

ABSTRACT

BAKER, MICHAEL AARON. Root Multiplicities of some Hyperbolic Kac-Moody Lie Algebras. (Under the direction of Kailash Misra).

Over 50 years ago, Victor Kac and Robert Moody introduced Kac-Moody Lie algebras as a natural generalization of semisimple Lie algebras that were completely classified already. These algebras have found many connections to phenomena in both mathematics and physics. The Kac-Moody algebras come in three types: finite, affine, and indefinite. Both finite and affine Kac-Moody algebras have had all root multiplicities calculated. The indefinite type has root multiplicities computed in some cases, but they are not completely known.

In this thesis, we have studied some root multiplicities for the hyperbolic Kac-Moody Lie algebras $\mathfrak{g} = HE_7^{(1)}, HE_8^{(1)}$. We realize these algebras as minimal graded Lie algebras whose local part is $V \oplus \mathfrak{gl}(n; \mathbb{C}) \oplus V'$ for suitably chosen $\mathfrak{gl}(n; \mathbb{C})$ -modules V and V' . This realization gives rise to a natural \mathbb{Z} -gradation $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_0 = \mathfrak{gl}(n; \mathbb{C})$, $\mathfrak{g}_{-1} = V$, and $\mathfrak{g}_1 = V'$. It is known that the multiplicity of root α is the same as $-\alpha$, so without loss of generality we focus on the multiplicity of negative roots. We say the negative root α is of degree $-j$ if the α -root space is contained in \mathfrak{g}_{-j} . Kang's multiplicity formula allows one to view the roots of \mathfrak{g} as some combination of weights in $\mathfrak{gl}(n; \mathbb{C})$ modules. Using this formula, we calculate the multiplicities of roots in \mathfrak{g} .

We determine the root multiplicities of all roots up to degree -7 in $HE_7^{(1)}$ and root multiplicities of all roots up to degree -8 and one special root of degree -9 for $HE_8^{(1)}$. This special root in $HE_8^{(1)}$ exceeds the proposed upper bound by Frenkel, which verifies the calculation done by Kac et al. (1988). Additionally, three of the roots of $HE_7^{(1)}$ have multiplicity that exceeds the proposed upper bound by Frenkel as well, which shows that Frenkel's conjecture does not hold for $HE_7^{(1)}$.

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Root Multiplicities of some Hyperbolic Kac-Moody Lie Algebras

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INTRODUCTION

Around 1870, Sophus Lie began his study of continuous transformation groups [O'Connor and Robertson (2000)]. This research would give rise to the objects known as Lie groups, smooth manifolds with a group structure. It was eventually shown that there is a correspondence between these groups and their tangent spaces near the identity, called the Lie algebra [Hall (2015)]. The Lie algebra, being a vector space with non-associative multiplication, is far easier to study than the whole Lie group. Thus, Lie algebras corresponding to Lie groups were studied in order to find out properties of the group.

One important class of Lie algebras is simple Lie algebras, which have no nontrivial ideals. Between 1888 and 1890, Wilhelm Killing offered the first classification of simple Lie algebras over the complex numbers, which separated them into four infinite groups A_n , B_n , C_n , D_n as well as five exceptions E_6 , E_7 , E_8 , F_4 , and G_2 [Humphreys (2000)]. In 1894, the Ph.D thesis of Élie Cartan both rigorously confirmed the classification made by Killing and extended it to the real numbers [Cartan (1894)]. The beauty and simplicity of the classification has made it one of the the most famous results in mathematics.

Over time, research began to focus on infinite dimensional Lie algebras. Considering the success that was achieved on the classification of simple Lie algebras, the search for a similar type of infinite dimensional Lie algebra was prioritized. Over 50 years later, Victor Kac and Robert Moody would independently discover Kac-Moody algebras, a generalization of semisimple Lie algebras which also contained a large class of infinite dimensional Lie algebras as well [Kac (1990)]. These Kac-Moody algebras, associated with generalized Cartan matrices $A = (a_{ij})_{i,j \in I}$, $I = \{1, 2, \dots, n\}$, fall into three types: finite, affine, and indefinite [Kac (1990)].

Let \mathfrak{g} be a Kac-Moody Lie algebra. For nonzero $\alpha \in Q$, define the root lattice of \mathfrak{g} as $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, h \in \mathfrak{h}\}$, where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} . If $\mathfrak{g}_\alpha \neq 0$, then α is a root and \mathfrak{g}_α is called the α root space whose dimension is the multiplicity of α denoted by $\text{mult}(\alpha)$ [Kac (1990)]. Roots can be classified as either real or imaginary, where roots of real type all have multiplicity equal to 1 (see Kac (1990)). The roots of finite Kac-Moody algebras are all real and so have multiplicity 1. The imaginary root multiplicities of affine Kac-Moody algebras are equal to the rank of the Generalized Cartan Matrix associated to that algebra [Kac (1990)]. The remaining category, indefinite Kac-Moody algebras, have imaginary root multiplicities which were studied only in specific cases. They have been calculated for $HA_1^{(1)}$ [Feingold and Frenkel (1983); Kang (1993b)], $HA_n^{(1)}$ [Kang (1994); Hontz and Misra

(1994)], $HC_n^{(1)}$ [Klima and Misra (2008)], $HD_n^{(1)}$ [Wilson (2012)], $HX_n^{(1)}$ for $X = A, B, C, D$ [Benkart et al. (1994)], $HG_2^{(1)}$ [Hansen (2016)], $HD_4^{(3)}$ [Erbacher (2012)], and $E_{10} = HE_8^{(1)}$ [Kac et al. (1988); Klima et al. (2014)]. For any indefinite type Kac-Moody Lie algebra, the root multiplicities are not known completely though.

In this thesis, we study the root multiplicities of the hyperbolic indefinite type Kac-Moody algebras $\mathfrak{g} = HE_7^{(1)}, HE_8^{(1)}$. Using a well-known construction [Benkart et al. (1993a)], we realize \mathfrak{g} as a minimal \mathbb{Z} -graded Lie algebra with local part $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where \mathfrak{g}_0 is the Lie algebra $\mathfrak{gl}(n; \mathbb{C})$ with suitable modules \mathfrak{g}_{-1} and \mathfrak{g}_1 . This realization of \mathfrak{g} allows us to use Kang's multiplicity formula [Benkart et al. (1993a)], which involves viewing the roots of \mathfrak{g} as combinations of weights in \mathfrak{g}_0 -modules. As the multiplicity of root α is the same as $-\alpha$, without loss of generality we can focus on determining the multiplicities of the negative roots.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable Generalized Cartan Matrix associated with \mathfrak{g} . Let $S \subset I$ and $\mathfrak{g}_S = \mathfrak{sl}(n; \mathbb{C})$ be the Kac-Moody Lie algebra with Cartan Matrix $A_S = (a_{ij})_{i,j \in S}$. Since $\mathfrak{gl}(n; \mathbb{C}) = \mathfrak{sl}(n; \mathbb{C}) \oplus \mathbb{C}I$, where the central element acts trivially, the representation theory of \mathfrak{g}_0 and \mathfrak{g}_S is the same and so we can look at combinations of weights of \mathfrak{g}_S modules.

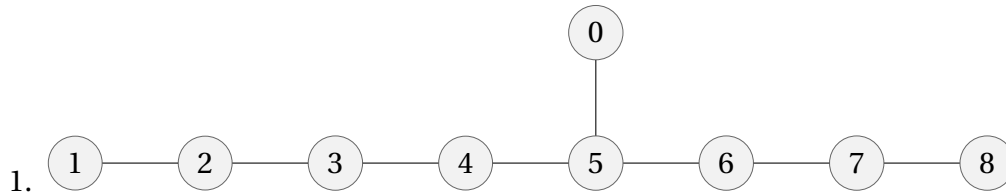
We say the degree of a root α is $-j$ if the α -root space is contained in \mathfrak{g}_{-j} . Since these modules depend on the gradation, the degree of a root depends on the choice of S . Root multiplicities in E_{10} of all roots up to degree -2 , for their choice of S , were determined in Kac et al. (1988). Root multiplicities in $E_{10} = HE_8^{(1)}$ were determined up to degree -5 in Klima et al. (2014), for their choice of S . In this thesis, we have determined root multiplicities for roots up to degree -8 for $HE_8^{(1)}$. Additionally, we have determined root multiplicities for roots up to degree -7 for $HE_7^{(1)}$.

As root multiplicities are still unknown completely for all indefinite type Kac-Moody algebras, finding an upper bound on the root multiplicities would be a natural first step towards completely determining them. One bound of interest is Frenkel's conjecture, which claims that $\text{mult}(\alpha) \leq p^{(\text{rank}-2)} \left(1 - \frac{|\alpha|}{2}\right)$ where p^k is the partition function in k colors. Kac et al. (1988) showed that Frenkel's conjecture fails for E_{10} in their degree -2 , which means it is not an upper bound for all indefinite Kac-Moody algebras. We calculate the multiplicity of the degree -9 root (which is degree -2 in Kac et al. (1988)) which violated Frenkel's conjecture, verifying their results. Additionally, we find three degree -5 roots in $HE_7^{(1)}$ which violate Frenkel's conjecture, showing that $HE_7^{(1)}$ is another indefinite Kac-Moody algebra for which Frenkel's conjecture does not hold.

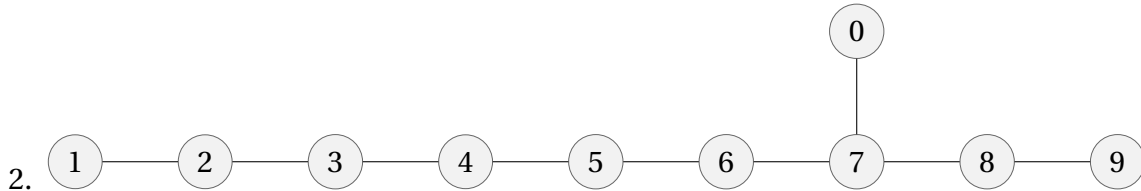
In this thesis, we begin with a review of the basic Kac-Moody Lie algebra theory in Chapter 1. We also discuss the construction of the algebras we are studying, as well as

Frenkel's conjecture. Chapter 2 reviews the representation theory of $\mathfrak{sl}(n; \mathbb{C})$, which is related to Kang's formula as mentioned earlier. In particular, we discuss weights and their multiplicities in the $\mathfrak{sl}(n; \mathbb{C})$ -modules which are related to Kostka numbers, a combinatorial object.

The last two chapters involve using Kang's multiplicity formula to calculate the root multiplicities for indefinite Kac-Moody Lie algebras 1. $HE_7^{(1)}$ and 2. $HE_8^{(1)}$ with the following Cartan matrices respectively:



(Note: $\mathfrak{g}_S = \mathfrak{sl}(9; \mathbb{C})$)



(Note: $\mathfrak{g}_S = \mathfrak{sl}(10; \mathbb{C})$)

In Chapter 3, we find all root multiplicities up to degree -7 in $HE_7^{(1)}$, which include three counterexamples to Frenkel's conjecture. In Chapter 4, we find all root multiplicities up to degree -8 in $HE_8^{(1)}$, as well as the multiplicity of the special degree -9 root which is the same counterexample that Kac et al. (1988) found.

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CHAPTER

1

SUMMARY OF KAC-MOODY ALGEBRAS AND THE FF-CONSTRUCTION

1.1 Kac-Moody Lie algebras

We begin with introducing the terminology of Kac-Moody Lie algebras. More details can be found in Humphreys (2000) and Kac (1990).

Definition 1.1.1 (Humphreys (2000)) *A vector space L over the field of complex numbers \mathbb{C} , with an operation $L \times L \rightarrow L$ denoted $(x, y) \rightarrow [x, y]$ (called ‘bracket’) is called a Lie algebra over \mathbb{C} if*

1. *The bracket operation is bilinear*
2. *$[x, x] = 0$ for all $x \in L$*
3. *The Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ holds for all $x, y, z \in L$*

Definition 1.1.2 (Kac (1990)) *A $n \times n$ integral matrix $A = (a_{ij})_{i,j \in I}$, where $I = 1, \dots, n$ is called a Generalized Cartan Matrix (GCM) if it satisfies the following:*

1. $a_{ii} = 2$ for all $i \in I$
2. a_{ij} are nonpositive integers when $i \neq j$
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$

Definition 1.1.3 (Kac (1990)) We say the matrix A is indecomposable if there is no partition of the set $\{1, 2, \dots, n\}$ into two nonempty subsets so that $a_{ij} = 0$ whenever i belongs to the first subset and j belongs to the second subset.

Definition 1.1.4 (Kac (1990)) We say the matrix A is symmetrizable if there exists an invertible diagonal matrix D such that DA is symmetric.

Definition 1.1.5 (Kac (1990)) A realization of A is a triple $(\mathfrak{h}, \pi, \check{\pi})$ where \mathfrak{h} is a complex vector space, $\pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, and $\check{\pi} = \{\check{\alpha}_1, \dots, \check{\alpha}_n\} = \{h_1, \dots, h_n\} \subset \mathfrak{h}$ satisfying

1. Both π and $\check{\pi}$ are linearly independent sets.
2. $\alpha_j(h_i) = \langle h_i, \alpha_j \rangle = a_{ij}$ for all i, j
3. $\dim(\mathfrak{h}) = 2n - \text{rank}(A)$

Definition 1.1.6 (Kac (1990)) Let A be a symmetrizable GCM and $(\mathfrak{h}, \pi, \check{\pi})$ be a realization of A . Then the Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra on generators e_i, f_i for $i \in I$ and \mathfrak{h} , with the following relations:

$$\begin{aligned}
[h, h'] &= 0 \text{ for } h, h' \in \mathfrak{h} \\
[h, f_j] &= -\langle h, \alpha_j \rangle f_j \text{ for } j \in I \\
[h, e_j] &= \langle h, \alpha_j \rangle e_j \text{ for } j \in I \\
[e_i, f_j] &= \delta_{i,j} h_i \text{ for } i, j \in I \\
(ad_{e_i})^{1-a_{ij}}(e_j) &= (ad_{f_i})^{1-a_{ij}}(f_j) = 0 \text{ for } i \neq j
\end{aligned}$$

Definition 1.1.7 (Kac (1990)) We define three classes of indecomposable GCMs associated with Kac-Moody Lie algebras.

1. A Kac-Moody Lie algebra of finite type has that $\det(A) \neq 0$. Thus, there exists $\mu > 0$ such that $A\mu > 0$; $Av \geq 0$ implies that $v > 0$ or $v = 0$.

2. A Kac-Moody Lie algebra of affine type has that $\det(A) = 0$ and the corank of A is 1. Thus, there exists a $\mu > 0$ such that $A\mu = 0$; $Av \geq 0$ implies that $Av = 0$.
3. A Kac-Moody Lie algebra of indefinite type is when there exists $\mu > 0$ such that $A\mu < 0$; $Av \geq 0, v \geq 0$ implies that $v = 0$.

Definition 1.1.8 (Kac (1990)) For each $i \in I$, the simple reflection r_i is an automorphism of \mathfrak{h}^* defined by $r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. The group W generated by $\{r_i \mid i \in I\}$ is called the Weyl group of $\mathfrak{g}(A)$.

Definition 1.1.9 (Kac (1990)) For $w \in W$, we say that $w = r_{i_1} r_{i_2} \dots r_{i_k}$ for $i_j \in I$ is reduced when k is minimal. The length of w , denoted $l(w)$, is defined as $l(w) = k$.

Definition 1.1.10 (Kac (1990)) We define the root lattice Q to be $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, and $Q_- = \bigoplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i$. We note that $Q_+ = -Q_-$.

Definition 1.1.11 (Kac (1990)) The Kac-Moody Lie algebra $\mathfrak{g}(A)$ has a root space decomposition $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } x \in \mathfrak{h}\}$$

is called the α -root space. An element $\alpha \in Q$ is called a root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The multiplicity of a root α is defined as $\text{mult}(\alpha) = \dim(\mathfrak{g}_\alpha)$.

Definition 1.1.12 (Kac (1990)) For a root $\alpha = \sum_{i \in I} k_i \alpha_i$, we define the height of α , denoted ht , to be $ht(\alpha) = \sum_{i \in I} k_i$. We define the principal gradation $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ by setting $\mathfrak{g}_j = \bigoplus_{\alpha: ht(\alpha)=j} \mathfrak{g}_\alpha$. Note that $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{-1} = \sum_{i \in I} \mathbb{C} f_i$, and $\mathfrak{g}_1 = \sum_{i \in I} \mathbb{C} e_i$.

Definition 1.1.13 (Kac (1990)) Let $\mathfrak{g}_\pm = \bigoplus_{j \geq 1} \mathfrak{g}_{\pm j}$. Then, the principal triangular decomposition of \mathfrak{g} is

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

Definition 1.1.14 (Kac (1990)) The set of roots of \mathfrak{g} is denoted as Δ , which can be written as $\Delta = \Delta^+ \sqcup \Delta^-$, where $\Delta^+ = \left\{ \sum_{i \in I} k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0} \right\}$ is called the set of positive roots and $\Delta^- = \left\{ \sum_{i \in I} k_i \alpha_i \mid k_i \in \mathbb{Z}_{\leq 0} \right\}$ is called the set of negative roots.

1.2 Integrable representations

Definition 1.2.1 (Kac (1990)) A \mathfrak{g} -module is \mathfrak{h} -diagonalizable if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \left\{ v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \right\}$$

is called the λ -weight space. If $V_\lambda \neq 0$, then λ is called a weight of V . The number $\text{mult}_V(\lambda) := \dim(V_\lambda)$ is called the multiplicity of λ in V .

Definition 1.2.2 (Kac (1990)) When all the weight spaces are finite-dimensional, we define the character of V to be

$$\text{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda) e^\lambda$$

where e^λ are basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ with the binary operation $e^\lambda e^\mu = e^{\lambda+\mu}$.

Definition 1.2.3 (Kac (1990)) A \mathfrak{g} -module V is called a highest-weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v \in V$ such that

1. $\mathfrak{g}_+ \cdot v = 0$
2. $h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}^*$
3. $U(\mathfrak{g}) \cdot v = V$, where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} .

The vector v is called the highest-weight vector.

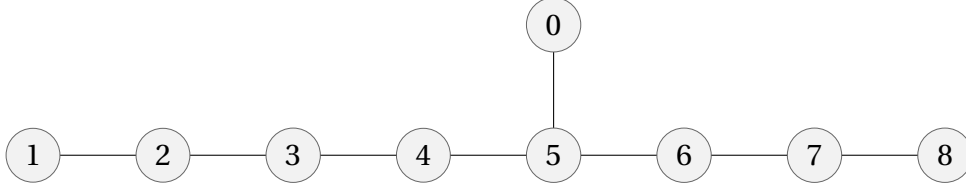
Definition 1.2.4 (Kac (1990)) An \mathfrak{h} -diagonalizable module over a Kac-Moody algebra $\mathfrak{g}(A)$ is called integrable if all e_i and f_i for all $i \in I$ are locally nilpotent on V .

Definition 1.2.5 (Kac (1990)) If $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i \in I$, then λ is called a dominant integral weight. If λ is dominant integral, then $V(\lambda)$ is integrable.

Definition 1.2.6 (Kac (1990)) A \mathfrak{g} -module V is called irreducible if it has only two \mathfrak{g} -submodules, itself and $\{0\}$. We say that V is completely reducible if V is a direct sum of irreducible \mathfrak{g} -submodules.

1.3 Construction of $HE_7^{(1)}$

The algebra $HE_7^{(1)}$ is the Kac-Moody Lie algebra associated with the Dynkin diagram



We follow the construction of the Lie algebra started by Feingold and Frenkel [Feingold and Frenkel (1983)] for some Lie algebras of hyperbolic type and expanded by Kang [Kang (1993a)] for use with his multiplicity formula. In honor of Feingold and Frenkel, we shall refer to this construction as the FF-construction henceforth. Consider $S \subset I$, where $S = \{0\}$ and $I = \{0, 1, \dots, 8\}$, and let $\mathfrak{g}_S = \mathfrak{g}(A_S)$ be the Kac-Moody Lie algebra associated to the Cartan Matrix $A_S = (a_{ij})_{i,j \in S}$. Denote the set of roots of \mathfrak{g}_S as Δ_S , the set of positive roots as Δ_S^+ , the set of negative roots as Δ_S^- , and the Weyl group of \mathfrak{g}_S as W_S . Define $\Delta^+(S) := \Delta^+ \setminus \Delta_S^+$ and $W(S) = \{w \in W \mid \Phi_w \subset \Delta^+(S)\}$ where $\Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in \Delta^-\}$.

For $\alpha \in Q$, define the generalized height of α with respect to S by $\text{ht}^S(\alpha) = \sum_{i \in I \setminus S} k_i$ and the degree of α as $\deg(\alpha) = \text{ht}^S(\alpha)$. Then we can define a \mathbb{Z} -gradation $\mathfrak{g}(A) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j^{(S)}$ induced by S (the S -gradation) by setting $\mathfrak{g}_j^{(S)} = \bigoplus_{\alpha: \text{ht}^S(\alpha)=j} \mathfrak{g}_\alpha$. Then, $\mathfrak{g}_0^{(S)} = \mathfrak{g}_S + \mathfrak{h}$ and all homogeneous subspaces $\mathfrak{g}_j^{(S)}$ are finite dimensional and so are completely reducible modules over $\mathfrak{g}_0^{(S)}$. If we let $\mathfrak{g}_+^{(S)} = \bigoplus_{j \geq 1} \mathfrak{g}_j^{(S)}$ and $\mathfrak{g}_-^{(S)} = \bigoplus_{j \geq 1} \mathfrak{g}_{-j}^{(S)}$, then we have a triangular decomposition $\mathfrak{g}(A) = \mathfrak{g}_-^{(S)} \oplus \mathfrak{g}_0^{(S)} \oplus \mathfrak{g}_+^{(S)}$.

In order to consider the subalgebra $\mathfrak{sl}(9; \mathbb{C}) \subset HE_7^{(1)}$, we choose our S -gradation sets as $I = \{0, 1, \dots, 8\}$ and $S = \{1, \dots, 8\}$. Thus, $A_S = (a_{ij})_{i,j \in S}$ is the Cartan Matrix of $\mathfrak{sl}(9; \mathbb{C})$. By $\mathfrak{sl}(n; \mathbb{C})$ representation theory, $\mathfrak{h} = \text{span}\{h_i = E_{i,i} - E_{i+1,i+1} \mid 1 \leq i \leq 8\}$, which means that $\mathfrak{g}_0^{(S)} = \mathfrak{sl}(9; \mathbb{C}) + \mathfrak{h} = \mathfrak{gl}(9; \mathbb{C})$. We can then define the maps $\epsilon_i(h) = i^{\text{th}}$ diagonal entry of $h \in \mathfrak{h}$, $\Lambda_i = \epsilon_1 + \dots + \epsilon_i$ for $i \in S$, and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in S$. The simple root $\alpha_0|_{\mathfrak{h}} = -\Lambda_5 = -\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5$, while the other simple roots remain as previously defined. Any root of $HE_7^{(1)}$ can be written in terms of the simple roots:

$$\alpha = \sum_{i=0}^8 k_i \alpha_i \text{ where } k_i \in \mathbb{Z}_{\geq 0} \text{ or } k_i \in \mathbb{Z}_{\leq 0}$$

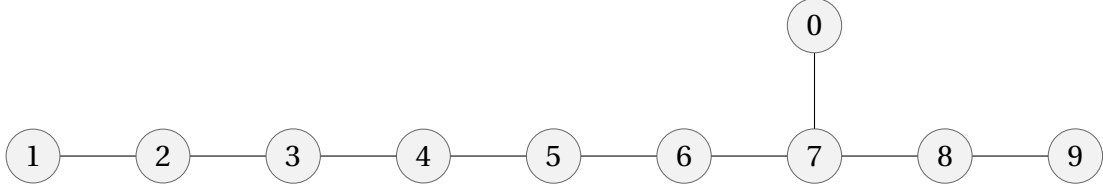
However, it is established that $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{-\alpha})$, so without loss of generality, we only

examine the negative roots

$$\alpha = -\sum_{i=0}^8 k_i \alpha_i \text{ where } k_i \in \mathbb{Z}_{\geq 0}$$

1.4 Construction of $HE_8^{(1)}$

The algebra $HE_8^{(1)}$ is the Kac-Moody Lie algebra associated with the Dynkin diagram



We consider another version of the FF-construction for $HE_8^{(1)}$. Consider $S \subset I$, where $S = \{0\}$ and $I = \{0, 1, \dots, 9\}$, and let $\mathfrak{g}_S = \mathfrak{g}(A_S)$ be the Kac-Moody Lie algebra associated to the Cartan Matrix $A_S = (a_{ij})_{i,j \in S}$. Denote the set of roots of \mathfrak{g}_S as Δ_S , the set of positive roots as Δ_S^+ , the set of negative roots as Δ_S^- , and the Weyl group of \mathfrak{g}_S as W_S . Define $\Delta^+(S) := \Delta^+ \setminus \Delta_S^+$ and $W(S) = \{w \in W \mid \Phi_w \subset \Delta^+(S)\}$ where $\Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in \Delta^-\}$.

For $\alpha \in Q$, define the generalized height of α with respect to S by $\text{ht}^S(\alpha) = \sum_{i \in I \setminus S} k_i$ and the degree of α as $\deg(\alpha) = \text{ht}^S(\alpha)$. Then we can define a \mathbb{Z} -gradation $\mathfrak{g}(A) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j^{(S)}$ induced by S (the S -gradation) by setting $\mathfrak{g}_j^{(S)} = \bigoplus_{\alpha: \text{ht}^S(\alpha)=j} \mathfrak{g}_\alpha$. Then, $\mathfrak{g}_0^{(S)} = \mathfrak{g}_S + \mathfrak{h}$ and all homogeneous subspaces $\mathfrak{g}_j^{(S)}$ are finite dimensional and so are completely reducible modules over $\mathfrak{g}_0^{(S)}$. If we let $\mathfrak{g}_+^{(S)} = \bigoplus_{j \geq 1} \mathfrak{g}_j^{(S)}$ and $\mathfrak{g}_-^{(S)} = \bigoplus_{j \geq 1} \mathfrak{g}_{-j}^{(S)}$, then we have a triangular decomposition $\mathfrak{g}(A) = \mathfrak{g}_-^{(S)} \oplus \mathfrak{g}_0^{(S)} \oplus \mathfrak{g}_+^{(S)}$.

In order to consider the subalgebra $\mathfrak{sl}(10; \mathbb{C}) \subset HE_8^{(1)}$, we choose our S -gradation sets as $I = \{0, 1, \dots, 9\}$ and $S = \{1, \dots, 9\}$. Thus, $A_S = (a_{ij})_{i,j \in S}$ is the Cartan Matrix of $\mathfrak{sl}(10; \mathbb{C})$. By $\mathfrak{sl}(n; \mathbb{C})$ representation theory, $\mathfrak{h} = \text{span}\{h_i = E_{i,i} - E_{i+1,i+1} \mid 1 \leq i \leq 9\}$. We can then define the maps $\epsilon_i(h) = i^{\text{th}}$ diagonal entry of $h \in \mathfrak{h}$, $\Lambda_i = \epsilon_1 + \dots + \epsilon_i$ for $i \in S$, and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in S$. The simple root $\alpha_0|_{\mathfrak{h}} = -\Lambda_7 = -\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7$, while the other simple roots remain as previously defined. Any root of $HE_8^{(1)}$ can be written in terms of the simple roots:

$$\alpha = \sum_{i=0}^9 k_i \alpha_i \text{ where } k_i \in \mathbb{Z}_{\geq 0} \text{ or } k_i \in \mathbb{Z}_{\leq 0}$$

However, it is established that $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{-\alpha})$, so without loss of generality, we only

examine the negative roots

$$\alpha = -\sum_{i=0}^9 k_i \alpha_i \text{ where } k_i \in \mathbb{Z}_{\geq 0}$$

Klima et al. (2014) used a method similar to ours to compute the multiplicities of dominant $HE_8^{(1)}$ roots up to degree -5 in this labeling. Our results match what they had found up to degree -5 .

1.5 Kang's Multiplicity Formula

Let \mathbb{C} be the trivial \mathfrak{g} -module. The homology modules $H_k(\mathfrak{g}_-^{(S)}) = H_k(\mathfrak{g}_-^{(S)}, \mathbb{C})$ are obtained from the $\mathfrak{g}_0^{(S)}$ -module complex

$$\dots \longrightarrow \Lambda^k(\mathfrak{g}_-^{(S)}) \xrightarrow{d_k} \Lambda^{k-1}(\mathfrak{g}_-^{(S)}) \dashrightarrow \Lambda^1(\mathfrak{g}_-^{(S)}) \xrightarrow{d_1} \Lambda^0(\mathfrak{g}_-^{(S)}) \xrightarrow{d_0} \mathbb{C} \longrightarrow 0$$

with the differentials d_k defined by $d_k : \Lambda^k(\mathfrak{g}_-^{(S)}) \rightarrow \Lambda^{k-1}(\mathfrak{g}_-^{(S)})$ defined by

$$d_k(x_1 \wedge \dots \wedge x_k) = \sum_{s < t} (-1)^{s+t} [x_s, x_t] \wedge x_1 \wedge \dots \wedge \hat{x}_s \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_k$$

for $k \geq 2$, $x_i \in \mathfrak{g}_-^{(S)}$, and $d_1 = d_0 = 0$ where $H_k(\mathfrak{g}_-^{(S)}) = \ker(d_k)/\text{Im}(d_{k+1})$. The terms \hat{x}_j represent that x_j is omitted from the wedge product. From the \mathbb{Z} -gradation of $\mathfrak{g}_-^{(S)}$, we have a \mathbb{Z} -gradation on $\Lambda^k(\mathfrak{g}_-^{(S)})$. For $j \geq 0$ and $x_i \in \mathfrak{g}_-^{(S)}$, we define $\Lambda^k(\mathfrak{g}_-^{(S)})_j$ to be the subspace of $\Lambda^k(\mathfrak{g}_-^{(S)})$ spanned by $x_1 \wedge \dots \wedge x_k$ such that $\deg(x_1) + \dots + \deg(x_k) = -j$. From the \mathbb{Z} -gradation on $\Lambda^k(\mathfrak{g}_-^{(S)})$, we have a \mathbb{Z} -gradation on $H_k(\mathfrak{g}_-^{(S)})$. From the definition of $\Lambda^k(\mathfrak{g}_-^{(S)})$, it is immediate that $\Lambda^k(\mathfrak{g}_-^{(S)})_{-j} = H_k(\mathfrak{g}_-^{(S)})_{-j} = 0$ for $k > j$. The $\mathfrak{g}_0^{(S)}$ -structure of the homology modules is made apparent by Kostant's formula.

Theorem 1.5.1 (Kostant's formula)

$$H_k(\mathfrak{g}_-^{(S)}) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=k}} V_S(w\rho - \rho)$$

where $V_S(\lambda)$ is the integrable highest-weight $\mathfrak{g}_0^{(S)}$ -module with highest weight λ . The weight $\rho \in \mathfrak{h}^*$ is defined by the property $\langle \rho, \alpha_i \rangle = \alpha_i(\rho) = 1$ for all $i \in I$.

Kostant proved this formula for finite-dimensional simple Lie algebras [Kostant (1959)]. It was first extended to symmetrizable Kac-Moody Lie algebras where A_S is of finite type by

Garland and Lepowsky [Garland and Lepowsky (1976)] and eventually to symmetrizable Kac-Moody Lie algebras where A_S is of arbitrary type [Liu (1992)].

Now, we proceed with applying Kostant's formula to the LL-construction when $\mathfrak{g}_S \cong \mathfrak{sl}(n)$, a finite-dimensional simple Lie algebra. In this case, $\dim((\mathfrak{g}_0^{(S)})_\alpha)$ is known for all $\alpha \in \Delta_S$ and $\dim(V_S(w\rho - \rho)_\tau)$ is known for all $w \in W(S)$ and $\tau \in \mathfrak{h}^*$. Then, the Euler-Poincare principle to our $\mathfrak{g}_0^{(S)}$ -module complex yields

$$\sum_{k=0}^{\infty} (-1)^k \text{ch } \Lambda^k(\mathfrak{g}_-^{(S)}) = \sum_{k=0}^{\infty} (-1)^k \text{ch } H_k(\mathfrak{g}_-^{(S)})$$

which expands to

$$1 - \text{ch } \mathfrak{g}_-^{(S)} + \sum_{k=2}^{\infty} (-1)^k \text{ch } \Lambda^k(\mathfrak{g}_-^{(S)}) = \text{ch } H_0(\mathfrak{g}_-^{(S)}) - \text{ch } H_1(\mathfrak{g}_-^{(S)}) + \sum_{k=2}^{\infty} (-1)^k \text{ch } H_k(\mathfrak{g}_-^{(S)})$$

Recall that $H_0(\mathfrak{g}_-^{(S)}) \cong \mathbb{C}$. Additionally,

$$H_1(\mathfrak{g}_-^{(S)}) \cong \bigoplus_{i \in I \setminus S} V_S(-\alpha_i) = \mathfrak{g}_{-1}^{(S)}$$

and, for $k > j$,

$$\text{ch } \Lambda^k(\mathfrak{g}_-^{(S)})_{-j} = \text{ch } H_k(\mathfrak{g}_-^{(S)})_{-j} = 0.$$

On the left-hand side, we have that

$$\text{ch } \Lambda^k(\mathfrak{g}_-^{(S)})_{-j} = \sum_{\substack{n_1 < \dots < n_r \\ k_1 + \dots + k_r = k \\ k_1 n_1 + \dots + k_r n_r = j}} \text{ch } \Lambda^{k_1}(\mathfrak{g}_{-n_1}^{(S)}) \dots \text{ch } \Lambda^{k_r}(\mathfrak{g}_{-n_r}^{(S)}).$$

Meanwhile, on the right-hand side, we can use Kostant's formula to show that

$$\text{ch } H_k(\mathfrak{g}_-^{(S)})_{-j} = \text{ch} \left(\sum_{\substack{w \in W(S) \\ l(w)=k}} V_S(w\rho - \rho) \right)_{-j} = \sum_{\substack{w \in W(S) \\ l(w)=k \\ \deg(w\rho - \rho) = -j}} \text{ch } V_S(w\rho - \rho).$$

Combining these yields a recursive formula for the $\mathfrak{g}_0^{(S)}$ -character of $\mathfrak{g}_{-j}^{(S)}$:

$$\begin{aligned} \text{ch } \mathfrak{g}_{-j}^{(S)} &= \sum_{k=2}^j (-1)^k \sum_{\substack{n_1 < \dots < n_r \\ k_1 + \dots + k_r = k \\ k_1 n_1 + \dots + k_r n_r = j}} \text{ch } \Lambda^{k_1}(\mathfrak{g}_{-n_1}^{(S)}) \dots \text{ch } \Lambda^{k_r}(\mathfrak{g}_{-n_r}^{(S)}) \\ &\quad - \sum_{k=2}^j (-1)^k \sum_{\substack{w \in W(S) \\ l(w) = k \\ \deg(w\rho - \rho) = -j}} \text{ch } V_S(w\rho - \rho) \end{aligned}$$

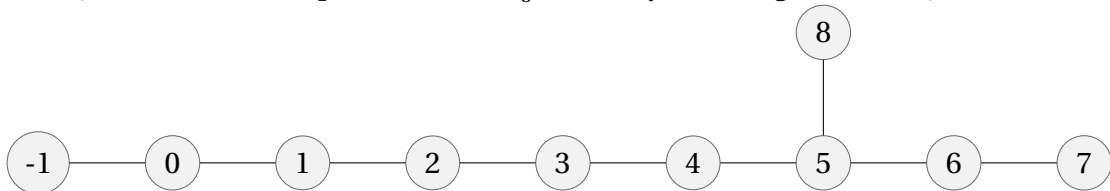
Suppose α is a negative root of degree $-j$, where $j \geq 2$. By matching coefficients in the expansions of the characters, we get

$$\begin{aligned} \dim \mathfrak{g}_\alpha &= \sum_{k=2}^j (-1)^k \sum_{\substack{n_1 < \dots < n_r \\ k_1 + \dots + k_r = k \\ k_1 n_1 + \dots + k_r n_r = j}} \dim \Lambda^{k_1}(\mathfrak{g}_{-n_1}^{(S)}) \dots \dim \Lambda^{k_r}(\mathfrak{g}_{-n_r}^{(S)}) \\ &\quad - \sum_{k=2}^j (-1)^k \sum_{\substack{w \in W(S) \\ l(w) = k \\ \deg(w\rho - \rho) = -j}} \dim V_S(w\rho - \rho), \end{aligned}$$

which we call Kang's multiplicity formula [Benkart et al. (1993a)].

1.6 Kac, Moody, and Wakimoto's Study of E_{10}

Kac et al. (1988) took a different approach to finding the multiplicities of roots in an algebra called E_{10} , which is isomorphic to our $HE_8^{(1)}$. The Dynkin diagram for E_{10} is



They found a function that generated all of the multiplicities for a certain level (degree). Define the function $\phi(q)$ by

$$\phi(q) := \prod_{j=1}^{\infty} (1 - q^j)$$

and the functions $p^{(8)}(n)$ and $\xi(n)$ by

$$\frac{1}{\phi(q)^8} = \sum_{n \geq 0} p^{(8)}(n) q^n$$

$$\frac{1}{\phi(q)^8} \left[1 - \frac{\phi(q^2)}{\phi(q^4)} \right] = \sum_{n \geq 0} \xi(n) q^n$$

For $\alpha \in \Delta$, we have that

$$\text{mult}(\alpha) = \dim(\mathfrak{g}^\alpha) = \begin{cases} p^{(8)}\left(1 - \frac{(\alpha|\alpha)}{2}\right) & \text{if } \alpha \text{ is of level 0 or 1} \\ \xi\left(3 - \frac{(\alpha|\alpha)}{2}\right) & \text{if } \alpha \text{ is of level 2} \end{cases}$$

As the functions $p^{(8)}(n)$ and $\xi(n)$ are defined via power series expansion, this gives an extremely powerful way of determining root multiplicities for E_{10} when they are level 0, 1, or 2 in this labeling. Additionally, they were able to find a root of level 2, α , that disproves Frenkel's conjecture

$$\dim(\mathfrak{g}^\alpha) \leq p^{(\text{rank}-2)}\left(1 - \frac{(\alpha|\alpha)}{2}\right)$$

However, the weakness lies in the roots that are missed, who are of higher level. Nevertheless, by converting any results obtained using our methodology to the labeling of Kac (1968) gives a way to check some of the root multiplicities we get for $HE_8^{(1)}$.

CHAPTER

2

REPRESENTATION THEORY OF $\mathfrak{sl}(n; \mathbb{C})$

2.1 Construction of $\mathfrak{sl}(n; \mathbb{C})$

A representation of a Lie algebra \mathfrak{g} on a vector space V gives a tool to associate each element in \mathfrak{g} with a linear transformation on V . In this case, a natural choice of bracket is the commutator bracket defined as $[A, B] = AB - BA$ for linear transformations A and B . Since these are linear transformations between finite spaces, we can represent them with matrices and so build Lie algebras whose elements are matrices. We define $\mathfrak{gl}(n; \mathbb{C})$ as the set of $n \times n$ matrices over \mathbb{C} . This is a very rich algebra consisting of many interesting subalgebras, so we take inspiration from group theory and focus on the “simple” subalgebras that have no nontrivial subalgebras. As mentioned in the Introduction, one such class is $A_{n-1} = \mathfrak{sl}(n; \mathbb{C})$, which are the trace zero matrices in $\mathfrak{gl}(n; \mathbb{C})$. As a Kac-Moody Lie algebra, there is an associated Cartan matrix $A = (a_{ij})_{i,j=1}^{n-1}$ where

$$a_{ij} = \begin{cases} 2, & i = j \\ -1, & |i - j| = 1 \\ 0, & \text{else} \end{cases}$$

We consider the Cartan subalgebra $\mathfrak{h} = \text{span}\{h_i = E_{i,i} - E_{i+1,i+1} \mid 1 \leq i \leq n-1\}$ contained in $\mathfrak{sl}(n; \mathbb{C})$, where $E_{i,j} = (\delta_{i,j})_{ij}$ are the standard basis elements of the space of square matrices and $\delta_{i,j}$ is the Kronecker delta. Define $\epsilon_i(h) = i^{\text{th}}$ diagonal entry of h for $h \in \mathfrak{h}$ and $i = 1, \dots, n-1$. The set of simple roots $\alpha_i \in \mathfrak{h}^*$ can be defined as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, n-1$. Notice that $\alpha_i(h_j) = a_{ji}$ by these definitions. Additionally, $\mathfrak{sl}(n; \mathbb{C})$ has S_n as its Weyl group, which is generated by the elements $r_i = (i, i+1)$ for $i = 1, \dots, n-1$, also called the simple reflections.

2.2 Weight Multiplicities

Denote the fundamental weights of $\mathfrak{sl}(n; \mathbb{C})$ as $\Lambda_i = \epsilon_1 + \dots + \epsilon_i$ for $i = 1, \dots, n-1$. Let $P = \bigoplus \mathbb{Z}\Lambda_i$ be the weight lattice and $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0\}$ be the set of dominant weights. For $\lambda \in P^+$, we have $\lambda = \sum_{i=1}^k \lambda_i \epsilon_i$ where $k \leq n$ and $\lambda_1 \geq \dots \geq \lambda_k > 0$ is a partition of $|\lambda| = \sum \lambda_i = m$. This means that the coefficients of dominant weights written in terms of ϵ_i can be seen as partitions, so there is a correspondence between dominant weights and partitions. The dominant weight $\mu = \mu_1 \epsilon_1 + \dots + \mu_k \epsilon_k$ is associated with a partition $\mu = \{\mu_1 \geq \dots \geq \mu_k > 0\}$. For the other direction, we can partition $|\lambda| = m$ with the dominant weight μ , $\mu \vdash |\lambda|$, if $m = \sum_{i=1}^k \mu_i$. The number of nonzero coefficients of the weight was assumed to be k , so the length of the partition $l(\mu) = k$.

As a module for $\mathfrak{g}^e = \mathfrak{g} \oplus \mathbb{C}I = \mathfrak{sl}(n; \mathbb{C}) \oplus \mathbb{C}I = \mathfrak{gl}(n; \mathbb{C})$, $V = V(\lambda)$ has that the central element I acts like the scalar 0. Since $\mathfrak{gl}(n; \mathbb{C}) = \mathfrak{sl}(n; \mathbb{C}) \oplus \mathbb{C}I$, where I acts like 0, the representation theories of $\mathfrak{sl}(n; \mathbb{C})$ and $\mathfrak{gl}(n; \mathbb{C})$ are the same and so we can examine the highest weight modules of $\mathfrak{sl}(n; \mathbb{C})$.

Now, we seek a description of the weights in each irreducible module $V(\lambda)$, along with the multiplicities of these weights. This is given in terms of the dominance order on partitions, which we define here. Let $\mu = \{\mu_1 \geq \dots \geq \mu_{l(\mu)} > 0\}$ be a partition and $P_i = \mu_1 + \dots + \mu_i$ be the i^{th} partial sum, where $P_i(\mu) = P_{l(\mu)}(\mu)$ for $i \geq l(\mu)$. Then, the dominance order on partitions, denoted $\lambda \geq \mu$, is defined as $P_i(\lambda) \geq P_i(\mu)$ for all i , and we say that λ dominates μ . Then, Benkart et al. (1993b) provides us with the following theorem:

Theorem 2.2.1 (Benkart et al. (1993b)) *When \mathfrak{g} is a simple Lie algebra of type A_{n-1} , the set of dominant weights of the \mathfrak{g}^e -irreducible module $V(\lambda)$ is $\{\mu \mid \mu \vdash |\lambda|, \lambda \geq \mu, \text{ and } l(\mu) \leq n\}$ where $\mu = \mu_1 \epsilon_1 + \dots + \mu_n \epsilon_n$ is the weight corresponding to partition $\mu = \{\mu_1 \geq \dots \geq \mu_{l(\mu)} > 0\}$.*

Recall that any dominant weight μ of $V(\lambda)$ can be acted upon by a Weyl group element $\omega \in S_n$ to give any other weight $\omega\mu$ of $V(\lambda)$. As the Weyl group is S_n , we obtain this action

by permutations of the coefficients of $\mu = \sum k_j \epsilon_j$, where the default decreasing order of k_j would be the dominant weight. Also, $\text{mult}(\mu) = \dim(V_\mu) = \dim(V_{\omega\mu}) = \text{mult}(\omega\mu)$, so we need only determine the dimension of dominant weight spaces of $V(\lambda)$, which is given by this theorem in Benkart et al. (1993b):

Theorem 2.2.2 (Benkart et al. (1993b)) *Let $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{C})$ and suppose that μ is a dominant weight of $V(\lambda)$. Then $\mu \vdash |\lambda|$ and the $\dim(V(\lambda)_\mu) = K_{\lambda, \mu}$, where $K_{\lambda, \mu}$ is the Kostka number, the number of column-strict tableaux of weight μ and shape λ .*

2.3 Kostka numbers

In order to determine the multiplicities of these dominant weights of A_{n-1} , we need to be able to compute these Kostka numbers. Associated with the partition $\lambda = \{\lambda_1 \geq \dots \geq \lambda_{l(\lambda)} > 0\}$ is its Ferrers diagram or Young frame having λ_i left-justified boxes in the i^{th} row for $i = 1, \dots, l(\lambda)$. Let us consider an example for a partition 7 called λ . For this partition $\lambda = \{5 \geq 2 > 0\} \vdash 7$, the frame of λ would be

$$F(\lambda) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Let us assume now that we are interested in computing the Kostka number $K_{\{5,2\}, \{3,2^2\}}$. The $\{5,2\}$ means we are dealing with the same frame as above, and the $\{3,2^2\}$ means that we are placing three 1s, two 2s, and two 3s into these boxes. Since the tableaux is semistandard, we require all placements of the 1s, 2s, and 3s which obey the following rules:

1. the numbers are non-decreasing down each row from left to right
2. the numbers are strictly increasing down each column from top to bottom

Under these considerations, there are only three such tableaux:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & & & \\ \hline \end{array}$$

which means that $K_{\{5,2\}, \{3,2^2\}} = 3$.

CHAPTER

3

$$H E_7^{(1)}$$

3.1 Roots of Degree -1

By our construction of $H E_7^{(1)}$, we know that $\mathfrak{g}_{-1}^{(S)} = V_S(-\alpha_0) = V_S(\Lambda_5) = V_S(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5)$. By $\mathfrak{sl}(9; \mathbb{C})$ representation theory, we know that the dominant weights of this module are the weights under it in the dominance order. In other words, we need weights $\mu = \mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \mu_3 \epsilon_3 + \mu_4 \epsilon_4 + \mu_5 \epsilon_5$ for which $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 \geq \mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \mu_3 \epsilon_3 + \mu_4 \epsilon_4 + \mu_5 \epsilon_5$ in the dominance order. One can then see that the only dominant weight satisfying this condition is $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5$ associated with the partition $\{1^5\}$. By Lemma 1.5.2, we have that $\text{mult}(\{1^5\}) = K_{\{1^5\}, \{1^5\}} = 1$, where the last equality follows immediately from the fact that there is only one semistandard Young tableaux of the same height and weight.

3.2 Roots of Degree -2

Recall that $W(S) = \{w \in W \mid \Phi_w \subset \Delta^+(S)\}$. By our Lemma 1.5.1, we know that we can get the elements of $W(S)$ of length 2 from those elements of length 1, r_0 . Then, we see with a little

calculation that, for $i \neq 0$,

$$\begin{aligned} r_0(\alpha_i) &= \alpha_i - \alpha_i(h_0)\alpha_0 \\ &= \begin{cases} \alpha_i, & i \neq 5 \\ \alpha_0 + \alpha_5, & i = 5 \end{cases} \end{aligned}$$

This means that the only element where $l(w) = 2$ in $W(S)$ is $w = r_0 r_5$. Now,

$$\begin{aligned} r_0 r_5 \rho - \rho &= r_0(\rho - \alpha_5) - \rho \\ &= r_0 \rho - r_0 \alpha_5 - \rho \\ &= \rho - \alpha_0 - (\alpha_5 - \alpha_5(h_0)\alpha_0) - \rho \\ &= -2\alpha_0 - \alpha_5 \\ &= 2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) - (\epsilon_5 - \epsilon_6) \\ &= \{2^4, 1^2\} \end{aligned}$$

which means that $\deg(r_0 r_5 \rho - \rho) = -2$ and so we can reduce our formula for the multiplicity in degree -2 to the following:

$$\begin{aligned} \text{mult}(\alpha) &= X_2 - \sum_{\substack{w \in W(S) \\ l(w)=2 \\ \deg(w\rho-\rho)=-2}} K_{w\rho-\rho, \alpha} \\ &= X_2 - K_{\{2^4, 1^2\}, \alpha} \end{aligned}$$

Now we can use Kang's multiplicity formula to determine the root multiplicities of all dominant roots of degree -2 , which will give us all root multiplicities of degree -2 via the Weyl group action. Therefore, we need all μ such that $\mu = (2 - k_1)\epsilon_1 + (2 + k_1 - k_2)\epsilon_2 + (2 + k_2 - k_3)\epsilon_3 + (2 + k_3 - k_4)\epsilon_4 + (2 + k_4 - k_5)\epsilon_5 + (k_5 - k_6)\epsilon_6 + (k_6 - k_7)\epsilon_7 + (k_7 - k_8)\epsilon_8 + k_8\epsilon_9$ where $2 - k_1 \geq 2 + k_1 - k_2 \geq 2 + k_2 - k_3 \geq 2 + k_3 - k_4 \geq 2 + k_4 - k_5 \geq k_5 - k_6 \geq k_6 - k_7 \geq k_7 - k_8 \geq k_8$, which corresponds to all partitions of 10 who have their largest summand not exceed 2 and with 9 or fewer summands. These partitions can be listed: $\{2, 1^8\}, \{2^2, 1^6\}, \{2^3, 1^4\}, \{2^4, 1^2\}, \{2^5\}$. The table below lists the dominant roots and important pieces of Kang's multiplicity formula used to determine the multiplicities. The full set of roots can be obtained from permutations

of the coefficients of ϵ_i for each dominant root.

Table 3.1: Degree -2 Dominant Root Multiplicities for $HE_7^{(1)}$

α	X_2	$K_{\{2^4, 1^2\}}$	$\text{mult}(\alpha)$
$\{2^2, 1^6\}$	10	9	1
$\{2, 1^8\}$	35	28	7

Example 3.2.1 Show that the degree -2 dominant root $\alpha = \{2^2, 1^6\}$ has multiplicity 1.

We recall the multiplicity formula for a degree -2 root in $HE_7^{(1)}$ is $\text{mult}(\alpha) = X_2 - K_{\{2^4, 1^2\}, \{2^2, 1^6\}}$. To find X_2 , we need to first find all pairs of permutations of degree -1 dominant roots which sum to α . In other words, all pairs of permutations of $\{1^5\}$ which sum to $\{2, 1^8\}$.

α	2	2	1	1	1	1	1	1	0
ϵ_1	1	1	1	1	1	0	0	0	0
ϵ_2	1	1	0	0	0	1	1	1	0

Notice that the first two columns of the sum must contain 1 and the last column can only contain 0. Additionally, there are only 3 ones left to place for each row and each column can only contain one 1. However, recall that the collection of all roots are permutations of the dominant roots, so we can preserve all columns summing to 1 with a permutation. There are $\frac{6!}{3!3!} = 20$ ways that these columns can be arranged. However, this overcounts the true value because we require that $\epsilon_1 > \epsilon_2$ in order to have no sums repeated, so we must divide by 2 to take out all pairs of permutations of ϵ_1 and ϵ_2 that repeat. Hence, $X_2 = \frac{20}{2} = 10$.

Now, we only need find $K_{\{2^4, 1^2\}, \{2^2, 1^6\}}$, which is the number of ways to fit two 1s, two 2s, one 3, one 4, one 5, one 6, one 7, and one 8 in the following frame:

$$F(\lambda) =$$

tableaux:	
-----------	--

1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	4	3	4	3	6	3	6	3	7	3	5	3	5
5	6	5	8	5	7	4	7	4	8	4	6	4	8
7		6		6		5		5		7		6	
8		7		8		8		7		6		8	

This means our Kostka number $K_{\{2^4, 1^2\}, \{2^2, 1^6\}} = 9$. Thus, we have

$$\text{mult}(\alpha) = X_2 - K_{\{2^4, 1^2\}, \{2^2, 1^6\}} = 10 - 9 = 1$$

Example 3.2.2 Show that the degree -2 dominant root $\alpha = \{2, 1^8\}$ has multiplicity 7.

The multiplicity formula for a degree -2 root in $HE_7^{(1)}$ is $\text{mult}(\alpha) = X_2 - K_{\{2^4, 1^2\}, \{2^2, 1^6\}}$. To find X_2 , we need to first find all pairs of permutations of degree -1 dominant roots which sum to α . In other words, all pairs of permutations of $\{1^5\}$ which sum to $\{2, 1^8\}$.

α	2	1	1	1	1	1	1	1	1
ϵ_1	1	1	1	1	1	0	0	0	0
ϵ_2	1	0	0	0	0	1	1	1	1

Notice that the first column of the sum must contain 1 and the last column can only contain 0. Additionally, there are only 4 ones left to place for each row and each column can only contain one 1. However, recall that the collection of all roots are permutations of the dominant roots, so we can preserve all columns summing to 1 with a permutation. There

are $\frac{8!}{4!4!} = 70$ ways that these columns can be arranged. However, this overcounts the true value because we require that $\epsilon_1 > \epsilon_2$ in order to have no sums repeated, so we must divide by 2 to take out all pairs of permutations of ϵ_1 and ϵ_2 that repeat. Hence, $X_2 = \frac{70}{2} = 35$.

Now, we only need find $K_{\{2^4, 1^2\}, \{2, 1^8\}}$, which is the number of ways to fit two 1s, one 2, one 3, one 4, one 5, one 6, one 7, one 8, and one 9 in the following frame:

$$F(\lambda) =$$

The top two boxes are forced to contain the two 1s and the second row, first column box has to contain 2. The 3 can be placed on either the third row, first column or the second row, second column. In the first case, only 4, 5, or 6 could go in the row below it, otherwise there would be no larger number left to fit in the fourth row, second column. This gives us, with $a \in \{4, 5, 6\}$,

1	1
2	
3	
a	

Since all the remaining numbers are distinct, we just need the number of unique pairs we can make with these numbers. For $a = 4$, we have 5 objects, so the number of pairs is $\binom{5}{2} = 10$. For $a = 5$, we have 4 objects and so $\binom{4}{2} = 6$ pairs. For $a = 6$, we have 3 objects and so $\binom{3}{2} = 3$ pairs, so in total there are 19 possibilities when 3 is placed in the third row, first column.

When 3 is placed in the second row, second column, we must have that 4 is placed in the third row, first column. Like before, this means that 5 can either be placed in the fourth row, first column or third row, second column. In the first case, we would have the following tableau:

1	1
2	3
4	
5	

Since all the remaining numbers are larger than 5, we need only count the number of pairs that fit in the empty first column boxes, as the remaining spots will be fixed then. We are making pairs from 4 numbers, so there will be $\binom{4}{2} = 6$ pairs that can be chosen. When 5 is placed in the third row, second column, we must have that 6 is placed in the fourth row, first column. Once again, we need only count the unique pairs that can be made in the first column and everything else will be determined. There are 3 numbers to choose, so we have $\binom{3}{2} = 3$ pairs. Thus, when we put 3 in the second row, second column, there are 9 possibilities.

This means our Kostka number $K_{\{2^4, 1^2\}, \{2, 1^8\}} = 19 + 9 = 28$. Thus, we have

$$\text{mult}(\alpha) = X_2 - K_{\{2^4, 1^2\}, \{2, 1^8\}} = 35 - 28 = 7$$

3.3 Roots of Degree -3 and Lower

As before, we require the set of $w \in W(S)$ where $2 \leq l(w) \leq 7$ in order to determine the root multiplicities up until degree -7 . One can construct a program in either Maple or MATLAB in order to do this manually, and receive Table 3.2 of these $w \in W(S)$ along with $w\rho - \rho$ in the ϵ -basis.

By our multiplicity formula, we have that the roots of degree less than or equal to -3 and greater than or equal to -7 only have contributions from Kostka numbers coming from entries whose length is equal to the absolute value of the degree. For instance, Kang's multiplicity formula is simplified to the following for degree -3 :

$$\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^3, 2^3\}, \alpha} + K_{\{3^4, 1^3\}, \alpha}$$

where we recall that

$$X_i = \sum_{\substack{\beta_1 < \dots < \beta_r \\ k_1 + \dots + k_r = k \\ k_1\beta_1 + \dots + k_r\beta_r = \alpha}} \binom{\dim \mathfrak{g}_{\beta_1}}{k_1} \cdots \binom{\dim \mathfrak{g}_{\beta_r}}{k_r}$$

Table 3.2: Set of $w \in W(S)$ with $2 \leq l(w) \leq 7$ in $HE_7^{(1)}$

w	$l(w)$	$\deg(w\rho - \rho)$	$w\rho - \rho$
$r_0 r_5$	2	-2	$\{2^4, 1^2\}$
$r_0 r_5 r_4$	3	-3	$\{3^3, 2^3\}$
$r_0 r_5 r_6$	3	-3	$\{3^4, 1^3\}$
$r_0 r_5 r_4 r_3$	4	-4	$\{4^2, 3^4\}$
$r_0 r_5 r_4 r_6$	4	-4	$\{4^3, 3, 2^2, 1\}$
$r_0 r_5 r_6 r_7$	4	-4	$\{4^4, 1^4\}$
$r_0 r_5 r_4 r_3 r_2$	5	-5	$\{5, 4^5\}$
$r_0 r_5 r_4 r_3 r_6$	5	-5	$\{5^2, 4^2, 3^2, 1\}$
$r_0 r_5 r_4 r_6 r_5$	5	-5	$\{5^3, 3^2, 2^2\}$
$r_0 r_5 r_4 r_6 r_7$	5	-5	$\{5^3, 4, 2^2, 1^2\}$
$r_0 r_5 r_6 r_7 r_8$	5	-5	$\{5^4, 1^5\}$
$r_0 r_5 r_4 r_3 r_2 r_1$	6	-6	$\{5^6\}$
$r_0 r_5 r_4 r_3 r_2 r_6$	6	-6	$\{6, 5^3, 4^2, 1\}$
$r_0 r_5 r_4 r_3 r_6 r_5$	6	-6	$\{6^2, 5, 4^2, 3, 2\}$
$r_0 r_5 r_4 r_3 r_6 r_7$	6	-6	$\{6^2, 5^2, 3^2, 1^2\}$
$r_0 r_5 r_4 r_6 r_5 r_0$	6	-6	$\{6^3, 3^4\}$
$r_0 r_5 r_4 r_6 r_5 r_7$	6	-6	$\{6^3, 4, 3, 2^2, 1\}$
$r_0 r_5 r_4 r_6 r_7 r_8$	6	-6	$\{6^3, 5, 2^2, 1^3\}$
$r_0 r_5 r_4 r_3 r_2 r_1 r_6$	7	-7	$\{6^4, 5^2, 1\}$
$r_0 r_5 r_4 r_3 r_2 r_6 r_5$	7	-7	$\{7, 6^2, 5^2, 4, 2\}$
$r_0 r_5 r_4 r_3 r_2 r_6 r_7$	7	-7	$\{7, 6^3, 4^2, 1^2\}$
$r_0 r_5 r_4 r_3 r_6 r_5 r_0$	7	-7	$\{7^2, 6, 4^3, 3\}$
$r_0 r_5 r_4 r_3 r_6 r_5 r_4$	7	-7	$\{7^2, 5^3, 3^2\}$
$r_0 r_5 r_4 r_3 r_6 r_5 r_7$	7	-7	$\{7^2, 6, 5, 4, 3, 2, 1\}$
$r_0 r_5 r_4 r_3 r_6 r_7 r_8$	7	-7	$\{7^2, 6^2, 3^2, 1^3\}$
$r_0 r_5 r_4 r_6 r_5 r_0 r_7$	7	-7	$\{7^3, 4, 3^3, 1\}$
$r_0 r_5 r_4 r_6 r_5 r_7 r_6$	7	-7	$\{7^3, 4^2, 2^3\}$
$r_0 r_5 r_4 r_6 r_5 r_7 r_8$	7	-7	$\{7^3, 5, 3, 2^2, 1^2\}$

For degree -4 , Kang's multiplicity formula simplifies to the following:

$$\text{mult}(\alpha) = X_2 - X_3 + X_4 - K_{\{4^2, 3^4\}, \alpha} - K_{\{4^3, 3, 2^2, 1\}, \alpha} - K_{\{4^4, 1^4\}, \alpha}$$

For degree -5 , Kang's multiplicity formula simplifies to the following:

$$\text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + K_{\{5,4^5\},\alpha} + K_{\{5^2,4^2,3^2,1\},\alpha} + K_{\{5^3,3^2,2^2\},\alpha} + K_{\{5^3,4,2^2,1^2\},\alpha} + K_{\{5^4,1^5\},\alpha}$$

For degree -6 , Kang's multiplicity formula simplifies to the following:

$$\begin{aligned} \text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + X_6 - K_{\{5^6\},\alpha} - K_{\{6,5^3,4^2,1\},\alpha} - K_{\{6^2,5,4^2,3,2\},\alpha} \\ - K_{\{6^2,5^2,3^2,1^2\},\alpha} - K_{\{6^3,3^4\},\alpha} - K_{\{6^3,4,3,2^2,1\},\alpha} - K_{\{6^3,5,2^2,1^3\},\alpha} \end{aligned}$$

For degree -7 , Kang's multiplicity formula simplifies to the following:

$$\begin{aligned} \text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + X_6 - X_7 + K_{\{6^4,5^2,1\},\alpha} + K_{\{7,6^2,5^2,4,2\},\alpha} + K_{\{7,6^3,4^2,1^2\},\alpha} + K_{\{7^2,6,4^3,3\},\alpha} + K_{\{7^2,5^3,3^2\},\alpha} \\ + K_{\{7^2,6,5,4,3,2,1\},\alpha} + K_{\{7^2,6^2,3^2,1^3\},\alpha} + K_{\{7^3,4,3^3,1\},\alpha} + K_{\{7^3,4^2,2^3\},\alpha} + K_{\{7^3,5,3,2^2,1^2\},\alpha} \end{aligned}$$

Now we can use Kang's multiplicity formula to determine the root multiplicities of all dominant roots of degree -3 , which will give us all root multiplicities of degree -3 via the Weyl group action. Therefore, we need all μ such that $\mu = (3 - k_1)\epsilon_1 + (3 + k_1 - k_2)\epsilon_2 + (3 + k_2 - k_3)\epsilon_3 + (3 + k_3 - k_4)\epsilon_4 + (3 + k_4 - k_5)\epsilon_5 + (k_5 - k_6)\epsilon_6 + (k_6 - k_7)\epsilon_7 + (k_7 - k_8)\epsilon_8 + k_8\epsilon_9$ where $3 - k_1 \geq 3 + k_1 - k_2 \geq 3 + k_2 - k_3 \geq 3 + k_3 - k_4 \geq 3 + k_4 - k_5 \geq k_5 - k_6 \geq k_6 - k_7 \geq k_7 - k_8 \geq k_8$ which corresponds to all partitions of 15 who have their largest summand not exceed 3 and with 9 or fewer summands. The table below lists the dominant roots and important pieces of Kang's multiplicity formula used to determine the multiplicities. The full set of roots can be obtained from permutations of the coefficients of ϵ_i for each dominant root.

Table 3.3: Degree -3 Dominant Root Multiplicities for $HE_7^{(1)}$

α	X_2	X_3	$K_{\{3^3,2^3\}}$	$K_{\{3^4,1^3\}}$	$\text{mult}(\alpha)$
$\{3, 2^4, 1^4\}$	23	106	42	42	1
$\{2^7, 1\}$	21	105	50	35	1
$\{2^6, 1^3\}$	87	285	110	95	7

Example 3.3.1 Show that the degree -3 dominant root $\alpha = \{3, 2^4, 1^4\}$ has multiplicity 1.

The multiplicity formula for a degree -3 root in $HE_7^{(1)}$ is $\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^3,2^3\},\alpha} +$

$K_{\{3^4, 1^3\}, \alpha}$. To find X_2 , we need to first find all pairs of permutations of one degree -2 dominant root and one degree -1 dominant root which sum to α . In other words, all pairs of permutations of $\{1^5\}$ and $\{2^2, 1^6\}$ or $\{2, 1^8\}$ which sum to $\{3, 2^4, 1^4\}$. Let us first start with permutations of $\{1^5\}$ and $\{2^2, 1^6\}$.

α	3	2	2	2	2	1	1	1	1
ϵ_1	2	2	1	1	1	1	1	1	0
ϵ_2	1	0	1	1	1	0	0	0	1

Notice that the first column of the sum must contain 2 and 1 and one of the next four columns can only contain 2 and 0. Additionally, the other three columns in the second through fifth column are forced to both contain 1. This leaves only columns with 0 or 1 for the last four columns, where three of the ones are in the first row and one is in the second row. Thus, the table above is the only way to place these two roots up to permutations of the columns which sum to the same number. There are $\binom{4}{1}\binom{4}{1} = 16$ ways that these columns can be arranged. Now, we look at the permutations of $\{1^5\}$ and $\{2, 1^8\}$.

α	3	2	2	2	2	1	1	1	1
ϵ_1	2	1	1	1	1	1	1	1	1
ϵ_2	1	1	1	1	1	0	0	0	0

Notice that the first column of the sum must contain 2 and 1 and the next four columns must contain 1 and 1. The remaining four columns can only contain 1 and 0 with all of the ones in the first row, so this is the only way to place these two roots up to permutations of the columns which sum to the same number. There is only 1 way to arrange these columns. Thus, we have that $X_2 = 16\binom{1}{1}\binom{1}{1} + 1\binom{7}{1}\binom{1}{1} = 23$.

To find X_3 , we need to find all pairs of permutations of three degree -1 dominant roots which sum to $\{3, 2^4, 1^4\}$. In order to simplify our calculation, we shall not worry about the dominance order on the rows in this case. If we find all possible cases with all possible permutations of the relevant columns, then we can simply divide by $3! = 6$ in order to get all the ordered possibilities. The first column can only contain 1s because it must sum

to 3. The next four columns must sum to 2, so these columns must contain two 1s and one 0. There are only 3 ways to make such a column: place the 0 in the first, second, or third row. Like before, we want to eventually take all permutations of these columns. Thus, we find all ways to place these 3 objects into 4 bins so that the order is fixed, and then we can take all permutations to obtain all possibilities without over-counting these four columns. By stars-and-bars, there are $\binom{4+3-1}{3-1} = 15$ ways to arrange these columns with order not mattering. The remaining four columns are determined by the previous five (up to permutation of the columns) so if we write out all allowed arrangements of second through fifth column, we will have all the possibilities. We note however that all 3 possibilities that repeat the same pattern in the 2 columns four times will not contribute to the multiplicity as that will be the same root added twice which yields $\binom{1}{2}\binom{1}{1} = 0$ to the multiplicity, so there are only 12 ways to arrange the columns in order, which we list below:

α	3	2	2	2	2	1	1	1	1
ϵ_1	1	1	1	1	1	0	0	0	0
ϵ_2	1	1	1	1	0	1	0	0	0
ϵ_3	1	0	0	0	1	0	1	1	1
ϵ_1	1	1	1	1	0	1	0	0	0
ϵ_2	1	1	1	1	1	0	0	0	0
ϵ_3	1	0	0	0	1	0	1	1	1
ϵ_1	1	1	1	1	1	0	0	0	0
ϵ_2	1	1	1	0	0	1	1	0	0
ϵ_3	1	0	0	1	1	0	0	1	1
ϵ_1	1	1	1	0	0	1	1	0	0
ϵ_2	1	1	1	1	1	0	0	0	0
ϵ_3	1	0	0	1	1	0	0	1	1
ϵ_1	1	1	1	1	1	0	0	0	0
ϵ_2	1	1	0	0	0	1	1	1	0
ϵ_3	1	0	1	1	1	0	0	0	1
ϵ_1	1	1	1	0	0	1	1	0	0
ϵ_2	1	1	0	1	1	0	0	1	0
ϵ_3	1	0	1	1	1	0	0	0	1
ϵ_1	1	1	0	0	0	1	1	1	0
ϵ_2	1	1	1	1	1	0	0	0	0
ϵ_3	1	0	1	1	1	0	0	0	1

α	3	2	2	2	2	1	1	1	1
ϵ_1	1	1	0	0	0	1	1	1	0
ϵ_2	1	1	1	1	1	0	0	0	0
ϵ_3	1	0	1	1	1	0	0	0	1
ϵ_1	1	1	1	1	0	1	0	0	0
ϵ_2	1	0	0	0	1	0	1	1	1
ϵ_3	1	1	1	1	1	0	0	0	0
ϵ_1	1	1	1	0	0	1	1	0	0
ϵ_2	1	0	0	1	1	0	0	1	1
ϵ_3	1	1	1	1	1	0	0	0	0
ϵ_1	1	1	0	0	0	1	1	1	0
ϵ_2	1	0	1	1	1	0	0	0	1
ϵ_3	1	1	1	1	1	0	0	0	0

When you go through each of these 12 possibilities and count all the possible ways to rearrange the columns (permutations of columns that sum to 2 times permutations of columns that sum to 1), you will get 636 possibilities. However, these are the unordered rows, so when we consider the rows as ordered and get rid of all permutations of the rows, we obtain that there are $\frac{636}{3!} = 106$ possible ways to arrange the rows and obtain α as the sum. As each possibility contributes $\binom{1}{1}\binom{1}{1}\binom{1}{1} = 1$ to the multiplicity, that means that $X_3 = 106$.

Now, we need to find $K_{\{3^3, 2^3\}, \{3, 2^4, 1^4\}}$, which is the number of ways to fit three 1s, two 2s, two 3s, two 4s, two 5s, one 6, one 7, one 8, and one 9 in the following frame:

$$F(\lambda) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

First, we notice that the first row can only contain 1s, the second row must contain 2s in the first and second column, and the third row must have 3 in the first column. There are two options to place the other 3: both 3s are in the third row, first and second columns or one 3 is in the second row, third column and one 3 is in the third row, first column.

Let us consider the first type of tableau:

1	1	1
2	2	
3	3	

We must place 2 numbers from the set $\{4, 4, 5, 5, 6, 7, 8, 9\}$ in order to fill out the top square. The numbers must be distinct, so we only have $\frac{5(6)}{2} = 15$ possibilities for the top square. We break this up into cases in order to determine how many tableaux have these 15 squares: either there is either both a 4 and a 5 in that square, there is one of 4 and 5 in that square, or there is neither in that square. In the first case, we have

1	1	1
2	2	4
3	3	5

Obviously, there is only one square has this form as all spaces are filled. There is only $\{4, 5, 6, 7, 8, 9\}$ left for the bottom part of the tableau, which can be arranged in 5 ways, so we have 5 possible tableaux. In the second case, we have

1	1	1
2	2	a
3	3	b

where $a \in \{4, 5\}$ and $b \in \{6, 7, 8, 9\}$, which has $2(4) = 8$ ways of occurring. There is only $\{4, a, 5, 6, 7, 8, 9\} \setminus \{b\}$ left. Considering the position of $\{4, 4\}$ and $\{5, 5\}$ are fixed in the bottom part of the tableau and all possible elements of b are distinct, without loss of generality we can say $a = 4$ and $b = 9$ as we will obtain the same amount of possibilities for each of these squares, so we have $\{4, 4, 5, 6, 7, 8\}$ left to fill the bottom part of the tableau. There are only 2 ways to fill this part of the tableau, so we have $8 * 2 = 16$ possible tableaux. In the final case, we have

1	1	1
2	2	b
3	3	c

where $b, c \in \{6, 7, 8, 9\}$. There are $\frac{3(4)}{2} = 6$ ways to make the upper square in this case. That means that we have $\{4, 4, 5, 5, 6, 7, 8, 9\} \setminus \{b, c\}$ left in our set, so we have $\{4, 4, 5, 5\}$ left. That means the fourth row must have all 4s and the fifth row must have all 5s, so there is only one way to place the remaining numbers. Thus, there are only 6 tableaux. In total, we have $16 + 5 + 6 = 27$ tableaux whose 3s are on the third row.

Now, let us consider the other type of tableau:

1	1	1
2	2	3
3		

where we only have $\{4, 4, 5, 5, 6, 7, 8, 9\}$ left to fill the tableau. At least one 4 must be in the third row, second column. If the other 4 is placed in the third row, third column, then we have $\{5, 5, 6, 7, 8, 9\}$ left for the bottom part of the tableau, which there are only 2 ways to arrange these numbers there and so there are 2 tableaux of this type. If we put a 5 in the third row, third column, then we have only $\{4, 5, 6, 7, 8, 9\}$ left for the bottom part of the tableau, which can be arranged in 5 ways. If we put $b \in \{6, 7, 8, 9\}$ in the third row, third column, the tableau looks like:

1	1	1
2	2	3
3	4	b

with 4 possible tableaux. The remaining numbers are $\{4, 5, 5, 6, 7, 8, 9\} \setminus \{b\}$ that can fill the bottom of the tableau. Since the numbers b can be are distinct, without loss of generality we can say $b = 9$ as the contribution will be the same regardless of what b is. Then, we have

1	1	1
2	2	3
3	4	9

with $\{4, 5, 5, 6, 7, 8\}$ left to fill out the tableau. There are 2 ways to arrange the numbers in this case, so we have 8 possible tableaux in total for this case. In total, we have $2 + 5 + 8 = 15$ tableaux. Thus, we have $K_{\{3^3, 2^3\}, \{3, 2^4, 1^4\}} = 27 + 15 = 42$.

Now, we need to find $K_{\{3^4, 1^3\}, \{3, 2^4, 1^4\}}$, which is the number of ways to fit three 1s, two 2s, two 3s, two 4s, two 5s, one 6, one 7, one 8, and one 9 in the following frame:

$F(\lambda) =$

First, we notice that the first row can only contain 1s, the second row must contain 2s in the first and second column, and the third row must have 3 in the first column. There are two options to place the other 3: both 3s are in the third row, first and second columns or one 3 is in the second row, third column and one 3 in the third row, first column.

Let us consider the first type of tableau:

1	1	1
2	2	3
3		

Now, we look at where to place the 4s. At least one 4 must go in the third row, second column. The other 4 can go in either the third row, third column or fourth row, first column. For the first of these subcategories, we must place the 5s in the fourth row, first and second columns. This leaves us with the following:

1	1	1
2	2	3
3	4	4
5	5	

Notice we are left with only {6, 7, 8, 9} left to place, which are all distinct. All the lower blocks can only have one order because of this, so we only have the freedom to choose which of these four to place in the fourth row, third column. Thus, there are only 4 of these type. For the second of these subcategories, we have the following tableau:

1	1	1
2	2	3
3	4	
4		

The first empty diagonal going from top to bottom must contain the two 5s and one of {6, 7, 8}. If we have 6 in that diagonal, there are 3 ways to arrange that diagonal and 3 numbers left to pick for the fourth row, third column, which leaves us with 9 tableaux. If we have 7 in that diagonal, there are 2 ways to arrange that diagonal which leaves us 2 numbers {8, 9} to pick for the fourth row, third column and there is 1 way to pick 6 as the number in the fourth row, third column, which leaves us with 5 tableaux. If we have 8 in that diagonal, there are only 2 ways to arrange that diagonal and 1 number left to pick for the fourth row, third column, which leaves us with 2 tableaux. Thus, for the first placement of the second 3, there are $4 + 9 + 5 + 2 = 20$ choices.

Let us consider the second type of tableau:

1	1	1
2	2	
3	3	

We are forced to place a 4 in the fourth row, first column. Additionally, we must either place a 4 or 5 in the second row, third column, so let us consider both subcategories of tableau. The first subcategory looks like:

1	1	1
2	2	4
3	3	
4		

We again require that the first free diagonal going from top to bottom contains both 5s, and there are 4 things we can place in the third row, third column, $\{5, 6, 7, 8\}$. If we pick 5 for the third row, third column, then we have $\frac{4(5)}{2} = 10$ ways of picking a pair of numbers from $\{5, 6, 7, 8, 9\}$ to fill out the fourth row and everything else will be fixed, so there are 10 tableaux for this case. If we pick 6, 7, or 8 for the third row, third column, then the other entries on that diagonal must be 5 as stated previously, so we need only worry about how many ways there are to pick the entry b in the fourth row, third column from $\{6, 7, 8, 9\} \setminus \{b\}$. There are 3 choices for 6, 2 choices for 7, and 1 choice for 8. In total, there are 16 tableaux for this subcategory. The other subcategory looks like:

1	1	1
2	2	5
3	3	
4	4	

This tableau only has $\{5, 6, 7, 8, 9\}$ left to place, which are all distinct. That means whatever is placed in the third column will determine the rest of the tableau. There are three numbers to place in the third row, third column as we cannot put 5 below 5 or 9 above the bottom of a column, so there are $\frac{3(4)}{2} = 6$ ways of picking pairs of numbers from $\{6, 7, 8, 9\}$ for the third column so the numbers are ordered, so there are 6 tableaux in this case. Thus, for the second placement of the second 3, there are $16 + 6 = 22$ choices. Thus, we have $K_{\{3^4, 1^3\}, \{3, 2^4, 1^4\}} = 20 + 22 = 42$.

From the above calculations, we have the multiplicity of the root $\alpha = \{3, 2^4, 1^4\}$ as

$$\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^3, 2^3\}, \alpha} + K_{\{3^4, 1^3\}, \alpha} = 23 - 106 + 42 + 42 = 1$$

As we can see from the previous example, the computation of root multiplicities for smaller degrees becomes more and more complicated. Additionally, the number of potential roots to check increases dramatically as well. In order to move forward, we will make use of the MATLAB program in Appendix A to simplify this process. The following table shows the roots of degree -4 , which are partitions of 20 which do not exceed 4 in the largest entry and do not exceed length 9.

Table 3.4: Degree -4 Dominant Root Multiplicities for $HE_7^{(1)}$

α	X_2	X_3	X_4	$K_{\{4^2, 3^4\}}$	$K_{\{4^3, 3, 2^2, 1\}}$	$K_{\{4^4, 1^4\}}$	$\text{mult}(\alpha)$
$\{4, 3, 2^6, 1\}$	33	315	900	71	475	71	1
$\{3^4, 2^3, 1^2\}$	41	381	1072	122	529	80	1
$\{4, 2^8\}$	119	805	1855	126	910	126	7
$\{3^3, 2^5, 1\}$	141	960	2205	206	1025	148	7
$\{3^2, 2^7\}$	455	2380	4550	357	1960	273	35

The following table shows the roots of degree -5 , which are partitions of 25 which do not exceed 5 in the largest entry and do not exceed length 9.

Table 3.5: Degree -5 Dominant Root Multiplicities for $HE_7^{(1)}$

α	X_2	X_3	X_4	X_5	$K_{\{5,4^5\}}$
$\{5,3^4,2^4\}$	46	648	3638	7029	45
$\{4^3,3,2^5\}$	60	855	4645	8800	118
$\{4^2,3^4,2^2,1\}$	59	840	4601	8811	127
$\{4^2,3^3,2^4\}$	214	2198	9863	16416	191
$\{4,3^6,2,1\}$	202	2133	9765	16445	202
$\{4,3^5,2^3\}$	682	5464	20785	30655	312
$\{3^8,1\}$	644	5306	20580	30660	322
$\{3^7,2^2\}$	2030	13265	43470	57225	511

α	$K_{\{5^2,4^2,3^2,1\}}$	$K_{\{5^3,3^2,2^2\}}$	$K_{\{5^3,4,2^2,1^2\}}$	$K_{\{5^4,1^5\}}$	$\text{mult}(\alpha)$
$\{5,3^4,2^4\}$	1332	1240	1332	45	1
$\{4^3,3,2^5\}$	1737	1366	1671	59	1
$\{4^2,3^4,2^2,1\}$	1965	1394	1458	48	1
$\{4^2,3^3,2^4\}$	3132	2432	2700	89	7
$\{4,3^6,2,1\}$	3456	2472	2412	76	7
$\{4,3^5,2^3\}$	5535	4262	4437	141	35
$\{3^8,1\}$	5985	4312	4032	126	35
$\{3^7,2^2\}$	9639	7378	7371	231	140

The following table shows the roots of degree -6 , which are partitions of 30 which do not exceed 6 in the largest entry and do not exceed length 9.

Table 3.6: Degree -6 Dominant Root Multiplicities for $HE_7^{(1)}$

α	X_2	X_3	X_4	X_5	X_6	$K_{\{5^6\}}$	$K_{\{6,5^3,4^2,1\}}$
$\{6,4,3^6,2\}$	56	1065	9005	34360	48060	0	1025
$\{6,3^8\}$	210	2758	18935	63385	80885	0	1540
$\{5^2,4,3^4,2^2\}$	82	1604	12888	47120	64696	44	2283
$\{5^2,3^6,2\}$	278	3977	26755	86825	109065	60	3325
$\{5,4^4,3,2^3\}$	99	1781	13794	49694	68007	53	2866
$\{5,4^3,3^4,1\}$	78	1522	12509	46658	64939	49	2736
$\{5,4^3,3^3,2^2\}$	308	4315	28548	91767	114889	72	4109
$\{5,4^2,3^5,2\}$	937	10333	58550	168340	193430	100	5955
$\{5,4,3^7\}$	2716	24297	119000	307370	325185	140	8715
$\{4^6,3,2,1\}$	80	1581	13130	49210	68750	59	3406
$\{4^6,2^3\}$	339	4650	30365	97005	121335	89	5086
$\{4^5,3^3,1\}$	281	4017	27530	90765	115410	79	4776
$\{4^5,3^2,2^2\}$	1005	11025	62050	177720	204090	119	7230
$\{4^4,3^4,2\}$	2866	25781	126007	324590	343072	166	10438
$\{4^3,3^6\}$	7822	59147	253605	590550	576400	236	15234

α	$K_{\{6^2,5,4^2,3,2\}}$	$K_{\{6^2,5^2,3^2,1^2\}}$	$K_{\{6^3,3^4\}}$	$K_{\{6^3,4,3,2^2,1\}}$	$K_{\{6^3,5,2^2,1^3\}}$	$\text{mult}(\alpha)$
$\{6,4,3^6,2\}$	6760	4895	1230	6760	1025	1
$\{6,3^8\}$	10500	7700	2100	10500	1540	7
$\{5^2,4,3^4,2^2\}$	8923	6783	1307	8256	1345	1
$\{5^2,3^6,2\}$	13915	10625	2290	13060	2014	7
$\{5,4^4,3,2^3\}$	10120	6938	1100	8018	1329	1
$\{5,4^3,3^4,1\}$	10317	6694	1392	7162	995	1
$\{5,4^3,3^3,2^2\}$	15706	10918	2010	12816	2025	7
$\{5,4^2,3^5,2\}$	24210	17081	3485	20295	3083	35
$\{5,4,3^7\}$	37170	26621	5845	31885	4718	140
$\{4^6,3,2,1\}$	11710	6830	1232	6960	971	1
$\{4^6,2^3\}$	17650	11234	1757	12580	1981	7
$\{4^5,3^3,1\}$	17820	10754	2172	11325	1506	7
$\{4^5,3^2,2^2\}$	27110	17598	3148	20095	3065	35
$\{4^4,3^4,2\}$	41408	27482	5382	31822	4736	140
$\{4^3,3^6\}$	63105	42773	8935	50025	7332	490

The following table shows the roots of degree -7 , which are partitions of 35 which do not exceed 7 in the largest entry and do not exceed length 9.

Table 3.7: Degree -7 Dominant Root Multiplicities for $HE_7^{(1)}$ (Part 1)

α	X_2	X_3	X_4	X_5	X_6	X_7	$K_{\{6^4, 5^2, 1\}}$
$\{7, 4^4, 3^4\}$	69	1632	18174	103197	284403	299072	0
$\{6^2, 4^2, 3^5\}$	102	2575	27950	151130	401975	416345	794
$\{6, 5^2, 4^2, 3^3, 2\}$	126	3032	31350	165030	433314	447262	1149
$\{6, 5^2, 4, 3^5\}$	409	7470	65670	308200	741980	715680	1528
$\{6, 5, 4^5, 2^2\}$	111	2875	30642	163735	433527	449715	1212
$\{6, 5, 4^4, 3^2, 2\}$	406	7380	64925	306129	740793	717375	1606
$\{6, 5, 4^3, 3^4\}$	1262	17931	134740	567615	1261807	1143984	2150
$\{6, 4^7, 1\}$	91	2310	25872	145040	397670	420525	1126
$\{6, 4^6, 3, 2\}$	1225	17544	132945	563905	1260165	1146755	2242
$\{6, 4^5, 3^3\}$	3614	41831	273213	1039530	2139040	1825355	3032
$\{5^4, 4^2, 3, 2^2\}$	144	3406	34394	178879	467568	483178	1669
$\{5^4, 4, 3^3, 2\}$	481	8503	72378	334068	799054	770779	2171
$\{5^4, 3^5\}$	1428	20330	149795	620245	1363735	1230800	2882
$\{5^3, 4^4, 3, 1\}$	102	2592	28515	157869	429604	453368	1534
$\{5^3, 4^4, 2^2\}$	431	8018	70391	330901	799988	776263	2274
$\{5^3, 4^3, 3^2, 2\}$	1385	19795	147173	615079	1362226	1235283	3021
$\{5^3, 4^2, 3^4\}$	3991	46652	301471	1133661	2313487	1966868	4049
$\{5^2, 4^6, 1\}$	348	6466	59575	293250	732945	724680	2081
$\{5^2, 4^5, 3, 2\}$	3857	45416	296417	1124530	2310280	1972935	4198
$\{5^2, 4^4, 3^3\}$	10650	105137	601575	2061730	3911823	3136929	5692
$\{5, 4^7, 2\}$	10339	102550	592018	2045260	3904250	3143910	5858
$\{5, 4^6, 3^2\}$	27337	232726	1189492	3730670	6595110	4995425	8024
$\{4^8, 3\}$	67753	506373	2331602	6720000	11092480	7946540	11360

Table 3.8: Degree -7 Dominant Root Multiplicities for $HE_7^{(1)}$ (Part 2)

α	$K_{\{7,6^2,5^2,4,2\}}$	$K_{\{7,6^3,4^2,1^2\}}$	$K_{\{7^2,6,4^3,3\}}$	$K_{\{7^2,5^3,3^2\}}$	$K_{\{7^2,6,5,4,3,2,1\}}$	$K_{\{7^2,6^2,3^2,1^3\}}$
$\{7,4^4,3^4\}$	7236	4592	9106	6622	46144	4592
$\{6^2,4^2,3^5\}$	12504	8825	10882	7865	61920	6740
$\{6,5^2,4^2,3^3,2\}$	17082	10546	11736	9965	65552	6759
$\{6,5^2,4,3^5\}$	23508	14805	16682	14360	97920	10325
$\{6,5,4^5,2^2\}$	18579	11165	13874	10550	65360	6225
$\{6,5,4^4,3^2,2\}$	25374	15657	19372	15076	97824	9577
$\{6,5,4^3,3^4\}$	34965	22033	27238	21582	145504	14660
$\{6,4^7,1\}$	18495	10550	15883	11200	61552	4710
$\{6,4^6,3,2\}$	37467	23085	31018	22720	145312	13645
$\{6,4^5,3^3\}$	51759	32635	43303	32330	215312	20945
$\{5^4,4^2,3,2^2\}$	22947	12646	13014	12282	69088	6650
$\{5^4,4,3^3,2\}$	31167	17594	18351	17652	103680	10262
$\{5^4,3^5\}$	42732	24710	25992	25260	154560	15770
$\{5^3,4^4,3,1\}$	22914	11839	15357	13070	65040	5004
$\{5^3,4^4,2^2\}$	33615	18519	21615	18524	103296	9473
$\{5^3,4^3,3^2,2\}$	45747	25983	30036	26300	154296	14606
$\{5^3,4^2,3^4\}$	62829	36585	42130	37447	229088	22434
$\{5^2,4^6,1\}$	33057	17270	24406	19350	97184	7140
$\{5^2,4^5,3,2\}$	66810	38120	47791	39090	228704	20885
$\{5^2,4^4,3^3\}$	92016	53910	66589	55390	338336	32123
$\{5,4^7,2\}$	97182	55735	74319	57820	337856	29980
$\{5,4^6,3^2\}$	134325	79150	103209	81700	498176	46195
$\{4^8,3\}$	195588	115900	157472	120120	731648	66660

Table 3.9: Degree -7 Dominant Root Multiplicities for $HE_7^{(1)}$ (Part 3)

α	$K_{\{7^3, 4, 3^3, 1\}}$	$K_{\{7^3, 4^2, 2^3\}}$	$K_{\{7^3, 5, 3, 2^2, 1^2\}}$	$\text{mult}(\alpha)$
$\{7, 4^4, 3^4\}$	9106	6622	7236	1
$\{6^2, 4^2, 3^5\}$	11980	8680	9834	1
$\{6, 5^2, 4^2, 3^3, 2\}$	10350	8072	9324	1
$\{6, 5^2, 4, 3^5\}$	17060	12560	14550	7
$\{6, 5, 4^5, 2^2\}$	9116	7610	8355	1
$\{6, 5, 4^4, 3^2, 2\}$	15284	11887	13110	7
$\{6, 5, 4^3, 3^4\}$	24817	18368	20439	35
$\{6, 4^7, 1\}$	9212	6070	5445	1
$\{6, 4^6, 3, 2\}$	22456	17410	18549	35
$\{6, 4^5, 3^3\}$	36016	26740	28917	140
$\{5^4, 4^2, 3, 2^2\}$	8896	7502	8664	1
$\{5^4, 4, 3^3, 2\}$	15076	11802	13689	7
$\{5^4, 3^5\}$	24700	18360	21486	35
$\{5^3, 4^4, 3, 1\}$	9142	6036	5673	1
$\{5^3, 4^4, 2^2\}$	13530	11194	12339	7
$\{5^3, 4^3, 3^2, 2\}$	22476	17515	19428	35
$\{5^3, 4^2, 3^4\}$	36294	27084	30432	140
$\{5^2, 4^6, 1\}$	13752	9150	8145	7
$\{5^2, 4^5, 3, 2\}$	33300	25870	27699	140
$\{5^2, 4^4, 3^3\}$	53092	39746	43344	490
$\{5, 4^7, 2\}$	49105	38095	39654	491
$\{5, 4^6, 3^2\}$	77449	58165	62037	1548
$\{4^8, 3\}$	112728	84960	89172	4530

The degree -7 roots $\alpha = \{5, 4^7, 2\}$, $\beta = \{5, 4^6, 3^2\}$, $\gamma = \{4^8, 3\}$ have that

$$p^{(7)}\left(1 - \frac{(\alpha|\alpha)}{2}\right) = 490 < 491 = \text{mult}(\alpha)$$

$$p^{(7)}\left(1 - \frac{(\beta|\beta)}{2}\right) = 1547 < 1548 = \text{mult}(\beta)$$

$$p^{(7)}\left(1 - \frac{(\gamma|\gamma)}{2}\right) = 4522 < 4530 = \text{mult}(\gamma)$$

which all disprove Frenkel's conjecture for $HE_7^{(1)}$.

CHAPTER

4

$HE_8^{(1)}$

4.1 Roots of Degree -1

By our construction of $HE_8^{(1)}$, we know that $\mathfrak{g}_{-1}^{(S)} = V_S(-\alpha_0) = V_S(\Lambda_5) = V_S(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7)$. By $\mathfrak{sl}(10; \mathbb{C})$ representation theory, we know that the dominant weights of this module are the weights under it in the dominance order. In other words, we need weights $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \mu_3\epsilon_3 + \mu_4\epsilon_4 + \mu_5\epsilon_5 + \mu_6\epsilon_6 + \mu_7\epsilon_7$ for which $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 \geq \mu_1\epsilon_1 + \mu_2\epsilon_2 + \mu_3\epsilon_3 + \mu_4\epsilon_4 + \mu_5\epsilon_5 + \mu_6\epsilon_6 + \mu_7\epsilon_7$ in the dominance order. One can then see that the only dominant weight satisfying this condition is $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7$ associated with the partition $\{1^7\}$. By Lemma 1.5.2, we have that $\text{mult}(\{1^7\}) = K_{\{1^7\}, \{1^7\}} = 1$, where the last equality follows immediately from the fact that there is only one semistandard Young tableaux of the same height and weight.

4.2 Roots of Degree -2

Recall that $W(S) = \{w \in W \mid \Phi_w \subset \Delta^+(S)\}$. By our Lemma 1.5.1, we know that we can get the elements of $W(S)$ of length 2 from those elements of length 1, r_0 . Then, we see with a little calculation that, for $i \neq 0$,

$$\begin{aligned} r_0(\alpha_i) &= \alpha_i - \alpha_i(h_0)\alpha_0 \\ &= \begin{cases} \alpha_i, & i \neq 7 \\ \alpha_0 + \alpha_7, & i = 7 \end{cases} \end{aligned}$$

This means that the only element where $l(w) = 2$ in $W(S)$ is $w = r_0 r_7$. Now,

$$\begin{aligned} r_0 r_7 \rho - \rho &= r_0(\rho - \alpha_7) - \rho \\ &= r_0 \rho - r_0 \alpha_7 - \rho \\ &= \rho - \alpha_0 - (\alpha_7 - \alpha_7(h_0)\alpha_0) - \rho \\ &= -2\alpha_0 - \alpha_7 \\ &= 2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7) - (\epsilon_7 - \epsilon_8) \\ &= \{2^6, 1^2\} \end{aligned}$$

which means that $\deg(r_0 r_7 \rho - \rho) = -2$ and so we can reduce our formula for the multiplicity in degree -2 to the following:

$$\begin{aligned} \text{mult}(\alpha) &= X_2 - \sum_{\substack{w \in W(S) \\ l(w)=2 \\ \deg(w\rho-\rho)=-2}} K_{w\rho-\rho, \alpha} \\ &= X_2 - K_{\{2^6, 1^2\}, \alpha} \end{aligned}$$

Now we can use Kang's multiplicity formula to determine the root multiplicities of all dominant roots of degree -2 , which will give us all root multiplicities of degree -2 via the Weyl group action. Therefore, we need all μ such that $\mu = (2 - k_1)\epsilon_1 + (2 + k_1 - k_2)\epsilon_2 + (2 + k_2 - k_3)\epsilon_3 + (2 + k_3 - k_4)\epsilon_4 + (2 + k_4 - k_5)\epsilon_5 + (2 + k_5 - k_6)\epsilon_6 + (2 + k_6 - k_7)\epsilon_7 + (k_7 - k_8)\epsilon_8 + (k_8 - k_9)\epsilon_9 + k_9\epsilon_{10}$ where $2 - k_1 \geq 2 + k_1 - k_2 \geq 2 + k_2 - k_3 \geq 2 + k_3 - k_4 \geq 2 + k_4 - k_5 \geq 2 + k_5 - k_6 \geq 2 + k_6 - k_7 \geq k_7 - k_8 \geq k_8 - k_9 \geq k_9$, which corresponds to all partitions of 14 who have their largest

summand not exceed 2 and with 10 or fewer summands. These partitions can be listed: $\{2^4, 1^6\}, \{2^5, 1^4\}, \{2^6, 1^2\}, \{2^7\}$. The table below lists the dominant roots and important pieces of Kang's multiplicity formula used to determine the multiplicities. The full set of roots can be obtained from permutations of the coefficients of ϵ_i for each dominant root.

Table 4.1: Degree -2 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	$K_{\{2^4, 1^2\}}$	$\text{mult}(\alpha)$
$\{2^4, 1^6\}$	10	9	1

Example 4.2.1 Show that the degree -2 dominant root $\alpha = \{2^4, 1^6\}$ has multiplicity 1.

We recall the multiplicity formula for a degree -2 root in $HE_7^{(1)}$ is $\text{mult}(\alpha) = X_2 - K_{\{2^6, 1^2\}, \{2^4, 1^6\}}$. To find X_2 , we need to first find all pairs of permutations of degree -1 dominant roots which sum to α . In other words, all pairs of permutations of $\{1^7\}$ which sum to $\{2^4, 1^6\}$.

α	2	2	2	2	1	1	1	1	1	1
ϵ_1	1	1	1	1	1	1	1	0	0	0
ϵ_2	1	1	1	1	0	0	0	1	1	1

Notice that the first four columns of the sum must contain 1. Additionally, there are only 3 ones left to place for each row and each column can only contain one 1. However, recall that the collection of all roots are permutations of the dominant roots, so we can preserve all columns summing to 1 with a permutation. There are $\frac{6!}{3!3!} = 20$ ways that these columns can be arranged. However, this overcounts the true value because we require that $\epsilon_1 > \epsilon_2$ in order to have no sums repeated, so we must divide by 2 to take out all pairs of permutations of ϵ_1 and ϵ_2 that repeat. Hence, $X_2 = \frac{20}{2} = 10$.

Now, we only need find $K_{\{2^6, 1^2\}, \{2^4, 1^6\}}$, which is the number of ways to fit two 1s, two 2s, two 3s, two 4s, one 5, one 6, one 7, one 8, one 9, and one 10 in the following frame:

$$F(\lambda) =$$

The top eight boxes are forced to contain the two 1s, two 2s, two 3s, and two 4s and the third row, first column box has to contain 5. Working through the remaining possibilities gives the following Young tableaux:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	6	5	6	5	6	5	7	5	7	5	8	5	8	5	9
7	8	7	9	7	10	6	8	6	9	6	10	6	10	6	10
9		8		8		9		8		8		7		7	
10		10		9		10		10		9		10		9	

This means our Kostka number $K_{\{26,12\},\{24,16\}} = 9$. Thus, we have

$$\text{mult}(\alpha) = X_2 - K_{\{26,14\},\{24,16\}} = 10 - 9 = 1$$

4.3 Roots of Degree -3 to -7

As before, we require the set of $w \in W(S)$ where $2 \leq l(w) \leq 7$ in order to determine the root multiplicities up until degree -7 . One can construct a program in either Maple or MATLAB in order to do this manually, and receive Table 4.2 of these $w \in W(S)$ along with $w\rho - \rho$ in the ϵ -basis.

By our multiplicity formula, we have that the roots of degree less than or equal to -3 and greater than or equal to -7 only have contributions from Kostka numbers coming from entries whose length is equal to the absolute value of the degree. For instance, Kang's multiplicity formula is simplified to the following for degree -3 :

$$\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^5, 2^3\}, \alpha} + K_{\{3^6, 1^3\}, \alpha}$$

Table 4.2: Set of $w \in W(S)$ with $2 \leq l(w) \leq 7$ in $HE_8^{(1)}$

w	$l(w)$	$\deg(w\rho - \rho)$	$w\rho - \rho$
$r_0 r_7$	2	-2	$\{2^6, 1^2\}$
$r_0 r_7 r_6$	3	-3	$\{3^5, 2^3\}$
$r_0 r_7 r_8$	3	-3	$\{3^6, 1^3\}$
$r_0 r_7 r_6 r_5$	4	-4	$\{4^4, 3^4\}$
$r_0 r_7 r_6 r_8$	4	-4	$\{4^5, 3, 2^2, 1\}$
$r_0 r_7 r_8 r_9$	4	-4	$\{4^6, 1^4\}$
$r_0 r_7 r_6 r_5 r_4$	5	-5	$\{5^3, 4^5\}$
$r_0 r_7 r_6 r_5 r_8$	5	-5	$\{5^4, 4^2, 3^2, 1\}$
$r_0 r_7 r_6 r_8 r_7$	5	-5	$\{5^5, 3^2, 2^2\}$
$r_0 r_7 r_6 r_8 r_9$	5	-5	$\{5^5, 4, 2^2, 1^2\}$
$r_0 r_7 r_6 r_5 r_4 r_3$	6	-6	$\{6^2, 5^6\}$
$r_0 r_7 r_6 r_5 r_4 r_8$	6	-6	$\{6^3, 5^3, 4^2, 1\}$
$r_0 r_7 r_6 r_5 r_8 r_7$	6	-6	$\{6^4, 5, 4^2, 3, 2\}$
$r_0 r_7 r_6 r_5 r_8 r_9$	6	-6	$\{6^4, 5^2, 3^2, 1^2\}$
$r_0 r_7 r_6 r_8 r_7 r_6$	6	-6	$\{6^5, 3^4\}$
$r_0 r_7 r_6 r_8 r_7 r_9$	6	-6	$\{6^5, 4, 3, 2^2, 1\}$
$r_0 r_7 r_6 r_5 r_4 r_3 r_2$	7	-7	$\{7, 6^7\}$
$r_0 r_7 r_6 r_5 r_4 r_3 r_8$	7	-7	$\{7^2, 6^4, 5^2, 1\}$
$r_0 r_7 r_6 r_5 r_4 r_8 r_7$	7	-7	$\{7^3, 6^2, 5^2, 4, 2\}$
$r_0 r_7 r_6 r_5 r_4 r_8 r_9$	7	-7	$\{7^3, 6^3, 4^2, 1^2\}$
$r_0 r_7 r_6 r_5 r_8 r_7 r_0$	7	-7	$\{7^4, 6, 4^3, 3\}$
$r_0 r_7 r_6 r_5 r_8 r_7 r_6$	7	-7	$\{7^4, 5^3, 3^2\}$
$r_0 r_7 r_6 r_5 r_8 r_7 r_9$	7	-7	$\{7^4, 6, 5, 4, 3, 2, 1\}$
$r_0 r_7 r_6 r_8 r_7 r_6 r_9$	7	-7	$\{7^5, 4, 3^3, 1\}$
$r_0 r_7 r_6 r_8 r_7 r_9 r_8$	7	-7	$\{7^5, 4^2, 2^3\}$

where we recall that

$$X_i = \sum_{\substack{\beta_1 < \dots < \beta_r \\ k_1 + \dots + k_r = k \\ k_1 \beta_1 + \dots + k_r \beta_r = \alpha}} \binom{\dim \mathfrak{g}_{\beta_1}}{k_1} \dots \binom{\dim \mathfrak{g}_{\beta_r}}{k_r}$$

For degree -4, Kang's multiplicity formula simplifies to the following:

$$\text{mult}(\alpha) = X_2 - X_3 + X_4 - K_{\{4^4, 3^4\}, \alpha} - K_{\{4^5, 3, 2^2, 1\}, \alpha} - K_{\{4^6, 1^4\}, \alpha}$$

For degree -5 , Kang's multiplicity formula simplifies to the following:

$$\text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + K_{\{5^3, 4^5\}, \alpha} + K_{\{5^4, 4^2, 3^2, 1\}, \alpha} + K_{\{5^5, 3^2, 2^2\}, \alpha} + K_{\{5^5, 4, 2^2, 1^2\}, \alpha}$$

For degree -6 , Kang's multiplicity formula simplifies to the following:

$$\begin{aligned} \text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + X_6 - K_{\{6^2, 5^6\}, \alpha} - K_{\{6^3, 5^3, 4^2, 1\}, \alpha} \\ - K_{\{6^4, 5, 4^2, 3, 2\}, \alpha} - K_{\{6^4, 5^2, 3^2, 1^2\}, \alpha} - K_{\{6^5, 3^4\}, \alpha} - K_{\{6^5, 4, 3, 2^2, 1\}, \alpha} \end{aligned}$$

For degree -7 , Kang's multiplicity formula simplifies to the following:

$$\begin{aligned} \text{mult}(\alpha) = X_2 - X_3 + X_4 - X_5 + X_6 - X_7 + K_{\{7, 6^7\}} + K_{\{7^2, 6^4, 5^2, 1\}, \alpha} + K_{\{7^3, 6^2, 5^2, 4, 2\}, \alpha} + K_{\{7^3, 6^3, 4^2, 1^2\}, \alpha} \\ + K_{\{7^4, 6, 4^3, 3\}, \alpha} + K_{\{7^4, 5^3, 3^2\}, \alpha} + K_{\{7^4, 6, 5, 4, 3, 2, 1\}, \alpha} + K_{\{7^5, 4, 3^3, 1\}, \alpha} + K_{\{7^5, 4^2, 2^3\}, \alpha} \end{aligned}$$

Now we can use Kang's multiplicity formula to determine the root multiplicities of all dominant roots of degree -3 , which will give us all root multiplicities of degree -3 via the Weyl group action. Therefore, we need all μ such that $\mu = (3 - k_1)\epsilon_1 + (3 + k_1 - k_2)\epsilon_2 + (3 + k_2 - k_3)\epsilon_3 + (3 + k_3 - k_4)\epsilon_4 + (3 + k_4 - k_5)\epsilon_5 + (3 + k_5 - k_6)\epsilon_6 + (3 + k_6 - k_7)\epsilon_7 + (k_7 - k_8)\epsilon_8 + (k_8 - k_9)\epsilon_9 + k_9\epsilon_{10}$ where $3 - k_1 \geq 3 + k_1 - k_2 \geq 3 + k_2 - k_3 \geq 3 + k_3 - k_4 \geq 3 + k_4 - k_5 \geq 3 + k_5 - k_6 \geq 3 + k_6 - k_7 \geq k_7 - k_8 \geq k_8 - k_9 \geq k_9$ which corresponds to all partitions of 21 who have their largest summand not exceed 3 and with 10 or fewer summands. The table below lists the dominant roots and important pieces of Kang's multiplicity formula used to determine the multiplicities. The full set of roots can be obtained from permutations of the coefficients of ϵ_i for each dominant root.

Table 4.3: Degree -3 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	X_3	$K_{\{3^5, 2^3\}}$	$K_{\{3^6, 1^3\}}$	$\text{mult}(\alpha)$
$\{3^2, 2^7, 1\}$	21	105	50	35	1
$\{3, 2^9\}$	84	280	120	84	8

Example 4.3.1 Show that the degree -3 dominant root $\alpha = \{3^2, 2^7, 1\}$ has multiplicity 1.

The multiplicity formula for a degree -3 root in $HE_7^{(1)}$ is $\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^5, 2^3\}, \alpha} +$

$K_{\{3^6, 1^3\}, \alpha}$. To find X_2 , we need to first find all pairs of permutations of one degree -2 dominant root and one degree -1 dominant root which sum to α . In other words, all pairs of permutations of $\{1^7\}$ and $\{2^4, 1^6\}$ which sum to $\{3^2, 2^7, 1\}$.

α	3	3	2	2	2	2	2	2	2	1
ϵ_1	2	2	2	2	1	1	1	1	1	1
ϵ_2	1	1	0	0	1	1	1	1	1	0

Notice that the first two columns of the sum must contain 2 and 1 and one of the next two columns can only contain 2 and 0. Additionally, the next five columns in the second through fifth column are forced to both contain 1. This leaves only columns with 1 in the top row and 0 in the bottom row for the last column. Thus, the table above is the only way to place these two roots up to permutations of the columns which sum to the same number. There are $\frac{7!}{2!5!} = 21$ ways that these columns can be arranged. Thus, we have that $X_2 = 21$.

To find X_3 , we need to find all pairs of permutations of three degree -1 dominant roots which sum to $\{3^2, 2^7, 1\}$. In order to simplify our calculation, we shall not worry about the dominance order on the rows in this case. If we find all possible cases with all possible permutations of the relevant columns, then we can simply divide by $3! = 6$ in order to get all the ordered possibilities. The first and second column can only contain 1s because it must sum to 3. The next seven columns must sum to 2, so these columns must contain two 1s and one 0. There are only 3 ways to make such a column: place the 0 in the first, second, or third row. Thus, the table below is the only way to place these three roots up to permutations of the columns which sum to the same number.

α	3	3	2	2	2	2	2	2	2	1
ϵ_1	1	1	1	1	0	1	1	1	0	0
ϵ_2	1	1	1	0	1	1	0	1	1	0
ϵ_3	1	1	0	1	1	0	1	0	1	1

As one can see in this table, having more than 3 of any one of the possible three columns that sum to 2 will lead to not enough 1s in the other rows for those to sum to 7. The same

argument will show that you must have three of one type of column which sums to 2, and two of the other types, so this table is the only possibility up to permutation. There are $\frac{7!}{3!2!2!} = 210$ ways to pick the columns in this case. However, we must be wary of overcounting as we did not consider the order of the weights. As we can see, one of the roots will have a 1 in the last column, which distinguishes it from the others so it cannot be the same as the other two roots. Therefore, we only need to order the other two roots, leading to $X_3 = \frac{1}{2!}(210) = 105$.

Now, we need to find $K_{\{3^5, 2^3\}, \{3^2, 2^7, 1\}}$, which is the number of ways to fit three 1s, three 2s, two 3s, two 4s, two 5s, two 6s, two 7s, two 8s, two 9s, and one 10 in the following frame:

$$F(\lambda) =$$

First, we notice that the first row can only contain 1s, the second row can only contain 2s, the third row must have 3s in the first and second column, and the fourth row must have 4 in the first column. There are two options to place the other 4: both 4s are in the fourth row, first and second columns or one 4 is in the third row, third column and one 4 is in the fourth row, first column.

Let us consider the first type of tableau:

1	1	1
2	2	2
3	3	
4	4	
5		

The other 5 must go in either the fifth row, second column or third row, third column. For the first option, we have two options to place the 6s:

1	1	1	1	1	1
2	2	2	2	2	2
3	3		3	3	6
4	4		4	4	
5	5		5	5	
6	6		6		

We can count the total numbers of these tableaux by considering all possible remaining columns for the first and third rows. The first column can only have $\{7, 8, 9\}$ while the third column can have $\{7, 8, 9, 10\}$. We must have at least one of $\{7, 8, 9\}$ in the first and third columns to avoid having duplicates in the second column. Using the above observations in both cases, we have $9 + 9 = 18$ of these tableaux. For the second option, we have three options to place the 6s. Let us list these options while also filling out any numbers whose positions are fixed in these tableaux:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	5	3	3	5	3	3	5
4	4	6	4	4	6	4	4	
5	6		5	7		5	6	
7			6	8		6		
8				9				
9				10				

The first and second tableaux can be counted by just cycling through the remaining entries, giving us $4 + 3$ tableaux. The third can be filled by considering all possible remaining pairs for the first and third columns, giving us 7 tableaux. Thus, we have $4 + 3 + 7 = 14$ of this type and so $18 + 14 = 32$ tableaux of the first type.

Now, let us consider the other type of tableau:

1	1	1
2	2	2
3	3	4
4	5	

The other 5 must go in either the fourth row, third column or fifth row, first column. For the first option, we have one option to place the 6s, so we shall list it after filling out the remaining spaces whose numbers are fixed:

1	1	1
2	2	2
3	3	4
4	5	5
6	6	
7		
8		
9		

There are 4 ways to fill the remaining spaces, so we have 4 tableaux. For the second option, we have three options to place the 6s, so we shall list them after filling out the remaining spaces whose numbers are fixed:

1	1	1
2	2	2
3	3	4
4	5	
5	6	
6		

1	1	1
2	2	2
3	3	4
4	5	6
5	6	
7		
8		
9		

1	1	1
2	2	2
3	3	4
4	5	6
5	7	
6	8	
	9	
	10	

The first tableaux can be filled by considering all possible remaining pairs for the first and third columns, giving us 7 tableaux. The second and third tableaux can be counted by cycling through the remaining entries, giving us 4 + 3 tableaux. Thus, we have $4 + 7 + 4 + 3 = 18$

Now, we need to find $K_{\{3^6, 1^3\}, \{3^2, 2^7, 1\}}$, which is the number of ways to fit three 1s, three 2s, two 3s, two 4s, two 5s, two 6, two 7s, two 8s, two 9s, and one 10 in the following frame:

$$F(\lambda) =$$

Let us consider the first type of tableau:

1	1	1
2	2	2
3	3	
4	4	

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1	1	1
2	2	2
3	3	
4	4	
5	5	

One 6 must go in the sixth row, first column. The other six can either go in the sixth row, second column or third row, third column. For the first option, we have one option to place the 7s, so we shall list it after filling out the remaining spaces whose numbers are fixed:

1	1	1
2	2	2
3	3	7
4	4	8
5	5	9
6	6	10
7		
8		
9		

This is the only way to fill this tableau, so we only have 1 tableau. For the second option, we have three options to place the 7s, so we shall list them after filling out the remaining spaces whose numbers are fixed:

1	1	1
2	2	2
3	3	6
4	4	7
5	5	8
6	7	9
8		
9		
10		

1	1	1
2	2	2
3	3	6
4	4	7
5	5	
6		
7		

1	1	1
2	2	2
3	3	6
4	4	8
5	5	9
6	7	10
7		
8		
9		

The first option only has that 1 tableau. The second option can be filled by considering all possible remaining pairs for the first and third columns, giving us 4 tableaux. The

third option only has that 1 tableau. Therefore, for the first subtype of tableau, we have $1 + 1 + 4 + 1 = 7$ tableaux.

For the second subtype of tableau, we have

1	1	1
2	2	2
3	3	5
4	4	
5		

Both 6s must be placed in the first free diagonal going from top to bottom. There are three ways to place them, so we shall list them after filling out the remaining spaces whose numbers are fixed:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	5	3	3	5	3	3	5
4	4	6	4	4	6	4	4	
5	6		5			5	6	
7		9	6			6		
8								
9								
10								

The first option can be counted by cycling through the remaining entries, giving 2 tableaux. The second option can be counted by filling out all possible pairs of columns for the first and third column and comparing them to see which are possible, which gives 5 tableaux. The third option can be counted by filling out all possible pairs of columns for the first and third column and comparing them to see which are possible, giving 6 tableaux. Therefore, for the second subtype of tableau, we have $2 + 5 + 6 = 13$ tableaux and so we have $7 + 13 = 20$ tableaux for the first type.

Let us consider the second type of tableau:

1	1	1
2	2	2
3	3	4
4		

One of the 5s must be placed in the fourth row, second column. The other 5 can be placed in either the fifth row, first column or fourth row, third column. After filling the remaining numbers which are fixed, let us consider the first subtype of tableau:

1	1	1
2	2	2
3	3	4
4	5	5
6	6	
7		9
8		
9		
10		

There are only two ways to place the remaining numbers, so there are only 2 tableaux of this subtype.

Let us consider the second subtype of tableau:

1	1	1
2	2	2
3	3	4
4	5	
5		

Both 6s must be placed in the first free diagonal going from top to bottom. There are three ways to place them, so we shall list them after filling out the remaining spaces whose numbers are fixed:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	4	3	3	4	3	3	4
4	5	6	4	5	6	4	5	
5	6		5			5	6	
7		9	6			6		
8								
9								
10								

The first option can be counted by cycling through the remaining entries, giving 2 tableaux. The second option can be counted by filling out all possible pairs of columns for the first and third column and comparing them to see which are possible, which gives 5 tableaux. The third option can be counted by filling out all possible pairs of columns for the first and third column and comparing the to see which are possible, giving 6 tableaux. Therefore, for the second subtype of tableau, we have $2+5+6 = 13$ tableaux and so we have $2+13 = 15$ tableaux for the second type. Therefore, we have $20 + 15 = 35$ tableaux in total, so $K_{\{3^6, 1^3\}, \alpha} = 35$.

From the above calculations, we have the multiplicity of the root $\alpha = \{3^2, 2^7, 1\}$ as

$$\text{mult}(\alpha) = X_2 - X_3 + K_{\{3^5, 2^3\}, \alpha} + K_{\{3^6, 1^3\}, \alpha} = 21 - 105 + 50 + 35 = 1$$

As we can see from the previous example, the computation of root multiplicities for smaller degrees becomes more and more complicated. Additionally, the number of potential roots to check increases dramatically as well. In order to move forward, we will make use of the MATLAB program in Appendix A to simplify this process. The following table shows the roots of degree -4 , which are partitions of 28 which do not exceed 4 in the largest entry and do not exceed length 10.

Table 4.4: Degree -4 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	X_3	X_4	$K_{\{4^4, 3^4\}}$	$K_{\{4^5, 3, 2^2, 1\}}$	$K_{\{4^6, 1^4\}}$	$\text{mult}(\alpha)$
$\{4, 3^6, 2^3\}$	36	353	1065	160	540	47	1
$\{3^9, 1\}$	36	378	1260	245	630	42	1
$\{3^8, 2^2\}$	155	1064	2590	385	1190	98	8

The following table shows the roots of degree -5 , which are partitions of 35 which do not exceed 5 in the largest entry and do not exceed length 10.

Table 4.5: Degree -5 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	X_3	X_4	X_5	$K_{\{5^3, 4^5\}}$	$K_{\{5^4, 4^2, 3^2, 1\}}$	$K_{\{5^5, 3^2, 2^2\}}$	$K_{\{5^5, 4, 2^2, 1^2\}}$	$\text{mult}(\alpha)$
$\{5, 4^3, 3^6\}$	46	703	4270	8805	207	2421	1530	1035	1
$\{4^6, 3^3, 2\}$	58	876	5293	11065	405	3222	1767	1197	1
$\{4^5, 3^5\}$	220	2360	11430	20490	576	5376	3138	2118	8

The following table shows the roots of degree -6 , which are partitions of 42 which do not exceed 6 in the largest entry and do not exceed length 10.

Table 4.6: Degree -6 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	X_3	X_4	X_5	X_6	$K_{\{6^2,5^6\}}$
$\{6,5,4^7,3\}$	57	1176	10927	45885	69195	147
$\{5^4,4^4,3^2\}$	78	1596	14272	58819	88722	383
$\{6,4^9\}$	232	3304	24626	89460	122220	210
$\{5^3,4^6,3\}$	294	4342	31881	114590	156870	530
$\{5^2,4^8\}$	1024	11522	70560	222460	276780	742

α	$K_{\{6^3,5^3,4^2,1\}}$	$K_{\{6^4,5,4^2,3,2\}}$	$K_{\{6^4,5^2,3^2,1^2\}}$	$K_{\{6^5,3^4\}}$	$K_{\{6^5,4,3,2^2,1\}}$	$\text{mult}(\alpha)$
$\{6,5,4^7,3\}$	4928	14525	6104	1533	5880	1
$\{5^4,4^4,3^2\}$	7556	17950	7904	1843	7020	1
$\{6,4^9\}$	7896	24150	9912	2478	9660	8
$\{5^3,4^6,3\}$	12040	30130	12804	3066	11535	8
$\{5^2,4^8\}$	18984	49770	20776	5026	19040	44

The following table shows the roots of degree -7 , which are partitions of 49 which do not exceed 7 in the largest entry and do not exceed length 10.

 Table 4.7: Degree -7 Dominant Root Multiplicities for $HE_8^{(1)}$

α	X_2	X_3	X_4	X_5	X_6	X_7
$\{7,5^6,4^3\}$	72	1834	22614	143000	437000	499000
$\{6^3,5^3,4^4\}$	99	2582	30782	188000	567000	646000
$\{6^2,5^6,4,3\}$	101	2682	32389	200000	609000	703000
$\{6^2,5^5,4^3\}$	390	7305	71402	382000	1040000	1110000
$\{6,5^8,3\}$	388	7484	74620	405000	1120000	1210000
$\{6,5^7,4^2\}$	1376	19846	162883	768782	1917720	1911315
$\{5^9,4\}$	4472	52152	366184	1537956	3511200	3276000

α	$K_{\{7,6^7\}}$	$K_{\{7^2,6^4,5^2,1\}}$	$K_{\{7^3,6^2,5^2,4,2\}}$	$K_{\{7^3,6^3,4^2,1^2\}}$	$K_{\{7^4,6,4^3,3\}}$	$K_{\{7^4,5^3,3^2\}}$
$\{7,5^6,4^3\}$	43	4545	41688	15040	21677	20000
$\{6^3,5^3,4^4\}$	191	8637	55998	21707	27190	23638
$\{6^2,5^6,4,3\}$	213	10685	67707	23750	28589	27880
$\{6^2,5^5,4^3\}$	265	13730	92826	34240	43498	39660
$\{6,5^8,3\}$	292	16640	111168	37400	46592	46600
$\{6,5^7,4^2\}$	369	21505	151488	53920	69405	65500
$\{5^9,4\}$	516	33300	243972	84720	110292	106800

α	$K_{\{7^4,6,5,4,3,2,1\}}$	$K_{\{7^5,4,3^3,1\}}$	$K_{\{7^5,4^2,2^3\}}$	$\text{mult}(\alpha)$
$\{7, 5^6, 4^3\}$	67552	8407	5530	1
$\{6^3, 5^3, 4^4\}$	85120	10384	6795	1
$\{6^2, 5^6, 4, 3\}$	89024	10088	6580	1
$\{6^2, 5^5, 4^3\}$	135104	16074	10540	8
$\{6, 5^8, 3\}$	141440	15904	10200	8
$\{6, 5^7, 4^2\}$	214240	25151	16430	44
$\{5^9, 4\}$	339456	39648	25740	192

4.4 Roots of Degree -8 and -9

The following table shows the roots of degree -8 , which are all partitions of 56 which do not exceed 8 in the largest entry and do not exceed length 10.

Table 4.8: Degree -8 Dominant Root Multiplicities for $HE_8^{(1)}$ (Part 1)

Weight α	X_2	X_3	X_4	X_5	X_6	X_7	X_8
$\{8, 6^3, 5^6\}$	82	2534	39224	331443	1530645	3564675	3181002
$\{7, 6^6, 5, 4^2\}$	151	4633	67338	538829	2406100	5538375	4978005
$\{7^3, 5^7\}$	112	3619	54852	449008	2024330	4655070	4143522
$\{7^2, 6^3, 5^4, 4\}$	126	4070	60714	491800	2206769	5075898	4538603
$\{7^2, 6^2, 5^6\}$	467	10829	133237	947327	3864285	8267010	6996969
$\{6^8, 5, 3\}$	128	4363	68012	567959	2611476	6161960	5667312
$\{7, 6^5, 5^3, 4\}$	519	12010	146424	1034706	4210488	9020760	7673220
$\{6^8, 4^2\}$	591	13408	161315	1132292	4595528	9859080	8429015
$\{7, 6^4, 5^5\}$	1722	30707	315914	1976873	7341070	14650815	11799102
$\{6^7, 5^2, 4\}$	1874	33618	344932	2153592	7994882	15995210	12950595
$\{6^6, 5^4\}$	5748	83238	733303	4084324	13883809	25912690	19869735

Weight α	$K_{\{7^8\}}$	$K_{\{8, 7^5, 6^2, 1\}}$	$K_{\{8^2, 7^3, 6^2, 5, 2\}}$	$K_{\{8^2, 7^4, 5^2, 1^2\}}$	$K_{\{8^3, 7^2, 5^3, 3\}}$	$K_{\{8^3, 7, 6^3, 4, 3\}}$
$\{8, 6^3, 5^6\}$	0	1395	43326	15165	77562	100080
$\{7, 6^6, 5, 4^2\}$	49	7083	109142	35445	122154	184221
$\{7^3, 5^7\}$	36	4221	68040	26460	103698	120960
$\{7^2, 6^3, 5^4, 4\}$	42	5487	86424	30474	112236	150318
$\{7^2, 6^2, 5^6\}$	51	6897	115281	41790	163884	207750
$\{6^8, 5, 3\}$	55	9045	146111	42915	161680	244685
$\{7, 6^5, 5^3, 4\}$	59	8814	144693	48270	179612	254391
$\{6^8, 4^2\}$	69	11229	180887	56355	197744	307741
$\{7, 6^4, 5^5\}$	72	11070	191769	65910	259128	348915
$\{6^7, 5^2, 4\}$	83	13938	238245	76200	286952	422527
$\{6^6, 5^4\}$	102	17514	314067	103680	409164	575643

Table 4.9: Degree -8 Dominant Root Multiplicities for $HE_8^{(1)}$ (Part 2)

Weight α	$K_{\{8^3, 7^2, 6, 5, 4, 2, 1\}}$	$K_{\{8^4, 6, 5^2, 4^2\}}$	$K_{\{8^4, 7, 5, 4^2, 3, 1\}}$
$\{8, 6^3, 5^6\}$	212985	65295	127890
$\{7, 6^6, 5, 4^2\}$	355495	88605	173850
$\{7^3, 5^7\}$	286965	80865	164025
$\{7^2, 6^3, 5^4, 4\}$	319380	84642	169071
$\{7^2, 6^2, 5^6\}$	454965	126360	249390
$\{6^8, 5, 3\}$	406875	113142	181270
$\{7, 6^5, 5^3, 4\}$	505730	134001	258555
$\{6^8, 4^2\}$	562555	141912	267580
$\{7, 6^4, 5^5\}$	718140	197730	380985
$\{6^7, 5^2, 4\}$	797265	212178	397220
$\{6^6, 5^4\}$	1128930	309474	584205

Weight α	$K_{\{8^4, 6^2, 5, 3^2, 1\}}$	$K_{\{8^4, 7, 5^2, 3, 2^2\}}$	$K_{\{8^5, 4^2, 3^2, 2\}}$	Multiplicity
$\{8, 6^3, 5^6\}$	112950	79848	15804	1
$\{7, 6^6, 5, 4^2\}$	164385	109299	20028	1
$\{7^3, 5^7\}$	138411	101556	19881	1
$\{7^2, 6^3, 5^4, 4\}$	151017	105276	20076	1
$\{7^2, 6^2, 5^6\}$	218871	154899	29646	8
$\{6^8, 5, 3\}$	180663	108612	17592	1
$\{7, 6^5, 5^3, 4\}$	238374	160788	29880	8
$\{6^8, 4^2\}$	258573	167160	29856	8
$\{7, 6^4, 5^5\}$	344532	236667	44451	44
$\{6^7, 5^2, 4\}$	374472	245931	44808	44
$\{6^6, 5^4\}$	540126	362205	67041	192

The degree -9 roots are all partitions of 63 which do not exceed 9 in the largest entry and do not exceed length 10. The degree -8 roots above provide enough information to calculate the root multiplicity of the degree -9 root $\{7^3, 6^7\}$, which will show a counterexample to Frenkel's conjecture for $HE_8^{(1)}$.

Table 4.10: Degree -9 Root Multiplicity for Dominant Root $\{7^3, 6^7\}$ in $HE_8^{(1)}$

X_2	20884
X_3	306217
X_4	2858282
X_5	17873842
X_6	74118576
X_7	193343885
X_8	282825270
X_9	172869354
$K_{\{8^6, 7^2, 1\}}$	4781
$K_{\{9, 8^4, 7^2, 6, 2\}}$	257915
$K_{\{9, 8^5, 6^2, 1^2\}}$	84231
$K_{\{9^2, 8^3, 6^3, 3\}}$	905520
$K_{\{9^2, 8^2, 7^3, 5, 3\}}$	1381996
$K_{\{9^2, 8^3, 7, 6, 5, 2, 1\}}$	2318176
$K_{\{9^3, 8, 7, 6^2, 5, 4\}}$	3112200
$K_{\{9^3, 8^2, 6, 5^2, 3, 1\}}$	3372894
$K_{\{9^3, 7^4, 4^2\}}$	541135
$K_{\{9^3, 8, 7^2, 6, 4, 3, 1\}}$	4718880
$K_{\{9^3, 8^2, 6^2, 4, 2^2\}}$	2014026
$K_{\{9^4, 6^2, 5^3\}}$	361725
$K_{\{9^4, 7, 6, 5, 4^2, 1\}}$	2520945
$K_{\{9^4, 8, 5^2, 4, 3, 2\}}$	1538313
$K_{\{9^4, 7, 6^2, 3^2, 2\}}$	1383375
$K_{\{9^5, 4^3, 3^2\}}$	54901
Multiplicity	727

The degree -9 root $\alpha = \{7^3, 6^7\}$ has $p^{(8)}\left(1 - \frac{(\alpha|\alpha)}{2}\right) = 726$ but $\text{mult}(\alpha) = 727$, which disproves Frenkel's conjecture.

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APPENDIX

APPENDIX

A

CODE

This code will be updated on Github with further improvements. The link is available at the following:

<https://github.com/mabaker1216/Improved-Kang-Root-Multiplicity-Algorithm>

```
ans1=0;
flag=0;
saverM=[];
l=1;
QB=AR8T; %Sorted array of roots for program to check
r=length(QB);
%checkagainer=[];
checkagainer=zeros(1000000000,8,'uint8');
chkagr=length(checkagainer);
counter2=1;
%doublecheckagainer=[0,0];
%array must be sorted for program to work
% y is the sum being added to
```

```

for i1=1:length(R1)
y2=y-RS(i1,:);
%outside loop does difference between root we are checking
%and all other possible roots
l=1;
r=length(QB);
%inside does two pointer method to check if difference is in array QB
while (l<=r)
    % If sum is greater
    if (isequal(QB(l,1:parsize)+QB(r,1:parsize),y2))
        ya=QB(r,1:parsize);
        yya=r;
        while (isequal(QB(r,1:parsize),ya))&&(r>=1)
            bet=[QB(l,dgr),QB(l,dgr+1),QB(l,dgr+2),QB(l,dgr+3),QB(r,dgr),
                QB(r,dgr+1),QB(r,dgr+2),QB(r,dgr+3),i1];
            bet(bet==0)=[];
            if length(bet)==8
                checkagainer(counter2,:)=sort(bet);
                counter2=counter2+1;
                %Next part removes duplicates in case memory is exceeded
                %based on size of array, saved in 'int8' because it
                %is the smallest number type that can store
                %the rows of the array QB
                if counter2>length(checkagainer)
                    checkagainer=unique(checkagainer,'rows');
                    chkagr2=length(checkagainer);
                    checkagainer=[checkagainer;zeros(chkagr-chkagr2,8,'int8')];
                    counter2=chkagr2+1;
                    flag=flag+1;
                end
            end
            r=r-1;
        end
    end
    r=yya;
end

```

```

        l=l+1;
    elseif (issortedrows([QB(l,1:parsize)+QB(r,1:parsize);y2]
,1:parsize,'ascend'))
        l=l+1; %if sum is too small,
        %move first pointer down
    else
        r=r-1; %if sum is too big,
        %move second pointer up
    end
end
end
i1
end

checkgainer( all(~checkgainer,2), : ) = [];
checkgainer=unique(checkgainer,'rows');

%Next part computes root multiplicities from roots
%stored in checkgainer, all duplicates
%were removed in last step

for l1=1:length(checkgainer(:,5))
    [C,ia,ic]=unique(checkgainer(l1,:));
    a_counts=accumarray(ic,1);
    ch=vector_counter_DM(Roots(C(:),mplc),a_counts(:));
    mult(c,8)=mult(c,8)+ch;
end

mult(c,8);

```