ABSTRACT

MAINELLIS, ERIK KARSTEN. Factor Systems and Schur Multipliers for the Algebras of Loday. (Under the Direction of Ernest Stitzinger).

The dissertation concerns extension theory and second cohomology for several classes of Loday algebras (diassociative, Leibniz, dendriform, and Zinbiel algebras). Using group and Lie theory as a guide, we first develop a theory of factor systems, or nonabelian 2-cocycles, for the algebras under consideration. Factor systems give rise to a characterization of the second cohomology group by extensions. We apply our theory of factor systems to prove a criterion for the nilpotence of certain related extensions. Notably, we obtain its associative analogue as a special case of diassociative algebras. We then narrow our focus to the case of central extensions and their corresponding central factor systems, or 2-cocycles. We investigate analogues of the group-theoretic Schur multiplier and the related notion of covers in the contexts of Leibniz and diassociative algebras. We start with the Leibniz setting and construct a Hochschild-Serre type cohomological sequence of low dimension. We then use this sequence to characterize the multiplier by the second cohomology group with coefficients in the field. Finally, we develop criteria for when the center of a cover maps onto the center of a Leibniz algebra. Along the way, we obtain a brief theory of unicentral algebras and stem extensions. We focus on diassociative algebras for the rest of the dissertation. Our first efforts here are to prove the uniqueness of the cover, as well as to obtain a characterization of the multiplier in terms of a free presentation. These results, which were already known in the Leibniz case, are used to develop diassociative analogues of our Leibniz investigation of multipliers and covers. We carry our diassociative theory further, focusing on the multipliers and covers of perfect algebras. Here, we examine universal central extensions and their relation with perfect algebras. We prove that the cover of a perfect diassociative algebra is itself perfect and has trivial multiplier. Finally, we consider the multipliers of nilpotent diassociative algebras. We first prove an alternative method for extending our Hochschild-Serre sequence. This result is applied to obtain another extension as well as a series of dimension bounds on the multiplier of a nilpotent diassociative algebra. We briefly explore the associative specialization of these results, and conclude with an example that demonstrates some of the dimension bounds. In particular, we consider an associative algebra and compute its multiplier as an associative algebra and as a diassociative algebra.

Factor Systems and Schur Multipliers for the Algebras of Loday

by
Erik Karsten Mainellis

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APPROVED BY:

Kailash Misra	Radmila Sazdanovic
Naihuan Jing	Ernest Stitzinger Chair of Advisory Committee

BIOGRAPHY

Erik Mainellis was born in 1996 in Elk Grove, IL to Scott and Kris Mainellis. He grew up in Bartlett, IL and attended South Elgin High School, graduating in 2014. He went to Loyola University Chicago in the fall of 2014 and earned a Bachelor of Science in Mathematics with a minor in philosophy, graduating in 2017. He went to North Carolina State University in the fall of 2018 to begin his doctoral studies in mathematics. Under the advisement of Dr. Ernest Stitzinger, he completed his PhD in 2022.

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1

INTRODUCTION

The objective of the research contained in this dissertation is to advance extension theory in the context of several classes of Loday algebras. In particular, the results herein concern noncentral factor systems, extensions of nilpotent algebras, multipliers and covers, stem extensions, unicentral and perfect algebras, and exact sequences involving the second cohomology group. In this introduction, we discuss several areas of context and preliminaries. Along the way, we establish an overview of the dissertation.

1.1 The Algebras of Loday

Let $\mathbb{F}$ be a field. Throughout this paper, all algebras will be $\mathbb{F}$-vector spaces equipped with bilinear multiplications that satisfy certain identities. We first recall that a *Lie algebra* L has multiplication that is *alternating* and satisfies the *Jacobi identity*. Respectively, this means that $xx = 0$ and that $(xy)z + (yz)x + (zx)y = 0$ for all $x, y, z \in L$. Lie algebras are the most famous example of nonassociative algebras, and have been studied since the 1800s. In the early 1990s, Jean-Louis Loday generated interest in another class of nonassociative algebras, *Leibniz algebras*, as a generalization of Lie algebras [10]. We define a (left) Leibniz algebra L to be equipped with a multiplication that satisfies the *(left) Leibniz identity* $x(yz) = (xy)z + y(xz)$ for all $x, y, z \in L$. Leibniz algebras are famously seen as the non-anticommutative generalization of Lie algebras, since the Leibniz identity can be rearranged to form the Jacobi identity under skew-symmetry. Loday also defined *dual Leibniz algebras* [11], which later took the name *Zinbiel algebras* based on Loday's pen name, G. W. Zinbiel, or "Leibniz" backwards. He wrote under this name in [24], an "encyclopedia" that lists various algebras and some of their properties from an operadic point of view. The operad that encodes Zinbiel algebras is dual to that of Leibniz algebras under the notion of Koszul duality, that was formulated for operads in the celebrated and highly-referenced work of Ginzburg and Kapranov [5]. For the present paper, we define a Zinbiel algebra Z as having multiplication that satisfies what we will call the *Zinbiel identity* $(xy)z = x(yz) + x(zy)$ for all $x, y, z \in Z$. Finally, Loday introduced the notions of *diassociative algebras* (or *associative dialgebras*) and *dendriform algebras* in the context of algebraic K-theory [12]. The operads that encode these algebras are also Koszul dual. Diassociative and dendriform algebras are classes of *dialgebras*, or algebras having two multiplications. In particular, a diassociative algebra D is a vector space equipped with two associative bilinear products $\dashv$ and $\vdash$ that satisfy

$$
\begin{aligned}
x \dashv (y \dashv z) &= x \dashv (y \vdash z) \quad \text{D1}\\
(x \vdash y) \dashv z &= x \vdash (y \dashv z) \quad \text{D2}\\
(x \dashv y) \vdash z &= (x \vdash y) \vdash z \quad \text{D3}
\end{aligned}
$$

for all $x, y, z \in D$. A dendriform algebra E is a vector space equipped with two bilinear products, that we will denote as $<$ and $>$, that satisfy

$$
\begin{aligned}
(x < y) < z &= x < (y < z) + x < (y > z) \quad \text{E1}\\
(x > y) < z &= x > (y < z) \quad \text{E2}\\
(x < y) > z + (x > y) > z &= x > (y > z) \quad \text{E3}
\end{aligned}
$$

for all $x, y, z \in E$.

These *algebras of Loday* fit nicely into a butterfly diagram (Figure 1.1) of inclusion functors between the categories of Zinbiel, dendriform, commutative, associative, diassociative, Lie, and Leibniz algebras. Said diagram depicts the symmetry of their corresponding operads under Koszul duality, reflected across a vertical line through As.

One powerful aspect of this diagram is that any northeast movement along its arrows corresponds to a generalization of algebra type. This fact can be reasoned by seeing each algebra as a special case of its northeast-adjacent category. Besides the aforementioned Lie to Leibniz comparison, any Zinbiel algebra can be seen as a dendriform algebra in which

$$\begin{tikzcd}
 & \text{Dend} \arrow[dr] & & \text{Dias} \arrow[dr] & \\
\text{Zinb} \arrow[ur] \arrow[dr] & & \text{As} \arrow[ur] \arrow[dr] & & \text{Leib} \\
 & \text{Com} \arrow[ur] & & \text{Lie} \arrow[ur] &
\end{tikzcd}$$

Figure 1.1 Butterfly diagram.

$x < y = y > x$. Next, any commutative algebra is simply an associative algebra in which $xy = yx$. Finally, any associative algebra can be seen as a diassociative algebra in which $x \dashv y = x \vdash y$. Thus, any result that holds for the Leibniz, diassociative, and dendriform cases must necessarily hold for *all seven algebras*.

We use the following notation throughout this dissertation. Given subsets A and B of a diassociative algebra D, denote by $A \Diamond B$ the ideal $A \dashv B + A \vdash B$ in D. In the dendriform context, we allow for the natural variant $A \Diamond B = A < B + A > B$. For any algebra L, we denote by L' the *derived subalgebra* generated by all products in L. In the cases of Leibniz, associative, and Zinbiel algebras, this takes the form $L' = LL$. In the dialgebraic cases, we require $L' = L \Diamond L$ to account for both products.

1.2 Extension Theory

Consider a pair of $\mathscr{P}$ algebras A and B. Here, $\mathscr{P}$ can be thought to range over the seven classes of algebras in the above butterfly diagram. An *extension* of A by B is a short exact sequence $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ of homomorphisms for which L is a $\mathscr{P}$ algebra. A *section* of the extension is a linear map $\mu : B \longrightarrow L$ such that $\pi\mu = \mathrm{id}_B$. The extension is called *central* if $\sigma(A)$ is contained in the center $Z(L)$ of L. The *extension problem* concerns the classification, or, more broadly, the investigation, of all L such that $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ is an extension. A formal approach to extension theory can be traced back to Otto Schreier's 1926 paper [18], as discussed in the introduction of [2]. In the former, Schreier developed *factor systems*, also known as nonabelian 2-cocycles, as a tool for working on the extension problem of groups. Beyond groups, they have since appeared in other algebraic contexts. Given our algebras A and B, factor systems are, loosely, a tuple of maps that allow B to interact with A, and that satisfy some identities unique to each class of algebra. They are naturally in one-to-one correspondence with extensions, and give rise to the second cohomology group $\mathscr{H}^2(B, A)$.

In Chapter 2 of this dissertation, we establish an explicit theory of factor systems for the algebras under consideration. This forms a foundation for the rest of the thesis. We take our methodology from a chapter in W. R. Scott's *Group Theory* [20], in which the author investigates the correspondence between factor systems and extensions of groups. It should be noted that the research contained in this dissertation began as an effort to develop a Leibniz analogue of said chapter in order to work on extensions of nilpotent algebras. We later discovered that Leibniz factor systems had already been obtained in a 2018 paper [9]. We then turned to the diassociative, dendriform, and Zinbiel cases. We have decided, however, to review the Leibniz case in explicit detail. The purpose is to provide a systematic approach to this theory as well as a self-contained paper with consistent notation. After the Leibniz case, we derive the diassociative analogue. We note that the dendriform case follows by a similar process, replacing $\dashv$ with $<$ and $\vdash$ with $>$, although the identities that appear are uniquely determined by the different algebra structures. We provide the definitions of all seven factor systems for the sake of structural comparison. These appear in the same sections as their generalizations. Unlike many results, factor systems are a case in which generalizing algebra type gives rise to considerably more complicated structures. For example, a Leibniz factor system consists of three maps and seven defining identities, while a Lie factor system has only two maps and three identities. Chapter 2 concludes with a brief discussion of cohomology and how $\mathscr{H}^2(B,A)$ arises from extension theory. We include a list of 2-cocycles and their defining identities.

The applications of factor systems are numerous and well-known. For example, their identities appear frequently in Lie theory. One appearance is in Nathan Jacobson's *Lie Algebras* [7], in its section on the theorems of Levi and Malcev-Harish-Chandra. For more appearances of Lie factor systems, see [21], and the references therein. More generally, applications of factor systems have primarily involved the special case of central extensions and their corresponding central factor systems. In [4], for instance, the author classifies nilpotent associative algebras of low dimension using central factor systems. Furthermore, in the present thesis, we use central factor systems to work with the multipliers of algebras as well as with their second cohomology in general.

There are also strong applications of the more general noncentral factor systems. In particular, let A and B be $\mathscr{P}$ algebras. An extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ is called *nilpotent* if L is nilpotent as an algebra. Generally, supposing A and B are nilpotent, an extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ is not necessarily nilpotent. However, it has been shown in [22] that, if A and B are Lie algebras, and if L_1 and L_2 are extensions that correspond to lifts of a map $\Phi : B \longrightarrow \mathrm{Out}(A)$, then L_1 is nilpotent if and only if L_2 is nilpotent. This criterion was based on the group analogue from [17], and its proof relies on noncentral factor systems of Lie algebras. In Chapter 3 of this dissertation, we prove analogous criteria for the algebras of Loday via the work of Chapter 2. As an important consequence, we thereby obtain its associative analogue as a special case of diassociative algebras. We detail several examples that highlight important intricacies in the results.

We now note that, given a pair of $\mathscr{P}$ algebras A and B, and an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ of A by B, one may assume that σ is the identity map. We make this assumption in Section 2.4, as well as in all chapters following Chapter 3, and think of A as being contained in L. Consequently, the extension is central if $A \subseteq Z(L)$.

1.3 The Schur Multiplier

In 1904, Schur introduced his multiplier in the study of group representations [19]. "Schur multipliers," along with the related notion of covers, have been of interest ever since. A great reference source is Karpilovsky's *The Schur Multiplier* [8]. Analogous notions have since been studied for Lie algebras, where there are major differences from the group case. Unlike the latter case, each algebra has a unique cover [1] and nilpotent algebras have nontrivial multiplier [15]. The uniqueness of covers has also been shown for Leibniz algebras [16].

For a $\mathscr{P}$ algebra L, multipliers and covers are defined as follows. A *defininining pair* (K, M) of L is itself a pair of $\mathscr{P}$ algebras that satisfies $K/M \cong L$ and $M \subseteq Z(K) \cap K'$. Such a pair is called a *maximal defining pair* if the dimension of K is maximal. In this case, we say that K is a *cover* of L and that M is the *multiplier* of L, denoted by $M(L)$. The multiplier is abelian and thus unique via dimension.

In Chapter 4 of this dissertation, we generalize a series of Lie-algebraic results from Chapters 3 and 4 in [1]. For a Leibniz algebra L, we begin by constructing a cohomological Hochschild-Serre type spectral sequence of low dimension

$$0 \longrightarrow \mathrm{Hom}(L/Z, A) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(L, A) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(Z, A) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/Z, A) \xrightarrow{\mathrm{Inf}_2} \mathscr{H}^2(L, A)$$

where Z is a central ideal of L and A is a central L-module. Specializing to $A = \mathbb{F}$, the sequence is used to characterize the multiplier in terms of the second cohomology group with coefficients in the field, and we obtain $M(L) \cong \mathscr{H}^2(L, \mathbb{F})$ when L is finite-dimensional. The sequence is then extended by a map

$$\delta : \mathscr{H}^2(L, \mathbb{F}) \longrightarrow L/L' \otimes Z \oplus Z \otimes L/L'$$

and an analogue of the Ganea sequence is constructed for Leibniz algebras. The maps involved with these exact sequences, as well as a characterization of the multiplier, are used to establish criteria for when a central ideal Z is contained in the set $Z^*(L)$, denoting the intersection of all images $\omega(Z(E))$ such that $0 \longrightarrow \ker \omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$ is a central extension of L. While it is easy to see that $Z^*(L) \subseteq Z(L)$, we say that a Leibniz algebra L is *unicentral* if $Z(L) = Z^*(L)$. The aforementioned criteria are specialized to the case $Z = Z(L)$, and we obtain conditions for when the center of a cover maps onto the center of the algebra. In particular, our criterion $Z \subseteq Z^*(L)$ becomes $Z(L) \subseteq Z^*(L)$, or when the algebra is unicentral.

We then turn to the diassociative setting. In Chapter 5, we establish a similar extension-theoretic crossroads between multipliers and covers, cohomology, and unicentral algebras. We first prove that covers of diassociative algebras are unique and obtain a characterization of the multiplier in terms of a free presentation via the methodology of [1]. In particular, it is shown that

$$M(L) \cong \frac{F' \cap R}{F \Diamond R + R \Diamond F}$$

where $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ is a free presentation of a diassociative algebra L. We then develop a diassociative analogue of the results from Chapter 4. In the diassociative context, our Hochschild-Serre sequence is extended by

$$\delta : \mathscr{H}^2(L, \mathbb{F}) \longrightarrow (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

because diassociative 2-cocycles are more complicated than Leibniz ones. It is remarkable that the diassociative analogue of Chapter 4 holds in its entirety, despite the significant structural differences between (di)associative algebras and Lie (or Leibniz) algebras. The same can be said for the final two chapters of this dissertation, which also take place in the diassociative setting. Notably, since diassociative algebras generalize associative algebras, we automatically obtain associative analogues of these three chapters.

In Chapter 6, we focus on the subclass of *perfect* diassociative algebras, i.e. algebras that are equal to their derived subalgebra. It is known that the multipliers, covers, and universal central extensions related to perfect Lie algebras have exceptional properties [1]. Some were generalized to Hom-Leibniz algebras in [3]. The objective of Chapter 6 in this dissertation is to obtain similar properties for diassociative algebras. Using Chapter 6 of [1] as a guide, we first establish a series of lemmas that relate universal central extensions to perfect algebras. For any universal central extension $0 \longrightarrow A \longrightarrow H \longrightarrow L \longrightarrow 0$ of diassociative algebras, it is shown that both L and H are perfect. Given a perfect diassociative algebra L, we also prove that the extension $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal if and only if every central extension of L splits. We then turn to the multipliers and covers of finite-dimensional perfect diassociative algebras. Given such an L, and using the characterization of $M(L)$ in terms of a free presentation $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ of L (from Chapter 5), we prove that $F'/(F \Diamond R + R \Diamond F)$ is a cover of L. It is then shown that the cover is perfect and that the extension

$$0 \longrightarrow M(L) \cong \frac{F' \cap R}{F \lozenge R + R \lozenge F} \longrightarrow \frac{F'}{F \lozenge R + R \lozenge F} \longrightarrow L \longrightarrow 0$$

is universal. Next, given a universal central extension $0 \longrightarrow A \longrightarrow L^* \longrightarrow L \longrightarrow 0$ of a perfect algebra L, we obtain $A \cong M(L)$, and prove that L^* is a cover of L. We also consider what happens when L has trivial multiplier. Finally, we use our extended Hochschild-Serre type spectral sequence, as established in Chapter 5, to prove that $C = C'$ and $M(C) = 0$ for any cover C of a finite-dimensional perfect diassociative algebra L.

In Chapter 7, we investigate the multipliers of nilpotent diassociative algebras. Following (primarily) a similar methodology to [23], we begin by extending our Hochschild-Serre sequence under alternative conditions. This automatically yields a new proof for the previous extension in the nilpotent case. As further applications, we obtain a handful of dimension bounds on the multiplier of a nilpotent diassociative algebra as well as another extension of our sequence. It is particularly interesting to consider an associative algebra and compare its multiplier *as* an associative algebra to its multiplier *as* a diassociative algebra. Such a phenomenon has been explored in the context of Lie and Leibniz multipliers [16]. We compute an example that highlights the associative to diassociative comparison as well as a couple of our dimension bounds.

CHAPTER

2

FACTOR SYSTEMS

Before studying factor systems, we first need to review some basic extension theory. Consider a pair of $\mathscr{P}$ algebras A and B. Two extensions $0 \longrightarrow A \xrightarrow{\sigma_1} L_1 \xrightarrow{\pi_1} B \longrightarrow 0$ and $0 \longrightarrow A \xrightarrow{\sigma_2} L_2 \xrightarrow{\pi_2} B \longrightarrow 0$ of A by B are called *equivalent* if there exists an isomorphism $\tau : L_1 \longrightarrow L_2$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\sigma_1"] \arrow[d, "\mathrm{id}_A"] & L_1 \arrow[r, "\pi_1"] \arrow[d, "\tau"] & B \arrow[r] \arrow[d, "\mathrm{id}_B"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\sigma_2"'] & L_2 \arrow[r, "\pi_2"'] & B \arrow[r] & 0 \end{tikzcd}$$

Figure 2.1 Equivalence of extensions.

commutes, i.e. such that $\tau\sigma_1 = \sigma_2$ and $\pi_2\tau = \pi_1$. An extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ of A by B is said to *split* if there exists a homomorphism $\mu : B \longrightarrow L$ that is also a section. An extension is called *abelian* if L is abelian. It is readily verified that equivalence of extensions is an equivalence relation. Furthermore, let $0 \longrightarrow A \xrightarrow{\sigma_1} L_1 \xrightarrow{\pi_1} B \longrightarrow 0$ be a split extension that is equivalent to another extension $0 \longrightarrow A \xrightarrow{\sigma_2} L_2 \xrightarrow{\pi_2} B \longrightarrow 0$ via the isomorphism τ. Then there is a homomorphism $\mu_1 : B \longrightarrow L_1$ such that $\pi_1\mu_1 = \mathrm{id}_B$, which implies that $\mu_2 = \tau\mu_1$ defines a homomorphism from B into L_2 satisfying $\pi_2\mu_2 = \pi_2\tau\mu_1 = \pi_1\mu_1 = \mathrm{id}_B$. Thus, if an extension splits, then so does every equivalent extension. Since an equivalence is an isomorphism, it is straightforward to verify that extensions equivalent to abelian extensions are abelian, and that extensions equivalent to central extensions are central.

2.1 Factor Systems of Leibniz Algebras

Recall that ad^l and ad^r denote the left and right multiplication operators respectively; ad^l is simply called ad in the Lie case since $\mathrm{ad}^r = -\mathrm{ad}$.

Definition 1. Let A and B be Leibniz algebras. A *factor system* of A by B is a tuple of maps (φ, φ', f) where

$\varphi : B \longrightarrow \mathrm{Der}(A)$ is linear,

$\varphi' : B \longrightarrow \mathscr{L}(A)$ is linear,

$f : B \times B \longrightarrow A$ is bilinear

such that

1. $m(\varphi(i)n) = (\varphi'(i)m)n + \varphi(i)(mn)$
2. $m(\varphi'(i)n) = \varphi'(i)(mn) + n(\varphi'(i)m)$
3. $\mathrm{ad}^r_{f(i,j)} + \varphi'(ij) = \varphi'(j)\varphi'(i) + \varphi(i)\varphi'(j)$
4. $\varphi(i)(mn) = (\varphi(i)m)n + m(\varphi(i)n)$
5. $\varphi(i)\varphi(j) = \varphi(ij) + \varphi(j)\varphi(i) + \mathrm{ad}^l_{f(i,j)}$
6. $\varphi(i)\varphi'(j) = \varphi'(j)\varphi(i) + \varphi'(ij) + \mathrm{ad}^r_{f(i,j)}$
7. $f(i, jk) + \varphi(i)f(j,k) = f(ij,k) + \varphi'(k)f(i,j) + f(j,ik) + \varphi(j)f(i,k)$

are satisfied for all $m, n \in A$ and $i, j, k \in B$. Note that the fourth identity allows for $\varphi : B \longrightarrow \mathrm{Der}(A)$.

Definition 2. Let A and B be Lie algebras and $\mathrm{Der}(A)$ be the Lie algebra under the commutator bracket. A *factor system* of A by B is a pair (φ, f) of functions

$\varphi : B \longrightarrow \mathrm{Der}(A)$ linear,

$f: B \times B \longrightarrow A$ bilinear

such that

1. $f(i,i) = 0$
2. $\varphi(ij) = [\varphi(i), \varphi(j)] - \mathrm{ad}_{f(i,j)}$
3. $\varphi(k)f(i,j) + \varphi(i)f(j,k) + \varphi(j)f(k,i) = f(ij,k) + f(jk,i) + f(ki,j)$

for all $i, j, k \in B$.

2.1.1 Belonging

Our first aim is to construct a correspondence between factor systems and extensions. Consider an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \rightarrow 0$ of A by B and a section $\mu: B \longrightarrow L$. Consider also the linear maps

$$\rho: L \longrightarrow \mathrm{Der}(\sigma(A)),$$
$$\rho': L \longrightarrow \mathscr{L}(\sigma(A))$$

defined by $\rho(x) = \mathrm{ad}^l_x|_{\sigma(A)}$ and $\rho'(x) = \mathrm{ad}^r_x|_{\sigma(A)}$ respectively for $x \in L$. Put simply, these maps denote the left and right multiplication operators that act on the image of σ in L. We next use ρ and ρ' to define the maps

$$P: L \longrightarrow \mathrm{Der}(A),$$
$$P': L \longrightarrow \mathscr{L}(A)$$

by $P(x) = \sigma^{-1}\rho(x)\sigma$ and $P'(x) = \sigma^{-1}\rho'(x)\sigma$ respectively, effectively formalizing a way for L to act on A. To work explicitly with these maps, one computes $P(x)m = \sigma^{-1}\rho(x)\sigma(m) = \sigma^{-1}(x\sigma(m))$ and $P'(x)m = \sigma^{-1}\rho'(x)\sigma(m) = \sigma^{-1}(\sigma(m)x)$ for any $m \in A$. The maps φ and φ' of a factor system are ways for B to act on A. It is thus natural to compose P and P' with μ, as well as to define f in terms of μ, which leads to Definition 3. The subsequent pair of converse results form the framework for our correspondence.

Definition 3. A factor system (φ, φ', f) of A by B *belongs* to the extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \rightarrow 0$ and μ if $\varphi = P\mu$, $\varphi' = P'\mu$, and $\sigma(f(i,j)) = \mu(i)\mu(j) - \mu(ij)$ for all $i, j \in B$.

Theorem 2.1.1. *Given an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \rightarrow 0$ of A by B and section $\mu: B \longrightarrow L$, there exists a unique factor system (φ, φ', f) of A by B belonging to the extension and μ.*

Proof. Let $\varphi = P\mu$ and $\varphi' = P'\mu$. To define f, one notes that $\mu(i)\mu(j) - \mu(ij) \in \ker \pi$ for any $i, j \in B$. By exactness, there exists an element $c_{i,j} \in A$ such that $\sigma(c_{i,j}) = \mu(i)\mu(j) - \mu(ij)$. Let f be defined by $f(i,j) = c_{i,j}$. One may verify that $f : B \times B \longrightarrow A$ is bilinear by applying σ to perform the computation and then applying σ^{-1}. It remains to verify that (φ, φ', f) is a factor system. The seven axioms follow by direct computation via the Leibniz identity. □

Theorem 2.1.2. *(Converse to Theorem 2.1.1) Let (φ, φ', f) be a factor system of A by B and let L denote the vector space $A \oplus B$ with multiplication $(m,i)(n,j) = (mn + \varphi(i)n + \varphi'(j)m + f(i,j)\,,\ ij)$ for $m, n \in A$ and $i, j \in B$. Let $\sigma : A \longrightarrow L$ by $\sigma(m) = (m, 0)$, $\pi : L \longrightarrow B$ by $\pi(m,i) = i$, and $\mu : B \longrightarrow L$ by $\mu(i) = (0, i)$. Then*

1. *L is a Leibniz algebra,*

2. *$0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ is an extension,*

3. *$\pi\mu = id_B$,*

4. *the factor system (φ, φ', f) belongs to the extension and μ.*

Proof. For part 1, the multiplication defined on L is clearly linear, and so it suffices to show that the Leibniz identity holds. One computes $(m,i)\big((n,j)(p,k)\big) = ((m,i)(n,j))(p,k) + (n,j)((m,i)(p,k))$ via the Leibniz identities on A and B and the axioms of the given factor system. Hence L is a Leibniz algebra.

For part 2, we first compute $\sigma(mn) = (mn, 0) = (m,0)(n,0) = \sigma(m)\sigma(n)$ and

$$\begin{aligned}\pi((m,i)(n,j)) &= \pi(mn + \varphi(i)n + \varphi'(j)m + f(i,j)\,,\ ij) \\ &= ij \\ &= \pi(m,i)\pi(n,j)\end{aligned}$$

which implies that σ and π are homomorphisms. Moreover, the exactness of $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ is trivial. Part 3 is also immediate. For part 4, let $m \in A$ and $i, j \in B$. Then

$$\begin{aligned}P\mu(i)m &= \sigma^{-1}((0,i)(m,0)) \\ &= \sigma^{-1}(\varphi(i)m + f(i,0)\,,\ 0) \\ &= \varphi(i)m\end{aligned}$$

implies that $P\mu = \varphi$. The equality $P'\mu = \varphi'$ holds by similar computation. Finally, $\sigma(f(i,j)) = (f(i,j), 0) = (0,i)(0,j) - (0, ij) = \mu(i)\mu(j) - \mu(ij)$. Hence our factor system belongs to the extension and μ. □

2.1.2 Equivalence

We now define a relation between factor systems under which a change in section (to an equivalent extension) corresponds to a change in factor system to an equivalent one. Such a notion establishes equivalence classes of factor systems that correspond to equivalence classes of extensions. Thus, Theorem 2.1.3 strengthens the correspondence of the first two theorems.

Definition 4. Factor systems (φ, φ', f) and (ψ, ψ', g) of A by B are called *equivalent* if there exists a linear transformation $\varepsilon : B \longrightarrow A$ such that

1. $\psi(i) = \varphi(i) + \mathrm{ad}^{l}_{\varepsilon(i)}$,

2. $\psi'(i) = \varphi'(i) + \mathrm{ad}^{r}_{\varepsilon(i)}$,

3. $g(i, j) = f(i, j) + \varphi'(j)\varepsilon(i) + \varphi(i)\varepsilon(j) + \varepsilon(i)\varepsilon(j) - \varepsilon(ij)$

for all $i, j \in B$. The function ε is called an *equivalence*.

Theorem 2.1.3. *If the factor system $(\varphi_1, \varphi'_1, f_1)$ belongs to the extension $0 \longrightarrow A \xrightarrow{\sigma_1} L_1 \xrightarrow{\pi_1} B \to 0$ and μ_1 and the factor system $(\varphi_2, \varphi'_2, f_2)$ belongs to the extension $0 \longrightarrow A \xrightarrow{\sigma_2} L_2 \xrightarrow{\pi_2} B \to 0$ and μ_2, then the factor systems are equivalent if and only if the extensions are equivalent.*

Proof. ($\Longrightarrow$) Assume the factor systems are equivalent and let ε be the corresponding equivalence. Recall that an equivalence of extensions requires an isomorphism $\tau : L_1 \longrightarrow L_2$ such that $\tau\sigma_1 = \sigma_2$ and $\pi_2\tau = \pi_1$. We know that any element in L_1 has a unique representation of the form $\mu_1(i) + \sigma_1(m)$ for $i \in B$ and $m \in A$. Define $\tau(\mu_1(i) + \sigma_1(m)) = \mu_2(i) + \sigma_2(-\varepsilon(i) + m)$. Clearly τ is linear. To show that τ preserves multiplication, consider elements $a, b \in L_1$ with unique representations $a = \mu_1(i) + \sigma_1(m)$ and $b = \mu_1(j) + \sigma_1(n)$. We first compute

$$
\begin{aligned}
\tau(ab) &= \tau\big(\mu_1(i)\mu_1(j) + \sigma_1(m)\mu_1(j) + \mu_1(i)\sigma_1(n) + \sigma_1(m)\sigma_1(n)\big) \\
&= \tau\Big(T_1(ij) + \sigma_1\big(f_1(i, j) + \varphi'_1(j)m + \varphi_1(i)n + mn\big)\Big) && \text{belonging} \\
&= T_2(ij) + \sigma_2\big(-\varepsilon(ij) + f_1(i, j) + \varphi'_1(j)m + \varphi_1(i)n + mn\big).
\end{aligned}
$$

On the other hand, one computes

$$\begin{aligned}\tau(a)\tau(b) &= \mu_2(i)\mu_2(j)+\mu_2(i)\sigma_2(-\varepsilon(j))+\mu_2(i)\sigma_2(n)+\sigma_2(-\varepsilon(i))\mu_2(j)\\ &\quad +\sigma_2(m)\mu_2(j)+\sigma_2(-\varepsilon(i))\sigma_2(-\varepsilon(j))+\sigma_2(m)\sigma_2(-\varepsilon(j))\\ &\quad +\sigma_2(-\varepsilon(i))\sigma_2(n)+\sigma_2(m)\sigma_2(n)\\ &= \mu_2(ij)+\sigma_2\big(f_2(i,j)-\varphi_2(i)\varepsilon(j)+\varphi_2(i)n-\varphi_2'(j)\varepsilon(i) \qquad \text{belonging}\\ &\qquad\qquad +\varphi_2'(j)m+\varepsilon(i)\varepsilon(j)-m\varepsilon(j)-\varepsilon(i)n+mn\big)\\ &= \mu_2(ij)+\sigma_2(p)\end{aligned}$$

where $p \in A$ is the expression in the argument of σ_2. Since $\mu_2(ij)$ is the only μ_2 term on both sides, it remains to check the σ_2 parts. Compute

$$\begin{aligned}p &= f_1(i,j)+\varphi_1'(j)\varepsilon(i)+\varphi_1(i)\varepsilon(j)+\varepsilon(i)\varepsilon(j)-\varepsilon(ij) && \text{equivalence axiom 3}\\ &\quad -\varphi_1(i)\varepsilon(j)-\varepsilon(i)\varepsilon(j)+\varphi_1(i)n+\varepsilon(i)n && \text{equivalence axiom 1}\\ &\quad -\varphi_1'(j)\varepsilon(i)-\varepsilon(i)\varepsilon(j)+\varphi_1'(j)m+m\varepsilon(j) && \text{equivalence axiom 2}\\ &\quad +\varepsilon(i)\varepsilon(j)-m\varepsilon(j)-\varepsilon(i)n+mn\\ &= f_1(i,j)-\varepsilon(ij)+\varphi_1(i)n+\varphi_1'(j)m+mn.\end{aligned}$$

Thus τ preserves multiplication. The computation

$$\begin{aligned}\pi(\mu_1(i)+\sigma_1(m)) = i &\\ &= \pi_2(\mu_2(i)+\sigma_2(-\varepsilon(i)+m))\\ &= \pi_2\tau(\mu_1(i)+\sigma_1(m))\end{aligned}$$

implies that $\pi_1=\pi_2\tau$. Finally, $\tau\sigma_1(m)=\sigma_2(m)$ for all $m\in A$ by the definition of τ. Hence $\tau\sigma_1=\sigma_2$ and the extensions are equivalent.

($\Longleftarrow$) Conversely, assume that the extensions are equivalent. Then there exists an isomorphism $\tau : L_1 \longrightarrow L_2$ such that $\tau\sigma_1 = \sigma_2$ and $\pi_2\tau = \pi_1$. The equality $\pi_1\tau^{-1}\mu_2(i) = \pi_2\mu_2(i) = \pi_1\mu_1(i)$ holds for any $i \in B$, yielding an element $\tau^{-1}\mu_2(i)-\mu_1(i) \in \ker\pi_1$. By exactness, $\ker\pi_1 = \operatorname{Im}\sigma_1$, and so there exists an element $n_i \in A$ such that $\tau^{-1}\mu_2(i)=\mu_1(i)+\sigma_1(n_i)$. Define $\varepsilon : B \longrightarrow A$ by $\varepsilon(i)=n_i$. By direct computation, ε is an equivalence. Thus the factor systems are equivalent. □

Two results follow easily from Theorem 2.1.3. The proofs are stated as one because they are so short.

Corollary 2.1.4. *Given an extension* $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$*, let* $\mu_1 : B \longrightarrow L$ *and* $\mu_2 : B \longrightarrow L$ *be linear maps such that* $\pi\mu_1 = id_B = \pi\mu_2$*. Suppose also that* (φ, φ', f) *is a factor system of A by B which belongs to the extension and* μ_1*, and* (ψ, ψ', g) *is a factor system of A by B which belongs to the extension and* μ_2*. Then* (φ, φ', f) *is equivalent to* (ψ, ψ', g)*.*

Corollary 2.1.5. *Equivalence of factor systems is an equivalence relation.*

Proof. For Corollary 2.1.4, note first that any extension of A by B is equivalent to itself. By Theorem 2.1.3, factor systems belonging to this extension (and differing μ_i) are equivalent. Corollary 2.1.5 follows from Theorem 2.1.3 and the fact that equivalence of extensions is an equivalence relation. □

We now look to ε. Given equivalent factor systems, there may be multiple equivalences between them. On the other hand, *any* linear transformation $\varepsilon : B \longrightarrow A$ defines an equivalence of factor systems, as demonstrated by Theorem 2.1.6.

Theorem 2.1.6. *If* (φ, φ', f) *is a factor system of A by B and* ε *is a linear transformation from B to A, then there exists a factor system* (ψ, ψ', g) *such that* ε *is an equivalence of* (φ, φ', f) *with* (ψ, ψ', g)*. Furthermore, if* ε *is an equivalence, then* (ψ, ψ', g) *is unique.*

Proof. Let (ψ, ψ', g) be defined by

i. $\psi(i) = \varphi(i) + \mathrm{ad}^l_{\varepsilon(i)}$,

ii. $\psi'(j) = \varphi'(j) + \mathrm{ad}^r_{\varepsilon(j)}$,

iii. $g(i,j) = f(i,j) + \varphi'(j)\varepsilon(i) + \varphi(i)\varepsilon(j) + \varepsilon(i)\varepsilon(j) - \varepsilon(ij)$

for $i, j \in B$. It is straightforward to check that $\psi, \psi' : B \longrightarrow \mathscr{L}(A)$ are linear transformations and that $g : B \times B \longrightarrow A$ is a bilinear form. By direct computation, (ψ, ψ', g) is a factor system. By construction, the two factor systems are equivalent with ε as their corresponding equivalence. It is straightforward to verify the uniqueness of (ψ, ψ', g). □

2.1.3 Split Extensions

We now discuss conditions under which $\varphi : B \longrightarrow \mathrm{Der}(A)$ is a homomorphism. Let (φ, φ', f) be a factor system of A by B. By axiom 5 of factor systems, we have

$$\varphi(i)\varphi(j) = \varphi(ij) + \varphi(j)\varphi(i) + \mathrm{ad}^l_{f(i,j)}$$

for all $i, j \in B$. Regarding multiplication on $\mathrm{Der}(A)$ as the usual commutator bracket, the equality $\varphi(ij) = [\varphi(i), \varphi(j)]$ holds if and only if $f(i,j) \in Z^l(A)$ for all $i, j \in B$. Hence φ is a homomorphism if and only if $f : B \times B \longrightarrow Z^l(A)$. Furthermore, if $Z^l(A) = 0$, then φ is a homomorphism if and only if $f = 0$. Finally, if A is abelian, this ensures that $\mathrm{ad}^l_m = 0$ for all $m \in A$. Hence axiom 5 of factor systems again implies that φ is a homomorphism.

Next, recall that if an extension splits, then so does every equivalent extension. We say that a factor system *splits* if and only if its corresponding extension splits. Therefore, if a factor system splits, then so does every equivalent factor system. Now consider a split extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ of A by B with associated homomorphism $\mu : B \longrightarrow L$ and let (φ, φ', f) be a factor system belonging to this extension. Then $\sigma(f(i,j)) = \mu(i)\mu(j) - \mu(ij) = 0$ for all $i, j \in B$, which implies that $f = 0$ since σ is injective. Axiom 5 of factor systems then implies that φ is a homomorphism.

The following theorem will be quite useful for later proofs.

Theorem 2.1.7. *Let (φ, φ', f) be a factor system of A by B. The following are equivalent:*

a. (φ, φ', f) splits,

b. (φ, φ', f) is equivalent to some factor system (ψ, ψ', g) such that $g = 0$,

c. there exists a linear transformation $\varepsilon : B \longrightarrow A$ such that $f(i,j) = -\varphi'(j)\varepsilon(i) - \varphi(i)\varepsilon(j) - \varepsilon(i)\varepsilon(j) + \varepsilon(ij)$ for all $i, j \in B$.

Proof. (a. $\Longrightarrow$ b.) We know (φ, φ', f) belongs to a split extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$. By definition, there is an associated homomorphism $\mu : B \longrightarrow L$ such that $\pi\mu = \mathrm{id}_B$. Hence there exists a factor system (ψ, ψ', g) belonging to $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ and μ which is equivalent to (φ, φ', f) by Corollary 2.1.4. Since μ is a homomorphism, we have $g = 0$.

(b. $\Longrightarrow$ c.) Let $\varepsilon : B \longrightarrow A$ be an equivalence of (φ, φ', f) with (ψ, ψ', g) where $g = 0$. The third axiom of equivalence gives $0 = g(i,j) = f(i,j) + \varphi'(j)\varepsilon(i) + \varphi(i)\varepsilon(j) + \varepsilon(i)\varepsilon(j) - \varepsilon(ij)$ for all $i, j \in B$, which implies the desired equality.

(c. $\Longrightarrow$ a.) Let ε be as in c. By Theorem 2.1.6, ε is an equivalence of (φ, φ', f) with another factor system (ψ, ψ', g) which belongs to an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ and $\mu : B \longrightarrow L$. One has $g(i,j) = f(i,j) + \varphi'(j)\varepsilon(i) + \varphi(i)\varepsilon(j) + \varepsilon(i)\varepsilon(j) - \varepsilon(ij) = 0$ by assumption. Then, since $\sigma(g(i,j)) = 0$ for all $i, j \in B$, the third axiom of belonging implies that μ is a homomorphism. Also, μ is injective since $\pi\mu = \mathrm{id}_B$. Hence the extension splits and, therefore, so does the original factor system. □

It is clear that every semidirect sum yields a split extension. The converse is also true in that every split extension of A by B is equivalent to a semidirect sum. Indeed, let $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ be a split extension of A by B. By Theorem 2.1.7, there is an equivalent extension $0 \longrightarrow A \longrightarrow L_2 \longrightarrow B \longrightarrow 0$ with associated linear map $\mu_2 : B \longrightarrow L_2$ and a factor system

(ψ, ψ', g) belonging to this extension and μ_2 such that $g(i, j) = 0$ for all $i, j \in B$. Thus ψ is a homomorphism. The Leibniz algebra construct in Theorem 2.1.2 is then a semidirect sum of A by B with factor system (ψ, ψ', g). By Theorem 2.1.3, the extension built in Theorem 2.1.2 is equivalent to $0 \longrightarrow A \longrightarrow L_2 \longrightarrow B \longrightarrow 0$ and hence to $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$.

2.1.4 Abelian A

Let A be an abelian Leibniz algebra and (φ, φ', f) be a factor system of A by B. Then φ is a homomorphism. Moreover, suppose a factor system (ψ, ψ', g) of A by B is equivalent to (φ, φ', f) via equivalence ε. Then

$$\varphi(i) = \psi(i) + \mathrm{ad}^l_{\varepsilon(i)} = \psi(i),$$
$$\varphi'(i) = \psi'(i) + \mathrm{ad}^r_{\varepsilon(i)} = \psi'(i)$$

for all $i \in B$, which implies that $\varphi = \psi$ and $\varphi' = \psi'$. We thus fix φ and φ' and define the following constructs.

Let $\mathrm{Fact}(B, A, \varphi, \varphi')$ be the set of bilinear maps $f : B \times B \longrightarrow A$ such that (φ, φ', f) is a factor system and let $\mathcal{T}(B, A, \varphi, \varphi')$ be the set of bilinear maps $f : B \times B \longrightarrow A$ such that (φ, φ', f) is a split factor system. We denote by $\mathrm{Ext}(B, A, \varphi, \varphi')$ the set of equivalence classes

$$\mathrm{Fact}(B, A, \varphi, \varphi')/\mathcal{T}(B, A, \varphi, \varphi')$$

with fixed φ and φ'.

Theorem 2.1.8. *If A is abelian, then*

1. $\mathrm{Fact}(B, A, \varphi, \varphi')$ *is an abelian Leibniz algebra,*
2. $\mathcal{T}(B, A, \varphi, \varphi')$ *is an ideal in* $\mathrm{Fact}(B, A, \varphi, \varphi')$*,*
3. *factor systems (φ, φ', f) and (φ, φ', g) are equivalent if and only if f and g are in the same coset of* $\mathrm{Fact}(B, A, \varphi, \varphi')$ *relative to* $\mathcal{T}(B, A, \varphi, \varphi')$*,*
4. *the quotient Leibniz algebra* $\mathrm{Ext}(B, A, \varphi, \varphi')$ *is in one-to-one correspondence with the set of equivalence classes of extensions to which φ and φ' belong.*

Proof. Let $f, g \in \mathrm{Fact}(B, A, \varphi, \varphi')$ and c be a scalar. We know $f - cg : B \times B \longrightarrow A$ and want to show that $(\varphi, \varphi', f - cg)$ is a factor system. Axioms 1, 2, and 4 are trivial since they do not involve f or g. Axioms 3, 5, and 6 hold since $\mathrm{ad}^l_{(f-cg)(i,j)} = 0$ and $\mathrm{ad}^r_{(f-cg)(i,j)} = 0$ for any

$i, j \in B$. Finally, axiom 7 holds by the following computation:

$$\begin{aligned}(f-cg)(i,jk)+\varphi(i)(f-cg)(j,k) &= f(i,jk)+\varphi(i)f(j,k)-c(g(i,jk)+\varphi(i)g(j,k))\\ &= f(ij,k)+\varphi'(k)f(i,j)+f(j,ik)+\varphi(j)f(i,k)\\ &\quad -c(g(ij,k)+\varphi'(k)g(i,j)+g(j,ik)+\varphi(j)g(i,k))\\ &=(f-cg)(ij,k)+\varphi'(k)(f-cg)(i,j)+(f-cg)(j,ik)\\ &\quad +\varphi(j)(f-cg)(i,k).\end{aligned}$$

Hence $\mathrm{Fact}(B, A, \varphi, \varphi')$ is a vector space. One easily checks that (φ, φ', fg) is a factor system (here juxtaposition denotes $f(i, j)g(i, j)$); indeed, $fg = 0$ since A is abelian. Thus $\mathrm{Fact}(B, A, \varphi, \varphi')$ is a Leibniz algebra with trivial multiplication.

To show that $\mathcal{T}(B, A, \varphi, \varphi')$ is an ideal, it suffices to verify that it is a subspace. Let $f, g \in \mathcal{T}(B, A, \varphi, \varphi')$. We want to show that $(\varphi, \varphi', f - cg)$ is a split factor system for scalar c. By Theorem 2.1.7, since (φ, φ', f) and (φ, φ', g) split, there exist linear maps $\varepsilon_1, \varepsilon_2 : B \longrightarrow A$ such that

$$\begin{aligned}f(i,j)&=-\varphi(j)\varepsilon_1(i)-\varphi(i)\varepsilon_1(j)-\varepsilon_1(i)\varepsilon_1(j)+\varepsilon_1(ij),\\ g(i,j)&=-\varphi(j)\varepsilon_2(i)-\varphi(i)\varepsilon_2(j)-\varepsilon_2(i)\varepsilon_2(j)+\varepsilon_2(ij)\end{aligned}$$

for any $i, j \in B$. Define $\varepsilon : B \longrightarrow A$ by $\varepsilon = \varepsilon_1 - c\varepsilon_2$. Then

$$(f-cg)(i,j)=-\varphi(j)\varepsilon(i)+\varphi'(i)\varepsilon(j)-\varepsilon(i)\varepsilon(j)+\varepsilon(ij)$$

and so $(\varphi, \varphi', f - cg)$ splits by Theorem 2.1.7.

Suppose factor systems (φ, φ', f) and (φ, φ', g) are equivalent via $\varepsilon : B \longrightarrow A$. Then $g(i, j) = f(i, j)+\varphi'(j)\varepsilon(i)+\varphi(i)\varepsilon(j)+\varepsilon(i)\varepsilon(j)-\varepsilon(ij)$ implies that

$$(f-g)(i,j)=-\varphi'(j)\varepsilon(i)-\varphi(i)\varepsilon(j)-\varepsilon(i)\varepsilon(j)+\varepsilon(ij).$$

By Theorem 2.1.7, factor system $(\varphi, \varphi', f - g)$ splits, and so

$$f+\mathcal{T}(B,A,\varphi,\varphi')=g+\mathcal{T}(B,A,\varphi,\varphi').$$

Conversely, if $(\varphi, \varphi', f - g)$ is a split factor system, then there exists a linear map $\varepsilon : B \longrightarrow A$ such that $(f - g)(i, j) = -\varphi'(j)\varepsilon(i)-\varphi(i)\varepsilon(j)-\varepsilon(i)\varepsilon(j)+\varepsilon(ij)$ for all $i, j \in B$ (by Theorem 2.1.7). Thus ε satisfies the third axiom of equivalence between factor systems (φ, φ', f) and (φ, φ', g). The first two axioms of equivalence hold trivially with $\varphi = \varphi$ and $\varphi' = \varphi'$ since

$\mathrm{ad}^l_m = 0$ and $\mathrm{ad}^r_m = 0$ for all $m \in A$.

The final statement follows from Theorem 2.1.3 and part 3 above. Indeed, part 3 says that two elements of $\mathrm{Ext}(B, A, \varphi, \varphi')$ are equal if and only if their factor systems with fixed φ, φ' are equivalent, and Theorem 2.1.3 guarantees that the latter statement is true if and only if the two extensions are equivalent. $\square$

2.1.5 Central Extensions

Recall that an extension which is equivalent to a central extension is itself central. One may thus refer to equivalence classes of central extensions and to *central factor systems*, i.e. factor systems that belong to central extensions. Once again, let A be an abelian Leibniz algebra and (φ, φ', f) be a factor system of A by B.

Theorem 2.1.9. *(φ, φ', f) is central if and only if $\varphi = 0$ and $\varphi' = 0$.*

Proof. By Theorem 2.1.2, (φ, φ', f) belongs to an extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ where $L = A \oplus B$ with multiplication $(m, i)(n, j) = (mn + \varphi(i)n + \varphi'(j)m + f(i, j)\,,\ ij)$. The extension is central if and only if $(m, i)(n, 0) = (mn + \varphi(i)n, 0) = (0, 0)$ and $(n, 0)(m, i) = (nm + \varphi'(i)n, 0) = (0, 0)$ for all $m, n \in A$ and $i \in B$. This happens if and only if φ and φ' are zero. $\square$

Theorem 2.1.10. *The classes of central extensions of A by B form a Leibniz algebra, denoted* $\mathrm{Cext}(B, A)$.

Proof. By Theorem 2.1.8 and Theorem 2.1.9; we set $\mathrm{Cext}(B, A) := \mathrm{Ext}(B, A, 0, 0)$. $\square$

Theorem 2.1.11. *Let A and B be abelian Leibniz algebras and let (φ, φ', f) be a central factor system of A by B. Then (φ, φ', f) belongs to an abelian extension if and only if $f = 0$.*

Proof. Since (φ, φ', f) is central, we know $\varphi = \varphi' = 0$. In the forward direction, the factor system belongs to an extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ and section μ. Since L and B are both abelian, one has $\sigma(f(i, j)) = \mu(i)\mu(j) - \mu(ij) = 0$ for all $i, j \in B$. Conversely, if $f = 0$, then the construction of L in Theorem 2.1.2 has multiplication $(m, i)(n, j) = (0, 0)$ for all $m, n \in A$ and $i, j \in B$. $\square$

2.2 Factor Systems of Diassociative Algebras

This section mimics the structure of the Leibniz case. We begin by stating the definition of factor systems for diassociative algebras as well as for the special cases of associative and commutative algebras. We proceed to construct diassociative analogues of the results from [20].

Definition 5. Let A and B be diassociative algebras. A *factor system* of A by B is a tuple $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ of maps such that

$\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash : B \longrightarrow \mathscr{L}(A)$ are linear,

$f_\dashv, f_\vdash : B \times B \longrightarrow A$ are bilinear,

and the following five sets of identities are satisfied for all $m, n, p \in A$ and $i, j, k \in B$:

1. Those resembling D1:
   (a) $m \dashv (\varphi_\dashv(j)p) = m \dashv (\varphi_\vdash(j)p)$
   (b) $m \dashv (\varphi'_\dashv(k)n) = m \dashv (\varphi'_\vdash(k)n)$
   (c) $m \dashv f_\dashv(j,k) + \varphi'_\dashv(j \dashv k)m = m \dashv f_\vdash(j,k) + \varphi'_\dashv(j \vdash k)m$
   (d) $\varphi_\dashv(i)(n \dashv p) = \varphi_\dashv(i)(n \vdash p)$
   (e) $\varphi_\dashv(i)(\varphi_\dashv(j)p) = \varphi_\dashv(i)(\varphi_\vdash(j)p)$
   (f) $\varphi_\dashv(i)(\varphi'_\dashv(k)n) = \varphi_\dashv(i)(\varphi'_\vdash(k)n)$
   (g) $\varphi_\dashv(i)f_\dashv(j,k) + f_\dashv(i, j \dashv k) = \varphi_\dashv(i)f_\vdash(j,k) + f_\dashv(i, j \vdash k)$
2. Those resembling D2:
   (a) $(\varphi_\vdash(i)n) \dashv p = \varphi_\vdash(i)(n \dashv p)$
   (b) $(\varphi'_\vdash(j)m) \dashv p = m \vdash (\varphi_\dashv(j)p)$
   (c) $f_\vdash(i,j) \dashv p + \varphi_\dashv(i \vdash j)p = \varphi_\vdash(i)(\varphi_\dashv(j)p)$
   (d) $\varphi'_\dashv(k)(m \vdash n) = m \vdash (\varphi'_\dashv(k)n)$
   (e) $\varphi'_\dashv(k)(\varphi_\vdash(i)n) = \varphi_\vdash(i)(\varphi'_\dashv(k)n)$
   (f) $\varphi'_\dashv(k)(\varphi'_\vdash(j)m) = \varphi'_\vdash(j \dashv k)m$
   (g) $\varphi'_\dashv(k)f_\vdash(i,j) + f_\dashv(i \vdash j, k) = \varphi_\vdash(i)f_\dashv(j,k) + f_\vdash(i, j \dashv k)$
3. Those resembling D3:
   (a) $(\varphi_\dashv(i)n) \vdash p = (\varphi_\vdash(i)n) \vdash p$
   (b) $(\varphi'_\dashv(j)m) \vdash p = (\varphi'_\vdash(j)m) \vdash p$
   (c) $f_\dashv(i,j) \vdash p + \varphi_\vdash(i \dashv j)p = f_\vdash(i,j) \vdash p + \varphi_\vdash(i \vdash j)p$
   (d) $\varphi'_\vdash(k)(m \dashv n) = \varphi'_\vdash(k)(m \vdash n)$
   (e) $\varphi'_\vdash(k)(\varphi_\dashv(i)n) = \varphi'_\vdash(k)(\varphi_\vdash(i)n)$

(f) $\varphi'_{\vdash}(k)(\varphi'_{\dashv}(j)m) = \varphi'_{\vdash}(k)(\varphi'_{\vdash}(j)m)$

(g) $\varphi'_{\vdash}(k)f_{\dashv}(i,j) + f_{\vdash}(i \dashv j, k) = \varphi'_{\vdash}(k)f_{\vdash}(i,j) + f_{\vdash}(i \vdash j, k)$

4. Those resembling the associativity of $\dashv$:

   (a) $m \dashv (\varphi_{\dashv}(j)p) = (\varphi'_{\dashv}(j)m) \dashv p$

   (b) $m \dashv (\varphi'_{\dashv}(k)n) = \varphi'_{\dashv}(k)(m \dashv n)$

   (c) $m \dashv f_{\dashv}(j,k) + \varphi'_{\dashv}(j \dashv k)m = \varphi'_{\dashv}(k)(\varphi'_{\dashv}(j)m)$

   (d) $\varphi_{\dashv}(i)(n \dashv p) = (\varphi_{\dashv}(i)n) \dashv p$

   (e) $\varphi_{\dashv}(i)(\varphi_{\dashv}(j)p) = \varphi_{\dashv}(i \dashv j)p + f_{\dashv}(i,j) \dashv p$

   (f) $\varphi_{\dashv}(i)(\varphi'_{\dashv}(k)n) = \varphi'_{\dashv}(k)(\varphi_{\dashv}(i)n)$

   (g) $\varphi_{\dashv}(i)f_{\dashv}(j,k) + f_{\dashv}(i, j \dashv k) = \varphi'_{\dashv}(k)f_{\dashv}(i,j) + f_{\dashv}(i \dashv j, k)$

5. Those resembling the associativity of $\vdash$:

   (a) $m \vdash (\varphi_{\vdash}(j)p) = (\varphi'_{\vdash}(j)m) \vdash p$

   (b) $m \vdash (\varphi'_{\vdash}(k)n) = \varphi'_{\vdash}(k)(m \vdash n)$

   (c) $m \vdash f_{\vdash}(j,k) + \varphi'_{\vdash}(j \vdash k)m = \varphi'_{\vdash}(k)(\varphi'_{\vdash}(j)m)$

   (d) $\varphi_{\vdash}(i)(n \vdash p) = (\varphi_{\vdash}(i)n) \vdash p$

   (e) $\varphi_{\vdash}(i)(\varphi_{\vdash}(j)p) = \varphi_{\vdash}(i \vdash j)p + f_{\vdash}(i,j) \vdash p$

   (f) $\varphi_{\vdash}(i)(\varphi'_{\vdash}(k)n) = \varphi'_{\vdash}(k)(\varphi_{\vdash}(i)n)$

   (g) $\varphi_{\vdash}(i)f_{\vdash}(j,k) + f_{\vdash}(i, j \vdash k) = \varphi'_{\vdash}(k)f_{\vdash}(i,j) + f_{\vdash}(i \vdash j, k)$

Definition 6. Let A and B be associative algebras. A *factor system* of A by B is a tuple of maps (φ, φ', f) where

$\varphi, \varphi' : B \longrightarrow \mathscr{L}(A)$ are linear,

$f : B \times B \longrightarrow A$ is bilinear

such that

1. $\varphi(i)\varphi(j) = \varphi(ij) + \mathrm{ad}^{l}_{f(i,j)}$
2. $\varphi'(i)\varphi'(j) = \varphi'(ji) + \mathrm{ad}^{r}_{f(i,j)}$
3. $\varphi(i)\varphi'(j) = \varphi'(j)\varphi(i)$

4. $\varphi'(i)(mn) = m(\varphi'(i)n)$

5. $\varphi(i)(mn) = (\varphi(i)m)n$

6. $(\varphi'(i)m)n = m(\varphi(i)n)$

7. $\varphi(i)f(j,k) + f(i,jk) = \varphi'(k)f(i,j) + f(ij,k)$

are satisfied for all $m, n \in A$ and $i, j, k \in B$.

Definition 7. Let A and B be commutative algebras. A *factor system* of A by B is a tuple of maps (φ, f) where

$\varphi : B \longrightarrow \mathscr{L}(A)$ is linear,

$f : B \times B \longrightarrow A$ is bilinear

such that

1. $f(i,j) = f(j,i)$

2. $\varphi(i)\varphi(j) = \varphi(ij) + \mathrm{ad}_{f(i,j)}$

3. $\varphi(i)(mn) = m(\varphi(i)n) = (\varphi(i)m)n$

4. $\varphi(i)f(j,k) + f(i,jk) = \varphi(k)f(i,j) + f(ij,k)$

are satisfied for all $m, n \in A$ and $i, j, k \in B$.

Definition 8. Let A and B be diassociative algebras. A factor system $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ of A by B *belongs* to an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ and section μ if

$$\varphi_\dashv = P_\dashv \mu, \qquad \varphi_\vdash = P_\vdash \mu,$$

$$\varphi'_\dashv = P'_\dashv \mu, \qquad \varphi'_\vdash = P'_\vdash \mu,$$

$$\sigma(f_\dashv(i,j)) = \mu(i) \dashv \mu(j) - \mu(i \dashv j), \qquad \sigma(f_\vdash(i,j)) = \mu(i) \vdash \mu(j) - \mu(i \vdash j)$$

for all $i, j \in B$, where

$$P_\dashv(x)m = \sigma^{-1}(x \dashv \sigma(m)), \qquad P_\vdash(x)m = \sigma^{-1}(x \vdash \sigma(m)),$$

$$P'_\dashv(x)m = \sigma^{-1}(\sigma(m) \dashv x), \qquad P'_\vdash(x)m = \sigma^{-1}(\sigma(m) \vdash x)$$

for any $x \in L$, $m \in A$.

Theorem 2.2.1. *Let A and B be diassociative algebras. Given an extension $0 \longrightarrow A \longrightarrow L \xrightarrow{\pi} B \longrightarrow 0$ of A by B and section $\mu : B \longrightarrow L$, there exists a unique factor system $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ of A by B belonging to the extension and μ.*

Proof. Set $\varphi_\dashv = P_\dashv \mu$, $\varphi_\vdash = P_\vdash \mu$, $\varphi'_\dashv = P'_\dashv \mu$, and $\varphi'_\vdash = P'_\vdash \mu$. Next, it is easily checked that $\mu(i) \dashv \mu(j) - \mu(i \dashv j)$ and $\mu(i) \vdash \mu(j) - \mu(i \vdash j)$ are in the kernel of π for all $i, j \in B$. We define $f_\dashv$ and $f_\vdash$ by $\sigma(f_\dashv(i,j)) = \mu(i) \dashv \mu(j) - \mu(i \dashv j)$ and $\sigma(f_\vdash(i,j)) = \mu(i) \vdash \mu(j) - \mu(i \vdash j)$ which are clearly bilinear maps $B \times B \longrightarrow A$. It is straightforward to verify that $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ is a factor system. □

Theorem 2.2.2. *Let $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ be a factor system of A by B and let L denote the vector space $A \oplus B$ with multiplications*

$$(m,i) \vdash (n,j) = (m \vdash n + \varphi_\vdash(i)n + \varphi'_\vdash(j)m + f_\vdash(i,j)\,,\ i \vdash j),$$
$$(m,i) \dashv (n,j) = (m \dashv n + \varphi_\dashv(i)n + \varphi'_\dashv(j)m + f_\dashv(i,j)\,,\ i \dashv j)$$

for $m, n \in A$ and $i, j \in B$. Let $\sigma : A \longrightarrow L$ by $\sigma(m) = (m, 0)$, $\pi : L \longrightarrow B$ by $\pi(m,i) = i$, and $\mu : B \longrightarrow L$ by $\mu(i) = (0, i)$. Then

1. *L is a diassociative algebra,*
2. *$0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ is an extension,*
3. *$\pi\mu = id_B$,*
4. *the factor system $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ belongs to the extension and μ.*

Proof. It takes five direct computations to verify that the vector space $L = A \oplus B$, with multiplications defined in the statement of the theorem, is a diassociative algebra. In particular, one must check D1, D2, D3, and the associativity of both $\dashv$ and $\vdash$. Said computations follow via the axioms of factor systems and the diassociative structures on A and B. □

We now define a notion of equivalence for factor systems so that equivalence classes of factor systems will correspond to those of extensions. The subsequent corollaries hold by the same logic as their Leibniz analogues.

Definition 9. Two factor systems $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ and $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$ of A by B are *equivalent* if there exists a linear transformation $\varepsilon : B \longrightarrow A$ such that

1. $\psi_\dashv(i) = \varphi_\dashv(i) + \mathrm{ad}^l_\dashv(\varepsilon(i))$,
2. $\psi'_\dashv(i) = \varphi'_\dashv(i) + \mathrm{ad}^r_\dashv(\varepsilon(i))$,

3. $\psi_\vdash(i) = \varphi_\vdash(i) + \mathrm{ad}^l_\vdash(\varepsilon(i))$,

4. $\psi'_\vdash(i) = \varphi'_\vdash(i) + \mathrm{ad}^r_\vdash(\varepsilon(i))$,

5. $g_\dashv(i,j) = f_\dashv(i,j) + \varphi'_\dashv(j)\varepsilon(i) + \varphi_\dashv(i)\varepsilon(j) + \varepsilon(i) \dashv \varepsilon(j) - \varepsilon(i \dashv j)$,

6. $g_\vdash(i,j) = f_\vdash(i,j) + \varphi'_\vdash(j)\varepsilon(i) + \varphi_\vdash(i)\varepsilon(j) + \varepsilon(i) \vdash \varepsilon(j) - \varepsilon(i \vdash j)$

for all $i, j \in B$ where $\mathrm{ad}^l_\dashv(\varepsilon(i))m = \varepsilon(i) \dashv m$, $\mathrm{ad}^l_\vdash(\varepsilon(i))m = \varepsilon(i) \vdash m$, $\mathrm{ad}^r_\dashv(\varepsilon(i))m = m \dashv \varepsilon(i)$, and $\mathrm{ad}^r_\vdash(\varepsilon(i))m = m \vdash \varepsilon(i)$ for all $m \in A$.

Theorem 2.2.3. *If the factor system* $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ *belongs to the extension* $0 \longrightarrow A \xrightarrow{\sigma_1} L_1 \xrightarrow{\pi_1} B \to 0$ *and* μ_1 *and the factor system* $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$ *belongs to the extension* $0 \longrightarrow A \xrightarrow{\sigma_2} L_2 \xrightarrow{\pi_2} B \to 0$ *and* μ_2*, then the factor systems are equivalent if and only if the extensions are equivalent.*

Proof. In the forward direction, one defines τ in the same way as the Leibniz case and computes $\tau(a \dashv b) = \tau(a) \dashv \tau(b)$ and $\tau(a \vdash b) = \tau(a) \vdash \tau(b)$ via the axioms of equivalence for diassociative factor systems. In the other direction, define $\varepsilon(i) = n_i$ where $\tau^{-1}\mu_2(i) = \mu_1(i) + \sigma_1(n_i)$. There are six axioms to check when verifying that ε is an equivalence of factor systems. Otherwise, the theorem follows by similar logic. □

Corollary 2.2.4. *Given an extension* $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$*, let* $\mu_1 : B \longrightarrow L$ *and* $\mu_2 : B \longrightarrow L$ *be linear maps such that* $\pi\mu_1 = id_B = \pi\mu_2$*. Suppose also that* $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ *is a factor system of A by B which belongs to the extension and* μ_1*, and* $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$ *is a factor system of A by B which belongs to the extension and* μ_2*. Then* $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ *is equivalent to* $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$.

Corollary 2.2.5. *Equivalence of factor systems is an equivalence relation.*

Theorem 2.2.6. *If* $(\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash, f_\dashv, f_\vdash)$ *is a factor system of A by B and* ε *is a linear transformation from B to A, then there exists a factor system* $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$ *such that* ε *is an equivalence between them. Furthermore, if* ε *is an equivalence, then* $(\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash, g_\dashv, g_\vdash)$ *is unique.*

Proof. Define

i. $\psi_\dashv(i) = \varphi_\dashv(i) + \mathrm{ad}^l_\dashv(\varepsilon(i))$,

ii. $\psi_\vdash(i) = \varphi_\vdash(i) + \mathrm{ad}^l_\vdash(\varepsilon(i))$,

iii. $\psi'_\dashv(i) = \varphi'_\dashv(i) + \mathrm{ad}^r_\dashv(\varepsilon(i))$,

iv. $\psi'_{\vdash}(i) = \varphi'_{\vdash}(i) + \mathrm{ad}^r_{\vdash}(\varepsilon(i))$,

v. $g_{\dashv}(i,j) = f_{\dashv}(i,j) + \varphi'_{\dashv}(j)\varepsilon(i) + \varphi_{\dashv}(i)\varepsilon(j) + \varepsilon(i) \dashv \varepsilon(j) - \varepsilon(i \dashv j)$,

vi. $g_{\vdash}(i,j) = f_{\vdash}(i,j) + \varphi'_{\vdash}(j)\varepsilon(i) + \varphi_{\vdash}(i)\varepsilon(j) + \varepsilon(i) \vdash \varepsilon(j) - \varepsilon(i \vdash j)$

for all $i, j \in B$. It is straightforward to verify that $\psi_{\dashv}$, $\psi_{\vdash}$, $\psi'_{\dashv}$, and $\psi'_{\vdash}$ are linear transformations and that $g_{\dashv}$ and $g_{\vdash}$ are bilinear forms. One checks that $(\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash}, g_{\dashv}, g_{\vdash})$ is a factor system via the identities of $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ and the axioms of diassociative algebras. By construction, the two factor systems are equivalent with ε as their corresponding equivalence. It is straightforward to verify uniqueness. □

Theorem 2.2.7. *Let $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ be a factor system of A by B. The following are equivalent:*

a. $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ *splits,*

b. $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ *is equivalent to some factor system* $(\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash}, g_{\dashv}, g_{\vdash})$ *such that* $g_{\dashv} = 0$ *and* $g_{\vdash} = 0$*,*

c. there exists a linear transformation $\varepsilon : B \longrightarrow A$ *such that*

$$f_{\dashv}(i,j) = -\varphi'_{\dashv}(j)\varepsilon(i) - \varphi_{\dashv}(i)\varepsilon(j) - \varepsilon(i) \dashv \varepsilon(j) + \varepsilon(i \dashv j),$$
$$f_{\vdash}(i,j) = -\varphi'_{\vdash}(j)\varepsilon(i) - \varphi_{\vdash}(i)\varepsilon(j) - \varepsilon(i) \vdash \varepsilon(j) + \varepsilon(i \vdash j).$$

Proof. (a. $\Longrightarrow$ b.) We know $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ belongs to a split extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$. By definition, there is an associated homomorphism $\mu : B \longrightarrow L$ such that $\pi\mu = \mathrm{id}_B$. Hence there exists a factor system $(\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash}, g_{\dashv}, g_{\vdash})$ belonging to the extension and μ which is equivalent to $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ by Corollary 2.2.4. Since μ is a homomorphism, we have $g_{\dashv} = g_{\vdash} = 0$.

(b. $\Longrightarrow$ c.) Let $\varepsilon : B \longrightarrow A$ be an equivalence of factor systems $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ and $(\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash}, g_{\dashv}, g_{\vdash})$ where $g_{\dashv} = g_{\vdash} = 0$. Then $0 = g_{\dashv}(i,j) = f_{\dashv}(i,j) + \varphi'_{\dashv}(j)\varepsilon(i) + \varphi_{\dashv}(i)\varepsilon(j) + \varepsilon(i) \dashv \varepsilon(j) - \varepsilon(i \dashv j)$ and $0 = g_{\vdash}(i,j) = f_{\vdash}(i,j) + \varphi'_{\vdash}(j)\varepsilon(i) + \varphi_{\vdash}(i)\varepsilon(j) + \varepsilon(i) \vdash \varepsilon(j) - \varepsilon(i \vdash j)$ for all $i, j \in B$ by the axioms of equivalence, which implies the desired equalities.

(c. $\Longrightarrow$ a.) Let ε be as in c. By Theorem 2.2.6, ε is an equivalence of $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ with another factor system $(\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash}, g_{\dashv}, g_{\vdash})$ which belongs to an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ and $\mu : B \longrightarrow L$. One has $g_{\dashv}(i,j) = f_{\dashv}(i,j) + \varphi'_{\dashv}(j)\varepsilon(i) + \varphi_{\dashv}(i)\varepsilon(j) + \varepsilon(i) \dashv \varepsilon(j) - \varepsilon(i \dashv j) = 0$ and $g_{\vdash}(i,j) = f_{\vdash}(i,j) + \varphi'_{\vdash}(j)\varepsilon(i) + \varphi_{\vdash}(i)\varepsilon(j) + \varepsilon(i) \vdash \varepsilon(j) - \varepsilon(i \vdash j) = 0$ by assumption.

Then, since $\sigma(g(i,j))=0$ for all $i,j\in B$, the axioms of belonging imply that μ is a homomorphism. Also, μ is injective since $\pi\mu=\mathrm{id}_B$. Hence the extension splits and, therefore, so does the original factor system. $\square$

Let A be an abelian diassociative algebra and let $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,f_\dashv,f_\vdash)$ be a factor system of A by B which is equivalent to another factor system $(\psi_\dashv,\psi_\vdash,\psi'_\dashv,\psi'_\vdash,g_\dashv,g_\vdash)$. Since A is abelian, all adjoint operators on A are equal to zero. Thus, by the axioms of equivalence for factor systems, $\varphi_\dashv=\psi_\dashv$, $\varphi_\vdash=\psi_\vdash$, $\varphi'_\dashv=\psi'_\dashv$, and $\varphi'_\vdash=\psi'_\vdash$. We now fix the first four maps of factor systems and narrow our focus to pairs of bilinear forms. Let $\mathrm{Fact}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ denote the set of all pairs $(f_\dashv,f_\vdash)$ such that $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,f_\dashv,f_\vdash)$ is a factor system and let $\mathscr{T}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ denote the set of all pairs $(f_\dashv,f_\vdash)$ such that $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,f_\dashv,f_\vdash)$ is a split factor system. For ease of notation, let φ denote the fixed tuple $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ and let $(\varphi,f_\dashv,f_\vdash)$ denote the factor system $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,f_\dashv,f_\vdash)$. We abbreviate the previous sets by Fact_φ and $\mathscr{T}_\varphi$ respectively and denote by

$$\mathrm{Ext}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$$

the set of equivalence classes $\mathrm{Fact}_\varphi/\mathscr{T}_\varphi$. For the rest of this subsection, A is abelian.

Theorem 2.2.8. *If A is abelian, then*

1. $\mathrm{Fact}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ *is an abelian diassociative algebra,*
2. $\mathscr{T}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ *is an ideal in* $\mathrm{Fact}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$*,*
3. *factor systems* $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,f_\dashv,f_\vdash)$ *and* $(\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash,g_\dashv,g_\vdash)$ *are equivalent if and only if* $(f_\dashv,f_\vdash)$ *and* $(g_\dashv,g_\vdash)$ *are in the same coset of* Fact_φ *relative to* $\mathscr{T}_\varphi$*,*
4. *the quotient diassociative algebra* $\mathrm{Ext}(B,A,\varphi_\dashv,\varphi_\vdash,\varphi'_\dashv,\varphi'_\vdash)$ *is in one-to-one correspondence with the set of equivalence classes of extensions to which* $\varphi_\dashv$*,* $\varphi_\vdash$*,* $\varphi'_\dashv$*, and* $\varphi'_\vdash$ *belong.*

Proof. For $(f_\dashv,f_\vdash)$ and $(g_\dashv,g_\vdash)$ in Fact_φ, one verifies $(f_\dashv-cg_\dashv,f_\vdash-cg_\vdash)\in\mathrm{Fact}_\varphi$ via the axioms of the factor systems $(\varphi,f_\dashv,f_\vdash)$ and $(\varphi,g_\dashv,g_\vdash)$ and the fact that multiplication in A is trivial. For the second statement, it suffices to verify that $\mathscr{T}_\varphi$ is a subspace. Consider elements $(f_\dashv,f_\vdash)$ and $(g_\dashv,g_\vdash)$ in $\mathscr{T}_\varphi$, which form split factor systems $(\varphi,f_\dashv,f_\vdash)$ and $(\varphi,g_\dashv,g_\vdash)$ respectively. By Theorem 2.2.7, there exist linear transformations $\varepsilon_f,\varepsilon_g:B\longrightarrow A$ such that

$$f_\dashv(i,j)=-\varphi'_\dashv(j)\varepsilon_f(i)-\varphi_\dashv(i)\varepsilon_f(j)-\varepsilon_f(i)\dashv\varepsilon_f(j)+\varepsilon_f(i\dashv j),$$
$$f_\vdash(i,j)=-\varphi'_\vdash(j)\varepsilon_f(i)-\varphi_\vdash(i)\varepsilon_f(j)-\varepsilon_f(i)\vdash\varepsilon_f(j)+\varepsilon_f(i\vdash j)$$

and

$$g_{\dashv}(i,j) = -\varphi'_{\dashv}(j)\varepsilon_g(i) - \varphi_{\dashv}(i)\varepsilon_g(j) - \varepsilon_g(i) \dashv \varepsilon_g(j) + \varepsilon_g(i \dashv j),$$
$$g_{\vdash}(i,j) = -\varphi'_{\vdash}(j)\varepsilon_g(i) - \varphi_{\vdash}(i)\varepsilon_g(j) - \varepsilon_g(i) \vdash \varepsilon_g(j) + \varepsilon_g(i \vdash j).$$

Letting $\varepsilon = \varepsilon_f - c\varepsilon_g$, one has

$$(f_{\dashv} - c g_{\dashv})(i,j) = -\varphi'_{\dashv}(j)\varepsilon(i) - \varphi_{\dashv}(i)\varepsilon(j) - \varepsilon(i) \dashv \varepsilon(j) + \varepsilon(i \dashv j)$$
$$(f_{\vdash} - c g_{\vdash})(i,j) = -\varphi'_{\vdash}(j)\varepsilon(i) - \varphi_{\vdash}(i)\varepsilon(j) - \varepsilon(i) \vdash \varepsilon(j) + \varepsilon(i \vdash j)$$

which implies that $(\varphi, f_{\dashv} - c g_{\dashv}, f_{\vdash} - c g_{\vdash})$ splits. For the third statement, one observes that the last two axioms of equivalence for factor systems hold if and only if the third condition of Theorem 2.2.7 holds for the factor system $(\varphi, f_{\dashv} - g_{\dashv}, f_{\vdash} - g_{\vdash})$. Since A is abelian, adjoint operators on A are trivial. The final statement holds as in the Leibniz analogue. □

Theorem 2.2.9. $(\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash}, f_{\dashv}, f_{\vdash})$ *is central if and only if* $\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash} = 0$.

Proof. By Theorem 2.2.2, the factor system belongs to an extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ that is central if and only if $(m,i) \dashv (n,0) = (n,0) \dashv (m,i) = (m,i) \vdash (n,0) = (n,0) \vdash (m,i) = (0,0)$ for all $m,n \in A$ and $i \in B$. But this happens if and only if $\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash} = 0$. □

Theorem 2.2.10. *The classes of central extensions of A by B form a diassociative algebra, denoted* $\mathrm{Cext}(B,A)$.

Proof. By Theorem 2.2.8 and Theorem 2.2.9; we set $\mathrm{Cext}(B,A) := \mathrm{Ext}(B,A,0,0,0,0)$. □

Theorem 2.2.11. *Let A and B be abelian diassociative algebras and let $(\varphi, f_{\dashv}, f_{\vdash})$ be a central factor system of A by B. Then $(\varphi, f_{\dashv}, f_{\vdash})$ belongs to an abelian extension if and only if $f_{\dashv} = 0$ and $f_{\vdash} = 0$.*

Proof. Since $(\varphi, f_{\dashv}, f_{\vdash})$ is central, we know all φ maps are zero. In the forward direction, $(\varphi, f_{\dashv}, f_{\vdash})$ belongs to an abelian extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ and section μ. Since L and B are both abelian, one has $\sigma(f_{\dashv}(i,j)) = \mu(i) \dashv \mu(j) - \mu(i \dashv j) = 0$ and $\sigma(f_{\vdash}(i,j)) = \mu(i) \vdash \mu(j) - \mu(i \vdash j) = 0$ for all $i,j \in B$. Conversely, if $f_{\dashv}$ and $f_{\vdash}$ are trivial, then the construction of L in Theorem 2.2.2 has trivial multiplications. □

2.3 Factor Systems of Dendriform Algebras

The dendriform versions of these results follow by the same logic as the diassociative case with the substitutions of $<$ and $>$ for multiplications $\dashv$ and $\vdash$ respectively.

Definition 10. Let A and B be dendriform algebras. A *factor system* of A by B is a tuple $(\varphi_<, \varphi_>, \varphi'_<, \varphi'_>, f_<, f_>)$ of maps such that

$\varphi_<, \varphi_>, \varphi'_<, \varphi'_> : B \longrightarrow \mathscr{L}(A)$ are linear,

$f_<, f_> : B \times B \longrightarrow A$ are bilinear,

and the following three sets of identities are satisfied for all $m, n, p \in A$ and $i, j, k \in B$:

1. Those resembling E1:
   (a) $(\varphi_<(i)n) < p = \varphi_<(i)(n < p) + \varphi_<(i)(n > p)$
   (b) $(\varphi'_<(j)m) < p = m < (\varphi_<(j)p) + m < (\varphi_>(j)p)$
   (c) $f_<(i, j) < p + \varphi_<(i < j)p = \varphi_<(i)(\varphi_<(j)p) + \varphi_<(i)(\varphi_>(j)p)$
   (d) $\varphi'_<(k)(m < n) = m < (\varphi'_<(k)n) + m < (\varphi'_>(k)n)$
   (e) $\varphi'_<(k)(\varphi_<(i)n) = \varphi_<(i)(\varphi'_<(k)n) + \varphi_<(i)(\varphi'_>(k)n)$
   (f) $\varphi'_<(k)(\varphi'_<(j)m) = m < f_<(j, k) + \varphi'_<(j < k)m + m < f_>(j, k) + \varphi'_<(j > k)m$
   (g) $\varphi'_<(k)f_<(i, j) + f_<(i < j, k) = \varphi_<(i)f_<(j, k) + f_<(i, j < k) + \varphi_<(i)f_>(j, k) + f_<(i, j > k)$
2. Those resembling E2:
   (a) $(\varphi_>(i)n) < p = \varphi_>(i)(n < p)$
   (b) $(\varphi'_>(j)m) < p = m > (\varphi_<(j)p)$
   (c) $f_>(i, j) < p + \varphi_<(i > j)p = \varphi_>(i)(\varphi_<(j)p)$
   (d) $\varphi'_<(k)(m > n) = m > (\varphi'_<(k)n)$
   (e) $\varphi'_<(k)(\varphi_>(i)n) = \varphi_>(i)(\varphi'_<(k)n)$
   (f) $\varphi'_<(k)(\varphi'_>(j)m) = \varphi'_>(j < k)m + m > f_<(j, k)$
   (g) $\varphi'_<(k)f_>(i, j) + f_<(i > j, k) = \varphi_>(i)f_<(j, k) + f_>(i, j < k)$
3. Those resembling E3:
   (a) $(\varphi_<(i)n) > p + (\varphi_>(i)n) > p = \varphi_>(i)(n > p)$
   (b) $(\varphi'_<(j)m) > p + (\varphi'_>(j)m) > p = m > (\varphi_>(j)p)$
   (c) $f_<(i, j) > p + \varphi_>(i < j)p + f_>(i, j) > p + \varphi_>(i > j)p = \varphi_>(i)(\varphi_>(j)p)$
   (d) $\varphi'_>(k)(m < n) + \varphi'_>(k)(m > n) = m > (\varphi'_>(k)n)$
   (e) $\varphi'_>(k)(\varphi_<(i)n) + \varphi'_>(k)(\varphi_>(i)n) = \varphi_>(i)(\varphi'_>(k)n)$

(f) $\varphi'_>(k)(\varphi'_<(j)m)+\varphi'_>(k)(\varphi'_>(j)m)=m>f_>(j,k)+\varphi'_>(j>k)m$

(g) $\varphi'_>(k)f_<(i,j)+f_>(i<j,k)+\varphi'_>(k)f_>(i,j)+f_>(i>j,k)=\varphi_>(i)f_>(j,k)+f_>(i,j>k)$

Definition 11. Let A and B be Zinbiel algebras. A *factor system* of A by B is a tuple of maps (φ,φ',f) where

$\varphi,\varphi' : B \longrightarrow \mathscr{L}(A)$ are linear,

$f : B\times B \longrightarrow A$ is bilinear

such that

1. $(\varphi(i)n)p=\varphi(i)(np)+\varphi(i)(pn)$
2. $(\varphi'(j)m)p=m(\varphi(j)p)+m(\varphi'(j)p)$
3. $\varphi'(k)(mn)=m(\varphi(k)n)+m(\varphi'(k)n)$
4. $f(i,j)p+\varphi(ij)p=\varphi(i)(\varphi(j)p))+\varphi(i)(\varphi'(j)p)$
5. $\varphi'(k)(\varphi(i)n)=\varphi(i)(\varphi(k)n)+\varphi(i)(\varphi'(k)n)$
6. $\varphi'(k)(\varphi'(j)m)=mf(j,k)+mf(k,j)+\varphi'(jk)m+\varphi'(kj)m$
7. $\varphi'(k)f(i,j)+f(ij,k)=\varphi(i)f(j,k)+\varphi(i)f(k,j)+f(i,jk)+f(i,kj)$

are satisfied for all $m,n,p\in A$ and $i,j,k\in B$.

2.4 Cohomology

We now discuss how second cohomology characterizes extensions, using the Leibniz case as a model example. Given a central extension $0\longrightarrow A\longrightarrow L\longrightarrow B\longrightarrow 0$ of Leibniz algebras A by B, the general construction of cohomology begins with the set $\mathscr{C}^n(B,A)$ of n-linear maps $f : B\times\cdots\times B\longrightarrow A$. Elements of $\mathscr{C}^n(B,A)=\mathrm{Mult}(B\times\cdots\times B,A)\cong\mathrm{Hom}_{\mathbb{F}}(B^{\otimes n},A)$ are called n*-cochains*. The usual Leibniz coboundary map $d^n : \mathscr{C}^n(B,A)\longrightarrow\mathscr{C}^{n+1}(B,A)$ is defined by

$$(d^nf)(x_1,\ldots,x_{n+1})=\sum_{1\le i<j\le n+1}(-1)^i f(x_1,\ldots,\hat{x}_i,\ldots,x_{j-1},x_ix_j,x_{j+1},\ldots,x_{n+1})$$

for $f\in\mathscr{C}^n(B,A)$. Note specifically that $(d^2f)(i,j,k)=-f(ij,k)+f(i,jk)-f(j,ik)$. We denote by $\mathscr{Z}^n(B,A)$ the set of all $f\in\mathscr{C}^n(B,A)$ such that $d^nf=0$ and by $\mathscr{B}^n(B,A)$ the set of all $f\in\mathscr{C}^n(B,A)$ such that $d^{n-1}\varepsilon=f$ for some $\varepsilon\in\mathscr{C}^{n-1}(B,A)$. Elements of $\mathscr{Z}^n(B,A)$ are called n*-cocycles*, while elements of $\mathscr{B}^n(B,A)$ are called n*-coboundaries*. It is well known that

$d^n d^{n-1} = 0$ and thus $\mathscr{B}^n(B,A) \subseteq \mathscr{Z}^n(B,A)$. Therefore $\mathscr{H}^n(B,A) = \mathscr{Z}^n(B,A)/\mathscr{B}^n(B,A)$ is the *n th cohomology group*. We refer the reader to Loday's [12] for constructions of (co)homology in the diassociative and dendriform settings.

Continuing with our Leibniz discussion, we narrow our focus to second cohomology and recall the construction $\text{Fact}(B,A,0,0)$ from Theorem 2.1.8. Given our central extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$, the axioms of its corresponding factor systems become trivial except for the seventh one, which reduces to $f(i, jk) = f(ij,k) + f(j,ik)$. Thus $\text{Fact}(B,A,0,0)$ is the set of all bilinear $f : B \times B \longrightarrow A$ such that $d^2 f = 0$. Moreover, $\mathscr{T}(B,A,0,0)$ is the set of all bilinear $f : B \times B \longrightarrow A$ such that $f(i,j) = -\varepsilon(ij)$ for some linear transformation $\varepsilon : B \longrightarrow A$. These sets are thus precisely the 2-cocycles and 2-coboundaries of our cohomology respectively, with $\mathscr{Z}^2(B,A) = \text{Fact}(B,A,0,0)$ and $\mathscr{B}^2(B,A) = \mathscr{T}(B,A,0,0)$, making $\text{Cext}(B,A)$ the second cohomology group $\mathscr{H}^2(B,A)$. For any section μ, we can thereby define a bilinear form $f : B \times B \longrightarrow A$ by $f(i,j) = \mu(i)\mu(j) - \mu(ij)$ that is automatically a 2-cocycle of Leibniz algebras. Furthermore, any f and g in $\mathscr{Z}^2(B,A)$ belong to equivalent extensions if and only if they differ by a 2-coboundary. Therefore, extensions of A by B are equivalent if and only if they give rise to the same element of $\mathscr{H}^2(B,A)$. Finally, the work of the current chapter also guarantees that each element $\overline{f} \in \mathscr{H}^2(B,A)$ gives rise to a central extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ and section μ such that $f(i,j) = \mu(i)\mu(j) - \mu(ij)$.

Similarly, *2-cocycle identities* for other classes of algebras are the central simplifications of their specific factor system identities. The following table lists these identities for each $\mathscr{P}$ algebra, as well as the total numbers $\varpi(\mathscr{P})$ of noncentral factor system identities. By construction, each set of cocycle identities resembles the defining identities of the corresponding $\mathscr{P}$ structure.

Table 2.1 2-cocycles.

$\mathscr{P}$	$\varpi(\mathscr{P})$	2-cocycle form	2-cocycle Identities
Associative	7	f	$f(ij,k)=f(i,jk)$
Leibniz	7	f	$f(i,jk)=f(ij,k)+f(j,ik)$
Zinbiel	7	f	$f(ij,k)=f(i,jk)+f(i,kj)$
Diassociative	35	$(f_{\dashv},f_{\vdash})$	$f_{\dashv}(i,j\dashv k)=f_{\dashv}(i,j\vdash k)$ $f_{\dashv}(i\vdash j,k)=f_{\vdash}(i,j\dashv k)$ $f_{\vdash}(i\dashv j,k)=f_{\vdash}(i\vdash j,k)$ $f_{\dashv}(i,j\dashv k)=f_{\dashv}(i\dashv j,k)$ $f_{\vdash}(i,j\vdash k)=f_{\vdash}(i\vdash j,k)$
Dendriform	21	$(f_{<},f_{>})$	$f_{<}(i<j,k)=f_{<}(i,j<k)+f_{<}(i,j>k)$ $f_{<}(i>j,k)=f_{>}(i,j<k)$ $f_{>}(i<j,k)+f_{>}(i>j,k)=f_{>}(i,j>k)$
Lie	3	f	$f(i,i)=0$ $f(ij,k)+f(jk,i)+f(ki,j)=0$
Commutative	4	f	$f(i,j)=f(j,i)$ $f(ij,k)=f(i,jk)$

CHAPTER

3

EXTENSIONS OF NILPOTENT ALGEBRAS

Recall that an extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ of $\mathscr{P}$ algebras is *nilpotent* if L is nilpotent as a $\mathscr{P}$ algebra. In [22], the author proved a criterion for the nilpotency of certain related extensions of Lie algebras. The objective of the present chapter is to develop analogues of this criterion for the algebras of Loday, and thus for all seven algebras under consideration. To this end, it suffices to prove the Leibniz and diassociative cases. In particular, the dendriform case of these results follows similarly to the diassociative case after replacing $\dashv$ and $\vdash$ by $<$ and $>$ respectively, as well as replacing Lemma 3.1.2 by the analogous Lemma 3.1.3. As mentioned in the introduction, the main result of the current chapter is a direct application of noncentral factor systems, and thus relies on Chapter 2. We begin by discussing notions of nilpotency.

3.1 Nilpotency

There is a well-known sequence of ideals called the *lower central series* that is defined recursively, for a Leibniz algebra L, by $L^0 = L$ and $L^{k+1} = LL^k$ for $k \geq 0$. We note that the product algebras LL^k are usually denoted by bracket algebras $[L, L^k]$ in the Lie case. A Leibniz algebra is called *nilpotent of class u*, denoted $\operatorname{nil} L = u$, if $L^u = 0$ and $L^{u-1} \neq 0$ for some $u \geq 0$. The following lemma holds via induction and repeated application of the

Leibniz identity.

Lemma 3.1.1. *Let L be a Leibniz algebra. Then $L^n L \subseteq LL^n$ for all n.*

For dialgebras, the definition of nilpotency is more involved. The following concepts and notations concerning the nilpotency of diassociative algebras are taken from [14]. Let A and B be subsets of a diassociative algebra D and recall the ideal $A \Diamond B = A \dashv B + A \vdash B$ in D. There are notions of left, right, and general nilpotency for D that are based on the $\Diamond$ operator. We define three sequences of ideals in D:

i. $D^{\{0\}} = D$, $D^{\{k+1\}} = D \Diamond D^{\{k\}}$,

ii. $D^{<0>} = D$, $D^{<k+1>} = D^{<k>} \Diamond D$,

iii. $D^0 = D$, $D^{k+1} = D^0 \Diamond D^k + D^1 \Diamond D^{k-1} + \cdots + D^k \Diamond D^0$.

A diassociative algebra D is called

i. *left nilpotent* if $D^{\{u\}} = 0$,

ii. *right nilpotent* if $D^{<u>} = 0$,

iii. *nilpotent* if $D^u = 0$

for some $u \geq 0$. We say D is *nilpotent of class u* if $D^u = 0$ and $D^{u-1} \neq 0$. The following lemma from [14] is crucial for the diassociative case in this section.

Lemma 3.1.2. *Let D be a diassociative algebra. For all $k \in \mathbb{N}$, $D^{\{k\}} = D^{<k>} = D^k$.*

The same definitions can be stated for dendriform algebras with the simple substitutions of $<$ and $>$ for $\dashv$ and $\vdash$ respectively. Let A and B be subsets of a dendriform algebra E. The dendriform analogue of Lemma 3.1.2 was shown in [13], where the same three sequences $E^{\{k\}}$, $E^{<k>}$, and E^k of ideals in E are defined based on $A \Diamond B = A < B + A > B$.

Lemma 3.1.3. *Let E be a dendriform algebra. For all $k \in \mathbb{N}$, $E^{\{k\}} = E^{<k>} = E^k$.*

3.2 Leibniz Case

Consider a pair of nilpotent Leibniz algebras A and B and let $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ be an extension of A by B with section $\mu : B \longrightarrow L$. We first define two ways for B to act on A. Let

$\varphi : B \longrightarrow \mathrm{Der}(A)$ by $\varphi(i)m = \sigma^{-1}(\mu(i)\sigma(m))$ and $\varphi' : B \longrightarrow \mathscr{L}(A)$ by $\varphi'(i)m = \sigma^{-1}(\sigma(m)\mu(i))$ for $i \in B$, $m \in A$. Next, let

$$q : \mathrm{Der}(A) \longrightarrow \mathrm{Der}(A)/\,\mathrm{ad}^l(A),$$
$$q' : \mathscr{L}(A) \longrightarrow \mathscr{L}(A)/\,\mathrm{ad}^r(A)$$

denote the natural projections and define a pair of maps $(\Phi, \Phi') = (q\varphi, q'\varphi')$. We say that the pair (φ, φ') is a *lift* of (Φ, Φ'). Any two lifts (φ, φ') and (ψ, ψ') of (Φ, Φ') are thus related by

$$\varphi(i) = \psi(i) + \mathrm{ad}^l_{m_i},$$
$$\varphi'(i) = \psi'(i) + \mathrm{ad}^r_{m'_i}$$

for $i \in B$, and some elements $m_i, m'_i \in A$ that depend on i. Our first proposition develops a criterion for when L is nilpotent that is based on the following recursive construction. Define $A_0 = A$ and $A_{k+1} = \sigma^{-1}(\sigma(A_k)L + L\sigma(A_k))$ for $k \geq 0$.

Proposition 3.2.1. *Let B be a nilpotent Leibniz algebra of class s. Then $L^{k+s} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$. Hence L is nilpotent if and only if $A_k = 0$ for some k.*

Proof. Since $\pi : L \longrightarrow B$ is a homomorphism, one computes $\pi(L^s) = B^s = 0$, which implies that $L^s \subseteq \ker \pi = \sigma(A) = \sigma(A_0)$. Also, $\sigma(A_0) = \sigma(A) \subseteq L = L^0$. We therefore have a base case $L^s \subseteq \sigma(A_0) \subseteq L^0$ for $k = 0$. Now suppose $L^{n+s} \subseteq \sigma(A_n) \subseteq L^n$ for some $n \geq 0$. Then

$$\begin{aligned}
L^{n+1+s} &= LL^{n+s} && \\
&\subseteq L\sigma(A_n) && \text{by induction}\\
&\subseteq \sigma(A_n)L + L\sigma(A_n) && \\
&\subseteq L^n L + LL^n && \text{by induction}\\
&\stackrel{*}{=} LL^n && \\
&= L^{n+1}
\end{aligned}$$

where $\sigma(A_n)L + L\sigma(A_n) = \sigma(A_{n+1})$ and the equality $*$ follows by Lemma 3.1.1. Thus $L^{s+k} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$ via induction. For the second statement, we first note that if L is nilpotent, then $\sigma(A_k) \subseteq L^k = 0$ for some $k \geq 0$. This means $A_k = 0$ since σ is injective. Conversely, if $A_k = 0$ for some $k \geq 0$, then $\sigma(A_k) = 0$ and thus $L^{k+s} = 0$. Hence L is nilpotent. $\square$

Again, let (φ, φ') be a lift of (Φ, Φ').

Definition 12. An ideal N of A is (φ,φ')-*invariant* if $\varphi(i)n, \varphi'(i)n \in N$ for all $i \in B$ and $n \in N$.

Lemma 3.2.2. *Let (φ,φ') and (ψ,ψ') be lifts of (Φ,Φ'). Then N is (φ,φ')-invariant if and only if N is (ψ,ψ')-invariant.*

Proof. Let $i \in B$. Since we have two lifts of the same pair, they are related by

$$\psi(i) = \varphi(i) + \mathrm{ad}^l_{m_i},$$
$$\psi'(i) = \varphi'(i) + \mathrm{ad}^r_{m'_i}$$

for some $m_i, m'_i \in A$. In one direction, assume N is (φ,φ')-invariant. Then $\varphi(i)n, \varphi'(i)n \in N$ for all $n \in N$ by definition. Also, $m_i n, n m'_i \in N$ for all $n \in N$ since N is an ideal. Thus $\psi(i)n, \psi'(i)n \in N$ and so N is (ψ,ψ')-invariant. The other direction is similar. □

Definition 13. An ideal N of A is B-*invariant* if N is (φ,φ')-invariant for some, and hence all, lifts of (Φ,Φ').

In particular, A itself is B-invariant since $\varphi(i), \varphi'(i) \in \mathscr{L}(A)$ for all $i \in B$. Consider a B-invariant ideal N of A and let (φ,φ') be a lift of (Φ,Φ'). We define $\Gamma(N,\varphi,\varphi')$ to be the B-invariant ideal of A generated by AN, NA, and $\{\varphi(i)n, \varphi'(i)n \mid i \in B, n \in N\}$. Then $\Gamma(N,\varphi,\varphi') \subseteq N$ and we reach the following lemma.

Lemma 3.2.3. *If (φ,φ') and (ψ,ψ') are lifts of (Φ,Φ'), then $\Gamma(N,\varphi,\varphi') = \Gamma(N,\psi,\psi')$.*

Proof. It again suffices to show one direction. First note that AN and NA are contained in both sides of the equality by definition. For $i \in B$ and $n \in N$, we know

$$\psi(i)n = \varphi(i)n + m_i n,$$
$$\psi'(i)n = \varphi'(i)n + n m'_i$$

for some $m_i, m'_i \in A$. These expressions clearly fall in $\Gamma(N,\varphi,\varphi')$ and therefore $\Gamma(N,\psi,\psi')$ is contained in $\Gamma(N,\varphi,\varphi')$. □

We now fix a lift (φ,φ') of (Φ,Φ') and denote $\Gamma N = \Gamma(N,\varphi,\varphi')$. Given B and A, as well as maps

$$\Phi : B \longrightarrow \mathrm{Der}(A)/\,\mathrm{ad}^l(A),$$
$$\Phi' : B \longrightarrow \mathscr{L}(A)/\,\mathrm{ad}^r(A),$$

and a B-invariant ideal N of A, define a descending sequence of B-invariant ideals $\Gamma_k^B N$ of N by $\Gamma_0^B N = N$ and $\Gamma_{k+1}^B N = \Gamma(\Gamma_k^B N)$ for $k \geq 0$.

Theorem 3.2.4. *Consider the extension* $0 \longrightarrow A \xrightarrow{\sigma} L \longrightarrow B \longrightarrow 0$ *and our pair of maps* (Φ, Φ'). *If* $A_0 = A$ *and* $A_{k+1} = \sigma^{-1}(\sigma(A_k)L + L\sigma(A_k))$, *then* $A_k = \Gamma_k^B A$ *for all* $k \geq 0$.

Proof. By Theorem 2.1.1, there exists a unique factor system (φ, φ', f) belonging to the extension $0 \longrightarrow A \xrightarrow{\sigma} L \longrightarrow B \longrightarrow 0$ and section μ. By construction, φ and φ' are the maps of our lift (φ, φ'). By Theorem 2.1.2, there exists another extension $0 \longrightarrow A \xrightarrow{\iota} L_2 \longrightarrow B \longrightarrow 0$ of A by B to which (φ, φ', f) belongs. Here, L_2 is the vector space $A \oplus B$ equipped with multiplication $(m, i)(n, j) = (mn + \varphi(i)n + \varphi'(j)m + f(i, j), ij)$, where $f : B \times B \longrightarrow A$ is a bilinear form. Also $\iota(m) = (m, 0)$. Since (φ, φ', f) is equivalent to itself, the extensions are equivalent, and thus there exists an isomorphism $\tau : L \longrightarrow L_2$ such that $\tau\sigma = \iota$.

We will now prove the statement via induction, first noting that the base case $A_0 = A = \Gamma_0^B A$ holds trivially. Assume that $A_n = \Gamma_n^B A$ for some $n \geq 0$. By definition, it suffices to show the inclusion of generating elements for each side of the equality. Generating elements of A_{n+1} have the forms $\sigma^{-1}(\sigma(m)x)$ and $\sigma^{-1}(x\sigma(m))$ for $x \in L$ and $m \in A_k$. Denote $\tau(x) = (m_x, i_x) \in L_2$. We compute

$$\begin{aligned}
\sigma^{-1}(\sigma(m)x) &= \sigma^{-1}\tau^{-1}(\tau\sigma(m)\tau(x)) \\
&= \iota^{-1}((m,0)(m_x, i_x)) \\
&= \iota^{-1}(mm_x + \varphi'(i_x)m, 0) \\
&= mm_x + \varphi'(i_x)m
\end{aligned}$$

and

$$\begin{aligned}
\sigma^{-1}(x\sigma(m)) &= \sigma^{-1}\tau^{-1}(\tau(x)\tau\sigma(m)) \\
&= \iota^{-1}((m_x, i_x)(m,0)) \\
&= \iota^{-1}(m_x m + \varphi(i_x)m, 0) \\
&= m_x m + \varphi(i_x)m.
\end{aligned}$$

Since $A_n = \Gamma_n^B A$, one has $m_x m \in A(\Gamma_n^B A)$ and $mm_x \in (\Gamma_n^B A)A$, which are both included in $\Gamma_{n+1}^B A$ since $\Gamma_{n+1}^B A$ is the B-invariant ideal generated by $(\Gamma_n^B A)A$, $A(\Gamma_n^B A)$, and

$$\{\varphi(i)m, \varphi'(i)m \mid m \in \Gamma_n^B A, i \in B\}.$$

Thus $\varphi'(i_x)m, \varphi(i_x)m \in \Gamma_{n+1}^B A$ as well, and so $A_{n+1} \subseteq \Gamma_{n+1}^B A$. Conversely, one computes

$$(\Gamma_n^B A)A = \sigma^{-1}(\sigma(\Gamma_n^B A)\sigma(A)) \subseteq \sigma^{-1}(\sigma(A_n)L) \subseteq A_{n+1},$$
$$A(\Gamma_n^B A) = \sigma^{-1}(\sigma(A)\sigma(\Gamma_n^B A)) \subseteq \sigma^{-1}(L\sigma(A_n)) \subseteq A_{n+1}.$$

Also, let $i \in B$ and $m \in \Gamma_n^B A = A_n$. Then $\varphi(i)m = \sigma^{-1}(\mu(i)\sigma(m)) \in A_{n+1}$ and $\varphi'(i)m = \sigma^{-1}(\sigma(m)\mu(i)) \in A_{n+1}$ since $\mu(i) \in L$. Therefore $\Gamma_{n+1}^B A \subseteq A_{n+1}$. $\square$

Given B, A, $\Phi : B \longrightarrow \mathrm{Der}(A)/\operatorname{ad}^l(A)$, and $\Phi' : B \longrightarrow \mathscr{L}(A)/\operatorname{ad}^r(A)$, we define a new notion of nilpotency for A.

Definition 14. A is *B-nilpotent of class u*, written $\mathrm{nil}_B A = u$, if $\Gamma_u^B A = 0$ and $\Gamma_{u-1}^B A \neq 0$ for some $u \geq 0$.

The following two corollaries hold similarly to the Lie case. For their proofs, simply replace Proposition 2.1 and Theorem 3.1 of [22] by the analogous Proposition 3.2.1 and Theorem 3.2.4 of the present work. The subsequent theorem is the main result, which follows from these corollaries and the same logic as the Lie proof.

Corollary 3.2.5. *L is nilpotent if and only if B is nilpotent and $\Gamma_u^B A = 0$ for some $u \geq 1$.*

Corollary 3.2.6. $\max(\mathrm{nil}_B A, \mathrm{nil}\, B) \leq \mathrm{nil}\, L \leq \mathrm{nil}_B A + \mathrm{nil}\, B$.

Theorem 3.2.7. *Let (φ, φ') and (ψ, ψ') be lifts of (Φ, Φ') corresponding to extensions $0 \longrightarrow A \longrightarrow L_{(\varphi,\varphi')} \longrightarrow B \longrightarrow 0$ and $0 \longrightarrow A \longrightarrow L_{(\psi,\psi')} \longrightarrow B \longrightarrow 0$ respectively. Then $L_{(\varphi,\varphi')}$ is nilpotent if and only if $L_{(\psi,\psi')}$ is nilpotent.*

3.3 Diassociative Case

Consider a pair of nilpotent diassociative algebras A and B and an extension $0 \longrightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \longrightarrow 0$ of A by B with section $\mu : B \longrightarrow L$. Throughout this subsection, we let $*$ range over $\dashv$ and $\vdash$ for the sake of brevity. We consider four natural ways for B to act on A. Define $\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash : B \longrightarrow \mathscr{L}(A)$ by $\varphi_*(i)m = \sigma^{-1}(\mu(i) * \sigma(m))$ and $\varphi'_*(i)m = \sigma^{-1}(\sigma(m) * \mu(i))$ for $i \in B$, $m \in A$. Let

$$q_* : \mathscr{L}(A) \longrightarrow \mathscr{L}(A)/\operatorname{ad}_*^l(A),$$
$$q'_* : \mathscr{L}(A) \longrightarrow \mathscr{L}(A)/\operatorname{ad}_*^r(A)$$

be the natural projections and define a tuple of maps $\Phi = (\Phi_\dashv, \Phi_\vdash, \Phi'_\dashv, \Phi'_\vdash)$ by $\Phi_* = q_*\varphi_*$ and $\Phi'_* = q'_*\varphi'_*$. We say that the tuple $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$ is a *lift* of Φ. Two lifts $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$

and $\psi = (\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash})$ of Φ are related by

$$\psi_*(i) = \varphi_*(i) + \mathrm{ad}^l_*(m_{*,i}),$$
$$\psi'_*(i) = \varphi'_*(i) + \mathrm{ad}^r_*(m'_{*,i})$$

for $i \in B$, and some $m_{*,i}, m'_{*,i} \in A$ that depend on i. Finally, let $A_0 = A$ and define $A_{k+1} = \sigma^{-1}(\sigma(A_k)\lozenge L + L\lozenge\sigma(A_k))$ for $k \geq 0$.

Proposition 3.3.1. *Let B be a nilpotent diassociative algebra of class s. Then $L^{k+s} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$. Hence L is nilpotent if and only if $A_k = 0$ for some k.*

Proof. As with the Leibniz case, the base case $k = 0$ follows by our definitions and the properties of extensions. Suppose $L^{n+s} \subseteq \sigma(A_n) \subseteq L^n$ for some $n \geq 0$. We recall that $L^n = L^{<n>} = L^{\{n\}}$ by Lemma 3.1.2, and thereby compute

$$\begin{aligned}
L^{n+1+s} &= L^{<n+1+s>} \\
&= L^{n+s}\lozenge L \\
&\subseteq \sigma(A_n)\lozenge L && \text{by induction} \\
&\subseteq \sigma(A_n)\lozenge L + L\lozenge\sigma(A_n) \\
&\subseteq L^{<n>}\lozenge L + L\lozenge L^{\{n\}} && \text{by induction} \\
&= L^{n+1}
\end{aligned}$$

where $\sigma(A_n)\lozenge L + L\lozenge\sigma(A_n) = \sigma(A_{n+1})$. Thus $L^{s+k} \subseteq \sigma(A_k) \subseteq L^k$ for $k \geq 0$ via induction. The second statement follows by the same logic as the Leibniz case. □

Once more, let $\varphi = (\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash})$ be a lift of Φ.

Definition 15. An ideal N of A is *φ-invariant* if $\varphi_*(i)n, \varphi'_*(i)n \in N$ for all $i \in B$, $n \in N$.

Lemma 3.3.2. *Let φ and ψ be lifts of Φ. Then N is φ-invariant if and only if N is ψ-invariant.*

Proof. Let $i \in B$. Since $\varphi = (\varphi_{\dashv}, \varphi_{\vdash}, \varphi'_{\dashv}, \varphi'_{\vdash})$ and $\psi = (\psi_{\dashv}, \psi_{\vdash}, \psi'_{\dashv}, \psi'_{\vdash})$ are lifts of the same tuple, they are related by

$$\psi_*(i) = \varphi_*(i) + \mathrm{ad}^l_*(m_{*,i}),$$
$$\psi'_*(i) = \varphi'_*(i) + \mathrm{ad}^r_*(m'_{*,i})$$

for some $m_{*,i}, m'_{*,i} \in A$. In one direction, suppose N is φ-invariant. Then $\psi_*(i)n, \psi'_*(i)n \in N$ for all $n \in N$ since N is a φ-invariant ideal in A. Therefore N is ψ-invariant. The converse is similar. □

Definition 16. An ideal N of A is B*-invariant* if N is φ-invariant for some, and hence all, lifts of Φ.

In particular, A is B-invariant since $\varphi_*(i), \varphi'_*(i) \in \mathscr{L}(A)$ for all $i \in B$. Now let N be a B-invariant ideal in A and φ be a lift of Φ. We denote by $\Gamma(N,\varphi)$ the B-invariant ideal generated by $N \dashv A$, $N \vdash A$, $A \dashv N$, $A \vdash N$, and the set $\{\varphi_*(i)n, \varphi'_*(i)n \mid i \in B, n \in N\}$. We thus have $\Gamma(N,\varphi) \subseteq N$ as well as the following lemma.

Lemma 3.3.3. *If φ and ψ are lifts of Φ, then $\Gamma(N,\varphi) = \Gamma(N,\psi)$.*

Proof. It suffices to show that $\Gamma(N,\psi) \subseteq \Gamma(N,\varphi)$. We first note that $N \dashv A$, $N \vdash A$, $A \dashv N$, and $A \vdash N$ are contained in both sides by definition. Similarly to the Leibniz case, the expressions for $\psi_*(i)n$ and $\psi'_*(i)n$ are clearly contained in $\Gamma(N,\varphi)$ for all $i \in B$ and $n \in N$. The converse holds without loss of generality. □

Fix a lift φ of Φ and denote $\Gamma N = \Gamma(N,\varphi)$. Given B, A, Φ, and a B-invariant ideal N of A, define a descending sequence of B-invariant ideals $\Gamma_k^B N$ of N by $\Gamma_0^B N := N$ and $\Gamma_{k+1}^B N := \Gamma(\Gamma_k^B N)$ for $k \geq 0$.

Theorem 3.3.4. *Consider $0 \longrightarrow A \xrightarrow{\sigma} L \longrightarrow B \longrightarrow 0$ and let Φ be defined as above. If $A_0 = A$ and $A_{k+1} = \sigma^{-1}(\sigma(A_k)\Diamond L + L\Diamond\sigma(A_k))$, then $A_k = \Gamma_k^B A$ for all $k \geq 0$.*

Proof. As in the Leibniz case, our work with factor systems in Chapter 2 yields an equivalent extension $0 \longrightarrow A \xrightarrow{\iota} L_2 \longrightarrow B \longrightarrow 0$. Let $\tau : L \longrightarrow L_2$ be the equivalence. Here, L_2 is the vector space $A \oplus B$ equipped with multiplications $(m,i)*(n,j) = (m*n + \varphi_*(i)n + \varphi'_*(j)m + f_*(i,j), i*j)$, and $\iota(m) = (m,0)$. Moreover, φ_* and φ'_* are the same maps as in our lift φ while $f_\dashv$ and $f_\vdash$ are the bilinear forms in some factor system of diassociative algebras.

The base case of this result is trivial since $A_0 = A = \Gamma_0^B A$ by definition. Now assume $A_n = \Gamma_n^B A$ for some $n \geq 0$. Also by definition, it suffices to show the inclusion of generating elements for each side of the equality. Generating elements in A_{n+1} have the forms $\sigma^{-1}(\sigma(m) * x)$ and $\sigma^{-1}(x * \sigma(m))$ for $m \in A_n$ and $x \in L$. Denote $\tau(x) = (m_x, i_x) \in L_2$. We compute

$$\begin{aligned}
\sigma^{-1}(\sigma(m) * x) &= \sigma^{-1}\tau^{-1}(\tau\sigma(m) * \tau(x)) \\
&= \iota^{-1}((m,0) * (m_x, i_x)) \\
&= m * m_x + \varphi'_*(i_x)m
\end{aligned}$$

and

$$\begin{aligned}\sigma^{-1}(x*\sigma(m)) &= \sigma^{-1}\tau^{-1}(\tau(x)*\tau\sigma(m))\\ &= \iota^{-1}((m_x,i_x)*(m,0))\\ &= m_x*m+\varphi_*(i_x)m.\end{aligned}$$

Since $A_n=\Gamma_n^B A$, one has $m_x*m\in A*(\Gamma_n^B A)$ and $m*m_x\in(\Gamma_n^B A)*A$, which are included in $\Gamma_{n+1}^B A$ since $\Gamma_{n+1}^B A$ is the B-invariant ideal generated by $(\Gamma_n^B A)*A$, $A*(\Gamma_n^B A)$, and

$$\{\varphi_*(i)m,\varphi'_*(i)m \mid m\in\Gamma_n^B A, i\in B\}.$$

Thus $\varphi'_*(i_x)m,\varphi_*(i_x)m\in\Gamma_{n+1}^B A$ as well. Therefore $A_{n+1}\subseteq\Gamma_{n+1}^B A$. Conversely, one computes

$$\begin{aligned}(\Gamma_n^B A)*A &= \sigma^{-1}(\sigma(\Gamma_n^B A)*\sigma(A))\subseteq\sigma^{-1}(\sigma(A_n)*L)\subseteq A_{n+1},\\ A*(\Gamma_n^B A) &= \sigma^{-1}(\sigma(A)*\sigma(\Gamma_n^B A))\subseteq\sigma^{-1}(L*\sigma(A_n))\subseteq A_{n+1}.\end{aligned}$$

Also, let $i\in B$ and $m\in\Gamma_n^B A=A_n$. Then $\varphi_*(i)m=\sigma^{-1}(\mu(i)*\sigma(m))\in A_{n+1}$ and $\varphi'_*(i)m=\sigma^{-1}(\sigma(m)*\mu(i))\in A_{n+1}$ since $\mu(i)\in L$. Therefore $\Gamma_{n+1}^B A\subseteq A_{n+1}$. □

Definition 17. Given B, A, and the tuple Φ, we say that A is *B-nilpotent of class u*, written $\operatorname{nil}_B A=u$, if $\Gamma_u^B A=0$ but $\Gamma_{u-1}^B A\neq 0$.

The following results hold similarly to the Lie and Leibniz cases. Here, $\operatorname{nil} L$ is used to denote the nilpotency class of a diassociative algebra L.

Corollary 3.3.5. *L is nilpotent if and only if B is nilpotent and $\Gamma_u^B A=0$ for some $u\geq 1$.*

Corollary 3.3.6. $\max(\operatorname{nil}_B A,\operatorname{nil} B)\leq\operatorname{nil} L\leq\operatorname{nil}_B A+\operatorname{nil} B$.

Theorem 3.3.7. *Let φ and ψ be lifts of $(\Phi_\dashv,\Phi_\vdash,\Phi'_\dashv,\Phi'_\vdash)$ corresponding to extensions $0\longrightarrow A\longrightarrow L_\varphi\longrightarrow B\longrightarrow 0$ and $0\longrightarrow A\longrightarrow L_\psi\longrightarrow B\longrightarrow 0$ respectively. Then L_φ is nilpotent if and only if L_ψ is nilpotent.*

We now state the associative case as a corollary. Let A and B be associative algebras and consider a pair of maps (Φ,Φ') such that $\Phi: B\longrightarrow\mathscr{L}(A)/\operatorname{ad}^l(A)$ and $\Phi': B\longrightarrow\mathscr{L}(A)/\operatorname{ad}^r(A)$. Let *lifts* (φ,φ') and (ψ,ψ') of (Φ,Φ') be defined similarly to the Leibniz case and consider their corresponding extensions $0\longrightarrow A\longrightarrow L_{(\varphi,\varphi')}\longrightarrow B\longrightarrow 0$ and $0\longrightarrow A\longrightarrow L_{(\psi,\psi')}\longrightarrow B\longrightarrow 0$ respectively.

Corollary 3.3.8. *$L_{(\varphi,\varphi')}$ is nilpotent if and only if $L_{(\psi,\psi')}$ is nilpotent.*

3.4 Examples

The first two examples demonstrate that extensions corresponding to lifts of the same tuple need not have the same nilpotency class. We provide an example for the non-Lie Leibniz case as well as for the diassociative case.

Example 3.4.1. Let $A = \langle x, y, z\rangle$ and $B = \langle w\rangle$ be abelian Leibniz algebras and consider two extensions L_1 and L_2 of A by B. Let $L_1 = \langle x, y, z, w\rangle$ have nonzero multiplications given by $w^2 = x$, $wx = y$, and $wy = z$. Then $L_1^2 = \langle x, y, z\rangle$, $L_1^3 = \langle y, z\rangle$, $L_1^4 = \langle z\rangle$, and $L_1^5 = 0$, making L_1 nilpotent of class 5. Now let $L_2 = \langle x, y, z, w\rangle$ have nonzero multiplications given by $wx = y$ and $wy = z$. Then $L_2^2 = \langle y, z\rangle$, $L_2^3 = \langle z\rangle$, and $L_2^4 = 0$, making L_2 nilpotent of class 4. Observe that L_1 and L_2 correspond to lifts of the same tuple, yet have different nilpotency classes. Indeed, A is abelian, and hence $\mathrm{ad}^l(M)$ and $\mathrm{ad}^r(M)$ are zero, making $(\Phi, \Phi') = (\varphi, \varphi')$ for any lift of (Φ, Φ'). In this case, $\Phi(w)x = \varphi(w)x = y$ and $\Phi(w)y = \varphi(w)y = z$ for both. Also $\Phi'(w) = 0$.

We would also like to compute A_k and $\Gamma_k^B A$. Note that, since $A^2 = 0$, one needs only consider the actions of φ and φ' on A when computing $\Gamma_k^B A$. As predicted, $A_k = \Gamma_k^B A$ for all k. One has

$$\begin{aligned}
A_0 &= A = \Gamma_0^B A,\\
A_1 &= \langle y, z\rangle = \Gamma_1^B A,\\
A_2 &= \langle z\rangle = \Gamma_2^B A,\\
A_3 &= 0 = \Gamma_3^B A,
\end{aligned}$$

and $A_k = 0 = \Gamma_k^B A$ otherwise.

Example 3.4.2. Now for a diassociative example. Let $A = \langle x, y\rangle$ and $B = \langle u, v\rangle$ be abelian algebras and L_φ be an extension of A by B having nonzero multiplications $u \dashv u = x$, $u \vdash u = x + y$, $v \dashv v = y$, $v \vdash v = x + y$, and $v \vdash u = x + y = u \vdash v$. This diassociative algebra is a special case of the isomorphism type $Dias_4^1$ in Theorem 4.2 of [14]. One computes $L_\varphi^2 = \langle x, y\rangle$ and $L_\varphi^3 = 0$; hence L_φ is nilpotent of class 3. We also note that the action of B on A is entirely zero, i.e. $\varphi_\dashv = \varphi_\vdash = \varphi'_\dashv = \varphi'_\vdash = 0$. Moreover, A is again abelian, and hence all lifts of the natural Φ tuple are equal. To finish the point, the abelian extension L_{ab} of A by B corresponds to the same zero-lift, but has nilpotency class 2.

We conclude with an example in which A is nonabelian and hence the lifts are allowed to vary by adjoint operators. In this example, however, our nilpotency classes turn out to be the same. We note that the algebras in this case are both associative and Leibniz.

Example 3.4.3. Let $A = \langle x, y, z\rangle$ and $B = \langle w\rangle$ be the associative algebras with only nonzero multiplications $x^2 = y^2 = z$. Consider two extensions $L_{(\varphi,\varphi')}$ and $L_{(\psi,\psi')}$ of A by B. Let $L_{(\varphi,\varphi')}$ have nonzero multiplications given by $x^2 = y^2 = xw = z$, $wx = -z$ and let $L_{(\psi,\psi')}$ have nonzero multiplications given by $x^2 = y^2 = z$. These algebras are clearly nilpotent of class 3 since both have center $\langle z\rangle$ equal to their derived subalgebras. One computes $\varphi(w)x = -z$, $\varphi'(w)x = z$, and $\varphi(w)y = \varphi'(w)y = \varphi(w)z = \varphi'(w)z = 0$. Also $\psi(w) = \psi'(w) = 0$. Thus $\varphi(w) = \psi(w) - \mathrm{ad}^l(x)$ and $\varphi'(w) = \psi'(w) + \mathrm{ad}^r(x)$, and so we have lifts (φ, φ') and (ψ, ψ') that vary by adjoint operators.

CHAPTER

4

MULTIPLIERS AND COVERS OF LEIBNIZ ALGEBRAS

Given a finite-dimensional Leibniz algebra L, the overarching objectives of this chapter are to characterize $M(L)$ by $\mathscr{H}^2(L,\mathbb{F})$ and then to obtain criteria for when the center of a cover maps onto the center of the algebra. We take our methodology from Chapters 3 and 4 of [1], in which the author developed the Lie case of these results. The work of the current chapter relies[1] on the Leibniz version of the culminating result from the first chapter of [1], as proven in [16]. This result guarantees the uniqueness of the cover, as well as characterizes the multiplier in terms of a free presentation. We state it here as Theorem 4.0.1. As in the Lie case, $C(L)$ is used to denote the set of all pairs (J,λ) such that $\lambda : J \longrightarrow L$ is a surjective homomorphism and $\ker\lambda \subseteq J' \cap Z(J)$. An element $(T,\tau) \in C(L)$ is called a *universal element* in $C(L)$ if, for any $(J,\lambda) \in C(L)$, there exists a homomorphism $\beta : T \longrightarrow J$ such that $\lambda\beta = \tau$.

Theorem 4.0.1. *Let L be a finite-dimensional Leibniz algebra and let* $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ *be a free presentation of L. Let*

$$B = \frac{R}{FR+RF} \qquad C = \frac{F}{FR+RF} \qquad D = \frac{F' \cap R}{FR+RF}$$

[1] The work of the present chapter also relies on the theory of factor systems. Specifically, the reader will recall Section 2.4 on cohomology and its relation with extensions.

Then

1. *all covers of L are isomorphic and have the form C/E where E is the complement to D in B,*
2. *the multiplier $M(L)$ of L is $D \cong B/E$,*
3. *the universal elements in $C(L)$ are the elements (K,λ) where K is a cover of L.*

4.1 Hochschild-Serre Spectral Sequence

Our first effort is to construct a five-term cohomological sequence that we refer to as the Hochschild-Serre spectral sequence of low dimension. The sequence is pivotal to this chapter. Let H be a central ideal of a Leibniz algebra L and

$$0 \longrightarrow H \longrightarrow L \xrightarrow{\beta} L/H \longrightarrow 0$$

be the natural central extension with section μ of β. Let A be a central L-module.

Theorem 4.1.1. *The sequence*

$$0 \longrightarrow \mathrm{Hom}(L/H,A) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(L,A) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(H,A) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/H,A) \xrightarrow{\mathrm{Inf}_2} \mathscr{H}^2(L,A)$$

is exact.

Before proving exactness, we need to define the maps of this sequence and check that they make sense. The first *inflation map* $\mathrm{Inf}_1 : \mathrm{Hom}(L/H,A) \longrightarrow \mathrm{Hom}(L,A)$ is defined by $\mathrm{Inf}_1(\chi) = \chi\beta$ for any homomorphism $\chi : L/H \longrightarrow A$. Next, the *restriction mapping* $\mathrm{Res} : \mathrm{Hom}(L,A) \longrightarrow \mathrm{Hom}(H,A)$ is defined by $\mathrm{Res}(\pi) = \pi\iota$ where $\iota : H \longrightarrow L$ is the inclusion map. It is readily verified that Inf_1 and Res are well-defined and linear.

Third is the *transgression map* $\mathrm{Tra} : \mathrm{Hom}(H,A) \longrightarrow \mathscr{H}^2(L/H,A)$. Let $f : L/H \times L/H \longrightarrow H$ be defined by $f(\overline{x},\overline{y}) = \mu(\overline{x})\mu(\overline{y}) - \mu(\overline{xy})$ and consider $\chi \in \mathrm{Hom}(H,A)$. Then $\chi f \in \mathscr{Z}^2(L/H,A)$ since $\chi f(\overline{x},\overline{yz}) - \chi f(\overline{xy},\overline{z}) - \chi f(\overline{y},\overline{xz}) = \chi(0) = 0$ for all $x,y,z \in L$. If ν is another section of β, let $g(\overline{x},\overline{y}) = \nu(\overline{x})\nu(\overline{y}) - \nu(\overline{xy})$. Then f and g are cohomologous in $\mathscr{H}^2(L/H,H)$, which implies that there exists a linear transformation $\varepsilon : L/H \longrightarrow H$ such that $f(\overline{x},\overline{y}) - g(\overline{x},\overline{y}) = -\varepsilon(\overline{xy})$. Clearly $\chi\varepsilon : L/H \longrightarrow A$ is also a linear transformation, and therefore χf and χg are cohomologous in $\mathscr{H}^2(L/H,A)$. Letting

$$\mathrm{Tra}(\chi) = \overline{\chi f},$$

we have shown that Tra is well-defined. It is straightforward to verify that Tra is linear.

Finally, let $\mathrm{Inf}_2 : \mathscr{H}^2(L/H, A) \longrightarrow \mathscr{H}^2(L, A)$ be defined by $\mathrm{Inf}_2(f + \mathscr{B}^2(L/H, A)) = f' + \mathscr{B}^2(L, A)$, where $f'(x, y) = f(\beta(x), \beta(y))$ for $x, y \in L$ and $f \in \mathscr{Z}^2(L/H, A)$. It is straightforward to verify that Inf_2 is linear. To check that Inf_2 maps cocycles to cocycles, one computes

$$\begin{aligned} 0 &= f(\beta(x), \beta(y)\beta(z)) - f(\beta(x)\beta(y), \beta(z)) - f(\beta(y), \beta(x)\beta(z)) \\ &= f'(x, yz) - f'(xy, z) - f'(y, xz) \end{aligned}$$

for all $x, y, z \in L$ since f is a 2-cocycle. Hence $f' \in \mathscr{Z}^2(L, A)$. To check that Inf_2 maps coboundaries to coboundaries, suppose $f \in \mathscr{B}^2(L/H, A)$. Then there exists a linear transformation $\varepsilon : L/H \longrightarrow A$ such that $f(\overline{x}, \overline{y}) = -\varepsilon(\overline{xy})$ for $x, y \in L$. Note that $\beta(x) = x + H = \overline{x}$ for any $x \in L$. Therefore $f'(x, y) = f(\beta(x), \beta(y)) = -\varepsilon\beta(xy)$, yielding $f' \in \mathscr{B}^2(L, A)$.

Proof. Once again, we are concerned with the central extension $0 \longrightarrow H \longrightarrow L \xrightarrow{\beta} L/H \longrightarrow 0$, a section μ of β, and a central L-module A. One has $f \in \mathscr{Z}^2(L/H, H)$ for $f(\overline{x}, \overline{y}) = \mu(\overline{x})\mu(\overline{y}) - \mu(\overline{xy})$. To show exactness at $\mathrm{Hom}(L/H, A)$, it suffices to show that Inf_1 is injective. Suppose $\mathrm{Inf}_1(\chi) = 0$ for $\chi \in \mathrm{Hom}(L/H, A)$. Then $\chi\beta(x) = 0$ for all $x \in L$, which means that $\chi = 0$ since β is surjective.

To prove exactness at $\mathrm{Hom}(L, A)$, first consider an element $\chi \in \mathrm{Hom}(L/H, A)$. One computes $\mathrm{Res}(\mathrm{Inf}_1(\chi)) = \mathrm{Res}(\chi\beta) = \chi\beta\iota = 0$ since ι includes H into L and β sends elements of H to zero in L/H. Thus $\mathrm{Im}(\mathrm{Inf}_1) \subseteq \ker(\mathrm{Res})$. Conversely, consider an element $\chi \in \ker(\mathrm{Res})$. Then $\chi\iota = 0$ implies that $H \subseteq \ker(\chi)$. By the fundamental theorem of homomorphisms, there exists $\hat{\chi} \in \mathrm{Hom}(L/H, A)$ such that $\hat{\chi}\beta = \chi$. But $\mathrm{Inf}_1(\hat{\chi}) = \hat{\chi}\beta = \chi$. Hence $\ker(\mathrm{Res}) \subseteq \mathrm{Im}(\mathrm{Inf}_1)$.

To show exactness at $\mathrm{Hom}(H, A)$, first consider a map $\chi \in \mathrm{Hom}(L, A)$. Then

$$\begin{aligned} \chi f(\overline{x}, \overline{y}) &= \chi\mu(\overline{x})\chi\mu(\overline{y}) - \chi\mu(\overline{xy}) \\ &= -\chi\mu(\overline{xy}) \end{aligned}$$

by centrality, which implies that $\chi f \in \mathscr{B}^2(L/H, A)$. Thus

$$\mathrm{Tra}(\mathrm{Res}(\chi)) = \mathrm{Tra}(\chi\iota) = \overline{\chi\iota f} = 0$$

and so $\mathrm{Im}(\mathrm{Res}) \subseteq \ker(\mathrm{Tra})$. Conversely, let $\theta \in \mathrm{Hom}(H, A)$ be such that $\mathrm{Tra}(\theta) = \overline{\theta f} = 0$. Then $\theta f \in \mathscr{B}^2(L/H, A)$ which implies that there exists a linear transformation $\varepsilon : L/H \longrightarrow A$ such that $\theta f(\overline{x}, \overline{y}) = -\varepsilon(\overline{xy})$. Let $x = \mu(\overline{x}) + h_x$ and $y = \mu(\overline{y}) + h_y$. Then $xy = \mu(\overline{xy}) + h_{xy} =$

$\mu(\overline{x})\mu(\overline{y})$ implies that

$$\theta(h_{xy}) = \theta(\mu(\overline{x})\mu(\overline{y}) - \mu(\overline{xy})) = \theta f(\overline{x}, \overline{y}) = -\varepsilon(\overline{xy}). \tag{4.1}$$

Now let $\sigma(x) = \theta(h_x) + \varepsilon(\overline{x})$. Since $\operatorname{Im}\sigma \subseteq A$, $\sigma(x)\sigma(y) = 0$ by centrality. By (4.1), $\sigma(xy) = \theta(h_{xy}) + \varepsilon(\overline{xy}) = 0$. Hence $\sigma \in \operatorname{Hom}(L, A)$ and $\sigma(h) = \theta(h) + \varepsilon(h + H) = \theta(h)$ for all $h \in H$, which means that $\operatorname{Res}(\sigma) = \theta$ and thus $\ker(\operatorname{Tra}) \subseteq \operatorname{Im}(\operatorname{Res})$.

To show exactness at $\mathscr{H}^2(L/H, A)$, first consider a map $\chi \in \operatorname{Hom}(H, A)$. Then $\operatorname{Tra}(\chi) = \overline{\chi f}$ where, as before, $f(\overline{x}, \overline{y}) = \mu(\overline{x})\mu(\overline{y}) - \mu(\overline{xy})$ and $\chi f \in \mathscr{Z}^2(L/H, A)$. By definition of Inf_2,

$$\operatorname{Inf}_2(\overline{\chi f}) = \overline{(\chi f)'}$$

where $(\chi f)'(x, y) = \chi f(\overline{x}, \overline{y})$. We want to show that $(\chi f)'$ is a coboundary in $\mathscr{H}^2(L, A)$. To this end, we once again consider $x = \mu(\overline{x}) + h_x$ and $y = \mu(\overline{y}) + h_y$ with product $xy = \mu(\overline{x})\mu(\overline{y}) = \mu(\overline{xy}) - h_{xy}$. Then $\chi f(\overline{x}, \overline{y}) = \chi(\mu(\overline{x})\mu(\overline{y}) - \mu(\overline{xy})) = \chi(h_{xy})$. Define $\varepsilon(x) = -\chi(h_x)$. Then $\varepsilon : L \longrightarrow A$ and is linear. One computes $\varepsilon(xy) = -\chi(h_{xy}) = -\chi f(\overline{x}, \overline{y}) = -(\chi f)'(x, y)$ which implies that $(\chi f)' \in \mathscr{B}^2(L, A)$. Therefore

$$\overline{(\chi f)'} = 0$$

and we have $\operatorname{Im}(\operatorname{Tra}) \subseteq \ker(\operatorname{Inf}_2)$. Conversely, suppose $g \in \mathscr{Z}^2(L/H, A)$ such that $\overline{g} \in \ker(\operatorname{Inf}_2)$. Then $g(\overline{x}, \overline{y}) = g'(x, y) = -\varepsilon(xy)$ for some linear $\varepsilon : L \longrightarrow A$. Since ε is linear, $\varepsilon f \in \mathscr{Z}^2(L/H, A)$. As before, let $x = \mu(\overline{x}) + h_x \in L$ with $xy = \mu(\overline{x})\mu(\overline{y})$ the product of two such elements. Then

$$\begin{aligned} g'(x, y) &= g(\overline{x}, \overline{y}) \\ &= -\varepsilon(\mu(\overline{x})\mu(\overline{y})) \\ &= -\varepsilon f(\overline{x}, \overline{y}) - \varepsilon\mu(\overline{xy}) \end{aligned}$$

where $\varepsilon\mu : L/H \longrightarrow A$. Thus $\overline{g} = \overline{-\varepsilon f} = -\operatorname{Tra}(\varepsilon)$ which implies that $\ker(\operatorname{Inf}_2) \subseteq \operatorname{Im}(\operatorname{Tra})$. □

4.2 Relation of Multipliers and Cohomology

The objective of this section is to prove that the multiplier $M(L)$ of a finite-dimensional Leibniz algebra L is isomorphic to the second cohomology group $\mathscr{H}^2(L, \mathbb{F})$, where $\mathbb{F}$ is considered as a central L-module.

Theorem 4.2.1. *Let Z be a central ideal in L. Then $L' \cap Z$ is isomorphic to the image of* $\operatorname{Hom}(Z, \mathbb{F})$ *under the transgression map. In particular, if* Tra *is surjective, then $L' \cap Z \cong$*

$\mathscr{H}^2(L/Z,\mathbb{F})$.

Proof. Let $0 \longrightarrow Z \longrightarrow L \longrightarrow L/Z \longrightarrow 0$ be the natural exact sequence for a central ideal Z in L. Then the sequence

$$\mathrm{Hom}(L,\mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(Z,\mathbb{F}) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/Z,\mathbb{F})$$

is exact by Theorem 4.1.1. Let J denote the set of all homomorphisms $\chi : Z \longrightarrow \mathbb{F}$ such that χ can be extended to an element of $\mathrm{Hom}(L,\mathbb{F})$. Then J is precisely the image of the restriction map in $\mathrm{Hom}(Z,\mathbb{F})$, which is equal to the kernel of the transgression map by exactness. This means that $\mathrm{Hom}(Z,\mathbb{F})/J \cong \mathrm{Im}(\mathrm{Tra})$ and thus it suffices to show that $\mathrm{Hom}(Z,\mathbb{F})/J \cong L' \cap Z$. Consider the natural restriction homomorphism

$$\mathrm{Hom}(Z,\mathbb{F}) \xrightarrow{\mathrm{Res}_2} \mathrm{Hom}(L' \cap Z,\mathbb{F}).$$

Since Z and $L' \cap Z$ are both abelian, Res_2 is surjective and $\mathrm{Hom}(L' \cap Z,\mathbb{F})$ is the dual space of $L' \cap Z$. Therefore

$$\frac{\mathrm{Hom}(Z,\mathbb{F})}{\ker(\mathrm{Res}_2)} \cong \mathrm{Hom}(L' \cap Z,\mathbb{F}) \cong L' \cap Z$$

and it remains to show that $J \cong \ker(\mathrm{Res}_2)$. For one direction, consider an element $\chi \in J$ with extension $\hat{\chi} \in \mathrm{Hom}(L,\mathbb{F})$. Then $L' \subseteq \ker \hat{\chi}$ since $\mathbb{F}$ is abelian, which implies that $L' \cap Z \subseteq \ker \chi$. Thus $\chi \in \ker(\mathrm{Res}_2)$ and we have $J \subseteq \ker(\mathrm{Res}_2)$. Conversely, let $\chi \in \ker(\mathrm{Res}_2)$. Then $\chi \in \mathrm{Hom}(Z,\mathbb{F})$ is such that $L' \cap Z \subseteq \ker \chi$, which implies that χ induces a homomorphism

$$\chi' : \frac{Z}{L' \cap Z} \longrightarrow \mathbb{F}$$

defined by $\chi'(z + (L' \cap Z)) = \chi(z)$. Since

$$\frac{Z}{L' \cap Z} \cong \frac{Z + L'}{L'},$$

there exists a homomorphism

$$\chi'' : \frac{Z + L'}{L'} \longrightarrow \mathbb{F}$$

defined by $\chi''(z + L') = \chi'(z + (L' \cap Z))$. But χ'' can be extended to a homomorphism $\chi''' : L/L' \longrightarrow \mathbb{F}$ that is defined by $\chi'''(x + L') = \chi''(x + L')$ for all $x \in Z$. Since L/L' is abelian, χ''' can be extended to a homomorphism $\hat{\chi} : L \longrightarrow \mathbb{F}$ that is defined by $\hat{\chi}(x) = \chi'''(x + L')$. Therefore $\chi \in J$ and the first statement holds. The second statement holds since Tra maps $\mathrm{Hom}(Z,\mathbb{F})$ to $\mathscr{H}^2(L/Z,\mathbb{F})$. □

Let L be a Leibniz algebra with free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\omega} L \longrightarrow 0$. The induced sequence

$$0 \longrightarrow \frac{R}{FR+RF} \longrightarrow \frac{F}{FR+RF} \longrightarrow L \longrightarrow 0$$

is a central extension since RF and FR are both contained in $FR+RF$. It is not unique, but has the following property.

Lemma 4.2.2. *Let* $0 \longrightarrow A \longrightarrow B \xrightarrow{\phi} C \longrightarrow 0$ *be a central extension and* $\alpha : L \longrightarrow C$ *be a homomorphism. Then there exists a homomorphism* $\beta : F/(FR+RF) \longrightarrow B$ *such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \frac{R}{FR+RF} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{FR+RF} \arrow[r] \arrow[d, "\beta"] & L \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Figure 4.1 Showing β.

is commutative, where γ *is the restriction of* β *to* $R/(FR+RF)$.

Proof. Since F is free, there exists a homomorphism $\sigma : F \longrightarrow B$ such that

$$\begin{tikzcd}
F \arrow[r, "\omega"] \arrow[d, "\sigma"'] & L \arrow[d, "\alpha"] \\
B \arrow[r, "\phi"] & C
\end{tikzcd}$$

Figure 4.2 Showing $\phi\sigma = \alpha\omega$.

is commutative. Let $r \in R \subseteq F$. Then $\omega(r) = 0$ since $\ker\omega = R$. Therefore $0 = \alpha\omega(r) = \phi\sigma(r)$ and so $\sigma(R) \subseteq \ker\phi$. We want to show that $FR+RF \subseteq \ker\sigma$. If $x \in F$ and $r \in R$, then $\sigma(xr) = \sigma(x)\sigma(r) = 0$ and $\sigma(rx) = \sigma(r)\sigma(x) = 0$ since $\sigma(r) \in \ker\phi = A \subseteq Z(B)$. Now σ induces a homomorphism $\beta : F/(FR+RF) \longrightarrow B$. The left diagram commutes since we may take $A \longrightarrow B$ to be the inclusion map. □

Lemma 4.2.3. *Let* $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ *be a free presentation of* L *and let* A *be a central* L*-module. Then the transgression map* $\mathrm{Tra} : \mathrm{Hom}(R/(FR+RF), A) \longrightarrow \mathscr{H}^2(L, A)$ *associated* *with*

$$0 \longrightarrow \frac{R}{FR+RF} \longrightarrow \frac{F}{FR+RF} \xrightarrow{\phi} L \longrightarrow 0$$

is surjective.

Proof. Consider $\overline{g} \in \mathscr{H}^2(L,A)$ and let $0 \longrightarrow A \longrightarrow E \xrightarrow{\varphi} L \longrightarrow 0$ be an associated central extension. By Lemma 4.2.2, there exists a homomorphism θ such that

$$\begin{tikzcd}
0 \arrow[r] & \frac{R}{FR+RF} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{FR+RF} \arrow[r, "\phi"] \arrow[d, "\theta"] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r, "\varphi"] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 4.3 Invoking Lemma 4.2.2.

is commutative and $\gamma = \theta|_{R/(FR+RF)}$. Let μ be a section of ϕ. Then $\varphi\theta\mu = \phi\mu = \mathrm{id}_L$ and so $\theta\mu$ is a section of φ. Let $\lambda = \theta\mu$ and define $\beta(x,y) = \lambda(x)\lambda(y) - \lambda(xy)$. Then $\beta \in \mathscr{Z}^2(L,A)$ and β is cohomologous with g since they are associated with the same extension. One computes

$$\begin{aligned}
\beta(x,y) &= \theta(\mu(x))\theta(\mu(y)) - \theta(\mu(xy)) \\
&= \theta(\mu(x)\mu(y) - \mu(xy)) \\
&= \gamma(\mu(x)\mu(y) - \mu(xy)) \\
&= \gamma(f(x,y))
\end{aligned}$$

where $f(x,y) = \mu(x)\mu(y) - \mu(xy)$ and since $\gamma = \theta|_{R/(FR+RF)}$. Thus $\mathrm{Tra}(\gamma) = \overline{\gamma f} = \overline{\beta} = \overline{g}$. □

Lemma 4.2.4. *If $C \subseteq A$ and $C \subseteq B$, then $A/C \cap B/C = (A \cap B)/C$.*

Proof. Clearly $(A \cap B)/C \subseteq A/C \cap B/C$. Let $x \in A/C \cap B/C$. Then $x = a + c_1 = b + c_2$ for $a \in A$, $b \in B$, and $c_1, c_2 \in C$. Since $C \subseteq B$, $a = b + c_2 - c_1 \in B$, which implies that $a \in A \cap B$. Then $x = a + c \in (A \cap B)/C$ and so $A/C \cap B/C \subseteq (A \cap B)/C$. □

Theorem 4.2.5. *Let L be a Leibniz algebra over a field $\mathbb{F}$ and $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L. Then*

$$\mathscr{H}^2(L,\mathbb{F}) \cong \frac{F' \cap R}{FR+RF}.$$

In particular, if L is finite-dimensional, then $M(L) \cong \mathscr{H}^2(L,\mathbb{F})$.

Proof. Denote

$$\overline{R} = \frac{R}{FR+RF} \qquad \overline{F} = \frac{F}{FR+RF}$$

Then $0 \longrightarrow \overline{R} \longrightarrow \overline{F} \longrightarrow L \longrightarrow 0$ is a central extension. By Lemma 4.2.3, Tra : $\mathrm{Hom}(\overline{R},\mathbb{F}) \longrightarrow \mathcal{H}^2(L,\mathbb{F})$ is surjective. By Theorem 4.2.1,

$$\overline{F}' \cap \overline{R} \cong \mathcal{H}^2(\overline{F}/\overline{R},\mathbb{F}) \cong \mathcal{H}^2(L,\mathbb{F}).$$

By Lemma 4.2.4,

$$\overline{F}' \cap \overline{R} \cong \frac{F'}{FR+RF} \cap \frac{R}{FR+RF} = \frac{F'\cap R}{FR\cap RF}.$$

Therefore, when L is finite-dimensional,

$$M(L) = \frac{F'\cap R}{FR+RF} \cong \mathcal{H}^2(L,\mathbb{F})$$

by the characterization of $M(L)$ from Theorem 4.0.1. □

We conclude this section with the Leibniz analogue of a corollary that appears at the end of Chapter 3 in [1].

Corollary 4.2.6. *For any cover E of L and any subalgebra A of E satisfying*

1. $A \subseteq Z(E) \cap E'$,
2. $A \cong M(L)$,
3. $L \cong E/A$,

the associated transgression map $\mathrm{Tra} : \mathrm{Hom}(A,\mathbb{F}) \longrightarrow M(L)$ *is bijective.*

Proof. First note that $0 \longrightarrow A \longrightarrow E \longrightarrow L \longrightarrow 0$ is a central extension of L. Invoking the Hochschild-Serre spectral sequence yields

$$0 \longrightarrow \mathrm{Hom}(L,\mathbb{F}) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(E,\mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(A,\mathbb{F}) \xrightarrow{\mathrm{Tra}} \mathcal{H}^2(L,\mathbb{F}) \xrightarrow{\mathrm{Inf}_2} \mathcal{H}^2(E,\mathbb{F})$$

with $\mathrm{Im(Res)} = \ker(\mathrm{Tra})$. Furthermore, any $\theta \in \mathrm{Hom}(E,\mathbb{F})$ yields $\mathrm{Res}(\theta) \in \mathrm{Hom}(A,\mathbb{F})$. Now let $a \in A \subseteq E'$. Then $a = e_1e_2$ for some $e_1, e_2 \in E$ which implies that $\mathrm{Res}(\theta(a)) = \mathrm{Res}(\theta(e_1)\theta(e_2)) = \mathrm{Res}(0) = 0$. Thus $\mathrm{Im(Res)} = 0$, making $\ker(\mathrm{Tra}) = 0$, and so Tra injective. Since $\mathrm{Hom}(A,\mathbb{F}) \cong A \cong M(L)$, Tra is bijective. □

4.3 Unicentral Leibniz Algebras

Let Z be a central ideal of a finite-dimensional Leibniz algebra L and let $Z^*(L)$ denote the intesection of all $\omega(Z(E))$ such that $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$ is a central extension of L. We recall that a Leibniz algebra L is *unicentral* if $Z(L) = Z^*(L)$.

The objective of this section is to determine a set of equivalent conditions for when the center of the cover maps onto the center of L. This result follows from a four-part theorem that gives equivalent statements to $Z \subseteq Z^*(L)$. The first of these statements involves a new map

$$\delta : \mathscr{H}^2(L, \mathbb{F}) \longrightarrow L/L' \otimes Z \oplus Z \otimes L/L'$$

that extends our Hochschild-Serre spectral sequence. The second involves another map $\beta : M(L) \longrightarrow M(L/Z)$, called the natural map, that appears in the *Ganea sequence*.[2] In this section, we will construct these sequences and then prove the equivalence of the following statements:

1. δ is the trivial map,
2. β is injective,
3. $M(L) \cong \frac{M(L/Z)}{L' \cap Z}$,
4. $Z \subseteq Z^*(L)$.

4.3.1 More Sequences

To extend our Hochschild-Serre sequence, consider the natural central extension $0 \longrightarrow Z \longrightarrow L \longrightarrow L/Z \longrightarrow 0$. To define our δ map, consider a cocycle $f' \in \mathscr{Z}^2(L, \mathbb{F})$ and define two bilinear forms $f_1'' : L/L' \times Z \longrightarrow \mathbb{F}$ and $f_2'' : Z \times L/L' \longrightarrow \mathbb{F}$ by

$$\begin{aligned} f_1''(x + L', z) &= f'(x, z), \\ f_2''(z, x + L') &= f'(z, x) \end{aligned}$$

for $x \in L$ and $z \in Z$. To check that they are well-defined, one computes

$$\begin{aligned} f_1''(xy + L', z) &= f'(xy, z) \\ &= f'(x, yz) - f'(y, xz) \\ &= 0 \end{aligned}$$

[2]The sequence was constructed for groups by Ganea in 1968 and for Lie algebras by Batten in 1993 [1].

and

$$\begin{aligned} f_2''(z, xy + L') &= f'(z, xy) \\ &= f'(x, zy) - f(xz, y) \\ &= 0 \end{aligned}$$

since $z \in Z(L)$. Hence $(f_1'', f_2'') \in \mathrm{Bil}(L/L' \times Z, \mathbb{F}) \oplus \mathrm{Bil}(Z \times L/L', \mathbb{F}) \cong L/L' \otimes Z \oplus Z \otimes L/L'$. Now consider a coboundary $f' \in \mathscr{B}^2(L, \mathbb{F})$. By definition, there exists a linear map $\varepsilon : L \longrightarrow \mathbb{F}$ such that $f'(x, y) = -\varepsilon(xy)$. One computes

$$\begin{aligned} f_1''(x + L', z) &= f'(x, z) = -\varepsilon(xz) = 0, \\ f_2''(z, x + L') &= f'(z, x) = -\varepsilon(zx) = 0 \end{aligned}$$

since $z \in Z(L)$. Hence, a map $\delta : f' + \mathscr{B}^2(L, \mathbb{F}) \mapsto (f_1'', f_2'')$ is induced which is clearly linear since f', f_1'', and f_2'' are all in vector spaces of bilinear forms and the latter two are defined by f'.

Theorem 4.3.1. *Let Z be a central ideal of a Leibniz algebra L. The sequence*

$$\mathscr{H}^2(L/Z, \mathbb{F}) \xrightarrow{\mathrm{Inf}} \mathscr{H}^2(L, \mathbb{F}) \xrightarrow{\delta} L/L' \otimes Z \oplus Z \otimes L/L'$$

is exact.

Proof. Let $f \in \mathscr{Z}^2(L/Z, \mathbb{F})$. Then $\mathrm{Inf}(f + \mathscr{B}^2(L/Z, \mathbb{F})) = f' + \mathscr{B}^2(L, \mathbb{F})$ where f' is the cocycle defined by $f'(x, y) = f(x + Z, y + Z)$. We also have $\delta(f' + \mathscr{B}^2(L, \mathbb{F})) = (f_1'', f_2'')$ where, for all $x \in L$ and $z \in Z$,

$$\begin{aligned} f_1''(x + L', z) &= f'(x, z) = f(x + Z, z + Z) = 0, \\ f_2''(z, x + L') &= f'(z, x) = f(z + Z, x + Z) = 0, \end{aligned}$$

which implies that $\delta(\mathrm{Inf}(f + \mathscr{B}^2(L/Z, \mathbb{F}))) = (f_1'', f_2'') = (0, 0)$. Therefore $\mathrm{Im}(\mathrm{Inf}) \subseteq \ker \delta$.

Conversely, suppose $f' \in \mathscr{Z}^2(L, \mathbb{F})$ is such that $\delta(f' + \mathscr{B}^2(L, \mathbb{F})) = (f_1'', f_2'') = (0, 0)$. Then, for all $x \in L$ and $z \in Z$, one has

$$\begin{aligned} 0 &= f_1''(x + L', z) = f'(x, z), \\ 0 &= f_2''(z, x + L') = f'(z, x). \end{aligned}$$

Hence, for all $z, z' \in Z$ and $x, y \in L$, one computes

$$\begin{aligned} f'(x+z, y+z') &= f'(x,y) + f_1''(x+L', z') + f_2''(z, y+L') + f_1''(z+L', z') \\ &= f'(x,y), \end{aligned}$$

which yields a bilinear form $g : L/Z \times L/Z \longrightarrow \mathbb{F}$, defined by $g(x+Z, y+Z) = f'(x,y)$, that is well-defined. Furthermore, $g \in \mathscr{Z}^2(L/Z, \mathbb{F})$ since f' is a cocycle. Thus $\mathrm{Inf}(g + \mathscr{B}^2(L/Z, \mathbb{F})) = f' + \mathscr{B}^2(L, \mathbb{F})$ and so $\ker\delta \subseteq \mathrm{Im}(\mathrm{Inf})$. □

Theorem 4.3.2. *(Ganea Sequence) Let Z be a central ideal in a finite-dimensional Leibniz algebra L. Then the sequence*

$$L/L' \otimes Z \oplus Z \otimes L/L' \longrightarrow M(L) \longrightarrow M(L/Z) \longrightarrow L' \cap Z \longrightarrow 0$$

is exact.

Proof. Let F be a free Leibniz algebra such that $L = F/R$ and $Z = T/R$ for some ideals T and R of F. Since $Z \subseteq Z(L)$, one has $T/R \subseteq Z(F/R)$ and $FT + TF \subseteq R$. Inclusion maps $\hat{\beta} : R \cap F' \longrightarrow T \cap F'$ and $\hat{\gamma} : T \cap F' \longrightarrow T \cap (F' + R)$ induce homomorphisms

$$\frac{R \cap F'}{FR + RF} \xrightarrow{\beta} \frac{T \cap F'}{FT + TF} \xrightarrow{\gamma} \frac{T \cap (F' + R)}{R} \longrightarrow 0.$$

Since $R \subseteq T$, one has

$$\frac{T \cap (F' + R)}{R} = \frac{(T + R) \cap (F' + R)}{R} \cong \frac{(T \cap F') + R}{R}$$

which implies that γ is surjective. By Theorem 4.0.1,

$$M(L) \cong \frac{R \cap F'}{FR + RF} \qquad \text{and} \qquad M(L/Z) \cong \frac{T \cap F'}{FT + TF}.$$

Also

$$L' \cap Z \cong (F/R)' \cap (T/R) \cong \frac{F' + R}{R} \cap \frac{T}{R} \cong \frac{(F' + R) \cap T}{R}.$$

Therefore, the sequence $M(L/Z) \xrightarrow{\gamma} L' \cap Z \longrightarrow 0$ is exact. Since

$$\ker\gamma = \frac{(T \cap F') \cap R}{FT + TF} = \frac{R \cap F'}{FT + TF} = \mathrm{Im}\,\beta,$$

the sequence $M(L) \xrightarrow{\beta} M(L/Z) \xrightarrow{\gamma} L' \cap Z$ is exact.

It remains to show that $L/L' \otimes Z \oplus Z \otimes L/L' \longrightarrow M(L) \xrightarrow{\beta} M(L/Z)$ is exact. Define a pair of maps

$$\theta_1 : \frac{F}{R+F'} \times \frac{T}{R} \longrightarrow \frac{R \cap F'}{FR+RF} \qquad \theta_2 : \frac{T}{R} \times \frac{F}{R+F'} \longrightarrow \frac{R \cap F'}{FR+RF}$$

by $\theta_1(f+(R+F'), t+R) = ft + (FR+RF)$ and $\theta_2(t+R, f+(R+F')) = tf + (FR+RF)$. Both are bilinear because multiplication is bilinear. To check that θ_1 and θ_2 are well-defined, suppose $(f+(R+F'), t+R) = (f'+(R+F'), t'+R)$ for $t, t' \in T$ and $f, f' \in F$. Then $t - t' \in R$ and $f - f' \in R + F'$ which implies that $t = t' + r$ for $r \in R$ and $f = f' + x$ for $x \in R + F'$. One computes

$$\begin{aligned} tf - t'f' &= (t'+r)(f'+x) - t'f' \\ &= t'x + rf' + rx \end{aligned}$$

and

$$\begin{aligned} ft - f't' &= (f'+x)(t'+r) - f't' \\ &= xt' + f'r + xr \end{aligned}$$

which both fall in $FR + RF$ by the Leibniz identity and the fact that $FT + TF \subseteq R$. Thus θ_1 and θ_2 are well-defined, and so induce linear maps

$$\overline{\theta_1} : \frac{F}{R+F'} \otimes \frac{T}{R} \longrightarrow \frac{R \cap F'}{FR+RF} \qquad \overline{\theta_2} : \frac{T}{R} \otimes \frac{F}{R+F'} \longrightarrow \frac{R \cap F'}{FR+RF}$$

These, in turn, yield a linear transformation

$$\overline{\theta} : \frac{F}{R+F'} \otimes \frac{T}{R} \oplus \frac{T}{R} \otimes \frac{F}{R+F'} \longrightarrow \frac{R \cap F'}{FR+RF}$$

defined by $\overline{\theta}(a,b) = \overline{\theta_1}(a) + \overline{\theta_2}(b)$. The image of $\overline{\theta}$ is

$$\frac{FT+TF}{FR+RF}$$

which is precisely equal to $\{x + (FR+RF) \mid x \in R \cap F',\ x \in FT + TF\} = \ker \beta$. Thus the sequence

$$\frac{F}{R+F'} \otimes \frac{T}{R} \oplus \frac{T}{R} \otimes \frac{F}{R+F'} \cong L/L' \otimes Z \oplus Z \otimes L/L' \longrightarrow \frac{R \cap F'}{FR+RF} \cong M(L) \longrightarrow \frac{F' \cap T}{FT+TF} \cong M(L/Z)$$

is exact. □

Corollary 4.3.3. *(Stallings Sequence) Let Z be a central ideal of a Leibniz algebra L. Then the sequence*

$$M(L) \longrightarrow M(L/Z) \longrightarrow Z \longrightarrow L/L' \longrightarrow \frac{L}{Z+L'} \longrightarrow 0$$

is exact.

Proof. Let F be a free Leibniz algebra such that $L = F/R$ and $Z = T/R$ for ideals T and R of F. Then $FT + TF \subseteq R$ since $Z \subseteq Z(L)$. The inclusion maps $R \cap F' \longrightarrow T \cap F' \longrightarrow T \longrightarrow F \longrightarrow F$ induce the following sequence of homomorphisms:

$$\frac{R \cap F'}{FR+RF} \xrightarrow{\beta} \frac{T \cap F'}{FT+TF} \xrightarrow{\theta} \frac{T}{R} \xrightarrow{\alpha} \frac{F}{R+F'} \xrightarrow{\omega} \frac{F}{T+F'} \xrightarrow{\gamma} 0$$

To prove exactness for our desired sequence, we make use of the following facts:

1. $M(L) \cong \frac{R\cap F'}{FR+RF}$,
2. $M(L/Z) \cong \frac{T\cap F'}{FT+TF}$,
3. $Z \cong T/R$,
4. $\frac{F}{R+F'} \cong L/L'$,
5. $\frac{F}{T+F'} = \frac{F}{T+F'+R} \cong \frac{(F/R)/(T+F'+R)}{R} \cong \frac{F/R}{T/R+(F'+R)/R} \cong \frac{L}{Z+L'}$.

Thus do the following equalities suffice for this proof:

i. $\ker\theta = \{x + (FT + TF) \mid x \in T \cap F',\ x \in R\} = \frac{T\cap F'\cap R}{FT+TF} = \frac{R\cap F'}{FT+TF} = \operatorname{Im}\beta$,

ii. $\ker\alpha = \{x + R \mid x \in T,\ x \in (R + F')\} = \frac{T\cap(R+F')}{R} = \frac{R+(T\cap F')}{R} = \operatorname{Im}\theta$,

iii. $\ker\omega = \{x + (R + F') \mid x \in F,\ x \in (T + F')\} = \frac{F\cap(T+F')}{R+F'} = \frac{T+F'}{R+F'} = \operatorname{Im}\alpha$,

iv. $\ker\gamma = \frac{F}{T+F'} = \operatorname{Im}\omega$.

□

4.3.2 The Main Result

The following pair of lemmas shows that our first three conditions are equivalent.

Lemma 4.3.4. *Let Z be a central ideal of a finite-dimensional Leibniz algebra L and consider the map*

$$\delta : M(L) \longrightarrow L/L' \otimes Z \oplus Z \otimes L/L'$$

from Theorem 4.3.1. Then

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if δ is the trivial map. Here, we have identitfied $L' \cap Z$ with its image in $M(L/Z)$.

Proof. We invoke Theorems 4.1.1 and 4.3.1, yielding an exact sequence

$$\mathrm{Hom}(L/Z, \mathbb{F}) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(L, \mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(Z, \mathbb{F}) \xrightarrow{\mathrm{Tra}} M(L/Z) \xrightarrow{\mathrm{Inf}_2} M(L) \xrightarrow{\delta} L/L' \otimes Z \oplus Z \otimes L/L'.$$

In one direction, suppose δ is the zero map. Then $M(L) \cong \ker \delta \cong \mathrm{Im}(\mathrm{Inf}_2)$. Since

$$\mathrm{Im}(\mathrm{Inf}_2) \cong \frac{M(L/Z)}{\ker(\mathrm{Inf}_2)}$$

and $\ker(\mathrm{Inf}_2) = \mathrm{Im}(\mathrm{Tra}) \cong L' \cap Z$ by Theorem 4.2.1, we have

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}.$$

Conversely, the isomorphism

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z} \cong \frac{M(L/Z)}{\ker(\mathrm{Inf}_2)} \cong \mathrm{Im}(\mathrm{Inf}_2) \cong \ker \delta$$

implies that δ is trivial. □

Lemma 4.3.5. *Let Z be a central ideal of a finite-dimensional Leibniz algebra L and consider the natural map $\beta : M(L) \longrightarrow M(L/Z)$ from Theorem 4.3.2. Then*

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if β is injective.

Proof. By Theorem 4.3.2, the sequence $M(L) \xrightarrow{\beta} M(L/Z) \xrightarrow{\alpha} L' \cap Z \xrightarrow{\omega} 0$ is exact. Suppose β is injective. Then $\ker \beta = 0$, which implies that

$$M(L) \cong \mathrm{Im}\,\beta \cong \ker \alpha \cong \frac{M(L/Z)}{\mathrm{Im}\,\alpha} = \frac{M(L/Z)}{\ker \omega} \cong \frac{M(L/Z)}{L' \cap Z}.$$

Conversely, the isomorphism

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z} \cong \mathrm{Im}\,\beta$$

implies that β is injective. □

Once more, our objective is to find conditions for when $\omega(Z(E)) = Z(L)$, where E is the cover of L and $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$ is a central extension. Such an extension is called a *stem extension*, i.e. a central extension $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in which $A \subseteq B'$. To this end, we will show that the preceding three conditions are equivalent to $Z \subseteq Z^*(L)$ for a general central ideal Z in L. The special case of $Z = Z(L)$ will lead to our main result.

Consider the free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ of L and let $\overline{X}$ denote the quotient algebra $\frac{X}{FR+RF}$ for any X such that $FR + RF \subseteq X \subseteq F$. Since $R = \ker\pi$ and $FR + RF \subseteq R$, π induces a homomorphism $\overline{\pi} : \overline{F} \longrightarrow L$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d] & L \\ \overline{F} \arrow[ur, "\overline{\pi}"'] & \end{tikzcd}$$

Figure 4.4 Induced $\overline{\pi}$.

commutes. Since $\overline{R} \subseteq Z(\overline{F})$, there exists a complement $\frac{S}{FR+RF}$ to $\frac{R\cap F'}{FR+RF}$ in $\frac{R}{FR+RF}$ yielding the diagram

$$\begin{tikzcd} & \frac{R}{FR+RF} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{S}{FR+RF} \arrow[dr, dash] & & \frac{R\cap F'}{FR+RF} \arrow[dl, dash] \\ & \overline{0} & \end{tikzcd}$$

Figure 4.5 Complement $\overline{S}$ in $\overline{R}$.

Here, $S \subseteq R \subseteq \ker\pi$ and $\overline{S} \subseteq \overline{R} \subseteq \ker\overline{\pi}$, and thus $\overline{\pi}$ induces a homomorphism $\pi_S : F/S \longrightarrow L$ such that the extension $0 \longrightarrow R/S \longrightarrow F/S \xrightarrow{\pi_S} L \longrightarrow 0$ is central. This extension is stem since $R/S \cong \frac{R\cap F'}{FR+RF} = \ker\pi_S$ implies that F/S is a cover of L.

Lemma 4.3.6. *For every free presentation* $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ *of* L *and every central extension* $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$*, one has* $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$.

Proof. Since the identity map $\mathrm{id} : L \longrightarrow L$ is a homomorphism, we can invoke Lemma 4.2.2, yielding a homomorphism $\beta : \overline{F} \longrightarrow E$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{R}{FR+RF} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{FR+RF} \arrow[r, "\overline{\pi}"] \arrow[d, "\beta"] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker\omega \arrow[r] & E \arrow[r, "\omega"] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 4.6 Invoking Lemma 4.2.2 again.

is commutative (where γ is the restriction of β to $\overline{R}$).

Let $A = \ker\omega$. Our first claim is that $E = A + \beta(\overline{F})$. Indeed, let $e \in E$. Then $\omega(e) = \overline{\pi}(f)$ for some $f \in \overline{F}$, and so $\omega(e) = \omega\beta(f)$ by diagram commutativity. This implies that $e - \beta(f) \in \ker\omega = A$, meaning $e - \beta(f) = a$ for some $a \in A$. Thus $e = a + \beta(f)$.

Our second claim is that $\beta(Z(\overline{F}))$ centralizes both A and $\beta(\overline{F})$. To see this, one first computes $\beta(Z(\overline{F}))\beta(\overline{F}) = \beta(Z(\overline{F})\overline{F}) = \beta(0) = 0$ and $\beta(\overline{F})\beta(Z(\overline{F})) = \beta(\overline{F}Z(\overline{F})) = \beta(0) = 0$. Next, we know that AE and EA are both zero, and so $A\beta(Z(\overline{F}))$ and $\beta(Z(\overline{F}))A$ are zero as well. But this implies that $\beta(Z(\overline{F}))$ centralizes E by the first claim. Hence $\beta(Z(\overline{F})) \subseteq Z(E)$ and $\omega\beta(Z(\overline{F})) \subseteq \omega(Z(E))$, which yields $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$. □

Theorem 4.3.7. *For every free presentation* $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ *of* L *and every stem extension* $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$*, one has* $Z^*(L) = \overline{\pi}(Z(\overline{F})) = \omega(Z(E))$.

Proof. By Lemma 4.3.6, $\overline{\pi}(Z(\overline{F}))$ is contained in $\omega'(Z(E'))$ for every central extension

$$0 \longrightarrow \ker\omega' \longrightarrow E' \xrightarrow{\omega'} L \longrightarrow 0$$

of L. Thus $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$ for our stem extension. We also know that $Z^*(L)$ is the intersection of all images $\omega'(Z(E'))$, and that $\overline{\pi}(Z(\overline{F}))$ is one of these images since $0 \longrightarrow \overline{R} \longrightarrow \overline{F} \xrightarrow{\overline{\pi}} L \longrightarrow 0$ is central. Therefore $\overline{\pi}(Z(\overline{F})) = Z^*(L)$. Since this equality holds for all F, we can assume that $0 \longrightarrow R/S \longrightarrow F/S \xrightarrow{\pi_S} L \longrightarrow 0$ is a stem extension where S is defined as above. Since the cover F/S is unique up to isomorphism, it now suffices to show that $\pi_S(Z(F/S)) = \overline{\pi}(Z(\overline{F}))$.

Let T be the inverse image of $Z(F/S)$ in F and consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\pi_3"] \arrow[d, "\pi_1"'] & F/S \\ \overline{F} \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Figure 4.7 Showing $\pi_3 = \pi_2\pi_1$.

where all mappings are the natural ones. Then $\overline{T} = \pi_1(T)$ by definition and

$$\pi_2(\overline{T}) = \pi_2\pi_1(T) = \pi_3(T) = Z(F/S),$$

yielding the diagram

$$\begin{tikzcd} T \arrow[r, "\pi_3"] \arrow[d, "\pi_1"'] & Z(F/S) \\ \overline{T} \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Figure 4.8 Restrictions of $\pi_3 = \pi_2\pi_1$.

where all maps denote their restrictions. Now let $x \in Z(\overline{F})$. Then $\pi_2(x) \in Z(F/S)$, which implies that there exists $y \in T$ such that $\pi_3(y) = \pi_2(x)$. The resulting equality $\pi_2\pi_1(y) = \pi_2(x)$ yields an element $\pi_1(y) - x \in \ker \pi_2 = \overline{S} \subseteq \overline{T}$, where $\overline{S} \subseteq \overline{T}$ since $S \subseteq T$. Therefore $x \in \overline{T}$ and $Z(\overline{F}) \subseteq \overline{T}$. For the reverse inclusion, we first note that $T/S = Z(F/S)$, and so $FT + TF \subseteq S$. Thus $\overline{FT} + \overline{TF} \subseteq \overline{S}$. Also $\overline{FT} + \overline{TF} \subseteq \overline{R}$ since $\overline{S} \subseteq \overline{R}$ and $\overline{FT} + \overline{TF} \subseteq \overline{F}'$ by definition. Hence $\overline{FT} + \overline{TF} \subseteq \overline{S} \cap (\overline{R} \cap \overline{F}') = \overline{0}$ which implies that $\overline{T} \subseteq Z(\overline{F})$ and thus $\overline{T} = Z(\overline{F})$. Hence, the commutative diagram

$$
\begin{tikzcd}
F \arrow[r, "\pi"] \arrow[d] & L \\
\overline{F} \arrow[ur, "\overline{\pi}"] \arrow[d] & \\
F/S \arrow[uur, "\pi_S"'] &
\end{tikzcd}
$$

Figure 4.9 Showing π, $\overline{\pi}$, and π_S.

yields the equality $\overline{\pi}(Z(\overline{F})) = \overline{\pi}(\overline{T}) = \pi_S(T/S) = \pi_S(Z(F/S))$ by the definition of T. □

Lemma 4.3.8. *Let Z be a central ideal of a finite-dimensional Leibniz algebra L and consider the map $\beta : M(L) \longrightarrow M(L/Z)$ from Theorem 4.3.2. Then $Z \subseteq Z^*(L)$ if and only if β is injective.*

Proof. In the proof of the Ganea sequence, we saw that $\ker \beta$ can be interpreted as $\overline{FT} + \overline{TF}$. If β is injective, then $\overline{FT} + \overline{TF} = 0$, which implies that $\overline{T} \subseteq Z(\overline{F})$. By the proof of Theorem 4.3.7, $Z \subseteq Z^*(L)$. Conversely, if $Z \subseteq Z^*(L)$, then $\overline{T} \subseteq Z(\overline{F})$, which implies that $\overline{FT} + \overline{TF} = 0$. Thus $\ker \beta = 0$ and β is injective. □

Theorem 4.3.9. *Let Z be a central ideal of a finite-dimensional Leibniz algebra L and*

$$\delta : M(L) \longrightarrow L/L' \otimes Z \oplus Z \otimes L/L'$$

be as in Theorem 4.3.1. Then the following are equivalent:

1. *δ is the trivial map,*
2. *the natural map β is injective,*
3. *$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$,*
4. *$Z \subseteq Z^*(L)$.*

We conclude this chapter by narrowing our focus to when the conditions of Theorem 4.3.9 hold for $Z = Z(L)$. Under this assumption, we show that the center of the cover maps onto the center of the algebra, which happens when L is unicentral.

Theorem 4.3.10. *Let L be a Leibniz algebra and $Z(L)$ be the center of L. If $Z(L) \subseteq Z^*(L)$, then $\omega(Z(E)) = Z(L)$ for every stem extension* $0 \longrightarrow \ker \omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$.

Proof. By definition, $Z^*(L) \subseteq \omega(Z(E)) \subseteq Z(L)$ for any stem extension $0 \longrightarrow \ker \omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$. By hypothesis, $Z(L) \subseteq Z^*(L)$. Therefore $Z^*(L) = \omega(Z(E)) = Z(L)$. □

CHAPTER

5

MULTIPLIERS AND COVERS OF DIASSOCIATIVE ALGEBRAS

Let L be a finite-dimensional diassociative algebra. In this chapter, we first prove the uniqueness of the cover and obtain a characterization of the multiplier $M(L)$ in terms of a free presentation. We then characterize the multiplier by the second cohomology group $\mathscr{H}^2(L,\mathbb{F})$. Finally, we establish a diassociative analogue of the four-part equivalence theorem from Chapter 4, and obtain conditions for when the center of the cover maps onto the center of L. Some of the results hold by the same logic as their Lie or Leibniz versions, and so we will sometimes refer to those proofs rather than rewriting them. As before, we use $A \Diamond B$ to denote the algebra $A \dashv B + A \vdash B$.

5.1 Existence of Universal Elements and Unique Covers

This section follows the methodology of Chapter 1 in [1], that was generalized to Leibniz algebras in [16]. Our initial dimension bounds are notably different from the Lie and Leibniz cases, as there are simply more possible multiplications for which to account.

Lemma 5.1.1. *For any diassociative algebra K, if* $\dim(K/Z(K)) = n$*, then* $\dim(K') \leq 2n^2$.

Proof. Let $\{\overline{x_1}, \overline{x_2}, \dots, \overline{x_n}\}$ be a basis for $K/Z(K)$. Then $\{x_i \dashv x_j, x_i \vdash x_j \mid 1 \le i, j \le n\}$ is a generating set for K'. Thus $\dim(K') \le 2n^2$. $\square$

Lemma 5.1.2. *Let L be a finite-dimensional diassociative algebra with* $\dim L = n$ *and let K be the first term in a defining pair for L. Then* $\dim K \le n(2n+1)$.

Proof. We know that $\dim(K/Z(K)) \le \dim(K/M) = \dim L = n$ since $M \subseteq Z(K)$. Therefore, $\dim M \le \dim(K') \le 2n^2$ via Lemma 5.1.1 since $M \subseteq K'$. We thus have $\dim K = \dim L + \dim M \le n + 2n^2 = n(2n+1)$. $\square$

These facts ensure that the members of any defining pair for a finite-dimensional diassociative algebra L have bounded dimension. Example 5.1.3 illustrates that the highest possible dimension bounds of Lemmas 5.1.1 and 5.1.2 can always be obtained.

Example 5.1.3. Let L be the n-dimensional abelian diassociative algebra with basis $\{x_i\}_{i=1,\dots,n}$ and let M be the $2n^2$-dimensional abelian diassociative algebra with basis $\{m_{ij}, s_{ij}\}_{i,j=1,\dots n}$. Let K denote the vector space $M \oplus L$ with only nonzero multiplications given by $x_i \dashv x_j = m_{ij}$ and $x_i \vdash x_j = s_{ij}$ for $i, j = 1, \dots, n$. Then K is a diassociative algebra of dimension $n+2n^2$ and $M = Z(K) = K'$. Clearly K is a cover of L and M is the multiplier since we have maximal possible dimension. Noting that $L = K/Z(K)$, we also obtain $\dim K' = 2n^2$.

Let $C(L)$ denote the set of all pairs (J, λ) such that $\lambda : J \longrightarrow L$ is a surjective homomorphism and $\ker \lambda \subseteq J' \cap Z(J)$. An element $(T, \tau) \in C(L)$ is called a *universal element* in $C(L)$ if, for any $(J, \lambda) \in C(L)$, there exists a homomorphism $\beta : T \longrightarrow J$ such that the diagram

$$\begin{tikzcd}
T \arrow[r, "\tau"] \arrow[d, "\beta"'] & L \\
J \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Figure 5.1 Showing $\lambda\beta = \tau$.

commutes, i.e. such that $\lambda\beta = \tau$.

Defining pairs for L correspond to elements of $C(L)$ in a natural way. Indeed, any $(K, \lambda) \in C(L)$ gives rise to a defining pair $(K, \ker \lambda)$. Conversely, any defining pair (K, M) yields a surjective homomorphism $\lambda : K \longrightarrow L$ such that $\ker \lambda = M \subseteq Z(K) \cap K'$, and thus $(K, \lambda) \in C(L)$. We will show that a pair $(T, \tau) \in C(L)$ is a universal element if and only if T is a cover.

Lemma 5.1.4. *Let K be a finite dimensional diassociative algebra. Then $Z(K)\cap K'$ is contained in every maximal subalgebra of K.*

Proof. Let M be a maximal subalgebra of K and let $A = Z(K)\cap K'$. Then $A+M$ is also subalgebra of K, which implies that $A+M = K$ or M. Suppose $A+M = K$. Then

$$\begin{aligned} K' &= K \dashv K + K \vdash K \\ &= A\Diamond A + A\Diamond M + M\Diamond A + M\Diamond M \\ &= M \dashv M + M \vdash M \\ &= M' \subseteq M. \end{aligned}$$

Therefore $Z(K)\cap K' \subseteq M$, a contradiction. □

Lemma 5.1.5. *Let $(J,\lambda)\in C(L)$ and $\mu : K \longrightarrow L$ be a surjective homomorphism. Suppose there is a homomorphism $\beta : K \longrightarrow J$ such that the diagram*

$$\begin{tikzcd} K \arrow[r, "\mu"] \arrow[d, "\beta"'] & L \\ J \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Figure 5.2 Showing $\lambda\beta = \mu$.

commutes, i.e. such that $\lambda\beta = \mu$. Then β is surjective.

Proof. Let $j \in J$ and $\lambda(j) = \mu(k)$ for some $k \in K$. Then the equality $\mu(k) = \lambda\beta(k) = \lambda(j)$ yields an element $\beta(k) - j \in \ker\lambda$ and so $J = \ker\lambda + \operatorname{Im}\beta$. By assumption, $\ker\lambda \subseteq Z(J)\cap J'$, where $Z(J)\cap J'$ is contained in every maximal subalgebra of J by Lemma 5.1.4. Suppose that $\operatorname{Im}\beta \neq J$. Then $\operatorname{Im}\beta$ is contained in some maximal subalgebra M of J that is not equal to J, and so $\operatorname{Im}\beta + (Z(J)\cap J') \subseteq M$. But this implies that $M = J$, a contradiction. Thus $\operatorname{Im}\beta = J$. □

Suppose there is an element $(K,\eta)\in C(L)$ such that, for all $(J,\lambda)\in C(L)$, there exists a homomorphism $\rho : K \longrightarrow J$ that satisfies $\lambda\rho = \eta$. Then Lemma 5.1.5 implies that ρ is surjective, and so $\dim J \leq \dim K$. Hence K is a cover of L since its dimension is maximal. Moreover, any other cover of L has the same dimension as K and is the homomorphic image of K, and so must be isomorphic to K. We have thus shown the following statement.

Lemma 5.1.6. *If there exists a universal element $(K,\eta) \in C(L)$, then all covers of L are isomorphic.*

It remains to show that universal elements exist in the diassociative setting. We begin by fixing a free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ of L and assigning

$$B = \frac{R}{F\Diamond R + R\Diamond F} \qquad C = \frac{F}{F\Diamond R + R\Diamond F} \qquad D = \frac{F' \cap R}{F\Diamond R + R\Diamond F}$$

for ease of notation. Then $C\Diamond B + B\Diamond C = 0$ and D is a central ideal in C. Thus π induces a homomorphism $\overline{\pi} : C \longrightarrow L$ with kernel B. We will show that there is a central ideal

$$E = \frac{S}{F\Diamond R + R\Diamond F}$$

of C, complementary to D in B, such that $(C/E, \pi_S) \in C(L)$ is a universal element, where π_S is induced by $\overline{\pi}$. Consider $(J,\lambda) \in C(L)$. By the universal property of our free diassociative algebra F, there exists a homomorphism $\sigma : F \longrightarrow J$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, "\sigma"'] & L \\ J \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Figure 5.3 Showing $\pi = \lambda\sigma$.

commutes, i.e. such that $\pi = \lambda\sigma$, as seen in the Lie and Leibniz cases. The following lemma yields a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\overline{\pi}"] \arrow[d, "\overline{\sigma}"'] & L \\ J \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Figure 5.4 Showing $\overline{\pi} = \lambda\overline{\sigma}$.

where $\overline{\sigma}$ is induced by σ.

Lemma 5.1.7. *Let $x \in F$. Then $x \in R$ if and only if $\sigma(x) \in \ker\lambda$. Also, $F\Diamond R + R\Diamond F \subseteq \ker\sigma$, and thus σ induces a homomorphism $\overline{\sigma} : C \longrightarrow J$ which is surjective and satisfies $\lambda\overline{\sigma} = \overline{\pi}$.*

Proof. Let $x \in F$. If $x \in R$, then $x \in \ker\pi$, which implies that $0 = \pi(x) = \lambda\sigma(x)$. Therefore $\sigma(x) \in \ker\lambda$. Conversely, if $\sigma(x) \in \ker\lambda$, then $0 = \lambda\sigma(x) = \pi(x)$, which implies that x is an element of $\ker\pi = R$. Now consider $r \dashv f \in F\Diamond R + R\Diamond F$. Then $\sigma(r \dashv f) = \sigma(r) \dashv \sigma(f) = 0$ since $\sigma(r) \in \ker\lambda \subseteq Z(J) \cap J'$ and $\sigma(f) \in J$. The cases of $r \vdash f$, $f \dashv r$, and $f \vdash r$ are similar, and so we have our homomorphism $\overline{\sigma} : C \longrightarrow J$, induced by σ, which is surjective since σ is surjective. One computes

$$\begin{aligned}\lambda\overline{\sigma}(f + (F\Diamond R + R\Diamond F)) &= \lambda\sigma(f)\\ &= \pi(f)\\ &= \overline{\pi}(f + (F\Diamond R + R\Diamond F))\end{aligned}$$

and thus $\lambda\overline{\sigma} = \overline{\pi}$. □

Lemma 5.1.8. $\overline{\sigma}(B) = \ker\lambda = \overline{\sigma}(D)$*, from which it follows $B = D + \ker\overline{\sigma}$.*

Proof. This lemma combines the results of Lemmas 1.7, 1.8, and 1.9 in [1], which follow similarly to the Lie case. □

Lemma 5.1.9. $(C/E, B/E)$ *is a defining pair for L, where E is a central ideal in C that is complementary to D in B.*

Proof. We first compute

$$\frac{C/E}{B/E} \cong C/B \cong F/R \cong L$$

and thus the first axiom of defining pairs is satisfied. Next, we know that $B \subseteq Z(C)$, and so

$$B/E \subseteq Z(C/E).$$

Finally,

$$D = \frac{F' \cap R}{F\Diamond R + R\Diamond F} \subseteq \frac{F'}{F\Diamond R + R\Diamond F} \cong \Big(\frac{F}{F\Diamond R + R\Diamond F}\Big)' = C'$$

implies that

$$B/E \cong \frac{D \oplus E}{E} \subseteq \frac{C' + E}{E} \cong (C/E)'.$$

Therefore $B/E \subseteq Z(C/E) \cap (C/E)'$. □

Lemma 5.1.8 allows us to choose a subspace E, complementary to D in B, which is contaied in $\ker\overline{\sigma}$. Thus, given an element $(J, \lambda) \in C(L)$, our $\overline{\sigma}$ induces a homomorphism $\sigma_S : C/E \longrightarrow J$ such that the diagram

$$\begin{tikzcd} C/E \arrow[r, "\pi_S"] \arrow[d, "\sigma_S"'] & L \\ J \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Figure 5.5 Showing $\lambda\sigma_S = \pi_S$.

commutes, i.e. such that $\lambda\sigma_S = \pi_S$. Furthermore, by Lemma 5.1.9, $(C/E, B/E)$ is a defining pair for L. Specializing this discussion to when J is a cover of L, we now prove that C/E is a cover and thereby obtain a characterization of the multiplier in terms of the free presentation.

Lemma 5.1.10. *Given a cover K of L, the corresponding C/E is also a cover of L and the multiplier of L is*

$$M(L) = \frac{F' \cap R}{F \lozenge R + R \lozenge F}.$$

Proof. If K is a cover of L, then $\dim K \geq \dim(C/E)$ since C/E is the first member of a defining pair for L. Since K is the homomorphic image of C/E, we also have $\dim K \leq \dim(C/E)$ and thus $\dim K = \dim(C/E)$. This means C/E is a cover of L. Finally, since $C/B \cong L$ and

$$B/E \cong \frac{F' \cap R}{F \lozenge R + R \lozenge F},$$

we have the desired expression for $M(L)$. □

Since $E \subseteq \ker\overline{\sigma}$, E necessarily depends on $\overline{\sigma}$ and thus on J. As in [16] and [1], we will show that there is a single C/E which works for all $(J, \lambda) \in C(L)$, i.e. a *universal* element of the form $(C/E, \pi_S) \in C(L)$. It suffices to show that all C/E's are isomorphic. To this end, we first state the following cancellation lemma, which holds by the same logic as its Lie analogue. We will then specialize F so that said lemma can be applied.

Lemma 5.1.11. *Let $L = B \oplus D = B_1 \oplus D_1$. If $B \cong B_1$ and B is finite-dimensional, then $D \cong D_1$.*

Denote $n = \dim L$. Noting that L is the homomorphic image of F, let F be generated by n elements. Then

$$E \cong B/D \cong \frac{R}{F' \cap R} \cong \frac{F' + R}{F'} \subseteq F/F'$$

where F/F' is abelian and generated by n elements, and thus E is finite dimensional. Next, consider $(J, \lambda) \in C(L)$. As above, one obtains a central ideal E_1 in C, complementary to D in B, and a homomorphism $\overline{\sigma} : C \longrightarrow J$ such that $E_1 \subseteq \ker\overline{\sigma}$ and $\lambda\overline{\sigma} = \overline{\pi}$. Since $E \cap C'$ and

$E_1 \cap C'$ are both zero, we may extend D in two different ways. First, extend D to a space G such that $C = E \oplus G$. Second, extend D to a space G_1 such that $C = E_1 \oplus G_1$. Since both E and E_1 have the same finite dimension and are abelian, we know $E \cong E_1$. By Lemma 5.1.11, $G \cong G_1$. Thus $C/E \cong C/E_1$.

In conclusion, we have shown that, given a free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ and any $(J, \lambda) \in C(L)$, one can choose a subspace E in C and induce a homomorphism $\sigma_S : C/E \longrightarrow J$ such that $\lambda\sigma_S = \pi_S$. By Lemma 5.1.5, σ_S is surjective. Thus $(C/E, \pi_S)$ is a universal element of $C(L)$ and C/E is a cover of L. Furthermore, each cover K of L is the homomorphic image of C/E and has the same dimension, and so every cover is isomorphic to C/E. Finally, for any $(J, \lambda) \in C(L)$ and a cover K of L, there exists a homomorphism $\beta : K \longrightarrow J$ such that $\lambda\beta = \tau$, where $(K, \tau) \in C(L)$. Thus, K is a cover of L if and only if (K, τ) is a universal element in $C(L)$.

Theorem 5.1.12. *Let L be a finite-dimensional diassociative algebra and let $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L. Let*

$$B = \frac{R}{F \Diamond R + R \Diamond F} \qquad C = \frac{F}{F \Diamond R + R \Diamond F} \qquad D = \frac{F' \cap R}{F \Diamond R + R \Diamond F}$$

Then

1. *all covers of L are isomorphic and have the form C/E where E is the complement to D in B,*
2. *the multiplier $M(L)$ of L is $D \cong B/E$,*
3. *the universal elements in $C(L)$ are the elements (K, λ) where K is a cover of L.*

5.2 Diassociative Cohomology

Given a pair of diassociative algebras A and B, consider a central extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ of A by B and section $\mu : B \longrightarrow L$. Define a pair of bilinear forms $(f_\dashv, f_\vdash)$ by $f_\dashv(i, j) = \mu(i) \dashv \mu(j) - \mu(i \dashv j)$ and $f_\vdash(i, j) = \mu(i) \vdash \mu(j) - \mu(i \vdash j)$ for $i, j \in B$. By our work on factor systems, $(f_\dashv, f_\vdash)$ is a 2-cocycle of diassociative algebras, meaning that these maps satisfy the 2-cocycle identities

$$\begin{aligned} f_{\dashv}(i, j \dashv k) &= f_{\dashv}(i, j \vdash k) \quad \text{C1} \\ f_{\dashv}(i \vdash j, k) &= f_{\vdash}(i, j \dashv k) \quad \text{C2} \\ f_{\vdash}(i \dashv j, k) &= f_{\vdash}(i \vdash j, k) \quad \text{C3} \\ f_{\dashv}(i, j \dashv k) &= f_{\dashv}(i \dashv j, k) \quad \text{C4} \\ f_{\vdash}(i, j \vdash k) &= f_{\vdash}(i \vdash j, k) \quad \text{C5} \end{aligned}$$

for all $i, j, k \in B$. We note that a 2-cocycle $(f_{\dashv}, f_{\vdash})$ is a 2-coboundary if there exists a linear transformation $\varepsilon : B \longrightarrow A$ such that $f_{\dashv}(i, j) = -\varepsilon(i \dashv j)$ and $f_{\vdash}(i, j) = -\varepsilon(i \vdash j)$. Furthermore, any elements $(f_{\dashv}, f_{\vdash})$ and $(g_{\dashv}, g_{\vdash})$ in $\mathscr{Z}^2(B, A)$ belong to equivalent extensions if and only if their corresponding bilinear forms differ by a coboundary, i.e. if there is a linear map $\varepsilon : B \longrightarrow A$ such that $f_{\dashv}(i, j) - g_{\dashv}(i, j) = -\varepsilon(i \dashv j)$ and $f_{\vdash}(i, j) - g_{\vdash}(i, j) = -\varepsilon(i \vdash j)$ for all $i, j \in B$. Therefore, extensions of A by B are equivalent if and only if they give rise to the same element of $\mathscr{H}^2(B, A)$. The work of Chapter 2 guarantees that each element

$$\overline{(f_{\dashv}, f_{\vdash})} \in \mathscr{H}^2(B, A)$$

gives rise to a central extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ with section μ such that $f_{\dashv}(i, j) = \mu(i) \dashv \mu(j) - \mu(i \dashv j)$ and $f_{\vdash}(i, j) = \mu(i) \vdash \mu(j) - \mu(i \vdash j)$.

5.3 Hochschild-Serre Spectral Sequence

The remainder of this chapter relies on the exactness of the following Hochschild-Serre type spectral sequence of low dimension. Let H be a central ideal of a diassociative algebra L and consider the natural central extension

$$0 \longrightarrow H \longrightarrow L \xrightarrow{\beta} L/H \longrightarrow 0$$

with section μ of β. Let A be a central L-module.

Theorem 5.3.1. *The sequence*

$$0 \longrightarrow \mathrm{Hom}(L/H, A) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(L, A) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(H, A) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/H, A) \xrightarrow{\mathrm{Inf}_2} \mathscr{H}^2(L, A)$$

is exact.

We first define the maps in the sequence and verify that they make sense. For any homomorphism $\chi : L/H \longrightarrow A$, define $\mathrm{Inf}_1 : \mathrm{Hom}(L/H, A) \longrightarrow \mathrm{Hom}(L, A)$ by $\mathrm{Inf}_1(\chi) = \chi\beta$.

Next, for $\pi \in \mathrm{Hom}(L,A)$, define $\mathrm{Res} : \mathrm{Hom}(L,A) \longrightarrow \mathrm{Hom}(H,A)$ by $\mathrm{Res}(\pi) = \pi\iota$, where $\iota : H \longrightarrow L$ is the inclusion map. It is readily verified that Inf_1 and Res are well-defined and linear. To define the transgression map, let $f_\dashv : L/H \times L/H \longrightarrow H$ and $f_\vdash : L/H \times L/H \longrightarrow H$ be defined by $f_\dashv(\overline{x},\overline{y}) = \mu(\overline{x}) \dashv \mu(\overline{y}) - \mu(\overline{x} \dashv \overline{y})$ and $f_\vdash(\overline{x},\overline{y}) = \mu(\overline{x}) \vdash \mu(\overline{y}) - \mu(\overline{x} \vdash \overline{y})$ for $x, y \in L$. Consider $\chi \in \mathrm{Hom}(H,A)$. Then $(\chi f_\dashv, \chi f_\vdash) \in \mathscr{Z}^2(L/H,A)$ since χ is a homomorphism. Given another section ν of β, define a pair $(g_\dashv, g_\vdash)$ of bilinear forms by $g_\dashv(\overline{x},\overline{y}) = \nu(\overline{x}) \dashv \nu(\overline{y}) - \nu(\overline{x} \dashv \overline{y})$ and $g_\vdash(\overline{x},\overline{y}) = \nu(\overline{x}) \vdash \nu(\overline{y}) - \nu(\overline{x} \vdash \overline{y})$ for $x, y \in L$. Then $(f_\dashv, f_\vdash)$ and $(g_\dashv, g_\vdash)$ are cohomologous in $\mathscr{H}^2(L/H,H)$, which implies that there exists a linear transformation $\varepsilon : L/H \longrightarrow H$ such that $f_\dashv(\overline{x},\overline{y}) - g_\dashv(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \dashv \overline{y})$ and $f_\vdash(\overline{x},\overline{y}) - g_\vdash(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \vdash \overline{y})$. Therefore, $\chi\varepsilon : L/H \longrightarrow A$ is a linear map by which $(\chi f_\dashv, \chi f_\vdash)$ and $(\chi g_\dashv, \chi g_\vdash)$ differ. In other words, $(\chi f_\dashv, \chi f_\vdash)$ and $(\chi g_\dashv, \chi g_\vdash)$ are cohomologous in $\mathscr{H}^2(L/H,A)$, and so we define

$$\mathrm{Tra}(\chi) = \overline{(\chi f_\dashv, \chi f_\vdash)}.$$

It is straightforward to verify that Tra is linear.

Finally, we define the second inflation map $\mathrm{Inf}_2 : \mathscr{H}^2(L/H,A) \longrightarrow \mathscr{H}^2(L,A)$ by

$$\mathrm{Inf}_2((f_\dashv, f_\vdash) + \mathscr{B}^2(L/H,A)) = (f'_\dashv, f'_\vdash) + \mathscr{B}^2(L,A)$$

where $f'_\dashv(x,y) = f_\dashv(\beta(x),\beta(y))$ and $f'_\vdash(x,y) = f_\vdash(\beta(x),\beta(y))$ for $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L/H,A)$ and $x, y \in L$. It is straightforward to verify that Inf_2 is linear. To check that Inf_2 maps cocycles to cocycles, we first compute

$$\begin{aligned}
f'_\dashv(x, y \dashv z) &= f_\dashv(\beta(x), \beta(y \dashv z)) \\
&= f_\dashv(\beta(x), \beta(y) \dashv \beta(z)) \\
&= f_\dashv(\beta(x), \beta(y) \vdash \beta(z)) \\
&= f_\dashv(\beta(x), \beta(y \vdash z)) \\
&= f'_\dashv(x, y \vdash z)
\end{aligned}$$

for all $x, y, z \in L$, which holds since $(f_\dashv, f_\vdash)$ is a 2-cocycle. Thus, $(f'_\dashv, f'_\vdash)$ satisfies the first axiom of 2-cocycles. The other axioms hold by similar computations, and hence $(f'_\dashv, f'_\vdash) \in \mathscr{Z}^2(L,A)$. To check that Inf_2 maps coboundaries to coboundaries, suppose $(f_\dashv, f_\vdash) \in \mathscr{B}^2(L/H,A)$. Then there is a linear transformation $\varepsilon : L/H \longrightarrow A$ such that $f_\dashv(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \dashv \overline{y})$ and

$f_\vdash(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \vdash \overline{y})$ for all $x, y \in L$. Here, $\beta(x) = x + H = \overline{x}$ for any $x \in L$. One has

$$\begin{aligned} f'_\dashv(x,y) &= f_\dashv(\beta(x),\beta(y)) \\ &= -\varepsilon(\beta(x) \dashv \beta(y)) \\ &= -\varepsilon\beta(x \dashv y) \end{aligned}$$

and, similarly, $f'_\vdash(x,y) = -\varepsilon\beta(x \vdash y)$. Therefore $(f'_\dashv, f'_\vdash) \in \mathscr{B}^2(L,A)$.

Proof. Given our section μ of $0 \longrightarrow H \longrightarrow L \xrightarrow{\beta} L/H \longrightarrow 0$, let $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L/H,H)$ be the cocycle defined by $f_\dashv(\overline{x},\overline{y}) = \mu(\overline{x}) \dashv \mu(\overline{y}) - \mu(\overline{x} \dashv \overline{y})$ and $f_\vdash(\overline{x},\overline{y}) = \mu(\overline{x}) \vdash \mu(\overline{y}) - \mu(\overline{x} \vdash \overline{y})$ for $x, y \in L$. We first note that Inf_1 is injective by the same logic as the Leibniz case. Thus the sequence is exact at $\mathrm{Hom}(L/H,A)$. Exactness at $\mathrm{Hom}(L,A)$ also follows similarly to the Leibniz case.

For exactness at $\mathrm{Hom}(H,A)$, first consider a homomorphism $\chi \in \mathrm{Hom}(L,A)$. Then

$$\begin{aligned} \chi f_\dashv(\overline{x},\overline{y}) &= \chi\mu(\overline{x}) \dashv \chi\mu(\overline{y}) - \chi\mu(\overline{x} \dashv \overline{y}) \\ &= -\chi\mu(\overline{x} \dashv \overline{y}) \end{aligned}$$

and, similarly, $\chi f_\vdash(\overline{x},\overline{y}) = -\chi\mu(\overline{x} \vdash \overline{y})$. This implies that $(\chi f_\dashv, \chi f_\vdash) \in \mathscr{B}^2(L/H,A)$. Thus

$$\mathrm{Tra}(\mathrm{Res}(\chi)) = \mathrm{Tra}(\chi\iota) = \overline{(\chi\iota f_\dashv, \chi\iota f_\vdash)} = 0$$

and so $\mathrm{Im(Res)} \subseteq \ker(\mathrm{Tra})$. Conversely, suppose there exists a homomorphism $\theta : H \longrightarrow A$ such that

$$\mathrm{Tra}(\theta) = \overline{(\theta f_\dashv, \theta f_\vdash)} = 0,$$

i.e. such that $(\theta f_\dashv, \theta f_\vdash) \in \mathscr{B}^2(L/H,A)$. Then there exists a linear transformation $\varepsilon : L/H \longrightarrow A$ such that $\theta f_\dashv(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \dashv \overline{y})$ and $\theta f_\vdash(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \vdash \overline{y})$. For any $x, y \in L$, we know that $x = \mu(\overline{x}) + h_x$ and $y = \mu(\overline{y}) + h_y$ for some $h_x, h_y \in H$. Thus, $x \dashv y = \mu(\overline{x} \dashv \overline{y}) + h_{x\dashv y} = \mu(\overline{x}) \dashv \mu(\overline{y})$ and $x \vdash y = \mu(\overline{x} \vdash \overline{y}) + h_{x \vdash y} = \mu(\overline{x}) \vdash \mu(\overline{y})$, which implies that

$$\begin{aligned} \theta(h_{x\dashv y}) &= \theta(\mu(\overline{x}) \dashv \mu(\overline{y}) - \mu(\overline{x} \dashv \overline{y})) = \theta f_\dashv(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \dashv \overline{y}), \\ \theta(h_{x\vdash y}) &= \theta(\mu(\overline{x}) \vdash \mu(\overline{y}) - \mu(\overline{x} \vdash \overline{y})) = \theta f_\vdash(\overline{x},\overline{y}) = -\varepsilon(\overline{x} \vdash \overline{y}). \end{aligned} \tag{5.1}$$

Define a linear map $\sigma : L \longrightarrow A$ by $\sigma(x) = \theta(h_x) + \varepsilon(\overline{x})$. Then $\sigma(x) \dashv \sigma(y) = 0$ and $\sigma(x) \vdash$

$\sigma(y)=0$ since $\operatorname{Im}\sigma\subseteq A$. By (5.1),

$$\sigma(x\dashv y)=\theta(h_{x\dashv y})+\varepsilon(x\dashv y)=0,$$
$$\sigma(x\vdash y)=\theta(h_{x\vdash y})+\varepsilon(x\vdash y)=0.$$

Thus, σ is a homomorphism. Moreover, $\sigma(h)=\theta(h)+\varepsilon(\overline{h})=\theta(h)$ for all $h\in H$, which implies that $\mathrm{Res}(\sigma)=\theta$. Hence, $\ker(\mathrm{Tra})\subseteq\mathrm{Im}(\mathrm{Res})$ and $\ker(\mathrm{Tra})=\mathrm{Im}(\mathrm{Res})$.

For exactness at $\mathscr{H}^2(L/H,A)$, first consider a map $\chi\in\mathrm{Hom}(H,A)$. Then

$$\mathrm{Tra}(\chi)=(\chi f_\dashv,\chi f_\vdash)+\mathscr{B}^2(L/H,A)$$

where $(\chi f_\dashv,\chi f_\vdash)\in\mathscr{Z}^2(L/H,A)$. One computes

$$\mathrm{Inf}_2((\chi f_\dashv,\chi f_\vdash)+\mathscr{B}^2(L/H,A))=((\chi f_\dashv)',(\chi f_\vdash)')+\mathscr{B}^2(L,A)$$

where

$$(\chi f_\dashv)'(x,y)=\chi f_\dashv(\overline{x},\overline{y}),$$
$$(\chi f_\vdash)'(x,y)=\chi f_\vdash(\overline{x},\overline{y})$$

for $x,y\in L$. To show that $\mathrm{Im}(\mathrm{Tra})\subseteq\ker(\mathrm{Inf}_2)$, we need to find a linear transformation $\varepsilon:L\longrightarrow A$ such that $(\chi f_\dashv)'(x,y)=-\varepsilon(x\dashv y)$ and $(\chi f_\vdash)'(x,y)=-\varepsilon(x\vdash y)$. Let $x=\mu(\overline{x})+h_x$ and $y=\mu(\overline{y})+h_y$. Again, the equalities $x\dashv y=\mu(\overline{x}\dashv\overline{y})+h_{x\dashv y}=\mu(\overline{x})\dashv\mu(\overline{y})$ and $x\vdash y=\mu(\overline{x}\vdash\overline{y})+h_{x\vdash y}=\mu(\overline{x})\vdash\mu(\overline{y})$ yield

$$\chi f_\dashv(\overline{x},\overline{y})=\chi(\mu(\overline{x}\dashv\overline{y}-\mu(\overline{x}\dashv\overline{y}))=\chi(h_{x\dashv y}),$$
$$\chi f_\vdash(\overline{x},\overline{y})=\chi(\mu(\overline{x}\vdash\overline{y}-\mu(\overline{x}\vdash\overline{y}))=\chi(h_{x\vdash y}).$$

Define $\varepsilon(x)=-\chi(h_x)$. Then ε is linear and

$$\varepsilon(x\dashv y)=-\chi(h_{x\dashv y})=-\chi f_\dashv(\overline{x},\overline{y})=-(\chi f_\dashv)'(x,y),$$
$$\varepsilon(x\vdash y)=-\chi(h_{x\vdash y})=-\chi f_\vdash(\overline{x},\overline{y})=-(\chi f_\vdash)'(x,y).$$

This implies that $((\chi f_\dashv)',(\chi f_\vdash)')\in\mathscr{B}^2(L,A)$, and hence $\mathrm{Im}(\mathrm{Tra})\subseteq\ker(\mathrm{Inf}_2)$.

Conversely, suppose $(g_\dashv,g_\vdash)\in\mathscr{Z}^2(L/H,A)$ is such that

$$\overline{(g_\dashv,g_\vdash)}\in\ker(\mathrm{Inf}_2).$$

Then there exists a linear transformation $\varepsilon : L \longrightarrow A$ such that $g_{\dashv}(\overline{x},\overline{y}) = g'_{\dashv}(x,y) = -\varepsilon(x \dashv y)$ and $g_{\vdash}(\overline{x},\overline{y}) = g'_{\vdash}(x,y) = -\varepsilon(x \vdash y)$ for all $x, y \in L$. Since ε is linear, $(\varepsilon f_{\dashv}, \varepsilon f_{\vdash}) \in \mathscr{Z}^2(L/H, A)$. As before, $x = \mu(\overline{x}) + h_x$ and $y = \mu(\overline{y}) + h_y$ for some $h_x, h_y \in H$. Therefore $x \dashv y = \mu(\overline{x}) \dashv \mu(\overline{y})$ and $x \vdash y = \mu(\overline{x}) \vdash \mu(\overline{y})$. Now

$$\begin{aligned} g'_{\dashv}(x,y) &= g_{\dashv}(\overline{x},\overline{y}) \\ &= -\varepsilon(x \dashv y) \\ &= -\varepsilon(\overline{x} \dashv \overline{y}) \\ &= -\varepsilon f_{\dashv}(\overline{x},\overline{y}) - \varepsilon\mu(\overline{x} \dashv \overline{y}) \end{aligned}$$

where $\varepsilon\mu : L/H \longrightarrow A$. Similarly, $g'_{\vdash}(x,y) = -\varepsilon f_{\vdash}(\overline{x},\overline{y}) - \varepsilon\mu(\overline{x} \vdash \overline{y})$. Therefore

$$\overline{(g_{\dashv}, g_{\vdash})} = \overline{(-\varepsilon f_{\dashv}, -\varepsilon f_{\vdash})} = -\mathrm{Tra}(\varepsilon)$$

which implies that $\ker(\mathrm{Inf}_2) \subseteq \mathrm{Im}(\mathrm{Tra})$. □

5.4 Relation of Multipliers and Cohomology

Let L be a diassociative algebra and let $\mathbb{F}$ be considered as a central L-module. The following theorem holds similarly to its Leibniz analogue.

Theorem 5.4.1. *Let Z be a central ideal in L. Then $L' \cap Z$ is isomorphic to the image of* $\mathrm{Hom}(Z,\mathbb{F})$ *under the transgression map. In particular, if* Tra *is surjective, then* $L' \cap Z \cong \mathscr{H}^2(L/Z,\mathbb{F})$.

Now consider a free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\omega} L \longrightarrow 0$ of L. The sequence

$$0 \longrightarrow \frac{R}{F \Diamond R + R \Diamond F} \longrightarrow \frac{F}{F \Diamond R + R \Diamond F} \longrightarrow L \longrightarrow 0$$

is a central extension since all of $R \dashv F$, $R \vdash F$, $F \dashv R$, and $F \vdash R$ are contained in $F \Diamond R + R \Diamond F$.

Lemma 5.4.2. *Let $0 \longrightarrow A \longrightarrow B \xrightarrow{\phi} C \longrightarrow 0$ be a central extension and $\alpha : L \longrightarrow C$ be a homomorphism. Then there exists a homomorphism $\beta : F/(F \Diamond R + R \Diamond F) \longrightarrow B$ such that*

$$\begin{tikzcd} 0 \arrow[r] & \frac{R}{F\lozenge R+R\lozenge F} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{F\lozenge R+R\lozenge F} \arrow[r] \arrow[d, "\beta"] & L \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Figure 5.6 Existence of β.

is commutative, where γ is the restriction of β to $R/(F\lozenge R+R\lozenge F)$.

Proof. Since F is free, there exists a homomorphism $\sigma : F \longrightarrow B$ such that

$$\begin{tikzcd} F \arrow[r, "\omega"] \arrow[d, "\sigma"'] & L \arrow[d, "\alpha"] \\ B \arrow[r, "\phi"] & C \end{tikzcd}$$

Figure 5.7 Showing $\phi\sigma = \alpha\omega$.

is commutative. Let $r \in R \subseteq F$. Then $\omega(r) = 0$ since $\ker\omega = R$. Therefore $0 = \alpha\omega(r) = \phi\sigma(r)$ and so $\sigma(R) \subseteq \ker\phi$. We want to show that $F\lozenge R + R\lozenge F \subseteq \ker\sigma$. If $x \in F$ and $r \in R$, then

$$\sigma(x \dashv r) = \sigma(x) \dashv \sigma(r) = 0, \qquad \sigma(x \vdash r) = \sigma(x) \vdash \sigma(r) = 0,$$
$$\sigma(r \dashv x) = \sigma(r) \dashv \sigma(x) = 0, \qquad \sigma(r \vdash x) = \sigma(r) \vdash \sigma(x) = 0$$

since $\sigma(r) \in \ker\phi = A \subseteq Z(B)$. Now σ induces a homomorphism $\beta : F/(F\lozenge R + R\lozenge F) \longrightarrow B$. The left diagram commutes since we may take $A \longrightarrow B$ to be the inclusion map. □

Lemma 5.4.3. *Let $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L. Let A be a central L-module. Then the transgression map* $\mathrm{Tra} : \mathrm{Hom}(R/(F\lozenge R + R\lozenge F), A) \longrightarrow \mathscr{H}^2(L, A)$ *associated with*

$$0 \longrightarrow \frac{R}{F\lozenge R + R\lozenge F} \longrightarrow \frac{F}{F\lozenge R + R\lozenge F} \xrightarrow{\phi} L \longrightarrow 0$$

is surjective.

Proof. Consider $\overline{(g_{\dashv}, g_{\vdash})} \in \mathscr{H}^2(L, A)$ and let $0 \longrightarrow A \longrightarrow E \xrightarrow{\varphi} L \longrightarrow 0$ be a central extension associated with $\overline{(g_{\dashv}, g_{\vdash})}$. By the previous lemma, there exists a homomorphism θ such that

$$\begin{tikzcd}
0 \arrow[r] & \frac{R}{F\Diamond R+R\Diamond F} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{F\Diamond R+R\Diamond F} \arrow[r, "\phi"] \arrow[d, "\theta"] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r, "\varphi"] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 5.8 Invoking Lemma 5.4.2.

is commutative and $\gamma = \theta|_{R/(F\Diamond R+R\Diamond F)}$. Let μ be a section of ϕ. Then $\varphi\theta\mu = \phi\mu = \mathrm{id}_L$, and so $\theta\mu$ is a section of φ. Let $\lambda = \theta\mu$ and define

$$h_\dashv(x,y) = \lambda(x) \dashv \lambda(y) - \lambda(x \dashv y),$$
$$h_\vdash(x,y) = \lambda(x) \vdash \lambda(y) - \lambda(x \vdash y).$$

Then $(h_\dashv, h_\vdash) \in \mathscr{Z}^2(L,A)$ and $(h_\dashv, h_\vdash)$ is cohomologous with $(g_\dashv, g_\vdash)$ since they are associated with the same extension. One computes

$$\begin{aligned}
h_\dashv(x,y) &= \theta(\mu(x)) \dashv \theta(\mu(y)) - \theta(\mu(x \dashv y)) \\
&= \theta(\mu(x) \dashv \mu(y) - \mu(x \dashv y)) \\
&= \gamma(\mu(x) \dashv \mu(y) - \mu(x \dashv y)) \\
&= \gamma(f_\dashv(x,y))
\end{aligned}$$

where $f_\dashv(x,y) = \mu(x) \dashv \mu(y) - \mu(x \dashv y)$, and since $\gamma = \theta|_{R/(F\Diamond R+R\Diamond F)}$. Similarly, one computes $h_\vdash(x,y) = \gamma(f_\vdash(x,y))$ for $f_\vdash(x,y) = \mu(x) \vdash \mu(y) - \mu(x \vdash y)$. Thus

$$\mathrm{Tra}(\gamma) = \overline{(\gamma f_\dashv, \gamma f_\vdash)} = \overline{(h_\dashv, h_\vdash)} = \overline{(g_\dashv, g_\vdash)}$$

and Tra is surjective. $\square$

Lemma 5.4.4. *If $C \subseteq A$ and $C \subseteq B$, then $A/C \cap B/C = (A \cap B)/C$.*

Proof. Follows by the same logic as Lemma 4.2.4. $\square$

Theorem 5.4.5. *Let L be a diassociative algebra over a field $\mathbb{F}$ and $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L. Then*

$$\mathscr{H}^2(L,\mathbb{F}) \cong \frac{F' \cap R}{F\Diamond R + R\Diamond F}.$$

In particular, if L is finite-dimensional, then $M(L) \cong \mathscr{H}^2(L,\mathbb{F})$.

Proof. Let

$$\overline{R} = \frac{R}{F\Diamond R + R\Diamond F} \qquad \overline{F} = \frac{F}{F\Diamond R + R\Diamond F}$$

Then $0 \longrightarrow \overline{R} \longrightarrow \overline{F} \longrightarrow L \longrightarrow 0$ is a central extension. By Lemma 5.4.3, $\mathrm{Tra} : \mathrm{Hom}(\overline{R}, \mathbb{F}) \longrightarrow \mathscr{H}^2(L, \mathbb{F})$ is surjective. By Theorem 5.4.1,

$$\overline{F}' \cap \overline{R} \cong \mathscr{H}^2(\overline{F}/\overline{R}, \mathbb{F}) \cong \mathscr{H}^2(L, \mathbb{F}).$$

By Lemma 5.4.4,

$$\overline{F}' \cap \overline{R} \cong \frac{F'}{F\Diamond R + R\Diamond F} \cap \frac{R}{F\Diamond R + R\Diamond F} = \frac{F' \cap R}{F\Diamond R + R\Diamond F}.$$

Therefore, when L is finite-dimensional,

$$M(L) = \frac{F' \cap R}{F\Diamond R + R\Diamond F} \cong \mathscr{H}^2(L, \mathbb{F})$$

by the characterization of $M(L)$ from Theorem 5.1.12. □

5.5 Unicentral Diassociative Algebras

For a diassociative algebra L, let $Z^*(L)$ denote the intersection of all images $\omega(Z(E))$ such that $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$ is a central extension of L. It is easy to see that $Z^*(L) \subseteq Z(L)$. We say that a diassociative algebra L is *unicentral* if $Z(L) = Z^*(L)$.

5.5.1 More Sequences

Given a central ideal Z in L, consider the natural extension $0 \longrightarrow Z \longrightarrow L \longrightarrow L/Z \longrightarrow 0$. We now extend our Hochschild-Serre sequence by a term

$$\delta : \mathscr{H}^2(L, \mathbb{F}) \longrightarrow (L/L' \otimes Z \oplus Z \otimes L/L')^2.$$

To define this δ map, consider a 2-cocycle $(f'_\dashv, f'_\vdash) \in \mathscr{Z}^2(L, \mathbb{F})$ and define four bilinear forms

$$f''_\dashv : L/L' \times Z \longrightarrow \mathbb{F}, \qquad f''_\vdash : L/L' \times Z \longrightarrow \mathbb{F},$$
$$g''_\dashv : Z \times L/L' \longrightarrow \mathbb{F}, \qquad g''_\vdash : Z \times L/L' \longrightarrow \mathbb{F}$$

by

$$f''_{\dashv}(x+L',z)=f'_{\dashv}(x,z), \qquad f''_{\vdash}(x+L',z)=f'_{\vdash}(x,z),$$
$$g''_{\dashv}(z,x+L')=f'_{\dashv}(z,x), \qquad g''_{\vdash}(z,x+L')=f'_{\vdash}(z,x)$$

for $x\in L$, $z\in Z$. To check that these four maps are well-defined, one computes

$$f''_{\dashv}(x\dashv y+L',z)=f'_{\dashv}(x\dashv y,z)\overset{\text{C4}}{=}f'_{\dashv}(x,y\dashv z)=0,$$
$$f''_{\dashv}(x\vdash y+L',z)=f'_{\dashv}(x\vdash y,z)\overset{\text{C2}}{=}f'_{\vdash}(x,y\dashv z)=0,$$

$$g''_{\dashv}(z,x\dashv y+L')=f'_{\dashv}(z,x\dashv y)\overset{\text{C4}}{=}f'_{\dashv}(z\dashv x,y)=0,$$
$$g''_{\dashv}(z,x\vdash y+L')=f'_{\dashv}(z,x\vdash y)\overset{\text{C1}}{=}f'_{\dashv}(z,x\dashv y)\overset{\text{C4}}{=}f'_{\dashv}(z\dashv x,y)=0,$$

$$f''_{\vdash}(x\dashv y+L',z)=f'_{\vdash}(x\dashv y,z)\overset{\text{C3}}{=}f'_{\vdash}(x\vdash y,z)\overset{\text{C5}}{=}f'_{\vdash}(x,y\vdash z)=0,$$
$$f''_{\vdash}(x\vdash y+L',z)=f'_{\vdash}(x\vdash y,z)\overset{\text{C5}}{=}f'_{\vdash}(x,y\vdash z)=0,$$

$$g''_{\vdash}(z,x\dashv y+L')=f'_{\vdash}(z,x\dashv y)\overset{\text{C2}}{=}f'_{\dashv}(z\vdash x,y)=0,$$
$$g''_{\vdash}(z,x\vdash y+L')=f'_{\vdash}(z,x\vdash y)\overset{\text{C5}}{=}f'_{\vdash}(z\vdash x,y)=0$$

since $z\in Z(L)$. Hence

$$\begin{aligned}(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})&\in(\mathrm{Bil}(L/L'\times Z,\mathbb{F})\oplus\mathrm{Bil}(Z\times L/L',\mathbb{F}))^2\\ &\cong(L/L'\otimes Z\oplus Z\otimes L/L')^2.\end{aligned}$$

Now let $(f'_{\dashv},f'_{\vdash})\in\mathscr{B}^2(L,\mathbb{F})$. Then there exists a linear transformation $\varepsilon:L\longrightarrow\mathbb{F}$ such that $f'_{\dashv}(x,y)=-\varepsilon(x\dashv y)$ and $f'_{\vdash}(x,y)=-\varepsilon(x\vdash y)$ for all $x,y\in L$. One computes

$$f''_{\dashv}(x+L',z)=f'_{\dashv}(x,z)=-\varepsilon(x\dashv z)=0, \qquad f''_{\vdash}(x+L',z)=f'_{\vdash}(x,z)=-\varepsilon(x\vdash z)=0,$$
$$g''_{\dashv}(z,x+L')=f'_{\dashv}(z,x)=-\varepsilon(z\dashv x)=0, \qquad g''_{\vdash}(z,x+L')=f'_{\vdash}(z,x)=-\varepsilon(z\vdash x)=0$$

since $z\in Z(L)$. Hence, a map $\delta:(f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F})\mapsto(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})$ is induced that is clearly linear since $f'_{\dashv}$, $f'_{\vdash}$, $f''_{\dashv}$, $g''_{\dashv}$, $f''_{\vdash}$, and $g''_{\vdash}$ are all in vector spaces of bilinear forms and the latter four are defined by the first two.

Theorem 5.5.1. *Let Z be a central ideal of a diassociative algebra L. The sequence*

$$\mathcal{H}^2(L/Z,\mathbb{F}) \xrightarrow{\text{Inf}} \mathcal{H}^2(L,\mathbb{F}) \xrightarrow{\delta} (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

is exact.

Proof. Let $(f_\dashv, f_\vdash) \in \mathcal{Z}^2(L/Z,\mathbb{F})$. Then $\text{Inf}((f_\dashv, f_\vdash) + \mathcal{B}^2(L/Z,\mathbb{F})) = (f'_\dashv, f'_\vdash) + \mathcal{B}^2(L,\mathbb{F})$ where $f'_\dashv(x,y) = f_\dashv(x+Z, y+Z)$ and $f'_\vdash(x,y) = f_\vdash(x+Z, y+Z)$ for $x, y \in L$. Moreover,

$$\delta((f'_\dashv, f'_\vdash) + \mathcal{B}^2(L,\mathbb{F})) = (f''_\dashv, g''_\dashv, f''_\vdash, g''_\vdash)$$

where

$$\begin{aligned}
f''_\dashv(x+L', z) &= f'_\dashv(x,z) = f_\dashv(x+Z, z+Z) = 0,\\
g''_\dashv(z, x+L') &= f'_\dashv(z,x) = f_\dashv(z+Z, x+Z) = 0,\\
f''_\vdash(x+L', z) &= f'_\vdash(x,z) = f_\vdash(x+Z, z+Z) = 0,\\
g''_\vdash(z, x+L') &= f'_\vdash(z,x) = f_\vdash(z+Z, x+Z) = 0
\end{aligned}$$

for all $x \in L$ and $z \in Z$. Thus,

$$\begin{aligned}
\delta(\text{Inf}((f_\dashv, f_\vdash) + \mathcal{B}^2(L/Z,\mathbb{F}))) &= \delta((f'_\dashv, f'_\vdash) + \mathcal{B}^2(L,\mathbb{F}))\\
&= (f''_\dashv, g''_\dashv, f''_\vdash, g''_\vdash)\\
&= (0,0,0,0)
\end{aligned}$$

which implies that $\text{Im(Inf)} \subseteq \ker\delta$.

Conversely, suppose that $\delta((f'_\dashv, f'_\vdash) + \mathcal{B}^2(L,\mathbb{F})) = (f''_\dashv, g''_\dashv, f''_\vdash, g''_\vdash) = (0,0,0,0)$ for some cocycle $(f'_\dashv, f'_\vdash) \in \mathcal{Z}^2(L,\mathbb{F})$. In other words,

$$\begin{aligned}
0 &= f''_\dashv(x+L', z) = f'_\dashv(x,z), & 0 &= f''_\vdash(x+L', z) = f'_\vdash(x,z),\\
0 &= g''_\dashv(z, x+L') = f'_\dashv(z,x), & 0 &= g''_\vdash(z, x+L') = f'_\vdash(z,x)
\end{aligned}$$

for all $x \in L$, $x \in Z$. One computes

$$\begin{aligned}
f'_\dashv(x+z, y+z') &= f'_\dashv(x,y) + f'_\dashv(x,z') + f'_\dashv(z,y) + f'_\dashv(z,z') = f'_\dashv(x,y),\\
f'_\vdash(x+z, y+z') &= f'_\vdash(x,y) + f'_\vdash(x,z') + f'_\vdash(z,y) + f'_\vdash(z,z') = f'_\vdash(x,y)
\end{aligned}$$

for all $z, z' \in Z$, which implies that the bilinear forms $g_\dashv : L/Z \times L/Z \longrightarrow \mathbb{F}$ and $g_\vdash : L/Z \times L/Z \longrightarrow \mathbb{F}$, defined by $g_\dashv(x+Z, y+Z) = f'_\dashv(x,y)$ and $g_\vdash(x+Z, y+Z) = f'_\vdash(x,y)$, are well-defined. Furthermore, $(g_{\dashv}, g_{\vdash}) \in \mathscr{Z}^2(L/Z, \mathbb{F})$ since $(f'_{\dashv}, f'_{\vdash})$ is a cocycle. Thus,

$$\mathrm{Inf}((g_{\dashv}, g_{\vdash}) + \mathscr{B}^2(L/Z, \mathbb{F})) = (f'_{\dashv}, f'_{\vdash}) + \mathscr{B}^2(L, \mathbb{F}),$$

which implies that $\ker\delta \subseteq \mathrm{Im}(\mathrm{Inf})$. □

Theorem 5.5.2. *(Ganea Sequence) Let Z be a central ideal of a finite-dimensional diassociative algebra L. Then the sequence*

$$(L/L' \otimes Z \oplus Z \otimes L/L')^2 \longrightarrow M(L) \longrightarrow M(L/Z) \longrightarrow L' \cap Z \longrightarrow 0$$

is exact.

Proof. Let F be a free diassociative algebra such that $L = F/R$ and $Z = T/R$ for ideals T and R of F. Since $Z \subseteq Z(L)$, we have $T/R \subseteq Z(F/R)$, and thus $F\lozenge T + T\lozenge F \subseteq R$. Now inclusion maps $\overline{\beta} : R \cap F' \longrightarrow T \cap F'$ and $\overline{\gamma} : T \cap F' \longrightarrow T \cap (F' + R)$ induce homomorphisms

$$\frac{R \cap F'}{F\lozenge R + R\lozenge F} \xrightarrow{\beta} \frac{T \cap F'}{F\lozenge T + T\lozenge F} \xrightarrow{\gamma} \frac{T \cap (F' + R)}{R} \longrightarrow 0.$$

Since $R \subseteq T$,

$$\frac{T \cap (F' + R)}{R} = \frac{(T + R) \cap (F' + R)}{R} \cong \frac{(T \cap F') + R}{R}$$

which implies that γ is surjective. By Theorem 5.1.12,

$$M(L) \cong \frac{R \cap F'}{F\lozenge R + R\lozenge F} \qquad \text{and} \qquad M(L/Z) \cong \frac{T \cap F'}{F\lozenge T + T\lozenge F}.$$

Also

$$L' \cap Z \cong (F/R)' \cap (T/R) \cong \frac{F' + R}{R} \cap \frac{T}{R} \cong \frac{(F' + R) \cap T}{R}.$$

Therefore, the sequence $M(L/Z) \xrightarrow{\gamma} L' \cap Z \longrightarrow 0$ is exact. Since

$$\ker\gamma = \frac{(T \cap F') \cap R}{F\lozenge T + T\lozenge F} = \frac{R \cap F'}{F\lozenge T + T\lozenge F} = \mathrm{Im}\,\beta,$$

the sequence $M(L) \xrightarrow{\beta} M(L/Z) \xrightarrow{\gamma} L' \cap Z$ is exact.

It remains to show that

$$(L/L' \otimes Z \oplus Z \otimes L/L')^2 \longrightarrow M(L) \xrightarrow{\beta} M(L/Z) \longrightarrow L' \cap Z$$

is exact. Define four maps

$$\theta_{\dashv}:\frac{T}{R}\times\frac{F}{R+F'}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F},\qquad \theta_{\vdash}:\frac{T}{R}\times\frac{F}{R+F'}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F},$$
$$\alpha_{\dashv}:\frac{F}{R+F'}\times\frac{T}{R}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F},\qquad \alpha_{\vdash}:\frac{F}{R+F'}\times\frac{T}{R}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F}$$

by

$$\theta_{\dashv}(\overline{t},\overline{f})=t\dashv f+(F\Diamond R+R\Diamond F),\qquad \theta_{\vdash}(\overline{t},\overline{f})=t\vdash f+(F\Diamond R+R\Diamond F),$$
$$\alpha_{\dashv}(\overline{f},\overline{t})=f\dashv t+(F\Diamond R+R\Diamond F),\qquad \alpha_{\vdash}(\overline{f},\overline{t})=f\vdash t+(F\Diamond R+R\Diamond F)$$

for $t\in T$, $f\in F$. These maps are bilinear since multiplication operations are bilinear. To check that they are well-defined, suppose $(t+R,f+(R+F'))=(t'+R,f'+(R+F'))$ for $t,t'\in T$ and $f,f'\in F$. Then $t-t'\in R$ and $f-f'\in R+F'$, which implies that $t=t'+r$ and $f=f'+x$ for some $r\in R$ and $x\in R+F'$. One computes

$$\begin{aligned} t\dashv f-t'\dashv f' &= (t'+r)\dashv(f'+x)-t'\dashv f'\\ &= r\dashv f'+r\dashv x+t'\dashv x\\ &\in (R\dashv F)+(R\dashv F)+(T\dashv R+T\dashv F') \end{aligned}$$

which is contained in $F\Diamond R+R\Diamond F$ since

$$\begin{aligned} T\dashv F' &= T\dashv(F\dashv F)+T\dashv(F\vdash F)\\ &= (T\dashv F)\dashv F+T\dashv(F\dashv F)\\ &= (T\dashv F)\dashv F+(T\dashv F)\dashv F \end{aligned}$$

and $T\dashv F\subseteq R$. Next,

$$\begin{aligned} t\vdash f-t'\vdash f' &= (t'+r)\vdash(f'+x)-t'\vdash f'\\ &= r\vdash f'+r\vdash x+t'\vdash x\\ &\in (R\vdash F)+(R\vdash F)+(T\vdash R+T\vdash F') \end{aligned}$$

which is also contained in $F\Diamond R+R\Diamond F$ since

$$\begin{aligned} T\vdash F' &= T\vdash(F\dashv F)+T\vdash(F\vdash F)\\ &= (T\vdash F)\dashv F+(T\vdash F)\vdash F \end{aligned}$$

and $T \vdash F \subseteq R$. Expressions $f \dashv t - f' \dashv t'$ and $f \vdash t - f' \vdash t'$ fall in $F\Diamond R + R\Diamond F$ by similar manipulations, via the identities of diassociative algebras and the fact that $F \dashv T$, $F \vdash T$, $T \dashv F$, and $T \vdash F$ are contained in R. Thus our bilinear forms $\theta_{\dashv}$, $\alpha_{\dashv}$, $\theta_{\vdash}$, and $\alpha_{\vdash}$ are well-defined, and so induce linear maps

$$\overline{\theta_{\dashv}}:\frac{T}{R}\otimes\frac{F}{R+F'}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F}, \qquad \overline{\theta_{\vdash}}:\frac{T}{R}\otimes\frac{F}{R+F'}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F},$$
$$\overline{\alpha_{\dashv}}:\frac{F}{R+F'}\otimes\frac{T}{R}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F}, \qquad \overline{\alpha_{\vdash}}:\frac{F}{R+F'}\otimes\frac{T}{R}\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F}.$$

These, in turn, yield a linear transformation

$$\overline{\theta}:\left(\frac{F}{R+F'}\otimes\frac{T}{R}\oplus\frac{T}{R}\otimes\frac{F}{R+F'}\right)^2\longrightarrow\frac{R\cap F'}{F\Diamond R+R\Diamond F}$$

defined by $\overline{\theta}(a,b,c,d)=\overline{\alpha_{\dashv}}(a)+\overline{\theta_{\dashv}}(b)+\overline{\alpha_{\vdash}}(c)+\overline{\theta_{\vdash}}(d)$. The image of $\overline{\theta}$ is

$$\frac{F\Diamond T+T\Diamond F}{F\Diamond R+R\Diamond F}$$

which is precisely equal to $\{x+(F\Diamond R+R\Diamond F)\mid x\in R\cap F',\ x\in F\Diamond T+T\Diamond F\}=\ker\beta$. Thus the final part of our sequence is exact. □

Corollary 5.5.3. *(Stallings Sequence) Let Z be a central ideal in a finite-dimensional diassociative algebra L. Then the sequence*

$$M(L)\longrightarrow M(L/Z)\longrightarrow Z\longrightarrow L/L'\longrightarrow\frac{L}{Z+L'}\longrightarrow 0$$

is exact.

Proof. Follows by the same logic as the Leibniz case with the replacements $F\Diamond R+R\Diamond F$ for $FR+RF$ and $F\Diamond T+T\Diamond F$ for $FT+TF$. □

5.5.2 The Main Result

We refer to the map β that appears in the Ganea sequence as the natural map. Let Z be a central ideal in a finite-dimensional diassociative algebra L. We will prove the equivalence of the following statements:

1. δ is the trivial map,

2. β is injective,

3. $M(L) \cong \frac{M(L/Z)}{L' \cap Z}$,

4. $Z \subseteq Z^*(L)$.

The following two lemmas form the equivalence of our first three statements. Both hold similarly to their Leibniz analogues.

Lemma 5.5.4. *Let Z be a central ideal in a finite-dimensional diassociative algebra L and consider the map*

$$\delta : M(L) \longrightarrow (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

from Theorem 5.5.1. Then

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if δ is the trivial map.

Lemma 5.5.5. *Let Z be a central ideal in a finite-dimensional diassociative algebra L and consider the natural map $\beta : M(L) \longrightarrow M(L/Z)$ from Theorem 5.5.2. Then*

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if β is injective.

It remains to show that these conditions are equivalent to $Z \subseteq Z^*(L)$. As in Chapter 4, we say that a central extension $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is *stem* if $A \subseteq B'$. Consider a free presentation $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ of L and let $\overline{X}$ denote the quotient algebra $\frac{X}{F \lozenge R + R \lozenge F}$ for any X such that $F \lozenge R + R \lozenge F \subseteq X \subseteq F$. Since $R = \ker \pi$ and $F \lozenge R + R \lozenge F \subseteq R$, π induces a homomorphism $\overline{\pi} : \overline{F} \longrightarrow L$ such that the diagram

$$\begin{tikzcd}
F \arrow[r, "\pi"] \arrow[d] & L \\
\overline{F} \arrow[ur, "\overline{\pi}"'] &
\end{tikzcd}$$

Figure 5.9 Induced $\overline{\pi}$.

commutes. Since $\overline{R} \subseteq Z(\overline{F})$, there exists a complement $\frac{S}{F \lozenge R + R \lozenge F}$ to $\frac{R \cap F'}{F \lozenge R + R \lozenge F}$ in $\frac{R}{F \lozenge R + R \lozenge F}$ where $S \subseteq R \subseteq \ker \pi$ and $\overline{S} \subseteq \overline{R} \subseteq \ker \overline{\pi}$. Thus $\overline{\pi}$ induces a homomorphism $\pi_S : F/S \longrightarrow L$ such that the extension $0 \longrightarrow R/S \longrightarrow F/S \xrightarrow{\pi_S} L \longrightarrow 0$ is central. This extension is stem since $R/S \cong \frac{R \cap F'}{F \lozenge R + R \lozenge F} = \ker \pi_S$ implies that F/S is a cover of L.

Lemma 5.5.6. *For every free presentation* $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ *of* L *and every central extension* $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$, *one has* $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$.

Proof. Since the identity map $\mathrm{id} : L \longrightarrow L$ is a homomorphism, we can invoke Lemma 5.4.2, yielding a homomorphism $\beta : \overline{F} \longrightarrow E$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{R}{F\Diamond R + R\Diamond F} \arrow[r] \arrow[d, "\gamma"] & \frac{F}{F\Diamond R + R\Diamond F} \arrow[r, "\overline{\pi}"] \arrow[d, "\beta"] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker\omega \arrow[r] & E \arrow[r, "\omega"] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 5.10 Invoking Lemma 5.4.2 again.

is commutative, where γ is the restriction of β to $\overline{R}$. As in the Leibniz case, we obtain $E = A + \beta(\overline{F})$, where $A = \ker\omega$. It remains to show that $\beta(Z(\overline{F}))$ centralizes A and $\beta(\overline{F})$. We first compute

$$\beta(Z(\overline{F})) * \beta(\overline{F}) = \beta(Z(\overline{F}) * \overline{F}) = \beta(0) = 0,$$
$$\beta(\overline{F}) * \beta(Z(\overline{F})) = \beta(\overline{F} * Z(\overline{F})) = \beta(0) = 0$$

where $*$ ranges over $\dashv$ and $\vdash$. Next, we know that $A * E$ and $E * A$ are zero, and so $A * \beta(Z(\overline{F}))$ and $\beta(Z(\overline{F})) * A$ must be zero as well. Thus, $\beta(Z(\overline{F}))$ centralizes E, which implies that $\beta(Z(\overline{F})) \subseteq Z(E)$. Therefore, $\omega\beta(Z(\overline{F})) \subseteq \omega(Z(E))$, or $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$. $\square$

The rest of this chapter follows similarly to its Leibniz case via simple replacements. By applying the preceding results and above discussion (concerning F/S as a cover), we thereby complete the diassociative analogue.

Theorem 5.5.7. *For every free presentation* $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ *of* L *and every stem extension* $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$, *one has* $Z^*(L) = \overline{\pi}(Z(\overline{F})) = \omega(Z(E))$.

Lemma 5.5.8. *Let* Z *be a central ideal of finite-dimensional diassociative algebra* L *and consider the map* $\beta : M(L) \longrightarrow M(L/Z)$ *from Theorem 5.5.2. Then* $Z \subseteq Z^*(L)$ *if and only if* β *is injective.*

Theorem 5.5.9. *Let* Z *be a central ideal of a finite-dimensional diassociative algebra* L *and*

$$\delta : M(L) \longrightarrow (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

be as in Theorem 5.5.1. Then the following are equivalent:

1. *δ is the trivial map,*

2. *the natural map β is injective,*

3. *$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$,*

4. *$Z \subseteq Z^*(L)$.*

We conclude this chapter by narrowing our focus to when the conditions of Theorem 5.5.9 hold for $Z = Z(L)$. Under this assumption, the center of the cover goes onto the center of the algebra.

Theorem 5.5.10. *Let L be a diassociative algebra and $Z(L)$ be the center of L. If $Z(L) \subseteq Z^*(L)$, then $\omega(Z(E)) = Z(L)$ for every stem extension $0 \longrightarrow \ker\omega \longrightarrow E \xrightarrow{\omega} L \longrightarrow 0$.*

CHAPTER

6

MULTIPLIERS AND COVERS OF PERFECT DIASSOCIATIVE ALGEBRAS

Recall that a diassociative algebra L is *perfect* if $L = L'$, where $L' = L \Diamond L$.

6.1 Universal Central Extensions

The aim of this section is to establish connections between perfect diassociative algebras and universal central extensions. Consider a finite-dimensional diassociative algebra L, an L-module A, and two central extensions $E : 0 \longrightarrow A \longrightarrow H \longrightarrow L \longrightarrow 0$ and $E_1 : 0 \longrightarrow A_1 \longrightarrow H_1 \longrightarrow L \longrightarrow 0$. We say that E *covers* E_1 if there exists a homomorphism $\tau : H \longrightarrow H_1$ such that the diagram

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
E_1 : 0 & \longrightarrow & A_1 & \longrightarrow & H_1 & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

Figure 6.1 Showing E covers E_1.

commutes, where the unmarked map is $\tau|_A$. If τ is unique, then E *uniquely covers* E_1. An extension E is *universal* if it uniquely covers any central extension of L.

Lemma 6.1.1. *If $E : 0 \longrightarrow A \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ and $E_1 : 0 \longrightarrow A_1 \longrightarrow H_1 \xrightarrow{\phi_1} L \longrightarrow 0$ are universal central extensions of L, then there exists an isomorphism $H \longrightarrow H_1$ which carries A onto A_1.*

Proof. Since both extensions are universal and central, there exist homomorphisms $\tau : H \longrightarrow H_1$ and $\tau_1 : H_1 \longrightarrow H$ such that the diagrams

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & H \arrow[r, "\phi"] \arrow[d, "\tau"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & H_1 \arrow[r, "\phi_1"'] & L \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & H_1 \arrow[r, "\phi_1"] \arrow[d, "\tau_1"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & H \arrow[r, "\phi"'] & L \arrow[r] & 0 \end{tikzcd}$$

Figure 6.2 Showing τ and τ_1.

commute. We thus have a mapping $\tau_1\tau : H \longrightarrow H$ such that $\phi\tau_1\tau = \phi_1\tau = \phi$. Since τ is unique, $\tau_1\tau = \mathrm{id}_H$. Similarly, $\tau\tau_1 = \mathrm{id}_{H_1}$. Therefore, τ is an isomorphism and $\tau|_A$ maps A onto A_1. □

Lemma 6.1.2. *If $E : 0 \longrightarrow A \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ is a universal central extension, then both H and L are perfect.*

Proof. Consider the central extension

$$0 \longrightarrow A \times H/H' \longrightarrow H \times H/H' \xrightarrow{\psi} L \longrightarrow 0$$

where $\psi(a,b) = \phi(a)$ for $a \in H$ and $b \in H/H'$. For $i = 1,2$, define homomorphisms

$$\tau_i : H \longrightarrow H \times H/H'$$

by $\tau_1(h) = (h,0)$ and $\tau_2(h) = (h, h + H/H')$. Then $\psi\tau_1(h) = \psi(h,0) = \phi(h)$ and $\psi\tau_2(h) = \psi(h, h + H/H') = \phi(h)$, which implies that $\psi\tau_i = \phi$ for $i = 1,2$. Since E is universal, we have $\tau_1 = \tau_2$. Thus $H/H' = 0$, and so $H = H'$. One computes

$$L' = (H/A)' = \frac{H' + A}{A} = \frac{H + A}{A} = H/A = L$$

and therefore H and L are perfect. □

Lemma 6.1.3. *Let $E : 0 \longrightarrow A \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ and $E_1 : 0 \longrightarrow A_1 \longrightarrow H_1 \xrightarrow{\phi_1} L \longrightarrow 0$ be central extensions and suppose H is perfect. Then E covers E_1 if and only if E uniquely covers E_1.*

Proof. The reverse direction is clear. In the forward direction, suppose E covers E_1. Then there exists a homomorphism $\tau : H \longrightarrow H_1$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & H \arrow[r, "\phi"] \arrow[d, "\tau"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & H_1 \arrow[r, "\phi_1"'] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 6.3 Showing τ.

commutes. Suppose there is another homomorphism $\beta : H \longrightarrow H_1$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & H \arrow[r, "\phi"] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & H_1 \arrow[r, "\phi_1"'] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 6.4 Showing β.

commutes. It remains to show that $\tau = \beta$. Let $x, y \in H$. Then

$$\begin{aligned}
\phi_1(\beta(x) - \tau(x)) &= \phi_1(\beta(x)) - \phi_1(\tau(x)) \\
&= \phi(x) - \phi(x) \\
&= 0
\end{aligned}$$

implies that $\beta(x) - \tau(x) \in \ker \phi_1 = A_1 \subseteq Z(H_1)$. Similarly, one obtains $\beta(y) - \tau(y) \in Z(H_1)$, and so $\beta(x) = \tau(x) + a$ and $\beta(y) = \tau(y) + b$ for some $a, b \in Z(H_1)$. We compute

$$\begin{aligned}\beta(x \dashv y) &= \beta(x) \dashv \beta(y)\\ &= \tau(x) \dashv \tau(y) + \tau(x) \dashv b + a \dashv \tau(y) + a \dashv b\\ &= \tau(x \dashv y)\end{aligned}$$

since $a, b \in Z(H_1)$. Similarly, $\beta(x \vdash y) = \tau(x \vdash y)$, and thus τ and β are equal on H'. Since H is perfect, we have $\tau = \beta$. □

Lemma 6.1.4. *Let L be a finite-dimensional perfect diassociative algebra. Then the extension $E : 0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal if and only if every central extension of L splits.*

Proof. In the forward direction, let $E_1 : 0 \longrightarrow A \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ be a central extension of L. Then there exists a unique homomorphism $\tau : L \longrightarrow H$ such that the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, "\mathrm{id}"] \arrow[d, "\tau"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ E_1 : 0 \arrow[r] & A \arrow[r] & H \arrow[r, "\phi"'] & L \arrow[r] & 0 \end{tikzcd}$$

Figure 6.5 Showing τ.

commutes. Therefore $\phi\tau = \mathrm{id}_L$, which implies that E_1 splits.

Conversely, suppose every central extension of L splits and let $E_1 : 0 \longrightarrow A \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ be a central extension. Then there exists a homomorphism $\beta : L \longrightarrow H$ such that $\phi\beta = \mathrm{id}_L$, which implies that the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, "\mathrm{id}"] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ E_1 : 0 \arrow[r] & A \arrow[r] & H \arrow[r, "\phi"'] & L \arrow[r] & 0 \end{tikzcd}$$

Figure 6.6 Showing β.

commutes. Thus E covers E_1. Since L is perfect, Lemma 6.1.3 guarantees that E uniquely covers E_1, and therefore E is universal. □

Lemma 6.1.5. *Let* $E_1 : 0 \longrightarrow B \longrightarrow G \xrightarrow{\phi} L \longrightarrow 0$ *and* $E_2 : 0 \longrightarrow C \longrightarrow L \xrightarrow{\psi} H \longrightarrow 0$ *be central extensions and let* $\pi = \psi\phi$ *and* $A = \ker \pi$*. If* G *is perfect, then* $E_3 : 0 \longrightarrow A \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$ *is a central extension.*

$$
\begin{tikzcd}
E_3 & & & E_1 & & \\
& 0 \arrow[dr] & & 0 \arrow[d] & & \\
& & A \arrow[dr] & B \arrow[d] & & \\
& & & G \arrow[d, "\phi"'] \arrow[dr, "\pi"] & & \\
E_2 & 0 \arrow[r] & C \arrow[r] & L \arrow[r, "\psi"'] \arrow[d] & H \arrow[r] \arrow[dr] & 0 \\
& & & 0 & & 0
\end{tikzcd}
$$

Figure 6.7 Interaction of E_1, E_2, and E_3.

Proof. For any $a \in A = \ker \pi$, one has $\psi\phi(a) = \pi(a) = 0$. Therefore, for any $x \in G$, $\phi(a \dashv x) = \phi(a) \dashv \phi(x) = 0$ since $\phi(a) \in \ker \psi$. Similarly, $\phi(a \dashv x)$, $\phi(x \dashv a)$, and $\phi(x \vdash a)$ are all zero. For $a \in A$, let $\lambda_a^{\dashv}$, $\lambda_a^{\vdash}$, $\rho_a^{\dashv}$, $\rho_a^{\vdash}$ be adjoint operators on G defined by $\lambda_a^*(x) = a * x$ and $\rho_a^*(x) = x * a$, where $*$ ranges over $\dashv$ and $\vdash$. Since $\phi(\lambda_a^*(x)) = 0$ and $\phi(\rho_a^*(x)) = 0$, we get

$\lambda_a^*(x), \rho_a^*(x) \in \ker\phi \subseteq Z(G)$. Now let $y, z \in G$. Then

$$\begin{aligned}
\lambda_a^{\dashv}(y \dashv z) &= a \dashv (y \dashv z) \overset{\text{As}}{=} (a \dashv y) \dashv z = 0,\\
\lambda_a^{\dashv}(y \vdash z) &= a \dashv (y \vdash z) \overset{\text{D1}}{=} a \dashv (y \dashv z) \overset{\text{As}}{=} (a \dashv y) \dashv z = 0,\\
\lambda_a^{\vdash}(y \dashv z) &= a \vdash (y \dashv z) \overset{\text{D2}}{=} (a \vdash y) \dashv z = 0,\\
\lambda_a^{\vdash}(y \vdash z) &= a \vdash (y \vdash z) \overset{\text{As}}{=} (a \vdash y) \vdash z = 0,
\end{aligned}$$

$$\begin{aligned}
\rho_a^{\dashv}(y \dashv z) &= (y \dashv z) \dashv a \overset{\text{As}}{=} y \dashv (z \dashv a) = 0,\\
\rho_a^{\dashv}(y \vdash z) &= (y \vdash z) \dashv a \overset{\text{D2}}{=} y \vdash (z \dashv a) = 0,\\
\rho_a^{\vdash}(y \dashv z) &= (y \dashv z) \vdash a \overset{\text{D3}}{=} (y \vdash z) \vdash a \overset{\text{As}}{=} y \vdash (z \vdash a) = 0,\\
\rho_a^{\vdash}(y \vdash z) &= (y \vdash z) \vdash a \overset{\text{As}}{=} y \vdash (z \vdash a) = 0
\end{aligned}$$

since $a \dashv y, a \vdash y, z \dashv a, z \vdash a \in Z(G)$, where "As" denotes the associativity of the diassociative multiplications. Therefore λ_a^* and ρ_a^* are trivial maps on $G' = G$, and so $a \in Z(G)$, which implies that E_3 is central. □

Lemma 6.1.6. *Let E_1, E_2, E_3, and their involved maps be as in Lemma 6.1.5. If E_1 is universal, then so is E_3.*

Proof. Suppose $E_1 : 0 \longrightarrow B \longrightarrow G \xrightarrow{\phi} L \longrightarrow 0$ is a universal extension. By Lemma 6.1.2, G and L are perfect. Since H is the homomorphic image of G, H is also perfect. Let $E_4 : 0 \longrightarrow D \longrightarrow S \xrightarrow{\omega} H \longrightarrow 0$ be another central extension of H. Let $T = \{(a,b) \in L \times S \mid \psi(a) = \omega(b)\}$ and define multiplications on T by $(a,b) \dashv (c,d) = (a \dashv c, b \dashv d)$ and $(a,b) \vdash (c,d) = (a \vdash c, b \vdash d)$. Then T is closed under multiplication since $\psi(a \dashv c) = \psi(a) \dashv \psi(c) = \omega(b) \dashv \omega(d) = \omega(b \dashv d)$ and $\psi(a \vdash c) = \omega(b \vdash d)$ similarly. Thus T is a subalgebra of $L \times S$. Let λ be the projection of T onto L. Since E_1 is universal, there exists a unique homomorphism $\alpha : G \longrightarrow T$ such that the diagram

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & B \arrow[r] \arrow[d] & G \arrow[r, "\phi"] \arrow[d, "\alpha"'] & L \arrow[r] \arrow[d, "\text{id}"] & 0\\
0 \arrow[r] & 0 \times D \arrow[r] & T \arrow[r, "\lambda"'] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 6.8 Showing α.

commutes, i.e. such that $\lambda\alpha = \phi$. Let $\gamma : T \longrightarrow S$ be the natural projection given by $\gamma(a,b) = b$. Let $\beta = \gamma\alpha$. Given $g \in G$, set $\alpha(g) = (a,b)$. Then $\beta(g) = \gamma\alpha(g) = \gamma(a,b) = b$ and $\phi(g) = \lambda\alpha(g) = \lambda(a,b) = a$, which implies that $\omega\beta(g) = \omega(b) = \psi(a) = \psi\phi(g) = \pi(g)$. Thus the diagram

$$\begin{array}{ccccccccc}
E_3 : 0 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
E_4 : 0 & \longrightarrow & D & \longrightarrow & S & \xrightarrow[\omega]{} & H & \longrightarrow & 0
\end{array}$$

Figure 6.9 Showing $\pi = \omega\beta$.

commutes, and so E_3 covers E_4. Since G is perfect, we know that E_3 uniquely covers E_4. Therefore, E_3 is universal. □

The proof of the following lemma holds by the same logic as its Lie analogue (Lemma 6.7 in [1]), but we provide it here for the sake of completeness and to detail how it fits in with the previous lemmas.

Lemma 6.1.7. *Let L be a finite-dimensional perfect diassociative algebra. If $0 \longrightarrow 0 \longrightarrow H \xrightarrow{\phi} L \longrightarrow 0$ is a universal extension, then so is $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$.*

Proof. Let $0 \longrightarrow A \longrightarrow L^* \xrightarrow{\psi} L \longrightarrow 0$ be an arbitrary central extension of L. Then there exists a unique homomorphism $\theta : H \longrightarrow L^*$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H & \xrightarrow{\phi} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \longrightarrow & L^* & \xrightarrow[\psi]{} & L & \longrightarrow & 0
\end{array}$$

Figure 6.10 Showing θ.

commutes. In other words, $\phi = \psi\theta$. However, since ϕ is an isomorphism, let $\beta = \theta\phi^{-1}$ be the homomorphism from L to L^*. One computes $\psi\beta = \psi\theta\phi^{-1} = \phi\phi^{-1} = \mathrm{id}_L$, which means

β is a section of ψ that is also a homomorphism. Thus, our central extension

$$0 \longrightarrow A \longrightarrow L^* \xrightarrow{\psi} L \longrightarrow 0$$

splits. By Lemma 6.1.4, and since L is perfect, we know that $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal. $\square$

6.2 Multipliers and Covers

We now return to multipliers and covers. Let L be a finite-dimensional perfect diassociative algebra with free presentation $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ and consider the central extension

$$0 \longrightarrow \frac{R}{F \Diamond R + R \Diamond F} \longrightarrow \frac{F}{F \Diamond R + R \Diamond F} \xrightarrow{\pi} L \longrightarrow 0.$$

We compute

$$\begin{aligned} L' &= \mathrm{Im}\left(\frac{F}{F \Diamond R + R \Diamond F}\right)' \\ &\cong \mathrm{Im}\left(\frac{F' + F \Diamond R + R \Diamond F}{F \Diamond R + R \Diamond F}\right) \\ &= \mathrm{Im}\left(\frac{F'}{F \Diamond R + R \Diamond F}\right) \end{aligned}$$

which implies that the restriction $\pi|_{F'/(F \Diamond R + R \Diamond F)}$ induces a central extension

$$0 \longrightarrow \frac{F' \cap R}{F \Diamond R + R \Diamond F} \longrightarrow \frac{F'}{F \Diamond R + R \Diamond F} \longrightarrow L \longrightarrow 0$$

since L is perfect. By the work in Chapter 5, we know that the algebra $(F' \cap R)/(F \Diamond R + R \Diamond F)$ is precisely the multiplier $M(L)$. Our aim is now to show that $F'/(F \Diamond R + R \Diamond F)$ is a cover of L and that the above extension is universal.

Theorem 6.2.1. *Let L be a finite-dimensional perfect diassociative algebra and $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L. Then $F'/(F \Diamond R + R \Diamond F)$ is a cover of L.*

Proof. Since $M(L) \cong (F' \cap R)/(F \Diamond R + R \Diamond F)$ and

$$L \cong \frac{F'/(F \Diamond R + R \Diamond F)}{(F' \cap R)/(F \Diamond R + R \Diamond F)},$$

it remains to prove that

$$\frac{F'\cap R}{F\Diamond R+R\Diamond F}\subseteq Z\left(\frac{F'}{F\Diamond R+R\Diamond F}\right)\cap\left(\frac{F'}{F\Diamond R+R\Diamond F}\right)'.$$

Clearly

$$\frac{F'\cap R}{F\Diamond R+R\Diamond F}\subseteq Z\left(\frac{F'}{F\Diamond R+R\Diamond F}\right)$$

and

$$\frac{F'\cap R}{F\Diamond R+R\Diamond F}\subseteq \frac{F'}{F\Diamond R+R\Diamond F}.$$

It thus suffices to show that

$$\left(\frac{F'}{F\Diamond R+R\Diamond F}\right)'=\frac{F'}{F\Diamond R+R\Diamond F}.$$

We first note that

$$\left(\frac{F'}{F\Diamond R+R\Diamond F}\right)'=\frac{F''+F\Diamond R+R\Diamond F}{F\Diamond R+R\Diamond F}$$

and that $F''+F\Diamond R+R\Diamond F\subseteq F'$. Since $L=L'$, we know that $F/R=(F/R)'=\frac{F'+R}{R}$. This implies that, for all $x_i\in F$, $x_i=y_i+r_i$ for some $y_i\in F'$ and $r_i\in R$. We compute

$$\begin{aligned}x_1\dashv x_2&=y_1\dashv y_2+y_1\dashv r_2+r_1\dashv y_2+r_1\dashv r_2,\\ x_1\vdash x_2&=y_1\vdash y_2+y_1\vdash r_2+r_1\vdash y_2+r_1\vdash r_2,\end{aligned}$$

which both fall in $F''+F\Diamond R+R\Diamond F$. Thus, $F'\subseteq F''+F\Diamond R+R\Diamond F$, and so $F'/(F\Diamond R+R\Diamond F)$ is a cover of L. □

Corollary 6.2.2. *Let L be a finite-dimensional perfect diassociative algebra with free presentation $0\longrightarrow R\longrightarrow F\longrightarrow L\longrightarrow 0$. Then $F'/(F\Diamond R+R\Diamond F)$ is perfect.*

Proof. By the proof of Theorem 6.2.1, we have $\big(F'/(F\Diamond R+R\Diamond F)\big)'=F'/(F\Diamond R+R\Diamond F)$. □

Theorem 6.2.3. *Let L be a finite-dimensional perfect diassociative algebra with free presentation $0\longrightarrow R\longrightarrow F\longrightarrow L\longrightarrow 0$. Then the extension*

$$E:0\longrightarrow\frac{F'\cap R}{F\Diamond R+R\Diamond F}\longrightarrow\frac{F'}{F\Diamond R+R\Diamond F}\longrightarrow L\longrightarrow 0$$

is universal.

Proof. Let $E_1:0\longrightarrow A\longrightarrow H\longrightarrow L\longrightarrow 0$ be a central extension of L. By Lemma 5.4.2, it is covered by a natural exact sequence, call it E_2, making the diagram

$$\begin{tikzcd}
E_2 : 0 \arrow[r] & \frac{R}{F\Diamond R+R\Diamond F} \arrow[r] \arrow[d] & \frac{F}{F\Diamond R+R\Diamond F} \arrow[r] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
E_1 : 0 \arrow[r] & A \arrow[r] & H \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 6.11 E_2 covers E_1.

commute. Since the diagram

$$\begin{tikzcd}
E : 0 \arrow[r] & \frac{F'\cap R}{F\Diamond R+R\Diamond F} \arrow[r] \arrow[d] & \frac{F'}{F\Diamond R+R\Diamond F} \arrow[r] \arrow[d, "\theta"'] & L \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
E_1 : 0 \arrow[r] & A \arrow[r] & H \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Figure 6.12 E covers E_1.

commutes, where $\theta = \beta|_{F'/(F\Diamond R+R\Diamond F)}$, we know that E_1 is covered by E. Since $F'/(F\Diamond R+R\Diamond F)$ is perfect, Lemma 6.1.3 implies that E uniquely covers E_1. Thus, E is universal. □

Theorem 6.2.4. *If $0 \longrightarrow A \longrightarrow L^* \longrightarrow L \longrightarrow 0$ is a universal central extension and L is perfect, then $A \cong M(L)$ and L^* is a cover of L.*

Proof. We know that

$$0 \longrightarrow \frac{F' \cap R}{F\Diamond R + R\Diamond F} \longrightarrow \frac{F'}{F\Diamond R + R\Diamond F} \longrightarrow L \longrightarrow 0$$

is universal. By Lemma 6.1.1, there exists an isomorphism

$$L^* \longrightarrow \frac{F'}{F\Diamond R + R\Diamond F}$$

which carries A onto

$$\frac{F' \cap R}{F\Diamond R + R\Diamond F} \cong M(L).$$

Thus $A \cong M(L)$ and L^* is a cover of L. □

Theorem 6.2.5. *Let L be a finite-dimensional perfect diassociative algebra and let $M(L) = 0$. Then $\mathscr{H}^2(L, A) = 0$ for any central module A of L.*

Proof. Since $M(L)=0$, Theorem 6.2.3 implies that the extension

$$0 \longrightarrow 0 \longrightarrow \frac{F'}{F\lozenge R + R\lozenge F} \longrightarrow L \longrightarrow 0$$

is universal. By Lemma 6.1.7, the extension $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is also universal. By Lemma 6.1.4, every central extension of L splits, and thus $\mathscr{H}^2(L,A)=0$. □

Theorem 6.2.6. *Let L be a finite-dimensional perfect diassociative algebra and let $M(L)=0$. If Z is a central ideal of L, then $Z \cong M(L/Z)$ and L is a cover of L/Z.*

Proof. By the proof of Theorem 6.2.5, the extension $0 \longrightarrow 0 \longrightarrow L \longrightarrow L \longrightarrow 0$ is universal. Since L is perfect, Lemma 6.1.5 implies that $0 \longrightarrow Z \longrightarrow L \longrightarrow L/Z \longrightarrow 0$ is also universal. Thus, by Theorem 6.2.4, $M(L/Z) \cong Z$, which implies that L is the cover of L/Z. □

Theorem 6.2.7. *Let L be a finite-dimensional perfect diassociative algebra and C be a cover of L. Then $C = C'$ and $M(C)=0$.*

Proof. Let $A = M(L)$. Then $L \cong C/A$ and $A \subseteq Z(C) \cap C'$. One computes

$$L/L' \cong \frac{C/A}{(C/A)'} \cong \frac{C/A}{(C'+A)/A} \cong \frac{C}{C'+A}.$$

Since $L = L'$ and $A \subseteq C'$, we have $C = C' + A = C'$. Thus C is perfect. We now invoke our extended Hochschild-Serre sequence that was obtained in Chapter 5.

$$0 \longrightarrow \mathrm{Hom}(L,\mathbb{F}) \xrightarrow{\mathrm{Inf}_1} \mathrm{Hom}(C,\mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(A,\mathbb{F}) \xrightarrow{\mathrm{Tra}} M(L) \xrightarrow{\mathrm{Inf}_2} M(C) \xrightarrow{\delta} (C/C' \otimes A \oplus A \otimes C/C')^2$$

Here, the term $(C/C' \otimes A \oplus A \otimes C/C')^2$ must be zero since $C = C'$, which yields $M(C) = \ker\delta = \mathrm{Im}(\mathrm{Inf}_2)$. Next, we also know that $\mathrm{Hom}(C,\mathbb{F}) = 0$ since C is perfect. This implies that $0 = \mathrm{Im}(\mathrm{Res}) = \ker(\mathrm{Tra})$. Then $\mathrm{Im}(\mathrm{Tra}) \cong \mathrm{Hom}(A,\mathbb{F}) \cong A \cong M(L)$ and therefore $M(C) = \mathrm{Im}(\mathrm{Inf}_2) \cong M(L)/\ker(\mathrm{Inf}_2) = M(L)/\mathrm{Im}(\mathrm{Tra}) = M(L)/M(L) \cong 0$. □

CHAPTER

7

MULTIPLIERS OF NILPOTENT DIASSOCIATIVE ALGEBRAS

In this chapter, we prove an alternative way to extend the Hochschild-Serre sequence from Chapter 5 and apply it to study the multipliers of nilpotent diassociative algebras. We compare this theory to its associative specialization and explicitly compute the multipliers of an associative algebra *as* an associative algebra and then *as* a diassociative algebra.

Recall the sequences of ideals from Section 3.1. For a diassociative algebra L, we will refer to L^k as the kth term in the *lower central series* of L. For the present section, we say that a diassociative algebra L is *nilpotent of class* n if $L^n \neq 0$ and $L^{n+1} = 0$. If L is nilpotent of class n, it induces a central extension $0 \longrightarrow L^n \longrightarrow L \longrightarrow L/L^n \longrightarrow 0$. We also define the *upper central series* of L by $Z_1 = Z(L)$ and

$$Z_{j+1} = \{x \in L \mid \forall l \in L,\ x \dashv l, x \vdash l, l \dashv x, l \vdash x \in Z_j\}$$

for $j \geq 1$. In particular, for a nilpotent diassociative algebra L of class n, one has $Z_n = L$, and thus $L' \subseteq Z_{n-1}$.

Lemma 7.0.1. *Let L be a diassociative algebra and Z_j denote the jth term in the upper* *central series of L. Then*

$$L^s \Diamond Z_i + Z_i \Diamond L^s \subseteq Z_{i-s}$$

for all $i \geq s$.

Proof. We proceed by induction on s. For the base case $s = 1$, one has $L \Diamond Z_i + Z_i \Diamond L \subseteq Z_{i-1}$ by definition. Now assume that the statement holds for some $s \geq 1$. We compute

$$\begin{aligned}
L^{s+1} \Diamond Z_i + Z_i \Diamond L^{s+1} &= (L \Diamond L^s) \Diamond Z_i + Z_i \Diamond (L^s \Diamond L) && \text{by Lemma 3.1.2} \\
&= L \Diamond (L^s \Diamond Z_i) + (Z_i \Diamond L^s) \Diamond L && (*) \\
&\subseteq L \Diamond Z_{i-s} + Z_{i-s} \Diamond L && \text{by induction} \\
&\subseteq Z_{i-s-1} && \text{by definition}
\end{aligned}$$

where $(*)$ follows via the diassociative identities and the associativity of the multiplications. □

7.1 The Main Result

The form $(X \otimes Y \oplus Y \otimes X)^2$ will continue to denote $(X \otimes Y \oplus Y \otimes X) \oplus (X \otimes Y \oplus Y \otimes X)$. This is not to be confused with the terms L^n in the lower central series of an algebra L. We will use L' to denote the second term L^2 in the lower central series.

Theorem 7.1.1. *Let L be a nilpotent diassociative algebra and let A and B be ideals in L such that $L' \subseteq A$ and $B \subseteq Z(L)$. If $f_\dashv(A,B) = 0$, $f_\dashv(B,A) = 0$, $f_\vdash(A,B) = 0$, and $f_\vdash(B,A) = 0$ for all $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L,\mathbb{F})$, then there exists a homomorphism δ such that*

$$\mathscr{H}^2(L/B,\mathbb{F}) \xrightarrow{\text{Inf}} \mathscr{H}^2(L,\mathbb{F}) \xrightarrow{\delta} (L/A \otimes B \oplus B \otimes L/A)^2$$

is exact.

Proof. Consider elements $x \in L$, $b \in B$, and $(f'_\dashv, f'_\vdash) \in \mathscr{Z}^2(L,\mathbb{F})$. We define the bilinear forms

$$f''_\dashv : L/A \times B \longrightarrow \mathbb{F}, \qquad g''_\dashv : B \times L/A \longrightarrow \mathbb{F},$$
$$f''_\vdash : L/A \times B \longrightarrow \mathbb{F}, \qquad g''_\vdash : B \times L/A \longrightarrow \mathbb{F}$$

by

$$f''_\dashv(x+A, b) = f'_\dashv(x,b), \qquad g''_\dashv(b, x+A) = f'_\dashv(b,x),$$
$$f''_\vdash(x+A, b) = f'_\vdash(x,b), \qquad g''_\vdash(b, x+A) = f'_\vdash(b,x).$$

Since $f'_{\dashv}(A,B)=0$, $f'_{\dashv}(B,A)=0$, $f'_{\vdash}(A,B)=0$, and $f'_{\vdash}(B,A)=0$, all of these maps are well-defined. We define $\delta':\mathscr{Z}^2(L,\mathbb{F})\longrightarrow(L/A\otimes B\oplus B\otimes L/A)^2$ by $\delta'(f'_{\dashv},f'_{\vdash})=(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})$. Now consider an element $(f'_{\dashv},f'_{\vdash})\in\mathscr{B}^2(L,\mathbb{F})$. Then there exists a linear transformation $\varepsilon:L\longrightarrow\mathbb{F}$ such that $f'_{\dashv}(x,y)=-\varepsilon(x\dashv y)$ and $f'_{\vdash}(x,y)=-\varepsilon(x\vdash y)$ for all $x,y\in L$. For $b\in B$, however, we compute

$$f''_{\dashv}(x+A,b)=f'_{\dashv}(x,b)=-\varepsilon(x\dashv b)=0,\qquad g''_{\dashv}(b,x+A)=f'_{\dashv}(b,x)=-\varepsilon(b\dashv x)=0,$$
$$f''_{\vdash}(x+A,b)=f'_{\vdash}(x,b)=-\varepsilon(x\vdash b)=0,\qquad g''_{\vdash}(b,x+A)=f'_{\vdash}(b,x)=-\varepsilon(b\vdash x)=0$$

since $B\subseteq Z(L)$. Thus, $\delta'(f'_{\dashv},f'_{\vdash})=0$ for any coboundary $(f'_{\dashv},f'_{\vdash})$, and so δ' induces a well-defined map $\delta:\mathscr{H}^2(L,\mathbb{F})\longrightarrow(L/A\otimes B\oplus B\otimes L/A)^2$ by $\delta((f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F}))=(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})$.

Now that we have established our δ, it remains to show that the sequence is exact. Consider a cocycle $(f_{\dashv},f_{\vdash})\in\mathscr{Z}^2(L/B,\mathbb{F})$ and set $f'_{\dashv}(x,y)=f_{\dashv}(x+B,y+B)$ and $f'_{\vdash}(x,y)=f_{\vdash}(x+B,y+B)$. We first recall (see Chapter 5) that $\mathrm{Inf}:\mathscr{H}^2(L/B,\mathbb{F})\longrightarrow\mathscr{H}^2(L,\mathbb{F})$ is defined by $\mathrm{Inf}((f_{\dashv},f_{\vdash})+\mathscr{B}^2(L/B,\mathbb{F}))=(f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F})$. To show that $\mathrm{Im}(\mathrm{Inf})\subseteq\ker\delta$, consider $(f_{\dashv},f_{\vdash})\in\mathscr{Z}^2(L/B,\mathbb{F})$. Then $(f_{\dashv},f_{\vdash})$ induces tuples $(f'_{\dashv},f'_{\vdash})$ and $(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})$, as defined previously. For $x\in L$ and $b\in B$, one computes

$$\begin{aligned}
f''_{\dashv}(x+A,b)&=f'_{\dashv}(x,b)=f_{\dashv}(x+B,b+B)=0,\\
g''_{\dashv}(b,x+A)&=f'_{\dashv}(b,x)=f_{\dashv}(b+B,x+B)=0,\\
f''_{\vdash}(x+A,b)&=f'_{\vdash}(x,b)=f_{\vdash}(x+B,b+B)=0,\\
g''_{\vdash}(b,x+A)&=f'_{\vdash}(b,x)=f_{\vdash}(b+B,x+B)=0
\end{aligned}$$

and so $\delta(\mathrm{Inf}((f_{\dashv},f_{\vdash})+\mathscr{B}^2(L/B,\mathbb{F})))=\delta((f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F}))=(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})=(0,0,0,0)$. Thus, $\mathrm{Im}(\mathrm{Inf})\subseteq\ker\delta$.

Conversely, consider a cocycle $(f'_{\dashv},f'_{\vdash})\in\mathscr{Z}^2(L,\mathbb{F})$ such that $(f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F})\in\ker\delta$. In other words, $\delta((f'_{\dashv},f'_{\vdash})+\mathscr{B}^2(L,\mathbb{F}))=(f''_{\dashv},g''_{\dashv},f''_{\vdash},g''_{\vdash})=(0,0,0,0)$, where $f''_{\dashv}$, $g''_{\dashv}$, $f''_{\vdash}$, and $g''_{\vdash}$ are defined using $(f'_{\dashv},f'_{\vdash})$ as above. This implies that

$$f'_{\dashv}(x,b)=f''_{\dashv}(x+A,b)=0,\qquad f'_{\dashv}(b,x)=g''_{\dashv}(b,x+A)=0,$$
$$f'_{\vdash}(x,b)=f''_{\vdash}(x+A,b)=0,\qquad f'_{\vdash}(b,x)=g''_{\vdash}(b,x+A)=0$$

for all $x\in L$, $b\in B$. Define a pair $(f_{\dashv},f_{\vdash})$ of bilinear forms $L/B\times L/B\longrightarrow\mathbb{F}$ by $f_{\dashv}(x+B,y+B)=f'_{\dashv}(x,y)$ and $f_{\vdash}(x+B,y+B)=f'_{\vdash}(x,y)$ for any $x,y\in L$. To show that $f_{\dashv}$ and $f_{\vdash}$ are well-defined, consider elements $x+B=x_1+B$ and $y+B=y_1+B$. Then $x_1=x+b$ and $y_1=y+c$ for some $b, c \in B$. We compute

$$\begin{aligned} f_\dashv(x_1 + B, y_1 + B) &= f'_\dashv(x_1, y_1) \\ &= f'_\dashv(x + b, y + c) \\ &= f'_\dashv(x, y) + f'_\dashv(x, c) + f'_\dashv(b, y) + f'_\dashv(b, c) \\ &= f'_\dashv(x, y) \\ &= f_\dashv(x + B, y + B) \end{aligned}$$

and, similarly, $f_\vdash(x_1 + B, y_1 + B) = f_\vdash(x + B, y + B)$. Moreover, $(f_\dashv, f_\vdash)$ satisfies the diassociative cocycle identities since $(f'_\dashv, f'_\vdash)$ does. We thus obtain an element $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L/B, \mathbb{F})$ such that $\mathrm{Inf}((f_\dashv, f_\vdash) + \mathscr{B}^2(L/B, \mathbb{F})) = (f'_\dashv, f'_\vdash) + \mathscr{B}^2(L, \mathbb{F})$. Therefore, $\ker\delta \subseteq \mathrm{Im(Inf)}$, and the sequence is exact. □

7.2 Applications

Our first application of Theorem 7.1.1 is an alternative proof of Theorem 5.5.1 in the case when L is a nilpotent diassociative algebra. Letting $A = L'$ and $B = Z \subseteq Z(L)$, we obtain the following.

Proposition 7.2.1. *Let L be a nilpotent diassociative algebra and Z be a central ideal in L. Then*

$$\mathscr{H}^2(L/Z, \mathbb{F}) \xrightarrow{\mathrm{Inf}} \mathscr{H}^2(L, \mathbb{F}) \xrightarrow{\delta} (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

is exact.

Proof. To invoke Theorem 7.1.1, it suffices to show that $f_\dashv(L', Z) = 0$, $f_\dashv(Z, L') = 0$, $f_\vdash(L', Z) = 0$, and $f_\vdash(Z, L') = 0$ for all $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L, \mathbb{F})$. But this holds by the diassociative cocycle identities and their ability to associate products within the bilinear forms. For example, we compute

$$\begin{aligned} f_\dashv(L', Z) &= f_\dashv(L \dashv L, Z) + f_\dashv(L \vdash L, Z) \\ &= f_\dashv(L, L \dashv Z) + f_\vdash(L, L \dashv Z) \\ &= 0 \end{aligned}$$

via C4 and C2 respectively. □

The following corollary is the diassociative analogue of a result that was proved in [6], and we use a similar approach to our proof.

Corollary 7.2.2. *Let L be a nilpotent, finite-dimensional diassociative algebra and $Z \subseteq Z(L) \cap L'$ be an ideal such that* $\dim Z = 1$. *Then*

$$\dim \mathscr{H}^2(L,\mathbb{F}) + 1 \le \dim \mathscr{H}^2(L/Z,\mathbb{F}) + 4\dim(L/L').$$

Proof. We may invoke our extended cohomological five-sequence regardless of $\dim Z$.

$$0 \longrightarrow \mathrm{Hom}(L/Z,\mathbb{F}) \longrightarrow \mathrm{Hom}(L,\mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(Z,\mathbb{F}) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/Z,\mathbb{F})$$

$$\xrightarrow{\mathrm{Inf}} \mathscr{H}^2(L,\mathbb{F}) \xrightarrow{\delta} (L/L' \otimes Z \oplus Z \otimes L/L')^2$$

Since $Z \subseteq L'$, we obtain Res $= 0$. This follows since Res simply restricts any homomorphism $\pi : L \longrightarrow \mathbb{F}$ to $\pi|_Z$, and any product in $\mathbb{F}$ is zero. By exactness, Tra is injective. Thus, $\dim(\mathrm{Im}(\mathrm{Tra})) = 1$ since $\dim(\mathrm{Hom}(Z,\mathbb{F})) = 1$. Also by exactness, we know that

$$\dim(\mathrm{Im}\,\delta) + \dim(\mathrm{Im}(\mathrm{Inf})) = \dim \mathscr{H}^2(L,\mathbb{F})$$

and

$$\dim(\mathrm{Im}(\mathrm{Inf})) + \dim(\mathrm{Im}(\mathrm{Tra})) = \dim \mathscr{H}^2(L/Z,\mathbb{F}).$$

We therefore compute

$$\begin{aligned}
\dim \mathscr{H}^2(L,\mathbb{F}) + 1 &= \dim \mathscr{H}^2(L,\mathbb{F}) + \dim(\mathrm{Im}(\mathrm{Tra})) \\
&= \dim(\mathrm{Im}\,\delta) + \dim(\mathrm{Im}(\mathrm{Inf})) + \dim(\mathrm{Im}(\mathrm{Tra})) \\
&= \dim(\mathrm{Im}\,\delta) + \dim \mathscr{H}^2(L/Z,\mathbb{F}) \\
&\le \dim((L/L' \otimes Z \oplus Z \otimes L/L')^2) + \dim \mathscr{H}^2(L/Z,\mathbb{F}) \\
&= 4\dim(L/L') + \dim \mathscr{H}^2(L/Z,\mathbb{F}).
\end{aligned}$$

□

Theorem 7.2.3. *Let L be a nilpotent diassociative algebra of class n. Then*

$$\mathscr{H}^2(L/L^n,\mathbb{F}) \xrightarrow{\mathrm{Inf}} \mathscr{H}^2(L,\mathbb{F}) \xrightarrow{\delta} (L/Z_{n-1} \otimes L^n \oplus L^n \otimes Z_{n-1})^2$$

is exact.

Proof. Supposing that L is nilpotent of class n, we first note that $L^n \subseteq Z(L)$ and $L' \subseteq Z_{n-1}$ via our preliminary discussion. To invoke Theorem 7.1.1, it suffices to show that $f_\dashv(Z_{n-1}, L^n) = f_\dashv(L^n, Z_{n-1}) = f_\vdash(Z_{n-1}, L^n) = f_\vdash(L^n, Z_{n-1}) = 0$ for all $(f_\dashv, f_\vdash) \in \mathscr{Z}^2(L,\mathbb{F})$ and $n \ge 1$. For $n = 1$, we have $f_{\dashv}(Z_0, L) = f_{\dashv}(0, L) = 0$ and, similarly, $f_{\dashv}(L, Z_0) = f_{\vdash}(Z_0, L) = f_{\vdash}(L, Z_0) = 0$. For $k \geq 1$, we compute

$$\begin{aligned}
f_{\dashv}(Z_k, L^{k+1}) &= f_{\dashv}(Z_k, L^k \Diamond L) \\
&= f_{\dashv}(Z_k, L^k \dashv L) + f_{\dashv}(Z_k, L^k \vdash L) \\
&= f_{\dashv}(Z_k \dashv L^k, L) + f_{\dashv}(Z_k, L^k \dashv L) \\
&\subseteq f_{\dashv}(Z_0, L) + f_{\dashv}(Z_k \dashv L^k, L) \\
&\subseteq f_{\dashv}(Z_0, L) \\
&= 0
\end{aligned}$$

via Lemma 3.1.2, C4, C1, and Lemma 7.0.1. The other computations follow similarly, and thus the result holds by Theorem 7.1.1. □

Remark. The Lie analogue of Theorem 7.2.3 relies on induction, but we note that the diassociative version is attainable without it. This reveals that, while the diassociative cocycle identities lack an anticommutative-type property, they are actually more powerful than the Lie conditions in some ways.

Corollary 7.2.4. *Let L be a nilpotent, finite-dimensional diassociative algebra of class n. Then*

$$\dim \mathscr{H}^2(L, \mathbb{F}) \leq \dim \mathscr{H}^2(L/L^n, \mathbb{F}) + 4\dim(L^n)\dim(L/Z_{n-1}) - \dim(L^n).$$

Proof. Consider the terms in our extended sequence

$$\mathrm{Hom}(L, \mathbb{F}) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(L^n, \mathbb{F}) \xrightarrow{\mathrm{Tra}} \mathscr{H}^2(L/L^n, \mathbb{F}) \xrightarrow{\mathrm{Inf}} \mathscr{H}^2(L, \mathbb{F}) \xrightarrow{\delta} (L/Z_{n-1} \otimes L^n \oplus L^n \otimes L/Z_{n-1})^2$$

and denote

$$\begin{aligned}
q &= \dim \mathrm{Hom}(L^n, \mathbb{F}), \\
r &= \dim \mathscr{H}^2(L/L^n, \mathbb{F}), \\
s &= \dim \mathscr{H}^2(L, \mathbb{F}), \\
t &= \dim((L/Z_{n-1} \otimes L^n \oplus L^n \otimes L/Z_{n-1})^2).
\end{aligned}$$

We first note that $\mathbb{F}^n = 0$ for $n \geq 2$, and thus any homomorphism $f : L^n \longrightarrow \mathbb{F}^n$, as the restriction of some $f \in \mathrm{Hom}(L, \mathbb{F})$, is the zero map. Therefore $\mathrm{Res} = 0$, and so Tra is injective. It follows that $q = \dim(\mathrm{Im}(\mathrm{Tra})) = \dim(\ker(\mathrm{Inf}))$, which implies that

$$r - q = \dim \mathscr{H}^2(L/L^n, \mathbb{F}) - \dim(\ker(\mathrm{Inf})) = \dim(\mathrm{Im}(\mathrm{Inf})) \leq s.$$

On the other hand, we know that

$$\dim \mathscr{H}^2(L,\mathbb{F}) - \dim(\ker\delta) = \dim(\operatorname{Im}\delta) \le t,$$

and so $s - \dim(\ker\delta) \le t$. Finally, the equality $r - q = \dim(\operatorname{Im}(\operatorname{Inf})) = \dim(\ker\delta)$ yields $s-(r-q) \le t$. We thus obtain

$$\begin{aligned}\dim \mathscr{H}^2(L,\mathbb{F}) &\le \dim \mathscr{H}^2(L/L^n,\mathbb{F}) + \dim((L/Z_{n-1} \otimes L^n \oplus L^n \otimes L/Z_{n-1})^2) - \dim \operatorname{Hom}(L^n,\mathbb{F}) \\ &= \dim \mathscr{H}^2(L/L^n,\mathbb{F}) + 4\dim(L/Z_{n-1})\dim(L^n) - \dim(L^n)\end{aligned}$$

from $s \le t + r - q$. □

Corollary 7.2.5. *Let L be a nilpotent, finite-dimensional diassociative algebra. Then*

$$\dim \mathscr{H}^2(L,\mathbb{F}) \le \dim \mathscr{H}^2(L/L',\mathbb{F}) + \dim(L')[4\dim(L/Z(L)) - 4\dim((L/Z(L))') - 1].$$

Proof. We proceed by induction on the nilpotency class of L. As a base case, if L is nilpotent of class 1, then $L' = 0$ and the result holds trivially. Suppose now that the result holds for all nilpotent diassociative algebras of class less than n. We note the following facts:

1. L/L^n is nilpotent of class $n-1$,
2. $L^n \subseteq Z(L)$,
3. $L' \subseteq Z_{n-1}(L)$,
4. $(L/L^n)' = L'/L^n$,
5. $Z(L)/L^n \subseteq Z(L/L^n)$.

Denote $A = (L/L^n)/Z(L/L^n)$ and $B = L/Z(L) = (L/L^n)/(Z(L)/L^n)$. By fact 5, A is a homomorphic image of B, and so $\dim(A/A') \le \dim(B/B')$. We thus have

$$\begin{aligned}\dim \mathscr{H}^2(L/L^n,\mathbb{F}) &\le \dim \mathscr{H}^2((L/L^n)/(L/L^n)',\mathbb{F}) + \dim((L/L^n)')[4\dim(A/A') - 1] \\ &\le \dim \mathscr{H}^2(L/L',\mathbb{F}) + \dim(L'/L^n)[4\dim(B/B') - 1]\end{aligned}$$

by induction, and

$$\dim \mathscr{H}^2(L,\mathbb{F}) \le \dim \mathscr{H}^2(L/L^n,\mathbb{F}) + 4\dim(L/Z_{n-1})\dim(L^n) - \dim(L^n)$$

by Corollary 7.2.4. Furthermore,

$$\dim(L/Z_{n-1}) \leq \dim(L/(L' + Z(L))) = \dim(B/B')$$

since $L' + Z(L) \subseteq Z_{n-1}$. Combining these inequalities, we compute

$$\begin{aligned}
\dim \mathscr{H}^2(L,\mathbb{F}) &\leq \dim \mathscr{H}^2(L/L^n,\mathbb{F}) + 4\dim(L/Z_{n-1})\dim(L^n) - \dim(L^n) \\
&\leq \dim \mathscr{H}^2(L/L',\mathbb{F}) + \dim(L'/L^n)[4\dim(B/B') - 1] \\
&\qquad + 4\dim(B/B')\dim(L^n) - \dim(L^n) \\
&= \dim \mathscr{H}^2(L/L',\mathbb{F}) + [\dim(L') - \dim(L^n)][4\dim(B/B') - 1] \\
&\qquad + 4\dim(B/B')\dim(L^n) - \dim(L^n) \\
&= \dim \mathscr{H}^2(L/L',\mathbb{F}) + \dim(L')[4\dim(B/B') - 1]
\end{aligned}$$

which yields the desired result. □

Noting that $\dim(B/B') \leq \dim(L/L')$, the next corollary is an immediate consequence of the previous one. What follows is an alternative way of writing our bound on $\dim \mathscr{H}^2(L,\mathbb{F})$ that is based on the dimensions of L and L/L'.

Corollary 7.2.6. $\dim \mathscr{H}^2(L,\mathbb{F}) \leq \dim \mathscr{H}^2(L/L',\mathbb{F}) + \dim(L')[4\dim(L/L') - 1]$.

Corollary 7.2.7. *Let $n = \dim L$ and $d = \dim(L/L')$. Then*

$$\dim \mathscr{H}^2(L,\mathbb{F}) \leq -2d^2 + d + 4nd - n.$$

Proof. We first note that, since L/L' is abelian, its multiplier $\mathscr{H}^2(L/L',\mathbb{F})$ has the maximal possible dimension of $2d^2$ (a bound obtained in the proof of Lemma 5.1.2). Using Corollary 7.2.6, we compute

$$\begin{aligned}
\dim \mathscr{H}^2(L,\mathbb{F}) &\leq 2d^2 + (n-d)[4d - 1] \\
&= 2d^2 + 4nd - n - 4d^2 + d
\end{aligned}$$

since $\dim(L/L') = \dim L - \dim(L')$ implies that $\dim(L') = n - d$. □

7.3 Associative Case

We now consider the special case of associative algebras, as any associative algebra L can be thought of as a diassociative algebra in which $x \dashv y = x \vdash y$. Indeed, this condition allows us to denote multiplication by xy without distinction, and the axioms of the diassociative structure condense down to $x(yz)=(xy)z$. Next, consider a pair of associative algebras A and B, and a central extension $0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0$ of A by B. A 2-cocycle $f \in \mathscr{Z}^2(B,A)$ is a bilinear form $f : B \times B \longrightarrow A$ that satisfies $f(i,jk)=f(ij,k)$ for all $i,j,k \in B$. Recall that any diassociative cocycle $(f_\dashv, f_\vdash)$ may be defined by a section $\mu : B \longrightarrow L_2$ of some equivalent extension. In particular, $f_\dashv(i,j)=\mu(i)\dashv\mu(j)-\mu(i\dashv j)$ and $f_\vdash(i,j)=\mu(i)\vdash\mu(j)-\mu(i\vdash j)$. In the associative case, one computes $f_\dashv(i,j)=f_\vdash(i,j)$, and we may thus think of our cocycle as a single bilinear form.

The five-term cohomological sequence is extended by $L/A \otimes B \oplus B \otimes L/A$ for the associative analogue of Theorem 7.1.1, which need only require that $f(A,B)=0$ and $f(B,A)=0$ for all cocycles $f \in \mathscr{Z}^2(L,\mathbb{F})$. Moreover, our δ map is defined by $\delta(f'+\mathscr{B}^2(L,\mathbb{F}))=(f'',g'')$, where $f'' : L/A \times B \longrightarrow \mathbb{F}$ and $g'' : B \times L/A \longrightarrow \mathbb{F}$. In the context of the diassociative to associative simplification, this pair would arise by computing equalities $f''_\dashv = f''_\vdash$ and $g''_\dashv = g''_\vdash$ via $f'_\dashv = f'_\vdash$. Similarly, our other results that extend the sequence by a term of the form $(X \otimes Y \oplus Y \otimes X)^2$ need only extend by the term $X \otimes Y \oplus Y \otimes X$ (as in the Leibniz sequences of Chapter 4). The associative analogue of Corollary 7.2.2 is thus the inequality

$$\dim \mathscr{H}^2(L,\mathbb{F})+1 \leq \dim \mathscr{H}^2(L/Z,\mathbb{F})+2\dim(L/L')$$

since $\dim(L/L' \otimes Z \oplus Z \otimes L/L') = 2\dim(L/L')$ in the case of $\dim Z = 1$. The associative analogue of Corollary 7.2.4 is

$$\dim \mathscr{H}^2(L,\mathbb{F}) \leq \mathscr{H}^2(L/L^n,\mathbb{F})+2\dim(L^n)\dim(L/Z_{n-1})-\dim(L^n),$$

that of Corollary 7.2.5 is

$$\dim \mathscr{H}^2(L,\mathbb{F}) \leq \dim \mathscr{H}^2(L/L',\mathbb{F})+\dim(L')[2\dim(L/Z(L))-2\dim((L/Z(L))')-1],$$

and that of Corollary 7.2.6 is

$$\dim \mathscr{H}^2(L,\mathbb{F}) \leq \dim \mathscr{H}^2(L/L',\mathbb{F})+\dim(L')[2\dim(L/L')-1].$$

For our last corollary, we obtain $\dim \mathscr{H}^2(L,\mathbb{F}) \leq -d^2+d+2nd-n$, where $n=\dim L$ and $d=\dim(L/L')$.

7.4 Example

We denote by $M_{\text{As}}(L)$ the multiplier of L as an associative algebra, and by $M_{\text{Dias}}(L)$ the same for L as a diassociative algebra. Recall that $M_*(X) = \mathscr{H}^2_*(X, \mathbb{F})$, where $*$ ranges over the categories As and Dias. As with the Leibniz multiplier, the dimension of the associative multiplier is bounded by n^2 for an algebra of dimension n (by the same logic used in Lemmas 2.0.2 and 2.0.3 of [16]). The dimension of the diassociative multiplier is bounded by $2n^2$ (see Chapter 5). These bounds are reached exactly when the algebra is abelian.

Example 7.4.1. Let L be the 2-dimensional associative algebra with basis $\{x_1, x\}$ and nonzero multiplication given solely by $x_1 x_1 = x$.

Associative Extension. We first compute the multiplier $M_{\text{As}}(L)$ of L as an associative algebra. Let K be the cover $M \oplus L$ of L with multiplications given by

$$\begin{aligned} x_1 x_1 &= x + m_{11}, \\ x_1 x &= m_{12}, \\ x x_1 &= m_{21}, \\ x x &= m_{22}. \end{aligned}$$

To simplify, we let $x_2 = x + m_{11} = x_1 x_1$, and thus multiplication in K becomes

$$\begin{aligned} x_1 x_1 &= x_2, \\ x_1 x_2 &= m_{12}, \\ x_2 x_1 &= m_{21}, \\ x_2 x_2 &= m_{22} \end{aligned}$$

where M is generated by m_{12}, m_{21}, m_{22} and K is generated by $m_{12}, m_{21}, m_{22}, x_1, x_2$. To find bases for our multiplier and cover, it remains to check linear relations between our generating elements. We note that any product of four or more elements in K is zero, and, in particular, that $m_{22} = x_2 x_2 = x_1 x_1 x_2 = x_1 m_{12} = 0$. It thus suffices to plug our x_1's into the associative identity. We compute

$$\begin{aligned}0 &= \mathrm{As}(x_1, x_1, x_1)\\ &= x_1(x_1x_1)-(x_1x_1)x_1\\ &= x_1x_2 - x_2x_1\\ &= m_{12}-m_{21}\end{aligned}$$

which implies that $m_{12} = m_{21}$. We let $m_{12} \neq 0$ to obtain the maximal possible dimension of our defining pair, and thus $\{m_{12}\}$ forms a basis for $M = M_{\mathrm{As}}(L)$. In other words, $\dim M_{\mathrm{As}}(L) = 1$.

We now verify that the inequality

$$\dim \mathscr{H}^2(L,\mathbb{F})+1 \le \dim \mathscr{H}^2(L/Z,\mathbb{F})+2\dim(L/L')$$

holds. Let $Z = \langle x\rangle$, noting that $Z \subseteq Z(L)\cap L'$ and $\dim Z = 1$. Since L/Z is abelian, we know that $\dim M_{\mathrm{As}}(L/Z) = \dim(L/Z)^2 = 1$. Moreover, $\dim(L/L') = 1$, and thus the inequality is computed as $1+1 \le 1+2(1)$, or $2 \le 3$. We can also check

$$\dim \mathscr{H}^2(L,\mathbb{F}) \le \dim \mathscr{H}^2(L/L',\mathbb{F})+2\dim(L')\dim(L/Z(L))-\dim(L')$$

for the associative analogue of Corollary 7.2.4, since L is nilpotent of class 2. Since $L' = Z(L) = \langle x\rangle$, we have $\dim M_{\mathrm{As}}(L/L') = (1)^2 = 1$ and $\dim(L') = \dim(L/Z(L)) = 1$. The inequality thus becomes $1 \le 1+2(1)(1)-1$, or $1 \le 2$.

Diassociative Extension. Our algebra L can be thought of as the diassociative algebra with basis $\{x_1, x\}$ and nonzero multiplications given solely by $x_1 \dashv x_1 = x = x_1 \vdash x_1$. Let K be the cover $M \oplus L$ of L with multiplications denoted by

$$\begin{aligned}
x_1 \dashv x_1 &= x + m_{11} & x_1 \vdash x_1 &= x + s_{11}\\
x_1 \dashv x &= m_{12} & x_1 \vdash x &= s_{12}\\
x \dashv x_1 &= m_{21} & x \vdash x_1 &= s_{21}\\
x \dashv x &= m_{22} & x \vdash x &= s_{22}.
\end{aligned}$$

Letting $x_2 = x_1 \dashv x_1 = x + m_{11}$, we obtain $x_1 \vdash x_1 = x + s_{11} = x_2 - m_{11} + s_{11} = x_2 + m$ for some

$m \in M$. Thus, multiplication in K is given by

$$x_1 \dashv x_1 = x_2 \qquad\qquad x_1 \vdash x_1 = x_2 + m$$
$$x_i \dashv x_j = m_{ij} \qquad\qquad x_i \vdash x_j = s_{ij}$$

for $(i,j) \neq (1,1)$. Now M is generated by $m, m_{12}, m_{21}, m_{22}, s_{12}, s_{21}, s_{22}$. As in the associative case, we need to verify linear relations in K based on the five axioms of diassociative algebras. Noting that any four-product is zero, we get $m_{22} = s_{22} = 0$, and it remains to plug x_1's into the diassociative identities. We compute

$$\begin{aligned} 0 &= \mathrm{As}_{\dashv}(x_1, x_1, x_1) \\ &= x_1 \dashv (x_1 \dashv x_1) - (x_1 \dashv x_1) \dashv x_1 \\ &= x_1 \dashv x_2 - x_2 \dashv x_1 \\ &= m_{12} - m_{21} \end{aligned}$$

and

$$\begin{aligned} 0 &= \mathrm{As}_{\vdash}(x_1, x_1, x_1) \\ &= x_1 \vdash (x_1 \vdash x_1) - (x_1 \vdash x_1) \vdash x_1 \\ &= x_1 \vdash (x_2 + m) - (x_2 + m) \vdash x_1 \\ &= s_{12} - s_{21} \end{aligned}$$

which yields $m_{12} = m_{21}$ and $s_{12} = s_{21}$. In a similar fashion, plugging x_1's into axioms D1 and D3 yields trivial equalities. Finally, axiom D2 yields $m_{21} = s_{12}$, and thus $\{m, m_{12}\}$ forms a maximal basis for M. Therefore, $\dim M_{\mathrm{Dias}}(L) = 2$, which is notably different from $M_{\mathrm{As}}(L)$.

To verify Corollary 7.2.2, let $Z = \langle x \rangle$, which is again 1-dimensional and falls in $Z(L) \cap L'$. Since L/Z is abelian, we have $\dim M_{\mathrm{Dias}}(L/Z) = 2\dim(L/Z)^2 = 2(1)^2 = 2$. Our inequality

$$\dim \mathscr{H}^2(L, \mathbb{F}) + 1 \leq \dim \mathscr{H}^2(L/Z, \mathbb{F}) + 4\dim(L/L')$$

is thus satisfied, with $2+1 \leq 2+4(1)$, or $3 \leq 6$. For Corollary 7.2.4, we want

$$\dim \mathscr{H}^2(L, \mathbb{F}) \leq \dim \mathscr{H}^2(L/L', \mathbb{F}) + 4\dim(L')\dim(L/Z(L)) - \dim(L')$$

since L is nilpotent of class 2. Since $L' = Z(L) = \langle x \rangle$, we have $\dim M_{\mathrm{Dias}}(L/L') = 2(1)^2 = 2$ and $\dim(L') = \dim(L/Z(L)) = 1$. The desired inequality is thus $2 \leq 2 + 4(1)(1) - 1$, or $2 \leq 5$.

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