_{23} - b^0_{23} = b'_{22} + b'_{23}$$

$$b'_{24} = b_{24} - b^0_{24} = b_{12} + b_{13} - a_0 - a_1 = b_{12} - b^0_{12} + b_{13} - b^0_{13} = b'_{12} + b'_{13}$$

Therefore, $\bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)} \subseteq B^{2,\infty}$.

Now we want to show that $B^{2,\infty} \subseteq \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. We know $b^\infty \in \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. So we just need to check other $b' \in B^{2,\infty}$. Let $b'_{ij} + b^0_{ij} = b_{ij}$. Also, we set $a_1 = \max\{-b'_{12}, -b'_{23}, -b'_{24} + b'_{13}, -b'_{11} + b'_{22}, 0\}$, $a_2 = \max\{-b'_{22}, -b'_{11} + b'_{12}, -b'_{11} - a_1\}$, and $a_0 = \max\{-b'_{13}, -b'_{24} + b'_{23}, -b'_{24} - a_1, 0\}$. We also again set $b^0_{11} = a_1 + a_2$, $b^0_{12} = a_1$, $b^0_{13} = a_0$, $b^0_{22} = a_2$, $b^0_{23} = a_1$, and $b^0_{24} = a_0 + a_1$. Then we check that each $b_{ij} \in \mathbb{Z}_{\geq 0}$

$$b_{11} = b'_{11} + a_1 + a_2 \geq 0 \Longleftrightarrow b'_{11} \geq -a_1 - a_2 \Longleftrightarrow a_1 + a_2 \geq -b_{11}$$

$$b_{12} = b'_{12} + a_1 \geq 0 \Longleftrightarrow a_1 \geq -b_{12}$$

$$b_{23} = b'_{23} + a_1 \geq 0 \Longleftrightarrow a_1 \geq -b_{23}$$

$$b_{13} = b'_{13} + a_0 \geq 0 \Longleftrightarrow a_0 \geq -b_{13}$$

$$b_{22} = b'_{22} + a_2 \geq 0 \Longleftrightarrow a_2 \geq -b_{22}$$

$$b_{24} = b'_{24} + a_0 + a_1 \geq 0 \Longleftrightarrow b'_{24} \geq -a_0 - a_1 \Longleftrightarrow a_0 + a_1 \geq -b_{24}$$

Therefore, $b = (b_{ij}) \in \mathbb{Z}_{\geq 0}$. Now we show that the other conditions for our crystal hold. First, we know that $\sum_{j=i}^{i+2} b_{ij} = \sum_{j=i}^{i+2} b'_{ij} + \sum_{j=i}^{i+2} b^0_{ij} = l + 0 = l$ Now we check the following relations

1. $b_{11} \geq b_{22}$
2. $b_{11} + b_{12} \geq b_{22} + b_{23}$
3. $b_{11} = b_{22} + b_{23}$
4. $b_{24} = b_{12} + b_{13}$

Now we prove these relations hold:

1. $b_{11} = b'_{11} + b^0_{11} = b'_{11} + a_1 + a_2 = b'_{11} + a_1 + b^0_{22} \geq b'_{11} - b'_{11} + b'_{22} + b^0_{22} = b_{22}$
2. $b_{11} + b_{12} = b'_{11} + b^0_{11} + b'_{12} + b^0_{12} = b'_{11} + a_1 + a_2 + b'_{12} + a_1$
   $\geq b'_{22} + b'_{23} + b^0_{22} + b^0_{23} + b'_{23} - b'_{23} = b_{22} + b_{23}$
3. $b_{11} = b'_{11} + b^0_{11} = b'_{22} + b'_{23} + a_1 + a_2 = b'_{22} + b'_{23} + b^0_{22} + b^0_{23} = b_{22} + b_{23}$
4. $b_{24} = b'_{24} + b^0_{24} = b'_{12} + b'_{13} + a_0 + a_1 = b'_{12} + b'_{13} + b^0_{12} + b^0_{13} = b_{12} + b_{13}$

Therefore, $b' \in \bigcup_{(l,b)\in J} \operatorname{Im} f_{(l,b)}$, so 3. is proven, and $B^{2,l}$ is a coherent family of perfect crystals with limit $B^{2,\infty}$. □

$C_3^{(1)}$ Perfect Crystal

The Langlands dual of $C_3^{(1)}$ is $D_4^{(2)}$. We give a parametrization for the perfect crystal associated with $V(l\Lambda_3)$ along with its associated limit below:

$$B^{3,l} = \left\{ (b_{ij})_{1\leq i\leq 3, i\leq j\leq i+3} \;\middle|\; \begin{array}{l} (b_{ij}) \in \mathbb{Z}^{\geq 0}, \sum_{j=i}^{i+3} b_{ij} = l, 1 \leq i \leq 3 \\ \sum_{j=i}^{t} b_{ij} \geq \sum_{j=i+1}^{t+1} b_{i+1,j}, 1 \leq i \leq 2, 1 \leq t \leq 4 \\ b_{11} = b_{33} + b_{34} + b_{35}, b_{36} = b_{12} + b_{13} + b_{14} \\ b_{22} = b_{33} + b_{34}, \;\; b_{25} = b_{13} + b_{14} \end{array} \right\}$$

$$B^{3,\infty}=\left\{(b_{ij})_{1\le i\le 3, i\le j\le i+3}\;\middle|\;\begin{array}{l}(b_{ij})\in\mathbb{Z},\sum_{j=i}^{i+3}b_{ij}=0,1\le i\le 3\\ b_{11}=b_{33}+b_{34}+b_{35}, b_{36}=b_{12}+b_{13}+b_{14}\\ b_{22}=b_{33}+b_{34},\ \ b_{25}=b_{13}+b_{14}\end{array}\right\}$$

We have the following actions and relations of this perfect crystal:
For $0\le k\le 3$, $b\in B^{3,\infty}$ $(B^{3,l})$, $\tilde{e}_k(b)=(b'_{ij})$, where

- $k=0$ for each case $b'_{11}=b_{11}-1, b'_{36}=b_{36}+1$

$$\begin{cases} b'_{12}=b_{12}+1, b'_{22}=b_{22}-1, b'_{23}=b_{23}+1, b'_{33}=b_{33}-1 & b_{23}\ge b_{34}, 2b_{12}\ge b_{23}+b_{34},\\ & b_{12}+b_{13}\ge b_{34}+b_{35},\\ & (b_{11},b_{22},b_{33}>0)\\ b'_{22}=b_{22}-1, b'_{12}=b_{12}+1, b'_{24}=b_{24}+1, b'_{34}=b_{34}-1 & b_{12}+b_{13}\ge b_{23}+b_{35}, b_{34}>b_{23},\\ & b_{12}+b_{24}\ge b_{23}+b_{35},\\ & (b_{11},b_{22},b_{34}>0)\\ b'_{13}=b_{13}+1, b'_{23}=b_{23}-1, b'_{25}=b_{25}+1, b'_{35}=b_{35}-1 & b_{13}\ge b_{24}, b_{35}+b_{34}>b_{12}+b_{24},\\ & b_{12}<b_{23}+b_{35},\\ & (b_{11},b_{23},b_{35}>0)\\ b'_{14}=b_{14}+1, b'_{24}=b_{24}-1, b'_{25}=b_{25}+1, b'_{35}=b_{35}-1 & b_{24}>b_{13}, 2b_{35}>b_{13}+b_{24},\\ & b_{35}+b_{34}>b_{12}+b_{13},\\ & (b_{11},b_{24},b_{35}>0)\end{cases}$$

- $k=1$

$$b'_{12}=b_{12}-1, b'_{11}=b_{11}+1, b'_{36}=b_{36}-1, b'_{35}=b_{35}+1,\ (b_{12},b_{36}>0)$$

- $k=2$

$$\begin{cases} b'_{12}=b_{12}+1, b'_{13}=b_{13}-1, b'_{24}=b_{24}+1, b'_{25}=b_{25}-1 & b_{23}+b_{35}\le b_{12}+b_{24},\\ & (b_{25},b_{13}>0)\\ b'_{22}=b_{22}+1, b'_{23}=b_{23}-1, b'_{34}=b_{34}+1, b'_{35}=b_{35}-1 & b_{23}+b_{35}>b_{12}+b_{24},\\ & (b_{23},b_{35}>0)\end{cases}$$

- $k=3$

$$\begin{cases} b'_{13}=b_{13}+1, b'_{14}=b_{14}-1 & b_{24}\le b_{13}, b_{13}+b_{23}\ge b_{24}+b_{34}, (b_{14}>0) \\ b'_{23}=b_{23}+1, b'_{24}=b_{24}-1 & b_{24}>b_{13}, b_{34}\le b_{23}, (b_{24}>0) \\ b'_{33}=b_{33}+1, b'_{34}=b_{34}-1 & b_{34}>b_{23}, b_{24}+b_{34}>b_{13}+b_{23}, (b_{34}>0) \end{cases}$$

and $b'_{ij}=b_{ij}$ otherwise.
For $0\le k\le 3$, $b\in B^{3,\infty}$ $(B^{3,l})$, $\tilde{f}_k(b)=(b'_{ij})$, where

- $k=0$ for each case $b'_{11}=b_{11}+1, b'_{36}=b_{36}-1$

$$\begin{cases} b'_{12}=b_{12}-1, b'_{22}=b_{22}+1, b'_{23}=b_{23}-1, b'_{33}=b_{33}+1 & b_{23}>b_{34}, b_{12}+b_{13}>b_{34}+b_{35}, \\ & 2b_{12}>b_{23}+b_{34}, \\ & b_{12}+b_{24}\ge b_{23}+b_{35}, \\ & (b_{36}, b_{12}, b_{23}>0) \\ b'_{22}=b_{22}+1, b'_{12}=b_{12}-1, b'_{24}=b_{24}-1, b'_{34}=b_{34}+1 & b_{12}+b_{13}>b_{23}+b_{35}, b_{34}\ge b_{23}, \\ & b_{35}+b_{23}<b_{24}+b_{12}, \\ & (b_{36}, b_{12}, b_{24}>0) \\ b'_{13}=b_{13}-1, b'_{23}=b_{23}+1, b'_{25}=b_{25}-1, b'_{35}=b_{35}+1 & b_{13}>b_{24}, b_{35}+b_{34}\ge b_{12}+b_{24} \\ & b_{12}\le b_{23}+b_{35}, \\ & (b_{36}, b_{13}, b_{25}>0) \\ b'_{14}=b_{14}-1, b'_{24}=b_{24}+1, b'_{25}=b_{25}-1, b'_{35}=b_{35}+1 & b_{24}\ge b_{13}, b_{35}+b_{34}\ge b_{12}+b_{13}, \\ & 2b_{35}\ge b_{24}+b_{13}, \\ & (b_{36}, b_{14}, b_{25}>0) \end{cases}$$

- $k=1$

$$b'_{12}=b_{12}+1, b'_{11}=b_{11}-1, b'_{36}=b_{36}+1, b'_{35}=b_{35}-1,\ (b_{11}, b_{35}>0)$$

- $k=2$

$$\begin{cases} b'_{12}=b_{12}-1, b'_{13}=b_{13}+1, b'_{24}=b_{24}-1, b'_{25}=b_{25}+1 & b_{23}+b_{35}<b_{12}+b_{24}, \\ & (b_{24}, b_{12}>0) \\ b'_{22}=b_{22}-1, b'_{23}=b_{23}+1, b'_{34}=b_{34}-1, b'_{35}=b_{35}+1 & b_{23}+b_{35}\geq b_{12}+b_{24}, \\ & (b_{22}, b_{34}>0) \end{cases}$$

- $k=3$

$$\begin{cases} b'_{13}=b_{13}-1, b'_{14}=b_{14}+1 & b_{13}>b_{24}, b_{13}+b_{23}>b_{24}+b_{34}, (b_{14}>0) \\ b'_{23}=b_{23}-1, b'_{24}=b_{24}+1 & b_{23}>b_{34}, b_{13}\leq b_{24}, (b_{23}>0) \\ b'_{33}=b_{33}-1, b'_{34}=b_{34}+1 & b_{23}\leq b_{34}, b_{24}+b_{34}\geq b_{13}+b_{23}, (b_{34}>0) \end{cases}$$

and $b'_{ij}=b_{ij}$ otherwise.

The actions ε_i, φ_i and wt_i are defined as follows: $\varepsilon_1(b)=b_{12}$
$\varepsilon_2(b)=b_{13}+(b_{23}-b_{12})_+$
$\varepsilon_3(b)=b_{14}+\max\{b_{24}-b_{13}, b_{34}+b_{24}-b_{13}-b_{23}, 0\}$

$$\varepsilon_0(b)=\begin{cases} l-b_{36}-\min\{b_{12}+b_{13}, b_{23}+b_{24}, b_{34}+b_{35}, b_{12}+b_{24}, b_{23}+b_{35}\} & b\in B^{2,l} \\ -b_{36}-\min\{b_{12}+b_{13}, b_{23}+b_{24}, b_{34}+b_{35}, b_{12}+b_{24}, b_{23}+b_{35}\} & b\in B^{2,\infty} \end{cases}$$

$$\varphi_0(b)=\begin{cases} l-b_{11}-\min\{b_{12}+b_{13}, b_{23}+b_{24}, b_{34}+b_{35}, b_{12}+b_{24}, b_{23}+b_{35}\} & b\in B^{2,l} \\ -b_{11}-\min\{b_{12}+b_{13}, b_{23}+b_{24}, b_{34}+b_{35}, b_{12}+b_{24}, b_{23}+b_{35}\} & b\in B^{2,\infty} \end{cases}$$

$\varphi_1(b)=b_{35}$
$\varphi_2(b)=b_{22}-b_{33}+(b_{12}-b_{23})_+$
$\varphi_3(b)=b_{33}+\max\{b_{23}-b_{34}, b_{13}+b_{23}-b_{24}-b_{34}, 0\}$
$\mathrm{wt}_0(b)=b_{36}-b_{11}$
$\mathrm{wt}_1(b)=b_{35}-b_{12}$
$\mathrm{wt}_2(b)=b_{22}-b_{33}-b_{25}+b_{14}+b_{12}-b_{23}$
$\mathrm{wt}_3(b)=b_{33}-b_{14}+b_{13}-b_{24}+b_{23}-b_{34}$
Note that $(B^{3,l})_{\min}=\{b\in B^{3,l} | b_{12}=b_{23}=b_{34}, b_{13}=b_{24}=b_{35}\}$

For $\lambda\in\bar{P}$, we consider the crystal $T_\lambda=\{t_\lambda\}$ with $\tilde{e}_k(t_\lambda)=\tilde{f}_k(t_\lambda)=0$, $\varepsilon_k(t_\lambda)=\varphi_k(t_\lambda)=-\infty$, and $\mathrm{wt}(t_\lambda)=\lambda$ for $k=0,1,2$. For $\lambda,\mu\in\bar{P}$, $T_\lambda\otimes B^{2,l}\otimes T_\mu$ is a crystal given by:

$\tilde{e}_k(t_\lambda \otimes b \otimes t_\lambda) = t_\lambda \otimes \tilde{e}_k(b) \otimes t_\lambda$ $\qquad$ $\tilde{f}_k(t_\lambda \otimes b \otimes t_\lambda) = t_\lambda \otimes \tilde{f}_k(b) \otimes t_\lambda$

$\varepsilon_k(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_k(b) - \langle \check{\alpha}_k, \lambda\rangle$ $\qquad$ $\varphi_k(t_\lambda \otimes b \otimes t_\mu) = \varphi_k(b) + \langle \check{\alpha}_k, \mu\rangle$

$\mathrm{wt}\,(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + \mathrm{wt}\,(b)$

Theorem 4.2.6. *$B^{3,l}$ is a coherent family of perfect crystals and the crystal $B^{3,\infty}$ is its limit with the vector $b_\infty = (0)_{3\times 6}$.*

Proof. We again check the 3 conditions in Definition 4.2.3.
1. For each $k = 0,1,2,3$, $\varepsilon_k(b_\infty) = \varphi_k(b_\infty) = 0$, so $\varphi(b_\infty) = \varepsilon(b_\infty) = \mathrm{wt}(b_\infty) = 0$
2. We need to check the embedding. To do this, we parametrize elements $b^0 = (b^0_{ij}) \in (B^{3,l})_{\min}$ with $a_k \in \mathbb{Z}_{\geq 0}$ s.t. $a_0 + 2a_1 + 2a_2 + a_3 = l$ and $\varphi(b^0) = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3$ and $\varepsilon(b^0) = a_3\Lambda_0 + a_2\Lambda_1 + 2a_1\Lambda_2 + a_0\Lambda_3$. Then we have the following:
$b^0_{11} = a_1 + a_2 + a_3$, $b^0_{12} = b^0_{23} = b^0_{34} = a_2$, $b^0_{13} = b^0_{24} = b^0_{35} = a_1$, $b^0_{14} = a_0$, $b^0_{22} = a_2 + a_3$, $b^0_{25} = a_0 + a_1$, $b^0_{33} = a_3$, and $b^0_{36} = a_0 + a_1 + a_2$. For $b \in B^{3,l}$, we define

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes B^{2,l} \otimes T_{-\varphi(b)} \to B^{2,\infty}$$

by $f_{(l,b)}(t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}) = b' = (b_{ij})'$ where $b'_{ij} = b_{ij} - b^0_{ij}$ We can see easily that
$\varepsilon_0(b') = \varepsilon_0(b) - \varepsilon_0(b^0) = \varepsilon_0(b) - a_3 = \varepsilon_0(b) - \langle \check{\alpha}_0, \varepsilon_0(b^0)\rangle$
$\varepsilon_1(b') = \varepsilon_1(b) - \varepsilon_1(b^0) = \varepsilon_1(b) - 2a_2 = \varepsilon_1(b) - \langle \check{\alpha}_1, \varepsilon_1(b^0)\rangle$
$\varepsilon_2(b') = \varepsilon_2(b) - \varepsilon_2(b^0) = \varepsilon_2(b) - 2a_1 = \varepsilon_2(b) - \langle \check{\alpha}_2, \varepsilon_2(b^0)\rangle$
$\varepsilon_3(b') = \varepsilon_3(b) - \varepsilon_3(b^0) = \varepsilon_3(b) - a_0 = \varepsilon_3(b) - \langle \check{\alpha}_3, \varepsilon_3(b^0)\rangle$
$\varphi_0(b') = \varphi_0(b) - \varphi_0(b^0) = \varphi_0(b) - a_0 = \varphi_0(b) + \langle \check{\alpha}_0, -\varphi_0(b^0)\rangle$
$\varphi_1(b') = \varphi_1(b) - \varphi_1(b^0) = \varphi_1(b) - 2a_1 = \varphi_1(b) + \langle \check{\alpha}_1, -\varphi_1(b^0)\rangle$
$\varphi_2(b') = \varphi_2(b) - \varphi_2(b^0) = \varphi_2(b) - 2a_2 = \varphi_2(b) + \langle \check{\alpha}_2, -\varphi_2(b^0)\rangle$
$\varphi_3(b') = \varphi_3(b) - \varphi_3(b^0) = \varphi_3(b) - a_3 = \varphi_3(b) + \langle \check{\alpha}_3, -\varphi_3(b^0)\rangle$
Additionally, $\mathrm{wt}\,(b') = \sum_k (\varphi(b') - \varepsilon(b'))\Lambda_k = \mathrm{wt}\,(b) + \varepsilon(b^0) - \varphi(b^0)$.

It can be easily shown that the conditions for the actions $\tilde{e}_k(b')$ and $\tilde{f}_k(b')$ only depend on the conditions for the actions $\tilde{e}_k(b)$ and $\tilde{f}_k(b)$. So $\tilde{e}_k(b') = \tilde{e}_k(b) - b^0$. Then we can see that $f_{(l,b^0)}(\tilde{e}_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})) = f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes \tilde{e}_k(b) \otimes t_{-\varphi(b^0)}) = \tilde{e}_k(b) - b^0 = \tilde{e}_k(b') = \tilde{e}_k(f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}))$. Similarly, it can be easily shown that the same is true for $\tilde{f}_k$ for $k = 0,1,2,3$. Finally, clearly $f_{(l,b^0)}$ is injective with $f_{(l,b^0)}((t_{\varepsilon(b^0)} \otimes b^0 \otimes t_{-\varphi(b^0)}) = b_\infty$. This proves 2.

Now we prove 3. First, we see that $\sum_{j=i}^{i+3} b'_{ij} = \sum_{j=i}^{i+3} b_{ij} - \sum_{j=i}^{i+3} b^0_{ij} = l - l = 0$ for $i = 1,2,3$. Now we check that the other relations of a perfect crystal hold.

$$b'_{11} = b_{11} - b^0_{11} = b_{33} + b_{34} + b_{35} - a_1 - a_2 - a_3 = b_{33} - b^0_{33} + b_{34} - b^0_{34} + b_{35} - b^0_{35} = b'_{33} + b'_{34} + b'_{35}$$

$$b'_{36} = b_{36} - b^0_{36} = b_{12} + b_{13} + b_{14} - a_0 - a_1 - a_2 = b_{12} - b^0_{12} + b_{13} - b^0_{13} + b_{14} - b^0_{14} = b'_{12} + b'_{13} + b'_{14}$$

$$b'_{25} = b_{25} - b^0_{25} = b_{13} + b_{14} - a_0 - a_1 = b_{13} - b^0_{13} + b_{14} - b^0_{14} = b'_{13} + b'_{14}$$

$$b'_{22} = b_{22} - b^0_{22} = b_{33} + b_{34} - a_2 - a_3 = b_{33} - b^0_{33} + b_{34} - b^0_{34} = b'_{33} + b'_{34}$$

Therefore, $\bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)} \subseteq B^{3,\infty}$.

Now we want to show that $B^{3,\infty} \subseteq \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. We know $b^\infty \in \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. So we just need to check other $b' \in B^{3,\infty}$. Let $b'_{ij} + b^0_{ij} = b_{ij}$. Also, we set

$$\begin{aligned}
a_1 =& \max\{-b'_{13}, -b'_{24}, -b'_{35}, -b'_{25} + b'_{14}, -b'_{11} + b'_{22}, -b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24},\\
&- b'_{11} - b'_{12} + b'_{22} + b'_{23}, -b'_{36} + b'_{14} + b'_{12}, -b'_{36} + b'_{14} + b'_{23}, -b'_{36} + b'_{14} + b'_{34},\\
&- b'_{11} + b'_{33} + b'_{12}, -b'_{11} + b'_{33} + b'_{23}, -b'_{11} + b'_{33} + b'_{34}, 0\}\\
a_2 =& \max\{-b'_{12}, -b'_{23}, -b'_{34}, -b'_{22} + b'_{33}, -b'_{22} - b'_{23} + b'_{33} + b'_{34}, -b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34} + b'_{35},\\
&- b'_{36} + b'_{25}, -b'_{11} + b'_{33} - a_1, -b'_{36} + b'_{14} - a_1, 0\}\\
a_3 =& \max\{-b'_{33}, -b'_{22} - a_2, -b'_{11} - a_1 - a_2, 0\}\\
a_0 =& \max\{-b'_{14}, -b'_{25} - a_1, -b'_{36} - a_1 - a_2, 0\}
\end{aligned}$$

We also again set $b^0_{11} = a_1 + a_2 + a_3$, $b^0_{12} = b^0_{23} = b^0_{34} = a_2$, $b^0_{13} = b^0_{24} = b^0_{35} = a_1$, $b^0_{14} = a_0$, $b^0_{22} = a_2 + a_3$, $b^0_{25} = a_0 + a_1$, $b^0_{33} = a_3$, and $b^0_{36} = a_0 + a_1 + a_2$.. Then we check that each $b_{ij} \in \mathbb{Z}_{\geq 0}$

$$b_{11} = b'_{11} + a_1 + a_2 + a_3 \geq 0 \Longleftrightarrow a_1 + a_2 + a_3 \geq -b_{11}$$

$$b_{12} = b'_{12} + a_2 \geq 0 \Longleftrightarrow a_2 \geq -b_{12}$$

$$b_{13} = b'_{13} + a_1 \geq 0 \Longleftrightarrow a_1 \geq -b'_{13}$$

$$b_{14} = b'_{14} + a_0 \geq 0 \Longleftrightarrow a_0 \geq -b_{14}$$

$$b_{22} = b'_{22} + a_2 + a_3 \geq 0 \Longleftrightarrow a_2 + a_3 \geq -b_{22}$$

$$b_{23} = b'_{23} + a_2 \geq 0 \Longleftrightarrow a_2 \geq -b_{23}$$

$$b_{24} = b'_{24} + a_1 \geq 0 \Longleftrightarrow a_1 \geq -b_{24}$$

$$b_{25} = b'_{25} + a_0 + a_1 \geq 0 \Longleftrightarrow a_0 + a_1 \geq -b'_{25}$$

$$b_{33} = b'_{33} + a_3 \geq 0 \Longleftrightarrow a_3 \geq -b_{33}$$

$$b_{34} = b'_{34} + a_2 \geq 0 \Longleftrightarrow a_2 \geq -b_{34}$$

$$b_{35} = b'_{35} + a_1 \geq 0 \Longleftrightarrow a_1 \geq -b'_{35}$$

$$b_{36} = b'_{36} + a_0 + a_1 + a_2 \geq 0 \Longleftrightarrow a_0 + a_1 + a_2 \geq -b'_{36}$$

Therefore, $b = (b_{ij}) \in \mathbb{Z}_{\geq 0}$. Now we show that the other conditions for our crystal hold. First, we know that $\sum_{j=i}^{i+3} b_{ij} = \sum_{j=i}^{i+3} b'_{ij} + \sum_{j=i}^{i+3} b^0_{ij} = l + 0 = l$ Now we check the following relations

1. $b_{11} \geq b_{22}$
2. $b_{11} + b_{12} \geq b_{22} + b_{23}$
3. $b_{11} + b_{12} + b_{13} \geq b_{22} + b_{23} + b_{24}$
4. $b_{22} \geq b_{33}$
5. $b_{22} + b_{23} \geq b_{33} + b_{34}$
6. $b_{22} + b_{23} + b_{24} \geq b_{33} + b_{34} + b_{35}$
7. $b_{11} = b_{33} + b_{34} + b_{35}$
8. $b_{36} = b_{12} + b_{13} + b_{14}$
9. $b_{22} = b_{33} + b_{34}$
10. $b_{25} = b_{13} + b_{14}$

Now we prove these relations hold:

1. $b_{11} = b'_{11} + b^0_{11} = b'_{11} + a_1 + a_2 + a_3 = b'_{11} + a_1 + b^0_{22} \geq b'_{11} - b'_{11} + b'_{22} + b^0_{22} = b_{22}$
2. $b_{11} + b_{12} = b'_{11} + b^0_{11} + b'_{12} + b^0_{12} = b'_{11} + a_1 + a_2 + a_3 + b'_{12} + a_2 = b'_{11} + b'_{12} + a_1 + b^0_{22} + b^0_{23}$ $\geq b'_{11} + b'_{12} - b'_{11} - b'_{12} + b'_{22} + b'_{23} + b^0_{22} + b^0_{23} = b_{22} + b_{23}$
3. $b_{11} + b_{12} + b_{13} = b'_{11} + b'_{12} + b'_{13} + 2a_1 + 2a_2 + a_3 = b'_{11} + b'_{12} + b'_{13} + a_1 + b^0_{22} + b^0_{23} + b^0_{24}$ $\geq b'_{11} + b'_{12} + b'_{13} - b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24} + b^0_{22} + b^0_{23} + b^0_{24} = b_{22} + b_{23} + b_{24}$
4. $b_{22} = b'_{22} + a_2 + a_3 \geq b'_{22} - b'_{22} + b'_{33} + b^0_{33} = b_{33}$
5. $b_{22} + b_{23} = b'_{22} + b'_{23} + 2a_2 + a_3 \geq b'_{22} + b'_{23} - b'_{22} - b'_{23} + b'_{33} + b'_{34} + b^0_{33} + b^0_{34} = b_{33} + b_{34}$

6. $b_{22}+b_{23}+b_{24}=b'_{22}+b'_{23}+b'_{24}+a_1+2a_2+a_3 \geq b'_{22}+b'_{23}+b'_{24}-b'_{22}-b'_{23}-b'_{24}+b'_{33}+b'_{34}+b'_{35}+b^0_{33}+b^0_{34}+b^0_{35}=b_{33}+b_{34}+b_{35}$

7. $b_{11}=b'_{11}+b^0_{11}=b'_{33}+b'_{34}+b'_{35}+a_1+a_2+a_3=b'_{33}+b'_{34}+b'_{35}+b^0_{33}+b^0_{34}+b^0_{35}=b_{33}+b_{34}+b_{35}$

8. $b_{36}=b'_{36}+b^0_{36}=b'_{12}+b'_{13}+b'_{14}+a_0+a_1+a_2=b'_{12}+b'_{13}+b'_{14}+b^0_{12}+b^0_{13}+b^0_{14}=b_{12}+b_{13}+b_{14}$

9. $b_{22}=b'_{22}+a_2+a_3=b'_{33}+b'_{34}+b^0_{33}+b^0_{34}=b_{33}+b_{34}$

10. $b_{25}=b'_{25}+a_0+a_1=b'_{13}+b'_{14}+b^0_{13}+b^0_{14}=b_{13}+b_{14}$

Therefore, $b' \in \bigcup_{(l,b)\in J} \operatorname{Im} f_{(l,b)}$, so 3. is proven. □

$C_4^{(1)}$ Perfect Crystal

The Langlands Dual of $C_4^{(1)}$ is $D_5^{(2)}$. We can parametrize the perfect crystal $B(l\Lambda_4)$ and its limit $B^{4,\infty}$ as follows:

$$B^{4,l}=\left\{ (b_{ij})_{1\leq i\leq 4, i\leq j\leq i+4} \;\middle|\; \begin{array}{l} (b_{ij})\in\mathbb{Z}^{\geq 0}, \sum_{j=i}^{i+4} b_{ij}=l, 1\leq i\leq 4 \\ \sum_{j=i}^{t} b_{ij} \geq \sum_{j=i+1}^{t+1} b_{i+1,j}, 1\leq i\leq 3, 1\leq t\leq 5 \\ b_{11}=b_{44}+b_{45}+b_{46}+b_{47}, b_{48}=b_{12}+b_{13}+b_{14}+b_{15}, \\ b_{22}=b_{44}+b_{45}+b_{46},\;\; b_{37}=b_{13}+b_{14}+b_{15}, \\ b_{22}+b_{23}=b_{33}+b_{34}+b_{35},\;\; b_{36}+b_{37}=b_{24}+b_{25}+b_{26}, \\ b_{33}=b_{44}+b_{45},\;\; b_{26}=b_{14}+b_{15} \end{array} \right\}$$

$$B^{4,\infty}=\left\{ (b_{ij})_{1\leq i\leq 4, i\leq j\leq i+4} \;\middle|\; \begin{array}{l} (b_{ij})\in\mathbb{Z}, \sum_{j=i}^{i+4} b_{ij}=0, 1\leq i\leq 4 \\ b_{11}=b_{44}+b_{45}+b_{46}+b_{47}, b_{48}=b_{12}+b_{13}+b_{14}+b_{15}, \\ b_{22}=b_{44}+b_{45}+b_{46},\;\; b_{37}=b_{13}+b_{14}+b_{15}, \\ b_{22}+b_{23}=b_{33}+b_{34}+b_{35},\;\; b_{36}+b_{37}=b_{24}+b_{25}+b_{26}, \\ b_{33}=b_{44}+b_{45},\;\; b_{26}=b_{14}+b_{15} \end{array} \right\}$$

We have the following actions and relations of this perfect crystal: For $0\leq k\leq 4$, $b\in B^{4,\infty}$ ($B^{4,l}$), $\tilde{e}_k(b)=(b'_{ij})$, where

For $k=0$ $b'_{11}=b_{11}-1$, $b'_{48}=b_{48}+1$ and

$$\begin{cases} b'_{12}=b_{12}+1,\ b'_{23}=b_{23}+1,\ b'_{34}=b_{34}+1,\ b'_{22}=b_{22}-1,\ b'_{33}=b_{33}-1, b'_{44}=b_{44}-1 & (A1)\\ b'_{13}=b_{13}+1,\ b'_{24}=b_{24}+1,\ b'_{37}=b_{37}+1,\ b'_{23}=b_{23}-1,\ b'_{34}=b_{34}-1, b'_{47}=b_{47}-1 & (A2)\\ b'_{12}=b_{12}+1,\ b'_{24}=b_{24}+1,\ b'_{36}=b_{36}+1,\ b'_{22}=b_{22}-1,\ b'_{34}=b_{34}-1,\ b'_{46}=b_{46}-1 & (A3)\\ b'_{12}=b_{12}+1,\ b'_{23}=b_{23}+1,\ b'_{35}=b_{35}+1,\ b'_{22}=b_{22}-1,\ b'_{33}=b_{33}-1,\ b'_{45}=b_{45}-1 & (A4)\\ b'_{12}=b_{12}+1,\ b'_{25}=b_{25}+1,\ b'_{36}=b_{36}+1,\ b'_{22}=b_{22}-1,\ b'_{35}=b_{35}-1,\ b'_{46}=b_{46}-1 & (A5)\\ b'_{14}=b_{14}+1,\ b'_{26}=b_{26}+1, b'_{37}=b_{37}+1,\ b'_{24}=b_{24}-1,\ b'_{36}=b_{36}-1,\ b'_{47}=b_{47}-1 & (A6)\\ b'_{13}=b_{13}+1,\ b'_{25}=b_{25}+1,\ b'_{37}=b_{37}+1,\ b'_{23}=b_{23}-1,\ b'_{35}=b_{35}-1,\ b'_{47}=b_{47}-1 & (A7)\\ b'_{15}=b_{15}+1,\ b'_{26}=b_{26}+1,\ b'_{37}=b_{37}+1,\ b'_{25}=b_{25}-1,\ b'_{36}=b_{36}-1,\ b'_{47}=b_{47}-1 & (A8) \end{cases}$$

$(A1) = b_{12}+b_{35}+b_{36} \ge b_{45}+b_{46}+b_{47},\ b_{23}+b_{35} \ge b_{45}+b_{46},\ b_{34} \ge b_{45},\ b_{23}+b_{24} \ge b_{45}+b_{46},$

$b_{12}+b_{13}+b_{25} \ge b_{45}+b_{46}+b_{47},\ b_{12}+b_{24}+b_{36} \ge b_{45}+b_{46}+b_{47},$

$b_{12}+b_{13}+b_{14} \ge b_{45}+b_{46}+b_{47}, (b_{11}>0, b_{22}>0, b_{33}>0, b_{44}>0)$

$(A2) = b_{45}+b_{46}+b_{47} > b_{12}+b_{35}+b_{36},\ b_{23}+b_{47} \ge b_{12}+b_{36},\ b_{34}+b_{46}+b_{47} \ge b_{12}+b_{35}+b_{36},$

$b_{23}+b_{24}+b_{47} \ge b_{12}+b_{35}+b_{36},\ b_{13}+b_{25} \ge b_{35}+b_{36},\ b_{24} \ge b_{35},\ b_{13}+b_{14} \ge b_{35}+b_{36},$

$b_{23}+b_{24}+b_{47} > b_{12}+b_{35}+b_{36}$ or $b_{23}+b_{47} > b_{12}+b_{36}, (b_{11}>0, b_{23}>0, b_{34}>0, b_{47}>0)$

$(A3) = b_{45}+b_{46} > b_{23}+b_{35},\ b_{12}+b_{36} \ge b_{23}+b_{47},\ b_{34}+b_{46} \ge b_{23}+b_{35},\ b_{24} \ge b_{35},$

$b_{12}+b_{13}+b_{25} \ge b_{23}+b_{35}+b_{47},\ b_{12}+b_{13}+b_{14} \ge b_{23}+b_{35}+b_{47},$

$b_{12}+b_{24}+b_{36} \ge b_{23}+b_{35}+b_{47},\ b_{12}+b_{36} \ge b_{23}+b_{47},\ b_{24}+b_{36} \ge b_{13}+b_{25},$

$(b_{11}, b_{22}, b_{34}, b_{46} > 0)$

$(A4) = b_{45} > b_{34},\ b_{12}+b_{35}+b_{36} > b_{34}+b_{46}+b_{47},\ b_{23}+b_{35} > b_{34}+b_{46},\ b_{23}+b_{24} \ge b_{34}+b_{46},$

$b_{12}+b_{13}+b_{25} \ge b_{34}+b_{46}+b_{47},\ b_{12}+b_{24}+b_{36} \ge b_{34}+b_{46}+b_{47},$

$b_{12}+b_{13}+b_{14} \ge b_{34}+b_{46}+b_{47}, (b_{11}, b_{22}, b_{33}, b_{45} > 0)$

$(A5) = b_{45}+b_{46} > b_{23}+b_{24},\ b_{12}+b_{35}+b_{36} > b_{23}+b_{24}+b_{47},\ b_{35} > b_{24},\ b_{34}+b_{46} > b_{23}+b_{24},$

$b_{12}+b_{13}+b_{25} \ge b_{23}+b_{24}+b_{47},\ b_{12}+b_{36} \ge b_{23}+b_{47},\ b_{12}+b_{13}+b_{14} \ge b_{23}+b_{24}+b_{47},$

$b_{12}+b_{36} \ge b_{23}+b_{47},$ or $b_{12}+b_{35}+b_{36} \ge b_{34}+b_{46}+b_{47}, (b_{11}, b_{22}, b_{35}, b_{46} > 0)$

$$(A6) = b_{45} + b_{46} + b_{47} > b_{12} + b_{13} + b_{25},\ b_{35} + b_{36} > b_{13} + b_{25},\ b_{23} + b_{35} + b_{47} > b_{12} + b_{13} + b_{25},$$
$$b_{34} + b_{46} + b_{47} > b_{12} + b_{13} + b_{25},\ b_{23} + b_{24} + b_{47} > b_{12} + b_{13} + b_{25},\ b_{24} + b_{36} \geq b_{13} + b_{25},$$
$$b_{14} \geq b_{25}, (b_{11}, b_{24}, b_{36}, b_{47} > 0)$$
$$(A7) = b_{45} + b_{46} + b_{47} > b_{12} + b_{24} + b_{36},\ b_{35} > b_{24},\ b_{23} + b_{35} + b_{47} > b_{12} + b_{24} + b_{36},$$
$$b_{34} + b_{46} + b_{47} > b_{12} + b_{24} + b_{36},\ b_{23} + b_{47} > b_{12} + b_{36},\ b_{13} + b_{25} > b_{24} + b_{36},$$
$$b_{13} + b_{14} \geq b_{24} + b_{36},\ b_{34} + b_{46} \leq b_{23} + b_{35}, (b_{11}, b_{23}, b_{35}, b_{47} > 0)$$
$$(A8) = b_{45} + b_{46} + b_{47} > b_{12} + b_{13} + b_{14},\ b_{35} + b_{36} > b_{13} + b_{14},\ b_{23} + b_{35} + b_{47} > b_{12} + b_{13} + b_{14},$$
$$b_{34} + b_{46} + b_{47} > b_{12} + b_{13} + b_{14},\ b_{23} + b_{24} + b_{47} > b_{12} + b_{13} + b_{14},\ b_{25} > b_{14},$$
$$b_{36} + b_{24} > b_{13} + b_{14}, (b_{11}, b_{25}, b_{36}, b_{47} > 0)$$

For $k = 1$, $b'_{11} = b_{11} + 1$, $b'_{12} = b_{12} - 1$, $b'_{47} = b_{47} + 1$, and $b'_{48} = b_{48} - 1$ and if $b \in B^{4,l}$ we have the additional conditions $b_{12}, b_{48} > 0$.

For $k = 2$,

$$\begin{cases} b'_{12} = b_{12} + 1,\ b'_{13} = b_{13} - 1,\ b'_{36} = b_{36} + 1,\ b'_{37} = b_{37} - 1 & b_{23} + b_{47} \leq b_{12} + b_{36}, (b_{13}, b_{37} > 0) \\ b'_{22} = b_{22} + 1,\ b'_{23} = b_{23} - 1,\ b'_{46} = b_{46} + 1,\ b'_{47} = b_{47} - 1 & b_{23} + b_{47} > b_{12} + b_{36}, (b_{23}, b_{47} > 0) \end{cases}$$

For $k = 3$,

$$\begin{cases} b'_{33} = b_{33} + 1,\ b'_{34} = b_{34} - 1,\ b'_{45} = b_{45} + 1,\ b'_{46} = b_{46} - 1 & b_{25} + b_{35} < b_{36} + b_{46} \\ & b_{35} < b_{46},\ (b_{34}, b_{46} > 0) \\ b'_{23} = b_{23} + 1,\ b'_{24} = b_{24} - 1,\ b'_{35} = b_{35} + 1,\ b'_{36} = b_{36} - 1 & b_{35} \geq b_{46},\ b_{13} < b_{24},\ (b_{24}, b_{36} > 0) \\ b'_{13} = b_{13} + 1,\ b'_{14} = b_{14} - 1,\ b'_{25} = b_{25} + 1,\ b'_{26} = b_{26} - 1 & b_{13} \geq b_{24} \\ & b_{25} + b_{35}, \geq b_{36} + b_{46},\ (b_{14}, b_{26} > 0) \end{cases}$$

For $k=4$,

$$\begin{cases} b'_{14}=b_{14}+1,\ b'_{15}=b_{15}-1 & b_{14}\geq b_{25},\ b_{14}+b_{24}\geq b_{25}+b_{35},\ b_{14}+b_{24}+b_{34}\geq b_{25}+b_{35}+b_{45}, \\ & (b_{15}>0) \\ b'_{24}=b_{24}+1,\ b'_{25}=b_{25}-1 & b_{14}<b_{25},\ b_{24}\geq b_{35},\ b_{24}+b_{34}\geq b_{35}+b_{45},\ (b_{25}>0) \\ b'_{34}=b_{34}+1,\ b'_{35}=b_{35}-1 & b_{14}+b_{24}<b_{25}+b_{35},\ b_{24}<b_{35},\ b_{34}\geq b_{45},\ (b_{35}>0) \\ b'_{44}=b_{44}+1,\ b'_{45}=b_{45}-1 & b_{14}+b_{24}+b_{34}<b_{25}+b_{35}+b_{45},\ b_{24}+b_{34}<b_{35}+b_{45},\ b_{34}<b_{45}, \\ & (b_{45}>0) \end{cases}$$

For $0\leq k\leq 4$, $b\in B^{4,\infty}$, $\tilde{f}_k(b)=(b'_{ij})$, where For $k=0$ $b'_{11}=b_{11}+1$, $b'_{48}=b_{48}-1$ and

$$\begin{cases} b'_{12}=b_{12}-1,\ b'_{23}=b_{23}-1,\ b'_{34}=b_{34}-1,\ b'_{22}=b_{22}+1,\ b'_{33}=b_{33}+1, b'_{44}=b_{44}+1 & (B1) \\ b'_{13}=b_{13}-1,\ b'_{24}=b_{24}-1,\ b'_{37}=b_{37}-1,\ b'_{23}=b_{23}+1,\ b'_{34}=b_{34}+1, b'_{47}=b_{47}+1 & (B2) \\ b'_{12}=b_{12}-1,\ b'_{24}=b_{24}-1,\ b'_{36}=b_{36}-1,\ b'_{22}=b_{22}+1,\ b'_{34}=b_{34}+1,\ b'_{46}=b_{46}+1 & (B3) \\ b'_{12}=b_{12}-1,\ b'_{23}=b_{23}-1,\ b'_{35}=b_{35}-1,\ b'_{22}=b_{22}+1,\ b'_{33}=b_{33}+1,\ b'_{45}=b_{45}+1 & (B4) \\ b'_{12}=b_{12}-1,\ b'_{25}=b_{25}-1,\ b'_{36}=b_{36}-1,\ b'_{22}=b_{22}+1,\ b'_{35}=b_{35}+1,\ b'_{46}=b_{46}+1 & (B5) \\ b'_{14}=b_{14}-1,\ b'_{26}=b_{26}-1, b'_{37}=b_{37}-1,\ b'_{24}=b_{24}+1,\ b'_{36}=b_{36}+1,\ b'_{47}=b_{47}+1 & (B6) \\ b'_{13}=b_{13}-1,\ b'_{25}=b_{25}-1,\ b'_{37}=b_{37}-1,\ b'_{23}=b_{23}+1,\ b'_{35}=b_{35}+1,\ b'_{47}=b_{47}+1 & (B7) \\ b'_{15}=b_{15}-1,\ b'_{26}=b_{26}-1,\ b'_{37}=b_{37}-1,\ b'_{25}=b_{25}+1,\ b'_{36}=b_{36}+1,\ b'_{47}=b_{47}+1 & (B8) \end{cases}$$

$$\begin{aligned} (B1)=\ & b_{12}+b_{35}+b_{36}>b_{45}+b_{46}+b_{47},\ b_{23}+b_{35}>b_{45}+b_{46},\ b_{34}>b_{45},\ b_{23}+b_{24}>b_{45}+b_{46}, \\ & b_{12}+b_{13}+b_{25}>b_{45}+b_{46}+b_{47},\ b_{12}+b_{24}+b_{36}>b_{45}+b_{46}+b_{47}, \\ & b_{12}+b_{13}+b_{14}>b_{45}+b_{46}+b_{47},\ (b_{48},b_{12},b_{23},b_{34}>0) \\ (B2)=\ & b_{45}+b_{46}+b_{47}\geq b_{12}+b_{35}+b_{36},\ b_{23}+b_{47}>b_{12}+b_{36},\ b_{34}+b_{46}+b_{47}>b_{12}+b_{35}+b_{36}, \\ & b_{23}+b_{24}+b_{47}>b_{12}+b_{35}+b_{36},\ b_{13}+b_{25}>b_{35}+b_{36},\ b_{24}>b_{35},\ b_{13}+b_{14}>b_{35}+b_{36}, \\ & b_{23}+b_{24}+b_{47}\geq b_{12}+b_{35}+b_{36} \text{ or } b_{23}+b_{47}\geq b_{12}+b_{36},\ (b_{48},b_{13},b_{24},b_{37}>0) \end{aligned}$$

$(B3) = b_{45} + b_{46} \geq b_{23} + b_{35},\ b_{12} + b_{36} > b_{23} + b_{47},\ b_{34} + b_{46} > b_{23} + b_{35},\ b_{24} > b_{35},$

$b_{12} + b_{13} + b_{25} > b_{23} + b_{35} + b_{47},\ b_{12} + b_{13} + b_{14} > b_{23} + b_{35} + b_{47},$

$b_{12} + b_{24} + b_{36} > b_{23} + b_{35} + b_{47},\ b_{12} + b_{36} > b_{23} + b_{47},\ b_{24} + b_{36} > b_{13} + b_{25},\ (b_{48},$

$b_{12}, b_{24}, b_{36} > 0)$

$(B4) = b_{45} \geq b_{34},\ b_{12} + b_{35} + b_{36} \geq b_{34} + b_{46} + b_{47},\ b_{23} + b_{35} \geq b_{34} + b_{46},\ b_{23} + b_{24} > b_{34} + b_{46},$

$b_{12} + b_{13} + b_{25} > b_{34} + b_{46} + b_{47},\ b_{12} + b_{24} + b_{36} > b_{34} + b_{46} + b_{47},$

$b_{12} + b_{13} + b_{14} > b_{34} + b_{46} + b_{47},\ (b_{48}, b_{12}, b_{23}, b_{35} > 0)$

$(B5) = b_{45} + b_{46} \geq b_{23} + b_{24},\ b_{12} + b_{35} + b_{36} \geq b_{23} + b_{24} + b_{47},\ b_{35} \geq b_{24},\ b_{34} + b_{46} \geq b_{23} + b_{24},$

$b_{12} + b_{13} + b_{25} > b_{23} + b_{24} + b_{47},\ b_{12} + b_{36} > b_{23} + b_{47},\ b_{12} + b_{13} + b_{14} > b_{23} + b_{24} + b_{47},$

$b_{12} + b_{36} > b_{23} + b_{47}$, or $b_{12} + b_{35} + b_{36} > b_{34} + b_{46} + b_{47},\ (b_{48}, b_{12}, b_{25}, b_{36} > 0)$

$(B6) = b_{45} + b_{46} + b_{47} \geq b_{12} + b_{13} + b_{25},\ b_{35} + b_{36} \geq b_{13} + b_{25},\ b_{23} + b_{35} + b_{47} \geq b_{12} + b_{13} + b_{25},$

$b_{34} + b_{46} + b_{47} \geq b_{12} + b_{13} + b_{25},\ b_{23} + b_{24} + b_{47} \geq b_{12} + b_{13} + b_{25},\ b_{24} + b_{36} > b_{13} + b_{25},$

$b_{14} > b_{25},\ (b_{48}, b_{14}, b_{26}, b_{37} > 0)$

$(B7) = b_{45} + b_{46} + b_{47} \geq b_{12} + b_{24} + b_{36},\ b_{35} \geq b_{24},\ b_{23} + b_{35} + b_{47} \geq b_{12} + b_{24} + b_{36},$

$b_{34} + b_{46} + b_{47} \geq b_{12} + b_{24} + b_{36},\ b_{23} + b_{47} \geq b_{12} + b_{36},\ b_{13} + b_{25} \geq b_{24} + b_{36},$

$b_{13} + b_{14} > b_{24} + b_{36},\ b_{34} + b_{46} < b_{23} + b_{35},\ (b_{48}, b_{13}, b_{25}, b_{37} > 0)$

$(B8) = b_{45} + b_{46} + b_{47} \geq b_{12} + b_{13} + b_{14},\ b_{35} + b_{36} \geq b_{13} + b_{14},\ b_{23} + b_{35} + b_{47} \geq b_{12} + b_{13} + b_{14},$

$b_{34} + b_{46} + b_{47} \geq b_{12} + b_{13} + b_{14},\ b_{23} + b_{24} + b_{47} \geq b_{12} + b_{13} + b_{14},\ b_{25} \geq b_{14},$

$b_{36} + b_{24} \geq b_{13} + b_{14},\ (b_{48}, b_{15}, b_{26}, b_{37} > 0)$

For $k = 1$, $b'_{11} = b_{11} - 1$, $b'_{12} = b_{12} + 1$, $b'_{47} = b_{47} - 1$, and $b'_{48} = b_{48} + 1$ and if $b \in B^{4,l}$ we have the additional conditions $b_{11}, b_{47} > 0$.

For $k = 2$,

$$\begin{cases} b'_{12} = b_{12} - 1,\ b'_{13} = b_{13} + 1,\ b'_{36} = b_{36} - 1,\ b'_{37} = b_{37} + 1 & b_{23} + b_{47} < b_{12} + b_{36} \\ & (b_{12}, b_{36} > 0) \\ b'_{22} = b_{22} - 1,\ b'_{23} = b_{23} + 1,\ b'_{46} = b_{46} - 1,\ b'_{47} = b_{47} + 1 & b_{23} + b_{47} \geq b_{12} + b_{36} \\ & (b_{22}, b_{46} > 0) \end{cases}$$

For $k=3$,

$$\begin{cases} b'_{33}=b_{33}-1,\ b'_{34}=b_{34}+1,\ b'_{45}=b_{45}-1,\ b'_{46}=b_{46}+1 & b_{25}+b_{35}\le b_{36}+b_{46} \\ & b_{35}\le b_{46},\ (b_{33},b_{45}>0) \\ b'_{23}=b_{23}-1,\ b'_{24}=b_{24}+1,\ b'_{35}=b_{35}-1,\ b'_{36}=b_{36}+1 & b_{35}>b_{46},\ b_{13}\le b_{24},\ (b_{23},b_{35}>0) \\ b'_{13}=b_{13}-1,\ b'_{14}=b_{14}+1,\ b'_{25}=b_{25}-1,\ b'_{26}=b_{26}+1 & b_{25}+b_{35}>b_{36}+b_{46} \\ & b_{13}>b_{24},\ (b_{13},b_{25}>0) \end{cases}$$

For $k=4$,

$$\begin{cases} b'_{14}=b_{14}-1,\ b'_{15}=b_{15}+1 & b_{14}>b_{25},\ b_{14}+b_{24}>b_{25}+b_{35},\ b_{14}+b_{24}+b_{34}>b_{25}+b_{35}+b_{45} \\ & (b_{14}>0) \\ b'_{24}=b_{24}-1,\ b'_{25}=b_{25}+1 & b_{14}\le b_{25},\ b_{24}>b_{35},\ b_{24}+b_{34}>b_{35}+b_{45},\ (b_{24}>0) \\ b'_{34}=b_{34}-1,\ b'_{35}=b_{35}+1 & b_{14}+b_{24}\le b_{25}+b_{35},\ b_{24}\le b_{35},\ b_{34}>b_{45},\ (b_{34}>0) \\ b'_{44}=b_{44}-1,\ b'_{45}=b_{45}+1 & b_{14}+b_{24}+b_{34}\le b_{25}+b_{35}+b_{45},\ b_{24}+b_{34}\le b_{35}+b_{45},\ b_{34}\le b_{45}, \\ & (b_{44}>0) \end{cases}$$

Then we have the following formulas for $\varphi_i(b)$.

$$\varphi_0(b)=\begin{cases} -b_{11}-\min\{b_{45}+b_{46}+b_{47}, b_{34}+b_{46}+b_{47}, b_{12}+b_{35}+b_{36}, b_{23}+b_{35} & b\in B^{4,\infty} \\ +b_{47}, b_{23}+b_{24}+b_{47}, b_{12}+b_{13}+b_{25}, b_{12}+b_{24}+b_{36}, b_{12}+b_{13}+b_{14}\} & \\ l-b_{11}-\min\{b_{45}+b_{46}+b_{47}, b_{34}+b_{46}+b_{47}, b_{12}+b_{35}+b_{36}, b_{23}+b_{35} & b\in B^{4,l} \\ +b_{47}, b_{23}+b_{24}+b_{47}, b_{12}+b_{13}+b_{25}, b_{12}+b_{24}+b_{36}, b_{12}+b_{13}+b_{14}\} & \end{cases}$$

$$\varphi_1(b)=b_{47}$$
$$\varphi_2(b)=b_{46}+\max\{b_{12}-b_{23},0\}$$
$$\varphi_3(b)=b_{45}+\max\{b_{13}-b_{24}, b_{13}+b_{23}-b_{24}-b_{34},0\}$$
$$\varphi_4(b)=b_{44}+\max\{b_{14}-b_{25}, b_{14}+b_{24}-b_{25}-b_{35}, b_{14}+b_{24}+b_{34}-b_{25}-b_{35}-b_{45},0\}$$

And the following formulas for $\varepsilon_i(b)$.

$$\varepsilon_0(b)=\begin{cases} -b_{48}-\min\{b_{45}+b_{46}+b_{47}, b_{34}+b_{46}+b_{47}, b_{12}+b_{35}+b_{36}, b_{23}+b_{35} & b\in B^{4,\infty} \\ +b_{47}, b_{23}+b_{24}+b_{47}, b_{12}+b_{13}+b_{25}, b_{12}+b_{24}+b_{36}, b_{12}+b_{13}+b_{14}\} & \\ l-b_{48}-\min\{b_{45}+b_{46}+b_{47}, b_{34}+b_{46}+b_{47}, b_{12}+b_{35}+b_{36}, b_{23}+b_{35} & b\in B^{4,l} \\ +b_{47}, b_{23}+b_{24}+b_{47}, b_{12}+b_{13}+b_{25}, b_{12}+b_{24}+b_{36}, b_{12}+b_{13}+b_{14}\} & \end{cases}$$

$$\varepsilon_1(b)=b_{12}$$
$$\varepsilon_2(b)=b_{13}+\max\{b_{23}-b_{12},0\}$$
$$\varepsilon_3(b)=b_{14}+\max\{b_{24}-b_{13}, b_{24}+b_{34}-b_{13}-b_{23},0\}$$
$$\varepsilon_4(b)=b_{15}+\max\{b_{25}-b_{14}, b_{25}+b_{35}-b_{24}-b_{14}, b_{25}+b_{35}+b_{45}-b_{14}-b_{24}-b_{34},0\}$$

Finally, the formulas for $\mathrm{wt}_i(b)$ are:

$$\mathrm{wt}_0(b)=b_{48}-b_{11}$$
$$\mathrm{wt}_1(b)=b_{47}-b_{12}$$
$$\mathrm{wt}_2(b)=b_{46}-b_{13}+b_{12}-b_{23}$$
$$\mathrm{wt}_3(b)=b_{45}-b_{14}+b_{13}-b_{24}+b_{23}-b_{34}$$
$$\mathrm{wt}_4(b)=b_{44}-b_{15}+b_{14}-b_{25}+b_{24}-b_{35}+b_{34}-b_{45}$$

Note that $(B^{4,l})_{\min}=\{b\in B^{4,l}|b_{12}=b_{23}=b_{34}=b_{45}, b_{13}=b_{24}=b_{35}=b_{46}, b_{14}=b_{25}=b_{36}=b_{47}\}$

For $\lambda\in\bar{P}$, we consider the crystal $T_\lambda=\{t_\lambda\}$ with $\tilde{e}_k(t_\lambda)=\tilde{f}_k(t_\lambda)=0$, $\varepsilon_k(t_\lambda)=\varphi_k(t_\lambda)=-\infty$, and $\mathrm{wt}(t_\lambda)=\lambda$ for $k=0,1,2,3,4$. For $\lambda,\mu\in\bar{P}$, $T_\lambda\otimes B^{4,l}\otimes T_\mu$ is a crystal given by:

$$\tilde{e}_k(t_\lambda\otimes b\otimes t_\lambda)=t_\lambda\otimes\tilde{e}_k(b)\otimes t_\lambda \qquad \tilde{f}_k(t_\lambda\otimes b\otimes t_\lambda)=t_\lambda\otimes\tilde{f}_k(b)\otimes t_\lambda$$
$$\varepsilon_k(t_\lambda\otimes b\otimes t_\mu)=\varepsilon_k(b)-\langle\check{\alpha}_k,\lambda\rangle \qquad \varphi_k(t_\lambda\otimes b\otimes t_\mu)=\varphi_k(b)+\langle\check{\alpha}_k,\mu\rangle$$
$$\mathrm{wt}\,(t_\lambda\otimes b\otimes t_\mu)=\lambda+\mu+\mathrm{wt}\,(b)$$

Theorem 4.2.7. *$B^{4,l}$ is a coherent family of perfect crystals and the crystal $B^{4,\infty}$ is its limit with the vector $b_\infty=(0)_{4\times 8}$.*

Proof. 1. For each $k=0,1,2,3,4$, $\varepsilon_k(b_\infty)=\varphi_k(b_\infty)=0$, so $\varphi(b_\infty)=\varepsilon(b_\infty)=\mathrm{wt}(b_\infty)=0$
2. We need to check the embedding. To do this, we parametrize elements $b^0=(b^0_{ij})\in(B^{4,l})_{\min}$ with $a_k\in\mathbb{Z}_{\geq 0}$ s.t. $a_0+2a_1+2a_2+2a_3+a_4=l$ and $\varphi(b^0)=a_0\Lambda_0+a_1\Lambda_1+a_2\Lambda_2+a_3\Lambda_3+a_4\Lambda_4$

and $\varepsilon(b^0) = a_4\Lambda_0 + a_3\Lambda_1 + a_2\Lambda_2 + a_1\Lambda_3 + a_0\Lambda_4$. Then we have the following:
$b^0_{11} = a_1 + a_2 + a_3 + a_4$, $b^0_{12} = b^0_{23} = b^0_{34} = b^0_{45} = a_3$, $b^0_{13} = b^0_{24} = b^0_{35} = b^0_{46} = a_2$, $b^0_{14} = b^0_{25} = b^0_{36} = b^0_{47} = a_1$, $b^0_{15} = a_0$, $b^0_{22} = a_2 + a_3 + a_4$, $b^0_{26} = a_0 + a_1$, $b^0_{33} = a_3 + a_4$, $b^0_{37} = a_0 + a_1 + a_2$, $b_{44} = a_4$, and $b_{48} = a_0 + a_1 + a_2 + a_3$.

For $b \in B^{4,l}$, we define

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes B^{4,l} \otimes T_{-\varphi(b)} \to B^{4,\infty}$$

by $f_{(l,b)}(t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}) = b' = (b_{ij})'$ where $b'_{ij} = b_{ij} - b^0_{ij}$ We can see easily that
$\varepsilon_0(b') = \varepsilon_0(b) - \varepsilon_0(b^0) = \varepsilon_0(b) - a_4 = \varepsilon_0(b) - \langle \check{\alpha}_0, \varepsilon_0(b^0)\rangle$
$\varepsilon_1(b') = \varepsilon_1(b) - \varepsilon_1(b^0) = \varepsilon_1(b) - 2a_3 = \varepsilon_1(b) - \langle \check{\alpha}_1, \varepsilon_1(b^0)\rangle$
$\varepsilon_2(b') = \varepsilon_2(b) - \varepsilon_2(b^0) = \varepsilon_2(b) - 2a_2 = \varepsilon_2(b) - \langle \check{\alpha}_2, \varepsilon_2(b^0)\rangle$
$\varepsilon_3(b') = \varepsilon_3(b) - \varepsilon_3(b^0) = \varepsilon_3(b) - 2a_1 = \varepsilon_3(b) - \langle \check{\alpha}_3, \varepsilon_3(b^0)\rangle$
$\varepsilon_4(b') = \varepsilon_4(b) - \varepsilon_4(b^0) = \varepsilon_4(b) - a_0 = \varepsilon_4(b) - \langle \check{\alpha}_4, \varepsilon_4(b^0)\rangle$
$\varphi_0(b') = \varphi_0(b) - \varphi_0(b^0) = \varphi_0(b) - a_0 = \varphi_0(b) + \langle \check{\alpha}_0, -\varphi_0(b^0)\rangle$
$\varphi_1(b') = \varphi_1(b) - \varphi_1(b^0) = \varphi_1(b) - 2a_1 = \varphi_1(b) + \langle \check{\alpha}_1, -\varphi_1(b^0)\rangle$
$\varphi_2(b') = \varphi_2(b) - \varphi_2(b^0) = \varphi_2(b) - 2a_2 = \varphi_2(b) + \langle \check{\alpha}_2, -\varphi_2(b^0)\rangle$
$\varphi_3(b') = \varphi_3(b) - \varphi_3(b^0) = \varphi_3(b) - 2a_3 = \varphi_3(b) + \langle \check{\alpha}_3, -\varphi_3(b^0)\rangle$
$\varphi_4(b') = \varphi_4(b) - \varphi_4(b^0) = \varphi_4(b) - a_4 = \varphi_4(b) + \langle \check{\alpha}_4, -\varphi_4(b^0)\rangle$
Additionally, $\mathrm{wt}\,(b') = \sum_k (\varphi(b') - \varepsilon(b'))\Lambda_k = \mathrm{wt}\,(b) + \varepsilon(b^0) - \varphi(b^0)$.

It can be easily shown that the conditions for the actions $\tilde{e}_k(b')$ and $\tilde{f}_k(b')$ only depend on the conditions for the actions $\tilde{e}_k(b)$ and $\tilde{f}_k(b)$. So $\tilde{e}_k(b') = \tilde{e}_k(b) - b^0$. Then we can see that $f_{(l,b^0)}(\tilde{e}_k(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)})) = f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes \tilde{e}_k(b) \otimes t_{-\varphi(b^0)}) = \tilde{e}_k(b) - b^0 = \tilde{e}_k(b') = \tilde{e}_k(f_{(l,b^0)}(t_{\varepsilon(b^0)} \otimes b \otimes t_{-\varphi(b^0)}))$. Similarly, we see the same is true for $\tilde{f}_k$ for $k = 0,1,2,3,4$. Finally, clearly $f_{(l,b^0)}$ is injective with $f_{(l,b^0)}((t_{\varepsilon(b^0)} \otimes b^0 \otimes t_{-\varphi(b^0)}) = b_\infty$. This proves 2.

Now we prove 3. First, we see that $\sum_{j=i}^{i+4} b'_{ij} = \sum_{j=i}^{i+4} b_{ij} - \sum_{j=i}^{i+4} b^0_{ij} = l - l = 0$ for $i = 1,2,3,4$.

Now we check that the other relations of a perfect crystal hold.

$$
\begin{aligned}
&b'_{11} = b_{11} - b^0_{11} = b_{44} + b_{45} + b_{46} + b_{47} - a_1 - a_2 - a_3 - a_4 = b_{44} - b^0_{44} + b_{45} - b^0_{45}\\
&+ b_{46} - b^0_{46} + b_{47} - b^0_{47} = b'_{44} + b'_{45} + b'_{46} + b'_{47}\\
&b'_{48} = b_{48} - b^0_{48} = b_{12} + b_{13} + b_{14} + b_{15} - a_0 - a_1 - a_2 - a_3 = b_{12} - b^0_{12} + b_{13} - b^0_{13}\\
&+ b_{14} - b^0_{14} + b_{15} - b^0_{15} = b'_{12} + b'_{13} + b'_{14} + b'_{15}\\
&b'_{22} = b_{22} - b^0_{22} = b_{44} + b_{45} + b_{46} - a_2 - a_3 - a_4 = b_{44} - b^0_{44} + b_{45} - b^0_{45} + b_{46} - b^0_{46}\\
&= b'_{44} + b'_{45} + b'_{46}\\
&b'_{37} = b_{37} - b^0_{37} = b_{13} + b_{14} + b_{15} - a_0 - a_1 - a_2 = b_{13} - b^0_{13} + b_{14} - b^0_{14} + b_{15} - b^0_{15}\\
&= b'_{13} + b'_{14} + b'_{15}\\
&b'_{33} = b_{33} - b^0_{33} = b_{44} + b_{45} - a_3 - a_4 = b_{44} + b_{45} - b^0_{44} - b^0_{45} = b'_{44} + b'_{45}\\
&b'_{26} = b_{26} - b^0_{26} = b_{14} + b_{15} - a_0 - a_1 = b_{14} + b_{15} - b^0_{14} - b^0_{15} = b'_{14} + b'_{15}\\
&b'_{22} + b'_{23} = b_{22} + b_{23} - b^0_{22} - b^0_{23} = b_{33} + b_{34} + b_{35} - a_2 - 2a_3 - a_4 = b_{33} - b^0_{33} + b_{34} - b^0_{34}\\
&+ b_{35} - b^0_{35} = b'_{33} + b'_{34} + b'_{35}\\
&b'_{36} + b'_{37} = b_{36} + b_{37} - b^0_{36} - b^0_{37} = b_{24} + b_{25} + b_{26} - a_0 - 2a_1 - a_2 = b_{24} - b^0_{24} + b_{25} - b^0_{25}\\
&+ b_{26} - b^0_{26} = b'_{24} + b'_{25} + b'_{26}
\end{aligned}
$$

Therefore, $\bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)} \subseteq B^{4,\infty}$.

Now we want to show that $B^{4,\infty} \subseteq \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. We know $b^{\infty} \in \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$. So we just need to check other $b' \in B^{4,\infty}$. Let $b'_{ij} + b^0_{ij} = b_{ij}$. Also, we set

$$
\begin{aligned}
a_1 = \max\{&-b'_{14}, -b'_{25}, -b'_{36}, , -b'_{47}, -b'_{26} + b'_{15}, -b'_{11} + b'_{22}, -b'_{37} + b'_{15} + b'_{13}, -b'_{37} + b'_{15} + b'_{24},\\
&- b'_{37} + b_{15} + b'_{35}, -b'_{37} + b_{15} + b'_{46}, -b'_{11} - b'_{12} + b'_{22} + b'_{23},\\
&- b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24}, -b'_{11} - b'_{12} - b'_{13} - b'_{14} + b'_{22} + b'_{23} + b'_{24} + b'_{25}, 0\}\\
a_2 = \max\{&-b'_{13}, -b'_{24}, -b'_{35}, -b'_{46}, -b'_{22} + b'_{33}, -b'_{22} - b'_{23} + b'_{33} + b'_{34}, -b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34}\\
&+ b'_{35}, -b'_{22} - b'_{23} - b'_{24} - b'_{25} + b'_{33} + b'_{34} + b'_{35} + b'_{36}, 0\}\\
a_3 = \max\{&-b'_{12}, -b'_{23}, -b'_{34}, -b'_{45}, -b'_{33} + b'_{44}, -b'_{33} - b'_{34} + b'_{44} + b'_{45}, -b'_{33} - b'_{34} - b'_{35}\\
&+ b'_{44} + b'_{45} + b'_{46}, -b'_{33} - b'_{34} - b'_{35} - b'_{36} + b'_{44} + b'_{45} + b'_{46} + b'_{47}, 0\}\\
a_4 = \max\{&-b'_{44}, -b'_{33} - a_3, , -b'_{22} - a_2 - a_3, -b'_{11} - a_1 - a_2 - a_3, 0\}\\
a_0 = \max\{&-b'_{15}, -b'_{26} - a_1, -b'_{37} - a_1 - a_2, , -b'_{48} - a_1 - a_2 - a_3, 0\}
\end{aligned}
$$

We also again set $b^0_{11}=a_1+a_2+a_3+a_4$, $b^0_{12}=b^0_{23}=b^0_{34}=b^0_{45}=a_3$, $b^0_{13}=b^0_{24}=b^0_{35}=b^0_{46}=a_2$,$b^0_{14}=b^0_{25}=b^0_{36}=b^0_{47}=a_1$, $b^0_{15}=a_0$, $b^0_{22}=a_2+a_3+a_4$, $b^0_{26}=a_0+a_1$, $b^0_{33}=a_3+a_4$, $b^0_{37}=a_0+a_1+a_2$, $b_{44}=a_4$, and $b_{48}=a_0+a_1+a_2+a_3$. Then we check that each $b_{ij}\in\mathbb{Z}_{\geq 0}$

$$b_{11}=b'_{11}+a_1+a_2+a_3+a_4\geq 0 \iff a_1+a_2+a_3+a_4\geq -b_{11}$$

$$b_{12}=b'_{12}+a_3\geq 0 \iff a_3\geq -b'_{12}$$

$$b_{13}=b'_{13}+a_2\geq 0 \iff a_2\geq -b_{13}$$

$$b_{14}=b'_{14}+a_1\geq 0 \iff a_1\geq -b'_{14}$$

$$b_{15}=b'_{15}+a_0\geq 0 \iff a_0\geq -b_{15}$$

$$b_{22}=b'_{22}+a_2+a_3+a_4\geq 0 \iff a_4\geq -b_{22}-a_2-a_3$$

$$b_{23}=b'_{23}+a_3\geq 0 \iff a_3\geq -b_{23}$$

$$b_{24}=b'_{24}+a_2\geq 0 \iff a_2\geq -b_{24}$$

$$b_{25}=b'_{25}+a_1\geq 0 \iff a_1\geq -b'_{25}$$

$$b_{26}=b'_{26}+a_0+a_1\geq 0 \iff a_0\geq -b'_{26}-a_1$$

$$b_{33}=b'_{33}+a_3+a_4\geq 0 \iff a_4\geq -b'_{33}-a_3$$

$$b_{34}=b'_{34}+a_3\geq 0 \iff a_3\geq -b'_{34}$$

$$b_{35}=b'_{35}+a_2\geq 0 \iff a_2\geq -b'_{35}$$

$$b_{36}=b'_{36}+a_1\geq 0 \iff a_1\geq -b'_{36}$$

$$b_{37}=b'_{37}+a_0+a_1+a_2\geq 0 \iff a_0\geq -b'_{37}-a_1-a_2$$

$$b_{44}=b'_{44}+a_4\geq 0 \iff a_4\geq -b'_{44}$$

$$b_{45}=b'_{45}+a_3\geq 0 \iff a_3\geq -b_{45}$$

$$b_{46}=b'_{35}+a_2\geq 0 \iff a_2\geq -b'_{46}$$

$$b_{47}=b'_{47}+a_1\geq 0 \iff a_1\geq -b'_{47}$$

$$b_{48}=b'_{48}+a_0+a_1+a_2+a_3\geq 0 \iff a_0\geq -b'_{48}-a_1-a_2-a_3$$

Therefore, $b=(b_{ij})\in\mathbb{Z}_{\geq 0}$. Now we show that the other conditions for our crystal hold. First, we know that $\sum_{j=i}^{i+3} b_{ij} = \sum_{j=i}^{i+3} b'_{ij} + \sum_{j=i}^{i+3} b^0_{ij} = l + 0 = l$ Now we check the following relations

1. $b_{11} \geq b_{22}$
2. $b_{11} + b_{12} \geq b_{22} + b_{23}$
3. $b_{11} + b_{12} + b_{13} \geq b_{22} + b_{23} + b_{24}$
4. $b_{11} + b_{12} + b_{13} + b_{14} \geq b_{22} + b_{23} + b_{24} + b_{25}$
5. $b_{22} \geq b_{33}$
6. $b_{22} + b_{23} \geq b_{33} + b_{34}$
7. $b_{22} + b_{23} + b_{24} \geq b_{33} + b_{34} + b_{35}$
8. $b_{22} + b_{23} + b_{24} + b_{25} \geq b_{33} + b_{34} + b_{35} + b_{36}$
9. $b_{33} \geq b_{44}$
10. $b_{33} + b_{34} \geq b_{44} + b_{45}$
11. $b_{33} + b_{34} + b_{35} \geq b_{44} + b_{45} + b_{46}$
12. $b_{33} + b_{34} + b_{35} + b_{36} \geq b_{44} + b_{45} + b_{46} + b_{47}$
13. $b_{11} = b_{44} + b_{45} + b_{46} + b_{47}$
14. $b_{48} = b_{12} + b_{13} + b_{14} + b_{15}$
15. $b_{22} = b_{44} + b_{45} + b_{46}$
16. $b_{37} = b_{13} + b_{14} + b_{15}$
17. $b_{22} + b_{23} = b_{33} + b_{34} + b_{35}$
18. $b_{36} + b_{37} = b_{24} + b_{25} + b_{26}$
19. $b_{33} = b_{44} + b_{45}$
20. $b_{26} = b_{14} + b_{15}$

Now we prove these relations hold:

1. $b_{11} = b'_{11} + b^0_{11} = b'_{11} + a_1 + a_2 + a_3 + a_4 = b'_{11} + a_1 + b^0_{22} \geq b'_{11} - b'_{11} + b'_{22} + b^0_{22} = b_{22}$

2. $b_{11} + b_{12} = b'_{11} + b^0_{11} + b'_{12} + b^0_{12} = b'_{11} + a_1 + a_2 + a_3 + a_4 + b'_{12} + a_3 = b'_{11} + b'_{12} + a_1 + b^0_{22} + b^0_{23} \geq b'_{11} + b'_{12} - b'_{11} - b'_{12} + b'_{22} + b'_{23} + b^0_{22} + b^0_{23} = b_{22} + b_{23}$

3. $b_{11} + b_{12} + b_{13} = b'_{11} + b'_{12} + b'_{13} + a_1 + 2a_2 + 2a_3 + a_4 = b'_{11} + b'_{12} + b'_{13} + a_1 + b^0_{22} + b^0_{23} + b^0_{24} \geq b'_{11} + b'_{12} + b'_{13} - b'_{11} - b'_{12} - b'_{13} + b'_{22} + b'_{23} + b'_{24} + b^0_{22} + b^0_{23} + b^0_{24} = b_{22} + b_{23} + b_{24}$

4. $b_{11} + b_{12} + b_{13} + b_{14} = b'_{11} + b'_{12} + b'_{13} + b'_{14} + 2a_1 + 2a_2 + 2a_3 + a_4 = b'_{11} + b'_{12} + b'_{13} + b_{14} + a_1 + b^0_{22} + b^0_{23} + b^0_{24} \geq b'_{11} + b'_{12} + b'_{13} + b'_{14} - b'_{11} - b'_{12} - b'_{13} - b'_{14} + b'_{22} + b'_{23} + b'_{24} + b'_{25} + b^0_{22} + b^0_{23} + b^0_{24} = b_{22} + b_{23} + b_{24} + b_{25}$

5. $b_{22} = b'_{22} + a_2 + a_3 + a_4 \geq b'_{22} - b'_{22} + b'_{33} + b^0_{33} = b_{33}$

6. $b_{22} + b_{23} = b'_{22} + b'_{23} + a_2 + 2a_3 + a_4 \geq b'_{22} + b'_{23} - b'_{22} - b'_{23} + b'_{33} + b'_{34} + b^0_{33} + b^0_{34} = b_{33} + b_{34}$

7. $b_{22} + b_{23} + b_{24} = b'_{22} + b'_{23} + b'_{24} + 2a_2 + 2a_3 + a_4 \geq b'_{22} + b'_{23} + b'_{24} - b'_{22} - b'_{23} - b'_{24} + b'_{33} + b'_{34} + b'_{35} + b^0_{33} + b^0_{34} + b^0_{35} = b_{33} + b_{34} + b_{35}$

8. $b_{22} + b_{23} + b_{24} + b_{25} = b'_{22} + b'_{23} + b'_{24} + b'_{25} + a_1 + 2a_2 + 2a_3 + a_4 \geq b'_{22} + b'_{23} + b'_{24} + b'_{25} - b'_{22} - b'_{23} - b'_{24} - b'_{25} + b'_{33} + b'_{34} + b'_{35} + b_{36} + b^0_{33} + b^0_{34} + b^0_{35} + b^0_{36} = b_{33} + b_{34} + b_{35} + b_{36}$

9. $b_{33} = b'_{33} + a_3 + a_4 \geq b'_{33} - b'_{33} + b'_{44} + b^0_{44} = b_{44}$

10. $b_{33} + b_{34} = b'_{33} + b'_{34} + 2a_3 + a_4 \geq b'_{33} + b'_{34} - b'_{33} - b'_{34} + b'_{44} + b'_{45} + b^0_{44} + b^0_{45} = b_{44} + b_{45}$

11. $b_{33} + b_{34} + b_{35} = b'_{33} + b'_{34} + b'_{35} + a_2 + 2a_3 + a_4 \geq b'_{33} + b'_{34} + b'_{35} - b'_{33} - b'_{34} - b'_{35} + b'_{44} + b'_{45} + b'_{46} + b^0_{44} + b^0_{45} + b^0_{46} = b_{44} + b_{45} + b_{46}$

12. $b_{33} + b_{34} + b_{35} + b_{36} = b'_{33} + b'_{34} + b'_{35} + b'_{36} + a_1 + a_2 + 2a_3 + a_4 \geq b'_{33} + b'_{34} + b'_{35} + b'_{36} - b'_{33} - b'_{34} - b'_{35} - b'_{36} + b'_{44} + b'_{45} + b'_{46} + b_{47} + b^0_{44} + b^0_{45} + b^0_{46} + b^0_{47} = b_{44} + b_{45} + b_{46} + b_{47}$

13. $b_{11} = b'_{11} + b^0_{11} = b'_{44} + b'_{45} + b'_{46} + b'_{47} + a_1 + a_2 + a_3 + a_4 = b'_{44} + b'_{45} + b'_{46} + b'_{47} + b^0_{44} + b^0_{45} + b^0_{46} + b^0_{47} = b_{44} + b_{45} + b_{46} + b_{47}$

14. $b_{48} = b'_{48} + b^0_{48} = b'_{12} + b'_{13} + b'_{14} + b'_{15} + a_0 + a_1 + a_2 + a_3 = b'_{12} + b'_{13} + b'_{14} + b'_{15} + b^0_{12} + b^0_{13} + b^0_{14} + b^0_{15} = b_{12} + b_{13} + b_{14} + b_{15}$

15. $b_{22} = b'_{22} + a_2 + a_3 + a_4 = b'_{44} + b'_{45} + b'_{46} + b^0_{44} + b^0_{45} + b^0_{46} = b_{44} + b_{45} + b_{46}$

16. $b_{37} = b'_{37} + a_0 + a_1 + a_2 = b'_{13} + b'_{14} + b'_{15} + b^0_{13} + b^0_{14} + b^0_{15} = b_{13} + b_{14} + b_{15}$

17. $b_{22}+b_{23}=b'_{22}+b'_{23}+a_2+2a_3+a_4=b'_{33}+b'_{34}+b'_{35}+b^0_{33}+b^0_{34}+b^0_{35}=b_{33}+b_{34}+b_{35}$

18. $b_{36}+b_{37}=b'_{36}+b'_{37}+a_0+2a_1+a_2=b'_{24}+b'_{25}+b'_{26}+b^0_{24}+b^0_{25}+b^0_{26}=b_{24}+b_{25}+b_{26}$

19. $b_{33}=b'_{33}+a_3+a_4=b'_{44}+b'_{45}+b^0_{44}+b^0_{45}=b_{44}+b_{45}$

20. $b_{26}=b_{26}+a_0+a_1=b'_{14}+b'_{15}+b^0_{14}+b^0_{15}=b_{14}+b_{15}$

Therefore, $b' \in \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$, so 3. is proven. $\square$

CHAPTER

5

GEOMETRIC CRYSTALS

5.1 Kac-Moody Groups

We will present the theory of Kac Moody groups and geometric crystals following [2, 19, 23, 30, 33]. Consider the Kac-Moody algebra $\mathfrak{g}$ with root data $(\mathfrak{h}, \Pi, \Pi^\vee)$. Associated with $\mathfrak{g}$ is a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ and an index set $I = \{0, 1, \dots n\}$. Recall that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. Also, let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$.

Let G be a Kac-Moody group associated with $\mathfrak{g}$ and the weight lattice P of $\mathfrak{g}$. One way to construct G is as follows: We start with G^*, which is the free product of $\mathfrak{g}_\alpha$ where $\alpha \in \Delta^{\mathrm{re}}$, the set of real roots. There is a clear inclusion map $i_\alpha : \mathfrak{g}_\alpha \hookrightarrow G^*$. Also, for every $\mathfrak{g}'$-module V, there exists a map $\pi : \mathfrak{g}' \to \mathrm{End}(V)$ and a corresponding map $\pi^* : G^* \to \mathrm{Aut}(V)$. Now $\pi^*(i_\alpha(e)) = \exp(\pi(e))$. If we let $N^* = \bigcap \ker \pi^*$ for all π^*, then $G = G^*/N^*$.

Now $U_\alpha := \exp(\mathfrak{g}_\alpha)$ where $\alpha \in \Delta^{\mathrm{re}}$ is defined to be the one parameter subgroup of G, which generates G. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$, where $\alpha \in \Delta^{\mathrm{re}}_+$. The group $\mathrm{SL}(2,\mathbb{C})$ is a subgroup of any Kac-Moody group, and so their exist unique homomorphisms $\phi_i : \mathrm{SL}(2,\mathbb{C}) \to G$ such that $\phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(t e_i) := x_i(t)$, $\phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(t f_i) := y_i(t)$

where $t \in \mathbb{C}$. Let $\check{\alpha}_i(c) = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$. Let $G_i := \phi_i(SL_2(\mathbb{C}))$, and let T be the subgroup of G with lattice P, which is called the maximal torus of G. Let $T_i := \phi_i(\{\text{diag}(c, c^{-1})|c \in \mathbb{C}^*\})$, and $N_i(T) := N_{G_i}(T_i)$, where $N_{G_i}(T_i)$ is the normalizer of T_i in G_i. Let N be a subgroup of G generated by the N_i's. $B^{\pm} = U^{\pm}T$ is the Borel subgroup of G. Then $\phi : W \to G/N$ defined by $\phi(s_i) = N_i T/T$ is an isomorphism between N/T and the Weyl group of $\mathfrak{g}$, W, and s_i is a simple reflection in W.

5.2 Definition

The simple reflections s_i, generate the Weyl group, and are defined on $\mathfrak{h}^*$ as $s_i(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i$. We define $R(w)$, the set of reduced expressions of w as follows:

$$R(w) := \{(i_1, i_2, ..., i_l) \in I^l | w = s_{i_1}...s_{i_l}\}$$

where l is the length of the reduced expression of w.

Now let X be an ind-variety, $\gamma_i : X \to \mathbb{C}$, and $\varepsilon_i : X \to \mathbb{C}$ be rational functions, and $e_i^c : \mathbb{C}^\times \times X \to X$ be $\mathbb{C}^\times$ actions on X.

Definition 5.2.1. *A quadruple $(X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a $\mathfrak{g}$-geometric crystal if:*

1. $\{1\} \times X \cap dom\{e_i\}$ *is open dense in* $\{1\} \times X$ *for any* $i \in I$
2. $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$
3. $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ *and* $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ *if* $a_{ij} = a_{ji} = 0$
4. *The e_i satisfy:*

$$\begin{aligned}
&e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} \quad \textit{if } a_{ij} = a_{ji} = 0 \\
&e_i^{c_1} e_j^{c_1c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1} \quad \textit{if } a_{ij} = a_{ji} = -1 \\
&e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^2c_2} e_i^{c_1} \quad \textit{if } a_{ij} = -2,\ a_{ji} = -1 \\
&e_i^{c_1} e_j^{c_1^3c_2} e_i^{c_1^2c_2} e_j^{c_1^3c_2^2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^3c_2^2} e_i^{c_1^2c_2} e_j^{c_1^3c_2} e_i^{c_1} \quad \textit{if } a_{ij} = -3,\ a_{ji} = -1
\end{aligned}$$

Now let us consider a specific instance where we can obtain explicit formulas, as presented in [2, 30]. Let $w = s_{i_1}...s_{i_k}$ be an element of the Weyl group with the given reduced expression. Let $X := G/B$ be the flag variety which is also an ind-variety and

$X_w := BwB/B \subset X$ be the Schubert cell associated with w. X_w has a natural geometric crystal structure and is isomorphic to $B_{\mathbf{i}}^-$ defined as follows:

$$B_{\mathbf{i}}^- = \{Y_{i_1}(c_1)\dots Y_{i_k}(c_k) | c_1,\dots,c_k \in \mathbb{C}^\times \subset B^-\}$$

where $\mathbf{i} = (i_1,\dots,i_k)$ and $Y_i(c) = y_i(\frac{1}{c})\check{\alpha}_i(c)$. Then the explicit formulas for e_i^c, ε_i, and γ_i are given by

$$e_i^c(Y_{\mathbf{i}}(c_1,\dots,c_k) = Y_{\mathbf{i}}(\mathcal{C}_1,\dots,\mathcal{C}_k)$$

where

$$\begin{aligned}
\mathcal{C}_j &:= c_j \frac{\displaystyle\sum_{1\le m\le j, i_m=i} \frac{c}{c_1^{a_{i_1,i}}\dots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j<m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\dots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1\le m<j, i_m=i} \frac{c}{c_1^{a_{i_1,i}}\dots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\dots c_{m-1}^{a_{i_{m-1},i}} c_m}} \\
\varepsilon_i(Y_{\mathbf{i}}(c_1,\dots,c_k)) &= \sum_{1\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\dots c_{m-1}^{a_{i_{m-1},i}} c_m} \\
\gamma_i(Y_{\mathbf{i}}(c_1,\dots,c_k)) &= c_1^{a_{i_1,i}}\dots c_k^{a_{i_k,i}}
\end{aligned}$$

Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ associated with the index set $I/\{0\}$ and $\mathfrak{g}_n$ be the subalgebra of $\mathfrak{g}$ associated with the index set $I/\{n\}$. Using the explicit formulas above, we can construct the geometric crystals associated with $\mathfrak{g}_0$ and $\mathfrak{g}_n$. With the geometric crystals corresponding to its subalgebras we can construct the geometric crystal associated with the affine algebra $\mathfrak{g}$. In the rest of the section, we give the explicit constructions for the affine geometric crystals corresponding to the affine algebras $C_2^{(1)}$, $C_3^{(1)}$, and $C_4^{(1)}$.

Notice that the element w of the Weyl group we are interested in is a translation corresponding to the highest weights of the $\mathfrak{g}_n$-module $V(\Lambda_0)$ and $\mathfrak{g}_0$-module $V(\Lambda_n)$.

5.3 Construction

First we consider the Affine algebra in question, $C_n^{(1)}$. The generalized Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -2 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -2 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

$C_n^{(1)}$ has simple roots $\alpha_0, \alpha_1, \dots, \alpha_n$, null root $\delta = \alpha_0 + 2\alpha_1 \dots + 2\alpha_{n-1} + \alpha_n$ and central element $c = h_0 + h_1 + \dots + h_{n-1} + h_n$. It has the following Dynkin diagram:

$$\circ \Rightarrow \circ — \cdots — \circ \Leftarrow \circ$$

We will compute the geometric crystals corresponding to $\mathfrak{g}_0, \mathfrak{g}_n$ and use them to compute the geometric crystal corresponding to $\mathfrak{g}$.

$C_2^{(1)}$ Geometric Crystal

First we compute the actions of the simple reflections s_0, s_1, and s_2 and then we will compute the translation. From the Cartan matrix, we have

$$\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$$

$$\alpha_1 = 2\Lambda_1 - \Lambda_0 - \Lambda_2$$

$$\alpha_2 = 2\Lambda_2 - 2\Lambda_1$$

From this, we know the simple reflections act as:

$$s_0(\lambda_0, \lambda_1, \lambda_2) = (\lambda_0, \lambda_1, \lambda_2) - (2, -2, 0)\lambda_0 = (-\lambda_0, 2\lambda_0 + \lambda_1, \lambda_2)$$

$$s_1(\lambda_0, \lambda_1, \lambda_2) = (\lambda_0, \lambda_1, \lambda_2) - (-1, 2, -1)\lambda_1 = (\lambda_0 + \lambda_1, -\lambda_1, \lambda_1 + \lambda_2)$$

$$s_2(\lambda_0, \lambda_1, \lambda_2) = (\lambda_0, \lambda_1, \lambda_2) - (0, -2, 2)\lambda_2 = (\lambda_0, \lambda_1 + 2\lambda_2, -\lambda_2)$$

We show below that $t(\omega_2) = \sigma s_2 s_1 s_2$ and $t(\check{\omega}_2) = \sigma s_0 s_1 s_0$ where σ is the diagram automorphism $0 \longleftrightarrow 2$.

$$\begin{aligned}
&\sigma s_2 s_1 s_2(\lambda_0, \lambda_1, \lambda_2) \\
&= \sigma s_2 s_1(\lambda_0, \lambda_1 + 2\lambda_2, -\lambda_2) \\
&= \sigma s_2(\lambda_0 + \lambda_1 + 2\lambda_2, , -\lambda_1 - 2\lambda_2, \lambda_1 + \lambda_2) \\
&= \sigma(\lambda_0 + \lambda_1 + 2\lambda_2, \lambda_1, -\lambda_1 - \lambda_2) \\
&= (-\lambda_1 - \lambda_2, \lambda_1, , \lambda_0 + \lambda_1 + 2\lambda_2) \\
&= (\lambda_0, \lambda_1, \lambda_2) + (\lambda_0 + \lambda_1 + \lambda_2)(\Lambda_2 - \Lambda_0)
\end{aligned}$$

$$\begin{aligned}
&\sigma s_0 s_1 s_0(\lambda_0, \lambda_1, \lambda_2) \\
&= \sigma s_0 s_1(-\lambda_0, \lambda_1 + 2\lambda_0, \lambda_2) \\
&= \sigma s_0(\lambda_0 + \lambda_1, -2\lambda_0 - \lambda_1, 2\lambda_0 + \lambda_1 + \lambda_2) \\
&= \sigma(-\lambda_0 - \lambda_1, \lambda_1, 2\lambda_0 + \lambda_1 + \lambda_2) \\
&= (2\lambda_0 + \lambda_1 + \lambda_2, \lambda_1, -\lambda_0 - \lambda_1) \\
&= (\lambda_0, \lambda_1, \lambda_2) + (\lambda_0 + \lambda_1 + \lambda_2)(\Lambda_0 - \Lambda_2)
\end{aligned}$$

The $C_2^{(1)}$ fundamental representation $V(\Lambda_2 - \Lambda_0) = V(\varpi_2)$ has basis $\{(1,2), (1,\overline{2}), (2,\overline{2}), (2,\overline{1}), (\overline{2},\overline{1})\}$ with respective weights $\{\Lambda_2 - \Lambda_0, 2\Lambda_1 - \Lambda_0 - \Lambda_1, 0, \Lambda_0 + \Lambda_1 - 2\Lambda_2, \Lambda_0 - \Lambda_2\}$. Therefore, for $\mathfrak{g}_n$, the highest weight vector is $(\overline{2},\overline{1})$ and for $\mathfrak{g}_0$ the highest weight vector is $(1,2)$. The fundamental representation crystal graph is pictured below.

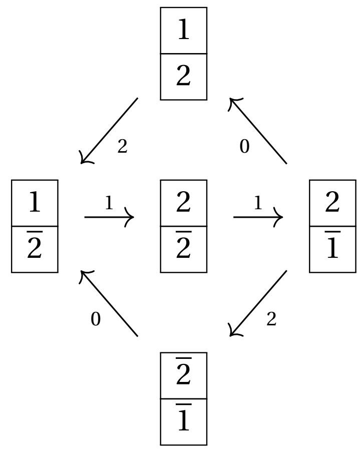

Figure 5.1: Crystal graph of $V(\Lambda_2 - \Lambda_0)$

We now calculate the varieties $V_1(x)$ and $V_2(y)$. We have

$$
\begin{aligned}
& Y_2(x_{22})Y_1(x_{11})Y_2(x_{21})(1,2) \\
&= Y_2(x_{22})Y_1(x_{11})\check{\alpha}_2(x_{21})(1+\frac{f_2}{x_{21}})(1,2) \\
&= Y_2(x_{22})Y_1(x_{11})[x_{21}(1,2)+(1,\overline{2})] \\
&= Y_2(x_{22})\check{\alpha}_1(x_{11})(1+\frac{f_1}{x_{21}}+\frac{f_1^2}{2x_{11}^2})[x_{21}(1,2)+(1,\overline{2})] \\
&= Y_2(x_{22})[x_{21}(1,2)+x_{11}^2(1,\overline{2})+x_{11}(2,\overline{2})+(2,\overline{1})] \\
&= \check{\alpha}_2(x_{22})(1+\frac{f_2}{x_{22}})[x_{21}(1,2)+x_{11}^2(1,\overline{2})+x_{11}(2,\overline{2})+(2,\overline{1})] \\
&= [x_{21}x_{22}(1,2)+(x_{21}+\frac{x_{11}^2}{x_{22}})(1,\overline{2})+x_{11}(2,\overline{2})+x_{22}(2,\overline{1})+(\overline{2},\overline{1})]
\end{aligned}
$$

and for $V_2(y)$ we have:

$$
\begin{aligned}
& Y_0(y_{02})Y_1(y_{11})Y_0(y_{01})(\overline{2},\overline{1}) \\
&= Y_0(y_{02})Y_1(y_{11})\check{\alpha}_0(x_{01})(1+\frac{f_0}{x_{01}})(\overline{2},\overline{1}) \\
&= Y_0(y_{02})Y_(y_{11})[y_{01}(\overline{2},\overline{1})+(1,\overline{2})] \\
&= Y_0(y_{02})\check{\alpha}_1(x_{11})(1+\frac{f_1}{x_{11}}+\frac{f_1^2}{2x_{11}^2})[y_{01}(\overline{2},\overline{1})+(1,\overline{2})] \\
&= Y_0(y_{02})[y_{01}(\overline{2},\overline{1})+y_{11}^2(1,\overline{2})+y_{11}(2,\overline{2})+(2,\overline{1})] \\
&= \check{\alpha}_0(x_{02})(1+\frac{f_0}{x_{02}})[y_{01}(\overline{2},\overline{1})+y_{11}^2(1,\overline{2})+y_{11}(2,\overline{2})+(2,\overline{1})] \\
&= [y_{01}y_{02}(\overline{2},\overline{1})+(y_{01}+\frac{y_{11}^2}{y_{02}})(1,\overline{2})+y_{11}(2,\overline{2})+y_{02}(2,\overline{1})+(1,2)]
\end{aligned}
$$

Setting $V_1(x)\cdot a(x)=V_2(y)$, we obtain the following relations:

$$1=a(x)\cdot x_{21}x_{22}$$

$$
\begin{aligned}
y_{01}y_{02} &= a(x) \\
(x_{21}+\frac{x_{11}^2}{x_{22}})a(x) &= y_{01}+\frac{y_{11}^2}{y_{02}} \\
x_{11}a(x) &= y_{11} \\
x_{22}a(x) &= y_{02}
\end{aligned}
$$

with the following solutions: $a(x)=\frac{1}{x_{21}x_{22}}$, $y_{11}=\frac{x_{11}}{x_{21}x_{22}}$, $y_{02}=\frac{1}{x_{21}}$ and $y_{01}=\frac{1}{x_{22}}$. We can also see that $x_{21}=\frac{1}{y_{02}}$, $x_{22}=\frac{1}{y_{01}}$ and $x_{11}=\frac{y_{11}}{y_{01}y_{02}}$. Let $\mathcal{V}_1$ be the geometric crystal corresponding to $\mathfrak{g}_0$ and $\mathcal{V}_2$ be the geometric crystal corresponding to $\mathfrak{g}_n$. This map is clearly birational and bipositive, and defines the isomorphism $\overline{\sigma}:\mathcal{V}_1\longrightarrow\mathcal{V}_2$ and $\overline{\sigma^{-1}}:\mathcal{V}_2\longrightarrow\mathcal{V}_1$.

Now we compute the actions e_i^c, ε_i, and γ_i from the general formula in section 5.2. First we compute e_1^c and e_2^c.

$$e_1^c(V_1(x_{22},x_{11},x_{21}))=\left(x_{22}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right),x_{11}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right),x_{21}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{c\,x_{22}}{x_{11}}}\right)\right)=(x_{21},c\,x_{11},x_{22})$$

$$e_2^c(V_1(x_{22},x_{11},x_{21}))=\left(x_{22}\left(\frac{\frac{c}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}{\frac{1}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}\right),x_{11}\left(\frac{\frac{c}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}{\frac{c}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}\right),x_{21}\left(\frac{\frac{c}{x_{22}}+\frac{c\,x_{11}^2}{x_{22}^2x_{21}}}{\frac{c}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}\right)\right)$$

We define

$$c_2:=\frac{\frac{c}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}{\frac{1}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}}=\frac{\frac{c\,x_{22}x_{21}+x_{11}^2}{x_{22}^2x_{21}}}{\frac{x_{22}x_{21}+x_{11}^2}{x_{22}^2x_{21}}}=\frac{c\,x_{21}x_{22}+x_{11}^2}{x_{21}x_{22}+x_{11}^2}$$

Then

$$e_2^c(V_1(x_{22},x_{11},x_{21})=\left(c_2x_{22},x_{11},\frac{c}{c_2}x_{21}\right)$$

Now the formulas for ε_1 and ε_2 are the following:

$$\varepsilon_1(x_{22},x_{11},x_{21})=\frac{x_{22}}{x_{11}}$$

$$\varepsilon_2(x_{22},x_{11},x_{21})=\frac{1}{x_{22}}+\frac{x_{11}^2}{x_{22}^2x_{21}}=\frac{x_{21}x_{22}+x_{11}^2}{x_{22}^2x_{21}}$$

And the formulas for γ_1 and γ_2 are as follows:

$$\gamma_1(x_{22},x_{11},x_{21})=\frac{x_{11}^2}{x_{21}x_{22}}$$

$$\gamma_2(x_{22},x_{11},x_{21})=\frac{x_{21}^2x_{22}^2}{x_{11}^2}$$

To get the formulas for e_0^c, ε_0 and γ_0 we first must get the formulas for $\overline{e_0^c}$, $\overline{\varepsilon_0}$ and $\overline{\gamma_0}$ in $\mathcal{V}_2(y)$. These are as follows:

$$\overline{e_0^c}(y_{02},y_{11},y_{01})=\left(y_{02}\left(\frac{\frac{c}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}{\frac{1}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}\right),y_{11}\left(\frac{\frac{c}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}{\frac{c}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}\right),y_{01}\left(\frac{\frac{c}{y_{02}}+\frac{c\,y_{11}^2}{y_{02}^2y_{01}}}{\frac{c}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}\right)\right)$$

We define

$$\overline{c}_0=\frac{\frac{c}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}{\frac{1}{y_{02}}+\frac{y_{11}^2}{y_{02}^2y_{01}}}=\frac{\frac{c\,y_{02}y_{01}+y_{11}^2}{y_{02}^2y_{01}}}{\frac{y_{02}y_{01}+y_{11}^2}{y_{02}^2y_{01}}}=\frac{c\,y_{01}y_{02}+y_{11}^2}{y_{01}y_{02}+y_{11}^2}$$

Then

$$\overline{e_0^c}(y_{02},y_{11},y_{01})=(\overline{c}_0y_{02},y_{11},\frac{c}{\overline{c}_0}y_{01})$$

And the formulas for $\overline{\varepsilon_0}$ and $\overline{\gamma_0}$ are

$$\overline{\varepsilon_0}(y_{02}, y_{11}, y_{01}) = \frac{1}{y_{02}} + \frac{y_{11}^2}{y_{02}^2 y_{01}} = \frac{y_{01} y_{02} + y_{11}^2}{y_{02}^2 y_{01}}$$

$$\overline{\gamma_0}(y_{02}, y_{11}, y_{01}) = \frac{y_{01}^2 y_{02}^2}{y_{11}^2}$$

Now e_0^c, ε_0 and γ_0 are defined as follows:

$$\begin{aligned} e_0^c(V_1(x)) &= \overline{\sigma^{-1}} \circ \overline{e_0^c} \circ \overline{\sigma}(V_1(x)) \\ \gamma_0(V_1(x)) &= \overline{\gamma_0}(\overline{\sigma}(V_1(x))) \\ \varepsilon_0(V_1(x)) &= \overline{\varepsilon_0}(\overline{\sigma}(V_1(x))) \end{aligned}$$

Now we find e_0^c

$$\begin{aligned} & e_0^c(x_{22}, x_{11}, x_{21}) = \overline{\sigma^{-1}} \circ \overline{e_0^c} \circ \overline{\sigma}(x_{22}, x_{11}, x_{21}) = \overline{\sigma^{-1}} \circ \overline{e_0^c}\left(\frac{1}{y_{01}}, \frac{y_{11}}{y_{01} y_{02}}, \frac{1}{y_{02}}\right) \\ &= \overline{\sigma^{-1}}\left(\frac{1}{y_{01}\frac{c}{\overline{\overline{c_0}}}}, \frac{y_{11}}{c\, y_{01} y_{02}}, \frac{1}{y_{02}\overline{c_0}}\right) = \overline{\sigma^{-1}}\left(\frac{c\, y_{01} y_{02} + y_{11}^2}{c\, y_{01}(y_{01} y_{02} + y_{11}^2)}, \frac{y_{11}}{c\, y_{01} y_{02}}, \frac{y_{01} y_{02} + y_{11}^2}{y_{02}(c\, y_{01} y_{02} + y_{11}^2)}\right) \\ &= \left(\frac{\frac{c}{x_{21} x_{22}} + \frac{x_{11}^2}{x_{21}^2 x_{22}^2}}{c\,\frac{1}{x_{22}}(\frac{1}{x_{21} x_{22}} + \frac{x_{11}^2}{x_{21}^2 x_{22}^2})}, \frac{\frac{x_{11}}{x_{21} x_{22}}}{\frac{c}{x_{21} x_{22}}}, \frac{\frac{1}{x_{21} x_{22}} + \frac{x_{11}^2}{x_{21}^2 x_{22}^2}}{\frac{1}{x_{21}}(\frac{c}{x_{21} x_{22}} + \frac{x_{11}^2}{x_{21}^2 x_{22}^2})}\right) = \left(x_{22}\frac{\frac{c\, x_{21} x_{22} + x_1 1^2}{x_{21}^2 x_{22}^2}}{\frac{c\, x_{21} x_{22} + c\, x_1 1^2}{x_{21}^2 x_{22}^2}}, \frac{x_{11}}{c}, x_{21}\frac{\frac{x_{21} x_{22} + x_1 1^2}{x_{21}^2 x_{22}^2}}{\frac{c\, x_{21} x_{22} + x_1 1^2}{x_{21}^2 x_{22}^2}}\right) \\ &= (\frac{c_2}{c} x_{22}, \frac{x_{11}}{c}, \frac{x_{21}}{c_2}) \end{aligned}$$

And then we compute ε_0 and γ_0

$$\begin{aligned} \varepsilon_0(V_1(x)) = \overline{\varepsilon_0}(\overline{\sigma}(V_1(x))) = \overline{\varepsilon_0}\left(\frac{1}{x_{21}}, \frac{x_{11}}{x_{21} x_{22}}, \frac{1}{x_{22}}\right) &= \frac{1}{\frac{1}{x_{22}}} + \frac{\frac{x_{11}^2}{x_{21}^2 x_{22}^2}}{\frac{1}{x_{21}^2 x_{22}}} = x_{21} + \frac{x_{11}^2}{x_{22}} \\ \gamma_0(V_1(x)) = \overline{\gamma_0}(\overline{\sigma}(V_1(x))) = \overline{\gamma_0}\left(\frac{1}{x_{22}}, \frac{x_{11}}{x_{21} x_{22}}, \frac{1}{x_{21}}\right) &= \frac{\frac{1}{x_{21}^2 x_{22}^2}}{\frac{x_{11}^2}{x_{21}^2 x_{22}^2}} = \frac{1}{x_{11}^2} \end{aligned}$$

Theorem 5.3.1. *$\mathcal{V}_1 = (V_1(x), e_i^c, \varepsilon_i, \varphi_i)$ for $i = 0, 1, 2$ is a positive geometric crystal.*

Proof. Clearly this is positive, since all coefficients are positive. Then we need to check the following relations:

$$\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$$
$$e_2^c e_0^d = e_0^d e_2^c$$
$$e_1^c e_0^{c^2 d} e_1^{cd} e_0^d = e_0^d e_1^{cd} e_0^{c^2 d} e_1^c$$
$$e_1^c e_2^{c^2 d} e_1^{cd} e_2^d = e_2^d e_1^{cd} e_2^{c^2 d} e_1^c$$
$$\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$$
$$\varepsilon_0(e_2^c(x)) = \varepsilon_0(x)$$
$$\varepsilon_2(e_0^c(x)) = \varepsilon_2(x)$$

We actually only need to check the relations involving e_0^c, φ_0, and ε_0, since by [2, 30] we already know $\mathcal{V}_1 = (V_1(x), e_i^c, \varepsilon_i, \varphi_i)$ for $i = 1,2$. First we check the relations $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$:

$$\gamma_0(e_0^c(x)) = \gamma_0(\frac{c_2}{c}x_{22}, \frac{x_{11}}{c}, \frac{1}{c_2}x_{21}) = \frac{c^2}{x_{11}^2} = c^2\gamma_0(x)$$
$$\gamma_0(e_1^c(x)) = \gamma_0(x_{22}, c\,x_{11}, x_{21}) = \frac{1}{c^2 x_{11}^2} = c^{-2}\gamma_0(x)$$
$$\gamma_0(e_2^c(x)) = \gamma_0(c_2 x_{22}, x_{11}, \frac{c}{c_2}x_{21}) = \frac{1}{x_{11}^2} = c^0\gamma_0(x)$$
$$\gamma_1(e_0^c(x)) = \gamma_1(\frac{c_2}{c}x_{22}, \frac{x_{11}}{c}, \frac{1}{c_2}x_{21}) = \frac{c\,x_{11}^2}{c^2 x_{21}x_{22}} = c^{-1}\gamma_1(x)$$
$$\gamma_2(e_0^c(x)) = \gamma_2(\frac{c_2}{c}x_{22}, \frac{x_{11}}{c}, \frac{1}{c_2}x_{21}) = \frac{c^2 x_{21}^2 x_{22}^2}{c^2 x_{11}^2} = c^0\gamma_2(x)$$

Then we will check the relations $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$, $\varepsilon_0(e_2^c(x)) = \varepsilon_0(x)$ and $\varepsilon_2(e_0^c(x)) = \varepsilon_2(x)$.

$$\varepsilon_0(e_0^c(x)) = \varepsilon_0(\frac{c_2}{c}x_{22}, \frac{x_{11}}{c}, \frac{x_{21}}{c_2}) = \frac{x_{21}}{c_2} + \frac{x_{11}^2}{c^2\frac{c_2}{c}x_{22}} = \frac{c\,x_{22}x_{21} + x_{11}^2}{c\,c_2 x_{22}}$$
$$= \frac{(x_{22}x_{21} + x_{11}^2)}{c\,x_{22}\frac{c\,x_{21}x_{22}+x_{11}^2}{x_{21}x_{22}+x_{11}^2}} = \frac{(x_{21}x_{22} + x_{11}^2)}{c\,x_{22}} = \frac{x_{21}}{c} + \frac{x_{11}^2}{c\,x_{22}} = c^{-1}\varepsilon_0(x)$$
$$\varepsilon_0(e_2^c(x)) = \varepsilon_2(c_2 x_{22}, x_{11}, \frac{c}{c_2}x_{21}) = \frac{c\,x_{21}}{c_2} + \frac{x_{11}^2}{c_2 x_{22}} = \frac{c\,x_{21}x_{22} + x_{11}^2}{x_{22}\frac{c\,x_{21}x_{22}+x_{11}^2}{x_{21}x_{22}+x_{11}^2}} = x_{21} + \frac{x_{11}^2}{x_{22}} = \varepsilon_0(x)$$
$$\varepsilon_2(e_0^c(x)) = \varepsilon_2(\frac{c_2}{c}x_{22}, \frac{x_{11}}{c}, \frac{x_{21}}{c_2}) = \frac{c\,x_{21}x_{22} + x_{11}^2}{x_{22}^2 x_{21}\frac{c\,x_{21}x_{22}+x_{11}^2}{x_{21}x_{22}+x_{11}^2}} = \frac{x_{21}x_{22} + x_{11}^2}{x_{22}^2 x_{21}} = \varepsilon_2(x)$$

Now we check the relation $e_0^c e_2^d = e_2^d e_0^c$

$$e_0^c e_2^d(x_{22}, x_{11}, x_{21}) = e_0^c(d_2 x_{22}, x_{11}, \frac{d}{d_2} x_{21}) = e_0^c\left(\frac{x_{22}(d x_{21} x_{22} + x_{11}^2)}{x_{21} x_{22} + x_{11}^2}, x_{11}, \frac{d x_{21}(x_{21} x_{22} + x_{11}^2)}{d x_{21} x_{22} + x_{11}^2}\right)$$
$$= \left(\frac{c_2^*}{c}\frac{x_{22}(d x_{21} x_{22} + x_{11}^2)}{x_{21} x_{22} + x_{11}^2}, \frac{x_{11}}{c}, \frac{1}{c_2^*}\frac{d x_{21}(x_{21} x_{22} + x_{11}^2)}{d x_{21} x_{22} + x_{11}^2}\right)$$

We will solve for c_2^* to substitute it in:

$$c_2^* = \frac{c(\frac{(d x_{21} x_{22} + x_{11}^2) x_{22}}{d x_{21} x_{22} + x_{11}^2} \cdot \frac{(d x_{21} x_{22} + x_{11}^2) x_{22}}{x_{21} x_{22} + x_{11}^2}) + x_{11}^2}{\frac{(d x_{21} x_{22} + x_{11}^2) d x_{21}}{x_{21} x_{22} + x_{11}^2} \cdot \frac{d x_{21}(x_{21} x_{22} + x_{11}^2)}{d x_{21} x_{22} + x_{11}^2} + x_{11}^2} = \frac{c d x_{21} x_{22} + x_{11}^2}{d x_{21} x_{22} + x_{11}^2}$$

Substituting this in, we obtain the following

$$\begin{aligned}
&e_0^c e_2^d(x_{22}, x_{11}, x_{21}) \\
&= \left(\frac{c d x_{21} x_{22} + x_{11}^2}{c(d x_{21} x_{22} + x_{11}^2)} \cdot \frac{x_{22}(d x_{21} x_{22} + x_{11}^2)}{x_{21} x_{22} + x_{11}^2}, \frac{x_{11}}{c}, \frac{d x_{21} x_{22} + x_{11}^2}{c d x_{21} x_{22} + x_{11}^2} \cdot \frac{d x_{21}(x_{21} x_{22} + x_{11}^2)}{d x_{21} x_{22} + x_{11}^2}\right) \\
&= \left(\frac{x_{22}(c d x_{21} x_{22} + x_{11}^2)}{c(x_{21} x_{22} + x_{11}^2)}, \frac{x_{11}}{c}, \frac{d x_{21}(x_{21} x_{22} + x_{11}^2)}{c d x_{21} x_{22} + x_{11}^2}\right)
\end{aligned}$$

Now we calculate the other side of the equation:

$$e_2^d e_0^c(x_{22}, x_{11}, x_{21}) = e_2^d\left(\frac{x_{22}(c x_{21} x_{22} + x_{11}^2)}{c(x_{21} x_{22} + x_{11}^2)}, \frac{x_{11}}{c}, \frac{x_{21}(x_{21} x_{22} + x_{11}^2)}{c x_{21} x_{22} + x_{11}^2}\right)$$
$$= \left(d_2^* \frac{x_{22}(c x_{21} x_{22} + x_{11}^2)}{c(x_{21} x_{22} + x_{11}^2)}, \frac{x_{11}}{c}, \frac{d}{d_2^*}\frac{x_{21}(x_{21} x_{22} + x_{11}^2)}{c x_{21} x_{22} + x_{11}^2}\right)$$

We solve for c_2^* to substitute it in:

$$d_2^* = \frac{d(\frac{(c x_{21} x_{22} + x_{11}^2) x_{22}}{c(x_{21} x_{22} + x_{11}^2)} \cdot \frac{(x_{21} x_{22} + x_{11}^2) x_{21}}{c x_{21} x_{22} + x_{11}^2}) + \frac{x_{11}^2}{c^2}}{\frac{(c x_{21} x_{22} + x_{11}^2) d x_{22}}{c(x_{21} x_{22} + x_{11}^2)} \cdot \frac{x_{21}(x_{21} x_{22} + x_{11}^2)}{c x_{21} x_{22} + x_{11}^2} + \frac{x_{11}^2}{c^2}} = \frac{c d x_{21} x_{22} + x_{11}^2}{c x_{21} x_{22} + x_{11}^2}$$

Then we have:

$$
\begin{aligned}
&e_2^d e_0^c(x_{22}, x_{11}, x_{21})\\
&=\left(\frac{c\,d\,x_{21}x_{22}+x_{11}^2}{c\,x_{21}x_{22}+x_{11}^2}\cdot\frac{x_{22}(c\,x_{21}x_{22}+x_{11}^2)}{c(x_{21}x_{22}+x_{11}^2)},\frac{x_{11}}{c},\frac{d(c\,x_{21}x_{22}+x_{11}^2)}{c\,d\,x_{21}x_{22}+x_{11}^2}\cdot\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{c\,x_{21}x_{22}+x_{11}^2}\right)\\
&=\left(\frac{x_{22}(c\,d\,x_{21}x_{22}+x_{11}^2)}{c(x_{21}x_{22}+x_{11}^2)},\frac{x_{11}}{c},\frac{d\,x_{21}(x_{21}x_{22}+x_{11}^2)}{c\,d\,x_{21}x_{22}+x_{11}^2}\right)
\end{aligned}
$$

Both sides are equal, so the relation holds. Finally, we check the last relation:

$$
\begin{aligned}
&e_1^c e_0^{c^2d} e_1^{cd} e_0^d(x_{22}, x_{11}, x_{21}) = e_1^c e_0^{c^2d} e_1^{cd}\left(\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)},\frac{x_{11}}{d},\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}\right)\\
&= e_1^c e_0^{c^2d}\left(\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)}, c\,x_{11},\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}\right)\\
&= e_1^c\left(\frac{(c^2d)^*}{c^2d}\cdot\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)},\frac{x_{11}}{c\,d},\frac{1}{(c^2d)^*}\cdot\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}\right)
\end{aligned}
$$

Then we solve for $(c^2d)^*$

$$
(c^2d)^* = \frac{c^2d\left(\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)}\cdot\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}\right)+c^2x_{11}^2}{\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)}\cdot\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}+c^2x_{11}^2} = \frac{\frac{c^2d\,x_{21}x_{22}}{d}+c^2x_{11}^2}{\frac{x_{21}x_{22}}{d}+c^2x_{11}^2} = \frac{c^2d(x_{21}x_{22}+x_{11}^2)}{x_{21}x_{22}+c^2d\,x_{11}^2}
$$

Then we have

$$
\begin{aligned}
&e_1^c\left(\frac{c^2d(x_{21}x_{22}+x_{11}^2)}{c^2d(x_{21}x_{22}+c^2d\,x_{11}^2)}\cdot\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+x_{11}^2)},\frac{x_{11}}{c\,d},\frac{x_{21}x_{22}+c^2d\,x_{11}^2}{c^2d(x_{21}x_{22}+x_{11}^2)}\cdot\frac{x_{21}(x_{21}x_{22}+x_{11}^2)}{d\,x_{21}x_{22}+x_{11}^2}\right)\\
&= e_1^c\left(\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+c^2d\,x_{11}^2)},\frac{x_{11}}{c\,d},\frac{x_{21}(x_{21}x_{22}+c^2d\,x_{11}^2)}{c^2d(d\,x_{21}x_{22}+x_{11}^2)}\right)\\
&=\left(\frac{x_{22}(d\,x_{21}x_{22}+x_{11}^2)}{d(x_{21}x_{22}+c^2d\,x_{11}^2)},\frac{x_{11}}{d},\frac{x_{21}(x_{21}x_{22}+c^2d\,x_{11}^2)}{c^2d(d\,x_{21}x_{22}+x_{11}^2)}\right)
\end{aligned}
$$

Now we evaluate the other side of the relation:

$$
\begin{aligned}
&e_0^d e_1^{cd} e_0^{c^2 d} e_1^c (x_{22}, x_{11}, x_{21}) = e_0^d e_1^{cd} e_0^{c^2 d} (x_{22}, c\,x_{11}, x_{21}) \\
&= e_0^d e_1^{cd} \left(\frac{x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)}, \frac{x_{11}}{c\,d}, \frac{x_{21}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} \right) \\
&= e_0^d \left(\frac{x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)}, x_{11}, \frac{x_{21}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} \right) \\
&= \left(\frac{d_2^*}{d} \cdot \frac{x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)}, \frac{x_{11}}{d}, \frac{1}{d_2^*} \cdot \frac{x_{21}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} \right)
\end{aligned}
$$

We solve for d_2^*:

$$
d_2^* = \frac{\frac{d\,x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)} \cdot \frac{x_{21}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} + x_{11}^2}{\frac{x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)} \cdot \frac{x_{21}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} + x_{11}^2} = \frac{\frac{d\,x_{21} x_{22}}{c^2 d} + x_{11}^2}{\frac{x_{21} x_{22}}{c^2 d} + x_{11}^2} = \frac{d\,x_{21} x_{22} + c^2 d\,x_{11}^2}{x_{21} x_{22} + c^2 d\,x_{11}^2}
$$

Substituting this expression in, we get:

$$
\begin{aligned}
&\left(\frac{d\,x_{21} x_{22} + c^2 d\,x_{11}^2}{d(x_{21} x_{22} + c^2 d\,x_{11}^2)} \cdot \frac{x_{21}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 x_{11}^2)}, \frac{x_{11}}{d}, \frac{x_{21} x_{22} + c^2 d\,x_{11}^2}{d\,x_{21} x_{22} + c^2 d\,x_{11}^2} \cdot \frac{x_{22}(x_{21} x_{22} + c^2 x_{11}^2)}{c^2(d\,x_{21} x_{22} + x_{11}^2)} \right) \\
&= \left(\frac{x_{22}(d\,x_{21} x_{22} + x_{11}^2)}{d(x_{21} x_{22} + c^2 d\,x_{11}^2)}, \frac{x_{11}}{d}, \frac{x_{21}(x_{21} x_{22} + c^2 d\,x_{11}^2)}{c^2 d(d\,x_{21} x_{22} + x_{11}^2)} \right)
\end{aligned}
$$

Since all of these as satisfied, $\mathcal{V}_1(x)$ along with the relations e_i^c, γ_i and ε_i for $i = 0, 1, 2$ is a $C_2^{(1)}$ geometric crystal. □

$C_3^{(1)}$ Geometric Crystal

We start by computing the following actions of simple reflections: s_0, s_1, s_2, s_3. From the Cartan matrix, we know that

$$\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$$

$$\alpha_1 = 2\Lambda_1 - \Lambda_0 - \Lambda_2$$

$$\alpha_2 = 2\Lambda_2 - \Lambda_1 - \Lambda_3$$

$$\alpha_3 = 2\Lambda_3 - 2\Lambda_2$$

Then the simple reflections act as:

$$s_0(\lambda_0,\lambda_1,\lambda_2,\lambda_3)=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)-(2,-2,0,0)\lambda_0=(-\lambda_0,2\lambda_0+\lambda_1,\lambda_2,\lambda_3)$$

$$s_1(\lambda_0,\lambda_1,\lambda_2,\lambda_3)=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)-(-1,2,-1,0)\lambda_1=(\lambda_0+\lambda_1,-\lambda_1,\lambda_1+\lambda_2,\lambda_3)$$

$$s_2(\lambda_0,\lambda_1,\lambda_2,\lambda_3)=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)-(0,-1,2,-1)\lambda_2=(\lambda_0,\lambda_1+\lambda_2,-\lambda_2,\lambda_2+\lambda_3)$$

$$s_3(\lambda_0,\lambda_1,\lambda_2,\lambda_3)=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)-(0,0,-2,2)\lambda_3=(\lambda_0,\lambda_1,\lambda_2++2\lambda_3,-\lambda_3)$$

We show below that $t(\omega_3)=\sigma s_3 s_2 s_3 s_1 s_2 s_3$ and $t(\check{\omega}_3)=\sigma s_0 s_1 s_0 s_2 s_1 s_0$ where σ is the diagram automorphism $0\longleftrightarrow 3$ and $1\longleftrightarrow 2$.

$$\begin{aligned}
&\sigma s_3 s_2 s_3 s_1 s_2 s_3(\lambda_0,\lambda_1,\lambda_2,\lambda_3)\\
&=\sigma s_3 s_2 s_3 s_1 s_2(\lambda_0,\lambda_1,\lambda_2+2\lambda_3,-\lambda_3)\\
&=\sigma s_3 s_2 s_3 s_1(\lambda_0,\lambda_1+\lambda_2+2\lambda_3,-\lambda_2-2\lambda_3,\lambda_2+\lambda_3)\\
&=\sigma s_3 s_2 s_3(\lambda_0+\lambda_1+\lambda_2+2\lambda_3,-\lambda_1-\lambda_2-2\lambda_3,\lambda_1,\lambda_2+\lambda_3)\\
&=\sigma s_3 s_2(\lambda_0+\lambda_1+\lambda_2+2\lambda_3,-\lambda_1-\lambda_2-2\lambda_3,\lambda_1+2\lambda_2+2\lambda_3,-\lambda_2-\lambda_3)\\
&=\sigma s_3(\lambda_0+\lambda_1+\lambda_2+2\lambda_3,\lambda_2,-\lambda_1-2\lambda_2-2\lambda_3,\lambda_1+\lambda_2+\lambda_3)\\
&=\sigma(\lambda_0+\lambda_1+\lambda_2+2\lambda_3,\lambda_2,\lambda_1,-\lambda_1-\lambda_2-\lambda_3)\\
&=(-\lambda_1-\lambda_2-\lambda_3,\lambda_1,\lambda_2,\lambda_0+\lambda_1+\lambda_2+2\lambda_3)\\
&=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)+(\lambda_0+\lambda_1+\lambda_2+\lambda_3)(\Lambda_3-\Lambda_0)
\end{aligned}$$

$$\begin{aligned}
&\sigma s_0 s_1 s_0 s_2 s_1 s_0(\lambda_0,\lambda_1,\lambda_2,\lambda_3)\\
&=\sigma s_0 s_1 s_0 s_2 s_1(-\lambda_0,\lambda_1+2\lambda_0,\lambda_2,\lambda_3)\\
&=\sigma s_0 s_1 s_0 s_2(\lambda_0+\lambda_1,-2\lambda_0-\lambda_1,2\lambda_0+\lambda_1+\lambda_2,\lambda_3)\\
&=\sigma s_0 s_1 s_0(\lambda_0+\lambda_1,\lambda_2,-2\lambda_0-\lambda_1-\lambda_2,2\lambda_0+\lambda_1+\lambda_2+\lambda_3)\\
&=\sigma s_0 s_1(-\lambda_0-\lambda_1,2\lambda_0+2\lambda_1+\lambda_2,-2\lambda_0-\lambda_1-\lambda_2,2\lambda_0+\lambda_1+\lambda_2+\lambda_3)\\
&=\sigma s_0(\lambda_0+\lambda_1+\lambda_2,-2\lambda_0-2\lambda_1-\lambda_2,\lambda_1,2\lambda_0+\lambda_1+\lambda_2+\lambda_3)\\
&=\sigma(-\lambda_0-\lambda_1-\lambda_2,\lambda_2,\lambda_1,2\lambda_0+\lambda_1+\lambda_2+\lambda_3)\\
&=(2\lambda_0+\lambda_1+\lambda_2+\lambda_3,\lambda_1,\lambda_2-\lambda_0-\lambda_1-\lambda_3)\\
&=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)+(\lambda_0+\lambda_1+\lambda_2+\lambda_3)(\Lambda_0-\Lambda_3)
\end{aligned}$$

Now we compute $V_1(x)$ and $V_2(y)$.

$$Y_3(x_{33})Y_2(x_{22})Y_3(x_{32})Y_1(x_{11})Y_2(x_{21})Y_3(x_{31})(1,2,3)$$
$$= Y_3(x_{33})Y_2(x_{22})Y_3(x_{32})Y_1(x_{11})Y_2(x_{21})[x_{31}(1,2,3)+(1,2,\overline{3})]$$
$$= Y_3(x_{33})Y_2(x_{22})Y_3(x_{32})Y_1(x_{11})[x_{31}(1,2,3)+x_{21}^2(1,2,\overline{3})$$
$$+x_{21}(1,3,\overline{3})+(1,3,\overline{2})]$$
$$= Y_3(x_{33})Y_2(x_{22})Y_3(x_{32})[x_{31}(1,2,3)+x_{21}^2(1,2,\overline{3})+x_{21}x_{11}(1,3,\overline{3})+x_{11}^2(1,3,\overline{2})+x_{21}(2,3,\overline{3})$$
$$+x_{11}(2,3,\overline{2})+(2,3,\overline{1})]$$
$$= Y_3(x_{33})Y_2(x_{22})[x_{31}x_{32}(1,2,3)+(x_{31}+\frac{x_{21}^2}{x_{32}})(1,2,\overline{3})+x_{21}x_{11}(1,3,\overline{3})+x_{11}^2x_{32}(1,3,\overline{2})+x_{11}^2(1,\overline{3},\overline{2})$$
$$+x_{21}(2,3,\overline{3})+x_{11}x_{32}(2,3,\overline{2})+x_{32}(2,3,\overline{1})+x_{11}(2,\overline{3},\overline{2})+(2,\overline{3},\overline{1})]$$
$$= Y_3(x_{33})[x_{31}x_{32}(1,2,3)+(x_{31}+\frac{x_{21}^2}{x_{32}})x_{22}^2(1,2,\overline{3})+(x_{21}x_{11}+\frac{x_{21}^2x_{22}}{x_{32}}+x_{31}x_{22})(1,3,\overline{3})$$
$$+(\frac{x_{11}^2x_{32}}{x_{22}^2}+x_{31}+\frac{x_{21}^2}{x_{32}}+\frac{2x_{21}x_{11}}{x_{22}})(1,3,\overline{2})+x_{11}^2(1,\overline{3},\overline{2})+x_{21}x_{22}(2,3,\overline{3})+(x_{21}+\frac{x_{11}x_{32}}{x_{22}})(2,3,\overline{2})$$
$$+x_{32}(2,3,\overline{1})+x_{11}x_{22}(2,\overline{3},\overline{2})+x_{11}(3,\overline{3},\overline{2})+x_{22}^2(2,\overline{3},\overline{1})+x_{22}(3,\overline{3},\overline{1})+(3,\overline{2},\overline{1})]$$
$$=[x_{31}x_{32}x_{33}(1,2,3)+(x_{31}x_{32}+\frac{x_{31}x_{22}^2}{x_{33}}+\frac{x_{21}^2x_{22}^2}{x_{32}x_{33}})(1,2,\overline{3})+(x_{21}x_{11}+\frac{x_{21}^2x_{22}}{x_{32}}+x_{31}x_{22})(1,3,\overline{3})+$$
$$(\frac{x_{11}^2x_{32}}{x_{22}^2}+x_{31}+\frac{x_{21}^2}{x_{32}}+\frac{2x_{21}x_{11}}{x_{22}})x_{33}(1,3,\overline{2})+(\frac{x_{11}^2x_{32}}{x_{22}^2}+x_{31}+\frac{x_{21}^2}{x_{32}}+\frac{2x_{21}x_{11}}{x_{22}}+\frac{x_{11}^2}{x_{33}})(1,\overline{3},\overline{2})$$
$$+x_{21}x_{22}(2,3,\overline{3})+(x_{21}+\frac{x_{11}x_{32}}{x_{22}})x_{33}(2,3,\overline{2})+(x_{21}+\frac{x_{11}x_{32}}{x_{22}}+\frac{x_{11}x_{22}}{x_{33}})(2,\overline{3},\overline{2})+x_{32}x_{33}(2,3,\overline{1})$$
$$+(x_{32}+\frac{x_{22}^2}{x_{33}})(2,\overline{3},\overline{1})+x_{11}(3,\overline{3},\overline{2})+x_{22}(3,\overline{3},\overline{1})+x_{33}(3,\overline{2},\overline{1})+(\overline{3},\overline{2},\overline{1})]$$

And $V_2(y)$:

$$
\begin{aligned}
& Y_0(y_{03})Y_1(y_{12})Y_0(y_{02})Y_2(y_{21})Y_1(y_{11})Y_0(y_{01})(\overline{3},\overline{2},\overline{1}) \\
&= Y_0(y_{03})Y_1(y_{12})Y_0(y_{02})Y_2(y_{21})Y_1(y_{11})[y_{01}(\overline{3},\overline{2},\overline{1})+(1,\overline{3},\overline{2})] \\
&= Y_0(y_{03})Y_1(y_{12})Y_0(y_{02})Y_2(y_{21})[y_{01}(\overline{3},\overline{2},\overline{1})+y_{11}^2(1,\overline{3},\overline{2})+y_{11}(2,\overline{3},\overline{2})+(2,\overline{3},\overline{1})] \\
&= Y_0(y_{03})Y_1(y_{12})Y_0(y_{02})[y_{01}(\overline{3},\overline{2},\overline{1})+y_{11}^2(1,\overline{3},\overline{2})+y_{11}y_{21}(2,\overline{3},\overline{2})+y_{21}^2(2,\overline{3},\overline{1})+y_{11}(3,\overline{3},\overline{2}) \\
&+ y_{21}(3,\overline{3},\overline{1})+(3,\overline{2},\overline{1})] \\
&= Y_0(y_{03})Y_1(y_{12})[y_{01}y_{02}(\overline{3},\overline{2},\overline{1})+(y_{01}+\frac{y_{11}^2}{y_{02}})(1,\overline{3},\overline{2})+y_{11}y_{21}(2,\overline{3},\overline{2})+y_{21}^2y_{02}(2,\overline{3},\overline{1})+y_{21}^2(1,2,\overline{3}) \\
&+ y_{11}(3,\overline{3},\overline{2})+y_{21}y_{02}(3,\overline{3},\overline{1})+y_{21}(1,3,\overline{3})+y_{02}(3,\overline{2},\overline{1})+(1,3,\overline{2})] \\
&= Y_0(y_{03})[y_{01}y_{02}(\overline{3},\overline{2},\overline{1})+(y_{01}+\frac{y_{11}^2}{y_{02}})y_{12}^2(1,\overline{3},\overline{2})+(y_{11}y_{21}+y_{01}y_{12}+\frac{y_{11}^2y_{12}}{y_{02}})(2,\overline{3},\overline{2}) \\
&+(\frac{2y_{11}y_{21}}{y_{12}}+y_{01}+\frac{y_{11}^2}{y_{02}}+\frac{y_{21}^2y_{02}}{y_{12}^2})(2,\overline{3},\overline{1})+y_{21}^2(1,2,\overline{3})+y_{11}y_{12}(3,\overline{3},\overline{2})+(y_{11}+\frac{y_{21}y_{02}}{y_{12}})(3,\overline{3},\overline{1}) \\
&+ y_{21}y_{12}(1,3,\overline{3})+y_{21}(2,3,\overline{3})+y_{02}(3,\overline{2},\overline{1})+y_{12}^2(1,3,\overline{2})+y_{12}(2,3,\overline{2})+(2,3,\overline{1})] \\
&=[y_{01}y_{02}y_{03}(\overline{3},\overline{2},\overline{1})+(y_{01}y_{02}+\frac{y_{01}y_{12}^2}{y_{03}}+\frac{y_{11}^2y_{12}^2}{y_{02}y_{03}})(1,\overline{3},\overline{2})+(y_{11}y_{21}+y_{01}y_{12}+\frac{y_{11}^2y_{12}}{y_{02}})(2,\overline{3},\overline{2}) \\
&+(\frac{2y_{11}y_{21}}{y_{12}}+y_{01}+\frac{y_{11}^2}{y_{02}}+\frac{y_{21}^2y_{02}}{y_{12}^2})y_{03}(2,\overline{3},\overline{1})+(\frac{2y_{11}y_{21}}{y_{12}}+y_{01}+\frac{y_{11}^2}{y_{02}}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{21}^2}{y_{03}})(1,2,\overline{3}) \\
&+ y_{11}y_{12}(3,\overline{3},\overline{2})+(y_{11}+\frac{y_{21}y_{02}}{y_{12}})y_{03}(3,\overline{3},\overline{1})+(\frac{y_{21}y_{12}}{y_{03}}+y_{11}+\frac{y_{21}y_{02}}{y_{12}})(1,3,\overline{3})+y_{21}(2,3,\overline{3}) \\
&+ y_{02}y_{03}(3,\overline{2},\overline{1})+(y_{02}+\frac{y_{12}^2}{y_{03}})(1,3,\overline{2})+y_{12}(2,3,\overline{2})+y_{03}(2,3,\overline{1})+(1,2,3)]
\end{aligned}
$$

Now we solve for the coefficients in terms of one another using the equation: $V_1(x)a(x)=V_2(y)$ We get the following solution: $a(x)=\frac{1}{x_{31}x_{32}x_{33}}$, $y_{21}=\frac{x_{21}x_{22}}{x_{31}x_{32}x_{33}}$, $y_{03}=\frac{1}{x_{31}}$ and $y_{02}=\frac{1}{x_{32}}$, $y_{01}=\frac{1}{x_{33}}$, $y_{12}=\frac{x_{21}x_{22}+x_{11}x_{32}}{x_{31}x_{32}x_{22}}$ and $y_{11}=\frac{x_{11}x_{22}}{x_{33}(x_{21}x_{22}+x_{11}x_{32})}$. We can also see that $x_{31}=\frac{1}{y_{03}}$, $x_{32}=\frac{1}{y_{02}}$, $x_{33}=\frac{1}{y_{01}}$, $x_{21}=\frac{y_{21}y_{12}}{y_{03}(y_{21}y_{02}+y_{11}y_{12})}$, $x_{22}=\frac{y_{21}y_{02}+y_{11}y_{12}}{y_{01}y_{02}y_{12}}$ and $x_{11}=\frac{y_{11}y_{12}}{y_{01}y_{02}y_{03}}$.
From the general formula, we need to compute the maps for this geometric crystal: e_i^c, ε_i and γ_i for $i=0,1,2,3$.

First we compute e_1^c, e_2^c and e_3^c.

$$
\begin{aligned}
e_1^c(x_{33},x_{22},x_{32},x_{11},x_{21},x_{31}) &= \left(x_{33}\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}},x_{22}\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}},x_{32}\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}},x_{11}\frac{\frac{cx_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}},x_{21}\frac{\frac{cx_{22}}{x_{11}}}{\frac{cx_{22}}{x_{11}}},x_{31}\frac{\frac{cx_{22}}{x_{11}}}{\frac{cx_{22}}{x_{11}}}\right) \\
&=(x_{33},x_{22},x_{32},cx_{11},x_{21},x_{31})
\end{aligned}
$$

$$
\begin{aligned}
&e_2^c(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) \\
&= \left(x_{33}, x_{22} \frac{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}{\frac{x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}, x_{32} \frac{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}, x_{11} \frac{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}, x_{21} \frac{\frac{c\,x_{33}}{x_{22}} + \frac{c\,x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}, x_{31} \right) \\
&= (x_{33}, c_2 x_{22}, x_{32}, x_{11}, \frac{c}{c_2} x_{21}, x_{31})
\end{aligned}
$$

where

$$
c_2 = \frac{\frac{c\,x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}}{\frac{x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}} = \frac{c\,x_{21} x_{22} + x_{32} x_{11}}{x_{21} x_{22} + x_{32} x_{11}}
$$

Now let $c_3 = x_{32}^2 x_{31} x_{33} + x_{22}^2 x_{31} x_{32} + x_{22}^2 x_{21}^2$, $c_{31} = c\,x_{32}^2 x_{31} x_{33} + x_{22}^2 x_{31} x_{32} + x_{22}^2 x_{21}^2$ and $c_{32} = c\,x_{32}^2 x_{31} x_{33} + c\,x_{22}^2 x_{31} x_{32} + x_{22}^2 x_{21}^2$. Then

$$
e_3^c(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \left(\frac{c_{31}}{c_3} x_{33}, x_{22}, \frac{c_{32}}{c_{31}} x_{32}, x_{11}, x_{21}, \frac{c \cdot c_3}{c_{32}} x_{31} \right)
$$

Then we compute the ε_i and γ_i actions for $i = 1, 2, 3$

$$
\varepsilon_1(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{x_{22}}{x_{11}}
$$

$$
\varepsilon_2(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{x_{22}}{x_{33}} + \frac{x_{32} x_{33} x_{11}}{x_{22}^2 x_{21}} = \frac{x_{33} x_{22} x_{21} + x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}
$$

$$
\varepsilon_3(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{1}{x_{33}} + \frac{x_{22}^2}{x_{33}^2 x_{32}} + \frac{x_{21}^2 x_{22}^2}{x_{33}^2 x_{32}^2 x_{31}} = \frac{x_{33} x_{32}^2 x_{31} + x_{22}^2 x_{32} x_{31} + x_{22}^2 x_{21}^2}{x_{33}^2 x_{32}^2 x_{31}}
$$

$$
\gamma_1(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{x_{11}^2}{x_{21} x_{22}}
$$

$$
\gamma_2(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{x_{21}^2 x_{22}^2}{x_{31} x_{32} x_{33} x_{11}}
$$

$$
\gamma_3(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) = \frac{x_{31}^2 x_{32}^2 x_{33}^2}{x_{21}^2 x_{22}^2}
$$

To get the formulas for e_0^c, ε_0 and γ_0 we first must get the formulas for $\overline{e_0^c}$, $\overline{\varepsilon_0}$ and $\overline{\gamma_0}$ in $\mathcal{V}_2(y)$. These are as follows:

$$
\overline{\varepsilon_0}(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = \frac{1}{y_{03}} + \frac{y_{12}^2}{y_{03}^2 y_{02}} + \frac{y_{12}^2 y_{11}^2}{y_{03}^2 y_{02}^2 y_{01}} = \frac{y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2}{y_{03}^2 y_{02}^2 y_{01}}
$$

$$
\overline{\gamma_0}(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = \frac{y_{01}^2 y_{02}^2 y_{03}^2}{y_{11}^2 y_{12}^2}
$$

Let $c_0 = y_{03}y_{02}^2y_{01} + y_{12}^2y_{02}y_{01} + y_{12}^2y_{11}^2$, $c_{01} = c\,y_{03}y_{02}^2y_{01} + y_{12}^2y_{02}y_{01} + y_{12}^2y_{11}^2$, $c_{02} = c\,y_{03}y_{02}^2y_{01} + c\,y_{12}^2y_{02}y_{01} + y_{12}^2y_{11}^2$. Then

$$\overline{e_0^c}(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = \left(\frac{c_{01}}{c_0}y_{03}, y_{12}, \frac{c_{02}}{c_{01}}y_{02}, y_{21}, y_{11}, \frac{c \cdot c_0}{c_{02}}y_{01}\right)$$

Now e_0^c, ε_0 and γ_0 are defined as follows:

$$\begin{aligned} e_0^c(V_1(x)) &= \overline{\sigma^{-1}} \circ \overline{e_0^c} \circ \overline{\sigma}(V_1(x)) \\ \gamma_0(V_1(x)) &= \overline{\gamma_0}(\overline{\sigma}(V_1(x))) \\ \varepsilon_0(V_1(x)) &= \overline{\varepsilon_0}(\overline{\sigma}(V_1(x))) \end{aligned}$$

Now we find γ_0

$$\gamma_0(V_1(x)) = \overline{\gamma_0}(\overline{\sigma}(V_1(x))) = \overline{\gamma_0}(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = \frac{y_{01}^2y_{02}^2y_{03}^2}{y_{11}^2y_{12}^2} = \frac{\frac{1}{x_{31}^2x_{32}^2x_{33}^2}}{\frac{x_{11}^2x_{22}^2}{x_{31}^2x_{32}^2x_{33}^2x_{22}^2}} = \frac{1}{x_{11}^2}$$

$$\varepsilon_0(V_1(x)) = \frac{1}{y_{03}} + \frac{y_{12}^2}{y_{03}^2y_{02}} + \frac{y_{11}^2y_{12}^2}{y_{03}^2y_{02}^2y_{01}} = x_{31} + \frac{(x_{21}x_{22} + x_{11}x_{32})^2}{x_{22}^2x_{32}} + \frac{x_{11}^2}{x_{33}}$$

$$e_0^c(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = \overline{\sigma^{-1}}\left(\frac{c_{01}}{c_0}y_{03}, y_{12}, \frac{c_{02}}{c_{01}}y_{02}, y_{21}, y_{11}, \frac{c \cdot c_0}{c_{02}}y_{01}\right)$$

We define $c_2' = x_{31} + \frac{x_{21}^2}{x_{32}} + \frac{2x_{21}x_{11}}{x_{22}} + \frac{x_{11}^2x_{32}}{x_{22}^2} + \frac{x_{11}^2}{x_{33}}$, $c_{21}' = c\,x_{31} + \frac{x_{21}^2}{x_{32}} + \frac{2x_{21}x_{11}}{x_{22}} + \frac{x_{11}^2x_{32}}{x_{22}^2} + \frac{x_{11}^2}{x_{33}}$, $c_{24}' = c\,x_{31} + c\,\frac{x_{21}^2}{x_{32}} + c\,\frac{2x_{21}x_{11}}{x_{22}} + c\,\frac{x_{11}^2x_{32}}{x_{22}^2} + \frac{x_{11}^2}{x_{33}}$.

$$\begin{aligned} &e_0^c(x_{33}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}) \\ &= \left(\frac{c_{24}'}{c \cdot c_2'}x_{33}, x_{22}\frac{x_{21}x_{22}c_{24}' + x_{11}x_{32}c_{21}'}{(x_{21}x_{22} + x_{11}x_{32})c \cdot c_2'}, \frac{c_{21}'}{c_{24}'}x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22} + x_{11}x_{32}) \cdot c_2'}{x_{21}x_{22}c_{24}' + x_{11}x_{32}c_{21}'}, \frac{c_2'}{c_{21}'}x_{31}\right) \end{aligned}$$

Theorem 5.3.2. *$\mathcal{V}_1 = (V_1(x), e_i^c, \varepsilon_i, \varphi_i)$ for $i = 0, 1, 2, 3$ is a positive geometric crystal associated with $C_3^{(1)}$.*

Proof. This is clearly positive as all the coefficients are positive. We already know that $\mathcal{V}_1$ is a geometric crystal without the additionally 0-actions, so we only need to check relations with the 0 index:

1. $\gamma_0(e_i^c(V_1(x))) = c^{a_{i0}}\gamma_0(V_1(x))$

2. $\gamma_i(e_0^c(V_1(x))) = c^{a_{0i}}\gamma_i(V_1(x))$

3. $\varepsilon_0(e_0^c(V_1(x))) = c^{-1}\varepsilon_0(V_1(x))$

4. $\varepsilon_2(e_0^c(V_1(x))) = \varepsilon_2(V_1(x))$

5. $\varepsilon_3(e_0^c(V_1(x))) = \varepsilon_3(V_1(x))$

6. $\varepsilon_0(e_2^c(V_1(x))) = \varepsilon_0(V_1(x))$

7. $\varepsilon_0(e_3^c(V_1(x))) = \varepsilon_0(V_1(x))$

8. $e_0^c e_2^d = e_2^d e_0^c$

9. $e_0^c e_3^d = e_3^d e_0^c$

10. $e_1^c e_0^{c^2 d} e_1^{cd} e_0^d = e_0^d e_1^{cd} e_0^{c^2 d} e_1^c$

1. $\gamma_0(e_0^c(V_1(x)))$

$$= \gamma_0\left(\frac{c'_{24}}{c \cdot c'_2} x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}, \frac{c'_{21}}{c'_{24}} x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}} x_{31}\right)$$

$$= \frac{1}{\frac{x_{11}^2}{c^2}} = \frac{c^2}{x_{11}^2} = c^2\varepsilon_0(V_1(x))$$

$$\gamma_0(e_1^c(V_1(x))) = \gamma_0(x_{33}, x_{22}, x_{32}, c\,x_{11}, x_{21}, x_{31}) = \frac{1}{c^2 x_{11}^2} = \frac{1}{c^2}\gamma_0(V_1(x))$$

$$\gamma_0(e_2^c(V_1(x)) = \gamma_0(x_{33}, c_2 x_{22}, x_{32}, x_{11}, \frac{c}{c_2} x_{21}, x_{31}) = \frac{1}{x_{11}^2} = \gamma_0(V_1(x))$$

$$\gamma_0(e_3^c(V_1(x))) = \gamma_0\left(\frac{c_{31}}{c_3} x_{33}, x_{22}, \frac{c_{32}}{c_{31}} x_{32}, x_{11}, x_{21}, \frac{c \cdot c_3}{c_{32}} x_{31}\right) = \frac{1}{x_{11}^2} = \gamma_0(V_1(x))$$

2. $\gamma_1(e_0^c(V_1(x)))$

$$= \gamma_1\left(\frac{c'_{24}}{c \cdot c'_2} x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}, \frac{c'_{21}}{c'_{24}} x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}} x_{31}\right)$$

$$= \frac{\frac{x_{11}^2}{c^2}}{\frac{x_{21}x_{22}}{c}} = c^{-1}\gamma_1(V_1(x))$$

$\gamma_2(e_0^c(V_1(x)))$

$$= \gamma_2\left(\frac{c'_{24}}{c \cdot c'_2} x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}, \frac{c'_{21}}{c'_{24}} x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}} x_{31}\right)$$

$$= \frac{\frac{x_{21}^2 x_{22}^2}{c^2}}{\frac{x_{11}x_{31}x_{32}x_{33}}{c^2}} = \gamma_2(V_1(x))$$

$\gamma_3(e_0^c(V_1(x)))$

$$= \gamma_3\left(\frac{c'_{24}}{c \cdot c'_2} x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}, \frac{c'_{21}}{c'_{24}} x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22} + x_{11}x_{32})c \cdot c'_2}{x_{21}x_{22}c'_{24} + x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}} x_{31}\right)$$

$$= \frac{\frac{x_{31}^2 x_{32}^2 x_{33}}{c^2}}{\frac{x_{21}^2 x_{22}^2}{c^2}} = \gamma_3(V_1(x))$$

Now we prove 3.:

$$\varepsilon_0(e_0^c(V_1(x))) = \overline{\varepsilon_0}\overline{\sigma}\overline{\sigma^{-1}}\overline{e_0}^c\overline{\sigma}(V_1(x)) = \overline{\varepsilon_0}\left(\frac{c_{01}}{c_0} y_{03}, y_{12}, \frac{c_{02}}{c_{01}} y_{02}, y_{21}, y_{11}, \frac{c \cdot c_0}{c_{02}} y_{01}\right)$$

$$= \frac{(\frac{y_{03}y_{02}^2 y_{01} c \cdot c_{02} + y_{12}^2 c \cdot c_0 y_{02} y_{01}}{c_{01}} + y_{12}^2 y_{11}^2)c_0}{c \cdot c_{02} y_{01} y_{02}^2 y_{03}^2} = \frac{(y_{03}y_{02}^2 y_{01} c \cdot c_{02} + y_{12}^2 c \cdot c_0 y_{02} y_{01} + y_{12}^2 y_{11}^2 c_{01})c_0}{c \cdot c_{02} c_{01} y_{01} y_{02}^2 y_{03}^2}$$

$$= \frac{(y_{03}y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2)[c\, y_{03} y_{02}^2 y_{01}(c\, y_{03} y_{02}^2 y_{01} + c\, y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2) +}{c\, y_{03}^2 y_{02}^2 y_{01}(c\, y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2)(c\, y_{03} y_{02}^2 y_{01} + c\, y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2)}$$

$$\frac{c\, y_{12}^2 y_{02} y_{01}(y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2) + y_{11}^2 y_{12}^2(c\, y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2)}{c\, y_{03}^2 y_{02}^2 y_{01}(c\, y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2)(c\, y_{03} y_{02}^2 y_{01} + c\, y_{12}^2 y_{02} y_{01} + y_{12}^2 y_{11}^2}\Big)$$

$$= \frac{c_0 K}{c K} = \frac{c_0}{c} = \frac{y_{03} y_{02}^2 y_{01} + y_{12}^2 y_{02} y_{01} + y_{11}^2 y_{12}^2}{c\, y_{03}^2 y_{02}^2 y_{01}} = c^{-1}\overline{\varepsilon_0}(V_2(y)) = c^{-1}\varepsilon_0(V_1(x))$$

Where $K = c^2 y_{03}^2 y_{02}^4 y_{01}^2 + (c^2 + c) y_{12}^2 y_{03} y_{02}^3 y_{01}^2 + 2c\, y_{12}^2 y_{11}^2 y_{03} y_{02}^2 y_{01} + (c+1) y_{11}^2 y_{12}^4 y_{02} y_{01} +$

$c\, y_{12}^4 y_{02}^2 y_{01}^2 + y_{11}^4 y_{12}^4$.

4. $\varepsilon_2(e_0^c(V_1(x)))$
$$= \varepsilon_2\left(\frac{c'_{24}}{c\cdot c'_2}x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21}}{(x_{21}x_{22}+x_{11}x_{32})c\cdot c'_2}, \frac{c'_{21}}{c'_{24}}x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22}+x_{11}x_{32})c\cdot c'_2}{x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}}x_{31}\right)$$
$$= \frac{\frac{c'_{24}x_{33}x_{21}x_{22}+c'_{21}x_{32}x_{33}x_{11}}{c^2 c'_2}}{\frac{x_{21}x_{22}^2(x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21})}{(x_{21}x_{22}+x_{11}x_{32})c^2\cdot c'_2}} = \frac{x_{33}(x_{21}x_{22}+x_{11}x_{32})}{x_{22}^2 x_{21}} = \varepsilon_2(V_1(x))$$

5. $\varepsilon_3(e_0^c(V_1(x)))$
$$= \varepsilon_3\left(\frac{c'_{24}}{c\cdot c'_2}x_{33}, x_{22}\frac{x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21}}{(x_{21}x_{22}+x_{11}x_{32})c\cdot c'_2}, \frac{c'_{21}}{c'_{24}}x_{32}, \frac{x_{11}}{c}, x_{21}\frac{(x_{21}x_{22}+x_{11}x_{32})c\cdot c'_2}{x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21}}, \frac{c'_2}{c'_{21}}x_{31}\right)$$
$$= \frac{\frac{c'_{21}x_{31}x_{32}^2x_{33}}{c'_{24}\cdot c} + \frac{x_{32}x_{31}x_{22}^2(x_{21}x_{22}c'_{24}+x_{11}x_{32}c'_{21})}{(x_{21}x_{22}+x_{11}x_{32})c'_{24}c'_2c^2} + \frac{x_{21}^2x_{22}^2}{c^2}}{\frac{x_{31}x_{32}^2x_{33}^2}{c^2c'_2}}$$

By substituting the expressions c_2, c_{21}, and c_{24} and factoring, we get ε_3. Now we move on to the proofs of 6. and 7.:

6. $\varepsilon_0(e_2^c(V_1(x))) = \varepsilon_0(x_{33}, c_2x_{22}, x_{32}, x_{11}, \frac{c}{c_2}x_{21}, x_{31}) = x_{31} + \frac{x_{11}^2}{x_{33}} + \frac{(c\,x_{21}x_{22}+x_{11}x_{32})^2}{\frac{x_{32}x_{22}^2(c\,x_{21}x_{22}+x_{11}x_{32})^2}{(x_{21}x_{22}+x_{11}x_{32})^2}}$
$$= x_{31} + \frac{x_{11}^2}{x_{33}} + \frac{(x_{21}x_{22}+x_{11}x_{32})^2}{x_{22}^2x_{32}} = \varepsilon_0(V_1(x))$$

7. $\varepsilon_0(e_3^c(V_1(x))) = \varepsilon_0\left(\frac{c_{31}}{c_3}x_{33}, x_{22}, \frac{c_{32}}{c_{31}}x_{32}, x_{11}, x_{21}, \frac{c\cdot c_3}{c_{32}}x_{31}\right)$
$$= \frac{c\cdot c_3}{c_{32}}x_{31} + \frac{x_{11}^2c_3}{x_{33}c_{31}} + \frac{x_{21}^2c_{31}}{c_{32}x_{32}} + \frac{2x_{11}x_{21}}{x_{22}} + \frac{x_{11}^2c_{32}x_{32}}{c_{31}x_{22}^2}$$

Now the term $\frac{2x_{11}x_{21}}{x_{22}}$ is already the same as it would be if we had $\varepsilon_0(V_1(x))$. So we focus only on the rest of the terms.

$$\frac{c\cdot c_3}{c_{32}}x_{31} + \frac{x_{11}^2c_3}{x_{33}c_{31}} + \frac{x_{21}^2c_{31}}{c_{32}x_{32}} + \frac{x_{11}^2c_{32}x_{32}}{c_{31}x_{22}^2}$$
$$= \frac{c\,c_3c_{31}x_{31}x_{32}x_{33}x_{22}^2 + x_{21}^2c_{31}^2x_{33}x_{22}^2 + x_{11}^2c_3c_{32}x_{32}x_{22}^2 + x_{11}^2c_{32}^2x_{32}^2x_{33}}{c_{31}c_{32}x_{32}x_{33}x_{22}^2}$$

If we expand these terms and factor, we get $x_{31} + \frac{x_{21}^2}{x_{32}} + \frac{x_{11}^2x_{32}}{x_{22}} + \frac{x_{11}^2}{x_{33}}$. Along with the term we removed earlier, $\frac{2x_{11}x_{21}}{x_{22}}$, we get $\varepsilon_0(V_1(x))$.

Before proving the last 3 identities, we prove a useful statement: $\overline{\sigma}\circ e_i^c = \overline{e_{3-i}}^c \circ \overline{\sigma}$ we prove this for $1 \leq i \leq 2$. For i=1:

$$\overline{\sigma}\circ e_1^c(V_1(x)) = \overline{\sigma}(x_{33}, x_{22}, x_{32}, c\,x_{11}, x_{21}, x_{31}) = (y_{03}, y_{12}, y_{02}, c\,y_{21}, y_{11}, y_{01})$$
$$\overline{e_2}^c \circ \overline{\sigma}(V_1(x)) = \overline{e_2}^c(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = (y_{03}, y_{12}, y_{02}, c\,y_{21}, y_{11}, y_{01})$$

For i=2:

$$\overline{\sigma}\circ e_2^c(V_1(x)) = \overline{\sigma}(x_{33}, c_2 x_{22}, x_{32}, x_{11}, \frac{c}{c_2} x_{21}, x_{31}) = (y_{03}, c_1' y_{12}, y_{02}, y_{21}, \frac{c}{c_1'} y_{11}, y_{01})$$
$$\overline{e_1}^c \circ \overline{\sigma}(V_1(x)) = \overline{e_1}^c(y_{03}, y_{12}, y_{02}, y_{21}, y_{11}, y_{01}) = (y_{03}, c_1' y_{12}, y_{02}, y_{21}, \frac{c}{c_1'} y_{11}, y_{01})$$

Where $c_1' = \frac{c\,y_{12}y_{11}+y_{02}y_{21}}{y_{12}y_{11}+y_{02}y_{21}}$ This also implies that $e_i^c \circ \overline{\sigma^{-1}} = \overline{\sigma^{-1}} \circ \overline{e_{(n-i)}}^c$ for i=1,2. Using these identities and the fact that $V_2(x)$ is a geometric crystal, we can prove 8. and 10.

$$e_0^c e_2^d = \overline{\sigma^{-1}}\,\overline{e_0}^c\,\overline{\sigma}\, e_2^d = \overline{\sigma^{-1}}\,\overline{e_0}^c\,\overline{e_1}^d\,\overline{\sigma} = \overline{\sigma^{-1}}\,\overline{e_1}^d\,\overline{e_0}^c\,\overline{\sigma} = e_2^d\,\overline{\sigma^{-1}}\,\overline{e_0}^c\,\overline{\sigma} = e_2^d e_0^c$$

For 9, we used a computer algebra system to confirm the identity held. And finally for 10.:

$$\begin{aligned}
&e_1^c\,\overline{\sigma^{-1}}\,\overline{e_0}^{c^2d}\,\overline{\sigma}\, e_1^{cd}\,\overline{\sigma^{-1}}\,\overline{e_0}^d\,\overline{\sigma} = \overline{\sigma^{-1}}\,\overline{e_2}^c\,\overline{e_0}^{c^2d}\,\overline{\sigma}\,\overline{\sigma^{-1}}\,\overline{e_2}^{cd}\,\overline{e_0}^d\,\overline{\sigma} = \overline{\sigma^{-1}}\,\overline{e_2}^c\,\overline{e_0}^{c^2d}\,\overline{e_2}^{cd}\,\overline{e_0}^d\,\overline{\sigma}\\
&= \overline{\sigma^{-1}}\,\overline{e_0}^d\,\overline{e_2}^{cd}\,\overline{e_0}^{c^2d}\,\overline{e_2}^c\,\overline{\sigma} = \overline{\sigma^{-1}}\,\overline{e_0}^d\,\overline{e_2}^{cd}\,\overline{\sigma}\,\overline{\sigma^{-1}}\,\overline{e_0}^{c^2d}\,\overline{\sigma}\, e_1^c\\
&= \overline{\sigma^{-1}}\,\overline{e_0}^d\,\overline{\sigma}\, e_1^{cd}\,\overline{\sigma^{-1}}\,\overline{e_0}^{c^2d}\,\overline{\sigma}\, e_1^c = e_0^d e_1^{cd} e_0^{c^2d} e_1^c
\end{aligned}$$

Therefore, all relations hold, so it is a geometric crystal. □

$C_4^{(1)}$ Geometric Crystal

This case is the first case where the representation is not multiplicity free. The two vectors $(3,4,\overline{4},\overline{3})$ and $(2,4,\overline{4},\overline{2})$ both have weight 0. As a result, we used the global base to consider the e and f actions. By direct computation, we see that $f_2(2,4,\overline{4},\overline{3}) = (2,4,\overline{4},\overline{2}) + (3,4,\overline{4},\overline{3})$. All other actions are the same for the global and crystal bases. Additionally, we note that if $f_i^2 \neq 0$, then we have $f_i^2(b) = 2b'$. Before we compute the variety, we must compute the translations. We first compute the following actions of simple reflections: s_0, s_1, s_2, s_3, and

s_4. From the Cartan matrix, we know that

$$\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$$

$$\alpha_1 = 2\Lambda_1 - \Lambda_0 - \Lambda_2$$

$$\alpha_2 = 2\Lambda_2 - \Lambda_1 - \Lambda_3$$

$$\alpha_3 = 2\Lambda_3 - \Lambda_2 - \Lambda_4$$

$$\alpha_4 = 2\Lambda_4 - 2\Lambda_3$$

From this, we know the simple reflections act as:

$$s_0(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - (2, -2, 0, 0, 0)\lambda_0 = (-\lambda_0, 2\lambda_0 + \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$s_1(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - (-1, 2, -1, 0, 0)\lambda_1 = (\lambda_0 + \lambda_1, -\lambda_1, \lambda_1 + \lambda_2, \lambda_3, \lambda_4)$$

$$s_2(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - (0, -1, 2, -1, 0)\lambda_2 = (\lambda_0, \lambda_1 + \lambda_2, -\lambda_2, \lambda_2 + \lambda_3, \lambda_4)$$

$$s_3(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - (0, 0, -1, 2, -1)\lambda_3 = (\lambda_0, \lambda_1, \lambda_2 + \lambda_3, -\lambda_3, \lambda_3 + \lambda_4)$$

$$s_4(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - (0, 0, 0, -2, 2)\lambda_4 = (\lambda_0, \lambda_1, \lambda_2, \lambda_3 + 2\lambda_4, -\lambda_4)$$

We show below that $t(\omega_3) = \sigma s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2 s_3 s_4$ and $t(\check{\omega}_3) = \sigma s_0 s_1 s_0 s_2 s_1 s_0 s_3 s_2 s_1 s_0$ where σ is the diagram automorphism $0 \longleftrightarrow 4$, $1 \longleftrightarrow 3$, and $2 \longleftrightarrow 2$.

$$\begin{aligned}
&\sigma s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2 s_3 s_4(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2 s_3(\lambda_0, \lambda_1, \lambda_2, \lambda_3 + 2\lambda_4, -\lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2(\lambda_0, \lambda_1, \lambda_2 + \lambda_3 + 2\lambda_4, -\lambda_3 - 2\lambda_4, \lambda_3 + \lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2 s_3 s_4 s_1(\lambda_0, \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4, -\lambda_2 - \lambda_3 - 2\lambda_4, \lambda_2, \lambda_3 + \lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2 s_3 s_4(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 - 2\lambda_4, \lambda_1, \lambda_2, \lambda_3 + \lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2 s_3(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 - 2\lambda_4, \lambda_1, \lambda_2 + 2\lambda_3 + 2\lambda_4, -\lambda_3 - \lambda_4) \\
&= \sigma s_4 s_3 s_4 s_2(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 - 2\lambda_4, \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4, \\
&- \lambda_2 - 2\lambda_3 - 2\lambda_4, \lambda_2 + \lambda_3 + \lambda_4)
\end{aligned}$$

$$= \sigma s_4 s_3 s_4(\lambda_0+\lambda_1+\lambda_2+\lambda_3+2\lambda_4, \lambda_3, -\lambda_1-\lambda_2-2\lambda_3-2\lambda_4, \lambda_1, \lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_4 s_3(\lambda_0+\lambda_1+\lambda_2+\lambda_3+2\lambda_4, \lambda_3, -\lambda_1-\lambda_2-2\lambda_3-2\lambda_4, \lambda_1+2\lambda_2+2\lambda_3+2\lambda_4, -\lambda_2-\lambda_3-\lambda_4)$$
$$= \sigma s_4(\lambda_0+\lambda_1+\lambda_2+\lambda_3+2\lambda_4, \lambda_3, \lambda_2, -\lambda_1-2\lambda_2-2\lambda_3-2\lambda_4, \lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma(\lambda_0+\lambda_1+\lambda_2+\lambda_3+2\lambda_4, \lambda_3, \lambda_2, \lambda_1, -\lambda_1-\lambda_2-\lambda_3-\lambda_4)$$
$$= (-\lambda_1-\lambda_2-\lambda_3-\lambda_4, \lambda_1, \lambda_2, \lambda_3, \lambda_0+\lambda_1+\lambda_2+\lambda_3+2\lambda_4)$$
$$= (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) + (\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\Lambda_4-\Lambda_0)$$

$$\sigma s_0 s_1 s_0 s_2 s_1 s_0 s_3 s_2 s_1 s_0(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2 s_1 s_0 s_3 s_2 s_1(-\lambda_0, \lambda_1+2\lambda_0, \lambda_2, \lambda_3, \lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2 s_1 s_0 s_3 s_2(\lambda_0+\lambda_1, -2\lambda_0-\lambda_1, 2\lambda_0+\lambda_1+\lambda_2, \lambda_3, \lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2 s_1 s_0 s_3(\lambda_0+\lambda_1, \lambda_2, -2\lambda_0-\lambda_1-\lambda_2, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3, \lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2 s_1 s_0(\lambda_0+\lambda_1, \lambda_2, \lambda_3, -2\lambda_0-\lambda_1-\lambda_2-\lambda_3, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2 s_1(-\lambda_0-\lambda_1, 2\lambda_0+2\lambda_1+\lambda_2, \lambda_3, -2\lambda_0-\lambda_1-\lambda_2-\lambda_3, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_0 s_1 s_0 s_2(\lambda_0+\lambda_1+\lambda_2, -2\lambda_0-2\lambda_1-\lambda_2, 2\lambda_0+2\lambda_1+\lambda_2+\lambda_3, -2\lambda_0-\lambda_1-\lambda_2-\lambda_3,$$
$$2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_0 s_1 s_0(\lambda_0+\lambda_1+\lambda_2, \lambda_3, -2\lambda_0-2\lambda_1-\lambda_2-\lambda_3, \lambda_1, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_0 s_1(-\lambda_0-\lambda_1-\lambda_2, 2\lambda_0+2\lambda_1+2\lambda_2+\lambda_3, -2\lambda_0-2\lambda_1-\lambda_2-\lambda_3, \lambda_1, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma s_0(\lambda_0+\lambda_1+\lambda_2+\lambda_3, -2\lambda_0-2\lambda_1-2\lambda_2-\lambda_3, \lambda_2, \lambda_1, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= \sigma(-\lambda_0-\lambda_1-\lambda_2-\lambda_3, \lambda_3, \lambda_2, \lambda_1, 2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$= (2\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4, \lambda_1, \lambda_2-\lambda_0-\lambda_1-\lambda_2-\lambda_3)$$
$$= (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) + (\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\Lambda_0-\Lambda_4)$$

Now we compute $V_1(x)$.

$$Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32})Y_4(x_{42})Y_1(x_{11})Y_2(x_{21})Y_3(x_{31})Y_4(x_{41})(1,2,3,4)$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32})Y_4(x_{42})Y_1(x_{11})Y_2(x_{21})Y_3(x_{31})[x_{41}(1,2,3,4)+(1,2,3,\overline{4})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32})Y_4(x_{42})Y_1(x_{11})Y_2(x_{21})[x_{41}(1,2,3,4)+x_{31}^2(1,2,3,\overline{4})$$
$$+x_{31}(1,2,4,\overline{4})+(1,2,4,\overline{3})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32})Y_4(x_{42})Y_1(x_{11})[x_{41}(1,2,3,4)+x_{31}^2(1,2,3,\overline{4})$$
$$+x_{31}x_{21}(1,2,4,\overline{4})+x_{21}^2(1,2,4,\overline{3})+x_{31}(1,3,4,\overline{4})+x_{21}(1,3,4,\overline{3})+(1,3,4,\overline{2})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32})Y_4(x_{42})[x_{41}(1,2,3,4)+x_{31}^2(1,2,3,\overline{4})$$
$$+x_{31}x_{21}(1,2,4,\overline{4})+x_{21}^2(1,2,4,\overline{3})+x_{31}x_{11}(1,3,4,\overline{4})+x_{21}x_{11}(1,3,4,\overline{3})+x_{11}^2(1,3,4,\overline{2})$$
$$+x_{31}(2,3,4,\overline{4})+x_{21}(2,3,4,\overline{3})+x_{11}(2,3,4,\overline{2})+(2,3,4,\overline{1})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})Y_3(x_{32}) = [x_{41}x_{42}(1,2,3,4)+(\frac{x_{31}^2}{x_{42}}+x_{41})(1,2,3,\overline{4})$$
$$+x_{31}x_{21}(1,2,4,\overline{4})+x_{21}^2x_{42}(1,2,4,\overline{3})+x_{31}x_{11}(1,3,4,\overline{4})+x_{21}x_{11}x_{42}(1,3,4,\overline{3})$$
$$+x_{11}^2x_{42}(1,3,4,\overline{2})+x_{31}(2,3,4,\overline{4})+x_{21}x_{42}(2,3,4,\overline{3})+x_{11}x_{42}(2,3,4,\overline{2})+x_{42}(2,3,4,\overline{1})+x_{21}^2$$
$$(1,2,\overline{4},\overline{3})+x_{21}x_{11}(1,3,\overline{4},\overline{3})+x_{11}^2(1,3,\overline{4},\overline{2})+x_{21}(2,3,\overline{4},\overline{3})+x_{11}(2,3,\overline{4},\overline{2})+(2,3,\overline{4},\overline{1})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})Y_2(x_{22})[x_{41}x_{42}(1,2,3,4)+(\frac{x_{31}^2}{x_{42}}+x_{41})x_{32}^2(1,2,3,\overline{4})+(x_{31}x_{21}$$
$$+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}})(1,2,4,\overline{4})+(\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})(1,2,4,\overline{3})+x_{31}x_{11}x_{32}(1,3,4,\overline{4})$$
$$+(\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11})(1,3,4,\overline{3})+x_{11}^2x_{42}(1,3,4,\overline{2})+x_{31}x_{32}(2,3,4,\overline{4})+(\frac{x_{21}x_{42}}{x_{32}}+x_{31})(2,3,4,\overline{3})$$
$$+x_{11}x_{42}(2,3,4,\overline{2})+x_{42}(2,3,4,\overline{1})+x_{21}^2(1,2,\overline{4},\overline{3})+x_{21}x_{11}x_{32}(1,3,\overline{4},\overline{3})+x_{11}^2x_{32}^2(1,3,\overline{4},\overline{2})$$
$$+x_{21}x_{32}(2,3,\overline{4},\overline{3})+x_{11}x_{32}^2(2,3,\overline{4},\overline{2})+x_{32}^2(2,3,\overline{4},\overline{1})+x_{21}x_{11}(1,4,\overline{4},\overline{3})+x_{11}^2x_{32}(1,4,\overline{4},\overline{2})$$
$$+x_{11}^2(1,4,\overline{3},\overline{2})+x_{21}(2,4,\overline{4},\overline{3})+x_{11}x_{32}(2,4,\overline{4},\overline{2})+x_{11}(2,4,\overline{3},\overline{2})+x_{32}(2,4,\overline{4},\overline{1})+(2,4,\overline{3},\overline{1})]$$
$$= Y_4(x_{44})Y_3(x_{33})Y_4(x_{43})[x_{41}x_{42}(1,2,3,4)+(\frac{x_{31}^2}{x_{42}}+x_{41})x_{32}^2(1,2,3,\overline{4})+(x_{31}x_{21}+x_{41}x_{32}$$
$$+\frac{x_{31}^2x_{32}}{x_{42}})x_{22}(1,2,4,\overline{4})+(\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})x_{22}^2(1,2,4,\overline{3})+(\frac{x_{31}x_{11}x_{32}}{x_{22}}+x_{31}x_{21}$$
$$+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}})(1,3,4,\overline{4})+(\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11}+\frac{x_{21}^2x_{42}x_{22}}{x_{32}^2}+x_{41}x_{22}+\frac{2x_{31}x_{21}x_{22}}{x_{32}}$$
$$+\frac{x_{31}^2x_{22}}{x_{42}})(1,3,4,\overline{3})+(\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})$$

$$(1,3,4,\overline{2})+x_{31}x_{32}(2,3,4,\overline{4})+(\frac{x_{21}x_{42}x_{22}}{x_{32}}+x_{31}x_{22})(2,3,4,\overline{3})+(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31})$$

$$(2,3,4,\overline{2})+x_{42}(2,3,4,\overline{1})+x_{21}^2x_{22}^2(1,2,\overline{4},\overline{3})+(x_{21}x_{11}x_{32}+x_{21}^2x_{22})(1,3,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}^2}{x_{22}^2}+x_{21}^2$$

$$+\frac{2x_{21}x_{11}x_{32}}{x_{22}})(1,3,\overline{4},\overline{2})+x_{21}x_{32}x_{22}(2,3,\overline{4},\overline{3})+(\frac{x_{11}x_{32}^2}{x_{22}}+x_{21}x_{32})(2,3,\overline{4},\overline{2})+x_{32}^2(2,3,\overline{4},\overline{1})$$

$$+x_{21}x_{11}x_{22}(1,4,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}}{x_{22}}+x_{11}x_{21})(1,4,\overline{4},\overline{2})+x_{11}^2(1,4,\overline{3},\overline{2})+x_{21}x_{22}^2(2,4,\overline{4},\overline{3})$$

$$+(x_{11}x_{32}+x_{21}x_{22})(2,4,\overline{4},\overline{2})$$

$$+x_{21}x_{22}(3,4,\overline{4},\overline{3})+x_{21}(3,4,\overline{4},\overline{2})+x_{11}x_{22}(2,4,\overline{3},\overline{2})+x_{11}(3,4,\overline{3},\overline{2})+x_{32}x_{22}(2,4,\overline{4},\overline{1})$$

$$+x_{32}(3,4,\overline{4},\overline{1})+x_{22}^2(2,4,\overline{3},\overline{1})+x_{22}(3,4,\overline{3},\overline{1})+(3,4,\overline{2},\overline{1})]$$

$$=Y_4(x_{44})Y_3(x_{33})[x_{41}x_{42}x_{43}(1,2,3,4)+(x_{41}x_{42}+\frac{x_{31}^2x_{32}^2}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2}{x_{43}})(1,2,3,\overline{4})+(x_{31}x_{21}+x_{41}x_{32}$$

$$+\frac{x_{31}^2x_{32}}{x_{42}})x_{22}(1,2,4,\overline{4})+(\frac{x_{21}^2x_{22}^2x_{42}}{x_{32}^2}+x_{41}x_{22}^2+\frac{2x_{31}x_{21}x_{22}^2}{x_{32}}+\frac{x_{31}^2x_{22}^2}{x_{42}})x_{43}(1,2,4,\overline{3})$$

$$+(\frac{x_{31}x_{11}x_{32}}{x_{22}}+x_{31}x_{21}+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}})(1,3,4,\overline{4})+(\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11}+\frac{x_{21}^2x_{42}x_{22}}{x_{32}^2}$$

$$+x_{41}x_{22}+\frac{2x_{31}x_{21}x_{22}}{x_{32}}+\frac{x_{31}^2x_{22}}{x_{42}})x_{43}(1,3,4,\overline{3})+(\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}$$

$$+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})x_{43}(1,3,4,\overline{2})+x_{31}x_{32}(2,3,4,\overline{4})+(\frac{x_{21}x_{42}x_{22}}{x_{32}}+x_{31}x_{22})x_{43}(2,3,4,\overline{3})$$

$$+(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31})x_{43}(2,3,4,\overline{2})+x_{42}x_{43}(2,3,4,\overline{1})+(\frac{x_{21}^2x_{22}^2x_{42}}{x_{32}^2}+x_{41}x_{22}^2+\frac{2x_{31}x_{21}x_{22}^2}{x_{32}}$$

$$+\frac{x_{31}^2x_{22}^2}{x_{42}}+\frac{x_{21}^2x_{22}^2}{x_{43}})(1,2,\overline{4},\overline{3})+(\frac{x_{21}x_{11}x_{32}}{x_{43}}+\frac{x_{21}^2x_{22}}{x_{43}}+\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11}+\frac{x_{21}^2x_{42}x_{22}}{x_{32}^2}+x_{41}x_{22}$$

$$+\frac{2x_{31}x_{21}x_{22}}{x_{32}}+\frac{x_{31}^2x_{22}}{x_{42}})(1,3,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}^2}{x_{22}^2x_{43}}+\frac{x_{21}^2}{x_{43}}+\frac{2x_{21}x_{11}x_{32}}{x_{22}x_{43}}+\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}$$

$$+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})(1,3,\overline{4},\overline{2})+(\frac{x_{21}x_{32}x_{22}}{x_{43}}+\frac{x_{21}x_{42}x_{22}}{x_{32}}+x_{31}x_{22})$$

$$(2,3,\overline{4},\overline{3})+(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31}+\frac{x_{11}x_{32}^2}{x_{22}x_{43}}+\frac{x_{21}x_{32}}{x_{43}})(2,3,\overline{4},\overline{2})+(\frac{x_{32}^2}{x_{43}}+x_{42})(2,3,\overline{4},\overline{1})$$

$$+x_{21}x_{11}x_{22}(1,4,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}}{x_{22}}+x_{11}x_{21})(1,4,\overline{4},\overline{2})+x_{11}^2x_{43}(1,4,\overline{3},\overline{2})+x_{21}x_{22}^2(2,4,\overline{4},\overline{3})$$

$$+(x_{11}x_{32}+x_{21}x_{22})(2,4,\overline{4},\overline{2})+x_{21}x_{22}(3,4,\overline{4},\overline{3})+x_{21}(3,4,\overline{4},\overline{2})+x_{11}x_{22}x_{43}(2,4,\overline{3},\overline{2})$$

$$+x_{11}x_{43}(3,4,\overline{3},\overline{2})+x_{32}x_{22}(2,4,\overline{4},\overline{1})+x_{32}(3,4,\overline{4},\overline{1})+x_{22}^2x_{43}(2,4,\overline{3},\overline{1})+x_{22}x_{43}(3,4,\overline{3},\overline{1})$$

$$+x_{43}(3,4,\overline{2},\overline{1})+x_{11}^2(1,\overline{4},\overline{3},\overline{2})+x_{11}x_{22}(2,\overline{4},\overline{3},\overline{2})+x_{11}(3,\overline{4},\overline{3},\overline{2})+x_{22}^2(2,\overline{4},\overline{3},\overline{1})+x_{22}(3,\overline{4},\overline{3},\overline{1})$$
$$+(3,\overline{4},\overline{2},\overline{1})]$$
$$=Y_4(x_{44})[x_{41}x_{42}x_{43}(1,2,3,4)+(x_{41}x_{42}+\frac{x_{31}^2x_{32}^2}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2}{x_{43}})x_{33}^2(1,2,3,\overline{4})+(x_{41}x_{42}x_{33}+\frac{x_{31}^2x_{32}^2x_{33}}{x_{42}x_{43}}$$
$$+\frac{x_{41}x_{32}^2x_{33}}{x_{43}}+x_{31}x_{21}x_{22}+x_{41}x_{32}x_{22}+\frac{x_{31}^2x_{32}x_{22}}{x_{42}})(1,2,4,\overline{4})+(x_{41}x_{42}+\frac{x_{31}^2x_{32}^2}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2}{x_{43}}$$
$$+\frac{2x_{31}x_{21}x_{22}}{x_{33}}+\frac{2x_{41}x_{32}x_{22}}{x_{33}}+\frac{2x_{31}^2x_{32}x_{22}}{x_{42}x_{33}}+\frac{x_{21}^2x_{22}^2x_{42}x_{43}}{x_{32}^2x_{33}^2}+\frac{x_{41}x_{22}^2x_{43}}{x_{33}^2}+\frac{2x_{31}x_{21}x_{22}^2x_{43}}{x_{32}x_{33}^2}$$
$$+\frac{x_{31}^2x_{22}^2x_{43}}{x_{42}x_{33}^2})(1,2,4,\overline{3})+(\frac{x_{31}x_{11}x_{32}}{x_{22}}+x_{31}x_{21}+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}})x_{33}(1,3,4,\overline{4})+(\frac{x_{31}x_{11}x_{32}}{x_{22}}$$
$$+x_{31}x_{21}+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}}+\frac{x_{21}x_{11}x_{42}x_{43}}{x_{32}x_{33}}+\frac{x_{31}x_{11}x_{43}}{x_{33}}+\frac{x_{21}^2x_{42}x_{22}x_{43}}{x_{32}^2x_{33}}+\frac{x_{41}x_{22}x_{43}}{x_{33}}$$
$$+\frac{2x_{31}x_{21}x_{22}x_{43}}{x_{32}x_{33}}+\frac{x_{31}^2x_{22}x_{43}}{x_{42}x_{33}})(1,3,4,\overline{3})+(\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}$$
$$+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})x_{43}(1,3,4,\overline{2})+x_{31}x_{32}x_{33}(2,3,4,\overline{4})+(\frac{x_{21}x_{42}x_{22}x_{43}}{x_{32}x_{33}}+\frac{x_{31}x_{22}x_{43}}{x_{33}}+x_{31}x_{32})$$
$$(2,3,4,\overline{3})+(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31})x_{43}(2,3,4,\overline{2})+x_{42}x_{43}(2,3,4,\overline{1})+(\frac{x_{21}^2x_{22}^2x_{42}}{x_{32}^2}+x_{41}x_{22}^2$$
$$+\frac{2x_{31}x_{21}x_{22}^2}{x_{32}}+\frac{x_{31}^2x_{22}^2}{x_{42}}+\frac{x_{21}^2x_{22}^2}{x_{43}})(1,2,\overline{4},\overline{3})+(\frac{x_{21}x_{11}x_{32}}{x_{43}}+\frac{x_{21}^2x_{22}}{x_{43}}+\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11}$$
$$+\frac{x_{21}^2x_{42}x_{22}}{x_{32}^2}+x_{41}x_{22}+\frac{2x_{31}x_{21}x_{22}}{x_{32}}+\frac{x_{31}^2x_{22}}{x_{42}})x_{33}(1,3,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}^2}{x_{22}^2x_{43}}+\frac{x_{21}^2}{x_{43}}+\frac{2x_{21}x_{11}x_{32}}{x_{22}x_{43}}$$
$$+\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})x_{33}^2(1,3,\overline{4},\overline{2})+(\frac{x_{21}x_{32}x_{22}}{x_{43}}$$
$$+\frac{x_{21}x_{42}x_{22}}{x_{32}}+x_{31}x_{22})x_{33}(2,3,\overline{4},\overline{3})+(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31}+\frac{x_{11}x_{32}^2}{x_{22}x_{43}}+\frac{x_{21}x_{32}}{x_{43}})x_{33}^2(2,3,\overline{4},\overline{2})$$
$$+(\frac{x_{32}^2}{x_{43}}+x_{42})x_{33}^2(2,3,\overline{4},\overline{1})+(\frac{x_{21}x_{11}x_{22}}{x_{33}}+\frac{x_{21}x_{11}x_{32}}{x_{43}}+\frac{x_{21}^2x_{22}}{x_{43}}+\frac{x_{21}x_{11}x_{42}}{x_{32}}+x_{31}x_{11}+\frac{x_{21}^2x_{42}x_{22}}{x_{32}^2}$$
$$+x_{41}x_{22}+\frac{2x_{31}x_{21}x_{22}}{x_{32}}+\frac{x_{31}^2x_{22}}{x_{42}})(1,4,\overline{4},\overline{3})+(\frac{x_{11}^2x_{32}^2x_{33}}{x_{22}^2x_{43}}+\frac{x_{21}^2x_{33}}{x_{43}}+\frac{2x_{21}x_{11}x_{32}x_{33}}{x_{22}x_{43}}+\frac{x_{11}^2x_{42}x_{33}}{x_{22}^2}$$
$$+\frac{2x_{21}x_{11}x_{42}x_{33}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}x_{33}}{x_{22}}+\frac{x_{21}^2x_{42}x_{33}}{x_{32}^2}+x_{41}x_{33}+\frac{2x_{31}x_{21}x_{33}}{x_{32}}+\frac{x_{31}^2x_{33}}{x_{42}}+\frac{x_{11}^2x_{32}}{x_{22}}+x_{11}x_{21})$$
$$(1,4,\overline{4},\overline{2})+(\frac{x_{11}^2x_{32}^2}{x_{22}^2x_{43}}+\frac{x_{21}^2}{x_{43}}+\frac{2x_{21}x_{11}x_{32}}{x_{22}x_{43}}+\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}$$
$$+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}}+\frac{2x_{11}^2x_{32}}{x_{22}x_{33}}+\frac{x_{11}x_{21}}{x_{33}}+\frac{x_{11}^2x_{43}}{x_{33}^2})(1,4,\overline{3},\overline{2})+(\frac{x_{21}x_{32}x_{22}}{x_{43}}+\frac{x_{21}x_{42}x_{22}}{x_{32}}+x_{31}x_{22}$$

$$+\frac{x_{21}x_{22}^2}{x_{33}})(2,4,\overline{4},\overline{3})+(x_{11}x_{32}+x_{21}x_{22}+\frac{x_{11}x_{42}x_{33}}{x_{22}}+\frac{x_{21}x_{42}x_{33}}{x_{32}}+x_{31}x_{33}+\frac{x_{11}x_{32}^2x_{33}}{x_{22}x_{43}}$$

$$+\frac{x_{21}x_{22}^2}{x_{33}})(2,4,\overline{4},\overline{3})+(x_{11}x_{32}+x_{21}x_{22}+\frac{x_{11}x_{42}x_{33}}{x_{22}}+\frac{x_{21}x_{42}x_{33}}{x_{32}}+x_{31}x_{33}+\frac{x_{11}x_{32}^2x_{33}}{x_{22}x_{43}}$$

$$+\frac{x_{21}x_{32}x_{33}}{x_{43}})(2,4,\overline{4},\overline{2})+x_{21}x_{22}(3,4,\overline{4},\overline{3})+x_{21}(3,4,\overline{4},\overline{2})+(\frac{x_{11}x_{22}x_{43}}{x_{33}^2}+\frac{x_{11}x_{32}}{x_{33}}+\frac{2x_{21}x_{22}}{x_{33}}$$

$$+\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31}+\frac{x_{11}x_{32}^2}{x_{22}x_{43}}+\frac{x_{21}x_{32}}{x_{43}})(2,4,\overline{3},\overline{2})+x_{11}x_{43}(3,4,\overline{3},\overline{2})+x_{32}x_{22}x_{33}(2,4,\overline{4},\overline{1})$$

$$+x_{32}x_{33}(3,4,\overline{4},\overline{1})+(\frac{x_{22}^2x_{43}}{x_{33}}+x_{32}x_{22})(2,4,\overline{3},\overline{1})+(\frac{x_{22}x_{43}}{x_{33}}+x_{32})(3,4,\overline{3},\overline{1})+x_{43}(3,4,\overline{2},\overline{1})$$

$$+x_{11}^2(1,\overline{4},\overline{3},\overline{2})+x_{11}x_{22}(2,\overline{4},\overline{3},\overline{2})+x_{11}x_{33}(3,\overline{4},\overline{3},\overline{2})+x_{22}^2(2,\overline{4},\overline{3},\overline{1})+x_{22}x_{33}(3,\overline{4},\overline{3},\overline{1})$$

$$+x_{33}^2(3,\overline{4},\overline{2},\overline{1})+x_{11}(4,\overline{4},\overline{3},\overline{2})+x_{22}(4,\overline{4},\overline{3},\overline{1})+x_{33}(4,\overline{4},\overline{2},\overline{1})+(4,\overline{3},\overline{2},\overline{1})]$$

$$=[x_{41}x_{42}x_{43}x_{44}(1,2,3,4)+(x_{41}x_{42}x_{43}+\frac{x_{41}x_{42}x_{33}^2}{x_{44}}+\frac{x_{31}^2x_{32}^2x_{33}^2}{x_{42}x_{43}x_{44}}+\frac{x_{41}x_{32}^2x_{33}^2}{x_{43}x_{44}})(1,2,3,\overline{4})$$

$$+(x_{41}x_{42}x_{33}+\frac{x_{31}^2x_{32}^2x_{33}}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2x_{33}}{x_{43}}+x_{31}x_{21}x_{22}+x_{41}x_{32}x_{22}+\frac{x_{31}^2x_{32}x_{22}}{x_{42}})(1,2,4,\overline{4})$$

$$+(x_{41}x_{42}+\frac{x_{31}^2x_{32}^2}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2}{x_{43}}+\frac{2x_{31}x_{21}x_{22}}{x_{33}}+\frac{2x_{41}x_{32}x_{22}}{x_{33}}+\frac{2x_{31}^2x_{32}x_{22}}{x_{42}x_{33}}+\frac{x_{21}^2x_{22}^2x_{42}x_{43}}{x_{32}^2x_{33}^2}$$

$$+\frac{x_{41}x_{22}^2x_{43}}{x_{33}^2}+\frac{2x_{31}x_{21}x_{22}^2x_{43}}{x_{32}x_{33}^2}+\frac{x_{31}^2x_{22}^2x_{43}}{x_{42}x_{33}^2})x_{44}(1,2,4,\overline{3})+(\frac{x_{31}x_{11}x_{32}}{x_{22}}+x_{31}x_{21}$$

$$+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}})x_{33}(1,3,4,\overline{4})+(\frac{x_{31}x_{11}x_{32}}{x_{22}}+x_{31}x_{21}+x_{41}x_{32}+\frac{x_{31}^2x_{32}}{x_{42}}+\frac{x_{21}x_{11}x_{42}x_{43}}{x_{32}x_{33}}$$

$$+\frac{x_{31}x_{11}x_{43}}{x_{33}}+\frac{x_{21}^2x_{42}x_{22}x_{43}}{x_{32}^2x_{33}}+\frac{x_{41}x_{22}x_{43}}{x_{33}}+\frac{2x_{31}x_{21}x_{22}x_{43}}{x_{32}x_{33}}+\frac{x_{31}^2x_{22}x_{43}}{x_{42}x_{33}})x_{44}(1,3,4,\overline{3})$$

$$+(\frac{x_{11}^2x_{42}}{x_{22}^2}+\frac{2x_{21}x_{11}x_{42}}{x_{22}x_{32}}+\frac{2x_{31}x_{11}}{x_{22}}+\frac{x_{21}^2x_{42}}{x_{32}^2}+x_{41}+\frac{2x_{31}x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}})x_{43}x_{44}(1,3,4,\overline{2})$$

$$+x_{31}x_{32}x_{33}(2,3,4,\overline{4})+(\frac{x_{21}x_{42}x_{22}x_{43}}{x_{32}x_{33}}+\frac{x_{31}x_{22}x_{43}}{x_{33}}+x_{31}x_{32})x_{44}(2,3,4,\overline{3})+(\frac{x_{11}x_{42}x_{43}}{x_{22}}$$

$$+\frac{x_{21}x_{42}x_{43}}{x_{32}}+x_{31}x_{43})x_{44}(2,3,4,\overline{2})+x_{42}x_{43}x_{44}(2,3,4,\overline{1})+(\frac{x_{21}^2x_{22}^2x_{42}}{x_{32}^2x_{44}}+\frac{x_{41}x_{22}^2}{x_{44}}+\frac{2x_{31}x_{21}x_{22}^2}{x_{32}x_{44}}$$

$$+\frac{x_{31}^2x_{22}^2}{x_{42}x_{44}}+\frac{x_{21}^2x_{22}^2}{x_{43}x_{44}}+x_{41}x_{42}+\frac{x_{31}^2x_{32}^2}{x_{42}x_{43}}+\frac{x_{41}x_{32}^2}{x_{43}}+\frac{2x_{31}x_{21}x_{22}}{x_{33}}+\frac{2x_{41}x_{32}x_{22}}{x_{33}}+\frac{2x_{31}^2x_{32}x_{22}}{x_{42}x_{33}}$$

$$+\frac{x_{21}^2x_{22}^2x_{42}x_{43}}{x_{32}^2x_{33}^2}+\frac{x_{41}x_{22}^2x_{43}}{x_{33}^2}+\frac{2x_{31}x_{21}x_{22}^2x_{43}}{x_{32}x_{33}^2}+\frac{x_{31}^2x_{22}^2x_{43}}{x_{42}x_{33}^2})(1,2,\overline{4},\overline{3})+(\frac{x_{21}x_{11}x_{32}x_{33}}{x_{43}x_{44}}$$

$$+\frac{x_{21}^2x_{22}x_{33}}{x_{43}x_{44}}+\frac{x_{21}x_{11}x_{42}x_{33}}{x_{32}x_{44}}+\frac{x_{31}x_{11}x_{33}}{x_{44}}+\frac{x_{21}^2x_{42}x_{22}x_{33}}{x_{32}^2x_{44}}+\frac{x_{41}x_{22}x_{33}}{x_{44}}+\frac{2x_{31}x_{21}x_{22}x_{33}}{x_{32}x_{44}}$$

$$+\frac{x_{31}^2 x_{22} x_{33}}{x_{42} x_{44}}+\frac{x_{31} x_{11} x_{32}}{x_{22}}+x_{31} x_{21}+x_{41} x_{32}+\frac{x_{31}^2 x_{32}}{x_{42}}+\frac{x_{21} x_{11} x_{42} x_{43}}{x_{32} x_{33}}+\frac{x_{31} x_{11} x_{43}}{x_{33}}$$

$$+\frac{x_{21}^2 x_{42} x_{22} x_{43}}{x_{32}^2 x_{33}}+\frac{x_{41} x_{22} x_{43}}{x_{33}}+\frac{2 x_{31} x_{21} x_{22} x_{43}}{x_{32} x_{33}}+\frac{x_{31}^2 x_{22} x_{43}}{x_{42} x_{33}})(1,3,\overline{4},\overline{3})+(\frac{x_{11}^2 x_{42} x_{43}}{x_{22}^2}$$

$$+\frac{2 x_{21} x_{11} x_{42} x_{43}}{x_{22} x_{32}}+\frac{2 x_{31} x_{11} x_{43}}{x_{22}}+\frac{x_{21}^2 x_{42} x_{43}}{x_{32}^2}+x_{41} x_{43}+\frac{2 x_{31} x_{21} x_{43}}{x_{32}}+\frac{x_{31}^2 x_{43}}{x_{42}}+\frac{x_{11}^2 x_{32}^2 x_{33}^2}{x_{22}^2 x_{43} x_{44}}$$

$$+\frac{x_{21}^2 x_{33}^2}{x_{43} x_{44}}+\frac{2 x_{21} x_{11} x_{32} x_{33}^2}{x_{22} x_{43} x_{44}}+\frac{x_{11}^2 x_{42} x_{33}^2}{x_{22}^2 x_{44}}+\frac{2 x_{21} x_{11} x_{42} x_{33}^2}{x_{22} x_{32} x_{44}}+\frac{2 x_{31} x_{11} x_{33}^2}{x_{22} x_{44}}+\frac{x_{21}^2 x_{42} x_{33}^2}{x_{32}^2 x_{44}}+\frac{x_{41} x_{33}^2}{x_{44}}$$

$$+\frac{2 x_{31} x_{21} x_{33}^2}{x_{32} x_{44}}+\frac{x_{31}^2 x_{33}^2}{x_{42} x_{44}})(1,3,\overline{4},\overline{2})+(\frac{x_{21} x_{42} x_{22} x_{43}}{x_{32} x_{33}}+\frac{x_{31} x_{22} x_{43}}{x_{33}}+x_{31} x_{32}$$

$$+\frac{x_{21} x_{32} x_{22} x_{33}}{x_{43} x_{44}}+\frac{x_{21} x_{42} x_{22} x_{33}}{x_{32} x_{44}}+\frac{x_{31} x_{22} x_{33}}{x_{44}})(2,3,\overline{4},\overline{3})+(\frac{x_{11} x_{42} x_{43}}{x_{22}}+\frac{x_{21} x_{42} x_{43}}{x_{32}}+x_{31} x_{43}$$

$$+\frac{x_{11} x_{42} x_{33}^2}{x_{22} x_{44}}+\frac{x_{21} x_{42} x_{33}^2}{x_{32} x_{44}}+\frac{x_{31} x_{33}^2}{x_{44}}+\frac{x_{11} x_{32}^2 x_{33}^2}{x_{22} x_{43} x_{44}}+\frac{x_{21} x_{32} x_{33}^2}{x_{43} x_{44}})(2,3,\overline{4},\overline{2})+(\frac{x_{32}^2 x_{33}^2}{x_{43} x_{44}}+\frac{x_{42} x_{33}^2}{x_{44}}$$

$$+x_{42} x_{43})(2,3,\overline{4},\overline{1})+(\frac{x_{21} x_{11} x_{22}}{x_{33}}+\frac{x_{21} x_{11} x_{32}}{x_{43}}+\frac{x_{21}^2 x_{22}}{x_{43}}+\frac{x_{21} x_{11} x_{42}}{x_{32}}+x_{31} x_{11}+\frac{x_{21}^2 x_{42} x_{22}}{x_{32}^2}$$

$$+x_{41} x_{22}+\frac{2 x_{31} x_{21} x_{22}}{x_{32}}+\frac{x_{31}^2 x_{22}}{x_{42}})(1,4,\overline{4},\overline{3})+(\frac{x_{11}^2 x_{32}^2 x_{33}}{x_{22}^2 x_{43}}+\frac{x_{21}^2 x_{33}}{x_{43}}+\frac{2 x_{21} x_{11} x_{32} x_{33}}{x_{22} x_{43}}+\frac{x_{11}^2 x_{42} x_{33}}{x_{22}^2}$$

$$+\frac{2 x_{21} x_{11} x_{42} x_{33}}{x_{22} x_{32}}+\frac{2 x_{31} x_{11} x_{33}}{x_{22}}+\frac{x_{21}^2 x_{42} x_{33}}{x_{32}^2}+x_{41} x_{33}+\frac{2 x_{31} x_{21} x_{33}}{x_{32}}+\frac{x_{31}^2 x_{33}}{x_{42}}+\frac{x_{11}^2 x_{32}}{x_{22}}+x_{11} x_{21})$$

$$(1,4,\overline{4},\overline{2})+(\frac{x_{11}^2 x_{32}^2}{x_{22}^2 x_{43}}+\frac{x_{21}^2}{x_{43}}+\frac{2 x_{21} x_{11} x_{32}}{x_{22} x_{43}}+\frac{x_{11}^2 x_{42}}{x_{22}^2}+\frac{2 x_{21} x_{11} x_{42}}{x_{22} x_{32}}+\frac{2 x_{31} x_{11}}{x_{22}}+\frac{x_{21}^2 x_{42}}{x_{32}^2}+x_{41}$$

$$+\frac{2 x_{31} x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}}+\frac{2 x_{11}^2 x_{32}}{x_{22} x_{33}}+\frac{x_{11} x_{21}}{x_{33}}+\frac{x_{11}^2 x_{43}}{x_{33}^2})x_{44}(1,4,\overline{3},\overline{2})+(\frac{x_{21} x_{32} x_{22}}{x_{43}}+\frac{x_{21} x_{42} x_{22}}{x_{32}}+x_{31} x_{22}$$

$$+\frac{x_{21} x_{22}^2}{x_{33}})(2,4,\overline{4},\overline{3})+(x_{11} x_{32}+x_{21} x_{22}+\frac{x_{11} x_{42} x_{33}}{x_{22}}+\frac{x_{21} x_{42} x_{33}}{x_{32}}+x_{31} x_{33}+\frac{x_{11} x_{32}^2 x_{33}}{x_{22} x_{43}}$$

$$+\frac{x_{21} x_{32} x_{33}}{x_{43}})(2,4,\overline{4},\overline{2})+x_{21} x_{22}(3,4,\overline{4},\overline{3})+x_{21}(3,4,\overline{4},\overline{2})+(\frac{x_{11} x_{22} x_{43}}{x_{33}^2}+\frac{x_{11} x_{32}}{x_{33}}+\frac{2 x_{21} x_{22}}{x_{33}}$$

$$+\frac{x_{11} x_{42}}{x_{22}}+\frac{x_{21} x_{42}}{x_{32}}+x_{31}+\frac{x_{11} x_{32}^2}{x_{22} x_{43}}+\frac{x_{21} x_{32}}{x_{43}})x_{44}(2,4,\overline{3},\overline{2})+x_{11} x_{43} x_{44}(3,4,\overline{3},\overline{2})+x_{32} x_{22} x_{33}$$

$$(2,4,\overline{4},\overline{1})+x_{32} x_{33}(3,4,\overline{4},\overline{1})+(\frac{x_{22}^2 x_{43}}{x_{33}}+x_{32} x_{22})x_{44}(2,4,\overline{3},\overline{1})+(\frac{x_{22} x_{43}}{x_{33}}+x_{32})x_{44}(3,4,\overline{3},\overline{1})$$

$$+x_{43} x_{44}(3,4,\overline{2},\overline{1})+(\frac{x_{11}^2}{x_{44}}+\frac{x_{11}^2 x_{32}^2}{x_{22}^2 x_{43}}+\frac{x_{21}^2}{x_{43}}+\frac{2 x_{21} x_{11} x_{32}}{x_{22} x_{43}}+\frac{x_{11}^2 x_{42}}{x_{22}^2}+\frac{2 x_{21} x_{11} x_{42}}{x_{22} x_{32}}+\frac{2 x_{31} x_{11}}{x_{22}}$$

$$+\frac{x_{21}^2 x_{42}}{x_{32}^2}+x_{41}+\frac{2 x_{31} x_{21}}{x_{32}}+\frac{x_{31}^2}{x_{42}}+\frac{2 x_{11}^2 x_{32}}{x_{22} x_{33}}+\frac{x_{11} x_{21}}{x_{33}}+\frac{x_{11}^2 x_{43}}{x_{33}^2})(1,\overline{4},\overline{3},\overline{2})+(\frac{x_{11} x_{22}}{x_{44}}+\frac{x_{11} x_{22} x_{43}}{x_{33}^2}$$

$$+\frac{x_{11} x_{32}}{x_{33}}+\frac{2 x_{21} x_{22}}{x_{33}}+\frac{x_{11} x_{42}}{x_{22}}+\frac{x_{21} x_{42}}{x_{32}}+x_{31}+\frac{x_{11} x_{32}^2}{x_{22} x_{43}}+\frac{x_{21} x_{32}}{x_{43}})(2,\overline{4},\overline{3},\overline{2})$$

$$+\frac{x_{11}x_{32}}{x_{33}}+\frac{2x_{21}x_{22}}{x_{33}}+\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31}+\frac{x_{11}x_{32}^2}{x_{22}x_{43}}+\frac{x_{21}x_{32}}{x_{43}})(2,\overline{4},\overline{3},\overline{2})+(\frac{x_{11}x_{33}}{x_{44}}+x_{11}x_{43})$$

$$(3,\overline{4},\overline{3},\overline{2})+(\frac{x_{22}^2}{x_{44}}+\frac{x_{22}^2x_{43}}{x_{33}}+x_{32}x_{22})(2,\overline{4},\overline{3},\overline{1})+(\frac{x_{22}x_{33}}{x_{44}}+\frac{x_{22}x_{43}}{x_{33}}+x_{32})(3,\overline{4},\overline{3},\overline{1})$$

$$+(\frac{x_{33}^2}{x_{44}}+x_{43})(3,\overline{4},\overline{2},\overline{1})+x_{11}(4,\overline{4},\overline{3},\overline{2})+x_{22}(4,\overline{4},\overline{3},\overline{1})+x_{33}(4,\overline{4},\overline{2},\overline{1})+x_{44}(4,\overline{3},\overline{2},\overline{1})+(\overline{4},\overline{3},\overline{2},\overline{1})]$$

Now we compute $V_2(y)$:

$$Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})Y_0(y_{02})Y_3(y_{31})Y_2(y_{21})Y_1(y_{11})Y_0(y_{01})(\overline{4},\overline{3},\overline{2},\overline{1})$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})Y_0(y_{02})Y_3(y_{31})Y_2(y_{21})Y_1(y_{11})[y_{01}(\overline{4},\overline{3},\overline{2},\overline{1})+(1,\overline{4},\overline{3},\overline{2})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})Y_0(y_{02})Y_3(y_{31})Y_2(y_{21})[y_{01}(\overline{4},\overline{3},\overline{2},\overline{1})+y_{11}^2(1,\overline{4},\overline{3},\overline{2})$$
$$+y_{11}(2,\overline{4},\overline{3},\overline{2})+(2,\overline{4},\overline{3},\overline{1})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})Y_0(y_{02})Y_3(y_{31})[y_{01}(\overline{4},\overline{3},\overline{2},\overline{1})+y_{11}^2(1,\overline{4},\overline{3},\overline{2})+y_{11}y_{21}(2,\overline{4},\overline{3},\overline{2})$$
$$+y_{21}^2(2,\overline{4},\overline{3},\overline{1})+y_{11}(3,\overline{4},\overline{3},\overline{2})+y_{21}(3,\overline{4},\overline{3},\overline{1})+(3,\overline{4},\overline{2},\overline{1})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})Y_0(y_{02})[y_{01}(\overline{4},\overline{3},\overline{2},\overline{1})+y_{11}^2(1,\overline{4},\overline{3},\overline{2})+y_{11}y_{21}(2,\overline{4},\overline{3},\overline{2})$$
$$+y_{21}^2(2,\overline{4},\overline{3},\overline{1})+y_{11}y_{31}(3,\overline{4},\overline{3},\overline{2})+y_{21}y_{31}(3,\overline{4},\overline{3},\overline{1})+y_{31}^2(3,\overline{4},\overline{2},\overline{1})+y_{11}(4,\overline{4},\overline{3},\overline{2})+y_{21}(4,\overline{4},\overline{3},\overline{1})$$
$$+y_{31}(4,\overline{4},\overline{2},\overline{1})+(4,\overline{3},\overline{2},\overline{1})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})Y_1(y_{12})[y_{01}y_{02}(\overline{4},\overline{3},\overline{2},\overline{1})+(\frac{y_{11}^2}{y_{02}}+y_{01})(1,\overline{4},\overline{3},\overline{2})+y_{11}y_{21}(2,\overline{4},\overline{3},\overline{2})$$
$$+y_{21}^2y_{02}(2,\overline{4},\overline{3},\overline{1})+y_{11}y_{31}(3,\overline{4},\overline{3},\overline{2})+y_{21}y_{31}y_{02}(3,\overline{4},\overline{3},\overline{1})+y_{31}^2y_{02}(3,\overline{4},\overline{2},\overline{1})+y_{11}(4,\overline{4},\overline{3},\overline{2})$$
$$+y_{21}y_{02}(4,\overline{4},\overline{3},\overline{1})+y_{31}y_{02}(4,\overline{4},\overline{2},\overline{1})+y_{02}(4,\overline{3},\overline{2},\overline{1})+y_{21}^2(1,2,\overline{4},\overline{3})+y_{21}y_{31}(1,3,\overline{4},\overline{3})$$
$$+y_{31}^2(1,3,\overline{4},\overline{2})+y_{21}(1,4,\overline{4},\overline{3})+y_{31}(1,4,\overline{4},\overline{2})+(1,4,\overline{3},\overline{2})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})Y_2(y_{22})[y_{01}y_{02}(\overline{4},\overline{3},\overline{2},\overline{1})+(\frac{y_{11}^2}{y_{02}}+y_{01})y_{12}^2(1,\overline{4},\overline{3},\overline{2})+(\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}$$
$$+y_{11}y_{21})(2,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}})(2,\overline{4},\overline{3},\overline{1})+y_{11}y_{31}y_{12}(3,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}y_{31}y_{02}}{y_{12}}$$
$$+y_{11}y_{31})(3,\overline{4},\overline{3},\overline{1})+y_{31}^2y_{02}(3,\overline{4},\overline{2},\overline{1})+y_{11}y_{12}(4,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}y_{02}}{y_{12}}+y_{11})(4,\overline{4},\overline{3},\overline{1})$$
$$+y_{31}y_{02}(4,\overline{4},\overline{2},\overline{1})+y_{02}(4,\overline{3},\overline{2},\overline{1})+y_{21}^2(1,2,\overline{4},\overline{3})+y_{21}y_{31}y_{12}(1,3,\overline{4},\overline{3})+y_{31}^2y_{12}^2(1,3,\overline{4},\overline{2})$$
$$+y_{21}y_{12}(1,4,\overline{4},\overline{3})+y_{31}y_{12}^2(1,4,\overline{4},\overline{2})+y_{12}^2(1,4,\overline{3},\overline{2})+y_{21}y_{31}(2,3,\overline{4},\overline{3})+y_{31}^2y_{12}(2,3,\overline{4},\overline{2})$$
$$+y_{31}^2(2,3,\overline{4},\overline{1})+y_{21}(2,4,\overline{4},\overline{3})+y_{31}y_{12}(2,4,\overline{4},\overline{2})+y_{31}(2,4,\overline{4},\overline{1})+y_{12}(2,4,\overline{3},\overline{2})+(2,4,\overline{3},\overline{1})]$$
$$= Y_0(y_{04})Y_1(y_{13})Y_0(y_{03})[y_{01}y_{02}(\overline{4},\overline{3},\overline{2},\overline{1})+(\frac{y_{11}^2}{y_{02}}+y_{01})y_{12}^2(1,\overline{4},\overline{3},\overline{2})+(\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}+y_{11}y_{21})$$

$$y_{22}(2,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}})y_{22}^2(2,\overline{4},\overline{3},\overline{1})+(\frac{y_{11}y_{31}y_{12}}{y_{22}}+\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}$$

$$+y_{11}y_{21})(3,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2}+\frac{y_{11}^2y_{22}}{y_{02}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31})(3,\overline{4},\overline{3},\overline{1})$$

$$+(\frac{y_{31}^2y_{02}}{y_{22}^2}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}})(3,\overline{4},\overline{2},\overline{1})+y_{11}y_{12}(4,\overline{4},\overline{3},\overline{2})$$

$$+(\frac{y_{21}y_{02}}{y_{12}}+y_{11})y_{22}(4,\overline{4},\overline{3},\overline{1})+(\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11})(4,\overline{4},\overline{2},\overline{1})+y_{02}(4,\overline{3},\overline{2},\overline{1})+y_{21}^2y_{22}^2(1,2,\overline{4},\overline{3})$$

$$+(y_{21}y_{31}y_{12}+y_{21}^2y_{22})(1,3,\overline{4},\overline{3})+(\frac{y_{31}^2y_{12}^2}{y_{22}^2}+\frac{2y_{21}y_{31}y_{12}}{y_{22}}+y_{21}^2)(1,3,\overline{4},\overline{2})+y_{21}y_{12}y_{22}(1,4,\overline{4},\overline{3})$$

$$+(\frac{y_{31}y_{12}^2}{y_{22}}+y_{21}y_{12})(1,4,\overline{4},\overline{2})+y_{12}^2(1,4,\overline{3},\overline{2})+y_{21}y_{31}y_{22}(2,3,\overline{4},\overline{3})+(\frac{y_{31}^2y_{12}}{y_{22}}+y_{21}y_{31})(2,3,\overline{4},\overline{2})$$

$$+y_{31}^2(2,3,\overline{4},\overline{1})+y_{21}y_{22}^2(2,4,\overline{4},\overline{3})+(y_{31}y_{12}+y_{21}y_{22})(2,4,\overline{4},\overline{2})+y_{21}y_{22}(3,4,\overline{4},\overline{3})+y_{21}(3,4,\overline{4},\overline{2})$$

$$+y_{31}y_{22}(2,4,\overline{4},\overline{1})+y_{31}(3,4,\overline{4},\overline{1})+y_{12}y_{22}(2,4,\overline{3},\overline{2})+y_{12}(3,4,\overline{3},\overline{2})+y_{22}^2(2,4,\overline{3},\overline{1})+y_{22}(3,4,\overline{3},\overline{1})$$

$$+(3,4,\overline{2},\overline{1})]$$

$$=Y_0(y_{04})Y_1(y_{13})[y_{01}y_{02}y_{03}(\overline{4},\overline{3},\overline{2},\overline{1})+(y_{01}y_{02}+\frac{y_{11}^2y_{12}^2}{y_{02}y_{03}}+\frac{y_{01}y_{12}^2}{y_{03}})(1,\overline{4},\overline{3},\overline{2})+(\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}$$

$$+y_{11}y_{21})y_{22}(2,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}^2y_{02}y_{22}^2}{y_{12}^2}+\frac{y_{11}^2y_{22}^2}{y_{02}}+y_{01}y_{22}^2+\frac{2y_{11}y_{21}y_{22}^2}{y_{12}})y_{03}(2,\overline{4},\overline{3},\overline{1})+(\frac{y_{11}y_{31}y_{12}}{y_{22}}$$

$$+\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}+y_{11}y_{21})(3,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2}+\frac{y_{11}^2y_{22}}{y_{02}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}$$

$$+y_{11}y_{31})y_{03}(3,\overline{4},\overline{3},\overline{1})+(\frac{y_{31}^2y_{02}}{y_{22}^2}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}})y_{03}$$

$$(3,\overline{4},\overline{2},\overline{1})+y_{11}y_{12}(4,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}y_{02}y_{22}}{y_{12}}+y_{11}y_{22})y_{03}(4,\overline{4},\overline{3},\overline{1})+(\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11})y_{03}$$

$$(4,\overline{4},\overline{2},\overline{1})+y_{02}y_{03}(4,\overline{3},\overline{2},\overline{1})+(\frac{y_{21}^2y_{22}^2}{y_{03}}+\frac{y_{21}^2y_{02}y_{22}^2}{y_{12}^2}+\frac{y_{11}^2y_{22}^2}{y_{02}}+y_{01}y_{22}^2+\frac{2y_{11}y_{21}y_{22}^2}{y_{12}})(1,2,\overline{4},\overline{3})$$

$$+(\frac{y_{21}y_{31}y_{12}}{y_{03}}+\frac{y_{21}^2y_{22}}{y_{03}}+\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2y_{03}}+\frac{y_{11}^2y_{22}}{y_{02}y_{03}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31})$$

$$(1,3,\overline{4},\overline{3})+(\frac{y_{31}^2y_{02}}{y_{22}^2}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}}+\frac{y_{31}^2y_{12}^2}{y_{22}^2y_{03}}$$

$$+\frac{2y_{21}y_{31}y_{12}}{y_{22}y_{03}}+\frac{y_{21}^2}{y_{03}})(1,3,\overline{4},\overline{2})+(\frac{y_{21}y_{12}y_{22}}{y_{03}}+\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2}+\frac{y_{11}^2y_{22}}{y_{02}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}$$

$$+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31})(1,4,\overline{4},\overline{3})+(\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11}+\frac{y_{31}y_{12}^2}{y_{22}y_{03}}+\frac{y_{21}y_{12}}{y_{03}})(1,4,\overline{4},\overline{2})+(y_{02}$$

$$+ \frac{y_{12}^2}{y_{03}})(1,4,\bar{3},\bar{2}) + y_{21}y_{31}y_{22}(2,3,\bar{4},\bar{3}) + (\frac{y_{31}^2 y_{12}}{y_{22}} + y_{21}y_{31})(2,3,\bar{4},\bar{2}) + y_{31}^2 y_{03}(2,3,\bar{4},\bar{1})$$

$$+ y_{21}y_{22}^2(2,4,\bar{4},\bar{3}) + (y_{31}y_{12}+y_{21}y_{22})(2,4,\bar{4},\bar{2}) + y_{21}y_{22}(3,4,\bar{4},\bar{3}) + y_{21}(3,4,\bar{4},\bar{2})$$

$$+ y_{31}y_{22}y_{03}(2,4,\bar{4},\bar{1}) + y_{31}y_{03}(3,4,\bar{4},\bar{1}) + y_{12}y_{22}(2,4,\bar{3},\bar{2}) + y_{12}(3,4,\bar{3},\bar{2}) + y_{22}^2 y_{03}(2,4,\bar{3},\bar{1})$$

$$+ y_{22}y_{03}(3,4,\bar{3},\bar{1}) + y_{03}(3,4,\bar{2},\bar{1}) + y_{31}^2(1,2,3,\bar{4}) + y_{31}y_{22}(1,2,4,\bar{4}) + y_{31}(1,3,4,\bar{4})$$

$$+ y_{22}^2(1,2,4,\bar{3}) + y_{22}(1,3,4,\bar{3}) + (1,3,4,\bar{2})]$$

$$= Y_0(y_{04})[y_{01}y_{02}y_{03}(\bar{4},\bar{3},\bar{2},\bar{1}) + (y_{01}y_{02} + \frac{y_{11}^2 y_{12}^2}{y_{02}y_{03}} + \frac{y_{01}y_{12}^2}{y_{03}})y_{13}^2(1,\bar{4},\bar{3},\bar{2}) + (y_{01}y_{02}y_{13} + \frac{y_{11}^2y_{12}^2y_{13}}{y_{02}y_{03}}$$

$$+ \frac{y_{01}y_{12}^2y_{13}}{y_{03}} + \frac{y_{11}^2y_{12}y_{22}}{y_{02}} + y_{01}y_{12}y_{22} + y_{11}y_{21}y_{22})(2,\bar{4},\bar{3},\bar{2}) + (y_{01}y_{02} + \frac{y_{11}^2y_{12}^2}{y_{02}y_{03}} + \frac{y_{01}y_{12}^2}{y_{03}}$$

$$+ \frac{2y_{11}^2y_{12}y_{22}}{y_{02}y_{13}} + \frac{2y_{01}y_{12}y_{22}}{y_{13}} + \frac{2y_{11}y_{21}y_{22}}{y_{13}} + \frac{y_{21}^2y_{02}y_{22}^2y_{03}}{y_{12}^2y_{13}^2} + \frac{y_{11}^2y_{22}^2y_{03}}{y_{02}y_{13}^2} + \frac{y_{01}y_{22}^2y_{03}}{y_{13}^2}$$

$$+ \frac{2y_{11}y_{21}y_{22}^2y_{03}}{y_{12}y_{13}^2})(2,\bar{4},\bar{3},\bar{1}) + (\frac{y_{11}y_{31}y_{12}}{y_{22}} + \frac{y_{11}^2y_{12}}{y_{02}} + y_{01}y_{12} + y_{11}y_{21})y_{13}(3,\bar{4},\bar{3},\bar{2}) + (\frac{y_{21}^2y_{02}y_{22}y_{03}}{y_{12}^2y_{13}}$$

$$+ \frac{y_{11}^2y_{22}y_{03}}{y_{02}y_{13}} + \frac{y_{01}y_{22}y_{03}}{y_{13}} + \frac{2y_{11}y_{21}y_{22}y_{03}}{y_{12}y_{13}} + \frac{y_{21}y_{31}y_{02}y_{03}}{y_{12}y_{13}} + \frac{y_{11}y_{31}y_{03}}{y_{13}} + \frac{y_{11}y_{31}y_{12}}{y_{22}} + \frac{y_{11}^2y_{12}}{y_{02}}$$

$$+ y_{01}y_{12} + y_{11}y_{21})(3,\bar{4},\bar{3},\bar{1}) + (\frac{y_{31}^2y_{02}}{y_{22}^2} + \frac{y_{21}^2y_{02}}{y_{12}^2} + \frac{y_{11}^2}{y_{02}} + y_{01} + \frac{2y_{11}y_{21}}{y_{12}} + \frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}} + \frac{2y_{11}y_{31}}{y_{22}})$$

$$y_{03}(3,\bar{4},\bar{2},\bar{1}) + y_{11}y_{12}y_{13}(4,\bar{4},\bar{3},\bar{2}) + (\frac{y_{21}y_{02}y_{22}y_{03}}{y_{12}y_{13}} + \frac{y_{11}y_{22}y_{03}}{y_{13}} + y_{11}y_{12})(4,\bar{4},\bar{3},\bar{1}) + (\frac{y_{31}y_{02}}{y_{22}}$$

$$+ \frac{y_{21}y_{02}}{y_{12}} + y_{11})y_{03}(4,\bar{4},\bar{2},\bar{1}) + y_{02}y_{03}(4,\bar{3},\bar{2},\bar{1}) + (\frac{y_{21}^2y_{22}^2}{y_{03}} + \frac{y_{21}^2y_{02}y_{22}^2}{y_{12}^2} + \frac{y_{11}^2y_{22}^2}{y_{02}} + y_{01}y_{22}^2$$

$$+ \frac{2y_{11}y_{21}y_{22}^2}{y_{12}})(1,2,\bar{4},\bar{3}) + (\frac{y_{21}y_{31}y_{12}}{y_{03}} + \frac{y_{21}^2y_{22}}{y_{03}} + \frac{y_{21}^2y_{02}y_{22}}{y_{12}^2y_{03}} + \frac{y_{11}^2y_{22}}{y_{02}y_{03}} + y_{01}y_{22} + \frac{2y_{11}y_{21}y_{22}}{y_{12}}$$

$$+ \frac{y_{21}y_{31}y_{02}}{y_{12}} + y_{11}y_{31})y_{13}(1,3,\bar{4},\bar{3}) + (\frac{y_{31}^2y_{02}}{y_{22}^2} + \frac{y_{21}^2y_{02}}{y_{12}^2} + \frac{y_{11}^2}{y_{02}} + y_{01} + \frac{2y_{11}y_{21}}{y_{12}} + \frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}$$

$$+ \frac{2y_{11}y_{31}}{y_{22}} + \frac{y_{31}^2y_{12}^2}{y_{22}^2y_{03}} + \frac{2y_{21}y_{31}y_{12}}{y_{22}y_{03}} + \frac{y_{21}^2}{y_{03}})y_{13}^2(1,3,\bar{4},\bar{2}) + (\frac{y_{21}y_{12}y_{22}}{y_{03}} + \frac{y_{21}^2y_{02}y_{22}}{y_{12}^2} + \frac{y_{11}^2y_{22}}{y_{02}} + y_{01}y_{22}$$

$$+ \frac{2y_{11}y_{21}y_{22}}{y_{12}} + \frac{y_{21}y_{31}y_{02}}{y_{12}} + y_{11}y_{31})y_{13}(1,4,\bar{4},\bar{3}) + (\frac{y_{31}y_{02}}{y_{22}} + \frac{y_{21}y_{02}}{y_{12}} + y_{11} + \frac{y_{31}y_{12}^2}{y_{22}y_{03}} + \frac{y_{21}y_{12}}{y_{03}})y_{13}^2$$

$$(1,4,\bar{4},\bar{2}) + (y_{02} + \frac{y_{12}^2}{y_{03}})y_{13}^2(1,4,\bar{3},\bar{2}) + (\frac{y_{21}y_{31}y_{22}}{y_{13}} + \frac{y_{21}y_{31}y_{12}}{y_{03}} + \frac{y_{21}^2y_{22}}{y_{03}} + \frac{y_{21}^2y_{02}y_{22}}{y_{12}^2y_{03}} + \frac{y_{11}^2y_{22}}{y_{02}y_{03}}$$

$$+ y_{01}y_{22} + \frac{2y_{11}y_{21}y_{22}}{y_{12}} + \frac{y_{21}y_{31}y_{02}}{y_{12}} + y_{11}y_{31})(2,3,\bar{4},\bar{3}) + (\frac{y_{31}^2y_{12}}{y_{22}} + y_{21}y_{31} + \frac{y_{31}^2y_{02}y_{13}}{y_{22}^2} + \frac{y_{21}^2y_{02}y_{13}}{y_{12}^2}$$

$$+ \frac{y_{11}^2y_{13}}{y_{02}} + y_{01}y_{13} + \frac{2y_{11}y_{21}y_{13}}{y_{12}} + \frac{2y_{21}y_{31}y_{02}y_{13}}{y_{12}y_{22}} + \frac{2y_{11}y_{31}y_{13}}{y_{22}} + \frac{y_{31}^2y_{12}^2y_{13}}{y_{22}^2y_{03}} + \frac{2y_{21}y_{31}y_{12}y_{13}}{y_{22}y_{03}}$$

$$+\frac{y_{21}^2y_{13}}{y_{03}})(2,3,\overline{4},\overline{2})+(\frac{y_{31}^2y_{03}}{y_{13}^2}+\frac{2y_{31}^2y_{12}}{y_{22}y_{13}}+\frac{2y_{21}y_{31}}{y_{13}}+\frac{y_{31}^2y_{02}}{y_{22}^2}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}$$

$$+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}}+\frac{y_{31}^2y_{12}^2}{y_{22}^2y_{03}}+\frac{2y_{21}y_{31}y_{12}}{y_{22}y_{03}}+\frac{y_{21}^2}{y_{03}})(2,3,\overline{4},\overline{1})+(\frac{y_{21}y_{12}y_{22}}{y_{03}}+\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2}+\frac{y_{11}^2y_{22}}{y_{02}}$$

$$+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31}+\frac{y_{21}y_{22}^2}{y_{13}})(2,4,\overline{4},\overline{3})+(\frac{y_{31}y_{02}y_{13}}{y_{22}}+\frac{y_{21}y_{02}y_{13}}{y_{12}}+y_{11}y_{13}$$

$$+\frac{y_{31}y_{12}^2y_{13}}{y_{22}y_{03}}+\frac{y_{21}y_{12}y_{13}}{y_{03}}+y_{31}y_{12}+y_{21}y_{22})(2,4,\overline{4},\overline{2})+y_{21}y_{22}(3,4,\overline{4},\overline{3})+y_{21}y_{13}(3,4,\overline{4},\overline{2})$$

$$+(\frac{y_{31}y_{22}y_{03}}{y_{13}^2}+\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11}+\frac{y_{31}y_{12}^2y}{y_{22}y_{03}}+\frac{y_{21}y_{12}}{y_{03}}+\frac{2y_{31}y_{12}}{y_{13}}+\frac{2y_{21}y_{22}}{y_{13}})(2,4,\overline{4},\overline{1})$$

$$+(\frac{y_{31}y_{03}}{y_{13}}+y_{21})(3,4,\overline{4},\overline{1})+(y_{12}y_{22}+y_{02}y_{13}+\frac{y_{12}^2y_{13}}{y_{03}})(2,4,\overline{3},\overline{2})+y_{12}y_{13}(3,4,\overline{3},\overline{2})+(\frac{y_{22}^2y_{03}}{y_{13}^2}$$

$$+\frac{2y_{12}y_{22}}{y_{13}}+y_{02}+\frac{y_{12}^2}{y_{03}})(2,4,\overline{3},\overline{1})+(\frac{y_{22}y_{03}}{y_{13}}+y_{12})(3,4,\overline{3},\overline{1})+y_{03}(3,4,\overline{2},\overline{1})+y_{31}^2(1,2,3,\overline{4})+y_{31}y_{22}$$

$$(1,2,4,\overline{4})+y_{31}y_{13}(1,3,4,\overline{4})+y_{31}(2,3,4,\overline{4})+y_{22}^2(1,2,4,\overline{3})+y_{22}y_{13}(1,3,4,\overline{3})+y_{22}(2,3,4,\overline{3})$$

$$+y_{13}^2(1,3,4,\overline{2})+y_{13}(2,3,4,\overline{2})+(2,3,4,\overline{1})]$$

$$=[y_{01}y_{02}y_{03}y_{04}(\overline{4},\overline{3},\overline{2},\overline{1})+(y_{01}y_{02}y_{03}+\frac{y_{01}y_{02}y_{13}^2}{y_{04}}+\frac{y_{11}^2y_{12}^2y_{13}^2}{y_{02}y_{03}y_{04}}+\frac{y_{01}y_{12}^2y_{13}^2}{y_{03}y_{04}})(1,\overline{4},\overline{3},\overline{2})+(y_{01}y_{02}y_{13}$$

$$+\frac{y_{11}^2y_{12}^2y_{13}}{y_{02}y_{03}}+\frac{y_{01}y_{12}^2y_{13}}{y_{03}}+\frac{y_{11}^2y_{12}y_{22}}{y_{02}}+y_{01}y_{12}y_{22}+y_{11}y_{21}y_{22})(2,\overline{4},\overline{3},\overline{2})+(y_{01}y_{02}+\frac{y_{11}^2y_{12}^2}{y_{02}y_{03}}$$

$$+\frac{y_{01}y_{12}^2}{y_{03}}+\frac{2y_{11}^2y_{12}y_{22}}{y_{02}y_{13}}+\frac{2y_{01}y_{12}y_{22}}{y_{13}}+\frac{2y_{11}y_{21}y_{22}}{y_{13}}+\frac{y_{21}^2y_{02}y_{22}^2y_{03}}{y_{12}^2y_{13}^2}+\frac{y_{11}^2y_{22}^2y_{03}}{y_{02}y_{13}^2}+\frac{y_{01}y_{22}^2y_{03}}{y_{13}^2}$$

$$+\frac{2y_{11}y_{21}y_{22}^2y_{03}}{y_{12}y_{13}^2})y_{04}(2,\overline{4},\overline{3},\overline{1})+(\frac{y_{11}y_{31}y_{12}}{y_{22}}+\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}+y_{11}y_{21})y_{13}(3,\overline{4},\overline{3},\overline{2})$$

$$+(\frac{y_{21}^2y_{02}y_{22}y_{03}}{y_{12}^2y_{13}}+\frac{y_{11}^2y_{22}y_{03}}{y_{02}y_{13}}+\frac{y_{01}y_{22}y_{03}}{y_{13}}+\frac{2y_{11}y_{21}y_{22}y_{03}}{y_{12}y_{13}}+\frac{y_{21}y_{31}y_{02}y_{03}}{y_{12}y_{13}}+\frac{y_{11}y_{31}y_{03}}{y_{13}}$$

$$+\frac{y_{11}y_{31}y_{12}}{y_{22}}+\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}+y_{11}y_{21})y_{04}(3,\overline{4},\overline{3},\overline{1})+(\frac{y_{31}^2y_{02}}{y_{22}^2}+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}$$

$$+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}})y_{03}y_{04}(3,\overline{4},\overline{2},\overline{1})+y_{11}y_{12}y_{13}(4,\overline{4},\overline{3},\overline{2})+(\frac{y_{21}y_{02}y_{22}y_{03}}{y_{12}y_{13}}+\frac{y_{11}y_{22}y_{03}}{y_{13}}$$

$$+y_{11}y_{12})y_{04}(4,\overline{4},\overline{3},\overline{1})+(\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11})y_{03}y_{04}(4,\overline{4},\overline{2},\overline{1})+y_{02}y_{03}y_{04}(4,\overline{3},\overline{2},\overline{1})+(\frac{y_{21}^2y_{22}^2}{y_{03}y_{04}}$$

$$+\frac{y_{21}^2y_{02}y_{22}^2}{y_{12}^2y_{04}}+\frac{y_{11}^2y_{22}^2}{y_{02}y_{04}}+\frac{y_{01}y_{22}^2}{y_{04}}+\frac{2y_{11}y_{21}y_{22}^2}{y_{12}y_{04}}+y_{01}y_{02}+\frac{y_{11}^2y_{12}^2}{y_{02}y_{03}}+\frac{y_{01}y_{12}^2}{y_{03}}+\frac{2y_{11}^2y_{12}y_{22}}{y_{02}y_{13}}$$

$$+\frac{2y_{01}y_{12}y_{22}}{y_{13}}+\frac{2y_{11}y_{21}y_{22}}{y_{13}}+\frac{y_{21}^2y_{02}y_{22}^2y_{03}}{y_{12}^2y_{13}^2}+\frac{y_{11}^2y_{22}^2y_{03}}{y_{02}y_{13}^2}+\frac{y_{01}y_{22}^2y_{03}}{y_{13}^2}+\frac{2y_{11}y_{21}y_{22}^2y_{03}}{y_{12}y_{13}^2})(1,2,\overline{4},\overline{3})$$

$$+\Big(\frac{y_{21}y_{31}y_{12}y_{13}}{y_{03}y_{04}}+\frac{y_{21}^2y_{22}y_{13}}{y_{03}y_{04}}+\frac{y_{21}^2y_{02}y_{22}y_{13}}{y_{12}^2y_{03}y_{04}}+\frac{y_{11}^2y_{22}y_{13}}{y_{02}y_{03}y_{04}}+\frac{y_{01}y_{22}y_{13}}{y_{04}}+\frac{2y_{11}y_{21}y_{22}y_{13}}{y_{12}y_{04}}$$

$$+\frac{y_{21}y_{31}y_{02}y_{13}}{y_{12}y_{04}}+\frac{y_{11}y_{31}y_{13}}{y_{04}}+\frac{y_{21}^2y_{02}y_{22}y_{03}}{y_{12}^2y_{13}}+\frac{y_{11}^2y_{22}y_{03}}{y_{02}y_{13}}+\frac{y_{01}y_{22}y_{03}}{y_{13}}+\frac{2y_{11}y_{21}y_{22}y_{03}}{y_{12}y_{13}}$$

$$+\frac{y_{21}y_{31}y_{02}y_{03}}{y_{12}y_{13}}+\frac{y_{11}y_{31}y_{03}}{y_{13}}+\frac{y_{11}y_{31}y_{12}}{y_{22}}+\frac{y_{11}^2y_{12}}{y_{02}}+y_{01}y_{12}+y_{11}y_{21}\Big)(1,3,\overline{4},\overline{3})+\Big(\frac{y_{31}^2y_{02}y_{03}}{y_{22}^2}$$

$$+\frac{y_{21}^2y_{02}y_{03}}{y_{12}^2}+\frac{y_{11}^2y_{03}}{y_{02}}+y_{01}y_{03}+\frac{2y_{11}y_{21}y_{03}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}y_{03}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}y_{03}}{y_{22}}+\frac{y_{31}^2y_{02}y_{13}^2}{y_{22}^2y_{04}}$$

$$+\frac{y_{21}^2y_{02}y_{13}^2}{y_{12}^2y_{04}}+\frac{y_{11}^2y_{13}^2}{y_{02}y_{04}}+\frac{y_{01}y_{13}^2}{y_{04}}+\frac{2y_{11}y_{21}y_{13}^2}{y_{12}y_{04}}+\frac{2y_{21}y_{31}y_{02}y_{13}^2}{y_{12}y_{22}y_{04}}+\frac{2y_{11}y_{31}y_{13}^2}{y_{22}y_{04}}+\frac{y_{31}^2y_{12}^2y_{13}^2}{y_{22}^2y_{03}y_{04}}$$

$$+\frac{2y_{21}y_{31}y_{12}y_{13}^2}{y_{22}y_{03}y_{04}}+\frac{y_{21}^2y_{13}^2}{y_{03}y_{04}}\Big)(1,3,\overline{4},\overline{2})+\Big(\frac{y_{21}y_{02}y_{22}y_{03}}{y_{12}y_{13}}+\frac{y_{11}y_{22}y_{03}}{y_{13}}+y_{11}y_{12}+\frac{y_{21}y_{12}y_{22}y_{13}}{y_{03}y_{04}}$$

$$+\frac{y_{21}^2y_{02}y_{22}y_{13}}{y_{12}^2y_{04}}+\frac{y_{11}^2y_{22}y_{13}}{y_{02}y_{04}}+\frac{y_{01}y_{22}y_{13}}{y_{04}}+\frac{2y_{11}y_{21}y_{22}y_{13}}{y_{12}y_{04}}+\frac{y_{21}y_{31}y_{02}y_{13}}{y_{12}y_{04}}+\frac{y_{11}y_{31}y_{13}}{y_{04}}\Big)$$

$$(1,4,\overline{4},\overline{3})+\Big(\frac{y_{31}y_{02}y_{03}}{y_{22}}+\frac{y_{21}y_{02}y_{03}}{y_{12}}+y_{11}y_{03}+\frac{y_{31}y_{02}y_{13}^2}{y_{22}y_{04}}+\frac{y_{21}y_{02}y_{13}^2}{y_{12}y_{04}}+\frac{y_{11}y_{13}^2}{y_{04}}+\frac{y_{31}y_{12}^2y_{13}^2}{y_{22}y_{03}y_{04}}$$

$$+\frac{y_{21}y_{12}y_{13}^2}{y_{03}y_{04}}\Big)(1,4,\overline{4},\overline{2})+\Big(y_{02}y_{03}+\frac{y_{02}y_{13}^2}{y_{04}}+\frac{y_{12}^2y_{13}^2}{y_{03}y_{04}}\Big)(1,4,\overline{3},\overline{2})+\Big(\frac{y_{21}y_{31}y_{22}}{y_{13}}+\frac{y_{21}y_{31}y_{12}}{y_{03}}+\frac{y_{21}^2y_{22}}{y_{03}}$$

$$+\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2y_{03}}+\frac{y_{11}^2y_{22}}{y_{02}y_{03}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31}\Big)(2,3,\overline{4},\overline{3})+\Big(\frac{y_{31}^2y_{12}}{y_{22}}+y_{21}y_{31}$$

$$+\frac{y_{31}^2y_{02}y_{13}}{y_{22}^2}+\frac{y_{21}^2y_{02}y_{13}}{y_{12}^2}+\frac{y_{11}^2y_{13}}{y_{02}}+y_{01}y_{13}+\frac{2y_{11}y_{21}y_{13}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}y_{13}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}y_{13}}{y_{22}}$$

$$+\frac{y_{31}^2y_{12}^2y_{13}}{y_{22}^2y_{03}}+\frac{2y_{21}y_{31}y_{12}y_{13}}{y_{22}y_{03}}+\frac{y_{21}^2y_{13}}{y_{03}}\Big)(2,3,\overline{4},\overline{2})+\Big(\frac{y_{31}^2y_{03}}{y_{13}^2}+\frac{2y_{31}^2y_{12}}{y_{22}y_{13}}+\frac{2y_{21}y_{31}}{y_{13}}+\frac{y_{31}^2y_{02}}{y_{22}^2}$$

$$+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}}+\frac{y_{31}^2y_{12}^2}{y_{22}^2y_{03}}+\frac{2y_{21}y_{31}y_{12}}{y_{22}y_{03}}+\frac{y_{21}^2}{y_{03}}\Big)y_{04}$$

$$(2,3,\overline{4},\overline{1})+\Big(\frac{y_{21}y_{12}y_{22}}{y_{03}}+\frac{y_{21}^2y_{02}y_{22}}{y_{12}^2}+\frac{y_{11}^2y_{22}}{y_{02}}+y_{01}y_{22}+\frac{2y_{11}y_{21}y_{22}}{y_{12}}+\frac{y_{21}y_{31}y_{02}}{y_{12}}+y_{11}y_{31}$$

$$+\frac{y_{21}y_{22}^2}{y_{13}}\Big)(2,4,\overline{4},\overline{3})+\Big(\frac{y_{31}y_{02}y_{13}}{y_{22}}+\frac{y_{21}y_{02}y_{13}}{y_{12}}+y_{11}y_{13}\Big)+\frac{y_{31}y_{12}^2y_{13}}{y_{22}y_{03}}+\frac{y_{21}y_{12}y_{13}}{y_{03}}+y_{31}y_{12}$$

$$+y_{21}y_{22}\Big)(2,4,\overline{4},\overline{2})+y_{21}y_{22}(3,4,\overline{4},\overline{3})+y_{21}y_{13}(3,4,\overline{4},\overline{2})+\Big(\frac{y_{31}y_{22}y_{03}}{y_{13}^2}+\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11}$$

$$+\frac{y_{31}y_{12}^2}{y_{22}y_{03}}+\frac{y_{21}y_{12}}{y_{03}}+\frac{2y_{31}y_{12}}{y_{13}}+\frac{2y_{21}y_{22}}{y_{13}}\Big)y_{04}(2,4,\overline{4},\overline{1})+\Big(\frac{y_{31}y_{03}}{y_{13}}+y_{21}\Big)y_{04}(3,4,\overline{4},\overline{1})+\Big(y_{12}y_{22}$$

$$+y_{02}y_{13}+\frac{y_{12}^2y_{13}}{y_{03}}\Big)(2,4,\overline{3},\overline{2})+y_{12}y_{13}(3,4,\overline{3},\overline{2})+\Big(\frac{y_{22}^2y_{03}}{y_{13}^2}+\frac{2y_{12}y_{22}}{y_{13}}+y_{02}+\frac{y_{12}^2}{y_{03}}\Big)y_{04}(2,4,\overline{3},\overline{1})$$

$$+\left(\frac{y_{22}y_{03}}{y_{13}}+y_{12}\right)y_{04}(3,4,\overline{3},\overline{1})+y_{03}y_{04}(3,4,\overline{2},\overline{1})+\left(\frac{y_{31}^2}{y_{04}}+\frac{y_{31}^2y_{03}}{y_{13}^2}+\frac{2y_{31}^2y_{12}}{y_{22}y_{13}}+\frac{2y_{21}y_{31}}{y_{13}}+\frac{y_{31}^2y_{02}}{y_{22}^2}\right.$$
$$+\frac{y_{21}^2y_{02}}{y_{12}^2}+\frac{y_{11}^2}{y_{02}}+y_{01}+\frac{2y_{11}y_{21}}{y_{12}}+\frac{2y_{21}y_{31}y_{02}}{y_{12}y_{22}}+\frac{2y_{11}y_{31}}{y_{22}}+\frac{y_{31}^2y_{12}^2}{y_{22}^2y_{03}}+\frac{2y_{21}y_{31}y_{12}}{y_{22}y_{03}}+\left.\frac{y_{21}^2}{y_{03}}\right)(1,2,3,\overline{4})$$
$$+\left(\frac{y_{31}y_{22}}{y_{04}}+\frac{y_{31}y_{22}y_{03}}{y_{13}^2}+\frac{y_{31}y_{02}}{y_{22}}+\frac{y_{21}y_{02}}{y_{12}}+y_{11}+\frac{y_{31}y_{12}^2}{y_{22}y_{03}}+\frac{y_{21}y_{12}}{y_{03}}+\frac{2y_{31}y_{12}}{y_{13}}+\frac{2y_{21}y_{22}}{y_{13}}\right)(1,2,4,\overline{4})$$
$$+\left(\frac{y_{31}y_{13}}{y_{04}}+\frac{y_{31}y_{03}}{y_{13}}+y_{21}\right)(1,3,4,\overline{4})+y_{31}(2,3,4,\overline{4})+\left(\frac{y_{22}^2}{y_{04}}+\frac{y_{22}^2y_{03}}{y_{13}^2}+\frac{2y_{12}y_{22}}{y_{13}}+y_{02}+\frac{y_{12}^2}{y_{03}}\right)$$
$$(1,2,4,\overline{3})+\left(\frac{y_{22}y_{13}}{y_{04}}+\frac{y_{22}y_{03}}{y_{13}}+y_{12}\right)(1,3,4,\overline{3})+y_{22}(2,3,4,\overline{3})+\left(\frac{y_{13}^2}{y_{04}}+y_{03}\right)(1,3,4,\overline{2})+y_{13}(2,3,4,\overline{2})$$
$$+y_{04}(2,3,4,\overline{1})+(1,2,3,4)]$$

We solve for the coefficients in terms of one another using the equation: $V_1(x)a(x)=V_2(y)$ and get the following solution:

$$a(x)=\frac{1}{x_{41}x_{42}x_{43}x_{44}},$$
$$y_{01}=\frac{1}{x_{44}},\ y_{02}=\frac{1}{x_{43}},\ y_{03}=\frac{1}{x_{42}},\ y_{04}=\frac{1}{x_{41}},$$
$$y_{11}=\frac{1}{x_{44}\left(\frac{x_{43}}{x_{33}}+\frac{x_{32}}{x_{22}}+\frac{x_{21}}{x_{11}}\right)},$$
$$y_{12}=\frac{x_{32}(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33})}{x_{33}x_{43}(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{31}x_{32}x_{22})},$$
$$y_{13}=\frac{x_{11}}{x_{41}x_{22}}+\frac{x_{21}}{x_{41}x_{32}}+\frac{x_{31}}{x_{41}x_{42}},$$
$$y_{21}=\frac{x_{21}x_{32}x_{33}x_{22}}{x_{44}(x_{21}x_{22}x_{42}x_{43}+x_{31}x_{22}x_{32}x_{43}+x_{31}x_{32}^2x_{33})},$$
$$y_{22}=\frac{x_{31}x_{32}}{x_{41}x_{42}x_{43}}+\frac{x_{31}x_{22}}{x_{33}x_{41}x_{42}}+\frac{x_{21}x_{22}}{x_{32}x_{33}x_{41}},$$
$$y_{31}=\frac{x_{31}x_{32}x_{33}}{x_{41}x_{42}x_{43}x_{44}}$$

We can also see that $x_{41}=\frac{1}{y_{04}}$, $x_{42}=\frac{1}{y_{03}}$, $x_{43}=\frac{1}{y_{02}}$, $x_{44}=\frac{1}{y_{01}}$, $x_{33}=\frac{y_{31}y_{02}y_{12}+y_{21}y_{22}y_{02}+y_{11}y_{12}y_{22}}{y_{12}y_{22}y_{01}y_{02}}$, $x_{32}=\frac{(y_{31}y_{03}y_{22}+y_{31}y_{12}y_{13}+y_{21}y_{22}y_{13})y_{12}}{y_{03}y_{13}(y_{31}y_{02}y_{12}+y_{21}y_{22}y_{02}+y_{11}y_{12}y_{22})}$, $x_{31}=\frac{y_{31}y_{22}y_{13}}{y_{04}(y_{31}y_{03}y_{22}+y_{31}y_{12}y_{13}+y_{21}y_{22}y_{13})}$, $x_{22}=\frac{y_{21}y_{22}y_{02}y_{03}+y_{11}y_{12}y_{22}y_{03}+y_{11}y_{12}^2y_{13})}{y_{12}y_{13}y_{01}y_{02}y_{03}}$, $x_{21}=\frac{y_{21}y_{22}y_{12}y_{13}}{y_{04}(y_{21}y_{22}y_{02}y_{03}+y_{11}y_{12}y_{22}y_{03}+y_{11}y_{12}^2y_{13})}$, and $x_{11}=\frac{y_{11}y_{12}y_{13}}{y_{01}y_{02}y_{03}y_{04}}$.

Now we need to compute the actions of the geometric crystal. First we compute e_1^c, e_2^c, e_3^c, and e_4^c.

$$\begin{aligned}
&e_1^c(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41})\\
&=(x_{44}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{33}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{43}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{22}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{32}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{42}\left(\frac{\frac{x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right), x_{11}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{x_{22}}{x_{11}}}\right),\\
&x_{21}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{c\,x_{22}}{x_{11}}}\right), x_{31}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{c\,x_{22}}{x_{11}}}\right), x_{41}\left(\frac{\frac{c\,x_{22}}{x_{11}}}{\frac{c\,x_{22}}{x_{11}}}\right))=(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, c\,x_{11}, x_{21}, x_{31}, x_{41})
\end{aligned}$$

$$\begin{aligned}
&e_2^c(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41})\\
&=(x_{44}\left(\frac{\frac{x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}{\frac{x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}\right), x_{33}\left(\frac{\frac{x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}{\frac{x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}\right), x_{43}, x_{22}\left(\frac{\frac{c\,x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}{\frac{x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}\right), x_{32}, x_{42}, x_{11},\\
&x_{21}\left(\frac{\frac{c\,x_{33}}{x_{22}}+\frac{c\,x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}{\frac{c\,x_{33}}{x_{22}}+\frac{x_{33}\,x_{32}\,x_{11}}{x_{22}^2\,x_{21}}}\right), x_{31}, x_{41})
\end{aligned}$$

Let $\frac{c_{21}}{c_2}=\frac{\frac{c\,x_{33}\,x_{22}\,x_{21}+x_{32}\,x_{33}\,x_{11}}{x_{22}^2\,x_{21}}}{\frac{x_{33}\,x_{22}\,x_{21}+x_{32}\,x_{33}\,x_{11}}{x_{22}^2\,x_{21}}}=\frac{c\,x_{22}\,x_{21}+x_{32}\,x_{11}}{x_{22}\,x_{21}+x_{32}\,x_{11}}$ Then

$$\begin{aligned}
&e_2^c(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41})\\
&=(x_{44}, x_{33}, x_{43}, \frac{c_{21}}{c_2}x_{22}, x_{32}, x_{42}, x_{11}, \frac{c\,c_2}{c_{21}}x_{21}, x_{31}, x_{41})
\end{aligned}$$

Let $c_3=\frac{x_{44}}{x_{33}}+\frac{x_{44}\,x_{43}\,x_{22}}{x_{33}^2\,x_{32}}+\frac{x_{44}\,x_{43}\,x_{42}\,x_{22}\,x_{21}}{x_{33}^2\,x_{32}^2\,x_{31}}=\frac{(x_{44}\,x_{33}\,x_{32}^2\,x_{31}+x_{44}\,x_{43}\,x_{22}\,x_{32}\,x_{31}+x_{44}\,x_{43}\,x_{42}\,x_{22}\,x_{21})}{x_{33}^2\,x_{32}^2\,x_{31}}$,
$c_{31}=\frac{(c\,x_{44}\,x_{33}\,x_{32}^2\,x_{31}+x_{44}\,x_{43}\,x_{22}\,x_{32}\,x_{31}+x_{44}\,x_{43}\,x_{42}\,x_{22}\,x_{21})}{x_{33}^2\,x_{32}^2\,x_{31}}$
and $c_{32}=\frac{(c\,x_{44}\,x_{33}\,x_{32}^2\,x_{31}+c\,x_{44}\,x_{43}\,x_{22}\,x_{32}\,x_{31}+x_{44}\,x_{43}\,x_{42}\,x_{22}\,x_{21})}{x_{33}^2\,x_{32}^2\,x_{31}}$. Then

$$\begin{aligned}
&e_3^c(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41})\\
&=(x_{44}, \frac{c_{31}}{c_3}x_{33}, x_{43}, x_{22}, \frac{c_{32}}{c_{31}}x_{32}, x_{42}, x_{11}, x_{21}, \frac{c\,c_3}{c_{32}}x_{31}, x_{41})
\end{aligned}$$

Let $c_4=\frac{1}{x_{44}}+\frac{x_{33}^2}{x_{43}\,x_{44}^2}+\frac{x_{33}^2\,x_{32}^2}{x_{44}^2\,x_{43}^2\,x_{42}}+\frac{x_{33}^2\,x_{32}^2\,x_{31}^2}{x_{44}^2\,x_{43}^2\,x_{42}^2\,x_{41}}=\frac{x_{41}\,x_{42}^2\,x_{43}^2\,x_{44}+x_{33}^2\,x_{43}\,x_{42}^2\,x_{41}+x_{33}^2\,x_{32}^2\,x_{42}\,x_{41}+x_{33}^2\,x_{32}^2\,x_{31}^2}{x_{44}^2\,x_{43}^2\,x_{42}^2\,x_{41}}$,
$c_{41}=\frac{c\,x_{41}\,x_{42}^2\,x_{43}^2\,x_{44}+x_{33}^2\,x_{43}\,x_{42}^2\,x_{41}+x_{33}^2\,x_{32}^2\,x_{42}\,x_{41}+x_{33}^2\,x_{32}^2\,x_{31}^2}{x_{44}^2\,x_{43}^2\,x_{42}^2\,x_{41}}$, $c_{42}=\frac{c\,x_{41}\,x_{42}^2\,x_{43}^2\,x_{44}+c\,x_{33}^2\,x_{43}\,x_{42}^2\,x_{41}+x_{33}^2\,x_{32}^2\,x_{42}\,x_{41}+x_{33}^2\,x_{32}^2\,x_{31}^2}{x_{44}^2\,x_{43}^2\,x_{42}^2\,x_{41}}$,

and $c_{43} = \frac{c\, x_{41} x_{42}^2 x_{43}^2 x_{44} + c\, x_{33}^2 x_{43} x_{42}^2 x_{41} + c\, x_{33}^2 x_{32}^2 x_{42} x_{41} + x_{33}^2 x_{32}^2 x_{31}^2}{x_{44}^2 x_{43}^2 x_{42}^2 x_{41}}$. Then

$$e_4^c(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \left(\frac{c_{41}}{c_4} x_{44}, x_{33}, \frac{c_{42}}{c_{41}} x_{43}, x_{22}, x_{32}, \frac{c_{43}}{c_{42}} x_{42}, x_{11}, x_{21}, x_{31}, \frac{c\, c_4}{c_{43}} x_{41}\right)$$

Now we define ε_i and γ_i for $i = 1..4$.

$$\varepsilon_1(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{22}}{x_{11}}$$

$$\varepsilon_2(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{33}}{x_{22}} + \frac{x_{33} x_{32} x_{11}}{x_{22}^2 x_{21}}$$

$$\varepsilon_3(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{44}}{x_{33}} + \frac{x_{44} x_{43} x_{22}}{x_{33}^2 x_{32}} + \frac{x_{44} x_{43} x_{42} x_{22} x_{21}}{x_{33}^2 x_{32}^2 x_{31}}$$

$$\varepsilon_4(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{1}{x_{44}} + \frac{x_{33}^2}{x_{43} x_{44}^2} + \frac{x_{33}^2 x_{32}^2}{x_{44}^2 x_{43}^2 x_{42}} + \frac{x_{33}^2 x_{32}^2 x_{31}^2}{x_{44}^2 x_{43}^2 x_{42}^2 x_{41}}$$

$$\gamma_1(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{11}^2}{x_{21} x_{22}}$$

$$\gamma_2(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{21}^2 x_{22}^2}{x_{11} x_{31} x_{32} x_{33}}$$

$$\gamma_3(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{31}^2 x_{32}^2 x_{33}^2}{x_{21} x_{22} x_{41} x_{42} x_{43} x_{44}}$$

$$\gamma_4(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) = \frac{x_{41}^2 x_{42}^2 x_{43}^2 x_{44}^2}{x_{31}^2 x_{32}^2 x_{33}^2}$$

To get the formulas for e_0^c, ε_0 and γ_0 we first must get the formulas for $\overline{e_0^c}$, $\overline{\varepsilon_0}$ and $\overline{\gamma_0}$ in $\mathcal{V}_2(y)$. These are as follows:

$$\overline{e_0^c}(y_{04}, y_{13}, y_{03}, y_{22}, y_{12}, y_{02}, y_{31}, y_{21}, y_{11}, y_{01}) = (\frac{c_{01}}{c_0} y_{04}, y_{13}, \frac{c_{02}}{c_{01}} y_{03}, y_{22}, y_{12}, \frac{c_{03}}{c_{02}} y_{02}, y_{31}, y_{21}, y_{11}, \frac{c\, c_0}{c_{03}} y_{01})$$

where $c_0 = \frac{1}{y_{04}} + \frac{y_{13}^2}{y_{04}^2 y_{03}} + \frac{y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}$, $c_{01} = \frac{c}{y_{04}} + \frac{y_{13}^2}{y_{04}^2 y_{03}} + \frac{y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}$, $c_{02} = \frac{c}{y_{04}} + \frac{c\, y_{13}^2}{y_{04}^2 y_{03}} +$

$\frac{y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}$ and $c_{03} = \frac{c}{y_{04}} + \frac{c\, y_{13}^2}{y_{04}^2 y_{03}} + \frac{c\, y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}$.

$$\overline{\varepsilon_0}(y_{04}, y_{13}, y_{03}, y_{22}, y_{12}, y_{02}, y_{31}, y_{21}, y_{11}, y_{01}) = \frac{1}{y_{04}} + \frac{y_{13}^2}{y_{04}^2 y_{03}} + \frac{y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}$$

$$\overline{\gamma_0}(y_{04}, y_{13}, y_{03}, y_{22}, y_{12}, y_{02}, y_{31}, y_{21}, y_{11}, y_{01}) = \frac{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2}{y_{11}^2 y_{12}^2 y_{13}^2}$$

Now e_0^c, ε_0 and γ_0 are defined as follows:

$$\begin{aligned} e_0^c(V_1(x)) &= \overline{\sigma^{-1}} \circ \overline{e_0^c} \circ \overline{\sigma}(V_1(x)) \\ \gamma_0(V_1(x)) &= \overline{\gamma_0}(\overline{\sigma}(V_1(x))) \\ \varepsilon_0(V_1(x)) &= \overline{\varepsilon_0}(\overline{\sigma}(V_1(x))) \end{aligned}$$

Using this formula, we obtain γ_0:

$$\gamma_0(V_1(x)) = \frac{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2}{y_{11}^2 y_{12}^2 y_{13}^2} = \frac{x_{44}^2 x_{43}^2 x_{42}^2 x_{41}^2}{x_{11}^2 x_{44}^2 x_{43}^2 x_{42}^2 x_{41}^2} = \frac{1}{x_{11}^2}$$

Next we obtain ε_0:

$$\begin{aligned} \varepsilon_0(V_1(x)) &= \frac{1}{y_{04}} + \frac{y_{13}^2}{y_{04}^2 y_{03}} + \frac{y_{13}^2 y_{12}^2}{y_{04}^2 y_{03}^2 y_{02}} + \frac{y_{13}^2 y_{12}^2 y_{11}^2}{y_{04}^2 y_{03}^2 y_{02}^2 y_{01}} \\ &= x_{41} + \frac{(\frac{x_{11} x_{42}}{x_{22}} + \frac{x_{21} x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11} x_{43}}{x_{33}} + \frac{x_{11} x_{32}}{x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}} \end{aligned}$$

Finally we compute e_0^c.

$$
\begin{aligned}
e_0^c(V_1(x)) = {} & \overline{\sigma^{-1}} \circ \overline{e_0^c}\Big(\frac{1}{y_{01}}, \frac{y_{11}}{y_{01}\, y_{02}} + \frac{y_{21}}{y_{12}\, y_{01}} + \frac{y_{31}}{y_{22}\, y_{01}}, \frac{1}{y_{02}}, \frac{y_{11}\, y_{12}}{y_{01}\, y_{02}\, y_{03}} + \frac{y_{11}\, y_{22}}{y_{13}\, y_{01}\, y_{02}} + \frac{y_{21}\, y_{22}}{y_{12}\, y_{13}\, y_{01}}, \\
& \frac{\frac{y_{21}}{y_{02}\, y_{03}} + \frac{y_{31}\, y_{12}}{y_{22}\, y_{02}\, y_{03}} + \frac{y_{31}}{y_{13}\, y_{02}}}{\frac{y_{11}}{y_{02}} + \frac{y_{21}}{y_{12}} + \frac{y_{31}}{y_{22}}}, \frac{1}{y_{03}}, \frac{y_{11}\, y_{12}\, y_{13}}{y_{01}\, y_{02}\, y_{03}\, y_{04}}, \frac{y_{21}\, y_{22}}{y_{04}(y_{11}\, y_{12} + \frac{y_{11}\, y_{22}\, y_{03}}{y_{13}} + \frac{y_{21}\, y_{22}\, y_{02}\, y_{03}}{y_{12}\, y_{13}})}, \\
& \frac{y_{31}}{y_{04}(y_{21} + \frac{y_{31}\, y_{12}}{y_{22}} + \frac{y_{31}\, y_{03}}{y_{13}})}, \frac{1}{y_{04}}\Big) \\
= {} & \overline{\sigma^{-1}}\Big(\frac{1}{\frac{c\, c_0}{c_{03}}\, y_{01}}, \frac{y_{11}\, c_{02}}{c\, c_0\, y_{01}\, y_{02}} + \frac{c_{03}\, y_{21}}{c\, c_0\, y_{12}\, y_{01}} + \frac{y_{31}\, c_{03}}{c\, c_0\, y_{22}\, y_{01}}, \frac{c_{02}}{c_{03}\, y_{02}}, \frac{c_{01}\, y_{11}\, y_{12}}{y_{01}\, y_{02}\, y_{03}} + \frac{y_{11}\, y_{22}\, c_{02}}{y_{13}\, y_{01}\, y_{02}} \\
& + \frac{y_{21}\, y_{22}\, c_{03}}{c\, c_0\, y_{12}\, y_{13}\, y_{01}}, \frac{\frac{y_{21}\, c_{01}}{c_{03}\, y_{02}\, y_{03}} + \frac{c_{01}\, y_{31}\, y_{12}}{c_{03}\, y_{22}\, y_{02}\, y_{03}} + \frac{y_{31}\, c_{02}}{y_{02}\, y_{13}\, c_{03}}}{\frac{y_{11}\, c_{02}}{c_{03}\, y_{02}} + \frac{y_{21}}{y_{12}} + \frac{y_{31}}{y_{22}}}, \frac{c_{01}}{c_{02}\, y_{03}}, \frac{y_{11}\, y_{12}\, y_{13}}{c\, y_{01}\, y_{02}\, y_{03}\, y_{04}}, \\
& \frac{y_{21}\, y_{22}\, c_{03}}{c\, c_0\, y_{04}(y_{11}\, y_{12} + \frac{y_{11}\, y_{22}\, y_{03}\, c_{02}}{c_{01}\, y_{13}} + \frac{y_{21}\, y_{22}\, y_{02}\, y_{03}\, c_{03}}{c_{01}\, y_{12}\, y_{13}})}, \frac{y_{31}\, c_{03}}{c\, c_0\, y_{04}(y_{21} + \frac{y_{31}\, y_{12}}{y_{22}} + \frac{y_{31}\, y_{03}\, c_{02}}{c_{01}\, y_{13}})}, \frac{c_0}{c_{01}\, y_{04}}\Big)
\end{aligned}
$$

Let $c_4' = x_{41} + \frac{(\frac{x_{11}\, x_{42}}{x_{22}} + \frac{x_{21}\, x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11}\, x_{43}}{x_{33}} + \frac{x_{11}\, x_{32}}{x_{22})})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$,
$c_{41}' = c\, x_{41} + \frac{(\frac{x_{11}\, x_{42}}{x_{22}} + \frac{x_{21}\, x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11}\, x_{43}}{x_{33}} + \frac{x_{11}\, x_{32}}{x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$,
$c_{42}' = c\, x_{41} + \frac{c(\frac{x_{11}\, x_{42}}{x_{22}} + \frac{x_{21}\, x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11}\, x_{43}}{x_{33}} + \frac{x_{11}\, x_{32}}{x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$, and
$c_{43}' = c\, x_{41} + \frac{c(\frac{x_{11}\, x_{42}}{x_{22}} + \frac{x_{21}\, x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{c(x_{21} + \frac{x_{11}\, x_{43}}{x_{33}} + \frac{x_{11}\, x_{32}}{x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$.

Then we have

$$e_0^c(V_1(x)) = (x_{44}\frac{c'_{43}}{c\,c'_4}, \frac{x_{33}}{c\,c'_4}\cdot\frac{x_{11}x_{22}x_{43}c'_{42}+(x_{21}x_{22}x_{33}+x_{11}x_{32}x_{33})c'_{43}}{x_{11}x_{22}x_{43}+x_{21}x_{22}x_{33}+x_{11}x_{32}x_{33}}, \frac{x_{43}c'_{42}}{c'_{43}},$$

$$x_{22}\frac{c'_{41}(x_{11}^2x_{22}x_{32}x_{42}x_{43}+x_{11}^2x_{32}^2x_{33}x_{42}+x_{11}x_{21}x_{22}x_{32}x_{33}x_{42})}{(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{32}x_{22}x_{31})(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33})c\,c'_4}$$

$$+\frac{c'_{42}(x_{11}x_{21}x_{22}^2x_{42}x_{43}+x_{11}x_{22}^2x_{31}x_{32}x_{43}+x_{11}x_{22}x_{31}x_{32}^2x_{33})}{(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{32}x_{22}x_{31})(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33})c\,c'_4}$$

$$+\frac{c'_{43}(x_{11}x_{21}x_{22}x_{32}x_{33}x_{42}+x_{21}^2x_{22}^2x_{33}x_{42}+x_{21}x_{22}^2x_{31}x_{32}x_{33})}{(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{32}x_{22}x_{31})(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33})c\,c'_4},$$

$$x_{32}\frac{(c'_{41}x_{11}x_{32}x_{42}+c'_{41}x_{21}x_{22}x_{42}+c'_{42}x_{32}x_{22}x_{31})(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33})}{(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{32}x_{22}x_{31})(c'_{42}x_{11}x_{22}x_{43}+c'_{43}x_{11}x_{32}x_{33}+c'_{43}x_{21}x_{22}x_{33})}, \frac{c'_{41}x_{42}}{c'_{42}}, \frac{x_{11}}{c},$$

$$x_{21}(c'_4(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{32}x_{22}x_{31})(x_{11}x_{22}x_{43}+x_{11}x_{32}x_{33}+x_{21}x_{22}x_{33}))$$

$$(c'_{41}(x_{11}^2x_{22}x_{32}x_{42}x_{43}+x_{11}^2x_{32}^2x_{33}x_{42}+x_{11}x_{21}x_{22}x_{32}x_{33}x_{42})+c'_{42}(x_{11}x_{21}x_{22}^2x_{42}x_{43}$$

$$+x_{11}x_{22}^2x_{31}x_{32}x_{43}+x_{11}x_{22}x_{31}x_{32}^2x_{33})+c'_{43}(x_{11}x_{21}x_{22}x_{32}x_{33}x_{42}+x_{21}^2x_{22}^2x_{33}x_{42}$$

$$+x_{21}x_{22}^2x_{31}x_{32}x_{33}))^{-1}, x_{31}\frac{c'_4(x_{11}x_{32}x_{42}+x_{21}x_{22}x_{42}+x_{22}x_{31}x_{32})}{c'_{41}x_{11}x_{32}x_{42}+c'_{41}x_{21}x_{22}x_{42}+c'_{42}x_{22}x_{31}x_{32}}, \frac{x_{41}c'_4}{c'_{41}})$$

Theorem 5.3.3. *$\mathcal{V}_1 = (V_1(x), e_i^c, \varepsilon_i, \varphi_i)$ for $i = 0, 1, 2, 3, 4$ is a positive geometric crystal associated with $C_4^{(1)}$.*

Proof. We know this is positive because all of the coefficients are positive. We already know that $\mathcal{V}_1$ is a geometric crystal without the additionally 0-actions, so we only need to check relations with the 0 index:

1. $\gamma_0(e_i^c(V_1(x))) = c^{a_{i0}}\gamma_0(V_1(x))$
2. $\gamma_i(e_0^c(V_1(x))) = c^{a_{0i}}\gamma_i(V_1(x))$
3. $\varepsilon_0(e_0^c(V_1(x))) = c^{-1}\varepsilon_0(V_1(x))$
4. $\varepsilon_2(e_0^c(V_1(x))) = \varepsilon_2(V_1(x))$
5. $\varepsilon_3(e_0^c(V_1(x))) = \varepsilon_3(V_1(x))$
6. $\varepsilon_4(e_0^c(V_1(x)) = \varepsilon_4(V_1(x))$
7. $\varepsilon_0(e_2^c(V_1(x))) = \varepsilon_0(V_1(x))$
8. $\varepsilon_0(e_3^c(V_1(x))) = \varepsilon_0(V_1(x))$

9. $\varepsilon_0(e_4^c(V_1(x))) = \varepsilon_0(V_1(x))$

10. $e_0^c e_2^d = e_2^d e_0^c$

11. $e_0^c e_3^d = e_3^d e_0^c$

12. $e_0^c e_4^d = e_4^d e_0^c$

13. $e_1^c e_0^{c^2 d} e_1^{cd} e_0^d = e_0^d e_1^{cd} e_0^{c^2 d} e_1^c$

We first check 1.

- $\gamma_0(e_0^c(V_1(x))) = \frac{c^2}{x_{11}^2} = c^2 \gamma_0(V_1(x))$
- $\gamma_0(e_1^c(V_1(x))) = \frac{1}{c^2 x_{11}^2} = c^{-2} \gamma_0(V_1(x))$
- $\gamma_0(e_2^c(V_1(x))) = \frac{1}{x_{11}^2} = \gamma_0(V_1(x))$
- $\gamma_0(e_3^c(V_1(x))) = \frac{1}{x_{11}^2} = \gamma_0(V_1(x))$
- $\gamma_0(e_4^c(V_1(x))) = \frac{1}{x_{11}^2} = \gamma_0(V_1(x))$

Thus, 1 is satisfied. Next we check 2:

- $\gamma_1(e_0^c(V_1(x))) = \frac{\frac{x_{11}^2}{c^2}}{\frac{x_{21} x_{22}}{c}} = c^{-1} \gamma_0(V_1(x))$
- $\gamma_2(e_0^c(V_1(x))) = \frac{\frac{x_{21}^2 x_{22}^2}{c^2}}{\frac{x_{11} x_{31} x_{32} x_{33}}{c^2}} = \gamma_2(V_1(x))$
- $\gamma_3(e_0^c(V_1(x))) = \frac{\frac{x_{31}^2 x_{32}^2 x_{33}^2}{c^2}}{\frac{x_{11} x_{41} x_{42} x_{43} x_{44}}{c^2}} = \gamma_3(V_1(x))$
- $\gamma_4(e_0^c(V_1(x))) = \frac{\frac{x_{41}^2 x_{42}^2 x_{43}^2 x_{44}^2}{c^2}}{\frac{x_{31}^2 x_{32}^2 x_{33}^2}{c^2}} = \gamma_4(V_1(x))$

Now we check 3. We can see that this relation is equivalent to another relation as follows: $\varepsilon_0(e_0^c(V_1(x))) = \overline{\varepsilon}_0 \circ \overline{\sigma} \circ \overline{\sigma^{-1}} \circ \overline{e}_0^c \circ \overline{\sigma} = \overline{\varepsilon}_0 \circ \overline{e}_0^c \circ \overline{\sigma}(V_1(x)) = \overline{\varepsilon}_0 \circ \overline{e}_0^c(V_2(y))$ So we can just consider $\overline{\varepsilon}_0 \circ \overline{e}_0^c(V_2(y))$.

$$\begin{aligned}
&\overline{\varepsilon_0}(\frac{c_{01} y_{04}}{c_0}, y_{13}, \frac{c_{02} y_{03}}{c_{01}}, y_{22}, y_{12}, \frac{c_{03} y_{02}}{c_{02}}, y_{31}, y_{21}, y_{11}, \frac{c\, c_0\, y_{01}}{c_{03}}) \\
&= \frac{\frac{c_{03}}{c_{01}} y_{04} y_{03}^2 y_{02}^2 y_{01} + \frac{c\, c_0 c_{03}}{c_{01} c_{02}} y_{13}^2 y_{03} y_{02}^2 y_{01} + \frac{c\, c_0}{c_{02}} y_{13}^2 y_{12}^2 y_{02} y_{01} + y_{11}^2 y_{12}^2 y_{13}^2}{\frac{c\, c_{03}}{c_0} y_{04}^2 y_{03}^2 y_{02}^2 y_{01}}
\end{aligned}$$

If we expand the c_{01}, c_{02}, c_{03} terms and factor, we obtain

$$= \frac{y_{04}y_{03}^2y_{02}^2y_{01} + y_{13}^2y_{03}y_{02}^2y_{01} + y_{13}^2y_{12}^2y_{02}y_{01} + y_{11}^2y_{12}^2y_{13}^2}{y_{04}^2y_{03}^2y_{02}^2y_{01}} = \overline{\varepsilon}_0(V_2(y))$$

Next we check 4. For the following relations we assign the following:

$$\begin{aligned}
A &= x_{11}x_{22}x_{43}c'_{42} + c'_{43}x_{21}x_{22}x_{33} + c'_{43}x_{11}x_{32}x_{33}\\
B &= x_{11}x_{22}x_{43} + x_{21}x_{22}x_{33} + x_{11}x_{32}x_{33}\\
C &= c'_{41}x_{11}x_{32}x_{42} + c'_{41}x_{21}x_{22}x_{42} + c'_{42}x_{22}x_{31}x_{32}\\
D &= x_{11}x_{32}x_{42} + x_{21}x_{22}x_{42} + x_{22}x_{31}x_{32}\\
G &= c'_{41}(x_{11}^2x_{22}x_{32}x_{42}x_{43} + x_{11}^2x_{32}^2x_{33}x_{42} + x_{11}x_{21}x_{22}x_{32}x_{33}x_{42})\\
&+ c'_{42}(x_{11}x_{21}x_{22}^2x_{42}x_{43} + x_{11}x_{22}^2x_{31}x_{32}x_{43} + x_{11}x_{22}x_{31}x_{32}^2x_{33})\\
&+ c'_{43}(x_{11}x_{21}x_{22}x_{32}x_{33}x_{42} + x_{21}^2x_{22}^2x_{33}x_{42} + x_{21}x_{22}^2x_{31}x_{32}x_{33})
\end{aligned}$$

$$\varepsilon_2(e_0^c(V_1(x))) = \frac{\frac{x_{33}x_{22}x_{21}A}{Bc^2c'_4} + \frac{x_{11}x_{32}x_{33}C}{c^2c'_4D}}{\frac{x_{21}x_{22}^2G}{c^2c'_4BD}} = \frac{x_{33}x_{22}x_{21}AD + x_{11}x_{32}x_{33}BC}{x_{21}x_{22}^2G}$$

$$\begin{aligned}
&= \frac{x_{33}}{x_{22}^2x_{21}}\cdot\Big(\frac{c'_{42}x_{11}^2x_{21}x_{22}^2x_{32}x_{42}x_{43} + c'_{42}x_{11}x_{21}^2x_{22}^3x_{42}x_{43} + c'_{42}x_{11}x_{21}x_{22}^3x_{31}x_{32}x_{43}}{G}\\
&+ \frac{c'_{43}x_{11}x_{21}^2x_{22}^2x_{32}x_{33}x_{42} + c'_{43}x_{21}^2x_{22}^3x_{31}x_{32}x_{33} + c'_{43}x_{11}x_{21}x_{22}^2x_{31}x_{32}^2x_{33} + c'_{43}x_{21}^3x_{22}^3x_{33}x_{42}}{G}\\
&+ \frac{c'_{43}x_{11}^2x_{21}x_{22}x_{32}^2x_{33}x_{42} + c'_{43}x_{11}x_{21}^2x_{22}^2x_{32}x_{33}x_{42} + c'_{41}x_{11}^3x_{22}x_{32}^2x_{42}x_{43} + c'_{41}x_{11}^3x_{32}^3x_{33}x_{42}}{G}\\
&+ \frac{c'_{41}x_{11}^2x_{21}x_{22}x_{32}^2x_{33}x_{42} + c'_{41}x_{11}^2x_{21}x_{22}^2x_{32}x_{42}x_{43} + c'_{41}x_{11}x_{21}^2x_{22}^2x_{32}x_{33}x_{42}}{G}\\
&+ \frac{c'_{42}x_{11}^2x_{22}^2x_{31}x_{32}^2x_{43} + c'_{41}x_{11}^2x_{21}x_{22}x_{32}^2x_{33}x_{42} + c'_{42}x_{11}x_{21}x_{22}^2x_{31}x_{32}^2x_{33} + c'_{42}x_{11}x_{22}x_{31}x_{32}^3x_{33}}{G}\Big)\\
&= \frac{x_{33}(x_{21}x_{22} + x_{11}x_{32})G}{Gx_{22}^2x_{21}} = \varepsilon_2(V_1(x))
\end{aligned}$$

Now we check the fifth relation.

$$\varepsilon_3(e_0^c(V_1(x))) = \frac{\frac{x_{44} c'_{43} x_{33} x_{32}^2 x_{31} C B}{c^2 c'_4 A D} + \frac{x_{44} x_{43} c'_{42} x_{32} x_{31} c'_4 B x_{22} G}{c c'_4 A B D c c'_4} + \frac{x_{44} x_{43} x_{42} c'_{41} x_{22} x_{21}}{c^2 c'_4}}{\frac{x_{31} x_{32}^2 x_{33}^2 C}{c^2 c'_4 D}}$$

$$= \frac{\frac{x_{44} c'_{43} x_{33} x_{32}^2 x_{31} C B}{A D} + \frac{x_{44} x_{43} c'_{42} x_{32} x_{31} x_{22} G}{A D} + x_{44} x_{43} x_{42} c'_{41} x_{22} x_{21}}{\frac{x_{31} x_{32}^2 x_{33}^2 C}{D}}$$

Expanding the first two terms in the numerator and factoring, we get

$$\frac{x_{44} c'_{43} x_{33} x_{32}^2 x_{31} C B + x_{44} x_{43} c'_{42} x_{32} x_{31} x_{22} G}{A D}$$

$$= \frac{A(c'_{41} x_{11} x_{22} x_{31} x_{32}^2 x_{42} x_{43} x_{44} + c'_{41} x_{11} x_{31} x_{32}^3 x_{33} x_{42} x_{44} + c'_{41} x_{21} x_{22} x_{31} x_{32}^2 x_{33} x_{42} x_{44})}{A D}$$

$$+ \frac{A(c'_{42} x_{21} x_{22}^2 x_{31} x_{32} x_{42} x_{43} x_{44} + c'_{42} x_{22}^2 x_{31}^2 x_{32}^2 x_{43} x_{44} + c'_{42} x_{22} x_{31}^2 x_{32}^3 x_{33} x_{44})}{A D}$$

$$= \frac{c'_{41} x_{11} x_{22} x_{31} x_{32}^2 x_{42} x_{43} x_{44} + c'_{41} x_{11} x_{31} x_{32}^3 x_{33} x_{42} x_{44} + c'_{41} x_{21} x_{22} x_{31} x_{32}^2 x_{33} x_{42} x_{44}}{D}$$

$$+ \frac{c'_{42} x_{21} x_{22}^2 x_{31} x_{32} x_{42} x_{43} x_{44} + c'_{42} x_{22}^2 x_{31}^2 x_{32}^2 x_{43} x_{44} + c'_{42} x_{22} x_{31}^2 x_{32}^3 x_{33} x_{44}}{D}$$

Putting this expression back in the above expression we get

$$\frac{c'_{41} x_{11} x_{22} x_{31} x_{32}^2 x_{42} x_{43} x_{44} + c'_{41} x_{11} x_{31} x_{32}^3 x_{33} x_{42} x_{44} + c'_{41} x_{21} x_{22} x_{31} x_{32}^2 x_{33} x_{42} x_{44}}{C x_{31} x_{32}^2 x_{33}^2}$$

$$+ \frac{c'_{42} x_{21} x_{22}^2 x_{31} x_{32} x_{42} x_{43} x_{44} + c'_{42} x_{22}^2 x_{31}^2 x_{32}^2 x_{43} x_{44} + c'_{42} x_{22} x_{31}^2 x_{32}^3 x_{33} x_{44}}{C x_{31} x_{32}^2 x_{33}^2}$$

$$+ \frac{x_{44} x_{43} x_{42} x_{22} x_{21} c'_{41}(x_{11} x_{32} x_{42} + x_{21} x_{22} x_{42} + x_{22} x_{31} x_{32})}{C x_{31} x_{32}^2 x_{33}^2}$$

$$= \frac{C(x_{22} x_{31} x_{32} x_{43} x_{44} + x_{31} x_{32}^2 x_{33} x_{44} + x_{21} x_{22} x_{42} x_{43} x_{44})}{C x_{31} x_{32}^2 x_{33}^2} = \varepsilon_3(V_1(x))$$

Next we check 6.

$$\varepsilon_4(e_0^c(V_1(x))) = \frac{\frac{x_{44} x_{43}^2 x_{42}^2 x_{41} c'_{41}}{c c'_{43}} + \frac{x_{33}^2 A^2 x_{43} x_{42}^2 x_{41} c'_{41} c'_4}{c^2 c_4'^2 B^2 c'_{43} c'_{42}} + \frac{x_{33}^2 x_{32}^2 C^2 x_{41} x_{42} c'_4}{c^2 c_4'^2 D^2 c'_{42}} + \frac{x_{31}^2 x_{32}^2 x_{33}^2}{c^2}}{\frac{x_{44}^2 x_{43}^2 x_{42}^2 x_{41} c'_{41}}{c^2 c'_4}}$$

$$= \frac{\frac{x_{44} x_{43}^2 x_{42}^2 x_{41} c'_{41}}{c'_{43}} + \frac{x_{33}^2 A^2 x_{43} x_{42}^2 x_{41} c'_{41}}{B^2 c'_{43} c'_{42} c} + \frac{x_{33}^2 x_{32}^2 C^2 x_{41} x_{42}}{c c'_4 D^2 c'_{42}} + \frac{x_{31}^2 x_{32}^2 x_{33}^2}{c}}{\frac{x_{44}^2 x_{43}^2 x_{42}^2 x_{41} c'_{41}}{c c'_4}} = \varepsilon_4(V_1(x))$$

We obtain $\varepsilon_4(V_1(x))$ by expanding the terms and factoring the expanded rational function.

Now we check 7.

$$\varepsilon_0(e_2^c(V_1(x))) = x_{41} + \frac{(\frac{c_2 x_{11} x_{42}}{c_{21} x_{22}} + \frac{x_{21} x_{42} c\, c_2}{c_{21} x_{32}} + x_{31})^2}{x_{42}} + \frac{(\frac{c\, c_2 x_{21}}{c_{21}} + \frac{x_{11} x_{43}}{x_{33}} + \frac{c_2 x_{11} x_{32}}{c_{21} x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$$

We focus on the terms $\frac{c_2 x_{11} x_{42}}{c_{21} x_{22}} + \frac{x_{21} x_{42} c\, c_2}{c_{21} x_{32}} + x_{31}$ and $\frac{c\, c_2 x_{21}}{c_{21}} + \frac{x_{11} x_{43}}{x_{33}} + \frac{c_2 x_{11} x_{32}}{c_{21} x_{22}}$.

Then we can simplify the first expression as follows:

$$\begin{aligned}
&\frac{c_2 x_{11} x_{42}}{c_{21} x_{22}} + \frac{x_{21} x_{42} c\, c_2}{c_{21} x_{32}} + x_{31} = \frac{c_2 x_{11} x_{32} x_{42} + c\, c_2 x_{21} x_{22} x_{42} + c_{21} x_{22} x_{31} x_{32}}{c_{21} x_{22} x_{32}} \\
&= \frac{(x_{21} x_{22} + x_{11} x_{32}) x_{11} x_{32} x_{42} + x_{21} x_{22} x_{42}(c\, x_{21} x_{22} + c\, x_{11} x_{32}) + x_{31} x_{32} x_{22}(c\, x_{21} x_{22} + x_{11} x_{32})}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{32}} \\
&= \frac{x_{11} x_{21} x_{22} x_{32} x_{42} + x_{11}^2 x_{32}^2 x_{42} + c\, x_{21}^2 x_{22}^2 x_{42} + c\, x_{11} x_{32} x_{21} x_{22} x_{42} + c\, x_{21} x_{22}^2 x_{31} x_{32}}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{32}} \\
&+ \frac{x_{11} x_{22} x_{31} x_{32}^2}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{32}} = \frac{(c\, x_{21} x_{22} + x_{11} x_{32})(x_{21} x_{22} x_{42} + x_{11} x_{32} x_{42} + x_{22} x_{31} x_{32})}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{32}} \\
&= \frac{x_{11} x_{42}}{x_{22}} + \frac{x_{21} x_{42}}{x_{32}} + x_{31}
\end{aligned}$$

And the second expression as follows:

$$\begin{aligned}
&\frac{c\, c_2 x_{21}}{c_{21}} + \frac{x_{11} x_{43}}{x_{33}} + \frac{c_2 x_{11} x_{32}}{c_{21} x_{22}} = \frac{c\, c_2 x_{21} x_{22} x_{33} + x_{11} x_{22} x_{43} c_{21} + c_2 x_{11} x_{32} x_{33}}{c_{21} x_{22} x_{33}} \\
&= \frac{c\, x_{21}^2 x_{22}^2 x_{33} + c\, x_{11} x_{21} x_{22} x_{32} x_{33} + c\, x_{11} x_{21} x_{22}^2 x_{43} + x_{11}^2 x_{22} x_{32} x_{43} + x_{11} x_{21} x_{22} x_{32} x_{33}}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{33}} \\
&+ \frac{x_{11}^2 x_{32}^2 x_{33}}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{33}} = \frac{(c\, x_{21} x_{22} + x_{11} x_{32})(x_{21} x_{22} x_{33} + x_{11} x_{32} x_{33} + x_{11} x_{22} x_{43})}{(c\, x_{21} x_{22} + x_{11} x_{32}) x_{22} x_{33}} \\
&= x_{21} + \frac{x_{11} x_{43}}{x_{22}} + \frac{x_{11} x_{32}}{x_{22}}
\end{aligned}$$

Substituting these expressions into our earlier computation we obtain

$$= x_{41} + \frac{(\frac{x_{11} x_{42}}{x_{22}} + \frac{x_{21} x_{42}}{x_{32}} + x_{31})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11} x_{43}}{x_{33}} + \frac{x_{11} x_{32}}{x_{22}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}} = \varepsilon_0(V_1(x))$$

Next we check 8.

$$\varepsilon_0(e_3^c(V_1(x))) = x_{41} + \frac{(\frac{x_{11} x_{42}}{x_{22}} + \frac{c_{31} x_{21} x_{42}}{x_{32} c_{32}} + \frac{c\, c_3 x_{31}}{c_{32}})^2}{x_{42}} + \frac{(x_{21} + \frac{x_{11} x_{43} c_3}{c_{31} x_{33}} + \frac{x_{11} x_{32} c_{32}}{x_{22} c_{31}})^2}{x_{43}} + \frac{x_{11}^2}{x_{44}}$$

The first and last terms are already equal to the terms in $\varepsilon_0(V_1(x))$, so we focus on the middle two terms. We will expand and simplify $\frac{x_{11}x_{42}}{x_{22}} + \frac{c_{31}x_{21}x_{42}}{x_{32}c_{32}} + \frac{c\,c_3x_{31}}{c_{32}}$ and $x_{21} + \frac{x_{11}x_{43}c_3}{c_{31}x_{33}} + \frac{x_{11}x_{32}c_{32}}{x_{22}c_{31}}$. We can look at only the components of these expressions that contain c_3, c_{31}, or c_{32}. Thus we will rewrite and simplify $\frac{c_{31}x_{21}x_{42}}{x_{32}c_{32}} + \frac{c\,c_3x_{31}}{c_{32}}$ and $\frac{x_{11}x_{43}c_3}{c_{31}x_{33}} + \frac{x_{11}x_{32}c_{32}}{x_{22}c_{31}}$. The first expression simplifies as follows:

$$
\begin{aligned}
&\frac{c_{31}x_{21}x_{42}}{x_{32}c_{32}} + \frac{c\,c_3x_{31}}{c_{32}} = \frac{c_{31}x_{21}x_{42} + c\,c_3x_{31}x_{32}}{x_{32}c_{32}}\\
&= \frac{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{21}x_{42}}{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{32}}\\
&+ \frac{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + c\,x_{44}x_{43}x_{42}x_{22}x_{21})x_{31}x_{32}}{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{32}}\\
&= \frac{c\,x_{44}x_{42}x_{33}x_{32}^2x_{31}x_{21} + x_{44}x_{43}x_{42}x_{21}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}^2x_{21}^2x_{22} + c\,x_{44}x_{33}x_{32}^3x_{31}^2}{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{32}}\\
&+ \frac{c\,x_{44}x_{43}x_{22}x_{32}^2x_{31}^2 + c\,x_{44}x_{43}x_{42}x_{32}x_{31}x_{22}x_{21}}{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{32}}\\
&= \frac{(x_{21}x_{42} + x_{31}x_{32})(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}{(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})x_{32}} = \frac{x_{21}x_{42}}{x_{32}} + x_{31}
\end{aligned}
$$

And the second simplifies as follows:

$$
\begin{aligned}
&\frac{x_{11}x_{43}c_3}{c_{31}x_{33}} + \frac{x_{11}x_{32}c_{32}}{x_{22}c_{31}} = \frac{x_{11}}{x_{33}x_{22}}\frac{x_{43}x_{22}c_3 + x_{32}x_{33}c_{32}}{c_{31}}\\
&= \frac{x_{11}}{x_{33}x_{22}}\cdot\Big(\frac{x_{22}x_{43}(x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}\\
&+ \frac{x_{32}x_{33}(c\,x_{44}x_{33}x_{32}^2x_{31} + c\,x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}\Big)\\
&= \frac{x_{11}}{x_{22}x_{33}}\cdot\Big(\frac{x_{44}x_{43}x_{33}x_{32}^2x_{31}x_{22} + x_{44}x_{43}^2x_{22}^2x_{32}x_{31} + x_{44}x_{43}^2x_{42}x_{22}^2x_{21} + c\,x_{44}x_{33}^2x_{32}^3x_{31}}{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}\\
&+ \frac{c\,x_{44}x_{43}x_{22}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{42}x_{33}x_{32}x_{22}x_{21}}{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}\Big)\\
&= \frac{x_{11}}{x_{22}x_{33}}\cdot\frac{(x_{22}x_{43} + x_{32}x_{33})(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}{(c\,x_{44}x_{33}x_{32}^2x_{31} + x_{44}x_{43}x_{22}x_{32}x_{31} + x_{44}x_{43}x_{42}x_{22}x_{21})}\\
&= \frac{x_{11}x_{43}}{x_{33}} + \frac{x_{32}x_{11}}{x_{22}}
\end{aligned}
$$

Substituting these reduced expressions into $\varepsilon_0(e_3^c)$ we get:

$$x_{41}+\frac{(\frac{x_{11}x_{42}}{x_{22}}+\frac{c_{31}x_{21}x_{42}}{x_{32}c_{32}}+\frac{c\,c_3x_{31}}{c_{32}})^2}{x_{42}}+\frac{(x_{21}+\frac{x_{11}x_{43}c_3}{c_{31}x_{33}}+\frac{x_{11}x_{32}c_{32}}{x_{22}c_{31}})^2}{x_{43}}+\frac{x_{11}^2}{x_{44}}$$
$$=x_{41}+\frac{(\frac{x_{11}x_{42}}{x_{22}}+\frac{x_{21}x_{42}}{x_{32}}+x_{31})^2}{x_{42}}+\frac{(x_{21}+\frac{x_{11}x_{43}}{x_{33}}+\frac{x_{11}x_{32}}{x_{22}})^2}{x_{43}}+\frac{x_{11}^2}{x_{44}}=\varepsilon_0(V_1(x))$$

This proves relation 8. Next we prove 9.

$$\varepsilon_0(e_4^c(V_1(x)))=\frac{c_4'x_{41}}{c_{41}'}+\frac{(\frac{x_{11}x_{42}c_{41}'}{c_{42}'x_{22}}+\frac{x_{21}x_{42}c_{41}'}{x_{32}c_{42}'}+x_{31})^2c_{42}'}{c_{41}'x_{42}}+\frac{(x_{21}+\frac{x_{11}x_{43}c_{42}'}{c_{43}'x_{33}}+\frac{x_{11}x_{32}}{x_{22}})^2c_{43}'}{x_{43}c_{42}'}+\frac{x_{11}^2c\,c_4'}{x_{44}c_{43}'}$$

If we expand and factor these terms, the result is $\varepsilon_0(V_1(x))$.

To prove 10, 11 and 13, we need the following identity: $\overline{\sigma}\circ e_i^c=\overline{e_{4-i}}^c\circ\overline{\sigma}$. It can be shown by direct calculation that this relation is true. As a result, $e_i^c\circ\overline{\sigma^{-1}}=\overline{\sigma^{-1}}\circ\overline{e_{4-i}}^c$. Using this identity, we can prove 10, 11 and 13.

- 10. $e_0^ce_2^d=\overline{\sigma^{-1}}\overline{e_0}^c\overline{\sigma}e_2^d=\overline{\sigma^{-1}}\overline{e_0}^c\overline{e_2}^d\overline{\sigma}=\overline{\sigma^{-1}}\overline{e_2}^d\overline{e_0}^c\overline{\sigma}=e_2^d\overline{\sigma^{-1}}\overline{e_0}^c\overline{\sigma}=e_2^de_0^c$
- 11.$e_0^ce_3^d=\overline{\sigma^{-1}}\overline{e_0}^c\overline{\sigma}e_3^d=\overline{\sigma^{-1}}\overline{e_0}^c\overline{e_1}^d\overline{\sigma}=\overline{\sigma^{-1}}\overline{e_1}^d\overline{e_0}^c\overline{\sigma}=e_3^d\overline{\sigma^{-1}}\overline{e_0}^c\overline{\sigma}=e_3^de_0^c$
- 13.

$$e_1^ce_0^{c^2d}e_1^{cd}e_0^d=e_1^c\overline{\sigma^{-1}}\overline{e_0}^{c^2d}\overline{\sigma}e_1^{cd}\overline{\sigma^{-1}}e\overline{e_0}^d\overline{\sigma}=\overline{\sigma^{-1}}\overline{e_3}^c\overline{e_0}^{c^2d}\overline{\sigma}\overline{\sigma^{-1}}\overline{e_3}^{cd}\overline{e_0}^d\overline{\sigma}$$
$$\overline{\sigma^{-1}}\overline{e_3}^c\overline{e_0}^{c^2d}\overline{e_3}^{cd}\overline{e_0}^d\overline{\sigma}=\overline{\sigma^{-1}}\overline{e_0}^d\overline{e_3}^{cd}\overline{e_0}^{c^2d}\overline{e_3}^c\overline{\sigma}=\overline{\sigma^{-1}}\overline{e_0}^d\overline{e_3}^{cd}\overline{\sigma}\overline{\sigma^{-1}}\overline{e_0}^{c^2d}\overline{e_3}^c\overline{\sigma}$$
$$\overline{\sigma^{-1}}\overline{e_0}^d\overline{\sigma}e_1^{cd}\overline{\sigma^{-1}}\overline{e_0}^{c^2d}\overline{\sigma}e_1^c=e_0^de_1^{cd}e_0^{c^2d}e_1^c$$

The final relation, $e_0^ce_4^d=e_4^de_0^c$ can be shown via direct computation. Thus, since all of the relations hold, $\mathcal{V}_1(x)$ is an affine geometric crystal. □

Now we have constructed the affine geometric crystals for each case. In the next chapter we will apply the ultra-discretization functor to these crystals.

CHAPTER

6

ULTRA-DISCRETIZATION

We will define the ultra-discretization functor following [20]. Let $R := \mathbb{C}(c)$ and define

$$\begin{aligned} v : & R/\{0\} \to \mathbb{Z} \\ & f(c) \mapsto \deg(f(c)) \end{aligned}$$

where deg is the degree of the pole at $c = \infty$ [20]. Notice for $f_1, f_2 \in R/\{0\}$ we have

$$\begin{aligned} v(f_1 f_2) &= v(f_1) + v(f_2) \\ v(\frac{f_1}{f_2}) &= v(f_1) - v(f_2) \end{aligned}$$

We can observe that $f(c)$ is positive if f can be expressed as the ratio of polynomials with positive coefficients. If $f_1, f_2 \in R$ are positive, then

$$v(f_1 + f_2) = \max(v(f_1), v(f_2))$$

Now let $T \cong (\mathbb{C}^{\times})^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^{\times})$ be the lattice of characters of T and $X_*(T) := \mathrm{Hom}(\mathbb{C}^{\times}, T)$ be the lattice of cocharacters of T [20]. A nonzero rational function on T is called positive if it can be written as g/h where g and h are positive linear combinations of characters of T [20].

Definition 6.0.1 ([2])**.** *Let $f : T \to T'$ be a rational mapping between 2 algebraic tori T and T'. We say f is positive if $\chi \circ f$ is positive for any character $\chi : T' \to \mathbb{C}$.*

We let $\mathrm{Mor}^{+}(T, T')$ be the set of positive rational mappings from T to T'.

Lemma 6.0.2 ([2])**.** *For any $f \in Mor^{+}(T_1, T_2)$ and $g \in Mor^{+}(T_2, T_3)$. Then the composition $g \circ f$ is well defined and $g \circ f \in Mor^{+}(T_1, T_3)$.*

As a consequence of this lemma, we can define $\mathcal{T}_{+}$ as the category of algebraic tori over $\mathbb{C}$ with positive rational mappings as morphisms.

Let $f : T \to T'$ be a positive rational mapping from T to T'. We define

$$\begin{aligned} \hat{f} :& X_*(T) \to X_*(T') \\ & \langle \chi, \hat{f}(\xi) \rangle = v(\chi \circ f \circ \xi) \end{aligned}$$

where $\chi \in X_*(T')$ and $\xi \in X_*(T)$ [20, 2].

Lemma 6.0.3 ([2])**.** *For any algebraic tori T_1, T_2, T_3 and positive rational mappings $f \in Mor^{+}(T_1, T_2)$ and $g \in Mor^{+}(T_2, T_3)$, then $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.*

From this lemma we obtain the functor

$$\begin{aligned} \mathcal{UD} :& \mathcal{T}_{+} \to \mathbf{Set} \\ & T \to X_*(T) \\ & (f : T \to T') \mapsto (\hat{f} : X_*(T) \to X_*(T')) \end{aligned}$$

Now that we have built this framework, we can connect these ideas to geometric crystals and Kac-Moody Groups presented in Chapter 5. Let G be a Kac-Moody group and T be its Cartan subgroup.

Definition 6.0.4 ([2])**.** *Let $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ for $i \in I$ be a G-geometric crystal, T' an algebraic torus and $\theta : T' \to X$ be a birational mapping. The mapping θ is called a positive structure on χ if it satisfies:*

1. *for any $i \in I$, $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive*

2. *for any $i \in I$ the rational mapping $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c,t) = \theta^{-1} \circ e_i^c \circ \theta(t)$ are positive*

Let $\theta : T' \to X$ be a positive structure on χ. Then applying $\mathcal{UD}$ to $e_{i,\theta}$, $\gamma_i \circ \theta$, and $\varepsilon_i \circ \theta$ we obtain

$$\begin{aligned}
\tilde{e}_i &= \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T') \to X_*(T') \\
\mathrm{wt}_i &= \mathcal{UD}(\gamma_i \circ \theta) : X_*(T') \to \mathbb{Z} \\
\varepsilon_i &= \mathcal{UD}(\varepsilon_i \circ \theta) : X_*(T') \to \mathbb{Z}
\end{aligned}$$

Thus, $(X_*(T'), \{\tilde{e}_i\}, \{\mathrm{wt}_i\}, \{\varepsilon_i\})$ is a precrystal structure which we will denote by $\mathcal{UD}_{\theta,T'}(\chi)$. This leads to the following theorem:

Theorem 6.0.5 ([2],[30])**.** *For any geometric crystal χ and a positive structure $\theta : T' \to X$, the associated precrystal $\mathcal{UD}_{\theta,T'}(\chi)$ is a crystal.*

Because of this theorem, we see that there is a clearly defined relationship between geometric crystals and crystals. Let $\mathcal{GC}^+$ be the category whose object is the triple (χ, T', θ) where χ is a geometric crystal, T' is an algebraic torus, and θ is a positive structure on χ. The morphism $f : (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)$ is given by the morphism of geometric crystals $\phi : X_1 \to X_2$ such that

$$f := \theta_2^{-1} \circ \phi \circ \theta_1 : T_1' \to T_2'$$

is a rational mapping. Let $\mathcal{CR}$ be the category of crystals. Then

Corollary 6.0.6 ([20])**.** *$\mathcal{UD}$ defines a functor*

$$\begin{aligned}
\mathcal{UD} : \mathcal{GC}^+ &\to \mathcal{CR} \\
(\chi, T', \theta) &\mapsto X_*(T') \\
(f : (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)) &\mapsto (\hat{f} : X_*(T_1') \to X_*(T_2'))
\end{aligned}$$

$\mathcal{UD}$ is called the ultra-discretization functor or tropicalization functor. In our context, the positive structure is $\theta : T' := (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \to \mathcal{V}$ where $x \mapsto V_1(x)$. From the corollary, we get that $\chi = \mathcal{UD}(\mathcal{V}, T', \theta)$ is a crystal which is isomorphic to $\mathbb{Z}^{\frac{n(n+1)}{2}}$ as sets. In this case, the Ultra-Discretization functor maps positive rational functions in the following way:

$$x \times y \to x + y, \ \frac{x}{y} \to x - y, \ x + y \to \max\{x, y\}$$

Then we have the following correspondence between geometric crystals and Kashiwara crystals as presented in [20]:

$$\begin{aligned} \mathrm{wt}_i(x) &= \mathcal{UD}(\gamma_i) \\ \varepsilon_i &= \mathcal{UD}(\varepsilon_i) \\ \tilde{e}_i(x) &= \mathcal{UD}(e_i^c)|_{c=1} \\ \tilde{f}_i(x) &= \mathcal{UD}(e_i^c)|_{c=-1} \end{aligned}$$

In the following sections we will apply the ultra-discretization functor to the positive geometric crystals constructed in Chapter 5.

6.1 Ultra-Discretization for n=2 Case

Explicitly, for $C_2^{(1)}$, if $\mathcal{X} = \mathcal{UD}(\mathcal{V})$ then $\mathcal{X} = \mathbb{Z}^3$ as sets, and $\mathcal{UD}(\mathcal{V})$ is equipped with the following functions:

$$\mathrm{wt}_i(x) = \begin{cases} -2x_{11} & i=0 \\ 2x_{11} - x_{21} - x_{22} & i=1 \\ 2x_{21} + 2x_{22} - 2x_{11} & i=2 \end{cases}$$

$$\varepsilon_i(x) = \begin{cases} \max\{x_{21}, 2x_{11} - x_{22}\} & i=0 \\ x_{22} - x_{11} & i=1 \\ \max\{-x_{22}, 2x_{11} - 2x_{22} - x_{21}\} & i=2 \end{cases}$$

$$e_i^c(x) = \begin{cases} (C_2 + x_{22} - c, x_{11} - c, x_{21} - C_2) & i=0 \\ (x_{22}, c + x_{11}, x_{21}) & i=1 \\ (C_2 + x_{22}, x_{11}, C + x_{21} - C_2) & i=2 \end{cases}$$

where $C_2 = \max\{c + x_{21} + x_{22}, 2x_{11}\} - \max\{x_{21} + x_{22}, 2x_{11}\}$.

When we restrict c to compute $\tilde{e}_k$ and $\tilde{f}_k$ for $k = 0, 1, 2$, we get the following actions:

$$\tilde{f}_0(x) = \begin{cases} (x_{22}+1, x_{11}+1, x_{21}) & \text{if } 2x_{11} - x_{22} \leq x_{21} \\ (x_{22}, x_{11}+1, x_{21}+1) & \text{if } 2x_{11} - x_{22} > x_{21} \end{cases}$$

$$\tilde{f}_1(x) = (x_{22}, x_{11}-1, x_{21})$$

$$\tilde{f}_2(x) = \begin{cases} (x_{22}, x_{11}, x_{21}-1) & \text{if } 2x_{11} - x_{22} \geq x_{21} \\ (x_{22}-1, x_{11}, x_{21}) & \text{if } 2x_{11} - x_{22} < x_{21} \end{cases}$$

$$\tilde{e}_0(x) = \begin{cases} (x_{22}-1, x_{11}-1, x_{21}) & \text{if } 2x_{11} - x_{22} > x_{21} \\ (x_{22}, x_{11}-1, x_{21}-1) & \text{if } 2x_{11} - x_{22} \leq x_{21} \end{cases}$$

$$\tilde{e}_1(x) = (x_{22}, x_{11}+1, x_{21})$$

$$\tilde{e}_2(x) = \begin{cases} (x_{22}, x_{11}, x_{21}+1) & \text{if } 2x_{11} - x_{22} > x_{21} \\ (x_{22}+1, x_{11}, x_{21}) & \text{if } 2x_{11} - x_{22} \leq x_{21} \end{cases}$$

6.2 Ultra-Discretization for n=3 Case

For $C_3^{(1)}$, if $\mathcal{X} = \mathcal{UD}(\mathcal{V})$ then $\mathcal{X} = \mathbb{Z}^6$ as sets, and $\mathcal{UD}(\mathcal{V})$ is equipped with the following functions:

$$\mathrm{wt}_i(x) = \begin{cases} -2x_{11} & i = 0 \\ 2x_{11} - x_{21} - x_{22} & i = 1 \\ 2x_{21} + 2x_{22} - x_{11} - x_{31} - x_{32} - x_{33} & i = 2 \\ 2x_{31} + 2x_{32} + 2x_{33} - 2x_{21} - 2x_{22} & i = 3 \end{cases}$$

$$\varepsilon_i(x) = \begin{cases} \max\{x_{31}, 2x_{21} - x_{32}, x_{21} + x_{11} - x_{22}, 2x_{11} + x_{32} - 2x_{22}, 2x_{11} - x_{33}\} & i = 0 \\ x_{22} - x_{11} & i = 1 \\ \max\{x_{33} - x_{22}, x_{33} + x_{32} + x_{11} - 2x_{22} - x_{21}\} & i = 2 \\ \max\{-x_{33}, 2x_{22} - 2x_{33} - x_{32}, 2x_{22} + 2x_{21} - 2x_{33} - 2x_{32} - x_{31}\} & i = 3 \end{cases}$$

$$e_0^c(x) = (C'_{24} + x_{33} - c - C'_2, x_{22} + \max\{x_{21} + x_{22} + C'_{24}, x_{11} + x_{32} + C'_{21}\} - c - C'_2 - \max\{x_{21} + x_{22}, x_{11} + x_{32}\}, C'_{21} + x_{32} - C'_{24}, x_{11} - c, x_{21} - \max\{x_{21} + x_{22} + C'_{24}, x_{11} + x_{32} + C'_{21}\} + C'_2 + \max\{x_{21} + x_{22}, x_{11} + x_{32}\}, C'_2 + x_{31} - C'_{21})$$

where $C'_2 = \max\{x_{31}, 2\max\{x_{21} + x_{22}, x_{11} + x_{32}\} - 2x_{22} - x_{32}, 2x_{11} - x_{33}\}$,
$C'_{21} = \max\{c + x_{31}, 2\max\{x_{21} + x_{22}, x_{11} + x_{32}\} - 2x_{22} - x_{32}, 2x_{11} - x_{33}\}$ and
$C'_{24} = \max\{c + x_{31}, c + 2\max\{x_{21} + x_{22}, x_{11} + x_{32}\} - 2x_{22} - x_{32}, 2x_{11} - x_{33}\}$.

$$e_1^c(x) = (x_{33}, x_{22}, x_{32}, x_{11} + c, x_{21}, x_{31})$$

$$e_2^c(x) = (x_{33}, C_2 + x_{22}, x_{32}, x_{11}, c + x_{21} - C_2, x_{31})$$

where $C_2 = \max\{c + x_{21} + x_{22}, x_{11} + x_{32}\} - \max\{x_{21} + x_{22}, x_{11} + x_{32}\}$.

$$e_3^c(x) = (C_{31} + x_{33} - C_3, x_{22}, C_{32} + x_{32} - C_{31}, x_{11}, x_{21}, c + C_3 - C_32 + x_{31})$$

where $C_3 = \max\{2x_{32} + x_{31} + x_{33}, 2x_{22} + x_{31} + x_{32}, 2x_{21} + 2x_{22}\}$, $C_{31} = \max\{c + 2x_{32} + x_{31} + x_{33}, 2x_{22} + x_{31} + x_{32}, 2x_{21} + 2x_{22}\}$, and $C_{32} = \max\{c + 2x_{32} + x_{31} + x_{33}, c + 2x_{22} + x_{31} + x_{32}, 2x_{21} + 2x_{22}\}$.

We show the simplification of $e_0^c(x)$ when we set $c = 1$. Plugging in, we get

$$(C'_{24} + x_{33} - 1 - C'_2, x_{22} + \max\{x_{11} + x_{32} + C'_{21}, x_{21} + x_{22} + C'_{24}\} - 1 - C'_2 - \max\{x_{21} + x_{22}, x_{11} + x_{32}\}, C'_{21} + x_{32} - C'_{24}, x_{11} - 1, x_{21} - \max\{x_{21} + x_{22} + C'_{24}, x_{11} + x_{32} + C'_{21}\} + C'_2 + \max\{x_{21} + x_{22}, x_{11} + x_{32}\}, C'_2 + x_{31} - C'_{21})$$

along with $C'_2 = \max\{x_{31}, 2x_{21} - x_{32}, 2x_{11} - 2x_{22} + x_{32}, 2x_{11} - x_{33}\}$, $C'_{21} = \max\{x_{31} + 1, 2x_{21} - x_{32}, 2x_{11} - 2x_{22} + x_{32}, 2x_{11} - x_{33}\}$ and $C'_{24} = \max\{x_{31} + 1, 2x_{21} - x_{32} + 1, 2x_{11} - 2x_{22} + x_{32} + 1, 2x_{11} -$

$x_{33}\}$. Then based on which of the 4 components is maximal, we get 4 cases for $\tilde{e}_0$.

$$\tilde{e}_0(x)=\begin{cases}(x_{33},x_{22},x_{32},x_{11}-1,x_{21}-1,x_{31}-1) & x_{31}>2x_{21}-x_{32},\, x_{31}>2x_{11}-2x_{22}+x_{32},\\ & x_{31}>2x_{11}-x_{33}\\ (x_{33},x_{22},x_{32}-1,x_{11}-1,x_{21}-1,x_{31}) & 2x_{21}-x_{32}\geq x_{31},\\ & 2x_{21}-x_{32}>2x_{11}-2x_{22}+x_{32},\\ & 2x_{21}-x_{32}>2x_{11}-x_{33}\\ (x_{33},x_{22}-1,x_{32}-1,x_{11}-1,x_{21},x_{31}) & 2x_{11}-2x_{22}+x_{32}\geq x_{31},\\ & 2x_{11}-2x_{22}+x_{32}\geq 2x_{21}-x_{32},\\ & 2x_{11}-2x_{22}+x_{32}>2x_{11}-x_{33}\\ (x_{33}-1,x_{22}-1,x_{32},x_{11}-1,x_{21},x_{31}) & 2x_{11}-x_{33}\geq x_{31}, 2x_{11}-x_{33}\geq 2x_{21}-x_{32},\\ & 2x_{11}-x_{33}\geq 2x_{11}-2x_{22}+x_{32}\end{cases}$$

When we restrict c to compute $\tilde{e}_k$ and $\tilde{f}_k$ for $k=0,1,2,3$, we get the following actions (besides $\tilde{e}_0$ which is given above):

$$\tilde{e}_1(x)=(x_{33},x_{22},x_{32},x_{11}+1,x_{21},x_{31})$$

$$\tilde{e}_2(x)=\begin{cases}(x_{33},x_{22}+1,x_{32},x_{11},x_{21},x_{31}) & x_{21}+x_{22}\geq x_{11}+x_{32}\\ (x_{33},x_{22},x_{32},x_{11},x_{21}+1,x_{31}) & x_{21}+x_{22}<x_{11}+x_{32}\end{cases}$$

$$\tilde{e}_3(x)=\begin{cases}(x_{33}+1,x_{22},x_{32},x_{11},x_{21},x_{31}) & 2x_{32}+x_{31}+x_{33}\geq 2x_{22}+x_{31}+x_{32},\\ & 2x_{32}+x_{31}+x_{33}\geq 2x_{22}+2x_{21}\\ (x_{33},x_{22},x_{32}+1,x_{11},x_{21},x_{31}) & 2x_{22}+x_{31}+x_{32}>2x_{32}+x_{31}+x_{33},\\ & 2x_{22}+x_{31}+x_{32}\geq 2x_{22}+2x_{21}\\ (x_{33},x_{22},x_{32},x_{11},x_{21},x_{31}+1) & 2x_{22}+2x_{21}>2x_{32}+x_{31}+x_{33},\\ & 2x_{22}+2x_{21}>2x_{22}+x_{31}+x_{32}\end{cases}$$

$$\tilde{f}_0(x)=\begin{cases}(x_{33},x_{22},x_{32},x_{11}+1,x_{21}+1,x_{31}+1) & x_{31}\geq 2x_{21}-x_{32}, x_{31}\geq 2x_{11}-2x_{22}+x_{32},\\ & x_{31}\geq 2x_{11}-x_{33}\\ (x_{33},x_{22},x_{32}+1,x_{11}+1,x_{21}+1,x_{31}) & 2x_{21}-x_{32}>x_{31}, 2x_{21}-x_{32}\geq 2x_{11}-2x_{22}+x_{32},\\ & 2x_{21}-x_{32}\geq 2x_{11}-x_{33}\\ (x_{33},x_{22}+1,x_{32}+1,x_{11}+1,x_{21},x_{31}) & 2x_{11}-2x_{22}+x_{32}>x_{31},\\ & 2x_{11}-2x_{22}+x_{32}>2x_{21}-x_{32},\\ & 2x_{11}-2x_{22}+x_{32}\geq 2x_{11}-x_{33}\\ (x_{33}+1,x_{22}+1,x_{32},x_{11}+1,x_{21},x_{31}) & 2x_{11}-x_{33}>x_{31}, 2x_{11}-x_{33}>2x_{21}-x_{32},\\ & 2x_{11}-x_{33}>2x_{11}-2x_{22}+x_{32},\end{cases}$$

$$\tilde{f}_1(x)=(x_{33},x_{22},x_{32},x_{11}-1,x_{21},x_{31})$$

$$\tilde{f}_2(x)=\begin{cases}(x_{33},x_{22},x_{32},x_{11},x_{21}-1,x_{31}) & x_{21}+x_{22}\geq x_{11}+x_{32}\\ (x_{33},x_{22}-1,x_{32},x_{11},x_{21},x_{31}) & x_{21}+x_{22}<x_{11}+x_{32}\end{cases}$$

$$\tilde{f}_3(x)=\begin{cases}(x_{33}-1,x_{22},x_{32},x_{11},x_{21},x_{31}) & 2x_{32}+x_{31}+x_{33}>2x_{22}+x_{31}+x_{32},\\ & 2x_{32}+x_{31}+x_{33}>2x_{22}+2x_{21}\\ (x_{33},x_{22},x_{32}-1,x_{11},x_{21},x_{31}) & 2x_{22}+x_{31}+x_{32}\geq 2x_{32}+x_{31}+x_{33},\\ & 2x_{22}+x_{31}+x_{32}>2x_{22}+2x_{21}\\ (x_{33},x_{22},x_{32},x_{11},x_{21},x_{31}-1) & 2x_{22}+x_{21}\geq 2x_{32}+x_{31}+x_{33},\\ & 2x_{22}+x_{21}\geq 2x_{22}+x_{31}+x_{32}\end{cases}$$

6.3 Ultra-Discretization for n=4 Case

For $C_4^{(1)}$, if $\mathcal{X}=\mathcal{UD}(\mathcal{V})$, then $\mathcal{X}=\mathbb{Z}^{10}$ as sets and $\mathcal{UD}(\mathcal{V})$ is equipped with the following functions:

$$\mathrm{wt}_i(x)=\begin{cases}-2x_{11} & i=0\\ 2x_{11}-x_{21}-x_{22} & i=1\\ 2x_{21}+2x_{22}-x_{11}-x_{31}-x_{32}-x_{33} & i=2\\ 2x_{31}+2x_{32}+2x_{33}-x_{21}-x_{22}-x_{41}-x_{42}-x_{43}-x_{44} & i=3\\ 2x_{41}+2x_{42}+2x_{43}+2x_{44}-2x_{31}-2x_{32}-2x_{33} & i=4\end{cases}$$

$$\varepsilon_i(x) = \begin{cases} \max\{x_{41}, 2x_{11} - x_{44}, 2x_{31} - x_{42}, 2x_{11} + x_{42} - 2x_{22}, & i = 0 \\ 2x_{21} + x_{42} - 2x_{32}, 2x_{21} - x_{43}, 2x_{11} + x_{43} - 2x_{33}, 2x_{11} + 2x_{32} - 2x_{22} - x_{43}\} & \\ x_{22} - x_{11} & i = 1 \\ \max\{x_{33} - x_{22}, x_{33} + x_{32} + x_{11} - 2x_{22} - x_{21}\} & i = 2 \\ \max\{x_{44} - x_{33}, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}, & i = 3 \\ x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\} & \\ \max\{-x_{44}, 2x_{33} - x_{43} - 2x_{44}, 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}, & i = 4 \\ 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}\} & \end{cases}$$

For $i = 1$, we compute the ultra-discretization of e_1^c

$$(UD)(e_1^c(x)) = (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11} + c, x_{21}, x_{31}, x_{41})$$

By restricting c to -1 and 1, we get $\tilde{f}_1$ and $\tilde{e}_1$.

$$\tilde{e}_1(x) = (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11} + 1, x_{21}, x_{31}, x_{41})$$
$$\tilde{f}_1(x) = (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11} - 1, x_{21}, x_{31}, x_{41})$$

Now for $i = 2$

$$\mathcal{UD}(e_2^c(x)) = (x_{44}, x_{33}, x_{43}, x_{22} + \max\{x_{22} + x_{21} + c, x_{11} + x_{32}\} - \max\{x_{22} + x_{21}, x_{11} + x_{32}\},$$
$$x_{32}, x_{42}, x_{11}, x_{21} + c + \max\{x_{22} + x_{21}, x_{11} + x_{32}\} - \max\{x_{22} + x_{21} + c, x_{11} + x_{32}\}, x_{31}, x_{41})$$

Again, restricting c we obtain $\tilde{f}_2$ and $\tilde{e}_2$

$$\tilde{e}_2(x) = \begin{cases} (x_{44}, x_{33}, x_{43}, x_{22} + 1, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & x_{21} + x_{22} \geq x_{11} + x_{32} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21} + 1, x_{31}, x_{41}) & x_{21} + x_{22} < x_{11} + x_{32} \end{cases}$$

$$\tilde{f}_2(x) = \begin{cases} (x_{44}, x_{33}, x_{43}, x_{22} - 1, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & x_{21} + x_{22} > x_{11} + x_{32} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21} - 1, x_{31}, x_{41}) & x_{21} + x_{22} \leq x_{11} + x_{32} \end{cases}$$

For $i=3$, we have

$$\mathcal{UD}(e_3^c(x)) = (x_{44}, x_{33} + \max\{x_{44} - x_{33} + c, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\} - \max\{x_{44} - x_{33}, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\}, x_{43}, x_{22}, x_{32} + \max\{x_{44} - x_{33} + c, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32} + c, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\} - \max\{x_{44} - x_{33} + c, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\}, x_{42}, x_{11}, x_{21}, x_{31} + c + \max\{x_{44} - x_{33}, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\} - \max\{x_{44} - x_{33} + c, x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32} + c, x_{44} + x_{43} + x_{42} + x_{22} + x_{21} - 2x_{33} - 2x_{32} - x_{31}\}, x_{41})$$

Restricting c we obtain $\tilde{f}_3$ and $\tilde{e}_3$

$$\tilde{e}_3(x) = \begin{cases} (x_{44}, x_{33}+1, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & 0 \geq x_{43} + x_{22} - x_{33} - x_{32}, \\ & 0 \geq x_{43} + x_{42} + x_{22} + x_{21} \\ & -x_{33} - 2x_{32} - x_{31} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}+1, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & 0 < x_{43} + x_{22} - x_{33} - x_{32}, \\ & 0 \geq x_{42} + x_{21} - x_{32} - x_{31} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}+1, x_{41}) & 0 < x_{43} + x_{42} + x_{22} + x_{21} \\ & -x_{33} - 2x_{32} - x_{31} \\ & 0 < x_{42} + x_{21} - x_{32} - x_{31} \end{cases}$$

$$\tilde{f}_3(x) = \begin{cases} (x_{44}, x_{33}-1, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & 0 > x_{43} + x_{22} - x_{33} - x_{32}, \\ & 0 > x_{43} + x_{42} + x_{22} + x_{21} \\ & -x_{33} - 2x_{32} - x_{31} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}-1, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & 0 \leq x_{43} + x_{22} - x_{33} - x_{32}, \\ & 0 > x_{42} + x_{21} - x_{32} - x_{31} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}-1, x_{41}) & 0 \leq x_{43} + x_{42} + x_{22} + x_{21} \\ & -x_{33} - 2x_{32} - x_{31} \\ & 0 \leq x_{42} + x_{21} - x_{32} - x_{31} \end{cases}$$

For $i=4$, let

$$\begin{aligned}
\mathcal{UD}(c_4) = \tilde{c}_4 = \max\{&-x_{44}, 2x_{33}-2x_{44}-x_{43}, 2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42},\\
&2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}\}\\
\mathcal{UD}(c_{41}) = \tilde{c}_{41} = \max\{&c-x_{44}, 2x_{33}-2x_{44}-x_{43}, 2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42},\\
&2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}\}\\
\mathcal{UD}(c_{42}) = \tilde{c}_{42} = \max\{&c-x_{44}, c+2x_{33}-2x_{44}-x_{43}, 2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42},\\
&2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}\}\\
\mathcal{UD}(c_{43}) = \tilde{c}_{43} = \max\{&c-x_{44}, c+2x_{33}-2x_{44}-x_{43}, c+2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42},\\
&2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}\}
\end{aligned}$$

Now we have

$$\mathcal{UD}(e_4^c(x)) = (\tilde{c}_{41}-\tilde{c}_4+x_{44}, x_{33}, \tilde{c}_{42}-\tilde{c}_{41}+x_{43}, x_{22}, x_{32}, \tilde{c}_{43}-\tilde{c}_{42}+x_{42}, x_{11}, x_{21}, x_{31}, c+\tilde{c}_4-\tilde{c}_{43}+x_{41})$$

Let $L=-x_{44}$, $M=2x_{33}-2x_{44}-x_{43}$, $N=2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42}$, and $P=2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}$. Restricting c we obtain $\tilde{e}_4$ and $\tilde{f}_4$

$$\tilde{e}_4(x) = \begin{cases}
(x_{44}+1, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & L \geq M, L \geq N, L \geq P\\
(x_{44}, x_{33}, x_{43}+1, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & M > L, M \geq N, M \geq P\\
(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}+1, x_{11}, x_{21}, x_{31}, x_{41}) & N > L, N > M, N \geq P\\
(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}+1) & P > L, P > M, P > N
\end{cases}$$

$$\tilde{f}_4(x) = \begin{cases}
(x_{44}-1, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & L > M, L > N, L > P\\
(x_{44}, x_{33}, x_{43}-1, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}) & M \geq L, M > N, M > P\\
(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}-1, x_{11}, x_{21}, x_{31}, x_{41}) & N \geq L, N \geq M, N > P\\
(x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}, x_{21}, x_{31}, x_{41}-1) & P \geq L, P \geq M, P \geq N
\end{cases}$$

Finally, we need to find $\tilde{e}_0(x)$ and $\tilde{f}_0(x)$. Let $A=2x_{11}-2x_{22}+x_{42}$, $B=2x_{21}-2x_{32}+x_{42}$, $C=2x_{31}-x_{42}$, $X=2x_{21}-x_{43}$, $Y=2x_{11}-2x_{33}+x_{43}$, and $Z=2x_{11}+2x_{32}-2x_{22}-x_{43}$. By simplifying the inequalities $A>B$ and $X>Z$, we can see that these inequalities are mutually exclusive.

$$
\begin{aligned}
\mathcal{UD}(e_0^c(x_{44}) &= x_{44} - c + \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{33}) &= x_{33} - c - \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad + \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{43}) &= x_{43} + \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{22}) &= x_{22} - c - \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad + \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{32})) &= x_{32} + \max\{x_{41} + c, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{42})) &= x_{42} + \max\{x_{41} + c, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{11})) &= x_{11} - c \\
\mathcal{UD}(e_0^c(x_{21})) &= x_{21} + \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\} + c, \max\{X, Y, Z\} + c, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{31})) &= x_{31} + \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
\mathcal{UD}(e_0^c(x_{41})) &= x_{41} + \max\{x_{41}, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\} \\
&\quad - \max\{x_{41} + c, \max\{A, B, C\}, \max\{X, Y, Z\}, 2x_{11} - x_{44}\}
\end{aligned}
$$

From this, we can consider the different cases that occur based off which elements are maximal. Coupled with the fact that certain options are mutually exclusive, 8 cases for $\tilde{e}_0$

and $\tilde{f}_0$ are possible. These are as follows:

$$\tilde{e}_0(x) = \begin{cases} (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}-1, x_{21}-1, x_{31}-1, x_{41}-1) & x_{41} \geq A, x_{41} \geq B, x_{41} \geq C, \\ & x_{41} \geq X, x_{41} \geq Y, x_{41} \geq Z, \\ & x_{41} \geq 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}-1, x_{42}-1, x_{11}-1, x_{21}-1, x_{31}, x_{41}) & B > x_{41}, B > A, B \geq C, \\ & B \geq X, B \geq Y, B \geq Z \\ & B \geq 2x_{11} - x_{44}, X \geq Y, \\ & X \geq Z \\ (x_{44}, x_{33}, x_{43}, x_{22}-1, x_{32}-1, x_{42}-1, x_{11}-1, x_{21}, x_{31}, x_{41}) & A > x_{41}, A \geq B, C, X, Y, Z, \\ & A \geq 2x_{11} - x_{44} \\ & Y > X \text{ or } Z \geq X \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}-1, x_{11}-1, x_{21}-1, x_{31}-1, x_{41}) & C > x_{41}, A, B, \\ & C \geq X, Y, Z, 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}-1, x_{22}, x_{32}-1, x_{42}, x_{11}-1, x_{21}-1, x_{31}, x_{41}) & X > x_{41}, A, B, C \\ & X \geq Y, Z, 2x_{11} - x_{44} \\ & B, C \geq A \\ (x_{44}, x_{33}-1, x_{43}-1, x_{22}-1, x_{32}, x_{42}, x_{11}-1, x_{21}, x_{31}, x_{41}) & Y > x_{41}, A, B, C, X \\ & Y \geq Z, 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}-1, x_{22}-1, x_{32}-1, x_{42}, x_{11}-1, x_{21}, x_{31}, x_{41}) & Z > x_{41}, A, B, C, X, Y \\ & Z \geq 2x_{11} - x_{44} \\ & A > B \text{ or } C \geq B \\ (x_{44}-1, x_{33}-1, x_{43}, x_{22}-1, x_{32}, x_{42}, x_{11}-1, x_{21}, x_{31}, x_{41}) & 2x_{11} - x_{44} > x_{41}, A, B, C, \\ & 2x_{11} - x_{44} > X, Y, Z \end{cases}$$

$$\tilde{f}_0(x) = \begin{cases} (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}, x_{11}+1, x_{21}+1, x_{31}+1, x_{41}+1) & x_{41} > A, B, C, X, Y, Z, \\ & x_{41} > 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}+1, x_{42}+1, x_{11}+1, x_{21}+1, x_{31}, x_{41}) & B \geq x_{41}, A, B > C, X, Y, Z \\ & B > 2x_{11} - x_{44}, X > Y, Z \\ (x_{44}, x_{33}, x_{43}, x_{22}+1, x_{32}+1, x_{42}+1, x_{11}+1, x_{21}, x_{31}, x_{41}) & A \geq x_{41}, A > B, C, X, Y, Z, \\ & A > 2x_{11} - x_{44} \\ & Y \geq X \text{ or } Z > X \\ (x_{44}, x_{33}, x_{43}, x_{22}, x_{32}, x_{42}+1, x_{11}+1, x_{21}+1, x_{31}+1, x_{41}) & C \geq x_{41}, A, B, \\ & C > X, Y, Z, 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}+1, x_{22}, x_{32}+1, x_{42}, x_{11}+1, x_{21}+1, x_{31}, x_{41}) & X \geq x_{41}, A, B, C \\ & X > Y, Z, 2x_{11} - x_{44} \\ & B, C > A \\ (x_{44}, x_{33}+1, x_{43}+1, x_{22}+1, x_{32}, x_{42}, x_{11}+1, x_{21}, x_{31}, x_{41}) & Y \geq x_{41}, A, B, C, X \\ & Y > Z, 2x_{11} - x_{44} \\ (x_{44}, x_{33}, x_{43}+1, x_{22}+1, x_{32}+1, x_{42}, x_{11}+1, x_{21}, x_{31}, x_{41}) & Z \geq x_{41}, A, B, C, X, Y \\ & Z > 2x_{11} - x_{44} \\ & A \geq B \text{ or } C > B \\ (x_{44}+1, x_{33}+1, x_{43}, x_{22}+1, x_{32}, x_{42}, x_{11}+1, x_{21}, x_{31}, x_{41}) & 2x_{11} - x_{44} \geq x_{41}, A, B, C, \\ & 2x_{11} - x_{44} \geq X, Y, Z \end{cases}$$

CHAPTER

7

ISOMORPHISM

7.1 Conjecture

In [20], Kashiwara, Nakashima and Okado conjectured the following:

Conjecture 7.1.1 (Kashiwara, Nakashima, Okado, 2005)**.** *For any $i \in I \backslash \{0\}$, there exist a unique variety X endowed with a positive $\mathfrak{g}$-geometric crystal structure such that the ultra-discretization of X is isomorphic to the crystal B^{∞} of the Langlands dual $\mathfrak{g}^L$.*

In Chapter 4 we have constructed the limit of the coherent family of perfect crystals $B(l\Lambda_n)$ corresponding to the Langlands dual $D_{n+1}^{(2)}$, $B^{n,\infty}$, for $n = 2,3,4$. In Chapter 5, we constructed the positive geometric crystals corresponding to the affine algebra $C_n^{(1)}$ for each of these cases, and in Chapter 6 we computed the Ultra-Discretization of the geometric crystals for each case. To prove Conjecture 7.1.1 for each of these cases, we need to find an isomorphism. In the following sections we construct these maps and prove they are isomorphisms for each case $n = 2,3,4$.

7.2 Isomorphsim for n=2 Case

Theorem 7.2.1. *Let* $\Omega : \mathcal{UD}(\mathcal{V}) \to B^{2,\infty}$ *be given by*

$$\begin{aligned} b_{11} = x_{11},\ \ b_{12} = x_{22} - x_{11} \\ b_{13} = -x_{22},\ \ b_{22} = x_{21} \\ b_{23} = x_{11} - x_{21},\ \ b_{24} = -x_{11} \end{aligned}$$

and $\Omega^{-1} : B^{2,\infty} \to \mathcal{UD}(\mathcal{V})$ *be given by*

$$x_{11} = b_{11},\ x_{21} = b_{22},\ x_{22} = b_{11} + b_{12}$$

Ω *is an isomorphism of crystals.*

Proof. Clearly we see that the map is bijective. We need to prove the following conditions to show that Ω is an isomorphism:

1. $\Omega(\tilde{f}_k(b) = \tilde{f}_k\Omega(b)$
2. $\Omega(\tilde{e}_k(b) = \tilde{e}_k\Omega(b)$
3. $\mathrm{wt}_k(\Omega(b)) = \mathrm{wt}_k(b)$
4. $\varepsilon_k(\Omega(b) = \varepsilon_k(b)$

Clearly if these hold, then $\varphi_k(\Omega(b)) = \varepsilon_k(\Omega(b)) + \mathrm{wt}_k(\Omega(b)) = \varepsilon_k(b) + \mathrm{wt}_k(b) = \varphi_k(b)$, which will prove the isomorphism. Now we check the above conditions for $k = 0, 1, 2$.

1. For $k = 0$

$$b'_{11} = b_{11} + 1, b'_{24} = b_{24} - 1 \text{ and } \begin{cases} b'_{23} = b_{23} + 1, b'_{13} = b_{13} - 1 & b_{23} \geq b_{12}, b_{23} > 0 \\ b'_{22} = b_{22} + 1, b'_{12} = b_{12} - 1 & b_{23} < b_{12}, b_{22} > 0 \end{cases}$$

Applying the isomorphism we get:

$$b'_{11} = x_{11} + 1, b'_{24} = -x_{11} - 1 \text{ and } \begin{cases} b'_{23} = x_{11} - x_{21} + 1, b'_{13} = -x_{22} - 1 & 2x_{11} - x_{22} \leq x_{21} \\ b'_{22} = x_{21} + 1, b'_{12} = x_{22} - x_{11} - 1 & 2x_{11} - x_{22} > x_{21} \end{cases}$$

This is equivalent to

$$\begin{cases} (x_{22}+1, x_{11}+1, x_{21}) & 2x_{11}-x_{22} \le x_{21} \\ (x_{22}, x_{11}+1, x_{21}+1) & 2x_{11}-x_{22} > x_{21} \end{cases}$$

For $k=1$, $\tilde{f}_1(b)=b'$ such that $b'_{11}=b_{11}-1, b'_{12}=b_{12}+1, b'_{23}=b_{23}-1, b'_{24}=b_{24}+1$ $b_{23}, b_{11}>0$. Applying the isomorphism, we get $b'_{11}=x_{11}-1, b'_{12}=x_{22}-x_{11}+1, b'_{23}=x_{11}-x_{21}-1, b'_{24}=-x_{11}+1$. This is equivalent to $\tilde{f}_1(x)=(x_{22}, x_{11}-1, x_{21})$.

For $k=2$, $\tilde{f}_2(b)=b'$ such that

$$\begin{cases} b'_{13}=b_{23}+1, b'_{12}=b_{12}-1 & b_{12}>b_{23}, b_{12}>0 \\ b'_{23}=b_{23}+1, b'_{22}=b_{22}-1 & b_{12}\le b_{23}, b_{22}>0 \end{cases}$$

Applying the isomorphism, we get

$$\begin{cases} b'_{13}=-x_{22}+1, b'_{12}=x_{22}-x_{11}-1 & 2x_{11}-x_{22}<x_{21} \\ b'_{23}=x_{11}-x_{21}+1, b'_{22}=x_{21}-1 & 2x_{11}-x_{22}\ge x_{21} \end{cases}$$

This is equivalent to the action we get from the ultra-discretized geometric crystal.

2. For $k=0$

$$b'_{11}=b_{11}-1, b'_{24}=b_{24}+1 \text{ and } \begin{cases} b'_{23}=b_{23}-1, b'_{13}=b_{13}+1 & b_{23}>b_{12}, b_{23}>0 \\ b'_{22}=b_{22}-1, b'_{12}=b_{12}+1 & b_{23}\le b_{12}, b_{22}>0 \end{cases}$$

Applying isomorphism we get:

$$b'_{11}=x_{11}-1, b'_{24}=-x_{11}+1 \text{ and } \begin{cases} b'_{23}=x_{11}-x_{21}-1, b'_{13}=-x_{22}+1 & 2x_{11}-x_{22}>x_{21} \\ b'_{22}=x_{21}-1, b'_{12}=x_{22}-x_{11}+1 & 2x_{11}-x_{22}\le x_{21} \end{cases}$$

This is equivalent to

$$\begin{cases} (x_{22}-1, x_{11}-1, x_{21}) & 2x_{11}-x_{22} > x_{21} \\ (x_{22}, x_{11}-1, x_{21}-1) & 2x_{11}-x_{22} \le x_{21} \end{cases}$$

For $k=1$, $\tilde{e}_1(b)=b'$ such that $b'_{11}=b_{11}+1, b'_{12}=b_{12}-1, b'_{23}=b_{23}+1, and\, b'_{24}=b_{24}-1$ with $b_{23}, b_{11}>0$. Applying the isomorphism, we get $b'_{11}=x_{11}+1, b'_{12}=x_{22}-x_{11}-1, b'_{23}=x_{11}-x_{21}+1, b'_{24}=-x_{11}-1$. This is equivalent to $\tilde{f}_1(x)=(x_{22}, x_{11}+1, x_{21})$.

For $k=2$, $\tilde{e}_2(b)=b'$ such that

$$\begin{cases} b'_{13}=b_{23}-1, b'_{12}=b_{12}+1 & b_{12}\geq b_{23}, b_{12}>0 \\ b'_{23}=b_{23}-1, b'_{22}=b_{22}+1 & b_{12}<b_{23}, b_{22}>0 \end{cases}$$

Applying the isomorphism, we get

$$\begin{cases} b'_{13}=-x_{22}-1, b'_{12}=x_{22}-x_{11}+1 & 2x_{11}-x_{22}\geq x_{21} \\ b'_{23}=x_{11}-x_{21}-1, b'_{22}=x_{21}+1 & 2x_{11}-x_{22}<x_{21} \end{cases}$$

This is equivalent to the action we get from the ultra-discretized geometric crystal.

3.

$$\Omega(\varepsilon_1(b))=\Omega(b_{12})=x_{22}-x_{11}$$

$$\varepsilon_2(b)=\begin{cases} b_{13} & b_{23}<b_{12} \\ b_{13}+b_{23}-b_{12} & b_{12}\leq b_{23} \end{cases}=\begin{cases} -x_{22} & 2x_{11}-x_{22}<x_{21} \\ 2x_{11}-2x_{22}-2x_{21} & 2x_{11}-x_{22}\geq x_{21} \end{cases}=\Omega(\varepsilon_2(b))$$

$$\varepsilon_0(b)=\begin{cases} -b_{24}-b_{12} & b_{23}>b_{12} \\ -b_{24}-b_{23} & b_{23}\leq b_{12} \end{cases}=\begin{cases} 2x_{11}-x_{22} & 2x_{11}-x_{22}>x_{21} \\ x_{21} & 2x_{11}-x_{22}\leq x_{21} \end{cases}=\Omega(\varepsilon_0(b))$$

4. Finally we check $\mathrm{wt}_k(b)$.

$$\Omega(\mathrm{wt}_0(b))=\Omega(b_{24}-b_{11})=-2x_{11}$$
$$\Omega(\mathrm{wt}_1(b))=\Omega(b_{23}-b_{12})=2x_{11}-x_{22}-x_{21}$$
$$\Omega(\mathrm{wt}_2(b))=\Omega(b_{12}+b_{22}-b_{13}-b_{23})=2x_{22}+2x_{21}-2x_{11}$$

Since all 4 conditions hold, Ω is an isomorphism. □

7.3 Isomorphism for n=3 Case

Theorem 7.3.1. *Let* $\Omega : \mathcal{UD}(\mathcal{V}) \to B^{3,\infty}$ *be given by*

$$b_{11} = x_{11},\ \ b_{12} = x_{22} - x_{11}$$
$$b_{13} = -x_{22} + x_{33},\ \ b_{14} = -x_{33}$$
$$b_{22} = x_{21},\ \ b_{23} = -x_{21} + x_{32}$$
$$b_{24} = x_{22} - x_{32},\ \ b_{25} = -x_{22}$$
$$b_{33} = x_{31},\ \ b_{34} = x_{21} - x_{31}$$
$$b_{35} = x_{11} - x_{21},\ \ b_{36} = -x_{11}$$

and $\Omega^{-1} : B^{3,\infty} \to \mathcal{UD}(\mathcal{V})$ *be given by*

$$x_{11} = b_{11},\ x_{21} = b_{22},\ x_{22} = b_{11} + b_{12},\ x_{31} = b_{33},\ x_{32} = b_{22} + b_{23},\ x_{33} = -b_{14}$$

Ω *is an isomorphism of crystals.*

Proof. Clearly we see that the map is bijective. We need to prove the following conditions to show that Ω is an isomorphism:

1. $\Omega(\tilde{f}_k(b) = \tilde{f}_k\Omega(b)$
2. $\Omega(\tilde{e}_k(b) = \tilde{e}_k\Omega(b)$
3. $\mathrm{wt}_k(\Omega(b)) = \mathrm{wt}_k(b)$
4. $\varepsilon_k(\Omega(b) = \varepsilon_k(b)$

Clearly if these hold, then $\varphi_k(\Omega(b)) = \varepsilon_k(\Omega(b)) + \mathrm{wt}_k(\Omega(b)) = \varepsilon_k(b) + \mathrm{wt}_k(b) = \varphi_k(b)$. Now we check the above conditions for each $k = 0, 1, 2, 3$. First we check 1. There are 4 cases for $\tilde{f}_0(b)$.

1. $b'_{11} = b_{11}+1,\ b'_{36} = b_{36}-1,\ b'_{12} = b_{12}-1,\ b'_{22} = b_{22}+1,\ b'_{23} = b_{23}-1,\ b'_{33} = b_{33}+1$ if $b_{23} > b_{34},\ b_{12}+b_{13} > b_{34}+b_{35},\ 2b_{12} > b_{23}+b_{24}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11}+1$, $x'_{21} = x_{21}+1$ and $x'_{31} = x_{31}+1$. The inequalities give us the following: $b_{23} > b_{34} \iff x_{31} > 2x_{21} - x_{32}$, $b_{12} + b_{13} > b_{34} + b_{35} \iff x_{31} > 2x_{11} - x_{33}$ and $2b_{12} > b_{23} + b_{34} \iff x_{31} > 2x_{11} - 2x_{22} + x_{32}$.

2. $b'_{11} = b_{11}+1,\ b'_{36} = b_{36}-1,\ b'_{12} = b_{12}-1,\ b'_{22} = b_{22}+1,\ b'_{24} = b_{24}-1,\ b'_{34} = b_{34}+1$ if $b_{23} \le b_{34},\ b_{12}+b_{13} > b_{23}+b_{35},\ b_{12}+b_{24} > b_{23}+b_{35}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11}+1$, $x'_{21} = x_{21}+1$ and $x'_{32} = x_{32}+1$. The inequalities give us the following: $b_{23} \le b_{34} \iff x_{31} \le 2x_{21} - x_{32}$, $b_{12}+b_{13} > b_{23}+b_{35} \iff 2x_{21}-x_{32} > 2x_{11}-x_{33}$ and $b_{12}+b_{24} > b_{23}+b_{35} \iff 2x_{21}-x_{32} > 2x_{11}-2x_{22}+x_{32}$.

3. $b'_{11} = b_{11}+1,\ b'_{36} = b_{36}-1,\ b'_{13} = b_{13}-1,\ b'_{23} = b_{23}+1,\ b'_{25} = b_{25}-1,\ b'_{35} = b_{35}+1$ if $b_{13} > b_{24},\ b_{12}+b_{24} \le b_{34}+b_{35},\ b_{12}+b_{24} \le b_{23}+b_{35}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11}+1$, $x'_{22} = x_{22}+1$ and $x'_{32} = x_{32}+1$. The inequalities give us the following: $b_{13} > b_{24} \iff 2x_{11}-2x_{22}+x_{32} > 2x_{11}-x_{33}$, $b_{12}+b_{24} \le b_{34}+b_{35} \iff 2x_{11}-2x_{22}+x_{32} \ge x_{31}$ and $b_{12}+b_{24} \le b_{23}+b_{35} \iff 2x_{21}-x_{32} \le 2x_{11}-2x_{22}+x_{32}$.

4. $b'_{11} = b_{11}+1,\ b'_{36} = b_{36}-1,\ b'_{14} = b_{14}-1,\ b'_{24} = b_{24}+1,\ b'_{25} = b_{25}-1,\ b'_{35} = b_{35}+1$ if $b_{13} \le b_{24},\ b_{12}+b_{13} \le b_{34}+b_{35},\ 2b_{35} \ge b_{13}+b_{24}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11}+1$, $x'_{22} = x_{22}+1$ and $x'_{33} = x_{33}+1$. The inequalities give us the following: $b_{13} \le b_{24} \iff 2x_{11}-2x_{22}+x_{32} \le 2x_{11}-x_{33}$, $b_{12}+b_{13} \le b_{34}+b_{35} \iff 2x_{11}-x_{33} \ge x_{31}$ and $2b_{35} \ge b_{13}+b_{24} \iff 2x_{11}-x_{33} \ge 2x_{21}-x_{32}$.

Next we look at $\tilde{f}_1(b)$. We have $b'_{11} = b_{11}-1,\ b'_{12} = b_{12}+1,\ b'_{35} = b_{35}-1,\ b'_{36} = b_{36}+1$. This gives $x'_{11} = x_{11}-1$. Every other x_{ij} stays the same. Next we consider the two cases of $\tilde{f}_2(b)$.

1. $b'_{12} = b_{12}-1,\ b'_{13} = b_{13}+1,\ b'_{24} = b_{24}-1,\ b'_{25} = b_{25}+1$ if $b_{12}+b_{24} \ge b_{23}+b_{35}$. Applying the isomorphism, we have $x_{22}-1$ all others remaining the same if $x_{21}+x_{22} \ge x_{11}+x_{32}$.

2. $b'_{22} = b_{22}-1,\ b'_{23} = b_{23}+1,\ b'_{34} = b_{34}-1,\ b'_{35} = b_{35}+1$ if $b_{12}+b_{24} < b_{23}+b_{35}$. Applying the isomorphism, we have $x_{22}-1$ all others remaining the same if $x_{21}+x_{22} < x_{11}+x_{32}$.

Finally we consider the three cases of $\tilde{f}_3(b)$.

1. $b'_{13} = b_{13}-1,\ b'_{14} = b_{14}+1$ if $b_{13}+b_{23} > b_{24}+b_{34}$ and $b_{13} > b_{24}$. Applying the isomorphism, we get $x'_{33} = x_{33}-1$ and everything else stays the same. The inequalities give us: $b_{13} > b_{24} \iff x_{32}+x_{33} > 2x_{22}$ and $b_{13}+b_{23} > b_{24}+b_{34} \iff x_{31}+2x_{32}+x_{33} > 2x_{21}+2x_{22}$.

2. $b'_{23} = b_{23}-1,\ b'_{24} = b_{24}+1$ if $b_{23} > b_{34}$ and $b_{13} \le b_{24}$. Applying the isomorphism, we get $x'_{32} = x_{32}-1$ and everything else stays the same. The inequalities give us: $b_{13} \le b_{24} \iff x_{32}+x_{33} \le 2x_{22}$ and $b_{23} > b_{34} \iff 2x_{21} < x_{31}+x_{32}$.

3. $b'_{33} = b_{33} - 1,\ b'_{34} = b_{34} + 1$ if $b_{13} + b_{23} \leq b_{24} + b_{34}$ and $b_{13} \leq b_{24}$. Applying the isomorphism, we get $x'_{33} = x_{33} - 1$ and everything else stays the same. The inequalities give us: $b_{23} \leq b_{34} \Longleftrightarrow x_{32} + x_{33} \leq 2x_{22}$ and $b_{13} + b_{23} \leq b_{24} + b_{34} \Longleftrightarrow x_{31} + 2x_{32} + x_{33} \leq 2x_{21} + 2x_{22}$.

This proves that $\Omega(\tilde{f}_k(b) = \tilde{f}_k\Omega(b)$.

Now we consider 2. There are 4 cases for $\tilde{e}_0(b)$.

1. $b'_{11} = b_{11} - 1,\ b'_{36} = b_{36} + 1,\ b'_{12} = b_{12} + 1,\ b'_{22} = b_{22} - 1,\ b'_{23} = b_{23} + 1,\ b'_{33} = b_{33} - 1$ if $b_{23} \geq b_{34},\ b_{12} + b_{13} \geq b_{34} + b_{35},\ 2b_{12} \geq b_{23} + b_{34}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11} - 1$, $x'_{21} = x_{21} - 1$ and $x'_{31} = x_{31} - 1$. The inequalities give us the following: $b_{23} > b_{34} \Longleftrightarrow x_{31} > 2x_{21} - x_{32}$, $b_{12} + b_{13} > b_{34} + b_{35} \Longleftrightarrow x_{31} > 2x_{11} - x_{33}$ and $2b_{12} \geq b_{23} + b_{34} \Longleftrightarrow x_{31} \geq 2x_{11} - 2x_{22} + x_{32}$. The final inequality repeats these conditions.

2. $b'_{11} = b_{11} - 1,\ b'_{36} = b_{36} + 1,\ b'_{12} = b_{12} + 1,\ b'_{22} = b_{22} - 1,\ b'_{24} = b_{24} + 1,\ b'_{34} = b_{34} - 1$ if $b_{34} > b_{23},\ b_{12} + b_{24} \geq b_{23} + b_{35},\ b_{12} + b_{13} \geq b_{23} + b_{35}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11} - 1$, $x'_{21} = x_{21} - 1$ and $x'_{32} = x_{32} - 1$. The inequalities give us the following: $b_{12} + b_{24} \geq b_{23} + b_{35} \Longleftrightarrow 2x_{21} - x_{32} \geq 2x_{11} - 2x_{22} + x_{32}$, $b_{12} + b_{13} \geq b_{23} + b_{35} \Longleftrightarrow 2x_{21} - x_{32} \geq 2x_{11} - x_{33}$ and $b_{34} > b_{23} \Longleftrightarrow 2x_{21} - x_{32} > x_{31}$.

3. $b'_{11} = b_{11} - 1,\ b'_{36} = b_{36} + 1,\ b'_{13} = b_{13} + 1,\ b'_{23} = b_{23} - 1,\ b'_{25} = b_{25} + 1,\ b'_{35} = b_{35} - 1$ if $b_{13} \geq b_{24},\ b_{12} + b_{24} < b_{34} + b_{35},\ b_{12} + b_{24} < b_{23} + b_{35}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11} - 1$, $x'_{22} = x_{22} - 1$ and $x'_{32} = x_{32} - 1$. The inequalities give us the following: $b_{13} \geq b_{24} \Longleftrightarrow 2x_{11} - 2x_{22} + x_{32} \geq 2x_{11} - x_{33}$, $b_{12} + b_{24} < b_{34} + b_{35} \Longleftrightarrow 2x_{11} - 2x_{22} + x_{32} > 2x_{21} - x_{32}$ and $b_{12} + b_{24} > b_{34} + b_{35} \Longleftrightarrow x_{31} < 2x_{11} - 2x_{22} + x_{32}$.

4. $b'_{11} = b_{11} - 1,\ b'_{36} = b_{36} + 1,\ b'_{14} = b_{14} + 1,\ b'_{24} = b_{24} - 1,\ b'_{25} = b_{25} + 1,\ b'_{35} = b_{35} - 1$ if $b_{13} < b_{24},\ b_{12} + b_{13} < b_{34} + b_{35},\ b_{13} + b_{24} < 2b_{35}$. Applying the isomorphism, we get that all x_{ij} stay the same except $x'_{11} = x_{11} - 1$, $x'_{22} = x_{22} - 1$ and $x'_{33} = x_{33} - 1$. The inequalities give us the following: $b_{13} < b_{24} \Longleftrightarrow 2x_{11} - 2x_{22} + x_{32} < 2x_{11} - x_{33}$, $b_{12} + b_{13} < b_{34} + b_{35} \Longleftrightarrow 2x_{11} - x_{33} > x_{31}$ and $b_{24} \leq b_{35} \Longleftrightarrow 2x_{11} - 2x_{22} + x_{32} < 2x_{11} - x_{33}$.

Next we look at $\tilde{e}_1(b)$. We have $b'_{11} = b_{11} + 1,\ b'_{12} = b_{12} - 1,\ b'_{35} = b_{35} + 1,\ b'_{36} = b_{36} - 1$. This gives $x'_{11} = x_{11} + 1$. Every other x_{ij} stays the same. Next we consider the two cases of $\tilde{e}_2(b)$.

1. $b'_{12} = b_{12} + 1,\ b'_{13} = b_{13} - 1,\ b'_{24} = b_{24} + 1,\ b'_{25} = b_{25} - 1$ if $b_{12} + b_{24} \geq b_{23} + b_{35}$. Applying the isomorphism, we have $x_{22} + 1$ all others remaining the same if $x_{21} + x_{22} \geq x_{11} + x_{32}$.

2. $b'_{22} = b_{22} + 1,\ b'_{23} = b_{23} - 1,\ b'_{34} = b_{34} + 1,\ b'_{35} = b_{35} - 1$ if $b_{12} + b_{24} < b_{23} + b_{35}$. Applying the isomorphism, we have $x_{22} - 1$ all others remaining the same if $x_{21} + x_{22} < x_{11} + x_{32}$.

Finally we consider the three cases of $\tilde{e}_3(b)$.

1. $b'_{13} = b_{13} + 1,\ b'_{14} = b_{14} - 1$ if $b_{23} \geq b_{34}$ and $b_{13} \geq b_{24}$. Applying the isomorphism, we get $x'_{33} = x_{33} + 1$ and everything else stays the same. The inequalities give us: $b_{13} \geq b_{24} \iff x_{32} + x_{33} \geq 2x_{22}$ and $b_{13} + b_{23} \geq b_{24} + b_{34} \iff x_{31} + x_{32} \geq 2x_{21}$.

2. $b'_{23} = b_{23} + 1,\ b'_{24} = b_{24} - 1$ if $b_{23} \geq b_{34}$ and $b_{13} < b_{24}$. Applying the isomorphism, we get $x'_{32} = x_{32} + 1$ and everything else stays the same. The inequalities give us: $b_{13} < b_{24} \iff x_{33} + x_{32} \geq 2x_{22}$ and $b_{23} \geq b_{34} \iff 2x_{21} \leq x_{31} + x_{32}$.

3. $b'_{33} = b_{33}+1,\ b'_{34} = b_{34}-1$ if $b_{13}+b_{23} < b_{24}+b_{34}$ and $b_{24} > b_{13}$. Applying the isomorphism, we get $x'_{33} = x_{33} + 1$ and everything else stays the same. The inequalities give us: $b_{13} + b_{23} < b_{24} + b_{34} \iff x_{31} + 2x_{32} + x_{33} \leq 2x_{22} + 2x_{21}$ and $b_{23} < b_{34} \iff x_{31} + x_{32} < 2x_{21}$.

Therefore, $\Omega(\tilde{e}_k(b) = \tilde{e}_k\Omega(b)$

Now we show 3.

$$\mathrm{wt}_0(b) = b_{36} - b_{11} = -2x_{11} = \mathrm{wt}_0(\Omega(b))$$
$$\mathrm{wt}_1(b) = b_{35} - b_{12} = 2x_{11} - x_{21} - x_{22} = \mathrm{wt}_1(\Omega(b))$$
$$\mathrm{wt}_2(b) = b_{22} + b_{14} + b_{12} - b_{33} - b_{25} - b_{23} = 2x_{21} + 2x_{22} - x_{11} - x_{31} - x_{32} - x_{33} = \mathrm{wt}_2(\Omega(b))$$
$$\mathrm{wt}_3(b) = b_{33} - b_{14} + b_{13} - b_{24} + b_{23} - b_{34} = 2x_{31} + 2x_{32} + 2x_{33} - 2x_{21} - 2x_{22} = \mathrm{wt}_3(\Omega(b))$$

This proves $\mathrm{wt}_k(\Omega(b)) = \mathrm{wt}_k(b)$. Finally we show 4.

$$\varepsilon_0(b) = \begin{cases} -b_{36} - b_{12} - b_{13} & A < B, C, D, E \\ -b_{36} - b_{23} - b_{24} & B \geq A,\ B < C, D, E \\ -b_{36} - b_{34} - b_{35} & C \geq A, B\ C > D, E \\ -b_{36} - b_{12} - b_{24} & D \geq A, B, C,\ D > E \\ -b_{36} - b_{23} - b_{35} & E \geq A, B, C, D \end{cases}$$

where $A = b_{12} + b_{13}$, $B = b_{23} + b_{24}$, $C = b_{34} + b_{35}$, $D = b_{12} + b_{24}$, and $E = b_{23} + b_{35}$.

Applying the isomorphism, we see this is equivalent to

$$\varepsilon_0(b)=\begin{cases}2x_{11}-x_{33} & (1)>(2),(3),(4),(5)\\ x_{11}+x_{21}-x_{22} & (2)\le(1),\ (2)>(3),(4),(5)\\ x_{31} & (3)\le(1),(2),\ (3)>(4),(5)\\ 2x_{11}-2x_{22}+x_{32} & (4)\le(1),(2),(3),\ (4)>(5)\\ 2x_{21}-x_{32} & (5)\le(1),(2),(3),(4)\end{cases}$$

Where $(1)=2x_{11}-x_{33}$, $(2)=x_{11}+x_{21}-x_{22}$, $(3)=x_{31}$, $(4)=2x_{11}-2x_{22}+x_{32}$, $(5)=2x_{21}-x_{32}$

$$\varepsilon_1(b)=b_{12}=x_{22}-x_{11}=\varepsilon_1(\Omega(b))$$

$$\varepsilon_2(b)=\begin{cases}b_{25}-b_{14} & b_{23}>b_{12}\\ b_{25}-b_{14}+b_{23}-b_{12} & b_{23}\le b_{12}\end{cases}$$

Applying the isomorphism, we see this is equivalent to

$$\varepsilon_2(b)=\begin{cases}x_{33}-x_{22} & x_{21}+x_{22}>x_{11}+x_{32}\\ x_{33}+x_{32}+x_{11}-2x_{22}-x_{21} & x_{21}+x_{22}\le x_{11}+x_{32}\end{cases}$$

$$\varepsilon_3(b)=\begin{cases}b_{14} & b_{24}\le b_{13},\ \ b_{34}\le b_{23}\\ b_{14}+b_{24}-b_{13} & b_{24}>b_{13},\ \ b_{34}\le b_{23}\\ b_{14}+b_{24}-b_{13}+b_{34}-b_{23} & b_{34}>b_{23},\ \ b_{24}>b_{13}\end{cases}$$

Applying the isomorphism, we see this equivalent to

$$\varepsilon_3(b)=\begin{cases}-x_{33} & x_{32}+x_{33}\ge 2x_{22},\ \ x_{31}+x_{32}\ge 2x_{21}\\ 2x_{22}-2x_{33}-x_{32} & x_{32}+x_{33}<2x_{22},\ \ x_{31}+x_{32}\ge 2x_{21}\\ -2x_{33}+2x_{21}+2x_{22}-2x_{32}-x_{31} & x_{32}+x_{33}<2x_{22},\ \ x_{31}+x_{32}<2x_{21}\end{cases}$$

Therefore all of the conditions are satisfied, so Ω is an isomorphism. □

7.4 Isomorphism for n=4 Case

Theorem 7.4.1. *Let* $\Omega : \mathcal{UD}(\mathcal{V}) \to B^{4,\infty}$ *be given by*

$$b_{11} = x_{11},\ b_{12} = x_{22} - x_{11},\ b_{13} = x_{33} - x_{22},\ b_{14} = x_{44} - x_{33}$$
$$b_{15} = -x_{44},\ b_{22} = x_{21},\ b_{23} = x_{32} - x_{21},\ b_{24} = x_{43} - x_{32}$$
$$b_{25} = x_{33} - x_{43},\ b_{26} = -x_{33},\ b_{33} = x_{31},\ b_{34} = x_{42} - x_{31}$$
$$b_{35} = x_{32} - x_{42},\ b_{36} = x_{22} - x_{32},\ b_{37} = -x_{22},\ b_{44} = x_{41}$$
$$b_{45} = x_{31} - x_{41},\ b_{46} = x_{21} - x_{31},\ b_{47} = x_{11} - x_{21},\ b_{48} = -x_{11}$$

and $\Omega^{-1} : B^{4,\infty} \to \mathcal{UD}(\mathcal{V})$ *be given by* $x_{11} = b_{11},\ x_{21} = b_{22},\ x_{22} = -b_{37},\ x_{31} = b_{33},\ x_{32} = b_{22} + b_{23},\ x_{33} = -b_{26}, x_{41} = b_{44},\ x_{42} = b_{33} + b_{34},\ x_{43} = b_{22} + b_{23} + b_{24},\ x_{44} = -b_{15}$
Ω *is an isomorphism of crystals.*

Proof. Clearly we see that the map is bijective. We need to prove the following conditions to show that Ω is an isomorphism:

1. $\Omega(\tilde{f}_k(b)) = \tilde{f}_k\Omega(b)$
2. $\Omega(\tilde{e}_k(b)) = \tilde{e}_k\Omega(b)$
3. $\mathrm{wt}_k(\Omega(b)) = \mathrm{wt}_k(b)$
4. $\varepsilon_k(\Omega(b) = \varepsilon_k(b)$

Clearly if these hold, then $\varphi_k(\Omega(b)) = \varepsilon_k(\Omega(b)) + \mathrm{wt}_k(\Omega(b)) = \varepsilon_k(b) + \mathrm{wt}_k(b) = \varphi_k(b)$. Now we check the above conditions for each $k = 0,1,2,3,4$. First we check 1. There are 8 cases for $\tilde{f}_0(b)$.

1. $b'_{11} = b_{11}+1,\ b'_{48} = b_{48}-1,\ b'_{12} = b_{12}-1,\ b'_{23} = b_{23}-1,\ b'_{34} = b_{34} = 1,\ b'_{22} = b_{22}+1,\ b'_{33} = b_{33}+1,\ b'_{44} = b_{44}+1$ if $b_{12}+b_{35}+b_{36} > b_{45}+b_{46}+b_{47}$, $b_{23}+b_{35} > b_{45}+b_{46}$, $b_{34} > b_{45}$, $b_{23}+b_{34} > b_{45}+b_{46}$, $b_{12}+b_{13}+b_{25} > b_{45}+b_{46}+b_{47}$, $b_{12}+b_{24}+b_{36} > b_{45}+b_{46}+b_{47}$, and $b_{12}+b_{13}+b_{14} > b_{45}+b_{46}+b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{21} = x_{21}+1$,, $x'_{31} = x_{31}+1$, and $x'_{41} = x_{41}+1$ along with the following conditions: $x_{41} > 2x_{11} - 2x_{22} + x_{42}$, $x_{41} > 2x_{21} - 2x_{32} + x_{42}$, $x_{41} > 2x_{31} - x_{42}$, $x_{41} > 2x_{21} - x_{43}$, $x_{41} > 2x_{11} - 2x_{33} + x_{43}$, $x_{41} > 2x_{11} - 2x_{22} + 2x_{32} - x_{43}$, and $x_{41} > 2x_{11} - x_{44}$. These exactly correspond to the first case of $\tilde{f}_0(\Omega(b))$.

2. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{13} = b_{13}-1$, $b'_{24} = b_{24}-1$, $b'_{37} = b_{37}-1$, $b'_{23} = b_{23}+1$, $b'_{34} = b_{34}+1$, $b'_{47} = b_{47}+1$ if $b_{12}+b_{35}+b_{36} \leq b_{45}+b_{46}+b_{47}$, $b_{23}+b_{47} > b_{12}+b_{36}$, $b_{34}+b_{46}+b_{47} > b_{12}+b_{35}+b_{36}$, $b_{23}+b_{34}+b_{47} > b_{12}+b_{35}+b_{36}$, $b_{13}+b_{25} > b_{35}+b_{36}$, $b_{24} > b_{35}$, $b_{13}+b_{14} > b_{35}+b_{36}$, and $b_{23}+b_{24}+b_{47} \geq b_{12}+b_{13}+b_{25}$ or $b_{23}+b_{47} \geq b_{12}+b_{36}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{22} = x_{22}+1$,, $x'_{32} = x_{32}+1$, and $x'_{42} = x_{42}+1$ along with the following conditions: $x_{41} \leq 2x_{11}-2x_{22}+x_{42}$, $2x_{11}-2x_{22}+x_{42} > 2x_{21}-2x_{32}+x_{42}$, $2x_{11}-2x_{22}+x_{42} > 2x_{31}-x_{42}$, $2x_{11}-2x_{22}+x_{42} > 2x_{21}-x_{43}$, $2x_{11}-2x_{22}+x_{42} > 2x_{11}-2x_{33}+x_{43}$, $2x_{11}-2x_{22}+x_{42} > 2x_{11}-2x_{22}+2x_{32}-x_{43}$, and $2x_{11}-2x_{22}+x_{42} > 2x_{11}-x_{44}$, and $2x_{11}-2x_{33}+x_{43} \geq 2x_{21}-x_{43}$ or $2x_{11}+2x_{32}-2x_{22}-x_{43} \geq 2x_{21}-x_{43}$. These exactly correspond to the second case of $\tilde{f}_0(\Omega(b))$.

3. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{12} = b_{12}-1$, $b'_{24} = b_{24}-1$, $b'_{36} = b_{36}-1$, $b'_{22} = b_{22}+1$, $b'_{34} = b_{34}+1$, $b'_{46} = b_{46}+1$ if $b_{45}+b_{46} \geq b_{23}+b_{35}$, $b_{12}+b_{36} > b_{23}+b_{47}$, $b_{34}+b_{46} > b_{23}+b_{35}$, $b_{24} > b_{35}$, $b_{12}+b_{13}+b_{25} > b_{23}+b_{35}+b_{47}$,$b_{12}+b_{24}+b_{36} > b_{23}+b_{35}+b_{47}$, $b_{12}+b_{13}+b_{14} > b_{23}+b_{35}+b_{47}$, $b_{24}+b_{36} > b_{13}+b_{25}$, and $b_{12}+b_{36} > b_{23}+b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{21} = x_{21}+1$,, $x'_{32} = x_{32}+1$, and $x'_{42} = x_{42}+1$ along with the following conditions: $x_{41} \leq 2x_{21}-2x_{32}+x_{42}$, $2x_{11}-2x_{22}+x_{42} \leq 2x_{21}-2x_{32}+x_{42}$, $2x_{21}-2x_{32}+x_{42} > 2x_{31}-x_{42}$, $2x_{21}-2x_{32}+x_{42} > 2x_{21}-x_{43}$, $2x_{21}-2x_{32}+x_{42} > 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-2x_{32}+x_{42} > 2x_{11}-2x_{22}+2x_{32}-x_{43}$, and $2x_{21}-2x_{32}+x_{42} > 2x_{11}-x_{44}$, $2x_{11}-2x_{33}+x_{43} \geq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, and $2x_{21}-x_{43} > 2x_{11}+2x_{32}-2x_{22}-x_{43}$. These exactly correspond to the third case of $\tilde{f}_0(\Omega(b))$.

4. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{12} = b_{12}-1$, $b'_{23} = b_{23}-1$, $b'_{35} = b_{35}-1$, $b'_{22} = b_{22}+1$, $b'_{33} = b_{33}+1$, $b'_{45} = b_{45}+1$ if $b_{45} \geq b_{34}$ $b_{12}+b_{35}+b_{36} \geq b_{34}+b_{46}+b_{47}$, $b_{23}+b_{35} \geq b_{34}+b_{46}$, $b_{23}+b_{34} > b_{34}+b_{46}$, $b_{12}+b_{13}+b_{25} > b_{34}+b_{46}+b_{47}$, $b_{12}+b_{24}+b_{36} > b_{34}+b_{46}+b_{47}$, and $b_{12}+b_{13}+b_{14} > b_{34}+b_{46}+b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{21} = x_{21}+1$,, $x'_{31} = x_{31}+1$, and $x'_{42} = x_{42}+1$ along with the following conditions: $x_{41} \leq 2x_{31}-x_{42}$, $2x_{11}-2x_{22}+x_{42} \leq 2x_{31}-x_{42}$, $2x_{21}-2x_{32}+x_{42} \leq 2x_{31}-x_{42}$, $2x_{31}-x_{42} > 2x_{21}-x_{43}$, $2x_{31}-x_{42} > 2x_{11}-2x_{33}+x_{43}$, $2x_{31}-x_{42} > 2x_{11}-2x_{22}+2x_{32}-x_{43}$, and $2x_{31}-x_{42} > 2x_{11}-x_{44}$. These exactly correspond to the fourth case of $\tilde{f}_0(\Omega(b))$.

5. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{12} = b_{12}-1$, $b'_{25} = b_{25}-1$, $b'_{36} = b_{36}-1$, $b'_{22} = b_{22}+1$, $b'_{35} = b_{35}+1$, $b'_{46} = b_{46}+1$ if $b_{45}+b_{46} \geq b_{23}+b_{24}$ $b_{12}+b_{35}+b_{36} \geq b_{23}+b_{24}+b_{47}$,

$b_{35} \geq b_{24}$, $b_{34}+b_{46} \geq b_{23}+b_{24}$, $b_{12}+b_{13}+b_{25} > b_{23}+b_{24}+b_{47}$, $b_{12}+b_{36} > b_{23}+b_{47}$, $b_{12}+b_{13}+b_{14} > b_{23}+b_{24}+b_{47}$, and $b_{12}+b_{36} > b_{23}+b_{47}$ or $b_{12}+b_{35}+b_{46} > b_{34}+b_{46}+b_{47}$.

Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{21} = x_{21}+1,$, $x'_{32} = x_{32}+1$, and $x'_{43} = x_{43}+1$ along with the following conditions: $x_{41} \leq 2x_{21}-x_{43}$, $2x_{11}-2x_{22}+x_{42} \leq 2x_{21}-x_{43}$, $2x_{21}-2x_{32}+x_{42} \leq 2x_{21}-x_{43}$, $2x_{31}-x_{42} \leq 2x_{21}-x_{43}$, $2x_{21}-x_{43} > 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-x_{43} > 2x_{11}-2x_{22}+2x_{32}-x_{43}$, $2x_{21}-x_{43} > 2x_{11}-x_{44}$, and $2x_{21}-2x_{32}+x_{42} > 2x_{11}-2x_{22}+x_{42}$ or $2x_{31}-x_{42} > 2x_{11}-2x_{22}+x_{42}$. These exactly correspond to the fifth case of $\tilde{f}_0(\Omega(b))$.

6. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{14} = b_{14}-1$, $b'_{26} = b_{26}-1$, $b'_{37} = b_{37}-1$, $b'_{24} = b_{24}+1$, $b'_{36} = b_{36}+1$, $b'_{47} = b_{47}+1$ if $b_{45}+b_{46}+b_{47} \geq b_{12}+b_{13}+b_{25}$ $b_{35}+b_{36} \geq b_{13}+b_{25}$, $b_{23}+b_{35}+b_{47} \geq b_{12}+b_{13}+b_{25}$, $b_{34}+b_{46}+b_{47} \geq b_{12}+b_{13}+b_{25}$, $b_{12}+b_{13}+b_{25} \leq b_{23}+b_{24}+b_{47}$, $b_{24}+b_{36} > b_{13}+b_{25}$, and $b_{14} > b_{25}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{22} = x_{22}+1,$, $x'_{33} = x_{33}+1$, and $x'_{43} = x_{43}+1$ along with the following conditions: $x_{41} \leq 2x_{11}-2x_{33}+x_{43}$, $2x_{11}-2x_{22}+x_{42} \leq 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-2x_{32}+x_{42} \leq 2x_{11}-2x_{33}+x_{43}$, $2x_{31}-x_{42} \leq 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-x_{43} \leq 2x_{11}-2x_{33}+x_{43}$, $2x_{11}-2x_{33}+x_{43} > 2x_{11}-2x_{22}+2x_{32}-x_{43}$, and $2x_{11}-2x_{33}+x_{43} > 2x_{11}-x_{44}$. These exactly correspond to the sixth case of $\tilde{f}_0(\Omega(b))$.

7. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{13} = b_{13}-1$, $b'_{25} = b_{25}-1$, $b'_{37} = b_{37}-1$, $b'_{23} = b_{23}+1$, $b'_{35} = b_{35}+1$, $b'_{47} = b_{47}+1$ if $b_{45}+b_{46}+b_{47} \geq b_{12}+b_{24}+b_{36}$ $b_{35} \geq b_{24}$, $b_{23}+b_{35}+b_{47} \geq b_{12}+b_{24}+b_{36}$, $b_{34}+b_{46}+b_{47} \geq b_{12}+b_{24}+b_{36}$, $b_{23}+b_{47} \geq b_{12}+b_{36}$, $b_{24}+b_{36} \leq b_{13}+b_{25}$, $b_{13}+b_{14} > b_{24}+b_{36}$, and $b_{23}+b_{35} > b_{34}+b_{46}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{22} = x_{22}+1,$, $x'_{32} = x_{32}+1$, and $x'_{43} = x_{43}+1$ along with the following conditions: $x_{41} \leq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{11}-2x_{22}+x_{42} \leq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{21}-2x_{32}+x_{42} \leq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{31}-x_{42} \leq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{21}-x_{43} \leq 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{11}-2x_{33}+x_{43} \leq 2x_{11}-2x_{22}+2x_{32}-x_{43}$, $2x_{11}+2x_{32}-2x_{22}-x_{43} > 2x_{11}-x_{44}$, and $2x_{31}-x_{42} > 2x_{21}-2x_{32}+x_{42}$. These exactly correspond to the seventh case of $\tilde{f}_0(\Omega(b))$.

8. $b'_{11} = b_{11}+1$, $b'_{48} = b_{48}-1$, $b'_{15} = b_{15}-1$, $b'_{26} = b_{26}-1$, $b'_{37} = b_{37}-1$, $b'_{25} = b_{25}+1$, $b'_{36} = b_{36}+1$, $b'_{47} = b_{47}+1$ if $b_{45}+b_{46}+b_{47} \geq b_{12}+b_{13}+b_{14}$ $b_{35}+b_{36} \geq b_{13}+b_{14}$, $b_{23}+b_{35}+b_{47} \geq b_{12}+b_{13}+b_{14}$, $b_{34}+b_{46}+b_{47} \geq b_{12}+b_{13}+b_{14}$, $b_{23}+b_{24}+b_{47} \geq b_{12}+b_{13}+b_{14}$, $b_{24}+b_{36} \leq b_{13}+b_{14}$, and $b_{25} \geq b_{14}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}+1, x'_{22} = x_{22}+1,$,

$x'_{33} = x_{33}+1$, and $x'_{44} = x_{44}+1$ along with the following conditions: $x_{41} \leq 2x_{11} - x_{44}$, $2x_{11} - 2x_{22} + x_{42} \leq 2x_{11} - x_{44}$, $2x_{21} - 2x_{32} + x_{42} \leq 2x_{11} - x_{44}$, $2x_{31} - x_{42} \leq 2x_{11} - x_{44}$, $2x_{21} - x_{43} \leq 2x_{11} - x_{44}$, $2x_{11} - 2x_{33} + x_{43} \leq 2x_{11} - x_{44}$, and $2x_{11} + 2x_{32} - 2x_{22} - x_{43} \leq 2x_{11} - x_{44}$. These exactly correspond to the eighth case of $\tilde{f}_0(\Omega(b))$.

Now we show that $\Omega(\tilde{f}_1(b)) = \tilde{f}_1\Omega(b)$. The perfect crystal has $b'_{11} = b_{11}-1$, $b'_{12} = b_{12}+1$, $b'_{47} = b_{47}-1$, and $b'_{48} = b_{48}+1$. Applying the isomorphism, we can see that this corresponds with $x'_{11} = x_{11}-1$ and all other x_{ij} remaining the same.

There are 2 cases for f_2:

1. $b'_{12} = b_{12}-1$, $b'_{36} = b_{36}-1$, $b'_{13} = b_{13}+1$, and $b'_{37} = b_{37}+1$ if $b_{12}+b_{36} > b_{23}+b_{47}$. Applying the isomorphism, we get that this case corresponds to $x'_{22} = x_{22}-1$ along with the condition $x_{21}+x_{22} > x_{11}+x_{32}$. This corresponds to the first case of $\tilde{f}_2(\Omega(b))$.

2. $b'_{22} = b_{22}-1$, $b'_{46} = b_{46}-1$, $b'_{23} = b_{23}+1$, and $b'_{47} = b_{47}+1$ if $b_{12}+b_{36} \leq b_{23}+b_{47}$. Applying the isomorphism, we get that this case corresponds to $x'_{21} = x_{21}-1$ along with the condition $x_{21}+x_{22} \leq x_{11}+x_{32}$. This corresponds to the second case of $\tilde{f}_2(\Omega(b))$.

There are 3 cases for f_3:

1. $b'_{33} = b_{33}-1$, $b'_{45} = b_{45}-1$, $b'_{34} = b_{34}+1$, and $b'_{46} = b_{46}+1$ if $b_{25}+b_{35} \leq b_{36}+b_{46}$ and $b_{35} \leq b_{46}$. Applying the isomorphism, we see that these correspond to $x'_{31} = x_{31}-1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} \geq x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} \geq x_{44}-x_{33}$.

2. $b'_{23} = b_{23}-1$, $b'_{35} = b_{35}-1$, $b'_{24} = b_{24}+1$, and $b'_{36} = b_{36}+1$ if $b_{35} > b_{46}$ and $b_{13} \leq b_{24}$ Applying the isomorphism, we see that these correspond to $x'_{32} = x_{32}-1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} < x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{22}-2x_{33}-x_{32} \geq x_{44}-x_{33}$.

3. $b'_{13} = b_{13}-1$, $b'_{25} = b_{25}-1$, $b'_{14} = b_{14}+1$, and $b'_{26} = b_{26}+1$ if $b_{25}+b_{35} > b_{36}+b_{46}$ and $b_{13} > b_{24}$ Applying the isomorphism, we see that these correspond to $x'_{33} = x_{33}-1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} > x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} > x_{44}-x_{33}$.

These all correspond to the cases of $\tilde{f}_3(\Omega(b))$

Finally we consider f_4, which has 4 cases:

1. $b'_{14} = b_{14} - 1$, $b'_{15} = b_{15} + 1$ if $b_{14} > b_{25}$, $b_{14} + b_{24} > b_{25} + b_{35}$, and $b_{14} + b_{24} + b_{34} > b_{25} + b_{35} + b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{44} = x_{44} - 1$ along with the conditions $-x_{44} > 2x_{33} - 2x_{44} - x_{43}$, $-x_{44} > 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, and $-x_{44} > 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

2. $b'_{24} = b_{24} - 1$, $b'_{25} = b_{25} + 1$ if $b_{14} \leq b_{25}$, $b_{24} > b_{35}$, and $b_{24} + b_{34} > b_{35} + b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{43} = x_{43} - 1$ along with the conditions $-x_{44} \leq 2x_{33} - 2x_{44} - x_{43}$, $2x_{33} - 2x_{44} - x_{43} > 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, and $2x_{33} - 2x_{44} - x_{43} > 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

3. $b'_{34} = b_{34} - 1$, $b'_{35} = b_{35} + 1$ if $b_{14} + b_{24} \leq b_{25} + b_{35}$, $b_{24} \leq b_{35}$, and $b_{34} > b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{42} = x_{42} - 1$ along with the conditions $-x_{44} \leq 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, $2x_{33} - 2x_{44} - x_{43} \leq 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, and $2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42} > 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

4. $b'_{44} = b_{44} - 1$, $b'_{45} = b_{45} + 1$ if $b_{14} + b_{24} + b_{34} \leq b_{25} + b_{35} + b_{45}$, $b_{24} + b_{34} \leq b_{35} + b_{45}$, and $b_{34} \leq b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{41} = x_{41} - 1$ along with the conditions $-x_{44} \leq 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$, $2x_{33} - 2x_{44} - x_{43} \leq 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$, and $2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42} \leq 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

This corresponds to all of the cases of $\tilde{f}_4(\Omega(b))$. Thus 1. is proved.

Next we check 2. There are 8 cases for $\tilde{\varepsilon}_0(b)$.

1. $b'_{11} = b_{11} - 1$, $b'_{48} = b_{48} + 1$, $b'_{12} = b_{12} + 1$, $b'_{23} = b_{23} + 1$, $b'_{34} = b_{34} + 1$, $b'_{22} = b_{22} - 1$, $b'_{33} = b_{33} - 1$, $b'_{44} = b_{44} - 1$ if $b_{12} + b_{35} + b_{36} \geq b_{45} + b_{46} + b_{47}$, $b_{23} + b_{35} \geq b_{45} + b_{46}$, $b_{34} \geq b_{45}$, $b_{23} + b_{34} \geq b_{45} + b_{46}$, $b_{12} + b_{13} + b_{25} \geq b_{45} + b_{46} + b_{47}$, $b_{12} + b_{24} + b_{36} \geq b_{45} + b_{46} + b_{47}$, and $b_{12} + b_{13} + b_{14} \geq b_{45} + b_{46} + b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11} - 1, x'_{21} = x_{21} - 1,$, $x'_{31} = x_{31} - 1$, and $x'_{41} = x_{41} - 1$ along with the following conditions: $x_{41} \geq 2x_{11} - 2x_{22} + x_{42}$, $x_{41} \geq 2x_{21} - 2x_{32} + x_{42}$, $x_{41} \geq 2x_{31} - x_{42}$, $x_{41} \geq 2x_{21} - x_{43}$, $x_{41} \geq 2x_{11} - 2x_{33} + x_{43}$, $x_{41} \geq 2x_{11} - 2x_{22} + 2x_{32} - x_{43}$, and $x_{41} \geq 2x_{11} - x_{44}$. These exactly correspond to the first case of $\tilde{e}_0(\Omega(b))$.

2. $b'_{11} = b_{11} - 1$, $b'_{48} = b_{48} + 1$, $b'_{13} = b_{13} + 1$, $b'_{24} = b_{24} + 1$, $b'_{37} = b_{37} + 1$, $b'_{23} = b_{23} - 1$, $b'_{34} = b_{34} - 1$, $b'_{47} = b_{47} - 1$ if $b_{12} + b_{35} + b_{36} < b_{45} + b_{46} + b_{47}$, $b_{23} + b_{47} \geq b_{12} + b_{36}$, $b_{34} + b_{46} + b_{47} \geq b_{12} + b_{35} + b_{36}$, $b_{23} + b_{34} + b_{47} \geq b_{12} + b_{35} + b_{36}$, $b_{13} + b_{25} \geq b_{35} + b_{36}$, $b_{24} \geq b_{35}$, $b_{13} + b_{14} \geq b_{35} + b_{36}$, and $b_{23} + b_{24} + b_{47} > b_{12} + b_{13} + b_{25}$ or $b_{23} + b_{47} > b_{12} + b_{36}$.

Applying the isomorphism, we end up with the action $x'_{11} = x_{11} - 1, x'_{22} = x_{22} - 1,$, $x'_{32} = x_{32} - 1$, and $x'_{42} = x_{42} - 1$ along with the following conditions: $x_{41} < 2x_{11} - 2x_{22} + x_{42}$, $2x_{11} - 2x_{22} + x_{42} \geq 2x_{21} - 2x_{32} + x_{42}$, $2x_{11} - 2x_{22} + x_{42} \geq 2x_{31} - x_{42}$, $2x_{11} - 2x_{22} + x_{42} \geq 2x_{21} - x_{43}$, $2x_{11} - 2x_{22} + x_{42} \geq 2x_{11} - 2x_{33} + x_{43}$, $2x_{11} - 2x_{22} + x_{42} \geq 2x_{11} - 2x_{22} + 2x_{32} - x_{43}$, and $2x_{11} - 2x_{22} + x_{42} \geq 2x_{11} - x_{44}$, and $2x_{11} - 2x_{33} + x_{43} > 2x_{21} - x_{43}$ or $2x_{11} + 2x_{32} - 2x_{22} - x_{43} > 2x_{21} - x_{43}$. These exactly correspond to the second case of $\tilde{e}_0(\Omega(b))$.

3. $b'_{11} = b_{11} - 1,\ b'_{48} = b_{48} + 1,\ b'_{12} = b_{12} + 1,\ b'_{24} = b_{24} + 1,\ b'_{36} = b_{36} + 1,\ b'_{22} = b_{22} - 1,\ b'_{34} = b_{34} - 1,\ b'_{46} = b_{46} - 1$ if $b_{45} + b_{46} > b_{23} + b_{35}$, $b_{12} + b_{36} \geq b_{23} + b_{47}$, $b_{34} + b_{46} \geq b_{23} + b_{35}$, $b_{24} \geq b_{35}$, $b_{12} + b_{13} + b_{25} \geq b_{23} + b_{35} + b_{47}$, $b_{12} + b_{24} + b_{36} \geq b_{23} + b_{35} + b_{47}$, $b_{12} + b_{13} + b_{14} \geq b_{23} + b_{35} + b_{47}$, $b_{24} + b_{36} \geq b_{13} + b_{25}$, and $b_{12} + b_{36} \geq b_{23} + b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11} - 1, x'_{21} = x_{21} - 1,$, $x'_{32} = x_{32} - 1$, and $x'_{42} = x_{42} - 1$ along with the following conditions: $x_{41} < 2x_{21} - 2x_{32} + x_{42}$, $2x_{11} - 2x_{22} + x_{42} < 2x_{21} - 2x_{32} + x_{42}$, $2x_{21} - 2x_{32} + x_{42} \geq 2x_{31} - x_{42}$, $2x_{21} - 2x_{32} + x_{42} \geq 2x_{21} - x_{43}$, $2x_{21} - 2x_{32} + x_{42} \geq 2x_{11} - 2x_{33} + x_{43}$, $2x_{21} - 2x_{32} + x_{42} \geq 2x_{11} - 2x_{22} + 2x_{32} - x_{43}$, and $2x_{21} - 2x_{32} + x_{42} \geq 2x_{11} - x_{44}$, $2x_{11} - 2x_{33} + x_{43} > 2x_{11} + 2x_{32} - 2x_{22} - x_{43}$, and $2x_{21} - x_{43} \geq 2x_{11} + 2x_{32} - 2x_{22} - x_{43}$. These exactly correspond to the third case of $\tilde{e}_0(\Omega(b))$.

4. $b'_{11} = b_{11} - 1,\ b'_{48} = b_{48} + 1,\ b'_{12} = b_{12} + 1,\ b'_{23} = b_{23} + 1,\ b'_{35} = b_{35} + 1,\ b'_{22} = b_{22} - 1,\ b'_{33} = b_{33} - 1,\ b'_{45} = b_{45} - 1$ if $b_{45} > b_{34}$ $b_{12} + b_{35} + b_{36} > b_{34} + b_{46} + b_{47}$, $b_{23} + b_{35} > b_{34} + b_{46}$, $b_{23} + b_{34} \geq b_{34} + b_{46}$, $b_{12} + b_{13} + b_{25} \geq b_{34} + b_{46} + b_{47}$, $b_{12} + b_{24} + b_{36} \geq b_{34} + b_{46} + b_{47}$, and $b_{12} + b_{13} + b_{14} \geq b_{34} + b_{46} + b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11} - 1, x'_{21} = x_{21} - 1,$, $x'_{31} = x_{31} - 1$, and $x'_{42} = x_{42} - 1$ along with the following conditions: $x_{41} < 2x_{31} - x_{42}$, $2x_{11} - 2x_{22} + x_{42} < 2x_{31} - x_{42}$, $2x_{21} - 2x_{32} + x_{42} < 2x_{31} - x_{42}$, $2x_{31} - x_{42} \geq 2x_{21} - x_{43}$, $2x_{31} - x_{42} \geq 2x_{11} - 2x_{33} + x_{43}$, $2x_{31} - x_{42} \geq 2x_{11} - 2x_{22} + 2x_{32} - x_{43}$, and $2x_{31} - x_{42} \geq 2x_{11} - x_{44}$. These exactly correspond to the fourth case of $\tilde{e}_0(\Omega(b))$.

5. $b'_{11} = b_{11} - 1,\ b'_{48} = b_{48} + 1,\ b'_{12} = b_{12} + 1,\ b'_{25} = b_{25} + 1,\ b'_{36} = b_{36} + 1,\ b'_{22} = b_{22} - 1,\ b'_{35} = b_{35} - 1,\ b'_{46} = b_{46} - 1$ if $b_{45} + b_{46} > b_{23} + b_{24}$ $b_{12} + b_{35} + b_{36} > b_{23} + b_{24} + b_{47}$, $b_{35} > b_{24}$, $b_{34} + b_{46} > b_{23} + b_{24}$, $b_{12} + b_{13} + b_{25} \geq b_{23} + b_{24} + b_{47}$, $b_{12} + b_{36} \geq b_{23} + b_{47}$, $b_{12} + b_{13} + b_{14} \geq b_{23} + b_{24} + b_{47}$, and $b_{12} + b_{36} \geq b_{23} + b_{47}$ or $b_{12} + b_{35} + b_{46} \geq b_{34} + b_{46} + b_{47}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11} - 1, x'_{21} = x_{21} - 1,$, $x'_{32} = x_{32} - 1$, and $x'_{43} = x_{43} - 1$ along with the following conditions: $x_{41} < 2x_{21} - x_{43}$,

$2x_{11}-2x_{22}+x_{42} < 2x_{21}-x_{43}$, $2x_{21}-2x_{32}+x_{42} < 2x_{21}-x_{43}$, $2x_{31}-x_{42} < 2x_{21}-x_{43}$, $2x_{21}-x_{43} \geq 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-x_{43} \geq 2x_{11}-2x_{22}+2x_{32}-x_{43}$, $2x_{21}-x_{43} \geq 2x_{11}-x_{44}$, and $2x_{21}-2x_{32}+x_{42} \geq 2x_{11}-2x_{22}+x_{42}$ or $2x_{31}-x_{42} \geq 2x_{11}-2x_{22}+x_{42}$. These exactly correspond to the fifth case of $\tilde{e}_0(\Omega(b))$.

6. $b'_{11} = b_{11}-1$, $b'_{48} = b_{48}+1$, $b'_{14} = b_{14}+1$, $b'_{26} = b_{26}+1$, $b'_{37} = b_{37}+1$, $b'_{24} = b_{24}-1$, $b'_{36} = b_{36}-1$, $b'_{47} = b_{47}-1$ if $b_{45}+b_{46}+b_{47} > b_{12}+b_{13}+b_{25}$ $b_{35}+b_{36} > b_{13}+b_{25}$, $b_{23}+b_{35}+b_{47} > b_{12}+b_{13}+b_{25}$, $b_{34}+b_{46}+b_{47} > b_{12}+b_{13}+b_{25}$, $b_{12}+b_{13}+b_{25} < b_{23}+b_{24}+b_{47}$, $b_{24}+b_{36} \geq b_{13}+b_{25}$, and $b_{14} \geq b_{25}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}-1$,$x'_{22} = x_{22}-1$,, $x'_{33} = x_{33}-1$, and $x'_{43} = x_{43}-1$ along with the following conditions: $x_{41} < 2x_{11}-2x_{33}+x_{43}$, $2x_{11}-2x_{22}+x_{42} < 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-2x_{32}+x_{42} < 2x_{11}-2x_{33}+x_{43}$, $2x_{31}-x_{42} < 2x_{11}-2x_{33}+x_{43}$, $2x_{21}-x_{43} < 2x_{11}-2x_{33}+x_{43}$, $2x_{11}-2x_{33}+x_{43} \geq 2x_{11}-2x_{22}+2x_{32}-x_{43}$, and $2x_{11}-2x_{33}+x_{43} \geq 2x_{11}-x_{44}$. These exactly correspond to the sixth case of $\tilde{e}_0(\Omega(b))$.

7. $b'_{11} = b_{11}-1$, $b'_{48} = b_{48}+1$, $b'_{13} = b_{13}+1$, $b'_{25} = b_{25}+1$, $b'_{37} = b_{37}+1$, $b'_{23} = b_{23}-1$, $b'_{35} = b_{35}-1$, $b'_{47} = b_{47}-1$ if $b_{45}+b_{46}+b_{47} > b_{12}+b_{24}+b_{36}$ $b_{35} > b_{24}$, $b_{23}+b_{35}+b_{47} > b_{12}+b_{24}+b_{36}$, $b_{34}+b_{46}+b_{47} > b_{12}+b_{24}+b_{36}$, $b_{23}+b_{47} > b_{12}+b_{36}$, $b_{24}+b_{36} < b_{13}+b_{25}$, $b_{13}+b_{14} \geq b_{24}+b_{36}$, and $b_{23}+b_{35} \geq b_{34}+b_{46}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}-1$,$x'_{22} = x_{22}-1$,, $x'_{32} = x_{32}-1$, and $x'_{43} = x_{43}-1$ along with the following conditions: $x_{41} < 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{11}-2x_{22}+x_{42} < 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{21}-2x_{32}+x_{42} < 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{31}-x_{42} < 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{21}-x_{43} < 2x_{11}+2x_{32}-2x_{22}-x_{43}$, $2x_{11}-2x_{33}+x_{43} < 2x_{11}-2x_{22}+2x_{32}-x_{43}$, $2x_{11}+2x_{32}-2x_{22}-x_{43} \geq 2x_{11}-x_{44}$, and $2x_{31}-x_{42} \geq 2x_{21}-2x_{32}+x_{42}$. These exactly correspond to the seventh case of $\tilde{e}_0(\Omega(b))$.

8. $b'_{11} = b_{11}-1$, $b'_{48} = b_{48}+1$, $b'_{15} = b_{15}+1$, $b'_{26} = b_{26}+1$, $b'_{37} = b_{37}+1$, $b'_{25} = b_{25}-1$, $b'_{36} = b_{36}-1$, $b'_{47} = b_{47}-1$ if $b_{45}+b_{46}+b_{47} > b_{12}+b_{13}+b_{14}$ $b_{35}+b_{36} > b_{13}+b_{14}$, $b_{23}+b_{35}+b_{47} > b_{12}+b_{13}+b_{14}$, $b_{34}+b_{46}+b_{47} > b_{12}+b_{13}+b_{14}$, $b_{23}+b_{24}+b_{47} > b_{12}+b_{13}+b_{14}$, $b_{24}+b_{36} > b_{13}+b_{14}$, and $b_{25} > b_{14}$.

   Applying the isomorphism, we end up with the action $x'_{11} = x_{11}-1$,$x'_{22} = x_{22}-1$,, $x'_{33} = x_{33}-1$, and $x'_{44} = x_{44}-1$ along with the following conditions: $x_{41} < 2x_{11}-x_{44}$, $2x_{11}-2x_{22}+x_{42} < 2x_{11}-x_{44}$, $2x_{21}-2x_{32}+x_{42} < 2x_{11}-x_{44}$, $2x_{31}-x_{42} < 2x_{11}-x_{44}$, $2x_{21}-x_{43} < 2x_{11}-x_{44}$, $2x_{11}-2x_{33}+x_{43} < 2x_{11}-x_{44}$, and $2x_{11}+2x_{32}-2x_{22}-x_{43} < 2x_{11}-x_{44}$. These exactly correspond to the eighth case of $\tilde{e}_0(\Omega(b))$.

Now we show that $\Omega(\tilde{e}_1(b)) = \tilde{e}_1\Omega(b)$. The perfect crystal has $b'_{11} = b_{11}+1$, $b'_{12} = b_{12}-1$, $b'_{47} = b_{47}+1$, and $b'_{48} = b_{48}-1$. Applying the isomorphism, we can see that this corresponds with $x'_{11} = x_{11}+1$ and all other x_{ij} remaining the same.

There are 2 cases for $\tilde{e}_2$:

1. $b'_{12} = b_{12}+1$, $b'_{36} = b_{36}+1$, $b'_{13} = b_{13}-1$, and $b'_{37} = b_{37}-1$ if $b_{12}+b_{36} \geq b_{23}+b_{47}$. Applying the isomorphism, we get that this case corresponds to $x'_{22} = x_{22}+1$ along with the condition $x_{21}+x_{22} \geq x_{11}+x_{32}$. This corresponds to the first case of $\tilde{e}_2(\Omega(b))$.

2. $b'_{22} = b_{22}+1$, $b'_{46} = b_{46}+1$, $b'_{23} = b_{23}-1$, and $b'_{47} = b_{47}-1$ if $b_{12}+b_{36} < b_{23}+b_{47}$. Applying the isomorphism, we get that this case corresponds to $x'_{21} = x_{21}+1$ along with the condition $x_{21}+x_{22} < x_{11}+x_{32}$. This corresponds to the second case of $\tilde{e}_2(\Omega(b))$.

There are 3 cases for $\tilde{e}_3$:

1. $b'_{33} = b_{33}+1$, $b'_{45} = b_{45}+1$, $b'_{34} = b_{34}-1$, and $b'_{46} = b_{46}-1$ if $b_{25}+b_{35} < b_{36}+b_{46}$ and $b_{35} < b_{46}$. Applying the isomorphism, we see that these correspond to $x'_{31} = x_{31}+1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} > x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} > x_{44}-x_{33}$.

2. $b'_{23} = b_{23}+1$, $b'_{35} = b_{35}+1$, $b'_{24} = b_{24}-1$, and $b'_{36} = b_{36}-1$ if $b_{35} \geq b_{46}$ and $b_{13} < b_{24}$. Applying the isomorphism, we see that these correspond to $x'_{32} = x_{32}+1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} \leq x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{22}-2x_{33}-x_{32} > x_{44}-x_{33}$.

3. $b'_{13} = b_{13}+1$, $b'_{25} = b_{25}+1$, $b'_{14} = b_{14}-1$, and $b'_{26} = b_{26}-1$ if $b_{25}+b_{35} \geq b_{36}+b_{46}$ and $b_{13} \geq b_{24}$ Applying the isomorphism, we see that these correspond to $x'_{33} = x_{33}+1$ along with the conditions $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} \geq x_{44}+x_{43}+x_{22}-2x_{33}-x_{32}$ and $x_{44}+x_{43}+x_{42}+x_{22}+x_{21}-2x_{33}-2x_{32}-x_{31} \geq x_{44}-x_{33}$.

These all correspond to the cases of $\tilde{e}_3(\Omega(b))$

Finally we consider $\tilde{e}_4$, which has 4 cases:

1. $b'_{14} = b_{14}+1$, $b'_{15} = b_{15}-1$ if $b_{14} \geq b_{25}$, $b_{14}+b_{24} \geq b_{25}+b_{35}$, and $b_{14}+b_{24}+b_{34} \geq b_{25}+b_{35}+b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{44} = x_{44}+1$ along with the conditions $-x_{44} \geq 2x_{33}-2x_{44}-x_{43}$, $-x_{44} \geq 2x_{32}+2x_{33}-2x_{44}-2x_{43}-x_{42}$, and $-x_{44} \geq 2x_{31}+2x_{32}+2x_{33}-2x_{44}-2x_{43}-2x_{42}-x_{41}$

2. $b'_{24} = b_{24} + 1$, $b'_{25} = b_{25} - 1$ if $b_{14} < b_{25}$, $b_{24} \geq b_{35}$, and $b_{24} + b_{34} \geq b_{35} + b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{43} = x_{43} + 1$ along with the conditions $-x_{44} < 2x_{33} - 2x_{44} - x_{43}$, $2x_{33} - 2x_{44} - x_{43} \geq 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, and $2x_{33} - 2x_{44} - x_{43} \geq 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

3. $b'_{34} = b_{34} + 1$, $b'_{35} = b_{35} - 1$ if $b_{14} + b_{24} < b_{25} + b_{35}$, $b_{24} < b_{35}$, and $b_{34} \geq b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{42} = x_{42} + 1$ along with the conditions $-x_{44} < 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, $2x_{33} - 2x_{44} - x_{43} < 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42}$, and $2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42} \geq 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

4. $b'_{44} = b_{44} + 1$, $b'_{45} = b_{45} - 1$ if $b_{14} + b_{24} + b_{34} < b_{25} + b_{35} + b_{45}$, $b_{24} + b_{34} < b_{35} + b_{45}$, and $b_{34} < b_{45}$. Applying the isomorphism, we see that this corresponds to $x'_{41} = x_{41} + 1$ along with the conditions $-x_{44} < 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$, $2x_{33} - 2x_{44} - x_{43} < 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$, and $2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - x_{42} < 2x_{31} + 2x_{32} + 2x_{33} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$

This corresponds to all of the cases of $\tilde{e}_4(\Omega(b))$. Thus 2. is proved. Now we prove $\mathrm{wt}_k(\Omega(b)) = \mathrm{wt}_k(b)$ for $k = 0, 1, 2, 3, 4$.

$$
\begin{aligned}
&\mathrm{wt}_0(b) = b_{48} - b_{11} = -2x_{11} = \mathrm{wt}_0(\Omega(b))\\
&\mathrm{wt}_1(b) = b_{47} - b_{12} = 2x_{11} - x_{21} - x_{22} = \mathrm{wt}_1(\Omega(b))\\
&\mathrm{wt}_2(b) = b_{12} - b_{23} + b_{46} - b_{13} = 2x_{21} + 2x_{22} - x_{11} - x_{31} - x_{32} - x_{33} = \mathrm{wt}_2(\Omega(b))\\
&\mathrm{wt}_3(b) = b_{45} - b_{14} + b_{13} - b_{24} + b_{23} - b_{34}\\
&= 2x_{31} + 2x_{32} + 2x_{33} - x_{21} - x_{22} - x_{41} - x_{42} - x_{43} - x_{44} = \mathrm{wt}_3(\Omega(b))\\
&\mathrm{wt}_4(b) = b_{44} - b_{15} + b_{34} - b_{45} + b_{24} - b_{35} + b_{14} - b_{25}\\
&= 2x_{41} + 2x_{42} + 2x_{43} + 2x_{44} - 2x_{31} - 2x_{32} - 2x_{33} = \mathrm{wt}_4(\Omega(b))
\end{aligned}
$$

This proves 3. Finally, we need to show $\varepsilon_k(\Omega(b)) = \varepsilon_k(b)$.

For $k=0$, we have

$$\varepsilon_k(b)=\begin{cases}-b_{48}-b_{47}-b_{46}-b_{45} & 1<2,3,4,5,6,7,8\\ -b_{48}-b_{12}-b_{35}-b_{36} & 2\geq 1,2<3,4,5,6,7,8\\ -b_{48}-b_{47}-b_{35}-b_{23} & 3\geq 1,2,3<4,5,6,7,8\\ -b_{48}-b_{47}-b_{46}-b_{34} & 4\geq 1,2,3,4<5,6,7,8\\ -b_{48}-b_{47}-b_{24}-b_{23} & 5\geq 1,2,3,4,5<6,7,8\\ -b_{48}-b_{25}-b_{13}-b_{12} & 6\geq 1,2,3,4,5,6<7,8\\ -b_{48}-b_{36}-b_{24}-b_{12} & 7\geq 1,2,3,4,5,6,7<8\\ -b_{48}-b_{14}-b_{13}-b_{12} & 8\geq 1,2,3,4,5,6,7\end{cases}$$

where $1=b_{45}+b_{46}+b_{47}$, $2=b_{12}+b_{35}+b_{36}$, $3=b_{23}+b_{35}+b_{47}$, $4=b_{34}+b_{46}+b_{47}$, $5=b_{23}+b_{24}+b_{47}$, $6=b_{12}+b_{13}+b_{25}$, $7=b_{12}+b_{24}+b_{36}$, and $8=b_{12}+b_{13}=b_{14}$. Applying the isomorphism, we get the following, which is the same as $\varepsilon_0(\Omega(b))$

$$\varepsilon_0(\Omega(b))=\begin{cases}x_{41} & x_{41}>A,B,C,X,Y,Z,2x_{11}-x_{44}\\ 2x_{11}-2x_{22}+x_{42} & A\geq x_{41},A>B,C,X,Y,Z,2x_{11}-x_{44}\\ 2x_{21}-2x_{32}+x_{42} & B\geq x_{41},A,B>C,X,Y,Z,2x_{11}-x_{44}\\ 2x_{31}-x_{42} & C\geq x_{41},A,B,C>X,Y,Z,2x_{11}-x_{44}\\ 2x_{21}-x_{43} & X\geq x_{41},A,B,C,X>Y,Z,2x_{11}-x_{44}\\ 2x_{11}-2x_{33}+x_{43} & Y\geq x_{41},A,B,C,X,Y>Z,2x_{11}-x_{44}\\ 2x_{11}+2x_{32}-2x_{22}-x_{43} & Z\geq x_{41},A,B,C,X,Y,Z>2x_{11}-x_{44}\\ 2x_{11}-x_{44} & 2x_{11}-x_{44}\geq x_{41},A,B,C,X,Y,Z\end{cases}$$

where $A=2x_{11}-2x_{22}+x_{42}$, $B=2x_{21}-2x_{32}+x_{42}$, $C=2x_{31}-x_{42}$, $X=2x_{21}-x_{43}$, $Y=2x_{11}-2x_{33}+x_{43}$, and $Z=2x_{11}+2x_{32}-2x_{22}-x_{43}$.

For $k=1$, $\varepsilon_1(b)=b_{12}$. Applying the isomorphism, we get $x_{22}-x_{11}$ which is equal to $\varepsilon_1(\Omega(b))$.

For $k=2$,

$$\varepsilon_2(b)=\begin{cases}b_{13} & b_{23}-b_{12}<0\\ b_{13}+b_{23}-b_{12} & b_{23}-b_{12}\geq 0\end{cases}$$

Applying the isomorphism, we get exactly $\varepsilon_2(\Omega(b))$ as follows:

$$\varepsilon_2(\Omega(b)) = \begin{cases} x_{33} - x_{22} & x_{33} - x_{22} > x_{33} + x_{32} + x_{11} - 2x_{22} - x_{21} \\ x_{33} + x_{32} + x_{11} - 2x_{22} - x_{21} & x_{33} - x_{22} \leq x_{33} + x_{32} + x_{11} - 2x_{22} - x_{21} \end{cases}$$

For $k = 3$,

$$\varepsilon_3(b) = \begin{cases} b_{14} & b_{24} - b_{13} < 0, \ \ b_{24} + b_{34} - b_{13} - b_{23} < 0 \\ b_{14} + b_{24} - b_{13} & b_{23} - b_{12} \geq 0, \ \ b_{34} - b_{23} < 0 \\ b_{14} + b_{24} - b_{13} + b_{34} - b_{23} & b_{34} - b_{23} \geq 0, \ \ b_{24} + b_{34} - b_{13} - b_{23} \geq 0 \end{cases}$$

Applying the isomorphism, we get exactly $\varepsilon_3(\Omega(b))$ as follows:

$$\varepsilon_3(\Omega(b)) = \begin{cases} x_{44} - x_{33} & D > E, F \\ x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32} & D \leq E, E > F \\ x_{44} + x_{43} + x_{42} + x_{21} + x_{22} - 2x_{33} - 2x_{32} - x_{31} & F \geq D, E \end{cases}$$

where $D = x_{44} - x_{33}$, $E = x_{44} + x_{43} + x_{22} - 2x_{33} - x_{32}$, and $F = x_{44} + x_{43} + x_{42} + x_{21} + x_{22} - 2x_{33} - 2x_{32} - x_{31}$.

Finally we check the relation for $k = 4$.

$$\varepsilon_4(b) = \begin{cases} b_{15} & (1) > (2), (3), (4) \\ b_{15} + b_{22} - b_{14} & (2) \geq (1), (2) > (3), (4) \\ b_{15} + b_{25} - b_{14} + b_{35} - b_{24} & (3) \geq (2), (1), (3) > (4) \\ b_{15} + b_{25} - b_{14} + b_{35} - b_{24} + b_{45} - b_{34} & (4) \geq (3), (2), (1) \end{cases}$$

where $(1) = b_{15}$, $(2) = b_{15} + b_{25} - b_{14}$, $(3) = b_{15} + b_{25} - b_{14} + b_{35} - b_{24}$, and $(4) = b_{15} + b_{25} - b_{14} + b_{35} - b_{24} + b_{45} - b_{34}$.

Applying the isomorphism, we get exactly $\varepsilon_4(\Omega(b))$ as follows:

$$\varepsilon_4(\Omega(b)) = \begin{cases} -x_{44} & A > B, C, D \\ 2x_{33} - 2x_{44} - x_{43} & B \geq A, B > C, D \\ 2x_{33} + 2x_{32} - 2x_{44} - 2x_{43} - x_{42} & C \geq A, B, C > D \\ 2x_{33} + 2x_{32} + 2x_{31} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41} & D \geq A, B, C \end{cases}$$

where $A = -x_{44}$, $B = 2x_{33} - 2x_{44} - x_{43}$, and $C = 2x_{33} + 2x_{32} - 2x_{44} - 2x_{43} - x_{42}$, $D = 2x_{33} + 2x_{32} + 2x_{31} - 2x_{44} - 2x_{43} - 2x_{42} - x_{41}$.

This proves 4. Therefore, Ω is an isomorphism. □

We have shown in this section that Conjecture 1 is true for the positive geometric crystal associated with Dynkin node $i = n$ for $C_n^{(1)}$ and $n = 2, 3, 4$.

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