ABSTRACT

PAVLECHKO, ELLA. Partial Travel Time Representation of a Compact Riemannian Manifold with Strictly Convex Boundary. (Under the direction of Teemu Saksala).

In this thesis a compact Riemannian manifold with strictly convex boundary is reconstructed from its partial travel time data. This data assumes that an open measurement region on the boundary is given, and that for every point in the manifold, the respective distance function to the points on the measurement region is known. This geometric inverse problem has many connections to seismology, in particular to microseismicity. The reconstruction is based on embedding the manifold in a function space. This requires the differentiation of the distance functions. Therefore this thesis also studies some global regularity properties of the distance function on a compact Riemannian manifold with strictly convex boundary.

Partial Travel Time Representation of a Compact Riemannian Manifold with Strictly Convex Boundary

by
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DEDICATION

In memory of Ronald J. Pavlechko.

BIOGRAPHY

Ella Pavlechko was raised in the small town of Lewisburg, Pennsylvania. She attended Sarah Lawrence College in Bronxville, New York where she received her Bachelor of Arts with a concentration in Mathematics. She then moved to Raleigh, North Carolina to pursue her Masters and Ph.D. at North Carolina State University.

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CHAPTER

1

INTRODUCTION

In this thesis we solve a geometric inverse problem arising from seismology. In general, inverse problems study how to convert measurements, referred to as data, to information about a physical system. They are of particular use when the desired information about a physical system may be difficult to access or cannot be obtained directly. This is the case with many medical imaging procedures, since doctors do not want to interfere with a patient's organs unless it is deemed necessary. Instead, a physical system is probed with various fields, such as X-rays, acoustic waves, or seismic waves. By probing the system we obtain data which could consist of the time it takes for the wave to reach a receiver, strength of the wave when it reaches a receiver, or the path a wave took through the system.

Let us consider a model in obstetric sonography, more commonly known as ultrasound imaging. In this type of imaging, an ultrasonic scanner produces a sound wave, which propagates in the body, reaches tissue boundaries, and then echoes back to the scanner. The propagation is mathematically modeled by the wave equation, which gives a connection between a wave's speed and specific tissues. Thus the goal of the ultrasound inverse problem is to determine the wave speed as a function of the position inside the body, which can then be used to determine the specific tissues. In practice the wave speed is not what is determined by an ultrasound. Instead the current imaging procedure uses the travel time of the wave, strength of the echo when it reaches the scanner, and location of the scanner to produce an ultrasound image. One point in the image corresponds to the location of the scanner and the time it took for the echo to return, and the shade of the point is determined by the strength of the echo [39, pg. 4]. As a result the image quality largely depends on the operator. Therefore exact measurements of tissue structures are not possible [64]. A better understanding of the related inverse problem would lead to improvements in image resolution.

The inverse problem in this thesis pertains to imaging the internal structures of a planet using seismic waves. This type of imaging is essential to many industries ranging from oil companies searching for pockets of oil or gas, to space agencies examining Mars' inner core to determine habitability [20]. In this type of imaging we probe the planet using seismic waves that are either natural (i.e. earthquakes) or artificial (i.e. explosions). Since these waves are strong enough to propagate through the planet's interior, we then measure the time it takes for a wave to reach a seismic sensor on the surface of the planet. In the case of artificial seismic waves, we can also record the time of wave emission and compute the total travel time. In practice, the emission times of natural sources are not known, but can be closely approximated using a network of seismic sensors to triangulate the source time [63, Section 7.2].

Since a seismic wave encounters variations in velocity due to changes in the physical properties of materials (i.e. composition, temperature, and pressure) [63, pg.69], it follows that a seismic wave's speed provides key insight and indirect information about the materials inside a planet. Thus the goal of this inverse problem is to use the travel time data to determine the wave speed, and consequently the materials inside the planet. Early successes of this inverse problem by Herglotz [23] and Wiechert and Zoeppritz [70] estimated Earth's diameter as well as the location of the mantle, crust, and core. In these papers they assume spherical symmetry of the Earth and that the seismic wave speed only depends on the depth.

A more realistic model is to assume that wave speed depends on the position [67]. This is due to Fermat's principle, which states the path chosen by a seismic wave will locally minimize the travel time [63, pg. 71]. As a consequence, the path taken by a seismic wave, called a ray, will vary in different types of media. For example an anisotropic medium is one in which the wave speed varies based on the position and direction of propagation.

This type of structure appears in crystals, cracks, pores, impurities, and sequences of thin layers [8, pg. 11-12]. On the other hand an isotropic medium is one where the wave speed is the same in all directions.

It has been shown that in an inhomogeneous isotropic medium the rays are geodesics of a conformally Euclidean metric [8, Section 3.1.3], or in other words, a Riemannian metric. As a result the travel time of a seismic wave in this medium is represented as a distance, where the distance between two points in the object of interest is the shortest time it takes for any waves to go from one point to the other. By modeling the propagation of a seismic wave with the wave equation on a manifold, then the wave speed is given by the Riemannian metric. The goal of this inverse problem is to determine the Riemannian manifold from the boundary distance function, which is the distance between points on the boundary of the manifold.

However, the solution to this problem is not unique, since the boundary distance function is invariant under any change of coordinates fixing the boundary. Thus the best that one can do is to recover the Riemannian manifold up to a boundary-preserving isometry. In the geometric community this problem is known as the **boundary rigidity problem** and has been extensively studied. As an example, manifolds with constant curvature conditions have been shown to be boundary rigid [10] as well as metrics close to the Euclidean metric [6].

In general, the boundary rigidity problem is false, since there may be regions 'unseen' by the data and so altering the metric in those regions leads to a negative result. For this reason we typically consider the boundary rigidity problem with additional geometric assumptions. Due to a conjecture by Michel [41], a common assumption is the simplicity of the manifold. By simplicity, we mean a manifold with strictly convex boundary and any two points are connected by a unique distance minimizing geodesic. It has been confirmed that in two dimensions a simple manifold is boundary rigid [48]. However, the question is still open for higher dimensional cases.

A promising alternative is to increase the amount of available data. The scattering data maps a point and direction of entrance of a geodesic to the point and direction of exit. This extends our knowledge from only the distance-minimizing geodesics to all geodesics connecting the boundary. Given the length of each geodesic along with the scattering data provides the lens data. Alternatively, since the distance function is non-linear, we could consider the linearization of the boundary rigidity problem, known as the tensor tomography problem.

Because symmetric 2-tensors that solve the tensor tomography problem provide insight to the solutions of the boundary rigidity problem, this linearization is the main tool that is used to solve the boundary rigidity problem. Tomography is also an important technique in many applications such as imaging the Sun's interior [33], ocean acoustics [42], and medical imaging [64].

Another set of data, and the one that is considered in this thesis, is comprised of the distances from points in the manifold to points on the boundary. That is, assuming an infinite number of point-sources, the **boundary distance function** is the distance from a source to the boundary. Physically the data is associated with deep-focus earthquakes, which are seismic events occurring deeper in Earth's mantle and whose waves are measured on the surface [24, 31, 71]. Because isometric Riemannian manifolds preserve distances and are indistinguishable in terms of geometry, the best one could hope for with this data is recovering a Riemannian manifold up to an isometry. The authors in [29, 34] were able to prove that with the boundary distance function one can recover the Riemannian manifold up to a Riemannian isometry. However, we note that their proof relies on accessing the closest boundary point. Physically this corresponds with placing seismic sensors everywhere on a planet's surface, which is infeasible for real-world applications.

In this thesis we study a more physically realistic scenario by restricting the region where seismic sensors are placed. We consider an open subset of the boundary to represent the measurement region. Thus, our data consists of the distances from any point in the manifold to the points in the measurement region. We will show that this partial travel time data determines the Riemannian manifold up to a Riemannian isometry under some geometric constraints. Specifically, we assume that the manifold has a strictly convex boundary, which allows for any point in the manifold to be accessed from the measurement region using a distance-minimizing geodesic inside the manifold.

The main part of the proof is to use an embedding of the manifold into a function space to reconstruct the topological, smooth, and Riemannian structures. This embedding maps a point to its respective boundary distance function, which is determined by the given data. However, showing this mapping is injective requires the differentiation of the distance functions. Therefore we must determine when the distance functions are smooth, and so we show some global regularity properties of distance functions on a compact Riemannian manifold with strictly convex boundary.

We note that without the assumption of a strictly convex boundary our method fails. For example, consider a horseshoe-shaped planar domain with a measurement region at the tip of one prong. A distance-minimizing curve starting from the measurement region and traveling to the other prong must touch the boundary at some x_0. Then from the point of view of the measurement region, all points on the other prong and equidistant from x_0 are indistinguishable using the data. Thus the mapping from a point to its respective boundary distance function is not injective, and we cannot apply the technique discussed in this thesis.

1.1 Overview of the Thesis

We will briefly introduce the most important geometric notations, definitions, and theorems related to this thesis in Chapter 2. The material presented in Sections 2.1 and 2.2 can be found in many textbooks on Riemannian geometry. Readers familiar with those topics are therefore encouraged to begin reading Sections 2.3 and 2.4, since they deal with more specialized material which is not usually addressed in standard Riemannian geometry courses.

In Chapter 3 of this thesis we review some related results in the field of geometric inverse problems. Section 3.1 formalizes the discourse about the boundary rigidity problem, as well as the scattering and tomography problems. We then discuss the work of [29, 34] in Section 3.2. Since this section is the most similar to the problem in this thesis, it is of particular importance to the proof of our result.

The main result of this thesis is proved in Chapter 4. It is an extended version of the paper *Uniqueness of the partial travel time representation of a compact Riemannian manifold with strictly convex boundary* [47]. To conclude, in Chapter 5 we outline some possible future research questions.

CHAPTER

2

GEOMETRIC PRELIMINARIES

In this chapter we define the fundamental objects and tools of differential and Riemannian geometry. We adapt much of our notation from [37] and [38].

2.1 Smooth Manifolds

We say that a topological space N is a **topological manifold of dimension** $n \in \mathbb{N}$ if it is a second countable Hausdorff space, where each point $p \in N$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. That is, N can be covered by a collection of open sets U_i which have a homeomorphism with some open subsets $U_i' \subseteq \mathbb{R}^n$,

$$\phi_i : U_i \to U_i', \qquad \phi_i(p) = (x_i^1(p), \dots, x_i^n(p)).$$

The sets U_i are called the **coordinate neighborhoods** and ϕ_i are the **coordinate mappings**. For a specific $p \in U_i$, then $(x_i^1(p), \dots, x_i^n(p))$ are called the **local coordinates on** U_i. The pair (U_i, ϕ_i) is called a **coordinate chart**. The coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) are **smoothly compatible coordinate charts** if $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a smooth mapping.

A **smooth atlas** $\mathscr{A} = \{(U_i, \phi_i)\}_i$ refers to the collection of smoothly compatible coordinate charts that cover N. There may be several atlases on a manifold, some of which are finer than others. Thus, we say an atlas is **maximal** if it is not properly contained in any other smooth atlas. This maximal atlas on N is what defines the **smooth structure on** N. A **smooth manifold** N is a topological manifold equipped with a smooth structure. As the next Lemma shows, we only need to specify one smooth atlas on N to get the smooth structure.

Lemma 1. *Let N be a topological manifold.*

(a) Every smooth atlas $\mathscr{A}$ for N is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathscr{A}$.

(b) Two smooth atlases for N determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Proved in [37, Proposition 1.17]. □

Let N be a smooth manifold. A map $f : N \to \mathbb{R}$ is said to be **smooth** if for all $p \in N$ there exists a chart (U, ϕ) in the atlas such that $p \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is a smooth function. The collection of all smooth functions on N is denoted $C^\infty(N)$. Let N_1 and N_2 be smooth manifolds. A map $f : N_1 \to N_2$ is **smooth** if for every $p \in N_1$ there exists smooth charts (U, ϕ) of N_1 containing p and (V, ψ) of N_2 containing $f(p)$ such that $f(U) \subseteq V$ and the map $\psi \circ f \circ \phi^{-1}$ is smooth from $\phi(U)$ to $\psi(V)$. A map $f : N_1 \to N_2$ is a **diffeomorphism** if it is smooth and has a smooth inverse.

The tangent space at p is denoted T_pN and is the n-dimensional vector space of all tangent vectors at p. The disjoint union of the tangent spaces,

$$TN = \coprod_{p \in N} T_pN$$

is the **tangent bundle of** N. Additionally, the **cotangent space** T_p^*N is the space of linear functionals on T_pN. For a smooth mapping $f : N_1 \to N_2$, there is an associated linear map between the tangent spaces, $Df|_p : T_pN_1 \to T_{f(p)}N_2$ called the **differential of** f **at** p.

A manifold is **compact** if for every open cover there exists a finite subcover. Thus if N is compact and smooth it has a finite atlas. Additionally, we say N is **connected** if there does not exist two disjoint, nonempty, open subsets of N whose union is N. Examples of compact, smooth, connected manifolds are the n-dimensional unit-sphere or the n-dimensional torus when $n \geq 1$. All manifolds in this thesis are assumed to be compact, smooth, and connected. We note that often the term **closed manifold** refers to a compact manifold without boundary, whereas an **open manifold** means a noncompact manifold without boundary. In Section 2.2 we survey the existing literature on closed manifolds and solidify notations.

In the remaining sections of this thesis we will turn our attention to a **compact manifold with boundary**, M, which is an n-dimensional smooth manifold whose points have neighborhoods that are diffeomorphic to open subsets of the closed upper-half space $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$. That is, M still has an atlas $\mathscr{A} = \{(U_i, \phi_i)\}_i$ where U_i are open subsets of M and $\phi_i : U_i \to \mathbb{H}^n$ are continuous maps such that $\phi_i(U_i)$ is an open subset of $\mathbb{H}^n$ and $\phi_i : U_i \to \phi_i(U_i)$ is a diffeomorphism. The **boundary of** M is denoted

$$\partial M = \{p \in M \ : \ \text{there exists } (U_i, \phi_i) \in \mathscr{A} \text{ such that } p \in U_i \text{ and } \phi_i(p) \in \partial \mathbb{H}^n\},$$

while the **interior of** M is

$$M^{int} = \{p \in M \ : \ \text{there exists } (U_i, \phi_i) \in \mathscr{A} \text{ such that } p \in U_i \text{ and } \phi_i(U_i) \in \mathbb{H}^n \setminus \partial \mathbb{H}^n\}.$$

Then $M = \partial M \cup M^{int}$ where $\partial M \cap M^{int} = \emptyset$. Note that ∂M can also be viewed as an $n-1$ dimensional manifold embedded in M. Examples of a compact smooth manifold with boundary are the unit disc, paraboloid, cylinder, and catenoid.

2.1.1 Vector & Tensor Fields

If N is a smooth manifold with or without boundary, a **smooth vector field on** N is a smooth map $X : N \to TN$ with the property that $X(p) \in T_pN$ for each $p \in N$. Equivalently, X is a smooth vector field on N if it is a section of the map $\pi : TN \to N$ that is smooth and satisfies $\pi \circ X = id_N$. Thus we can visualize smooth vector fields as tangent vectors attached to N which vary smoothly as they move from point to point. We will denote the collection of all smooth vector fields on N as

$$\mathscr{X}(N) = \{X : N \to TN \ : \ X(p) \in T_pN \text{ for all } p \in N, X \text{ is smooth}\}.$$

Locally, if we let $(E_1,...,E_n)$ be a collection of smooth vector fields that are defined in an open neighborhood $U \subseteq N$, and such that for all $p \in U$, $(E_1(p),...,E_n(p))$ is a basis of T_pN, then $(E_1,...,E_n)$ is called a **local frame**.

Example 2. *Suppose N is a smooth n-manifold with or without boundary. If $(U,(x^i))$ is any smooth coordinate chart for N around p then we can say that any point in U near p is given by the local coordinates $(x^1,...,x^n)$. Thus, $(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n})$ forms a local frame. Then any vector $v \in T_pN$ at p can be given in Einstein summation notation as*

$$v(p) = v^i \frac{\partial}{\partial x^i}\Big|_p,$$

where $v^i \in \mathbb{R}$ are called the components of v with respect to the coordinate basis [37, Proposition 3.15]. A vector field $X \in \mathscr{X}(N)$ at p is given locally as

$$X(p) \;=\; X^i(p)\frac{\partial}{\partial x^i}\Big|_p$$

where $X^i : U \to \mathbb{R}$ are called the component functions of X.

Analogously to vector fields, we define a **smooth covector field** (or the differential 1-form) to be the smooth mapping $\omega : N \to T^*N$ such that $\omega(p) \in T_p^*N$. Equivalently, a covector field is a section of T^*N.

Example 3. *Suppose N is a smooth n-manifold with or without boundary. Recall if $f \in T_p^*N$ then $f : T_pN \to \mathbb{R}$ is a linear function, and if $(U,(x^i))$ is any smooth coordinate chart for N then*

$$f(v) \;=\; f\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) \;=\; v^i f\left(\frac{\partial}{\partial x^i}\Big|_p\right) \;=\; \omega_i|_p\, v^i, \qquad v \in T_pN$$

where $\omega_i|_p = f\left(\frac{\partial}{\partial x^i}|_p\right) \in \mathbb{R}$. Thus if the v^i form a basis for T_p^*N then the corresponding n-tuple $(\omega_1,...,\omega_n)$ determines a unique linear function, and we would consider the mapping $\omega : N \to T^*N$ to map a point p to the uniquely determined $f \in T_p^*N$.

However, the v^i may not be an appropriate choice of basis, so define the ***dual coframe*** *to be the set of vectors $(\xi^1,...,\xi^n)$ that form a basis for T_p^*N and $\xi^i(\frac{\partial}{\partial x^j}) = \delta^i_j$. Then the smooth covector field ω at p is given locally as*

$$\omega(p) \;=\; \omega_i|_p\; \xi^i \;=\; \omega\left(\frac{\partial}{\partial x^i}\Big|_p\right)\xi^i$$

where $\omega_i : U \to \mathbb{R}$ are called the component functions of ω.

Notice ω maps a point p to a linear function of T_pN, whose input is a vector, so there is a natural pairing between a vector field and a covector field. Let $\omega(X): T_p^*N \times T_pN \mapsto \mathbb{R}$ be such a pairing, given by

$$(\omega(p), X(p)) \mapsto \omega(p)(X(p)).$$

Using local coordinates $(U, (x^i))$ on N such that $\omega = \omega_i \xi^i$ and $X = X^j \frac{\partial}{\partial x^j}$ then $\omega(X) = \omega_i X^i$. Abstracting this notion of pairing vector and covector fields, we define a multilinear mapping

$$F : \underbrace{T_pN \times \cdots \times T_pN}_{k} \times \underbrace{T_p^*N \times \cdots \times T_p^*N}_{\ell} \to \mathbb{R}$$

to be a (k,ℓ)-**tensor** or k-covariant and ℓ-contravariant tensor. Then the collection of all (k,ℓ)-tensors on T_pN is denoted $T^{(k,\ell)}(T_pN)$. A k-covariant tensor is **symmetric** if it is unchanged by interchanging any pair of arguments:

$$F(v_1, ..., v_i, ..., v_j, ..., v_k) = F(v_1, ..., v_j, ..., v_i, ..., v_k), \qquad v_1, ..., v_k \in T_pN.$$

A k-covariant tensor is **antisymmetric** (also called skew symmetric or alternating) if interchanging any pair of arguments makes it differ by a sign,

$$F(v_1, ..., v_i, ..., v_j, ..., v_k) = -F(v_1, ..., v_j, ..., v_i, ..., v_k).$$

Example 4. *If F is a $(1,1)$-tensor then it is a multilinear mapping $F : T_pN \times T_p^*N \to \mathbb{R}$. In local coordinates, let $(E_1, ..., E_n)$ be the local frame and $(\xi^1, ..., \xi^n)$ be the dual coframe. For $v := v^i E_i \in T_pN$ and $\omega := \omega_j \xi^j \in T_p^*N$ then*

$$\begin{aligned}
F(v,\omega) &= F_i^{\ j} \xi^i(v) E_j(\omega) \\
&= F_i^{\ j} v^k \xi^i(E_k) \omega_\ell E_j(\xi^\ell) \\
&= F_i^{\ j} v^k \delta^i_k \omega_\ell \delta^\ell_j \\
&= F_i^{\ j} v^i \omega_j.
\end{aligned}$$

Thus, $F_i^{\ j}$ is treated as a matrix. This leads to the intuition that tensors are multi-dimensional matrices.

For a $(1,1)$-tensor denoted F the trace $\mathrm{Tr}(F)$ would then be the sum of the diagonal entries of the matrix $F_i^{\ j}$, so as an operator it is expressed $\mathrm{Tr}: T^{(1,1)}(T_pN) \to \mathbb{R}$ where $\mathrm{Tr}(F) = F_i^{\ i}$. In general, for (k,ℓ)-tensors with $k,\ell \geq 1$, we define the **trace** as $\mathrm{Tr}: T^{(k,\ell)}(T_pN) \to T^{(k-1,\ell-1)}(T_pN)$

by letting Tr(F) at $(v_1,...,v_{k-1},\omega^1,...,\omega^{\ell-1})$ be the trace of the $(1,1)$-tensor

$$F(v_1,...,v_{k-1},\cdot,\omega^1,...,\omega^{\ell-1}\cdot) \in T^{(1,1)}(T_pN).$$

We denote the (k,ℓ) **tensor-bundle of** N as

$$T^{(k,\ell)}(N) = \coprod_{p\in N} T^{(k,\ell)}(T_pN)$$

A section of a tensor bundle is called a **tensor field on** N. That is, a tensor field assigns a tensor to each point of a manifold.

Example 5.

- *A vector field X is a $(0,1)$-tensor field.*
- *A covector field ω is a $(1,0)$-tensor field.*
- *Since a $(0,0)$-tensor is just a real number, a $(0,0)$-tensor field is a continuous real-valued function.*

For more information on multilinear mappings and tensors, we refer the reader to [19, Chapter 4] or [38, Appendix B].

2.2 Riemannian Manifolds

We define a **Riemannian manifold** (N,g) to be a smooth manifold N equipped with a smooth **Riemannian metric** g, which is a symmetric and positive definite 2-covariant tensor field. We note that on any compact n-dimensional smooth manifold there may be several Riemannian metrics possible. Thus, when defining a Riemannian manifold we must specify our g.

Example 6. *Take D^2 to be the 2-dimensional unit disc. Consider these three possible Riemannian metrics for $p \in D^2$: Euclidean g_E, hyperbolic g_H, and cylindrical g_C (when p is away from the origin). To define these metrics, first specify a coordinate system (x^1,x^2) for a point $p \in D^2$. This induces a local frame $(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}) \in T_pD^2$ and dual coframe $(dx^1,dx^2) \in T_p^*D^2$ where $dx^i(dx_j) = \delta^i_j$. Then any $\omega \in T_p^*D^2$ is given as $\omega = \omega_i dx^i$, and we define the metric* *using Einstein summation notation*

$$g : T_pD^2 \times T_pD^2 \to \mathbb{R}, \qquad g = g_{ij}\, dx^i \otimes dx^j,$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. *In Table 2.1 we summarize the coordinates, resulting coframes, and definitions of some different metrics.*

	Euclidean	Hyperbolic	Cylindrical
(x^1, x^2)	(x, y)	(x, y)	$(r, \theta), r > 0$
(dx^1, dx^2)	(dx, dy)	(dx, dy)	$(dr, d\theta)$
g	$g_E = dx^2 + dy^2$	$g_H = \frac{dx^2+dy^2}{1-x^2-y^2}$	$g_C = dr^2 + r^2 d\theta^2$

Table 2.1 Three possible Riemannian metrics g on the 2-dimensional unit disc

Notice that the g_{ij} *for each metric can be expressed as a* 2×2 *matrix,*

$$g_E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_H = \frac{1}{1-x^2-y^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_C = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}.$$

For any $p \in N$ the metric tensor g defines an inner product on T_pN, denoted $\langle v, w\rangle_g$ for $v, w \in T_pN$. Thus the **length of a vector** $v \in T_pN$ is given by $|v|_g = \sqrt{\langle v, v\rangle_g}$. The **unit-sphere at** p is denoted $S_pN = \{v \in T_pN \ : \ |v|_g = 1\} \subset T_pN$.

Example 7. *Continuing with the previous example on the unit disc, if a vector* $v \in T_pD^2$ *is given in local coordinates as* $v = v^i \frac{\partial}{\partial x^i}$, *then the length of* v *in the different metrics are*

$$|v|_{g_E} = \sqrt{(v^1)^2 + (v^2)^2}, \qquad |v|_{g_H} = \frac{\sqrt{(v^1)^2 + (v^2)^2}}{1-x^2-y^2}, \qquad |v|_{g_C} = \sqrt{(v^1)^2 + (r\,v^2)^2}.$$

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds. A **Riemannian isometry** is a diffeomorphism $\psi : N_1 \to N_2$ that preserves the Riemannian inner product:

$$\langle v, w\rangle_{g_1} = \langle D\psi|_p v, D\psi|_p w\rangle_{g_2}, \qquad v, w \in T_pN.$$

We say that (N_1, g_1) and (N_2, g_2) are **Riemannian isometric** if there exists a Riemannian isometry between them. The **isometry class** of (N, g) is the collection of Riemannian man-

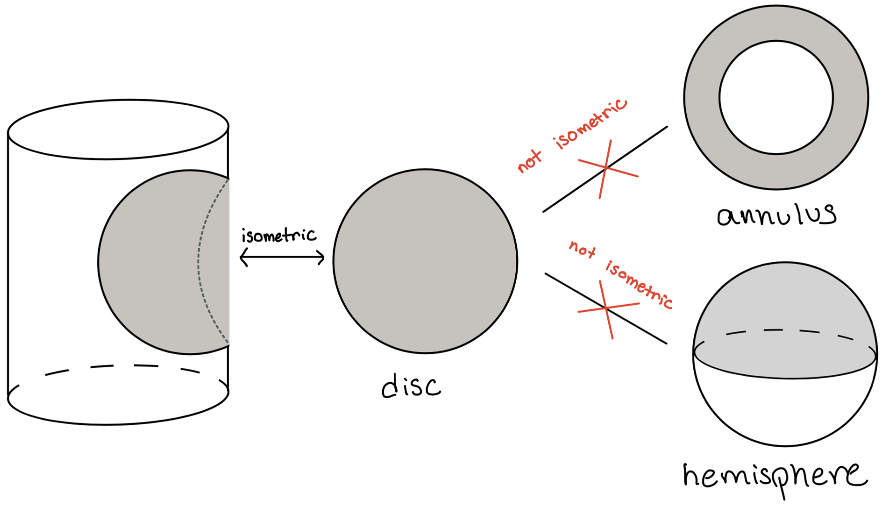

Figure 2.1 Examples of isometries

ifolds that are isometric to (N, g). From the point of view of Riemannian geometry, two isometric manifolds are the same.

Example 8. *Continuing with our previous example, the 2-dimensional unit disc with Euclidean metric g_E is isometric to the 2-dimensonal unit disc embedded on the cylinder. This is demonstrated in Figure 2.1. However, we note that the 2-dimensonal unit disc is not isometric to a hemisphere. The reason for this is that the sphere has constant positive curvature and the disc has constant zero curvature. These quantities are preserved in isometries. The 2-dimensional unit disc is also not isometric to the annulus, since they have different fundamental groups, which means they are not homeomorphic.*

2.2.1 Levi-Civita Connection

Now that we can compute the lengths of vectors using the metric g, the natural question arises how a vector changes if the base point is moved, which generalizes the concept of derivative. That is, for a vector field X on the manifold N, we need an invariant way to compare the values of vector fields at different points. In this section we define a coordinate-invariant set of rules for taking directional derivatives of vector fields.

Let $\mathscr{X}(N)$ be the collection of smooth vector fields on N. We define an **affine connection** ∇

(also known as a linear connection) as a map

$$\nabla : \mathscr{X}(N) \times \mathscr{X}(N) \to \mathscr{X}(N), \qquad \nabla : (X, Y) \mapsto \nabla_X Y,$$

such that it has the following properties:

(a) (Linearity in X) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \qquad f, g \in C^\infty(N)$.

(b) (Linearity in Y) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \qquad a, b \in \mathbb{R}$.

(c) (Product Rule) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \qquad f \in C^\infty(N)$.

We note that property (b) only has linearity over constants in Y, so an affine connection need not be linear over $C^\infty(N)$ in Y. Thus, ∇ may not be a (2,1)-tensor.

Locally, let $(E_1, ..., E_n)$ be a local frame (Defined in Section 2.1.1). Using Einstein summation notation, then components of an affine connection are

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k. \tag{2.1}$$

where local functions Γ_{ij}^k are called the **Christoffel symbols**. In local coordinates this affine connection took the vector E_j and saw how much it changed as it was moved in the E_i-th direction, resulting in a vector given by the (E_k) frame. By the properties of an affine connection if $X, Y \in \mathscr{X}(N)$ such that $X = X^i E_i$ and $Y = Y^j E_j$ then

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma_{ij}^k)E_k,$$

is called the **covariant derivative of** Y **in the direction of** X. Intuitively, we understand the covariant derivative is like the directional derivative of Y in the direction of X.

We now use the covariant derivative of a (k, ℓ)-tensor field T to define a $(k, \ell+1)$-tensor field. Let $Y_i \in \mathscr{X}(N)$ and ω_i are smooth covector fields. Then the **total covariant derivative** of a tensor field T is

$$\nabla T(\omega^1, ..., \omega^k, Y_1, ..., Y_\ell, X) = \nabla_X T(\omega^1, ..., \omega^k, Y_1, ..., Y_\ell), \qquad X \in \mathscr{X}(N).$$

Example 9. *If $f \in C^\infty(N)$ then f is a $(0,0)$-tensor field. Then*

$$Df(X) = \nabla f(X) = \nabla_X f = Xf, \qquad X \in \mathscr{X}(N).$$

is a $(1,0)$*-tensor field called the* ***differential of*** f. *The* ***covariant Hessian*** *of* f *is a* $(2,0)$*-tensor field defined by applying the total covariant derivative* ∇ *twice, so Hess*$(f) = \nabla^2 f$. *Using [38, Proposition 4.21], the covariant Hessian at* p *becomes*

$$Hess_p(f)(X,Y) = \nabla^2 f(Y,X) = \nabla_X(\nabla_Y f) - \nabla_{(\nabla_X Y)} f = Y(Xf) - (\nabla_Y X) f.$$

For a smooth curve $\gamma : I \to N$ the connection determines a unique operator $D_t : \mathscr{X}(\gamma) \to \mathscr{X}(\gamma)$ called the **covariant derivative along** γ [38, Theorem 4.24]. If $V \in \mathscr{X}(\gamma)$ is induced by a vector field $Y \in \mathscr{X}(N)$ (i.e. $V(t) = Y(\gamma(t))$) then

$$D_t V(t) = \nabla_{\gamma'(t)} Y.$$

Furthermore, consider the smooth local coordinates $(x^1, ..., x^n)$ near $\gamma(t_0)$, and local frame $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$. We write $V(t) = V^i(t) \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$ for t near t_0, where $V^1, ..., V^n$ are smooth real-valued functions defined on some neighborhood of t_0 in I. Then,

$$D_t V(t) = \left(\dot{V}^k(t) + \dot{\gamma}^i(t) V^j(t) \Gamma^k_{ij}(\gamma(t))\right) \frac{\partial}{\partial x^k}\Big|_{\gamma(t)}. \tag{2.2}$$

We note that there are many possible connections on the given manifold N. However, when we specify the metric g on N then there is a unique linear connection compatible with g.

Lemma 10 (Fundamental Lemma of Riemannian Geometry)**.** *Let* (N, g) *be a Riemannian manifold, and* $X, Y, Z \in \mathscr{X}(N)$. *There exists a unique affine connection* ∇ *on* N *that satisfies properties (a)-(c) as well as:*

(d) (Compatibility with metric tensor) $\nabla_X \langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$.

(e) (Symmetry/Torsion free) $\nabla_X Y - \nabla_Y X = [X, Y]$, *where* $[X, Y]$ *is the Lie bracket.*

This connection is called the ***Levi-Civita connection of*** g.

Proof. Proven in [38, Theorem 5.4]. □

Informally, the compatibility condition indicates we are able to differentiate the inner product by the usual 'product rule'. Then the symmetry implies in local coordinates $(x^1, ..., x^n)$

with associated frame $(dx^1, ..., dx^n)$

$$\nabla_{dx^i} dx^j - \nabla_{dx^j} dx^i = [dx^i, dx^j] = 0, \qquad i, j = 1, ..., n.$$

The statement above is equivalent to the fact that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

We consider a point $p \in N$ where $(x^1, ..., x^n)$ are the local coordinates near p, this gives rise to a local frame $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$. Then any vector $v \in T_p N$ can be given as $v = (v^1 \frac{\partial}{\partial x^1}, ..., v^n \frac{\partial}{\partial x^n})$. Thus the Levi-Civita connection near p has the form

$$\nabla_v Y = v^i \frac{\partial}{\partial x^i} Y^k(p) \frac{\partial}{\partial x^k} + \Gamma^k_{ij}(p) v^i Y^j(p) \frac{\partial}{\partial x^k}, \tag{2.3}$$

where

$$\Gamma^k_{ij}(p) = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{2.4}$$

2.2.2 Gradient, Divergence, and Laplace-Beltrami Operators

We note, in a typical Calculus sequence ∇ is used to denote the gradient operator. Then for $f \in C^\infty(\mathbb{R}^3)$ we understand the gradient of f to be a vector field representing the direction of maximum increase along a surface. However, in this thesis we use ∇ to represent the Levi-Civita connection, which is a geometric notion of directional derivative. Thus in this section we generalize the gradient to a Riemannian manifold. Let $f \in C^\infty(N)$ then the **gradient** is defined by the equation

$$\langle \mathrm{grad} f, X \rangle_g = Xf = Df(X), \qquad \text{for all } X \in \mathscr{X}(N). \tag{2.5}$$

On the Riemannian manifold (N, g) the **divergence operator**, div : $\mathscr{X}(N) \to C^\infty(N)$, is defined by

$$\mathrm{div}(X) = \mathrm{Tr}(\nabla X),$$

where Tr is the trace operator (see Section 2.1.1) and ∇ is the total covariant derivative for the Levi-Civita connection.

Example 11. *Let $N = \mathbb{R}^n$, then for vector field given by $X = X^k e_k$ we find*

$$div(X) = grad \cdot X = \frac{\partial}{\partial x^k} X^k,$$

which agrees with our Euclidean sense of divergence operator.

Using the divergence operator we create a second order linear elliptic partial differential operator of a Riemannian manifold (N, g) called the **Laplace-Beltrami operator** $\Delta_g : C^\infty(N) \to C^\infty(N)$, defined by

$$\Delta_g f = \text{div}(\text{grad} f), \qquad \text{for all } f \in C^\infty(N). \tag{2.6}$$

(Note that many books define $\Delta_g f = -\text{div}(\text{grad} f)$ so the operator has nonnegative eigenvalues.) In local coordinates $(x^1, ..., x^n)$ the gradient, divergence, and Laplace-Beltrami operator are given by

$$\begin{aligned}
\text{grad} f &= \left(g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j}, \\
\text{div}\left(X^i \frac{\partial}{\partial x^i} \right) &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(X^i \sqrt{\det(g)} \right), \\
\Delta_g f &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial f}{\partial x^j} \right),
\end{aligned}$$

where g^{ij} is the inverse matrix of g [38, Proposition 2.46].

2.2.3 Distance Function

The metric tensor g also defines the length of any path in N. Let $\gamma : [a, b] \to N$ be a piecewise smooth curve segment, then the **length of the path** γ is denoted

$$\mathscr{L}(\gamma) = \int_a^b \left| \frac{d\gamma(t)}{dt} \right|_g dt. \tag{2.7}$$

We note that the length of the path $\gamma([a, b])$ is invariant of the parametrization.

Lemma 12. *Let $\alpha : [\tilde{a}, \tilde{b}] \to [a, b]$ be C^1 and $\alpha' \neq 0$. We say $\tilde{\gamma} = \gamma \circ \alpha$ is the reparametrization of γ. Then*

$$\mathscr{L}(\tilde{\gamma}) = \mathscr{L}(\gamma).$$

Proof. This is shown in [37, Proposition 13.25]. The key idea of the proof is to let γ be smooth and either $\alpha' > 0$ or $\alpha' < 0$, then use a change of variables with $\alpha(t) = s$ so the following holds,

$$\mathscr{L}(\tilde{\gamma}) = \int_{\tilde{a}}^{\tilde{b}} \left| \frac{d\tilde{\gamma}(\alpha(t))}{dt} \right|_g dt = \int_{\tilde{a}}^{\tilde{b}} |\alpha'(t)\gamma'(\alpha(t))|_g \, dt = \int_a^b |\gamma'(s)|_g \, ds = \mathscr{L}(\gamma).$$

Then for any piecewise smooth curve, apply the same argument on each subinterval where γ is smooth. □

This lets us define the **distance function on** N**,**

$$d(x,y) = \inf_{\gamma}\{\, \mathscr{L}(\gamma) \, : \, \gamma \text{ is a piecewise smooth curve starting at } x \text{ and ending at } y\}.$$

From this definition, the following Lemma shows the compact Riemannian manifold (N,g) is a compact metric space, which induces a metric topology. However, recall by definition that the manifold N already has a pre-existing topology. So we must also show the existing topology agrees with the metric topology induced on N.

Lemma 13. *Let (N,g) be a connected Riemannian manifold with or without boundary. With the distance function d then N is a metric space whose metric topology is the same as the given manifold topology.*

Proof. See the proof in [38, Proposition 2.55]. □

We say that a path $\gamma : [a,b] \to N$ with $\gamma(a) \neq \gamma(b)$ is **distance minimizing** if $\mathscr{L}(\gamma) = d(\gamma(a), \gamma(b))$ (i.e. it is a shortest path between its endpoints).

Example 14. *Consider $x, y \in S^2$ to be points on the equator that are not anitpodal, shown in Figure 2.2. There are several possible curves connecting x and y, however, a distance minimizing curve γ is the shortest path along the equator which connects them. All other curves like γ_1 (which goes around the back of the sphere) and γ_2 (which does not lay on the equator) are longer than γ.*

2.2.4 Geodesics

Consider a smooth path $\gamma : [a,b] \to N$ and let $\gamma(t) = (x^1(t), ..., x^n(t))$ be its representation in local coordinates on (U,ϕ). Then γ is said to be a **geodesic** if it satisfies the second-order

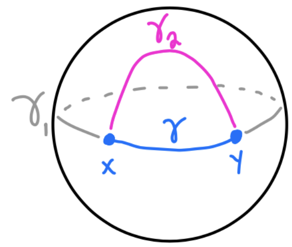

Figure 2.2 Distance minimizing curve γ shown on the sphere

differential equation,

$$D_t\dot{\gamma}(t) = \frac{d^2x^k(t)}{dt^2} + \frac{dx^i(t)}{dt}\frac{dx^j(t)}{dt}\Gamma^k_{ij}(x(t)) = 0 \tag{2.8}$$

known as the **geodesic ODE**. Here, Γ^k_{ij} are the Christoffel symbols given in equation (2.4) and $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$. Recall in Euclidean space that straight lines have an acceleration, or second derivative, equal to zero. In the Riemannian sense, equation (2.8) is the generalization of the acceleration being zero. Thus, we consider geodesics to be the generalization of 'straight' lines on the manifold.

We now derive the geodesic ODE. If $\gamma : [a,b] \to N$ is a smooth path, then a **variation of** γ is a family of curves $\Gamma : I \times [a,b] \to N$ such that I is an open interval containing 0 and $\Gamma(0,t) = \gamma(t)$. It is called a **proper variation** if in addition, $\Gamma(s,a) = \gamma(a)$ and $\Gamma(s,b) = \gamma(b)$ for all $s \in I$. Notice that when s is fixed then $\Gamma(s,\cdot)$ is a smooth curve as depicted in Figure 2.3. We say the **variation field of** Γ is the piecewise smooth vector field $V(t) = \partial_s\Gamma(0,t)$ along γ. We say that a vector field V along γ is **proper** if $V(a) = 0$ and $V(b) = 0$. Clearly by the definitions, the variation field of every proper variation is itself proper.

Lemma 15 (First Variation Formula)**.** *Let (N,g) be a Riemannian manifold. Suppose $\gamma : [a,b] \to N$ is a smooth curve, $\Gamma : I \times [a,b] \to N$ is a proper variation of γ, and V is its variation field. Then $\mathscr{L}_g(\Gamma(s,\cdot))$ is a smooth function of s and*

$$\frac{d}{ds}\bigg|_{s=0}\mathscr{L}_g(\Gamma(s,\cdot)) = -\int_a^b \langle V, D_t\dot{\gamma}\rangle_g \, dt. \tag{2.9}$$

Here $D_t\gamma$ refers to the covariant derivative along γ, defined in Section 2.2.1.

Proof. The proof is in [38, Theorem 6.3] and thus omitted here. □

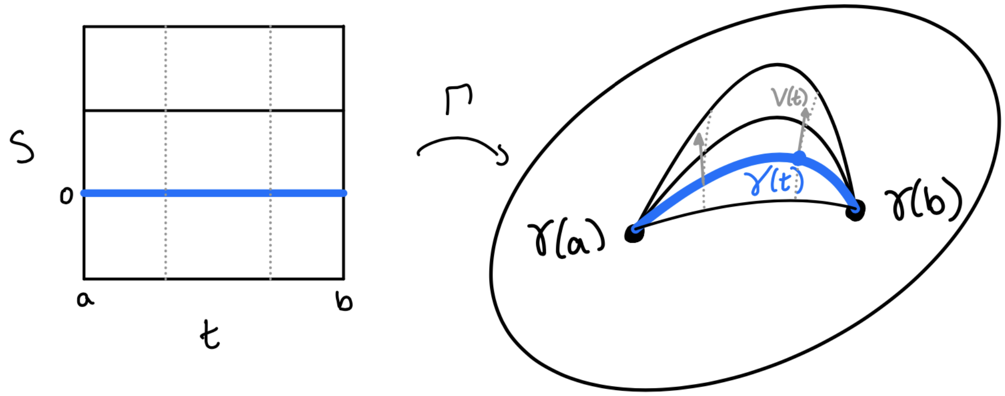

Figure 2.3 A proper variation of γ, denoted Γ.

We say that a smooth curve γ is a **critical point of** $\mathscr{L}_g$ if for every proper variation $\Gamma(s,\cdot)$ of γ the derivative of $\mathscr{L}_g(\Gamma(s,\cdot))$ with respect to s is zero at $s = 0$.

Lemma 16. *A smooth curve γ is a critical point for $\mathscr{L}_g$ if and only if it is a geodesic.*

Proof. If $\left.\frac{d}{ds}\right|_{s=0} \mathscr{L}_g(\Gamma(s,\cdot)) = 0$ for every proper variation Γ, then by (2.9) it follows that $\langle V, D_t\dot{\gamma}\rangle_g = 0$ for all proper variation fields V. Then from [38, Theorem 6.4] it follows that $D_t\dot{\gamma} = 0$, which is equivalent to (2.8).

If γ is a geodesic it satisfies (2.8). Thus, $D_t\dot{\gamma} = 0$ and by (2.9) we have $\left.\frac{d}{ds}\right|_{s=0} \mathscr{L}_g(\Gamma(s,\cdot)) = 0$ for every proper variation Γ. Thus γ is a critical point of $\mathscr{L}_g$. $\square$

Thus geodesics are critical points of $\mathscr{L}_g$, which gives rise to equation (2.8). We now provide a few examples of how to use this equation to compute geodesics on different manifolds.

Example 17. *(Geodesics on a sphere) Recall that a sphere of radius $r > 0$ is a 2-dimensional Riemannian manifold defined by the set of points $p \in \mathbb{R}^3$ such that $\|p\|_{g_E} = r$. Consider the coordinate neighborhood on the sphere parameterized by spherical coordinates*

$$X(\theta,\phi) = (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi)), \qquad \theta \in (0,2\pi), \phi \in (0,\pi).$$

All points on the sphere are given by similar coordinate neighborhoods. So we consider the local coordinates for this manifold to be $(x^1, x^2) = (\theta,\phi)$. We first calculate the metric, g on *the sphere.*

$$
\begin{aligned}
X_\theta &= (-r\sin(\theta)\sin(\phi), r\cos(\theta)\sin(\phi), 0) \\
X_\phi &= (r\cos(\theta)\cos(\phi), r\sin(\theta)\cos(\phi), -r\sin(\phi)) \\
g &= \begin{bmatrix} X_\theta \cdot X_\theta & X_\theta \cdot X_\phi \\ X_\phi \cdot X_\theta & X_\phi \cdot X_\phi \end{bmatrix} = \begin{bmatrix} r^2\sin^2(\phi) & 0 \\ 0 & r^2 \end{bmatrix}
\end{aligned}
$$

and if $\sin(\phi) \neq 0$ *it has the inverse*

$$
g^{-1} = \begin{bmatrix} \frac{1}{r^2\sin^2(\phi)} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}.
$$

Using formula (2.4) *then we find* Γ_{ij}^k *for all* $i, j, k \in \{1, 2\}$.

$$
\begin{aligned}
\Gamma_{11}^1 &= 0 = \Gamma_{22}^1 \\
\Gamma_{12}^1 &= \frac{\cos(\phi)}{\sin(\phi)} = \Gamma_{21}^1 \\
\Gamma_{11}^2 &= -\cos(\phi)\sin(\phi) \\
\Gamma_{12}^2 &= 0 = \Gamma_{21}^2 \\
\Gamma_{22}^2 &= 0
\end{aligned}
$$

Thus, the geodesic ODE gives the system of equations

$$
\begin{cases} \ddot{\theta} + 2\frac{\cos(\phi)}{\sin(\phi)}\dot{\theta}\dot{\phi} = 0 \\ \ddot{\phi} - \cos(\phi)\sin(\phi)(\dot{\theta})^2 = 0. \end{cases} \tag{2.10}
$$

We now show that curves with constant θ *are solutions to the system. If* $\theta = c$, *then the first equation of* (2.10) *holds trivially, while the second equation becomes* $\ddot{\phi} = 0$, *implying it is affine, or* $\phi(t) = at + b$ *for* $a, b \in \mathbb{R}$. *Thus, the curve on the sphere given by*

$$
\gamma(t) = (r\cos(c)\sin(\phi(t)), r\sin(c)\sin(\phi(t)), r\cos(\phi(t))), \qquad c \in (0, 2\pi)
$$

is a geodesic. These curves are known as great circles since it is the intersection of the sphere and a plane through the center of the sphere. By the rotational symmetry of the sphere, all great circles on the sphere are geodesics.

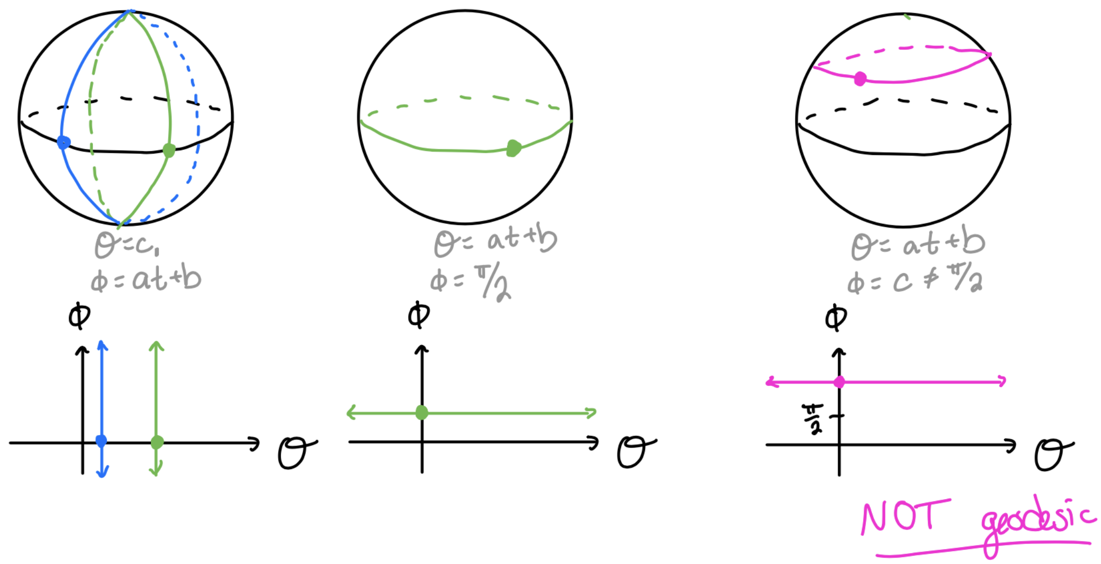

Figure 2.4 Some geodesics on a sphere of radius r.

However, we note that the only 'parallel' (i.e. curves with constant ϕ) of the sphere that is a geodesic is the equator (when $\phi = \frac{\pi}{2}$). Let $\phi = c$, then the first equation of (2.10) *becomes $\ddot{\theta} = 0$, implying $\theta(t) = at + b$ for $a, b \in \mathbb{R}$. Using this in the second equation of* (2.10)*, then it follows $-\cos(c)\sin(c)(a)^2 = 0$. Since $a = 0$ implies the geodesic is a point, it is an uninteresting case, and ignored. Taking $a \neq 0$ then either $\cos(c) = 0$ or $\sin(c) = 0$. However, to have g^{-1} and the Christoffel symbols defined, we assumed $\sin(\phi) \neq 0$. Thus, $\cos(c) = 0$, implying $c = \frac{\pi(2n+1)}{2}$ for $n \in \mathbb{Z}$. The resulting geodesic is given by*

$$\gamma(t) = (r\cos(\theta(t)), r\sin(\theta(t)), 0)$$

which is the equator of the sphere, and also a great circle.

Example 18. *(Geodesics on a catenoid [16, pg. 258-261]) We now consider an example which has rotational symmetry across only one axis. Define the catenoid to be a surface of revolution generated by rotating the curve $(v^2 + 1, 0, v)$ about the z-axis. Consider the coordinate neighborhood on the catenoid which is parameterized as*

$$X(u, v) = ((v^2 + 1)\cos(u), (v^2 + 1)\sin(u), v) \in \mathbb{R}^3, \qquad u \in (0, 2\pi), v \in \mathbb{R}.$$

All points on the catenoid are given by similar coordinate neighborhoods. For this example, *our local coordinates become* (u, v). *Finding the metric on the catenoid,*

$$\begin{aligned}
X_u &= (-(v^2+1)\sin(u), (v^2+1)\cos(u), 0) \\
X_v &= (2v\cos(u), 2v\sin(u), 1) \\
g &= \begin{bmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_v \cdot X_u & X_v \cdot X_v \end{bmatrix} = \begin{bmatrix} (v^2+1)^2 & 0 \\ 0 & 4v^2+1 \end{bmatrix}
\end{aligned}$$

which has the inverse

$$g^{-1} = \begin{bmatrix} \frac{1}{(v^2+1)^2} & 0 \\ 0 & \frac{1}{4v^2+1} \end{bmatrix}.$$

Using equation (2.4) *we find the Christoffel symbols are given by*

$$\begin{aligned}
\Gamma^1_{11} &= 0 = \Gamma^1_{22} \\
\Gamma^1_{12} &= \frac{2v}{v^2+1} = \Gamma^1_{21} \\
\Gamma^2_{11} &= -\frac{2v^3+2v}{4v^2+1} \\
\Gamma^2_{12} &= 0 = \Gamma^2_{21} \\
\Gamma^2_{22} &= \frac{4v}{4v^2+1}
\end{aligned}$$

Thus, the geodesic ODE gives the system of equations

$$\begin{cases} \ddot{u} + \frac{4v}{v^2+1}\dot{u}\dot{v} = 0 \\ \ddot{v} - \frac{2v^3+2v}{4v^2+1}(\dot{u})^2 + \frac{4v}{4v^2+1}(\dot{v})^2 = 0. \end{cases} \tag{2.11}$$

Instead of solving this system, we verify that meridians/longitudes parametrized by arclength s *and the 'equator'/'waist' are geodesics, whereas all other parallels are* <u>*not*</u> *geodesics.*

We define a meridian/longitude to have $u(s)$ = *constant and* $v(s)$ *is parametrized by arclength* s *so that* $\gamma(s) = ((v(s)^2+1)\cos(c), (v(s)^2+1)\sin(c), v(s))$ *is a meridian and* $\|\dot{\gamma}(s)\| = 1$. *Since the first equation of* (2.11) *is trivially satisfied by* u = *constant, it causes the second equation to be*

$$\ddot{v} + \frac{4v}{4v^2+1}(\dot{v})^2 = 0.$$

Since $\frac{d}{ds}\gamma(s) = (2v\dot{v}\cos(c), 2v\dot{v}\sin(c), \dot{v})$ *and* $\|\frac{d}{ds}\gamma(s)\|_g = 1$ *when parametrized by arclength,*

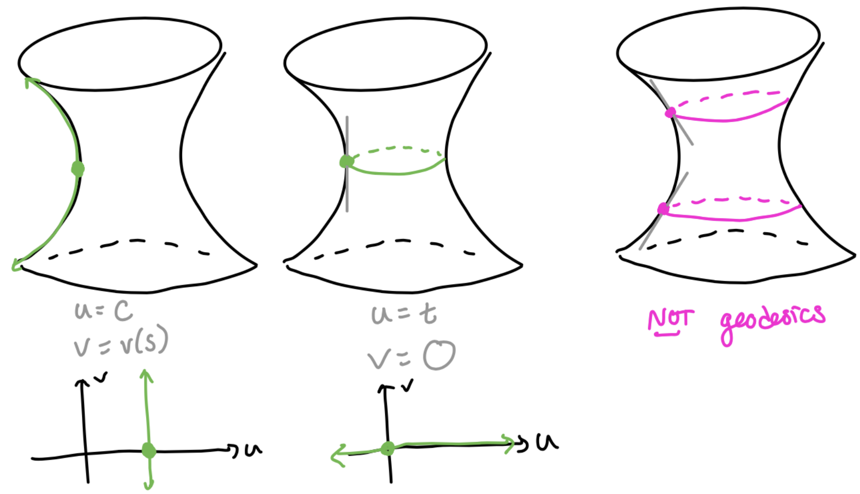

Figure 2.5 Some geodesics on a catenoid

we have

$$\left\| \frac{d}{ds}\gamma(s) \right\|_g = (4v^2+1)(\dot{v})^2 = 1.$$

Therefore, $(\dot{v})^2 = \frac{1}{4v^2+1}$. *By derivation,*

$$2\dot{v}\ddot{v} = -\frac{8v}{(4v^2+1)^2}\dot{v} = -\frac{8v}{4v^2+1}(\dot{v})^3$$

and since $\dot{v} \neq 0$,

$$\ddot{v} = -\frac{4v}{4v^2+1}(\dot{v})^2.$$

Thus, meridians satisfy the geodesic ODE.

Now consider the parallels/latitudes, which are defined to have $u(t) = t$ *and* $v(t) = $ *constant. Plugging in* $v(t) = $ *constant to the first equation of* (2.11), *it gives* $\ddot{u}(t) = 0$, *which holds for* $u(t) = t$. *Plugging* $v(t) = $ *constant into the second one, it becomes*

$$\frac{2v^2+2v}{4v^2+1}(\dot{u})^2 = 0.$$

Since $\dot{u} \neq 0$ *and* $4v^2+1 \neq 0$ *then we must have* $v = 0$, *which on this catenoid corresponds to the 'equator'/'waist'. Thus, the only parallel on the catenoid that is a geodesic is the 'equator'/ 'waist'.*

As the next Lemma shows, given any point $p \in N$ and vector $v \in T_pN$ there exists a unique geodesic which starts at p and has initial velocity v. We denote this geodesic as $\gamma_{p,v} : \mathbb{R} \to N$.

Lemma 19 (Existence and Uniqueness of Geodesics)**.** *Let N be a smooth manifold and ∇ a connection in TN. For any $p \in N$, any $v \in T_pN$ there exists an open maximal interval $I \subset \mathbb{R}$ containing 0 and a geodesic $\gamma_{p,v} : I \to N$ satisfying $\gamma_{p,v}(0) = p, \dot{\gamma}_{p,v}(0) = v$. Any two such geodesics agree on their common domain.*

Proof. This was shown in [38, Theorem 4.27], and thus omitted here. □

Example 20. *Going back to Example 17, we saw that great circles on a sphere were geodesics. For the case of the sphere, through each point and tangent to each direction there passes exactly one great circle. Thus by the uniqueness of the geodesic, the great circles are the only geodesics of a sphere.*

We now have defined two important types of curves, geodesics and distance-minimizing curves. The natural question arises about the relationship between them. In fact, we can say that if there is a distance minimizing curve between two points in N then the curve is a geodesic. However, the converse doesn't apply, and geodesics may not be distance-minimizing in a global sense.

Example 21. *As an example, we revisit the sphere in Figure 2.2 where x and y are points on the equator and are not antipodal. The curve γ_1 is a geodesic connecting x to y since it is a great circle, but this is not a distance minimizing curve. Only the curve γ is distance minimizing.*

The following Lemma summarizes this result.

Lemma 22. *Let (N, g) be a connected Riemannian manifold,*

(a) For any two points in N, if there is a distance minimizing curve between them then this curve is a geodesic.

(b) Geodesics are locally distance-minimizing.
Let $\gamma([a, b])$ be a geodesic between its endpoints. If $a < t_0 < b$, then there exists an $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset [a, b]$ and $\gamma([t - \varepsilon, t + \varepsilon])$ is distance minimizing.

Proof. This was shown in [38, Proposition 6.10 and Theorem 6.12] and [15, Proposition 3.6 and Remark 3.8]. □

As the following Lemma shows, any two points in a compact Riemannian manifold can be connected with a distance minimizing curve.

Lemma 23 (Hopf-Rinow Theorem)**.** *Let (N,g) be a compact and connected Riemannian manifold and $x, y \in N$. Then there exists a distance minimizing path between x and y.*

Proof. When N does not have a boundary this is a Corollary of [38, Theorem 6.19]. For a manifold with boundary, the same result is shown in [5, Proposition 2.5.19]. □

If (N,g) is a compact and connected Riemannian manifold without boundary, then applying Lemma 23 (the Hopf-Rinow Theorem) we can connect any two points in N with a distance minimizing curve. It then follows from Lemma 22(a) that any two points in N can be connected with a geodesic. However for a pair of points there may be several geodesics of the same length connecting them.

Example 24. *For example take N to be the unit sphere in Example 17. There are an infinite number of great circles/meridians of length π connecting the north pole to the south pole.*

We further investigate the property of several geodesics between points in the next section.

2.2.5 Critical Distances

Consider a closed Riemannian manifold (N,g). The set of geodesics starting at a point p determines the mapping,

$$\exp_p : T_pN \to N, \qquad \exp_p(v) = \gamma_{p,v}(1)$$

which is called the **exponential mapping**. Because there may be several geodesics of the same length connecting a pair of points, we note that the exponential mapping may not be injective on T_pN. Although, because $\exp_p$ is a smooth map between manifolds of the same dimension, the Inverse Function Theorem guarantees that it is a local diffeomorphism of each point $v \in T_pN$ where $D\exp_p|_v$ is invertible [38, pg. 297]. A point $q = \gamma_{p,v}(1)$ is called **a conjugate point** of p along $\gamma_{p,v}$ if $D\exp_p|_v$ is singular (i.e. it is degenerate). Equivalently, the conjugate point q is a critical point of $\exp_p$.

From the degeneracy of $D\exp_p|_v$ that means q is a point along $\gamma_{p,v}$ where the geodesic fails to be distance minimizing. To denote the time we can travel along $\gamma_{p,v}$ until it fails to be distance minimizing, define the **cut distance** function:

$$\tau_{\text{cut}}: SN \to \mathbb{R}, \quad \tau_{\text{cut}}(p,v) = \sup\{t > 0 : d(p, \gamma_{p,v}(t)) = t\}. \tag{2.12}$$

Thus the geodesic segment $\gamma_{p,v}: [0,t] \to M$ is a distance minimizing curve for any $t \in [0, \tau_{\text{cut}}(p,v)]$.

Traditionally on a closed Riemannian manifold (N,g) the set

$$\text{cut}_N(p) := \{\gamma_{p,v}(\tau_{\text{cut}}(p,v)) \in N : v \in S_pN\} \tag{2.13}$$

is known as the **cut locus** of the point $p \in N$ and each point in this set is called a **cut point** of p. The following Lemma details what the points in the cut locus look like.

Lemma 25 (Klingenberg's Lemma)**.** *Suppose (N,g) is a complete, connected Riemannian manifold, $p \in N$ and $v \in S_pN$. Then at least one of the following holds for $q = \gamma_{p,v}(\tau_{cut}(p,v))$:*

(a) There exists another distance minimizing geodesic from p to q

(b) q is the first conjugate point to p along $\gamma_{p,v}$.

Furthermore, for any $t_0 \in (0, \tau_{cut}(p,v))$ the geodesic $\gamma_{p,v}|_{[0,t_0]}$ has no conjugate points and is the unique unit-speed minimizing curve between its endpoints.

Proof. Proven in [38, Proposition 10.32]. □

Example 26. *The cut locus of a point on the ellipsoid can be seen in Figure 2.6. As demonstrated, most of the points in the cut locus can be connected to p with equal length geodesics, one across the front of the ellipsoid and the other around the back.*

We now consider some properties of the cut locus and the cut distance function.

Lemma 27. *The cut distance function τ_{cut} is continuous on SN.*

Proof. Proven in [38, Theorem 10.33] and [32, Lemma 2.1.5]. □

Lemma 28. *$d(p,\cdot)$ is smooth on $N \setminus (\{p\} \cup cut_N(p))$ but not at any $q \in (\{p\} \cup cut_N(p))$.*

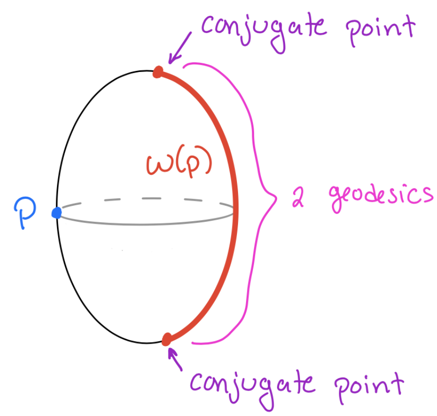

Figure 2.6 The cut locus of the ellipsoid.

Proof. Proven in [49, Section 9.1]. □

Much of our work in Section 4.2 is done generalizing Lemmas 25, 27, and 28 to the case of manifolds with strictly convex boundary.

2.2.6 Second Fundamental Form

The Riemannian metric tensor g is often referred to as the *First Fundamental Form*, which is an intrinsic property of the Riemannian manifold. In this section we discuss the extrinsic properties of a Riemannian manifold. In other words, we want to see how 'curvy' our manifold is in the ambient space. Let (N, g) be an oriented Riemannian submanifold of $(\tilde{N}, \tilde{g})$. In particular then N is an embedded submanifold of $\tilde{N}$ and g is the induced metric from $\tilde{N}$, with $g = i_N^* \tilde{g} = \tilde{g} \circ i_N$ such that $i_N : N \hookrightarrow \tilde{N}$ is the inclusion map. We consider ∇ to be the Levi-Civita connection of (N, g) and $\tilde{\nabla}$ to be the Levi-Civita connection of $(\tilde{N}, \tilde{g})$. By studying the extrinsic properties of N in $\tilde{N}$ we must then consider the relationship between these two Levi-Civita connections.

With this ambient space $\tilde{N}$ equipped, we decompose the ambient tangent bundle $T\tilde{N}|_N$ into tangential and orthogonal components, denoted $T^\top \tilde{N}$ and $T^\perp \tilde{N}$ respectively, using orthogonal projections. Consider vector fields X, Y in $\mathscr{X}(N)$ that can be extended locally to $\mathscr{X}(\tilde{N})$, then the covariant derivative $\tilde{\nabla}_X Y$ is a vector field on $\tilde{N}$ (see Section 2.2.1). Decomposing the covariant derivative into its tangential and normal components, we denote

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^{\top} + (\tilde{\nabla}_X Y)^{\perp}.$$

Here the shorthand notations $X^{\top}$ and $X^{\perp}$ are used for the orthogonal projections onto the tangential and orthogonal components.

Let us turn our attention to the normal component. We define the **Second Fundamental form of M** to be the map $\Pi : \mathscr{X}(N) \times \mathscr{X}(N) \to T^{\perp}\tilde{N}$ given by

$$\Pi(X, Y) = (\tilde{\nabla}_X Y)^{\perp},$$

where X and Y are extended arbitrarily to an open subset of $\tilde{N}$.

We note a few key properties of the Second Fundamental Form:

(a) This operator is symmetric since,

$$\Pi(X, Y) = (\tilde{\nabla}_X Y)^{\perp} = (\tilde{\nabla}_Y X + [X, Y])^{\perp} = (\tilde{\nabla}_Y X)^{\perp} + \underbrace{[X, Y]^{\perp}}_{=0} = (\tilde{\nabla}_Y X)^{\perp} = \Pi(Y, X).$$

The equation above follows from the symmetry of the Levi-Civita connection, linearity of the projection, and the extension of $[X, Y]$ to a vector field on $\tilde{N}$ (i.e. it is tangential to N).

(b) (The Gauss Formula) If $X, Y \in \mathscr{X}(N)$ are extended arbitrarily to smooth vector fields on a neighborhood of N in $\tilde{N}$ the following formula holds along N,

$$\tilde{\nabla}_X Y = \nabla_X Y + \Pi(X, Y).$$

The proof of this formula can be found in [38, Theorem 8.2].

Since $\Pi(X, Y)$ is considered to be the 'normal component' of $\tilde{\nabla}_X Y$, then by the decomposition $\nabla_X Y$ is the 'tangential component'.

Example 29. *Consider the smooth surface N embedded in $\mathbb{R}^3$ depicted in Figure 2.7. There we see a depiction of Gauss' formula, and decompose $\tilde{\nabla}_X Y$ into its tangential and normal components. The normal component of $\tilde{\nabla}_X Y$ is given by the Second Fundamental Form* $\Pi(X, Y)$.

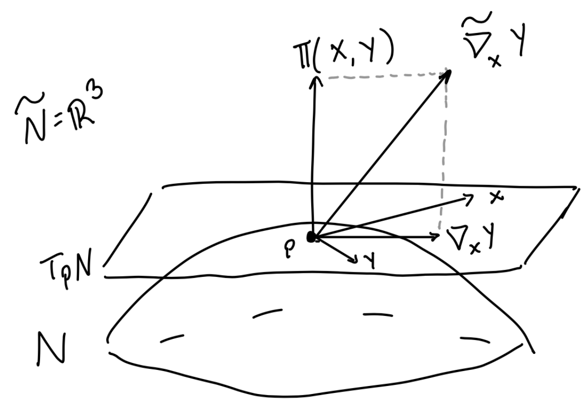

Figure 2.7 The second fundamental form of a smooth surface embedded in $\mathbb{R}^3$.

In this thesis, we are interested in the case when N is an oriented submanifold of $\tilde{N}$ of codimension 1. That is, if η is the outward-pointing vector field on N, we can rewrite the second fundamental form as

$$\Pi(X, Y) = \langle \tilde{\nabla}_X Y, \eta \rangle_g \ \eta.$$

We note that due to the uniqueness of the outer normal direction many textbooks choose to omit η on the far right of the expression, in which case Π is referred to as the *scalar second fundamental form*.

It follows that for $p \in N$ and vectors $v, w \in T_pN$ that $\eta(p)$ is normal to T_pN (i.e. $\eta(p) \in T_p^{\perp}N$). Then $\Pi : T_pN \times T_pN \to T_p^{\perp}N$ and

$$\Pi_p(v, w) = \langle \tilde{\nabla}_v w, \eta(p) \rangle_g \ \eta(p).$$

That way $\Pi_p(v, w)$ denotes the value of $\Pi(X, Y)$ at a specific $p \in N$. Recall that $\tilde{\nabla}_X Y|_p$ only depends on $X|_p$ and the value of Y along any curve $\gamma(t)$ with $\dot{\gamma}(0) = X|_p$ so that $\Pi_p(v, w)$ does not depend on the choice of the extension of η.

Now that we can decompose derivatives of the ambient space into meaningful tangential and normal components, we can analyse how quickly the space N curves away from the tangential components. For a point $p \in N$ we define the **shape operator** (also referred to as the Weingarten operator) as the operator $S_p : T_pN \to T_pN$ which satisfies

$$\langle \Pi_p(v, w), \eta(p) \rangle_g = \langle S_p(v), w \rangle_g, \qquad \text{for all } w \in T_pN. \tag{2.14}$$

So the shape operator returns a vector whose inner product with w gives the length of the second fundamental form. While it may not initially seem like much, the shape operator has many important properties:

(a) If η is a normal vector field to N defined in a neighborhood of p then $S_p(v) = -\nabla_v \eta$.

(b) S_p is a linear operator, which follows from the properties of the Levi-Civita connection.

(c) The shape operator is self-adjoint. That is, for all $v, w \in T_pM$ then $\langle S_p(v), w \rangle_g = \langle v, S_p(w) \rangle_g$. This follows from the symmetry of Π.

(d) For $p \in N$, then S_p has real eigenvalues $\lambda_1, ..., \lambda_n$ which represent the *principal curvatures* of N at p. The corresponding eigenvectors v_i are called *principal directions*. For more details we refer to [16] and [38].

Applying these properties means we can also think of the shape operator as measuring how much N pulls away from T_pN in a neighborhood of p. This is demonstrated in the following example.

Example 30. *Let N be a sphere of radius $r > 0$, embedded in $\mathbb{R}^3$. We consider the coordinate neighborhoods expressed in spherical coordinates, similar to*

$$X(\theta, \phi) = (r \sin(\theta)\cos(\phi), r \sin(\theta)\sin(\phi), r \cos(\theta)), \qquad \theta \in (0, 2\pi), \phi \in (0, \pi).$$

So for each $p \in N$ the vectors $(X_\theta, X_\phi) = (\frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial \phi})$, calculated in Example 17, form a frame for T_pN. Then for any point $p \in N$ the outer unit normal is given by

$$\eta = \frac{X_\theta \times X_\phi}{|X_\theta \times X_\phi|} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \; = \; \frac{X(\theta, \phi)}{r}.$$

Applying the shape operator to the basis vectors (X_θ, X_ϕ), we see from property (a) of the shape operator,

$$S_p(X_\theta) = -\nabla_{X_\theta} \eta = -\frac{\partial}{\partial \theta} \eta = -\frac{X_\theta}{r}$$
$$S_p(X_\phi) = -\nabla_{X_\phi} \eta = -\frac{\partial}{\partial \phi} \eta = -\frac{X_\phi}{r}.$$

For any $v \in T_pN$ we denote $v = (v^1 X_\theta + v^2 X_\phi)$, and it follows that

$$S_p(v) = -\nabla_v \eta = -\nabla_{v^1 X_\theta + v^2 X_\phi} \eta = -(v^1 \frac{X_\theta}{r} + v^2 \frac{X_\phi}{r}) = -\frac{v}{r}.$$

Specifically, the shape operator in the basis (X_θ, X_ϕ) is given as

$$S_p = -\frac{1}{r} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Due to rotational symmetry the sphere has the same curvature at every point, so we can disreguard the basepoint.

Observe that a circle in the plane with radius r parametrized by arclength has a curvature of $-\frac{1}{r}$ (when the unit normal is outward) [16, Section 1-5]. In Example 17 we saw that geodesics on the sphere with arclength parametrization correspond to great circles of radius r. Then the shape operator indicates that for each $p \in X$ the surface is curving away from the tangent plane in a manner corresponding to a circle in every direction.

2.3 Closed Extension of Manifolds with Boundary

From this point onward in this thesis, we consider (M, g) to be a compact Riemannian manifold with nonempty boundary ∂M. Specifically in this section we will construct a closed manifold, called the double of (M, g), such that (M, g) is isometrically embedded in its double. In the construction we will always need to be cautious near the boundary of M, so we begin by establishing a neighborhood of the boundary, called the collar neighborhood.

If M is a compact manifold with boundary, there exists a smooth vector field $\eta \in \mathscr{X}(M)$ whose restriction to ∂M is the outer unit-normal. Define the **boundary exponential mapping**, $\exp_{\partial M} : \partial M \times \mathbb{R} \to M$ by

$$\exp_{\partial M}(z, t) = \gamma_{z, -\eta(z)}(t),$$

where $\gamma_{z,-\eta(z)}$ is the geodesic of (M, g) with the initial conditions $(z, -\eta(z)) \in SM$. For sufficiently small $\varepsilon > 0$ then $\exp_{\partial M}(z, \varepsilon) \in M$. Denote

$$C_\varepsilon = \partial M \times [0, \varepsilon)$$

to be a subset of the boundary cylinder $\partial M \times [0,\infty)$. Then define $\Omega_\varepsilon := \exp_{\partial M}(C_\varepsilon)$. If there exists an $\varepsilon > 0$ such that Ω_ε is diffeomorphic to C_ε with $\partial M \times \{0\} \mapsto \partial M$ then Ω_ε is called a **collar neighborhood**. The following lemma shows that such an $\varepsilon > 0$ exists on compact manifolds.

Lemma 31. *Let (M,g) be a compact manifold with smooth boundary. Then ∂M has a collar neighborhood.*

Proof. Modification of [37, Theorem 9.24 and 9.25] and [22, Proposition 3.42]. Let $z_0 \in \partial M$. Define local coordinates on the boundary as $(z^1,\dots,z^{n-1})$ so then $(z^1,\dots,z^{n-1},t)$ are local coordinates on $\partial M \times \mathbb{R}$. Consider the map $E : \partial M \times \mathbb{R} \to M$ so that $E : (z,t) \mapsto \exp_{\partial M}(z,t)$. Then $E(z_0,0) = \gamma_{z_0,-\eta(z_0)}(0) = z_0$. Moreover,

$$DE\Big|_{(z_0,0)} \frac{\partial}{\partial z^i} = \frac{\partial}{\partial z^i}, \qquad DE\Big|_{(z_0,0)} \frac{\partial}{\partial t} = \dot{\gamma}_{z_0,-\eta(z_0)}(0) = -\eta(z_0).$$

Since the vectors $\frac{\partial}{\partial z^i}$ and $-\eta(z)$ are linearly independent the differential of E at $(z_0,0)$ is invertible. Then, by the Inverse Function Theorem, there is an open neighborhood $V_0 \subseteq \partial M$ containing z_0 and $\varepsilon_0 > 0$ such that $E|_{([0,\varepsilon_0)\times V_0)} = \exp_{\partial M}([0,\varepsilon_0)\times V_0)$ is a diffeomorphism onto its image, which we will denote $U_0 \subseteq M$.

Since M is compact, then ∂M is also compact. Thus, there exists a finite number of points $(z_1,\dots,z_N)$ in ∂M with open neighborhoods $(V_i)_{i=0}^N$ covering ∂M and $\varepsilon_i > 0$ such that $U_i := \exp_{\partial M}([0,\varepsilon_i)\times V_i) \subset M$ is diffeomorphic to V_i. If we set $\tilde{\varepsilon} = \min\{\varepsilon_i\}$ then $\exp_{\partial M}([0,\tilde{\varepsilon})\times \partial M)$ is a local diffeomorphism onto its image, by definition. Since injective local diffeomorphisms are diffeomorphisms, we would like to show there exists an $\varepsilon < \tilde{\varepsilon}$ such that $\exp_{\partial M}([0,\varepsilon)\times\partial M)$ is injective.

Suppose there is no such $\varepsilon > 0$ that makes it injective. Define sequences $\{z_n, w_n\} \in \partial M$ and $a_n, b_n \in [0,\tilde{\varepsilon})$ such that $(z_n,a_n) \neq (w_n,b_n)$ and $\exp_{\partial M}(z_n,a_n) = \exp_{\partial M}(w_n,b_n)$ for $a_n, b_n \to 0$ and, passing to subsequences if necessary, $z_n \to z$ and $w_n \to w$. Then $\exp_{\partial M}(z_n,a_n) \to \exp_{\partial M}(z,0) = z$ and $\exp_{\partial M}(w_n,b_n) \to \exp_{\partial M}(w,0) = w$, so we must have $z = w$. However this implies there exists a neighborhood $[0,\tilde{\varepsilon})\times V$ that contains $(z_k,a_k) \neq (w_k,b_k)$ and $\exp_{\partial M}(z_k,a_k) = \exp_{\partial M}(w_k,b_k)$ for sufficiently large values of k, which violates the fact that $[0,\tilde{\varepsilon})\times V$ is locally diffeomorphic and thus injective. This provides a contradiction.

Then $\exp_{\partial M}([0,\varepsilon)\times\partial M)$ is a diffeomorphism onto its image, which is the desired collar neighborhood Ω_ε. $\square$

We now discuss some properties of the maximal collar neighborhood, which we will refer to as **the collar neighborhood of** ∂M, where Ω_ε is the collar neighborhood with the maximum ε value. The next Lemma shows each point in M has a closest boundary point, and these points are unique inside the collar neighborhood.

Lemma 32. *Let* (M,g) *be a compact Riemannian manifold with smooth boundary. For every* $p \in M$ *there exists a (possibly non-unique) closest boundary point* $z_p \in \partial M$ *which satisfies*

$$d(z_p, p) = \inf_{z \in \partial M} d(p,z) = d(p, \partial M)$$

and $p = \exp_{\partial M}(z_p, d(p, \partial M))$.

Proof. Using the reverse triangle inequality, we see

$$|d(p,z_1) - d(p,z_2)| \le d(z_1, z_2), \qquad p \in M, z_1, z_2 \in \partial M$$

so the function $d(p,\cdot) : \partial M \to \mathbb{R}$ is a continuous function on the boundary. Since M is assumed to be compact, then ∂M must be compact as well. Thus $d(p, \partial M)$ is a compact subset of $\mathbb{R}$, meaning that it has a minimum value. So there must be a point $z_p \in \partial M$ such that $d(p, z_p)$ is the minimum distance from p to the boundary.

We consider the case that $d(z,p)$ is smooth at z_p. Since ∂M has codimension 1, and z_p is a minimal point it follows that $\operatorname{grad}_{\partial M} d(z,p)|_{z=z_p} = 0$. From the Hopf-Rinow Theorem on length-spaces (see Lemma 23 and [5, Proposition 2.5.19]) there exists a distance minimizing curve from $z \in \partial M$ to $p \in M$. This curve does not touch the boundary at any other $\tilde{z} \in \partial M$, since if it did then by the triangle inequality $d(\tilde{z}, p) < d(z,p)$ meaning it is not distance minimizing. Thus the distance minimizing curve from z to p is a geodesic and is denoted $\gamma_{z,v}$. Let $\dot{\gamma}_{z,v}(0) = v = v^\perp + v^\top$ where $v^\top$ is parallel to $T_z \partial M$ and $v^\perp$ is normal to $T_z \partial M$. Since the gradient of the distance function is the velocity of the geodesic $\gamma_{z,v}$,

$$v = \operatorname{grad} d(z,p) = \operatorname{grad}_{\partial M} d(z,p) + v^\perp.$$

If $z = z_p$ then v is normal to $T_z \partial M$, or $v = -\eta(z_p)$. Thus

$$p = \gamma_{z_p, -\eta(z_p)}(t) = \exp_{\partial M}(z_p, d(p, \partial M)).$$

In the case that $d(z,p)$ is not smooth at z_p along $\gamma_{z_p,v}$, let $t \in (0, d(z_p, p))$ and define the point $p' := \gamma_{z_p,v}(t)$ so that p and p' are both on $\gamma_{z_p,v}$. Using the Inverse Function Theorem then $d(z, p')$ is smooth. The same process as above then shows the geodesic from z to p' is still normal to the boundary, and gives the desired result. □

It follows from Lemma 32 that we can express any point $p \in \Omega_\varepsilon$ using **boundary normal coordinates**,

$$\exp^{-1}_{\partial M}(p) = (z_p, d(p, \partial M)) \tag{2.15}$$

where z_p denotes the unique boundary point such that $d(p, z_p) = d(p, \partial M)$. For any $z \in \partial M$ near z_p it has local coordinates $(z^1, ..., z^{n-1})$, so the local coordinates around p are understood as $(z^1(z_p), ..., z^{n-1}(z_p), d(p, \partial M))$.

From the proof of Lemma 31 we see $\gamma_{z,-\eta(z)}([0, t])$ is the unique shortest geodesic to ∂M when $t < \varepsilon$. However, as t becomes larger than ε, the geodesic $\gamma_{z,-\eta(z)}([0, t])$ fails to be the shortest geodesic from $\gamma_{z,-\eta(z)}(t)$ to ∂M. Thus there is a critical value where each geodesic $\gamma_{z,-\eta(z)}$ fails to be the unique shortest geodesic to ∂M. Define that value to be the **boundary cut distance** function, $\tau_{\partial M}(z)$ for $z \in \partial M$. Thus when $t < \tau_{\partial M}(z)$ we have $\gamma_{z,-\eta(z)}([0, t])$ is the unique shortest geodesic to ∂M. Yet if $t > \tau_{\partial M}(z)$ then $\gamma_{z,-\eta(z)}([0, t])$ is not a shortest geodesic from $\gamma_{z,-\eta(z)}(t)$ to ∂M.

Lemma 33. *Let (M, g) be a complete, connected, Riemannian manifold with smooth boundary, ∂M. The function $\tau_{\partial M}$ is continuous on ∂M.*

Proof. This is done similarly to the proof of [38, Theorem 10.33]. It is also discussed in [29, Theorem 2.12]. □

2.3.1 The Double Construction

We now utilise these collar neighborhoods to attach two manifolds along their boundaries. A common way of attaching two spaces together is by identifying the boundaries in an equivalence relation. A relation $\sim$ on a set X is called an **equivalence relation** if it is reflexive ($x \sim x$ for all $x \in X$), symmetric ($x \sim y$ implies $y \sim x$), and transitive ($x \sim y$ and $y \sim z$ imply $x \sim z$). For each $x \in X$ the **equivalence class of** x, denoted $[x]$, is the set of all $y \in X$ such that $y \sim x$. Let $X/\sim$ denote the set of equivalence classes in X so that $\pi : X \to X/\sim$ maps each point to its equivalence class.

If X and Y are topological spaces, a surjective map $f : X \to Y$ is said to be a **quotient map** provided a subset $V \subset Y$ is open in Y if and only if $f^{-1}(V)$ is open in X. If π is a surjective map then π is a quotient map when $X/\sim$ has the topology that declares $V \subset X/\sim$ to be open if and only if $\pi^{-1}(V)$ is open in X. This topology on $X/\sim$ is called the **quotient topology** of π. If $X/\sim$ is equipped with the quotient topology of π, then $X/\sim$ is called the **quotient space**.

Example 34. *As an example, let M and N be compact n-manifolds with nonempty boundaries, and suppose $h : \partial N \to \partial M$ is a diffeomorphism. Define the **disjoint union**,*

$$M \coprod N = \{(x,1),(y,2) \ : \ x \in M, y \in N\}$$

*so that each point in the disjoint union indicates which set it comes from. When it is clear which manifold a point $p \in M \coprod N$ originates from, we omit the index. Then the **disjoint union topology** declares a subset of $M \coprod N$ to be open if and only if its intersection with M is open in M, and its intersection with N is open in N. Moreover, the injections $i_M : M \to M \coprod N$ and $i_N : N \to M \coprod N$ are topological embeddings.*

The equivalence relation $\sim$ on $M \coprod N$ identifies points on ∂N with the corresponding image on ∂M from the map $h : \partial N \to \partial M$. That is, for $y \in \partial N$ then $y \sim h(y)$, and similarly for $x \in \partial M$ then $x \sim h^{-1}(x)$. Points in the interiors of the manifolds are not identified with any other points. The quotient space determined by this equivalence relation then 'glues' together the boundaries. All equivalence classes on $M \coprod N$ are given by

$$\begin{aligned} M \cup_h N := \left(M \coprod N\right)/\sim \ &= \{\{x\} \ : \ x \in M^{int}\} \cup \{\{y\} \ : \ y \in N^{int}\} \\ &\cup \{\{y, h(y)\} \ : \ y \in \partial N\} \\ &\cup \{\{x, h^{-1}(x)\} \ : \ x \in \partial M\}. \end{aligned}$$

*This space is known as an **adjunction space**, and can be thought of as attaching M to N along h. The following Lemma outlines some important properties of this space.*

Lemma 35. *Let M and N be compact smooth n-manifolds with nonempty boundaries, and suppose $h : \partial N \to \partial M$ is a diffeomorphism. Then:*

(a) $M \cup_h N$ is a topological manifold (without boundary).

(b) $M \cup_h N$ has a smooth structure such that there are regular domains (i.e. properly embedded codimension 0 submanifolds with boundary) $M', N' \subseteq M \cup_h N$ diffeomorphic

to M and N respectively, and satisfying

$$M' \cup N' = M \cup_h N, \qquad M' \cap N' = \partial M' = \partial N'.$$

(c) $M \cup_h N$ is compact.

(d) If M and N are connected then $M \cup_h N$ is connected.

Proof. Adapted from [37, Theorem 9.29].

(a) Let Ω_M and Ω_N be the collar neighborhoods of ∂M and ∂N respectively. For each $p \in \Omega_M$ we express it locally in boundary normal coordinates $p = (x, t)$ where $x \in \partial M$ and $t \in [0, \infty)$, and similarly for points in Ω_N. Now define a function $\phi : \Omega_M \coprod \Omega_N \to (-\varepsilon_M, \varepsilon_N) \times \partial M$ such that

$$\phi(p) = \begin{cases} (x, -t), & p = (x, t) \in \Omega_M \\ (h(y), t), & p = (y, t) \in \Omega_N \end{cases}.$$

This map ϕ can be seen as the 'gluing' of the collar neighborhoods. We will start by showing this map is continuous. We emphasize that Ω_M, Ω_N and $(-\varepsilon_M, \varepsilon_N) \times \partial M$ are metrizable topological spaces, so we consider a sequence $\{\tilde{y}_n := (y_n, t_n)\}_{n=1}^{\infty}$ in Ω_N which converges to $\tilde{y} := (y, t)$ where $y \in \partial N$. Then $\phi(\tilde{y}_n) = (h(y_n), t_n)$, and since h is a diffeomorphism then $\phi(\tilde{y}_n) \to (h(y), t) = \phi(\tilde{y})$. This proves $\phi \circ i_N|_{\Omega_N}$ is continuous and similarly $\phi \circ i_M|_{\Omega_M}$ is continuous. By [37, Proposition A.25 (a)] then ϕ must be continuous.

Additionally note that the restriction of ϕ to Ω_M or Ω_N is a topological embedding with closed images. Thus for any closed set $A \subset \Omega_M \coprod \Omega_N$ then $A \cap \Omega_M$ is closed in Ω_M and $A \cap \Omega_N$ is closed in Ω_N. It then follows that $\phi(A)$ is closed, making ϕ a closed map. For $q \in (-\varepsilon_M, \varepsilon_N) \times \partial M$ observe that $\phi^{-1}(q)$ corresponds to $(x, t) \in \Omega_M$ or $(y, t) \in \Omega_N$, then by the diffeomorphism h it follows that ϕ is surjective. Recall that a closed, surjective, and continuous map is a quotient map [43, Section 22], meaning that ϕ is a quotient map. From [43, Corollary 22.3(a)] then ϕ induces a homeomorphism $\psi : \pi(\Omega_M \coprod \Omega_N) \to (-\varepsilon_M, \varepsilon_N) \times \partial M$ such that $\phi = \psi \circ \pi$. Thus, $\pi(\Omega_M \coprod \Omega_N)$ is a topological n-manifold.

We must also show the region away from $\pi(\Omega_M \coprod \Omega_N)$ is a topological n-manifold. For any $[p] \in \big((M \cup_h N) \setminus \pi(\Omega_M \coprod \Omega_N)\big) \subset \pi(M^{int} \coprod N^{int})$ it corresponds to one point

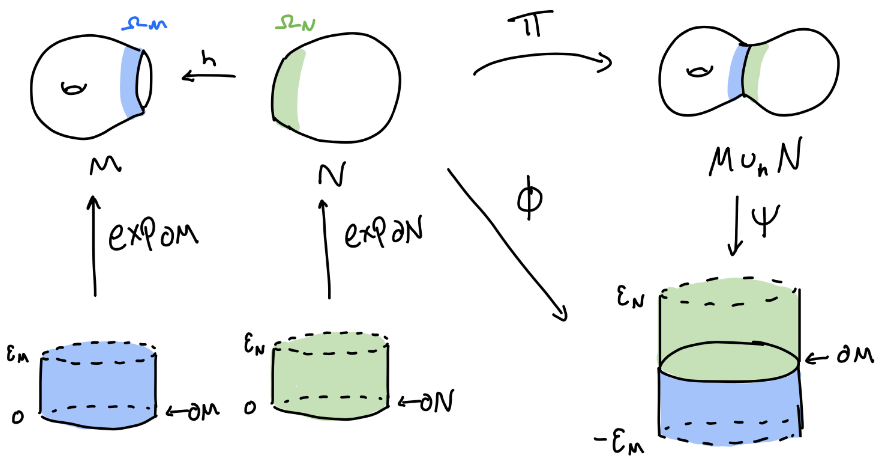

Figure 2.8 Attaching collar neighborhoods together

$x \in M^{int}$ or $y \in N^{int}$ which naturally implies $\pi|_{M^{int} \coprod N^{int}}$ is injective. Then by the definition of the quotient map, $\pi|_{M^{int} \coprod N^{int}}$ is a homeomorphism onto its image. This shows that $\pi(M^{int} \coprod N^{int})$ is locally Euclidean and of dimension n.

Since M and N are smooth manifolds they are second countable, so from [37, Proposition A.25(g)] then $M \coprod N$ is second countable. Since $\pi|_{M^{int} \coprod N^{int}}$ is a homeomorphism onto its image, then $\pi(M^{int} \coprod N^{int})$ is second countable. Moreover, since ψ is a homeomorphism then $\pi(\Omega_M \coprod \Omega_N) = \psi^{-1}((-\varepsilon_M, \varepsilon_N) \times \partial M)$ is second countable. Together, then

$$M \cup_h N = \pi(\Omega_m \coprod \Omega_N) \cup \pi(M^{int} \coprod N^{int})$$

is second countable and locally Euclidean with dimension n.

We now show that $M \cup_h N$ is also Hausdorff. Let $[p] \neq [q]$ be in $M \cup_h N$. If they are both in $\pi(\Omega_M \coprod \Omega_N)$ then take $U_p = \psi^{-1}(\tilde{U})$ and $V_q = \psi^{-1}(\tilde{V})$, where $\tilde{U}$ and $\tilde{V}$ are disjoint neighborhoods of $\psi([p])$ and $\psi([q])$ in $(-\varepsilon_M, \varepsilon_N) \times \partial M$. Then by the homeomorphism ψ the neighborhoods of $[p]$ and $[q]$, U_p and V_q respectively, are disjoint. Similarly if $[p]$ and $[q]$ are both in $\pi(M^{int} \coprod N^{int})$ then there exists disjoint neighborhoods $U_p = \pi(\tilde{U})$ and $V_q = \pi(\tilde{V})$ where $\tilde{U}$ and $\tilde{V}$ are neighborhoods of $\pi^{-1}([p])$ and $\pi^{-1}([q])$ in M^{int} or N^{int}. Now, without loss of generality assume $[p] \in \pi(\Omega_M \coprod \Omega_N)$ and $[q] \in \pi(M^{int} \coprod N^{int})$. Specifically of interest is when $[p] \in \pi(\partial M \coprod \partial N)$, which is when there exists some $x \in \partial M$ and $y \in \partial N$ such that $\pi(h(y)) = \pi(x) = [p]$. Let $\pi^{-1}([q]) \in M^{int}$ and take $x \in \partial M$ so that $\pi(x) = [p]$. Since M is Hausdorff there exists disjoint neighborhoods $\tilde{U}$ and $\tilde{V}$ of x and $\pi^{-1}([q])$ respectively in M. Now consider the neighborhood $\tilde{W}$ of $\phi(x)$ in $(-\varepsilon_M, \varepsilon_N) \times \partial M$ such that $\tilde{W} \cap (-\varepsilon_M, 0] \times \partial M \subset \phi(\tilde{U})$.

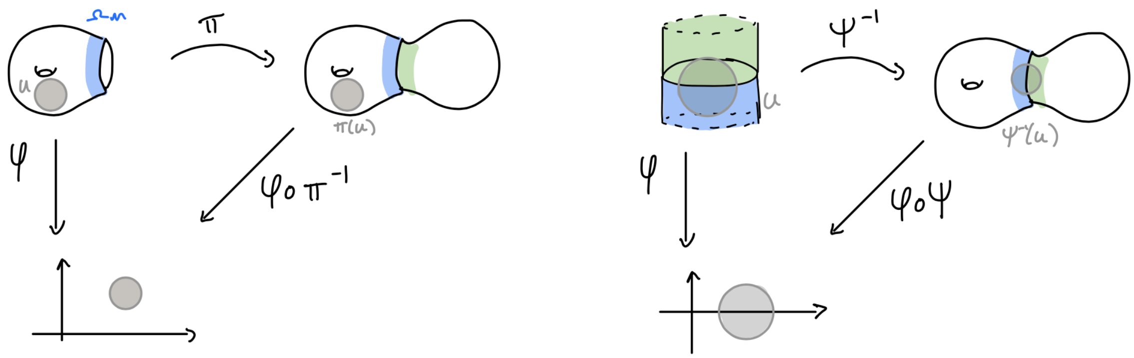

Figure 2.9 Local coordinates for the space $M \cup_h N$.

It follows that $U_p = \psi^{-1}(\tilde{W})$ and $V_q = \pi(\tilde{V})$ are disjoint neighborhoods of p and q in $M \cup_h N$. A similar construction for $\pi^{-1}([q]) \in N^{int}$ provides the required disjoint sets, making $M \cup_h N$ a Hausdorff space. Therefore $M \cup_h N$ is an n-dimensional topological manifold without boundary.

(b) Define charts on $M \cup_h N$ to be,

$$\begin{aligned}
&\left(\pi(U), \varphi \circ \pi^{-1}|_{\pi(U)}\right), && \text{for each smooth chart } (U, \varphi) \text{ on } M^{int} \text{ or } N^{int} \\
&\left(\psi^{-1}(U), \varphi \circ \psi|_{\psi^{-1}(U)}\right), && \text{for each smooth chart } (U, \varphi) \text{ on } (-\varepsilon_M, \varepsilon_N) \times \partial M,
\end{aligned}$$

as depicted in Figure 2.9. Since they are compositions of homeomorphisms they define coordinate charts on $M \cup_h N$. To determine the smooth compatibility of the charts, we must prove the following are smooth mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$,

$$\begin{aligned}
(\varphi_i \circ \pi^{-1}) \circ (\varphi_j \circ \psi)^{-1} &: (\varphi_j \circ \psi)|_{\pi(U_i) \cap \psi^{-1}(U_j)} \longrightarrow (\varphi_i \circ \pi^{-1})|_{\pi(U_i) \cap \psi^{-1}(U_j)} \\
(\varphi_j \circ \psi) \circ (\varphi_i \circ \pi^{-1})^{-1} &: (\varphi_i \circ \pi^{-1})|_{\pi(U_i) \cap \psi^{-1}(U_j)} \longrightarrow (\varphi_j \circ \psi)|_{\pi(U_i) \cap \psi^{-1}(U_j)} \\
(\varphi_i \circ \pi^{-1}) \circ (\varphi_j \circ \pi^{-1})^{-1} &: (\varphi_j \circ \pi^{-1})|_{\pi(U_i) \cap \pi(U_j)} \longrightarrow (\varphi_i \circ \pi^{-1})|_{\pi(U_i) \cap \pi(U_j)} \\
(\varphi_i \circ \psi) \circ (\varphi_j \circ \psi)^{-1} &: (\varphi_j \circ \psi)|_{\psi^{-1}(U_i) \cap \psi^{-1}(U_j)} \longrightarrow (\varphi_i \circ \psi)|_{\psi^{-1}(U_i) \cap \psi^{-1}(U_j)}.
\end{aligned}$$

Taking a look at these first two mappings, notice that $\pi(U_i) \cap \psi^{-1}(U_j)$ is entirely contained in $\pi(\Omega_M \coprod \Omega_N)$, but is away from the boundary ∂M. Thus, for the first mapping

$$(\varphi_i \circ \pi^{-1}) \circ (\varphi_j \circ \psi)^{-1} \;=\; \varphi_i \circ \pi^{-1} \circ \psi^{-1} \circ \varphi_j^{-1} \;=\; \varphi_i \circ \phi^{-1} \circ \varphi_j^{-1}$$

and is a smooth coordinate map based on the existing smooth coordinates in M and

N. In the third mapping, notice that $\pi(U_i) \cap \pi(U_j)$ is contained in M^{int} or N^{int}, so the smoothness follows from the definition of smooth manifolds M and N. As for the last mapping, since ψ is a homeomorphism, it becomes $\varphi_i \circ \varphi_j|_{U_i \cap U_j}$, which is smooth on $(-\varepsilon_M, \varepsilon_N) \times \partial M$. These smoothly compatible charts provide a smooth structure on $M \cup_h N$.

We now need to show that $\pi \circ i_M : M \to M \cup_h N$ is a smooth embedding. Define $M' = \pi \circ i_M(M)$ and $N' = \pi \circ i_N(N)$. Observe that in terms of the local coordinates defined above, $\pi \circ i_M|_{M^{int}}$ is the identity map and $\pi \circ i_M|_{\partial M}$ is the inclusion map. Because h is a diffeomorphism, then $\pi \circ i_M$ is smooth up to the boundary of M. Thus, $\pi \circ i_M$ is a smooth embedding, M' is a regular domain in $M \cup_h N$, and M' is diffeomorphic to M. Similar considerations apply to N.

Now let $\partial M' = \pi \circ i_M(\partial M)$ and $\partial N' = \pi \circ i_N(\partial N)$. Due to the identification of points on the boundaries, then $\partial M' = \partial N'$. Additionally, the smooth embeddings imply that $M' \cap N' \supset \partial M' = \partial N'$. The opposite inclusion follows from the diffeomorphism $h : \partial M \to \partial N$ and the quotient map identifying elements of the boundary together. It follows that

$$M' \cup N' = M \cup_h N, \qquad M' \cap N' = \partial M' = \partial N'.$$

(c) Since M and N are compact and π is a continuous map, then $\pi(M) = M'$ and $\pi(N) = N'$ are compact. Then from part (b) since $M \cup_h N = M' \cup N'$ it follows that $M \cup_h N$ is compact.

(d) Since M and N are connected and π is a continuous map, then $\pi(M) = M'$ and $\pi(N) = N'$ are connected. Then from part (b) since $M \cup_h N = M' \cup N'$ and $M' \cap N' \neq \emptyset$ it follows that $M \cup_h N$ is connected.

$\square$

The **double of** (M, g) is the manifold

$$D(M) = M \cup_{Id} M$$

where $Id : \partial M \to \partial M$ is the identity map of ∂M. Thus another way to view $D(M)$ is as the quotient space formed by taking the disjoint union $M \coprod M$ and identifying a point on the boundary with itself in the identical copy of M.

Corollary 36. *If (M,g) is a compact and connected smooth manifold with boundary, then the following hold:*

(a) *$D(M)$ is a compact and connected smooth manifold without boundary.*

(b) *There are regular domains $M_+, M_- \subset D(M)$ diffeomorphic to M such that*

$$D(M) = M_+ \cup M_-, \qquad M_+ \cap M_- = \partial M_+ = \partial M_-.$$

We denote the topological embeddings $i_\pm : M \to M \coprod M$ so that $F_\pm = \pi \circ i_\pm$ and $M_\pm = F_\pm(M)$.

Proof. This is a direct application of Lemma 35. □

Thus, we have the topological and local coordinate structures on $D(M)$. It remains to determine the metric structure on $D(M)$. Doing so will require extending the metric across ∂M so that it preserves smoothness. We say that a metric $\tilde{g}$ on $D(M)$ satisfies the extension property of g on M if $F_+ = \pi \circ i_+$ is an isometric embedding.

Lemma 37. *Let (M,g) be a compact connected n-dimensional Riemannian manifold with smooth boundary. There exists a metric $\tilde{g}$ on $D(M)$ which satisfies the extension property of g on M.*

Proof. To simplify the proof we represent the metric g with a smooth function f. We then show if $f \in C^\infty(M)$ then there exists an extension $\tilde{f} \in C^\infty(D(M))$ such that $\tilde{f}|_{M_+} = f \circ F_+^{-1}$.

We defined $f \in C^\infty(M)$ so that for each $x_0 \in M$ there exists coordinate chart (U_0, ϕ_0) of x_0 where $f \circ \phi_0^{-1}$ is smooth. In the case that $\phi_0(U_0)$ is an open subset of $\mathbb{H}^n$ we interpret the smoothness of $f \circ \phi_0^{-1}$ to mean that each point of $\phi_0(U_0)$ has a neighborhood in $\mathbb{R}^n$ on which $f \circ \phi_0^{-1}$ extends to a smooth function [37, pg. 33]. Thus for $y_0 := \phi_0(x_0)$ there exists an open neighborhood $V_0 \subset \mathbb{R}^n$ around y_0, and a smooth extension of $f \circ \phi_0^{-1}$, denoted $\ell_0 \in C^\infty(V_0)$, such that $f \circ \phi_0^{-1} = \ell_0|_{\mathbb{H}^n \cap V_0}$. This is depicted in Figure 2.10. Taking an intersection if necessary, we embed the set V_0 onto $(-\varepsilon_M, \varepsilon_M) \times \partial M$ where ε_M is the constant that defines the collar neighborhood of ∂M. It follows for $z_0 := F_+(x_0)$ that it is contained in the set $B_0 := \psi^{-1}(V_0)$ and there exists a function $L_0 \in C^\infty(B_0)$ where $L_0|_{B_0 \cap M_+} = f \circ F_+^{-1}$.

We will now create the smooth extension of the double $D(M)$. Since $D(M)$ is compact there exists a finite number of points $(z_1, ..., z_N)$ in $F_+(\partial M)$ with neighborhoods $\{B_i\}_{i=1}^N$ which

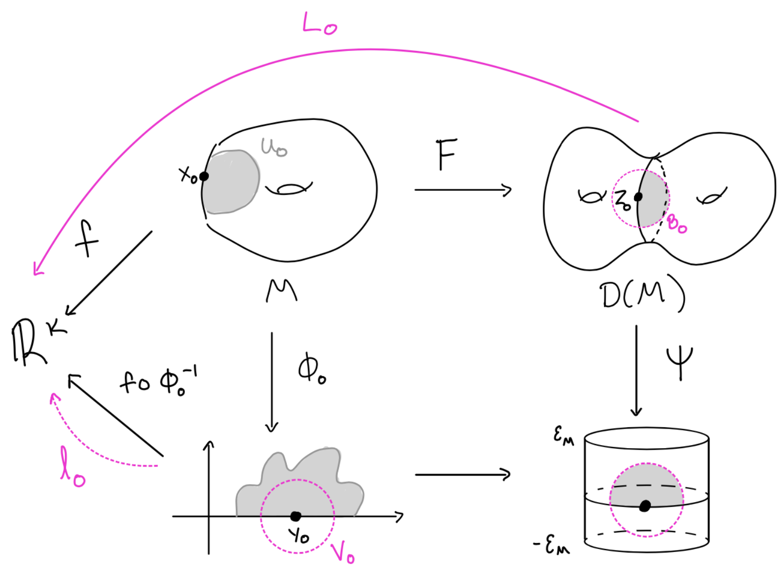

Figure 2.10 Extension of a smooth function f on M to a smooth function $\tilde{f}$ on the double $D(M)$.

cover $F_+(\partial M)$ and $L_i \in C^\infty(B_i)$ such that $L_i|_{B_i \cap M_+} = f \circ F_+^{-1}$. Consider a partition of unity $\{\xi_i\}_{i=1}^N$ subordinate to this cover. Then we define an extension of f on $\bigcup_{i=1}^N B_i \subset D(M)$ as $\hat{f} \in C^\infty(\bigcup_{i=1}^N B_i)$ where

$$\hat{f} = \sum_{i=1}^N L_i \xi_i.$$

For $x \in M_+ \cap (\bigcup_{i=1}^N B_i)$ then

$$\hat{f}(x) = \sum_{i=1}^N L_i(x)\xi_i(x) = f \circ F_+^{-1}(x)\left(\sum_{i=1}^N \xi_i(x)\right) = f \circ F_+^{-1}(x).$$

Now define the constant $\varepsilon > 0$ so that $\Psi^{-1}((-\varepsilon, \varepsilon) \times \partial M) \subset \bigcup_{i=1}^N B_i$. We set

$$\mathscr{B} := \Psi^{-1}((-\varepsilon, \varepsilon) \times \partial M)$$

We also define the sets $B^\pm := F_\pm(M \setminus \Omega_{\varepsilon/2})$ where $\Omega_{\varepsilon/2}$ is a collar neighborhood of M. Then $\{B^-, \mathscr{B}, B^+\}$ forms an open cover of $D(M)$. Consider the partition of unity $\{\alpha_1, \alpha_2, \alpha_3\}$ subordinate to the cover so that $\alpha_i \in [0,1]$ on $D(M)$. A visualization of this partition can be seen in Figure 2.11. Now let $f^+ = f \circ F_+^{-1} : M_+ \to \mathbb{R}^k$ and $f^- = f \circ F_-^{-1} : M_- \to \mathbb{R}^k$ and define the

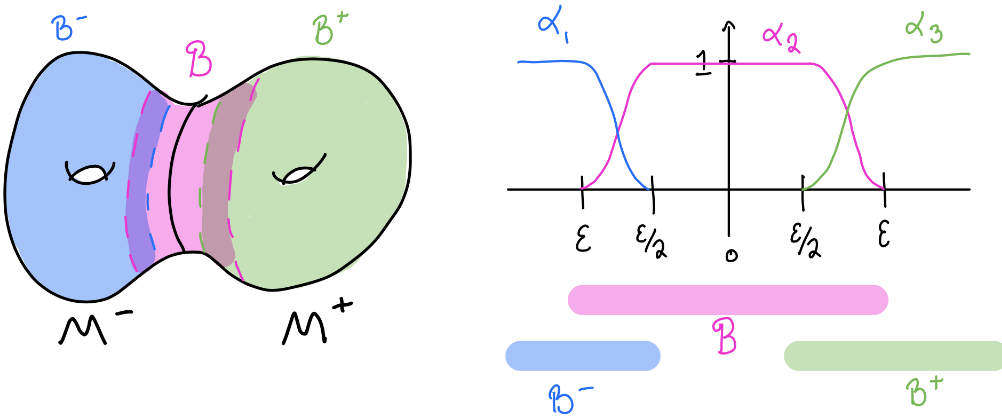

Figure 2.11 A partition of unity subordinate to the cover of $D(M)$.

function

$$\tilde{f} = \alpha_1 f^- + \alpha_2 \hat{f} + \alpha_3 f^+.$$

By construction, $\tilde{f} \in C^\infty(D(M))$ and $\tilde{f}|_{M^+} = f \circ F_+^{-1}$.

$\square$

As a consequence of Lemma 37 there exists a metric $\tilde{g}$ such that (M, g) is isometrically embedded in the closed and connected manifold $(D(M), \tilde{g})$. We say that $(D(M), \tilde{g})$ is a **closed extension of** (M, g).

2.3.2 Properties of the Manifold Resulting From Its Closed Extension

Since (M, g) is isometrically embedded in the closed extension $(D(M), \tilde{g})$, by studying some geometric properties of $D(M)$ it provides extrinsic properties on M. In this Section we introduce several of the properties that will be used later in this thesis.

Lemma 38. *Let (M, g) be a compact manifold with smooth boundary and let $(D(M), \tilde{g})$ be its closed extension. There is a function $\rho \in C^\infty(D(M))$, so that $\rho(x) = d(x, \partial M)$ near ∂M in $D(M) \setminus M$, and grad$\rho(z) = \eta(z)$ for all $z \in \partial M$ where η is the outer pointing unit normal vector field.*

Proof. Since ∂M is a compact codimension 1 subspace of $D(M)$ there exists the unit vector field η which is outer pointing on ∂M in $D(M)$. Consider the flow-out of this vector field, defined by $\Theta : \mathbb{R} \times \partial M \to D(M)$ where

$$\Theta(t, z) = \gamma_{z,\eta(z)}(t).$$

Similarly to the construction of the collar neighborhood in Lemma 31 then there is a collar neighborhood of ∂M in $D(M)$, denoted $\Omega \subset D(M)$. That is, there are values $\varepsilon, \tilde{\varepsilon} > 0$ where Θ is a diffeomorphism from Ω to $(-\varepsilon, \tilde{\varepsilon}) \times \partial M$.

Let the function π_t be the projection of $(t, z) \in \Omega$ to the $\mathbb{R}$ component. Define $f : D(M) \to \mathbb{R}$ by $f(x) = \pi_t \circ \Theta^{-1}(x)$. Then f is negative on M_+ (the original copy of M) and f is positive on M_- (the reflected copy of M). Let $C = \{x \in \Omega \ : \ d(x, \partial M) < \frac{3}{4} \min\{\varepsilon, \tilde{\varepsilon}\}\}$ so that C is another collar neighborhood of ∂M that is a strict subset of Ω. Denote $A^{\pm} = M_{\pm} \setminus \overline{C}$ so that $\{A^+, \Omega, A^-\}$ forms an open cover of $D(M)$. Consider the partition of unity $\{\alpha_1, \alpha_2, \alpha_3\}$ subordinate to the cover so that $\alpha_i \in [0, 1]$ on $D(M)$. Define the smooth function

$$\rho = -\varepsilon\alpha_1 + f\alpha_2 + \tilde{\varepsilon}\alpha_3.$$

For x in the collar neighborhood C then $\rho(x) = d(x, \partial M)$. By the definition of the collar neighborhood of the boundary, then $\operatorname{grad}\rho(z) = \eta(z)$ when $z \in \partial M$. Moreover,

$$\begin{aligned} M &= \{x \in D(M) \ : \ \rho \leq 0\} \\ \partial M &= \{x \in D(M) \ : \ \rho = 0\} \\ D(M) \setminus M &= \{x \in D(M) \ : \ \rho > 0\}. \end{aligned}$$

□

Let (M, g) be a compact manifold with smooth boundary, by Lemma 37 every manifold of this type has a closed extension, (N, g). Let ρ be defined as in Lemma 38. Now we use ρ to determine when a geodesic reaches the boundary of M. Define the function

$$h : SN \times \mathbb{R} \to \mathbb{R}, \qquad h(p, v, t) = \rho(\gamma_{p,v}(t)).$$

Due to the smoothness of ρ near ∂M, by the chain rule the function h is smooth in the collar neighborhood of ∂M.

Lemma 39. *Let (M, g) be a compact manifold with smooth boundary and closed extension*

(N, g). *If* $(p,v) \in SN$ *and if* $t_0 > 0$ *is such that* $x_0 = \gamma_{p,v}(t_0) \in \partial M$ *then*

$$\begin{aligned} h(p,v,t)\Big|_{t=t_0} &= 0 \\ \frac{\partial h}{\partial t}(p,v,t)\Big|_{t=t_0} &= \left\langle \eta(x_0), \frac{\partial \gamma_{p,v}}{\partial t}|_{t=t_0} \right\rangle_g \\ \frac{\partial^2 h}{\partial t^2}(p,v,t)\Big|_{t=t_0} &= Hess_{x_0}(\rho)\left(\frac{\partial \gamma_{p,v}}{\partial t}|_{t=t_0}, \frac{\partial \gamma_{p,v}}{\partial t}|_{t=t_0} \right). \end{aligned}$$

Proof. This was shown in [46, Lemma 3.1.13] and [4, pg. 198]. Since $\rho|_{\partial M} = 0$ then $h(p,v,t)\big|_{t=t_0} = 0$. Also, using that $\mathrm{grad}\rho|_{\partial M} = \eta$ then

$$\frac{\partial h}{\partial t}(p,v,t)\Big|_{t=t_0} = D\rho\Big|_{x_0}\left(\frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0} \right) = \left\langle \mathrm{grad}\rho(x_0), \frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0} \right\rangle_g = \left\langle \eta(x_0), \frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0} \right\rangle_g .$$

Finally, from the compatibility with the metric tensor

$$\begin{aligned} \frac{\partial^2 h}{\partial t^2}(p,v,t)\Big|_{t=t_0} &= \frac{d}{dt}\left\langle \mathrm{grad}\rho\Big|_{\gamma_{p,v}(t)}, \frac{\partial \gamma_{p,v}}{\partial t} \right\rangle_g \Big|_{t=t_0} \\ &= \left\langle \nabla_{\frac{\partial \gamma_{p,v}}{\partial t}} \mathrm{grad}\rho, \frac{\partial \gamma_{p,v}}{\partial t} \right\rangle_g \Big|_{t=t_0} + \left\langle \mathrm{grad}\rho, \nabla_{\frac{\partial \gamma_{p,v}}{\partial t}} \frac{\partial \gamma_{p,v}}{\partial t} \right\rangle_g \Big|_{t=t_0} \end{aligned}$$

and $\gamma_{p,v}$ being a geodesic implies $\nabla_{\frac{\partial \gamma_{p,v}}{\partial t}} \frac{\partial \gamma_{p,v}}{\partial t} = 0$. The rest follows by Example 9 and (2.5),

$$\begin{aligned} \frac{\partial^2 h}{\partial t^2}(p,v,t)\Big|_{t=t_0} &= \left\langle \nabla_{\frac{\partial \gamma_{p,v}}{\partial t}} \mathrm{grad}\rho, \frac{\partial \gamma_{p,v}}{\partial t} \right\rangle_g \Big|_{t=t_0} \\ &= \nabla^2 \rho \left(\frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0}, \frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0} \right) \\ &= \mathrm{Hess}_{x_0}(\rho)\left(\frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0}, \frac{\partial \gamma_{p,v}}{\partial t}\Big|_{t=t_0} \right). \end{aligned}$$

□

Corollary 40. *If* (M, g) *is a compact manifold with smooth boundary, then for any* $(x, v) \in T\partial M$ *one has*

$$-\left\langle \Pi_x(v,v), \eta(x) \right\rangle_g = Hess_x(\rho)(v,v) = \frac{d^2}{dt^2}\rho(\gamma_{x,v}(t))|_{t=0}.$$

Proof. This was shown in [46, Lemma 3.1.12]. Let $(x,v) \in T\partial M$, then by the previous

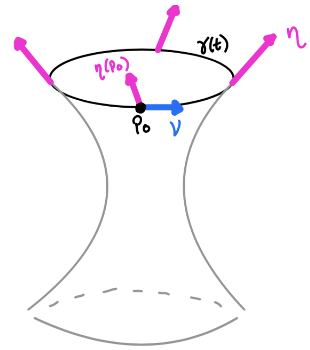

Figure 2.12 Computing the shape operator of the boundary of the catenoid

Lemma

$$
\begin{aligned}
h(x, v, 0) &= 0 \\
\frac{\partial h}{\partial t}(x, v, 0) &= \langle \eta(x), v \rangle = 0 \\
\frac{\partial^2 h}{\partial t^2}(x, v, 0) &= \langle \nabla_v \mathrm{grad}\rho, v \rangle \ = \ \mathrm{Hess}_x(\rho)(v, v).
\end{aligned}
$$

Since $\mathrm{grad}\rho|_{\partial M} = \eta$ then by (2.14),

$$\langle \nabla_v \mathrm{grad}\rho, v \rangle = \langle \nabla_v \eta, v \rangle = \langle -S_x(v), v \rangle = -\left\langle \Pi_x(v, v), \eta(x) \right\rangle. \tag{2.16}$$

$\square$

2.4 Strictly convex boundary

In this section we consider the Riemannian manifold (M, g) and its double $D(M)$, so that M is embedded in $D(M)$. Then the boundary of M, denoted ∂M, is an oriented Riemannian submanifold of $D(M)$. We say that a Riemannian manifold (M, g) has a **strictly convex boundary** ∂M if for all $x \in \partial M$ the shape operator $S_x : T_x\partial M \to T_x\partial M$ is negative definite. This means that

$$\langle \Pi_x(v, w), \eta(x) \rangle = \langle S_x(v), w \rangle_g,$$

is strictly negative whenever $v, w \in T_x\partial M$ agree, but does not vanish.

Example 41. *The catenoid whose boundary is given by a 'parallel' is a manifold with strictly* *convex boundary. Consider the catenoid M with a parametrization*

$$X(u,v)=((v^2+1)\cos(u),(v^2+1)\sin(u),v), \qquad u\in(-\pi,\pi), v\in[-2,2],$$

so for each $p\in M$ the vectors (X_u,X_v), given in Example 18, form a frame for T_pM. The boundary of the catenoid is given by parallels located at ± 2. It will suffice to prove that one of these parallels is strictly convex. Up to reparametrization, the upper parallel is given by

$$\gamma(t)=X(t,2)=(5\cos(t),5\sin(t),2), \qquad t\in(-\pi,\pi).$$

Observe that $X_u|_{v=2}$ spans $T_p\gamma$. Yet due to the rotational symmetry of the catenoid, we only need to compute the shape operator at one point. Consider $p_0:=\gamma(0)=(5,0,2)$. Then, the tangent vector along γ at p_0 is given by

$$w:=\frac{d\gamma}{dt}\Big|_{t=0}=(0\ ,\ 5\ ,\ 0).$$

Since the vectors (X_u,X_v) are orthogonal spanning sets of $T_{\gamma(t)}M$ we have the outer normal to ∂M is

$$\eta(t)\ =\ (4\cos(t),4\sin(t),1)=X_v|_{v=2}, \qquad t\in(-\pi,\pi).$$

Specifically at p_0, then $t=0$ and $\eta(0)=(4,0,1)$. Thus,

$$S_{p_0}(w)\ =\ -\nabla_w\eta\ =\ -(\Gamma^1_{12}(1)(1)+\Gamma^1_{22}(1)^2)X_u.$$

Note that

$$\Gamma^1_{12}=\frac{2v}{v^2+1},\Gamma^1_{22}=0$$

and at p_0 then $v=2$ and $u=0$, yielding

$$S_{p_0}(w)=-(0,4,0).$$

Clearly then $\langle S_{p_0}(w),w\rangle_g=-20$ which is strictly negative, meaning that the boundary is strictly convex at p_0. By rotational symmetry, this means the catenoid has strictly convex boundary.

An implication of strict convexity of the boundary is that any geodesic meeting the boundary tangentially immediately exits M.

Lemma 42. *Let (M,g) be a Riemannian manifold with strictly convex boundary. Let $p\in\partial M$*

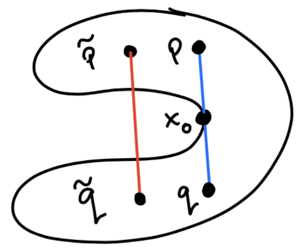

Figure 2.13 A non-geodesically convex manifold.

and $v \in T_p\partial M$. Then the geodesic $\gamma_{p,v}$ locally stays outside ∂M.

Proof. (Proved in [46] and earlier in [4]) By strict convexity of ∂M we also have $\langle \Pi_p(v,v), \eta(p)\rangle_g < 0$. From Corollary 40 then $h(p,v,0) = 0$, $\frac{\partial}{\partial t}h(p,v,0) = 0$, and $\frac{\partial^2}{\partial t^2}h(p,v,0) > 0$. Taylor expanding h we observe

$$h(p,v,t) = \underbrace{h(p,v,0)}_{=0} + \underbrace{\left(\frac{\partial}{\partial t}h(p,v,0)\right)}_{=0} t + \frac{1}{2}\underbrace{\left(\frac{\partial^2}{\partial t^2}h(p,v,0)\right)}_{>0} t^2 + O(t^3),$$

then for small t, $h(p,v,t) > 0$ and so $\gamma_{p,v}(t) \notin M$. $\square$

Another commonly used notion of convexity is the **geodesic convexity**, which assumes that any pair of points $p, q \in M$ can be connected by a distance minimizing geodesic (not necessarily unique) which is contained in the interior M^{int} of M modulo the terminal points.

Example 43. *Consider the horseshoe-shaped domain Ω of $\mathbb{R}^2$ shown in Figure 2.13. The points $\tilde{p}$ and $\tilde{q}$ cannot be connected by a line contained in $\overline{\Omega}$. Thus $\overline{\Omega}$ is <u>not</u> geodesically convex. Additionally, because the points p and q in Ω are connected with a straight line that intersects the boundary tangentially and stays inside $\overline{\Omega}$, then $\overline{\Omega}$ does <u>not</u> have strictly convex boundary.*

As [2] and [65] discuss, there are several other related notions of convexity on a Riemannian manifold, like variational convexity, However, for the purposes of this thesis we will concentrate on geodesic convexity and strict convexity. We relate these two notions of convexity by showing that strictly convex boundary implies geodesic convexity.

Lemma 44. *Let (M, g) be a compact manifold with nonempty boundary. If the boundary of M is strictly convex then M is geodesically convex.*

Proof. Let $p \in \partial M \subset N$, where N is the closed extension of M. By [38, Theorem 6.17] there is $r_1 > 0$ such that every metric ball $B(p, r)$ of N for $r \in (0, r_1)$ is geodesically convex. By this we mean that for each $x, y \in B(p, r)$ the distance minimizing geodesic segment connecting these points is contained in the ball $B(p, r)$.

Choose $r > 0$ small enough so that we can use boundary normal coordinates on $B(p, r)$. Let $x, y \in B(p, r) \cap M$ and let γ be the distance minimizing geodesic of N from x to y, so that $\gamma(0) = x$ and $\gamma(1) = y$. We write

$$\gamma(t) = (z(t), \tilde{h}(t)),$$

where $z(t)$ is the projection of γ to ∂M and $\tilde{h}(t) = \rho(\gamma(t)) = d(\gamma(t), z(t))$. If $\tilde{h}(t) < 0$ then $\gamma(t) \in M^{int}$ for $t \in (0, 1)$ and the claim holds.

Assume otherwise. Assume there exists $a, b \in (0, d(x, y))$ so that $\tilde{h}(a) = \tilde{h}(b) = 0$ and $\tilde{h}(t) \geq 0$ on the interval $[a, b]$. Then consider a variation of $\gamma(t)$ on the interval $[a, b]$,

$$\Gamma(s, t) = (z(t), (1+s)\tilde{h}(t)), \qquad t \in [a, b],$$

where s is close to zero. Observe that when s is held constant then $\Gamma(s, t)$ traces out curves connecting $\gamma(a)$ and $\gamma(b)$ in N, as depicted in Figure 2.14. Thus the variation is proper since $\Gamma(s, a) = \gamma(a)$ and $\Gamma(s, b) = \gamma(b)$.

By the first variation formula (see Lemma 15), we get

$$\left.\frac{d}{ds}\right|_{s=0} \mathscr{L}(\Gamma(s, t)) = -\int_a^b \langle V(t), D_t \dot{\gamma}(t) \rangle_g \, dt$$

where $V(t)$ is the variation field of Γ given by

$$V(t) := \left.\left(\frac{\partial}{\partial s}\Gamma(s, t)\right)\right|_{s=0} = (0, \tilde{h}(t)) \mathrm{grad}\rho(\gamma(t)) = \tilde{h}(t) \mathrm{grad}\rho(\gamma(t)).$$

Note that V is a proper variation field since $V(a) = V(b) = 0$. Moreover, since

$$\frac{d}{dt}\left\langle \dot{\gamma}(t), \mathrm{grad}\rho(\gamma(t)) \right\rangle_g = \left\langle D_t \dot{\gamma}, \mathrm{grad}\rho(\gamma(t)) \right\rangle_g + \left\langle \dot{\gamma}(t), D_t \mathrm{grad}\rho(\gamma(t)) \right\rangle_g,$$

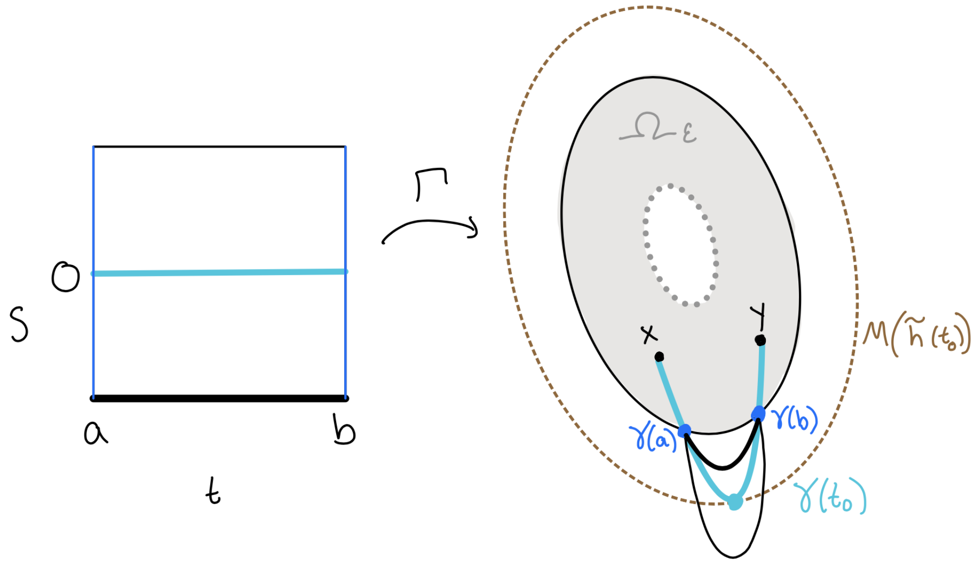

Figure 2.14 Variation $\Gamma(s,t)$ in a strictly convex planar domain.

it follows that

$$\begin{aligned}
&\left\langle V(t), D_t\dot{\gamma}(t)\right\rangle_g \\
&\quad=\tilde{h}(t)\left\langle \operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g-\left\langle D_t\left(\tilde{h}(t)\operatorname{grad}\rho(\gamma(t))\right),\dot{\gamma}(t)\right\rangle_g \\
&\quad=\tilde{h}(t)\left\langle \operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g-\tilde{h}(t)\left\langle D_t\operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g-\dot{\tilde{h}}(t)\left\langle \operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g \\
&\quad=\left(\tilde{h}(t)-\dot{\tilde{h}}(t)\right)\left\langle \operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g-\tilde{h}(t)\left\langle D_t\operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g.
\end{aligned}$$

Since $\tilde{h}(t)$ is a smooth function on $[a,b]$ that vanishes at the end points of this interval it attains its positive maximum value $h(t_0)>0$ at a point $t_0\in(a,b)$. It follows from the strict convexity of M and Proposition 60 that for small enough $\varepsilon>0$ we have that $M(\varepsilon):=\rho^{-1}(-\infty,\varepsilon]$ is a subset of N that contains M and has strictly convex boundary. Since $\dot{\gamma}(t_0)$ is a unit vector that is tangential to the boundary of $M(\tilde{h}(t_0))$ we get

$$\langle\dot{\tilde{h}}(t),\dot{\gamma}(t)\rangle_g\Big|_{t=t_0}=\langle\operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\rangle_g\Big|_{t=t_0}=0.$$

Additionally, by (2.16), we have

$$\langle V(t),D_t\dot{\gamma}(t)\rangle_g\Big|_{t=t_0}=\tilde{h}(t_0)\left\langle D_t\operatorname{grad}\rho(\gamma(t)),\dot{\gamma}(t)\right\rangle_g\Big|_{t=t_0}=-\tilde{h}(t_0)\langle\Pi(\dot{\gamma}(t_0),\dot{\gamma}(t_0)),\eta(t_0)\rangle_g>0.$$

However since γ is a geodesic, its covariant derivative $D_t\dot{\gamma}$ vanishes. Thus

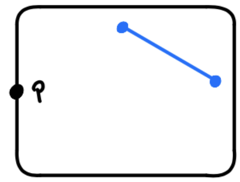

Figure 2.15 A geodesically convex manifold that does not have strictly convex boundary.

$\langle V(t), D_t\dot{\gamma}(t)\rangle_g\big|_{t=t_0} = 0$. This provides a contradiction. Therefore $\tilde{h}$ cannot be positive on the interval (a, b). This implies that $\gamma(t)$ is in M for every $t \in [0, d(p, q)]$. □

The following example shows that the converse of Lemma 44 need not be true.

Example 45. *Consider M to be a convex body of $\mathbb{R}^2$ obtained by rounding the edges of a square so that ∂M is smooth, as in Figure 2.15. Since M is convex for any $x, y \in M$ there exists a line connecting them. However, the boundary of this manifold is not strictly convex since for a point p on the 'flat' part of the boundary and $v \in T_p\partial M$ there exists small $t > 0$ where the geodesic $\gamma_{p,v}(t)$ is in M.*

2.4.1 Exit Time Function

Let (M, g) be a compact Riemannian manifold with strictly convex boundary. Define the **exit time function**,

$$\tau_{\text{exit}} : SM \to \mathbb{R} \cup \{\infty\}, \quad \tau_{\text{exit}}(p, v) = \sup\{t > 0 : \gamma_{p,v}(t) \in M\}.$$

Here $\gamma_{p,v}$ is the geodesic of (M, g) with the initial conditions $(p, v) \in SM$. Since the boundary of M is strictly convex, Lemma 42 gives that geodesics hit the boundary transversally, so $\tau_{\text{exit}}(p, v)$ is the first time when the geodesic $\gamma_{p,v}$ hits the boundary and $(-\tau_{\text{exit}}(p, -v), \tau_{\text{exit}}(p, v))$ is the maximal interval where the geodesic $\gamma_{p,v}$ is defined.

We say that (M, g) is **nontrapping** if $\gamma_{p,v}$ reaches ∂M in a finite amount of time for all $(p, v) \in SM$. Otherwise, the manifold is **trapping.**

Example 46. *As we showed in Example 18, the 'equator'/'waist' on the catenoid is an example of a geodesic which never hits the boundary. Thus, the 'equator'/'waist' is a trapped geodesic on the catenoid.*

In this thesis we do not assume that $\tau_{\text{exit}}(p,v)<\infty$ for all $(p,v)\in SM$. That is, (M,g) may have trapped geodesics. However, as the following lemma shows, τ_{exit} is smooth on inward pointing and non-trapping directions.

Lemma 47. *Let (M,g) be a compact Riemannian manifold with strictly convex boundary, then for $(p_0,v_0)\in SM\setminus T\partial M$ with $\tau_{exit}(p_0,v_0)<\infty$ then there exists a neighborhood $U\subset SM\setminus T\partial M$ of (p_0,v_0) such that $\tau_{exit}(p,v)<\infty$ for all $(p,v)\in U$ and τ_{exit} is smooth in U.*

Proof. (Proof based on [52, Lemma 4.1.1] and [46, Lemma 3.2.3].) Assume $(p,v)\in SM$ such that $\tau_{\text{exit}}(p,v)<\infty$. This implies $\gamma_{p,v}$ is non-trapping and $(p,v)\notin T\partial M$ so v is non-tangential to ∂M. Denote $q:=\gamma_{p,v}(\tau_{\text{exit}}(p,v))$ so that $q\in\partial M$. Then if $w:=\dot\gamma_{p,v}(\tau_{\text{exit}}(p,v))$, by strict convexity $w\notin T\partial M$ (otherwise $\tau_{\text{exit}}(p,v)=0$ which implies $(x,v)\in T\partial M$). Thus $\langle w,\eta\rangle>0$ where η is the outward-pointing normal vector. From Lemma 38 since $y\in\partial M$ we know $\text{grad}\rho(y)=\eta(y)$. Thus by strict convexity and Lemma 39,

$$\frac{dh}{dt}(p,v,\tau_{\text{exit}}(p,v)) \;=\; \langle \text{grad}\rho(y),w\rangle \;=\; \langle \eta(y),w\rangle \;>\; 0. \tag{2.17}$$

Recall the *Implicit Function Theorem* [37, Theorem C.40] says:

Let $U\subseteq\mathbb{R}^m\times\mathbb{R}^k$ be an open subset and let $(x,y)=(x^1,\dots,x^m,y^1,\dots,y^k)$ denote the coordinates on U. Suppose $\phi:U\to\mathbb{R}^k$ is a smooth function, $(a,b)\in U$ and $c=\phi(a,b)$. If the $k\times k$ matrix $\left(\frac{\partial\phi^i}{\partial y^j}\Big|_{(a,b)}\right)$ is nonsingular, then there exists neighborhoods $V_0\subseteq\mathbb{R}^m$ of a and $W_0\subseteq\mathbb{R}^k$ of b and a smooth function $F:V_0\to W_0$ such that $\phi(x,y)=c$ for $(x,y)\in W_0$ if and only if $y=F(x)$.

In Lemma 39 we showed there exists a neighborhood U such that $h:U\to\mathbb{R}$ is a smooth function with $0=h(p,v,t_0)$. From (2.17) then $\frac{\partial h}{\partial t}\neq 0$, so by the Implicit Function Theorem then there exists neighborhoods V_0 of (p,v) and W_0 of t_0 such that $\tau_{\text{exit}}:V_0\to W_0$ is smooth and $h(p,v,t)=0$ if and only if $t=\tau_{\text{exit}}(p,v)$. □

Thus the smoothness of τ_{exit} holds on nontrapping and non-tangential directions. Yet there is no such restriction on SM for continuity of τ_{exit}, and we show that $\tau_{\text{exit}}:SM\to\mathbb{R}\cup\{\infty\}$ is continuous if ∂M is strictly convex.

Lemma 48. *If (M,g) is a compact Riemannian manifold with strictly convex boundary, then τ_{exit} is continuous.*

Proof. Let $(p, v) \in SM$. To show that τ_{exit} is continuous at (p, v) we consider three cases:

<u>Case 1:</u> If $\tau_{\text{exit}}(p, v) = 0$ then v is tangential or points outwards and the geodesic $\gamma_{p,v}$ exits immediately (see Lemma 42). By the assumption of strict convexity and the continuity of geodesics, for any geodesic $\gamma(t)$ starting near (p, v) the distance from the boundary to $\gamma(t)$ is of order t^2 [60, Section D.8.1]. Thus geodesics close to tangent to ∂M will be short, proving that τ_{exit} is continuous at v.

<u>Case 2:</u> If $0 < \tau_{\text{exit}}(p, v) < \infty$ then $\gamma_{p,v}$ is non-trapping and $(p, v) \notin T\partial M$ so v is non-tangential to ∂M. Then continuity follows from the smoothness of τ_{exit} in a neighborhood of (p, v) from Lemma 47.

<u>Case 3:</u> If $\tau_{\text{exit}}(p, v) = \infty$ then $\gamma_{p,v}$ is a trapped geodesic. We would like to show for every $m \in \mathbb{N}$ there exists a neighborhood $U_m \subset SM$ of (p, v) such that $\tau_{\text{exit}}(q, w) \geq m$ for all $(q, w) \in U_m$. Assume otherwise. Let $(p_n, v_n) \to (p, v)$ in SM such that $\tau_{\text{exit}}(p_n, v_n) < C < \infty$. Then define $y_n := \gamma_{p_n,v_n}(\tau_{\text{exit}}(p_n, v_n)) \in \partial M$. For a sequence on a compact set ∂M there exists a convergent subsequence, which we will denote $y_{n_k} \to y$ where $y \in \partial M$. Since $(\tau_{\text{exit}}(p_n, v_n))_{n=1}^{\infty}$ is bounded we may assume after passing to a subsequence that $\tau_{\text{exit}}(p_{n_k}, v_{n_k}) \to B \leq C$. By continuity of the exponential map,

$$\gamma_{p,v}(B) = \lim_{n\to\infty} \gamma_{p_{n_k},v_{n_k}}(\tau_{\text{exit}}(p_{n_k}, v_{n_k})) = y.$$

This means $\gamma_{p,v}$ is not trapped, which provides a contradiction. □

CHAPTER

3

REVIEW OF SOME GEOMETRIC INVERSE PROBLEMS

In this chapter we focus on **geometric inverse problems**, which are named due to the geometric nature of the information in these inverse problems, as well as the techniques used to solve them. We begin by describing the components of an inverse problem. A model which is defined by some **parameters**, produces a set of observable information, which we call **data**. In abstract terms, a **Forward Problem** can be stated as:

Given the parameters of a model, what is the corresponding data?

Now the associated **Inverse Problem** is stated in the following manner:

Given a set of equivalent data for a model, are the parameters equivalent?

For each problem the notion of *equivalence* depends on the context of the problem and must be interpreted for each setting independently. This will be demonstrated in the next example. Additionally when solving an inverse problem, we must be conscious of the following:

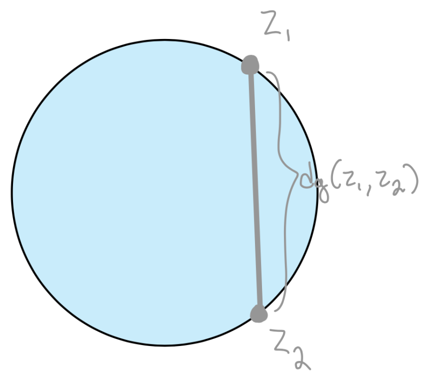

Figure 3.1 The boundary distance function on the unit-disc.

1. (Existence) Is there a set of parameters for the given data?
2. (Uniqueness) Is there a unique set of parameters for the given data?
3. (Stability) If two data sets are 'close' then are the respective parameters 'close'?

If we are able to answer affirmatively for all three of the above questions, then according to Hadamard [21] the inverse problem is said to be **well posed**. If it fails one of those conditions, it is called **ill-posed.** Many inverse problems are ill-posed, see [28] for a comprehensive list across many disciplines.

Example 49. *Let (M, g) be a compact Riemannian manifold with boundary. The metric g induces a function*

$$d_g : \partial M \times \partial M \to \mathbb{R}, \qquad d_g(z_1, z_2) = \inf\{\mathscr{L}_g(\gamma) \,:\, \gamma(0) = z_1, \gamma(1) = z_2, \gamma \text{ is a smooth curve}\} \tag{3.1}$$

*called the **boundary distance function**. Depicted in Figure 3.1, this function gives a distance between two points on the boundary, measured on a curve going through the manifold.*

*For these models, the parameters are M and g while the data is the boundary distance function. In this case two metrics on M are considered equivalent if there is a **boundary preserving isometry**, which is defined as a Riemannian isometry $\Psi : (M, g) \to (M, \tilde{g})$ (defined in Section 2.2) with the following properties,*

$$\begin{cases} \Psi|_{\partial M} = Id \\ \tilde{g} = \Psi^* g = (D\Psi \circ g \circ (D\Psi)^T) \circ \Psi. \end{cases}$$

It follows that

$$d_g(z_1, z_2) = d_{\Psi^* g}(z_1, z_2) = d_{\tilde{g}}(z_1, z_2), \qquad \textit{for all } z_1, z_2 \in \partial M.$$

Thus the boundary distance function is preserved under boundary preserving isometries. The forward problem can be stated as follows:

> *Given a Riemannian manifold* (M, g)*, what is the respective boundary distance function?*

The associated inverse problem is:

> *Given a smooth manifold* M *with boundary and two metrics* g_1, g_2 *on* M *for which* $d_{g_1}|_{\partial M \times \partial M} = d_{g_2}|_{\partial M \times \partial M}$*, are the associated Riemannian metrics the same up to boundary-preserving isometry?*

In the remainder of this chapter, we will survey several inverse problems whose data is related to the distance function.

3.1 Boundary Rigidity

One of the most extensively studied geometric inverse problems formulated with the distance functions is the boundary rigidity problem. This problem arose as a problem in geophysics in an attempt to determine the inner structures of the Earth using seismic waves, known as the **inverse kinematic problem** [50]. Because the speed of seismic waves vary based on the material it is passing through [63], it is a key insight to determine the composition of Earth's interior. Initial estimates for the location of the mantle, crust, and core were provided by Herglotz [23] and Weichert and Zoeppritz [70] (see also [69]). In these papers they assume spherical symmetry of the Earth in the sense that the seismic wave speed depends only on the depth. Mathematically they modeled the Earth by a 3-dimensional disc D^3 with boundary S^2, and a Riemannian metric given by $ds^2 = \frac{1}{c^2(|x|)} dx^2$, where c is a positive function representing the wave speed. The inverse problem becomes:

> Given $d_{ds_1^2} = d_{ds_2^2}$ on D^3, are the associated wave speeds c_1 and c_2 the same?

They provided an affirmative answer for the problem under the assumption that $\frac{d}{dx}\left(\frac{|x|}{c(|x|)}\right) > 0$.

However, Earth deviates from perfect spherical symmetry. In geophysics these deviations are called *horizontal nonhomogeneities* [53]. If the nonhomogeneities are small, then we consider the wave speed $\tilde{c}(x)$ which depends on all position variables $x \in D^3$, and determine this wave speed. This probem has been extensively studied, and is popular in practical geophysics (see for example [50]).

When the medium is no longer spherically symmetric, we generalize this type of inverse problem to any compact Riemannian manifold (M, g). Due to Fermat's principle we consider the wave speed to be given by a Riemannian metric. Thus if the parameters of the model are (M, g) the data is the travel time $d_g(z_1, z_2)$ for all $z_1, z_2 \in \partial M$. We note this was exactly the inverse problem posed in Example 49, and for convenience will restate it here. The **boundary rigidity problem** poses:

> Given a smooth manifold M with boundary and two metrics g_1, g_2 on M for which $d_{g_1}|_{\partial M \times \partial M} = d_{g_2}|_{\partial M \times \partial M}$, are the associated Riemannian metrics the same up to boundary-preserving isometry?

If the answer is affirmative then (M, g) is said to be **boundary rigid**.

For a general Riemannian manifold the answer is negative. Since the boundary distance function only takes into account the length of shortest geodesics between points on the boundary, there may be regions 'unseen' by the data. This is exemplified in Figure 3.2, where no distance minimizing geodesics connecting boundary points will travel into the regions Ω_1 or Ω_2. Thus, we do not have information from these regions, and it is impossible to distinguish the non-isometric surfaces in Figure 3.2 from each other solely from the boundary distance function.

For that reason we consider the boundary rigidity problem under some restrictions on the geometry of the manifold. One such restriction is the simplicity of the manifold. A compact and connected Riemannian manifold (M, g) with smooth boundary ∂M is called **simple** if the boundary ∂M is strictly convex and any two points $x, y \in M$ are joined by a unique distance minimizing geodesic that depends smoothly on these points. This second condition is equivalent to saying the exponential map $\exp_x$ is a diffeomorphism from the set $M_x = \{v \in T_x M \,:\, v = 0, \text{ or } \|v\|_g \leq \tau_{\text{exit}}(x, \frac{v}{\|v\|_g})\}$ onto M for every $x \in M$. Thus all distances in M are realized by geodesics, and it follows that a simple manifold is diffeomorphic to an n-dimensional disc, denoted D^n [26]. Thus when considering a simple manifold (M, g), the metric g is often referred to as a **simple metric** on D^n.

Example 50. *Some examples of simple domains are convex subsets of $\mathbb{R}^n$, convex subsets*

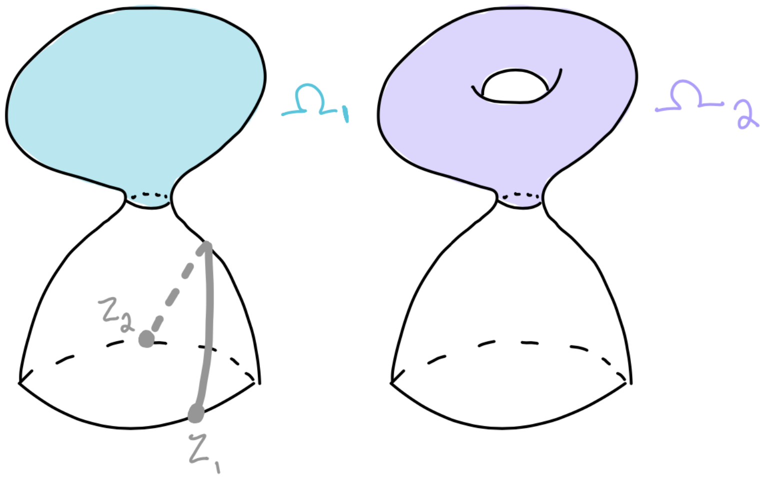

Figure 3.2 An example of manifolds that are not boundary rigid.

of $\mathbb{H}^n$, and open hemispheres of S^n (i.e they do not contain the equator). Although these examples all have constant curvature, because simplicity is preserved under small variations of the metric, we can use these three examples to construct many other simple domains.

Michel conjectured in [41] that every simple Riemannian manifold is boundary rigid. In two dimensions this was verified in [48], meaning simple surfaces with boundary are boundary rigid. This proof relies on a connection between the boundary distance function and the Dirichlet-to-Neumann map of the Laplace-Beltrami operator Δ_g for a Riemannian metric in 2 dimensions. It has not been generalized to other dimensions. In fact, for higher dimensional cases the boundary rigidity of simple manifolds is still open.

However, without assuming simplicity there have been many significant results in dimension 3 and higher for manifolds that satisfy constant curvature conditions. If (M, g) is a compact subdomain with smooth boundary of any of the following:

- Euclidean space $\mathbb{R}^n$,
- Hyperbolic space $\mathbb{H}^n$,
- open hemispheres of S^n,

then (M, g) is boundary rigid [10]. Additionally, non-trapping and geodesically convex subdomains of the flat torus are boundary rigid [9]. It was also shown in [3] if a manifold

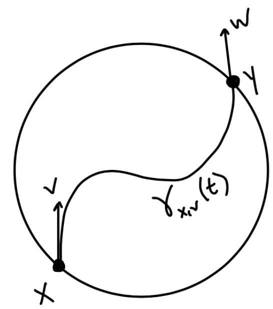

Figure 3.3 The scattering relation on a planar domain.

(M,g) admits an isometric immersion into the same dimensional Euclidean space, then M is boundary rigid. Surveys of further results can be found in [10, 62].

It was shown in [6] that metrics that are C^2-close to the Euclidean metric are boundary rigid, which is an extension of the semiglobal version of the result shown in [36]. Similarly, metrics that are C^3-close to the Hyperbolic metric of a region in $\mathbb{H}^n$ are boundary rigid [7]. A generic simple metric is also boundary rigid [56], which is obtained from linearizing the boundary rigidity problem, and is discussed further in Section 3.1.2. Surveys of further results can be found in [26] and [58].

In applications, one rarely has access to the whole boundary, thus partial data and local reconstructions are important to consider. It was shown in [61] that one can recover locally the manifold by the travel times of waves joining points close to a convex point on the boundary. Manifolds with 'some' positive curvature have also been shown to locally be boundary rigid [12]. Local results near the Euclidean metric are also known [55].

3.1.1 Scattering & Lens Rigidity

In light of the difficulties presented by the boundary rigidity problem, we turn our attention to related problems with more data. Since the boundary distance function is determined by the distance minimizing geodesics going through the manifold, an alternative problem could consider the behavior of all geodesics going through the manifold. This information is captured in the **scattering relation**, which is defined in the following manner. Let η be the outer pointing unit normal to the boundary, and define

$$\partial_{in}SM = \{(x,v) \in S\partial M \ : \ x \in \partial M, \langle \eta(x), v\rangle_g \leq 0\},$$
$$\partial_{out}SM = \{(x,v) \in S\partial M \ : \ x \in \partial M, \langle \eta(x), v\rangle_g \geq 0\}.$$

The scattering relation on a non-trapping Riemannian manifold (M,g) is the map

$$\Sigma_g : \partial_{in}SM \to \overline{\partial_{out}SM}, \qquad \Sigma_g(x,v) = (y,w) = (\gamma_{x,v}(\tau_{\text{exit}}(x,v)), \dot{\gamma}_{x,v}(\tau_{\text{exit}}(x,v))).$$

An example of this mapping is shown in Figure 3.3. The **scattering rigidity problem** poses:

> Given a smooth non-trapping manifold M with boundary and two metrics g_1, g_2 on M for which $\Sigma_{g_1} = \Sigma_{g_2}$, are the associated Riemannian metrics the same up to boundary-preserving isometry?

If the answer is affirmative, the manifold is said to be **scattering rigid**. As it was shown in [13], without the non-trapping assumption this problem is ill-posed. This is because changes in the metric near trapped directions is unseen by the data. So if there exists an open $U \subset M$ where SU consists of trapped directions then take $\alpha \in C^\infty(M)$ such that $\alpha > 0$ and $\text{supp}(1-\alpha) \subset U$. By defining $\tilde{g} = \alpha g$ then $\text{supp}(g - \tilde{g}) \subset U$ and g and $\tilde{g}$ are not isometric but $\Sigma_g = \Sigma_{\tilde{g}}$. Thus, manifolds with trapped geodesics and the same scattering data need not determine isometric manifolds.

If in addition to the scattering data we also have knowledge about the length of the geodesics, we have the **lens data**. The related **lens rigidity problem** poses:

> Given a smooth non-trapping manifold M with boundary, two metrics g_1, g_2 on M for which $\Sigma_{g_1} = \Sigma_{g_2}$, and the lengths of the geodesics in M, are the associated Riemannian metrics the same up to boundary-preserving isometry?

If we are able to answer affirmatively, the manifold is said to be **lens rigid**.

On simple manifolds the lens rigidity, scattering rigidity, and boundary rigidity problems are equivalent [41]. This is because on a simple manifold each pair of points is connected by a unique geodesic, whose direction is determined by differentiating the distance function. Thus, the boundary distance data determines an initial point and direction for a geodesic. By doing this for both points on the boundary we attain the scattering data.

However, there are few results for non-simple manifolds. It has been shown that if a manifold (M,g) with boundary ∂M is lens rigid and G is a finite group acting on (M,g) freely then

$(M/G, g)$ with boundary $\partial M/G$ is also lens rigid [11]. There has also been some progress showing lens rigidity near a generic class of non-simple manifolds [59], but it has only been shown locally.

3.1.2 Geodesic Ray Transform

Recall the goal of the boundary rigidity and lens rigidity problems was to recover the metric tensor g up to boundary-preserving isometry. The linearization of these problems is the tensor tomography problem, and it seeks to recover a symmetric (2,0)-tensor field 'up to natural obstruction' from the integrals of the (2,0)-tensor field along geodesics [61]. In this Section we will follow [54] to define the relevant terms, linearize the boundary rigidity problem, and then show how the tomography problem can be used to solve a variant of the boundary rigidity problem on simple manifolds with nonpositive sectional curvature.

Let (M, g) be a simple Riemannian manifold, where g^τ is a family of simple metrics on M smoothly depending on a parameter $\tau \in (-\varepsilon, \varepsilon)$ with $g^0 = g$. Fix $p, q \in \partial M, p \neq q$, and set $A = d_{g^0}(p, q)$. Let $\gamma_\tau : [0, A] \to M$ be the geodesic of the metric g^τ, such that $\gamma_\tau(0) = p$ and $\gamma_\tau(A) = q$. Since M is simple it is diffeomorphic to a disc, so we have global coordinates where $\gamma_\tau(t) = (\gamma_\tau^1(t), ..., \gamma_\tau^n(t))$ and $g^\tau = (g_{ij}^\tau)$. The simplicity of metric g^τ implies the smoothness of the functions $\gamma_\tau^i(t)$. It follows from (2.7) that

$$d_{g^\tau}(p, q) = \int_0^A \sqrt{g_{ij}^\tau(\gamma_\tau(t))\dot{\gamma}_\tau^i(t)\dot{\gamma}_\tau^j(t)}\, dt$$

and so

$$\frac{1}{A}[d_{g^\tau}(p, q)]^2 = \int_0^A g_{ij}^\tau(\gamma_\tau(t))\dot{\gamma}_\tau^i(t)\dot{\gamma}_\tau^j(t)\, dt.$$

Differentiating with respect to τ, and setting $\tau = 0$ then

$$\frac{1}{A}\frac{\partial}{\partial\tau}\Big|_{\tau=0}[d_{g^\tau}(p, q)]^2 = \int_0^A f_{ij}(\gamma_0(t))\dot{\gamma}_0^i(t)\dot{\gamma}_0^j(t)\, dt + \int_0^A \frac{\partial}{\partial\tau}\Big|_{\tau=0}\Big[g_{ij}^0(\gamma_\tau(t))\dot{\gamma}_\tau^i(t)\dot{\gamma}_\tau^j(t)\Big]\, dt,$$

where $f_{ij} = \frac{\partial}{\partial\tau}|_{\tau=0}\, g_{ij}^\tau$. That is, $f = (f_{ij})$ is a symmetric tensor of order 2. Since γ_0 is a geodesic in a simple metric then it is distance minimizing and thus an extremal value of the function $E_0(\gamma) = \int_0^A g_{ij}^0(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\, dt$. Thus the right-most integral is zero and we define a function

$$If(\gamma_0) := \frac{1}{A}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}[d_{g^\tau}(p,q)]^2 = \int_0^A f_{ij}(\gamma_0(t))\dot{\gamma}_0^i(t)\dot{\gamma}_0^j(t)\,dt.$$

The function If on the set of geodesics joining boundary points is called the **ray transform of f**. If d_{g^τ} does not depend on τ then $If(\gamma_0) = 0$.

On the other hand, if each of the metrics g^τ is boundary rigid there exists a family of boundary-preserving isometries Ψ_τ, where $g^\tau = (\Psi_\tau)^* g^0$. Define a vector field $V(x) = \frac{d}{d\tau}|_{\tau=0}\Psi_\tau(x)$ on M, so that it has flow Ψ_τ. Differentiating g^τ with respect to τ and putting $\tau = 0$ produces the Lie derivative of g^0 with respect to V, as seen in [37, pg. 321],

$$L_V g^0 := \frac{\partial}{\partial\tau}\bigg|_{\tau=0}(\Psi_\tau)^* g^0 = \frac{\partial}{\partial\tau}\bigg|_{\tau=0} g^\tau.$$

Let $(x^1, ..., x^n)$ be local coordinates on M, so if $\Psi_\tau(x) = (\Psi_\tau^1(x), ..., \Psi_\tau^n(x))$ then

$$g_{ij}^\tau = (g_{kl}^0 \circ \Psi_\tau)\frac{\partial \Psi_\tau^k(x)}{\partial x^i}\frac{\partial \Psi_\tau^l(x)}{\partial x^j}. \tag{3.2}$$

In local coordinates then [37, Example 12.35] shows

$$\begin{aligned} f_{ij} = \frac{\partial}{\partial\tau}\bigg|_{\tau=0} g_{ij}^\tau = (L_V g^0)_{ij} \\ &= V g_{ij}^0 + g_{il}^0\frac{\partial V^l}{\partial x^j} + g_{kj}^0\frac{\partial V^k}{\partial x^i} \\ &= V^k\left(\frac{\partial}{\partial x^k}g_{ij}^0 - \Gamma_{ik}^l g_{lj}^0 - \Gamma_{kj}^l g_{il}^0\right) + \left(g_{ik}^0\frac{\partial V^k}{\partial x^j} + g_{ik}^0\Gamma_{il}^k V^l\right) \\ &\quad + \left(g_{kj}^0\frac{\partial V^k}{\partial x^i} + g_{kj}^0\Gamma_{il}^k V^l\right). \end{aligned}$$

Letting $V_i = g_{ij}^0 V^j$, then

$$f_{ij} = \nabla_V g_{ij}^0 + \left(\frac{\partial V_i}{\partial x^j} - \Gamma_{ij}^k V_k\right) + \left(\frac{\partial V_j}{\partial x^i} - \Gamma_{ji}^k V_k\right).$$

Due to the definition of the metric tensor, the first term vanishes. Finally, let $\nabla_i V_j = \frac{\partial V_i}{\partial x^j} - \Gamma_{ij}^k V_k$, which represents the components of the total covariant derivatives of the field V in the metric g^0,

$$f_{ij} = \nabla_i V_j + \nabla_j V_i.$$

We then define,

$$(DV)_{ij} := \frac{1}{2}\left(\nabla_i V_j + \nabla_j V_i\right) = \frac{1}{2} f_{ij}.$$

Observe that the condition $\Psi_\tau|_{\partial M} = Id$ implies that $V|_{\partial M} = 0$, and thus by the Fundamental Theorem of Calculus

$$I(DV)(\gamma_0) = \int_0^A \frac{d}{dt}[V_i\,\dot{\gamma}_0^i]\,dt = V_i\,\dot{\gamma}_0^i\Big|_0^A = 0.$$

The inverse problem associated to this realization is the **tensor tomography problem**:

> Let $f \in C^\infty(T^{(2,0)}(M))$, if $If = 0$ for all geodesics connecting boundary points in g^0, does that imply the existence of a vector field V such that $V|_{\partial M} = 0$ and $DV = f$?

By $C^\infty(T^{(2,0)}(M))$ we denote the space of smooth covariant tensor fields of rank 2 on M. If we can answer the above problem affirmatively, then the ray transform I of (M, g) is **s-injective**.

It follows from [54, Theorem 2.4] that any symmetric tensor $f \in C^\infty(T^{(2,0)}(M))$ admits an orthogonal decomposition into **solenoidal** and **potential** parts, $f^s \in C^\infty(T^{(2,0)}(M))$ and $V \in C^\infty(T^{(1,0)}(M))$ respectively, such that

$$f = f^s + DV, \qquad \delta f^s = 0, \qquad V|_{\partial M} = 0.$$

Here δ is the operator such that $[\delta f]_i = g^{jk}\nabla_k f_{ij}$. Therefore, if I is injective on the space of solenoidal tensors then I is s-injective [57, pg. 2]. We are now ready to connect back to the boundary rigidity problem.

Lemma 51. *For all $\tau \in (-\varepsilon, \varepsilon)$ let g^τ be a family of simple metrics on a compact manifold M which induce distances d_{g^τ}, and I_τ is the ray transform corresponding to the metric g^τ. If I_τ is s-injective for every τ, and the boundary distance functions d_{g^τ} are independent of τ, then for each τ there exists a boundary preserving isometry Ψ_τ such that $\Psi|_{\partial M} = Id$ and $\Psi^* g^0 = g^\tau$.*

Proof. We summarize the key ideas of the proof presented in [52, Lemma 4.8.3]. Start by observing $I_\tau(\frac{\partial g^\tau}{\partial \tau}) = 0$ for all τ. As I_τ is s-injective then for all $\tau \in [0,1]$ there exists a field V^τ

which is a solution to the system

$$\begin{cases} D^\tau V^\tau = \frac{\partial g^\tau}{\partial \tau} \\ V^\tau|_{\partial M} = 0. \end{cases}$$

Here D^τ is the operator corresponding to the inner covariant differential, which depends on τ. Thus if g^τ is smooth, by the regularity theorems of the elliptic operator D^τ, then V^τ is smooth in $(x,\tau) \in M \times (-\varepsilon, \varepsilon)$.

Now let $y(\tau) := (y^1(\tau), ..., y^n(\tau))$ where $y^i(\tau)$ are real valued functions, and consider the system of ODE's on M,

$$\begin{cases} \frac{d}{d\tau} y^i = (g^\tau)^{ij}(y)\ V_j^\tau(y,\tau), & (y,\tau) \in M \times (-\varepsilon, \varepsilon) \\ y^i(0) = y^i, & y \in M. \end{cases} \tag{3.3}$$

It follows from [37, Theorem D.6] that because $(g^\tau)^{ij}(x)\ V_j^\tau(x,\tau)$ is smooth for any $(\tau_0, x_0) \in (-\varepsilon,\varepsilon) \times M$ there exists an open interval $J_0 \subset (-\varepsilon, \varepsilon)$ containing τ_0 and a neighborhood U_0 containing x_0 such that for each $\tau \in J_0$ and $x \in U_0$ there is a C^1 map $y^i : J_0 \to M$ that solves (3.3). Moreover, there exists a smooth map $\phi^i : J_0 \times J_0 \times U_0 \to M$ such that $\phi^i : (\tau_1, \tau_2, x) \mapsto y^i(\tau_1)$. Thus the map $\phi = (\phi^1, ..., \phi^n)$ is smooth and it satisfies $\phi(0,\tau,x) = x$. Using [37, Theorem 9.48(c)] the map $\psi_\tau : x \mapsto \phi(0,\tau,x)$ is a diffeomorphism onto its image for all $\tau \in (-\varepsilon, \varepsilon)$. Moreover, since $V^\tau|_{\partial M} = 0$ then ψ_τ is the identity on ∂M, so ψ_τ is a family of boundary preserving isometries. Because d_{g^τ} is independent of τ, this equivalently means the family of metrics are boundary rigid, and thus $g^0 = (\Psi_\tau)^* g^\tau$. □

Example 52. *Let Ω be a convex domain on a hyperbolic surface. By [52, Lemma 4.3.3] for every tensor field $f \in C^\infty(T^{(2,0)}(M))$ the solenoidal part f^s is uniquely determined by the ray transform If. Thus, I is injective on the space of solenoidal tensors, and consequently I is s-injective. After applying Lemma 51 then we see that small perturbations of the hyperbolic metric on Ω are boundary rigid.*

Similar work in [52] showed the boundary rigidity of metrics with small curvature. This work also provided conditional and non-sharp stability estimates for metrics with small curvature. Moreover, [12] used this to get local uniqueness results for the boundary rigidity problem.

We note that to simplify the narrative and make the connection to the boundary rigidity question more clear, in this section we only introduced tensors of order 2. The definition of If and its analysis can be generalized to a wider class of metrics and symmetric tensors of any order (see for example [52, 57]). Additional surveys on this problem can be found in [58, 66, 10].

3.2 Boundary Distance Data

Given the difficulty of the boundary rigidity problem, it is worth considering some alternative/modified versions of the problem. In this next section we consider a related inverse problem that has more data. Specifically, we assume sources of seismic activity can be anywhere in the planet, and not just on the surface.

For $x \in M$, a compact Riemannian manifold with boundary, define the **boundary distance function** of a point x as

$$r_x : \partial M \to \mathbb{R}, \qquad r_x(z) = d(x,z).$$

Then the **boundary distance data**, is given by

$$\partial M \qquad \text{and} \qquad \{r_x \in C(\partial M) : x \in M\}. \tag{3.4}$$

Observe that if $\Psi : (M_1, g_1) \to (M_2, g_2)$ is a Riemannian isometry (defined in Section 2.2) then

$$r_p(z) = d(p,z) = d(\Psi(p), \Psi(z)) = r_{\Psi(p)}(\Psi(z)).$$

Thus the boundary distance function is preserved under isometries. In this model, the inverse problem becomes:

> Given $\partial M_1, \partial M_2$, a diffeomorphism $\phi : \partial M_1 \to \partial M_2$, and $\{r_x \circ \phi^{-1} \in C(\partial M_1) : x \in M_1\} = \{r_y \in C(\partial M_2) : y \in M_2\}$, are the associated Riemannian manifolds the same up to isometry?

This question has been answered affirmatively in [34] and [29]. The rest of this section summarizes the proof of this result.

For any $x \in M$ it follows that r_x is a continuous function on the boundary. Since ∂M is assumed to be compact there must be a point $z_x \in \partial M$ such that $r_x(z_x)$ is the minimum distance from x to the boundary. Additionally, by the proof of Lemma 32, then x and z_x are connected by a distance minimizing geodesic γ that is normal to ∂M. The normality of the geodesics yields the following critical insight for their proof.

Lemma 53. *Let (M,g) be a compact Riemannian manifold with smooth boundary. For $x_0 \in M$ denote z_0 to be a nearest point to x_0 on the boundary. Then there are neighborhoods $U \subset M$ of x_0 and $V \subset \partial M$ of z_0 such that*

(a) $d(\cdot,\cdot) \in C^\infty(U \times V)$

(b) $grad_x d(x,z)|_{x=x_0}$*, considered as a function of z, is a diffeomorphism from V to its image in $S_{x_0}M$. In particular, this means that this image is an open set in $S_{x_0}M$.*

Proof. The proof can be found in [29, Lemma 2.15]. □

From the smoothness of the distance in Lemma 53(a), we can now use the travel time data to distinguish points from one another. In other words, for p_1 and p_2 in M satisfying $r_{p_1} = r_{p_2}$ on ∂M then p_1 is p_2. This is because having the same travel time data for p_1 and p_2 will have the same point z on the boundary which minimizes r_{p_1} and r_{p_2}. Consequently if $-\eta(z)$ is an inward pointing vector at z, due Lemma 32 then the geodesic $\gamma_{z,-\eta(z)}$ contains p_1 and p_2 in its image. Since they are also the same distance away from z along $\gamma_{z,-\eta(z)}$, then they are the same points.

We are now ready to reconstruct the topological structure on M. For $f \in C(\partial M)$ let $\|f\|_\infty = \sup_{z\in\partial M} |f(z)|$ and define the mapping

$$R : (M,g) \to (C(\partial M), \|\cdot\|_\infty), \qquad R : x \mapsto r_x.$$

From our ability to separate the data it follows that R is injective. Using the triangle inequality we see that R is continuous. By the compactness of M it follows that R is also a closed map. Thus R is a topological embedding, dictating the topological structure on M.

Now we consider the local coordinate structure for all points in our manifold. For points sufficiently close to the boundary, or points inside the collar neighborhood of ∂M, we identify them using the boundary normal coordinates (defined in (2.15)). So for each $x_0 \in \Omega_\varepsilon$ let z_0 be the unique boundary point such that $d(x_0, z_0) = d(x_0, \partial M)$. By the construction of

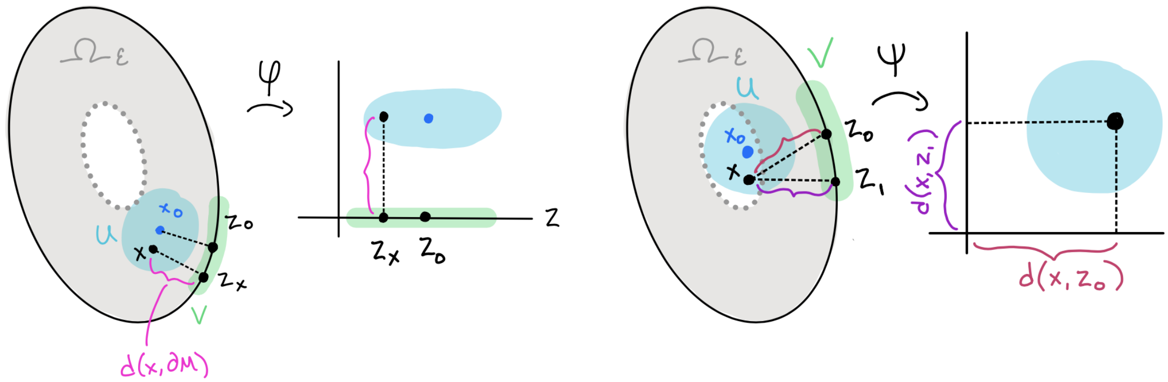

Figure 3.4 Local coordinates for the full data case.

the collar neighborhood (Section 2.3) there is a coordinate mapping φ defined by

$$\varphi : \Omega_\varepsilon \to \mathbb{R} \times \partial M, \qquad \varphi : x \mapsto (d(x, \partial M), z_x).$$

Here z_x is the closest boundary point to $x \in \Omega_\varepsilon$, in other words it is the point which minimizes the distance $d(x,\cdot)|_{\partial M}$. Since the function $d(x,\cdot)|_{\partial M}$ is given by the data, then its minimizer is determined by the data, meaning that the function φ is data-driven.

For points further away from the boundary we introduce the **boundary distance coordinates**. In general, these are constructed by taking a point $x_0 \in M$ and finding $z_0 \in \partial M$ to be a closest boundary point. By Lemma 53 there is a neighborhood $U \subset M$ of x_0 and $V \subset \partial M$ of z_0 such that the distances between points in these neighborhoods are smooth. It follows that there are curves $c_i(t)$, $i = 1, \dots, n-1$, in ∂M such that $c_i(0) = z_0$ and vectors $\frac{dc_i}{dt}(0) = v_i$ that form an orthonormal basis of $T_{z_0}\partial M$. For sufficiently small $t > 0$ the points $z_1 = c_1(t)$, ... $z_{n-1} = c_{n-1}(t)$ are in V. The boundary distance coordinates are given by the mapping ψ defined by

$$\psi : x \mapsto (d(x,z_0), d(x,z_1), \dots, d(x,z_{n-1})), \qquad x \in U. \tag{3.5}$$

To show ψ is a coordinate mapping let w_0 be the unit vector where $\exp_{x_0}(d(x_0,z_0)w_0) = z_0$. Due to [29, Lemma 2.13] x_0 and z_0 are not conjugate points and thus the mapping $D\exp_{x_0} : T_{d(x_0,z_0)w_0}T_{x_0}M \to T_{z_0}M$ is an isomorphism. Now for $i = 1, \dots, n-1$ define the vectors $w_i := \left(D\exp_{x_0}|_{d(x_0,z_0)w_0}\right)^{-1} v_i$, so that $\{w_0, w_1, \dots, w_{n-1}\}$ forms a basis of $T_{x_0}M$. For sufficiently small s there are vectors $\tilde{w}_i(s) \in T_{x_0}M$ such that $\tilde{w}_i(0) = w_0$ and $\frac{d\tilde{w}_i(s)}{ds}|_{s=0} = w_i$

for $i = 1, ..., n-1$. Then $z_i = \exp_{x_0}(d(x_0, z_i)\tilde{w}_i(s_0))$ for some sufficiently small s_0. Moreover,

$$\operatorname{grad}_x d(x,z)|_{x=x_0}(z_i) = -\frac{\tilde{w}_i(s_0)}{\|\tilde{w}_i(s_0)\|_g}$$

and by construction they are linearly independent. Taking the musical isomorphism of the gradients then $D_x d(x,z)|_{x=x_0} = \left(\operatorname{grad}_x d(x,z)|_{x=x_0}\right)^\flat$ where D_x is the differential with respect to x, and evaluated at z_i they are linearly independent. Thus the differential,

$$D\psi|_{x=x_0} = \Big[D_x d(x,z)|_{x=x_0}(z_0)\,,\ \cdots\,,\ D_x d(x,z)|_{x=x_0}(z_{n-1})\Big]^T$$

has rank n. It follows by the inverse function theorem [37, Theorem C.34] that there exist neighborhoods such that ψ^{-1} is a local diffeomorphism around x_0. Thus ψ provides a local coordinate structure around x_0, when x_0 is away from the boundary.

Together, the boundary normal coordinates and boundary distance coordinates give local coordinates for all points in M. Since the coordinate charts are smoothly compatible, we have a smooth atlas on M. This can be extended to the unique maximal atlas of the Riemannian manifold M as in Lemma 1, and thus defines the smooth structure.

It is left to reconstruct the Riemannian structure on M. For a point $x_0 \in M^{int}$ with closest boundary point $z_0 \in \partial M$, by construction of the local coordinates there are neighborhoods U of x_0 and V of z_0 such that $d(x,z)$ is smooth for all $(x,z) \in U \times V$. Thus consider the gradient $\operatorname{grad}_x d(x,z)$ for $x \in U$ which is the velocity of the distance minimizing unit speed geodesic from z to x. In particular, the map

$$\tilde{H}_{x_0} : z \mapsto \operatorname{grad}_x d(x,z)|_{x=x_0}, \qquad z \in V$$

is well defined. However, this map is unknown from the given data.

Instead, we work with its sister map

$$H_{x_0} : z \mapsto D_x d(x,z)|_{x=x_0}, \qquad z \in V$$

where D_x is the differential of the distance function with respect to x. This map is known from the given data. Moreover, $H_{x_0}(z)^\sharp = \tilde{H}_{x_0}(z)$ where $\sharp$ is the musical isomorphism which maps co-vectors to vectors. Denote

$$W^* = H_{x_0}(V),$$

so that W^* is an open subset of the unit co-sphere at x_0 with respect to the metric tensor g^{-1}. Denote the elements of M in local coordinates as $(x^1, \ldots x^n)$ so that the differentials $dx^1, \ldots, dx^n$ form a basis in W^*. Then for every $\omega, \nu \in W^*$ we can write

$$g^{-1}(\omega, \nu) = g^{ij}\omega_i \nu_j, \quad \text{where } g^{ij} = g^{-1}(dx^i, dx^j),\ i, j \in \{1, \ldots, n\}.$$

Since $W^* \subset S^*_{x_0}M$ is open it holds that we know the open cone,

$$C(W^*) = \{t\omega \in T^*_{x_0}M : \omega \in W^* : t > 0\}$$

and the smooth function $F : C(W^*) \to \mathbb{R}$, such that for $\xi = t\omega \in C(W^*)$ then

$$F(\xi) := \frac{1}{2}\|\xi\|^2_g = \frac{1}{2}g^{ij}\xi_i\xi_j = \frac{1}{2}t^2 g^{ij}dx^i dx^j = \frac{1}{2}t^2.$$

Therefore g^{ij} is the Hessian of F. This determines the inverse metric g^{-1} in the corresponding coordinates. Taking the inverse produces the metric g on M^{int}. To determine the metric on the whole M, express the metric tensor in terms of the boundary normal coordinates and use the smoothness of these coordinates as one approaches ∂M. An alternative method for reconstructing the metric on the boundary is done in [36, 72].

CHAPTER

4

EXTENDED VERSION OF THE PAPER: *UNIQUENESS OF THE PARTIAL TRAVEL TIME REPRESENTATION OF A COMPACT RIEMANNIAN MANIFOLD WITH STRICTLY CONVEX BOUNDARY*

4.1 Main Theorem and the Geometric Assumptions

We consider a compact n-dimensional smooth manifold M with smooth boundary ∂M, equipped with a smooth Riemannian metric g. For points $p, q \in M$ the Riemannian distance between them is denoted by $d(p,q)$. Then for $p \in M$ we define the **boundary distance function** $\hat{r}_p : \partial M \to \mathbb{R}$ given by $\hat{r}_p(z) = d(p,z)$. Let Γ be a non-empty open subset of the boundary ∂M. We denote the restriction of the boundary distance function on this set as

$r_p := \hat{r}_p\big|_\Gamma$. The collection

$$\Gamma \qquad \text{and} \qquad \{r_p : \Gamma \to \mathbb{R} \ : \ r_p(z) = d(p,z), p \in M\}, \tag{4.1}$$

are called the **partial travel time data of** $\Gamma \subset \partial M$. With these data we seek to recover the Riemannian manifold (M,g) up to a Riemannian isometry. The following definition explains when two Riemannian manifolds have the same partial travel time data (4.1).

Definition 54. *Let (M_1, g_1) and (M_2, g_2) be compact, connected, and oriented Riemannian manifolds of dimension $n \in \mathbb{N}$, $n \geq 2$ with smooth boundaries ∂M_1 and ∂M_2 and open non-empty regions $\Gamma_i \subset \partial M_i$ respectively. We say that the partial travel time data of (M_1, g_1) and (M_2, g_2) coincide if there exists a diffeomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that*

$$\{r_p \circ \phi^{-1} \ : \ p \in M_1\} = \{r_q \ : \ q \in M_2\}. \tag{4.2}$$

We want to emphasize that the equality (4.2) is for the non-indexed sets of travel time functions. Thus, for any $p \in M_1$ there exists a point $q \in M_2$ such that $r_p(\phi^{-1}(z)) = r_q(z)$ for every $z \in \Gamma_2$. We do not know *a priori* where the point $p \in M_1$ is or if there are several points $q \in M_2$ that satisfy this equation.

We use the notations TM and SM for the tangent and unit sphere bundles of M. Their respective fibers, for each point $p \in M$, are denoted by T_pM and S_pM. In order to show that the data (4.1) determine (M,g), up to an isometry or in other words that the Riemannian manifolds (M_1, g_1) and (M_2, g_2) of Definition 54 are Riemanian isometric, we need to place an additional geometric restriction. We assume that (M,g) has a **strictly convex boundary** ∂M which means that the shape operator $S \colon T\partial M \to T\partial M$ as a linear operator on each tangent space $T_x \partial M$ of the boundary ∂M for a point $x \in \partial M$ is negative definite (see Section 2.4).

It was shown in Lemma 44 that the strict convexity of the boundary implies the **geodesic convexity** of (M,g). That is any pair of points $p,q \in M$ can be connected by a distance minimizing geodesic (not necessarily unique) which is contained in the interior M^{int} of M modulo the terminal points. In particular any geodesic of M that hits the boundary exits immediately.

The main theorem of this Chapter is the following:

Theorem 55. *Let (M_1, g_1) and (M_2, g_2) be compact, connected, and oriented Riemannian* *manifolds of dimension $n \in \mathbb{N}$, $n \geq 2$ with smooth and strictly convex boundaries ∂M_1 and ∂M_2 and open non-empty measurement regions $\Gamma_i \subset \partial M_i$ respectively. If the travel time data of (M_1, g_1) and (M_2, g_2) coincide, in the sense of Definition 54, then the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are Riemannian isometric.*

Remark 56. *Our assumptions in Theorem 55 do not prevent the existence of the conjugate points. Actually quite a lot of work in this Chapter is needed to handle their existence. We also allow the manifolds to have trapped geodesics.*

4.1.1 Outline of the Proof of Theorem 55

The main tool of proving Theorem 55 is to differentiate the travel time functions given in equation (4.1). As these functions are defined only on a small open subset of the boundary we need to develop some regularity theorem for them. For this reason in Section 4.2 we study the regularity properties of the distance function on Riemannian manifolds satisfying the geometric constraints of Theorem 55. Section 4.2 has two main results. Theorem 57 is the aforementioned regularity result and the key of the proof of Theorem 55. In order to prove Theorem 57 we need to study, for each point in our manifold, the properties of its cut locus. This is the set past which the geodesics shot from the chosen point are not anymore distance minimizers. Theorem 65 collects the needed properties of these sets. Up to the best of our knowledge the material presented in Section 4.2 does not exist or is not easily accessible in the literature. Nevertheless, the corresponding results for manifolds without boundaries are well known.

In Section 4.3 we apply Theorem 57 to reconstruct the Riemannian manifold from its partial travel time data (4.1). This is done in five parts. Firstly we recover the geometry of the measurement region. As the second step we recover the topological structure by embedding the unknown manifold into a function space. Then we determine the boundary. The fourth step is to find local coordinates. Since our manifold has a boundary, we need different types of local coordinates for the interior and boundary points. Lastly we reconstruct the Riemannian metric. All the steps in Section 4.3 are fully data driven. Finally, in Section 4.4 we show that if two Riemannian manifolds, as in Theorem 55, have coinciding partial travel time data, in the sense of the Definition 54, then they are isometric.

4.1.2 The Convexity of the Domain in Theorem 55 is Necessary

Let us construct an explicit example of a surface M and a subset $\Gamma \subset \partial M$ so that our results fail with data recorded only on Γ (this example was originally presented in [14]). We recall that every pair of points on a smooth compact Riemannian manifold with boundary is always connected by a C^1-smooth distance minimizing curve [1]. We choose our a manifold to be the horseshoe-shaped domain of Figure 4.1. We split the domain M into two pieces M_1 and M_2 with respect to the line (red dotted line) that is normal to ∂M at $x_0 \in \partial M$ (blue dot). Then we choose a domain $\Gamma \subset \partial M_1$ (red arch) so that any minimizing curve joining a point on Γ and a point in M_2 touches the boundary near x_0. The curve $P \subset M$ is any involute of the boundary, meaning that the distance from all points on P to x_0 is the same. Because $d(z,p) = d(z,q)$ for any $z \in \Gamma$ and $p, q \in P$, from the point of view of our data (4.1), the set P appears to collapse to a point.

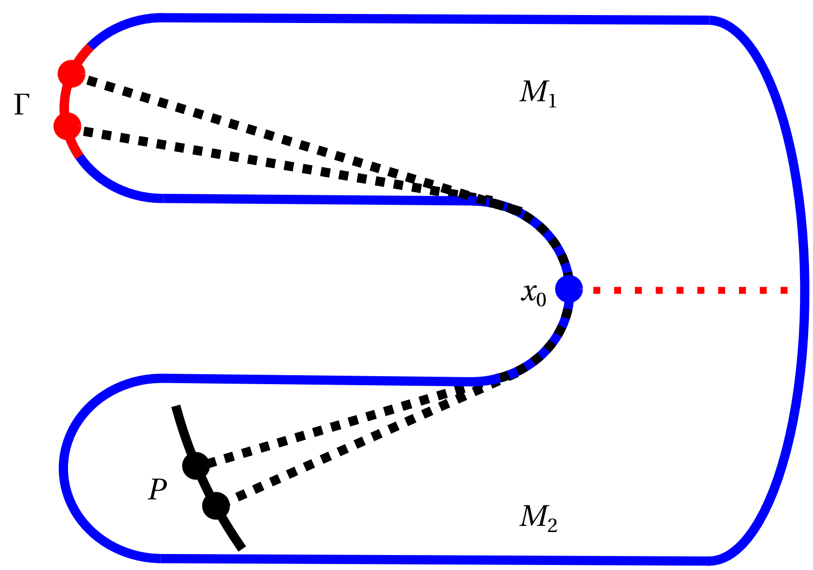

Figure 4.1 A domain where partial data is insufficient.

4.2 Distance Functions on Compact Manifolds with Strictly Convex Boundary

The aim of this section is to prove the following regularity result for the Riemannian distance function.

Theorem 57. *Let (M,g) be a smooth, compact, connected, and oriented Riemannian manifold of dimension $n \in \mathbb{N}$, $n \geq 2$ with smooth and strictly convex boundary. For any $p_0 \in M$ there exists an open and dense set $W_{p_0} \subset \partial M$ such that for every $z_0 \in W_{p_0}$ there are neighborhoods $U_{p_0} \subset M$ of p_0 and $V_{p_0} \subset M$ of z_0 such that the distance function $d(\cdot,\cdot)$ is smooth in the product set $U_{p_0} \times V_{p_0}$.*

This result is the key of the proof of Theorem 55.

4.2.1 Critical Distances, Extensions and the Cut Locus

In this section we consider a Riemannian manifold (M,g) as in Theorem 57, and study the properties of several critical distance functions. We define the *exit time function*

$$\tau_{\text{exit}}: SM \to \mathbb{R} \cup \{\infty\}, \quad \tau_{\text{exit}}(p,v) = \sup\{t > 0 : \gamma_{p,v}(t) \in M^{int}\},$$

where $\gamma_{p,v}$ is the geodesic of (M,g) with the initial conditions $(p,v) \in SM$. Since the boundary of M is strictly convex, $\tau_{\text{exit}}(p,v)$ is the first time when the geodesic $\gamma_{p,v}$ hits the boundary, and $(-\tau_{\text{exit}}(p,-v), \tau_{\text{exit}}(p,v))$ is the maximal interval where the geodesic is defined. We do not assume that $\tau_{\text{exit}}(p,v) < \infty$ for all $(p,v) \in SM$. That is, (M,g) may have trapped geodesics. Here we denote by $J \subset SM$ the set of all non-trapped directions, that are those $(p,v) \in SM$ for which $\tau_{\text{exit}}(p,v) < \infty$. It is shown in Section 2.4.1 that on compact Riemannian manifolds with strictly convex boundary the set J is open in SM, the exit time function τ_{exit} is continuous in J, and smooth on $J \setminus T\partial M$.

For any $p \in M$ we define a star shaped set

$$M_p := \left\{ v \in T_pM : \; v = 0, \; \text{or } \|v\|_g \leq \tau_{\text{exit}}\left(p, \frac{v}{\|v\|_g}\right)\right\}. \tag{4.3}$$

Thus M_p is the largest subset of T_pM where the exponential map of p

$$\exp_p : M_p \to M, \quad \exp_p(v) = \gamma_{p,v}(1)$$

is defined. Since ∂M is strictly convex this map is onto, but it does not need to be one-to-one, since there can be several geodesics of the same length connecting p to some common point. This leads to the following definition of the *cut distance* function:

$$\tau_{\text{cut}} : SM \to \mathbb{R}, \quad \tau_{\text{cut}}(p,v) = \sup\{t \in (0, \tau_{\text{exit}}(p,v)] : \ d(p, \gamma_{p,v}(t)) = t\}. \tag{4.4}$$

Thus the geodesic segment $\gamma_{p,v} : [0,t] \to M$ is a distance minimizing curve for any $t \in [0, \tau_{\text{cut}}(p,v)]$.

Traditionally on a closed Riemannian manifold (N,g) the set

$$\text{cut}_N(p) := \{\gamma_{p,v}(\tau_{\text{cut}}(p,v)) \in N : \ v \in S_pN\} \tag{4.5}$$

is known as the *cut locus* of the point $p \in N$ and each point in this set is called a *cut point* of p. Moreover, the cut locus of p coincides with the closure of the set of those points $q \in N$ such that there is more than one distance minimizing geodesic from p to q (see for instance [32, Theorem 2.1.14]). It has been also shown in [49, Section 9.1] that $d(p,\cdot)$ is smooth in $N \setminus (\{p\} \cup \text{cut}_N(p))$ but not at any $q \in (\{p\} \cup \text{cut}_N(p))$. In order to understand the smoothness properties of the distance function on a Riemannian manifold (M,g) with a strictly convex boundary, our aim is to define the set analogous to the one in (2.13) in this context.

If N is a closed manifold and $(p,v) \in SN$ then by Klingenberg's lemma [38, Proposition 10.32] either there is a second distance minimizing geodesic from p to $\gamma_{p,v}(\tau_{\text{cut}}(p,v))$ or these points are conjugate to each other along $\gamma_{p,v}$. In particular, the geodesic $\gamma_{p,v}$ is not a distance minimizer beyond the interval $[0, \tau_{\text{cut}}(p,v)]$. The following lemma extends this result in our case.

Lemma 58. *Let Riemannian manifold (M,g) be as in Theorem 57 and $(p,v) \in SM$. If*

$$\tau_{cut}(p,v) < \tau_{exit}(p,v)$$

then at least one of the following holds for $q := \gamma_{p,v}(\tau_{cut}(p,v))$:

- *There exists another distance minimizing geodesic from p to q.*

- *q is the first conjugate point to p along $\gamma_{p,v}$.*

Moreover, for any $t_0 \in (0, \tau_{cut}(p, v))$ the geodesic segment $\gamma_{p,v} : [0, t_0] \to M$ has no conjugate points and is the unique unit-speed distance minimizing curve between its endpoints.

Proof. Define $C = \tau_{\text{cut}}(p, v)$. By the assumption, $C < \tau_{\text{exit}}(p, v)$ we can extend the geodesic $\gamma_{p,v}$ beyond the point q. Let (B_i) be a sequence of real numbers such that $C < B_i < \tau_{\text{exit}}(p, v)$ and $\lim_{i\to\infty} B_i = C$. From the definition of the cut distance function, $\gamma_{p,v} : [0, B_i] \to M$ is not distance minimizing. Then for each i there is a unit vector $w_i \neq v$ such that $\tilde{\gamma}_{p,w_i} : [0, A_i] \to M$ is a unit-speed distance minimizing geodesic with the properties $\tilde{\gamma}_{p,w_i}(0) = p$ and $\tilde{\gamma}_{p,w_i}(A_i) = \gamma_{p,v}(B_i)$ and $A_i < B_i$. Using the continuity of the distance function, we have

$$C = d(p, \gamma_{p,v}(C)) = \lim_{i\to\infty} d(p, \gamma_{p,v}(B_i)) = \lim_{i\to\infty} A_i$$

and so $\lim_{i\to\infty} A_i = C$.

By compactness of the unit sphere, and passing to a subsequence if necessary, we can assume that $w_i \to w \in S_pM$. We need to ensure that the $\tilde{\gamma}(t) = \exp_p(t\,w)$ stays inside the manifold M and is a distance minimizer between p and q. Since $\tilde{\gamma}_{p,w_i} : (0, A_i] \to M$ is in M^{int}, we know $\tau_{\text{exit}}(p, w_i) > A_i$. Then by the continuity of the exit time function near the non-trapped directions

$$\tau_{\text{exit}}(p, w) = \lim_{i\to\infty} \tau_{\text{exit}}(p, w_i) \geq \lim_{i\to\infty} A_i = C.$$

Using this in combination with $A_i \to C$ and

$$q = \lim_{i\to\infty} \exp_p(B_i\,v) = \lim_{i\to\infty} \exp_p(A_i\,w_i) = \exp_p(C\,w)$$

means $\tilde{\gamma}(t)$ given by $\exp_p(t\,w)$ will also be a unit-speed minimizing geodesic from p to $q = \exp_p(C\,w)$. Since q is an interior point we also get $\tau_{\text{exit}}(p, w) > C$.

First we assume that $w = v$, which implies $C\,w = C\,v$, and hence for every neighborhood $U \subset TM$ of $C\,v$ there exists an index $k \in \mathbb{N}$ such that $A_i\,w_i, B_i\,v \in U$ for all $i \geq k$. Furthermore, we have

$$\exp_p(A_i\,w_i) = \exp_p(B_i\,v).$$

Therefore $\exp_p$ cannot be a local injection around $C\,v$, hence $D\exp_p(C\,v)$ must be singular. This implies p and q are conjugate along $\gamma_{p,v}$.

Now we assume that q is not a conjugate point to p along $\gamma_{p,v}$. All that is left to show is that $w \neq v$. We notice for each $i \in \mathbb{N}$ that $A_i w_i \neq B_i v$ in T_pM while $\exp_p(A_i w_i) = \exp_p(B_i v)$. However, since $q = \exp_p(C v)$ is assumed to not be a conjugate point to p along $\gamma_{p,v}$, the exponential map $\exp_p$ cannot have a critical point at $C v$ (see for instance [38, Proposition 10.20]). By the Inverse Function theorem there is $V \subset T_pM$, a neighborhood of $C v$, in which $\exp_p$ is injective. Since $B_i v \to C v$ there exists a value $i_N > 0$ such that $B_i v \in V$ for $i \geq i_N$. Due to the injectivity we can also conclude that $A_i w_i \notin V$ for any $i \geq i_N$. Thus, $C w \neq C v$, which implies $w \neq v$.

Considering the case when $t_0 \in (0, \tau_{\text{cut}}(p, v))$ is identical to the proof in [38, Proposition 10.32a]. □

Since the manifold M has a non-empty boundary ∂M it holds that both the tangent bundle TM and the unit sphere bundle SM are manifolds with boundaries ∂TM and ∂SM respectively.

$$(p, v) \in \partial TM, ((p, v) \in \partial SM) \quad \text{if and only if} \quad p \in \partial M.$$

We equip TM with the Sasaki metric g_S. Thus we can consider TM, and its submanifold SM, as Riemannian manifolds. In the following the convergence and other metric properties in TM or SM will be considered with respect to this metric.

Lemma 59. *Let the Riemannian manifold (M, g) be as in Theorem 57. The cut distance function τ_{cut} is continuous in SM.*

Proof. Let $(p, v) \in SM$ and $C = \tau_{\text{cut}}(p, v)$. By the definition of $\tau_{\text{cut}}(p, v)$ there exists a sequence $t_i \to C$ such that $\gamma_{p,v} : [0, t_i] \to M$ is distance minimizing. Then by the continuity of the exponential map and the distance function,

$$d(p, \gamma_{p,v}(C)) = \lim_{i\to\infty} d(p, \gamma_{p,v}(t_i)) = \lim_{i\to\infty} t_i = C. \tag{4.6}$$

Thus, $\gamma_{p,v}$ is distance minimizing on $[0, C]$.

Let (p_i, v_i) be a sequence in SM such that $(p_i, v_i) \to (p, v)$. Defining $C_i = \tau_{\text{cut}}(p_i, v_i) \leq \tau_{\text{exit}}(p_i, v_i)$, then we would like to show $C_i \to C$. For this it suffices to show

$$A := \limsup_{i\to\infty} C_i \leq C \leq \liminf_{i\to\infty} C_i =: B.$$

We will start by showing $A \leq C$. Passing to a subsequence, we have $C_{i_k} \leq \tau_{\text{exit}}(p_{i_k}, v_{i_k})$

and $C_{i_k} \to A$. Using Equation (4.6), we know $\gamma_{p_{i_k}, v_{i_k}}$ is minimizing on $[0, C_{i_k}]$. Then by the continuity of the exponential map and the distance function we have:

$$d(p, \gamma_{p,v}(A)) = \lim_{k\to\infty} d(p, \gamma_{p_{i_k}, v_{i_k}}(C_{i_k})) = \lim_{k\to\infty} C_{i_k} = A.$$

This makes $\gamma_{p,v}$ distance minimizing on $[0, A]$. Thus, $A \leq C$.

Next, we will show that $C \leq B$. Suppose first that $B = \tau_{\text{exit}}(p, v)$. In this case we have by the definition of the cut time $C \leq B$. To finish the proof we assume from here onwards that $B < \tau_{\text{exit}}(p, v)$. After passing to a subsequence, we have $(p_{i_k}, v_{i_k}) \to (p, v)$ and $C_{i_k} \to B$ as $k \to \infty$. Let $q_{i_k} = \exp_{p_{i_k}}(C_{i_k} v_{i_k})$.

Since $\tau_{\text{cut}}(p_{i_k}, v_{i_k}) \leq \tau_{\text{exit}}(p_{i_k}, v_{i_k})$ and

$$\lim_{k\to\infty} \tau_{\text{exit}}(p_{i_k}, v_{i_k}) = \tau_{\text{exit}}(p, v) > B = \lim_{k\to\infty} C_{i_k} = \lim_{k\to\infty} \tau_{\text{cut}}(p_{i_k}, v_{i_k})$$

there must exist $K \in \mathbb{N}$ such that

$$\tau_{\text{cut}}(p_{i_k}, v_{i_k}) < \tau_{\text{exit}}(p_{i_k}, v_{i_k}), \quad \text{for every } k \geq K.$$

Therefore, after possibly discarding the K first indices, it follows from the Lemma 58 that q_{i_k} is either a conjugate point to p_{i_k} along $\gamma_{p_{i_k}, v_{i_k}}$ or there exists another distance minimizing geodesic from p_{i_k} to q_{i_k}. Moreover, by the continuity of the exponential map

$$q := \exp_p(Bv) = \lim_{k\to\infty} \exp_{p_{i_k}}(C_{i_k} v_{i_k}) = \lim_{k\to\infty} q_{i_k}.$$

By $B < \tau_{\text{exit}}(p, v)$, the point q is contained in M, and we aim to show that either q is a conjugate point to p along $\gamma_{p,v}$, or that there is another geodesic of length B from p to q. By Lemma 58 either of these yield $C \leq B$ and ends the proof.

Consider the first case, and assume q_{i_k} and p_{i_k} are conjugate to each other for each i_k. Recall conjugate points are critical values of the exponential map. To utilize this, we consider a neighborhood $U \subset M$ of p with local coordinates $(x_1, ..., x_n)$, and define local coordinates of $TU = U \times \mathbb{R}^n$ as $(x_i, y_j)_{i,j=1}^n$. We consider the map

$$\exp\colon TU \to M, \quad (x, y) \mapsto \exp_x(y)$$

near (p, Bv). Since $(p_{i_k}, C_{i_k} v_{i_k})$ converges to (p, Bv) in TU and as q_{i_k} and p_{i_k} are assumed to be conjugate to each other, we have

$$\det\Big(D_y \exp_{p_{i_k}}|_{C_{i_k} v_{i_k}}\Big) = 0.$$

Thus by the continuity of the map $(x, y) \mapsto \det(D_y \exp_x |_y)$ this implies,

$$\det\Big(D_y \exp_p |_{B v}\Big) = \lim_{k\to\infty} \det\Big(D_y \exp_{p_{i_k}}|_{C_{i_k} v_{i_k}}\Big) = 0.$$

Therefore, $q = \exp_p(B v)$ is a conjugate point to p along $\gamma_{p,v}$.

Consider the second case, and assume that for each i_k there are 2 distance minimizing geodesics from p_{i_k} to q_{i_k}. The first geodesic is γ_k, where $\gamma_k(0) = p_{i_k}$ and $\dot\gamma_k(0) = v_{i_k}$. Let the second geodesic be σ_k where $\sigma_k(0) = p_{i_k}$ and $\dot\sigma_k(0) = w_k \in S_{p_{i_k}} M$. Moreover these geodesics satisfy

$$\gamma_k(C_{i_k}) = \sigma_k(C_{i_k}) = q_{i_k}, \quad \text{and} \quad v_k \neq w_k, \quad \text{for every } k \in \mathbb{N}. \tag{4.7}$$

After passing to a subsequence, we have by the compactness of SM that there is $w \in S_p M$ such that $(p_{i_k}, w_{i_k}) \to (p, w)$. Thus $(p_{i_k}, C_{i_k} w_{i_k}) \to (p, B w)$ and (4.7), with the continuity of the exponential map, yields

$$\exp_p(B w) = q = \exp_p(B v).$$

If v and w do not agree then $\gamma_{p,v}$ and $\gamma_{p,w}$ are two different geodesics of the length B connecting p to q.

To conclude the proof we choose a neighborhood $U \subset M$ of p and local coordinates $(x_1, \ldots, x_n)$ in U. Using the coordinates $(x_i, y_j)_{i,j=1}^n$ in TU and $(x_1', \ldots, x_n')$ in a neighborhood U' of q we define the map Φ in TU, near $(p, B v)$, by the formula

$$\Phi(x, y) = (x, \exp_x(y)) \in U \times U' \subset M \times M.$$

The differential of this map can be written as

$$D\Phi = \begin{bmatrix} \frac{dx}{dx} & \frac{dx}{dy} \\ \\ \frac{d\ \exp_x}{dx} & \frac{d\ \exp_x}{dy} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \\ \frac{d\ \exp_x}{dx} & D_y \exp_x \end{bmatrix},$$

and making $\det(D\Phi) = \det(D_y \exp_x)$. If p and q are not conjugates along $\gamma_{p,v}$ (otherwise we would be dealing with the previous case), we must have $\det(D\Phi) \neq 0$ at the point $(p, B v)$.

Thus the Inverse Function Theorem, implies that Φ has a local inverse near (p, Bv).

Finally we take $V \subset TU$ to be a neighborhood of (p, Bv) so that Φ is injective on V. For the sake of contradiction, assume $w = v$. That would make both $(p_k, C_{i_k} w_k)$ and $(p_{i_k}, C_{i_k} v_{i_k})$ converge to (p, Bv), so for a sufficiently large N we must have $(p_N, C_N v_N), (p_N, C_N w_N) \in V$. As $w_N \neq v_N$ we have distinct points $(p_N, C_N w_N)$ and $(p_N, C_N v_N)$ in V whose images under Φ coincide. This is a contradiction to the injectivity of Φ in V. Thus $w \neq v$, as desired. □

As M has a boundary, the definition of the cut time function τ_{cut}, in the equation (2.12), has an issue. Namely if $\tau_{\text{cut}}(p, v) = \tau_{\text{exit}}(p, v)$ for some $(p, v) \in SM$ we do not know *a priori* if the geodesic $\gamma_{p,v}$ just hits the boundary at $\gamma_{p,v}(\tau_{\text{cut}}(p, v))$ or if it is possible to find an extension of (M, g) such that $\gamma_{p,v}$ also extends as a distance minimizer.

To address this question, from here onwards we assume that (M, g) has been isometrically embedded in some closed Riemannian manifold (N, g). This can be done for instance by constructing the double of the manifold M as explained in Lemma 37 and extending the metric g smoothly across the boundary ∂M. The issue with this extension is that it might create 'short cuts' in the sense that there can be a curve in N, connecting some points of M, which is shorter than any curve entirely contained in M. Therefore we always have

$$d_M(p, q) \geq d_N(p, q), \quad \text{for all } p, q \in M,$$

where $d_M(\cdot, \cdot)$ and $d_N(\cdot, \cdot)$ are the distance functions of M and N respectively. The following proposition shows that while we stay close enough to M we do not need to worry about these short cuts.

Proposition 60. *Let (N, g) be a smooth, connected, orientable, and closed Riemannian manifold and $M \subset N$ an open set with closure $\bar{M}$ such that ∂M is smooth and has negative definite shape operator for an outward-pointing vector field on $\bar{M}$ in N. We say ∂M is a smooth strictly convex hyper-surface of (N, g). Then there exists an open subset $\hat{M}$ of N that contains $\bar{M}$, and the boundary of $\hat{M}$ is a smooth, strictly convex hyper-surface of N.*

Moreover

$$d_{\hat{M}}(p, q) = d_M(p, q), \quad \textit{for all } p, q \in \bar{M}. \tag{4.8}$$

Proof. Since ∂M is a smooth hyper-surface of N there exists a smooth function $s \colon N \to \mathbb{R}$

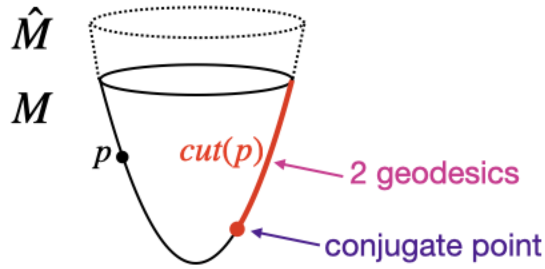

Figure 4.2 Extension of M to $\hat{M}$.

and a neighborhood U of ∂M such that

$$|s(x)| = \text{dist}(x, \partial M) := \inf\{d_N(x,z) : z \in \partial M\}, \quad \text{and} \quad \|\text{grad}\, s(x)\|_g \equiv 1,$$

for every $x \in U$. Moreover for each $x \in U$ there exists a unique $z \in \partial M$ such that $d_N(x,z) = |s(x)|$. We choose the sign convention of s such that $s(x) \geq 0$ for $x \in U \setminus M$. Then on ∂M the gradient of the function $s(\cdot)$ agrees with the outward pointing unit normal vector field of ∂M. The existence of this function is shown in Lemma 38.

By this construction, each $p \in U$ can be written uniquely as

$$p = (z(p), s(p)) \in \partial M \times \mathbb{R},$$

where $z(p)$ is the closest point of ∂M to p. Thus on U we write the Riemannian metric as a function of $(z,\varepsilon) \in \partial M \times \mathbb{R}$ in the form $\mathrm{d}s^2 + \tilde{g}(\varepsilon, z)$, where $\tilde{g}(\varepsilon, z)$ is the first fundamental form of the smooth hyper-surface $\Omega(\varepsilon) := s^{-1}\{\varepsilon\}$. By [38, Proposition 8.18] we can then write the second fundamental form of $\Omega(\varepsilon)$ as a bi-linear form

$$\Pi_{(z,\varepsilon)}(X,Y) = -\frac{1}{2}\frac{\partial}{\partial \varepsilon}\tilde{g}_{\alpha\beta}(\varepsilon, z) X^\alpha Y^\beta \in \mathbb{R}$$

on $T\Omega(s)$. Thus the eigenvalues $\lambda_1(z,\varepsilon), \ldots, \lambda_{n-1}(z,\varepsilon)$ of $\Pi_{(z,\varepsilon)}$ are continuous functions of (z,ε) [68, Appendix V, Section 4, Theorem 4A]. Since $\Omega(0)$ coincides with ∂M, which is strictly convex, we have that $\lambda_\alpha(z,0) < 0$ for every $\alpha \in \{1,\ldots,n-1\}$. Thus there exists $\varepsilon_0 > 0$ so that

$$\lambda_\alpha(z,\varepsilon) < 0, \quad \text{for every } \alpha \in \{1,\ldots,n-1\} \text{ and } |\varepsilon| < \varepsilon_0.$$

Therefore, for small enough $\varepsilon > 0$, we have that

$$M(\varepsilon) := s^{-1}(-\infty, \varepsilon) \subset M \cup U$$

is an open set of N that contains $\bar{M}$, and whose boundary $\partial M(\varepsilon) = \Omega(\varepsilon)$ is a smooth strictly convex hyper-surface of N. We choose $\varepsilon \in (0, \varepsilon_0)$ and set $\hat{M} = M(\varepsilon)$.

Let $p, q \in \bar{M}$ and choose a distance minimizing unit speed geodesic $\gamma : [0, d_{\hat{M}}(p,q)] \to \hat{M}$ that connects these points. Now without loss of generality we may assume that $\gamma(\tilde{t}) \in U$ for some $\tilde{t} \in [0, d_{\hat{M}}(p,q)]$. If this is not true then the trace of γ is contained in M and we are done.

Since U is open and $\gamma(\tilde{t}) \in U$ we can choose an interval $[a,b] \subset [0, d_{\hat{M}}(p,q)]$ such that $\gamma([a,b]) \subset U$ and define a smooth function

$$\tilde{s} : [a,b] \to \mathbb{R}, \quad \tilde{s}(t) := s(\gamma(t)).$$

Since p and q are in $\bar{M}$ we may without loss of generality assume that $\tilde{s}(a), \tilde{s}(b) \le 0$.

We aim to verify that $\tilde{s}$ is always non-positive. To establish this we show that the maximum value $m \in \mathbb{R}$ of $\tilde{s}$ is attained at the endpoints of the domain interval. So suppose that $m = \tilde{s}(t_0)$ is attained in some interior point $t_0 \in (a,b)$. As t_0 is a maximum point of $\tilde{s}$, laying in the interior of the domain interval, it must hold that $\dot{\tilde{s}}(t_0) = 0$ and $\ddot{\tilde{s}}(t_0) \le 0$. On the other hand since γ is a geodesic, we have by the Weingarten equation [38, Theorem 8.13 (c)] that

$$\dot{\tilde{s}}(t_0) = \langle \operatorname{grad} s(\gamma(t_0)), \dot{\gamma}(t_0)\rangle_g, \text{ and } \ddot{\tilde{s}}(t_0) = \langle D_t \operatorname{grad} s(\gamma(t_0)), \dot{\gamma}(t_0)\rangle_g = -\Pi_{\gamma(t_0)}(\dot{\gamma}(t_0), \dot{\gamma}(t_0)). \quad (4.9)$$

Here D_t stands for the covariant differentiation along the curve γ. Therefore $\dot{\gamma}(t_0)$ is tangential to the strictly convex hyper-surface $\Omega(m)$ which implies that $\Pi_{\gamma(t_0)}(\dot{\gamma}(t_0), \dot{\gamma}(t_0)) < 0$. This in conjunction with (4.9) leads into a contradiction with $\ddot{\tilde{s}}(t_0) \le 0$. We have verified that for all $p, q \in \bar{M}$ any distance minimizing geodesic in $\hat{M}$, between these points, is contained in $\bar{M}$. Therefore the equation (4.8) is true. □

By Proposition 60 we may assume that M is contained in the interior of some compact, Riemannian manifold $(\hat{M}, g)$ with a smooth strictly convex boundary. Moreover the distance function of $\hat{M}$ restricts to the one of M. Thus for every $(p, v) \in S\hat{M}$ where p is in M we always have that

$$\tau_{\text{cut}}(p,v) \le \widehat{\tau}_{\text{cut}}(p,v), \quad \text{and} \quad \tau_{\text{exit}}(p,v) < \widehat{\tau}_{\text{exit}}(p,v), \tag{4.10}$$

where $\widehat{\tau}_{\text{cut}}$ and $\widehat{\tau}_{\text{exit}}$ are the cut distance and the exit time functions of $(\widehat{M}, g)$ respectively. Motivated by this observation we define the cut locus of a point $p \in M$ as

$$\text{cut}(p) := \{\gamma_{p,v}(\tau_{\text{cut}}(p,v)) \in M : v \in S_pM,\ \tau_{\text{cut}}(p,v) = \widehat{\tau}_{\text{cut}}(p,v)\}. \tag{4.11}$$

The following result summarizes the basic properties of these sets.

Proposition 61. *Let the Riemannian manifold (M,g) be as in Theorem 57. Let $p \in M$.*

- *The cut locus cut(p) of the point p is a closed set of measure zero.*
- *If $q \in$ cut(p) and γ is a unit speed distance minimizing geodesic of M between p and q then at least one of the following holds:*
  1. *There exists another distance minimizing geodesic from p to q.*
  2. *q is the first conjugate point to p along γ.*

Proof.

- Let $q \in M$ belong to the closure of cut(p), and choose a sequence $q_i \in$ cut(p) that converges to q. Then for each $i \in \mathbb{N}$ we choose $v_i \in S_pM$ such that

  $$q_i = \gamma_{p,v_i}(\tau_{\text{cut}}(p,v_i)) = \gamma_{p,v_i}(\widehat{\tau}_{\text{cut}}(p,v_i)).$$

  After passing to the subsequence we may choose $v \in S_pM$ so that $v_i \to v$ as $i \to \infty$. Thus due to continuity of the cut distance function given in Lemma 59 and the continuity of the geodesic flow we arrive in

  $$\tau_{\text{cut}}(p,v) = \widehat{\tau}_{\text{cut}}(p,v), \text{ and } q = \gamma_{p,v_i}(\tau_{\text{cut}}(p,v)) = \gamma_{p,v_i}(\widehat{\tau}_{\text{cut}}(p,v)).$$

  This implies that $q \in$ cut(p) as claimed.

  In order to prove that cut(p) has zero measure we note that

  $$\text{cut}(p) \subset \{\exp_p(\tau_{\text{cut}}(p,v)v) \in M : v \in S_pM\} =: \text{CUT}(p).$$

Since the cut distance is continuous on S_pM a similar proof to the one given in [38, Theorem 10.34 (a)] yields that CUT(p) has a measure zero. Therefore also cut(p) is of measure zero.

- If $q \in \text{cut}(p)$ then by (4.10) there is $v \in S_pM$ such that

$$q = \gamma_{p,v}(\tau_{\text{cut}}(p,v)), \quad \tau_{\text{cut}}(p,v) = \widehat{\tau}_{\text{cut}}(p,v) \leq \tau_{\text{exit}}(p,v) < \widehat{\tau}_{\text{exit}}(p,v),$$

Thus Lemma 58 and Proposition 60 yield the second claim.

□

The following result introduces an open and dense subset of M where the distance function of an interior point is smooth.

Lemma 62. *Let the Riemannian manifold (M,g) be as in Theorem 57. Let $p \in M$. The distance function $d(p,\cdot)\colon M \to \mathbb{R}$, is smooth precisely in the open and dense set $M \setminus (\{p\} \cup cut(p))$.*

Proof. Let $q_0 \in M \setminus (\{p\} \cup \text{cut}(p))$. Since cut$(p)$ is closed the point q_0 has a neighborhood $U \subset M$ that is contained outside of cut(p). Let $q \in U$. Since M is geodesically convex there is $v(q) \in S_pM$ such that $q = \exp_p(t(q)v(q))$ for some $t(q) \in (0, \tau_{\text{exit}}(p, v(q))]$. Since q is not in the cut-locus of p we have by equation (4.10) and the definition of the cut locus of p that $t(q) < \widehat{\tau}_{\text{cut}}(p, v(q))$. By Lemma 58 and Proposition 60 this implies that $\gamma_{p,v(q)}$ is the unique distance minimizing geodesic from p to q. In particular $d(p,q) = t(q)$.

Since p and q_0 are not conjugate to each other along the geodesic $\gamma_{p,v(q_0)}$, the exponential map of p has an invertible differential at $t(q_0)v(q_0) \in T_pM$. As q_0 is not p, the Inverse function theorem implies that there is a neighborhood $V \subset U$ of q_0 such that the function

$$q \mapsto \|\exp_p^{-1}(q)\|_g = t(q) = d(p,q)$$

is smooth on V. We have proven that the distance function $d(p,\cdot)$ is smooth outside the set $\{p\} \cup \text{cut}(p)$. □

Proposition 63. *Let the Riemannian manifold (M,g) be as in Theorem 57. Let $p \in M$ and $q \in M \setminus (\{p\} \cup cut(p))$. There exist neighborhoods $U \subset M$ of p and $V \subset M$ of q such that the distance function $d(\cdot,\cdot)$ is smooth in the product set $U \times V$.*

Proof. Let (N,g) be a closed extension of (M,g) as in Proposition 60. We define a smooth map

$$F:(x,v)\in TN\mapsto (x,\exp_x(v))\in N\times N.$$

Then the differential of this map can be written as

$$\mathrm{D}F(x,v)=\begin{bmatrix}\mathrm{Id} & 0\\ * & \mathrm{D}\exp_x(v)\end{bmatrix}.$$

Since q is not in the cut locus of p there is a $v_0\in T_pN$ such that $\|v_0\|_g=d_M(p,q)$, $\exp_p(v_0)=q$ and $\mathrm{D}\exp_p(v_0)$ is not singular. Therefore $\det(\mathrm{D}F(p,v_0))=\det(\mathrm{D}\exp_p(v_0))$ does not vanish. Thus the Inverse function theorem implies that there are neighborhoods $\widetilde{W}\subset TN$ of (p,v_0) and $W\subset N\times N$ of (p,q) such that the local inverse function of F,

$$F^{-1}\colon W\to\widetilde{W},\quad F^{-1}(x,y)=(x,\exp_x^{-1}(y)),$$

is a diffeomorphism.

Since q is not in the cut locus of p we have $\|F^{-1}(p,q)\|_g<\widehat{\tau}_{\mathrm{cut}}\left(p,\frac{v_0}{\|v_0\|_g}\right)$. Thus by the continuity of the cut distance function $\widehat{\tau}_{\mathrm{cut}}(\cdot,\cdot)$ of (N,g) we can choose a neighborhood $W_1\subset W$ of (p,q) such that

$$\|F^{-1}(x,y)\|_g<\widehat{\tau}_{\mathrm{cut}}\left(x,\frac{F^{-1}(x,y)}{\|F^{-1}(x,y)\|_g}\right),\quad \text{for all } (x,y)\in W_1.$$

This gives $d_N(x,y)=\|F^{-1}(x,y)\|_g$, for all $(x,y)\in W_1$.

Finally we choose disjoint neighborhoods $U\subset M$ of p and $V\subset M$ of q such that $U\times V$ is contained in W_1. Let $(x,y)\in U\times V$ then $\gamma(t):=\exp_x(tF^{-1}(x,y))$ for $t\in[0,1]$ is a geodesic of N that connects x to y having the length of $d_N(x,y)$. Since both x and y are in M, we get from the proof of Proposition 60 that $\gamma(t)$ is contained in M. This yields

$$d_M(x,y)=\|F^{-1}(x,y)\|_g,\ \text{for all } (x,y)\in U\times V. \tag{4.12}$$

Since the sets U and V are disjoint we have that F^{-1} does not vanish in $U\times V$. Hence equation (4.12) gives the smoothness of $d_M(\cdot,\cdot)$ on $U\times V$. □

Recall that we have assumed that M is isometrically embedded in a closed Riemannian manifold (N,g). Thus any geodesic starting from M can be extended to the entire $\mathbb{R}$. Let

$p \in N$. We define the *conjugate distance function* $\tau_{\text{con}}: S_pN \to \mathbb{R} \cup \infty$ by the formula:

$$\tau_{\text{con}}(p, v) = \inf\{t > 0 : \gamma_{p,v}(t) \text{ is a conjugate point to } p\}.$$

As the infimum of the empty set is positive infinity we set $\tau_{\text{con}}(p, v) = \infty$ in the case when the geodesic $\gamma_{p,v}$ does not have any conjugate points to p. Since geodesics do not minimize the distance beyond the first conjugate point it holds that

$$\tau_{\text{cut}}(p, v) \leq \tau_{\text{con}}(p, v), \quad \text{if } (p, v) \in SM.$$

The following result is well known, but we could not find its proof in the existing literature, so we provide one below.

Lemma 64. *Let (N, g) be a closed Riemannian manifold and $p \in N$. The conjugate distance function is continuous on S_pN.*

Proof. Let $v_i \in S_pN$ for $i \in \mathbb{N}$ converge to v.

We set

$$C = \tau_{\text{con}}(p, v), \quad B = \liminf_{i\to\infty} \tau_{\text{con}}(p, v_i), \quad \text{and} \quad A = \limsup_{i\to\infty} \tau_{\text{con}}(p, v_i).$$

It suffices to show that $A \leq C \leq B$.

We assume first that $C = \infty$. If $A < \infty$ we choose a sub-sequence v_{i_k} of v_i such that $\tau_{\text{con}}(p, v_{i_k})$ converges to A. Then $\det(\mathrm{D}\exp_p(\tau_{\text{con}}(p, v_{i_k})v_{i_k})) = 0$, and the smoothness of the exponential map gives $\det(\mathrm{D}\exp_p(Av)) = 0$ yielding that $\exp_p(Av)$ is conjugate to p along $\gamma_{p,v}$. This implies that $\tau_{\text{con}}(p, v) \leq A$, which is impossible. By the same argument we see that $B = \infty$.

Let $C < \infty$, and by the same limiting argument as above we get $C \leq B$. Then we show that $A \leq C$. Choose a sub-sequence v_{i_k} such that $\tau_{\text{con}}(p, v_{i_k}) \to A$ as $k \to \infty$. For the sake of contradiction we suppose that $A > C$. By the definition of the conjugate distance function we have that $q = \gamma_{p,v}(C)$ is the first conjugate to p along $\gamma_{p,v}$. By [38, Theorem 10.26] for any $\varepsilon \in (0, A - C)$ there exists a piecewise smooth vector field X on the geodesic segment $\gamma_{p,v}: [0, C + \varepsilon] \to N$, that vanishes on 0 and $C + \varepsilon$ such that the index form of $\gamma_{p,v}$ over X is strictly negative. That is

$$I_v(X,X) := \int_0^{C+\varepsilon} \langle \mathrm{D}_t X, \mathrm{D}_t X\rangle_g + \langle R(\dot\gamma_{p,v}, X)\dot\gamma_{p,v}, X\rangle_g \,\mathrm{d}t < 0. \tag{4.13}$$

Here we used the notation D_t for the covariant differentiation along $\gamma_{p,v}$. The capital R stands for the Riemannian curvature tensor.

We choose vectors $E_1, \dots, E_n$ of T_pN that form a basis of T_pN and extend them on $\gamma_{p,v}(t)$ for $t \in [0, C+\varepsilon]$ via the parallel transport. Since parallel transport is an isomorphism the vector fields $\{E_1(t), \dots, E_n(t)\}$ constitute a basis of $T_{\gamma_{p,v}(t)}M$. We write $X(t) = X^j(t)E_j(t)$. Since X is piecewise smooth it holds that the component functions $X^j(t)$ are piecewise smooth. This lets us 'extend' X on $\gamma_{p,v_{i_k}}$ by the formula

$$X_k(t) = X^j(t)E_j^k(t), \tag{4.14}$$

where the vector field $E_j^k(t)$ is the parallel transport of E_j along $\gamma_{p,v_{i_k}}$. Thus X_k is a piecewise smooth vector field on $\gamma_{p,v_{i_k}}$ that vanishes at $t=0$ and $t = C+\varepsilon$.

Since $v_{i_k} \to v$, as $k \to \infty$, it holds that

$$\gamma_{p,v_{i_k}}(t) \to \gamma_{p,v}(t), \quad \text{and} \quad E_j^k(t) \to E_j(t), \quad \text{uniformly in } t \in [0, C+\varepsilon] \text{ as } k \to \infty.$$

Therefore by (4.13), (4.14), the continuity of the Levi-Civita connection, and the Riemannian curvature tensors we have

$$I_{v_{i_k}}(X_k, X_k) < 0, \quad \text{for large enough } k \in \mathbb{N}.$$

By [38, Theorem 10.28] there exists $s_k \in (0, C+\varepsilon]$ so that $\gamma_{p,v_{i_k}}(0)$ and $\gamma_{p,v_{i_k}}(s_k)$ are conjugate points. Therefore we must have $s_k \ge \tau_{\mathrm{con}}(p, v_{i_k})$ and we arrive at a contradiction $\tau_{\mathrm{con}}(p, v_{i_k}) < A$. This ends the proof. □

4.2.2 The Hausdorff Dimension of the Cut Locus

We fix a point $p \in M$ for this sub-section. By Lemma 62 we know that for any $p \in M$ the distance function $d(p, \cdot)$ is smooth in the open set $M \setminus (\mathrm{cut}(p) \cup \{p\})$. Moreover by Proposition 63 for each q in this open set there are neighborhoods U of p and V of q such that the distance function is smooth in the product set $U \times V$. As we are interested in the inverse problem where we study distance function $d(p,\cdot)$ restricted on some open subset Γ of the boundary, we do not know *a priori* if this function is smooth on Γ. In particular we do not know the size of the set $\mathrm{cut}(p) \cap \partial M$ yet. In this sub-section we show that the set $\partial M \setminus \mathrm{cut}(p)$, where $d(p,\cdot)$ is smooth, is always an open and dense subset of ∂M.

Proposition 61 yields that $\mathrm{cut}(p)$ can be written as a disjoint union of

- *Conjugate cut points:*

  $$Q(p) := \{\gamma_{p,v}(t) \in M : v \in S_pM,\ t = \tau_{\mathrm{cut}}(p,v) = \tau_{\mathrm{con}}(p,v)\} \subset \mathrm{cut}(p),$$

  that are those points $q \in \mathrm{cut}(p)$ such that there exists a distance minimizing geodesic from p to q along which these points are conjugate to each other. By Proposition 61 and Lemma 64 the set $Q(p)$ is closed in M.

- *Typical cut points:* $T(p) \subset (\tau_{\mathrm{cut}}(p) \setminus Q(p))$ that can be connected to p with exactly two distance minimizing geodesics.

- *A-typical cut points:* $L(p) \subset (\tau_{\mathrm{cut}}(p) \setminus Q(p))$ that can be connected to p with more than two distance minimizing geodesics. Thus an a-typical cut point is both non-conjugate and non-typical.

It was proven in [25] that the Hausdorff dimension of the cut locus on a closed Riemannian manifold (N,g) is locally an integer that does not exceed $\dim N - 1$. Moreover $T(p)$ is a smooth hyper-surface of N and the Hausdorff dimension of $Q(p) \cup L(p)$ does not exceed $\dim N - 2$. In this Chapter we will extended these results for manifolds with strictly convex boundary. The main result of this section is as follows:

Theorem 65. *Let (M,g) be a smooth, compact, connected, and oriented Riemannian manifold of dimension $n \in \mathbb{N}$, $n \geq 2$ with smooth and strictly convex boundary. If $p \in M$ then*

1. *The set $T(p)$ of typical cut points is a smooth hyper-surface of M that is transverse to ∂M.*
2. *The Hausdorff dimension of $Q(p) \cup L(p)$ does not exceed $n-2$.*
3. *The Hausdorff dimension of cut(p) does not exceed $n-1$.*

4. *The set $\partial M \setminus cut(p)$ is open and dense in ∂M.*

For the readers who want to learn more about Hausdorff measure and dimension we suggest to have look at [5, 40]. Some basic properties of the Hausdorff dimension $\dim_{\mathscr{H}}$ are collected in the following lemma.

Lemma 66. *Basic properties of the Hausdorff dimension are:*

- *If X is a metric space and $A \subset X$ then* $\dim_{\mathscr{H}}(A) \leq \dim_{\mathscr{H}}(X)$.
- *If X is a metric space and $\mathscr{X}$ is a countable cover of X then* $\dim_{\mathscr{H}}(X) = \sup_{A \in \mathscr{X}} \dim_{\mathscr{H}}(A)$.
- *If X, Y are metric spaces and $f: X \to Y$ is a bi-Lipschitz map then* $\dim_{\mathscr{H}}(A) = \dim_{\mathscr{H}}(f(A))$ *for any $A \subset X$.*
- *If $U \subset M$ is open and M is a Riemannian manifold of dimension $n \in \mathbb{N}$ then* $\dim_{\mathscr{H}}(U) = n$.

From here onwards we follow the steps of [25, 44] and develop machinery needed for the proof of Theorem 65. We recall that we have isometrically embedded M into the closed Riemannian manifold (N, g). The maximal subset $M_p \subset T_pM$ where the exponential map $\exp_p: M_p \to M$ of (M, g) is well defined was given in (4.3) as

$$M_p = \left\{ v \in T_pM : \ v = 0 \text{ or } \|v\|_g \leq \tau_{\text{exit}}\left(p, \frac{v}{\|v\|_g}\right) \right\}.$$

Thus the exponential function $\exp_p: T_pN \to N$ of N agrees with that of M in M_p.

Let $v_0 \in M_p$ be such that the exponential map $\exp_p$ is not singular at v_0. The Inverse function theorem yields that there are neighborhoods $U \subset T_pN$ of v_0 and $\tilde{V} \subset N$ of $x_0 = \exp_p(v_0) \in M$ such that $\exp_p: U \to \tilde{V}$ is a diffeomorphism. We want to emphasize that even when $v_0 \in M_p$ the set $U \subset T_pN$ does not need to be contained in M_p. If we equate T_pN and $T_{v_0}(T_pN)$ we note that the formula

$$Y(x) = \mathrm{D}\exp_p\Big|_{\exp_p^{-1}(x)} \exp_p^{-1}(x)$$

defines a smooth vector field on $\tilde{V}$ that satisfies the following properties

$$Y(x) = \dot{\gamma}_{p,\exp_p^{-1}(x)}(1), \quad \text{and} \quad \|Y(x)\|_g = \|\exp_p^{-1}(x)\|_g. \tag{4.15}$$

The vector field Y is called a *distance vector field* related to p and U. Let $x \in \tilde{V}$ and $X \in T_x N$. It holds by a similar proof to [44, Lemma 2.2.] that

$$X\|Y(x)\|_g = \frac{\langle X, Y(x)\rangle_g}{\|Y(x)\|_g}. \tag{4.16}$$

In what follows we will always consider $\mathrm{cut}(p)$ as defined for (M,g) in equation (4.11). Let $q \in \mathrm{cut}(p) \setminus Q(p)$ and $\lambda = d(p,q)$. By Proposition 61 it holds that there are at least two M-distance minimizing geodesics from p to q. Thus the set

$$E_{p,q} := \exp_p^{-1}\{q\} \cap S_\lambda M, \quad \text{where } S_\lambda M = \{w \in T_p M : \|w\|_g = \lambda\},$$

contains at least two points.

It was proven in [44] that the set $E_{p,q}$ is finite. We repeat the argument here as it is short. Suppose that $E_{p,q}$ is not finite. Then by the compactness of $S_\lambda M$ there is $w \in S_\lambda M$ that is an accumulation point of $E_{p,q}$. We choose a sequence $w_i \in E_{p,q}$ that converges to w. Since $\lambda \le \tau_{\mathrm{exit}}(p, \frac{w_i}{\lambda})$ for every $i \in \mathbb{N}$ we have that $\lambda \le \tau_{\mathrm{exit}}(p, \frac{w}{\lambda})$. Then $\exp_p(w) = q$ implies that $\exp_p$ is not an injection in some neighborhood of w in $T_p M$. By the Inverse function theorem $\exp_p$ cannot be of the full rank at w. Thus $q \in Q(p)$ which is a contradiction.

We write

$$E_{p,q} = \{w_i : i \in \{1, \dots, k_p(q)\}\},$$

where $k_p(q) \in \mathbb{N}$ is the number of distance minimizing geodesics from p to q. Since the set $Q(p)$ is closed in M, the complement $\mathrm{cut}(p) \setminus Q(p)$ is relatively open in $\mathrm{cut}(p)$, and there exists an open neighborhood $W \subset M$ of q such that $Q(p) \cap W = \emptyset$. Thus by the previous discussion for any $x \in \mathrm{cut}(p) \cap W$ there are only $k_p(x) \in \mathbb{N}$ many distance minimizing geodesics connecting p to x. Moreover, as the following lemma shows, q is a local maximum of the function k_p defined on $\mathrm{cut}(p) \cap W$. This statement is an adaptation of the analogous result given in [44].

Lemma 67. *Let (M,g) be a Riemannian manifold as in Theorem 65. Let $p \in M$ and $q \in$ cut$(p) \setminus Q(p)$. Let the closed manifold (N,g) be as in Proposition 60. Then there is a neighborhood V of q in M such that*

$$k_p(x) \le k_p(q), \quad \text{for every } x \in \mathit{cut}(p) \cap V. \tag{4.17}$$

Proof. Since the set $E_{p,q}$ is finite we can choose disjoint neighborhoods $U_i \subset T_pN$ for each $w_i \in E_{p,q}$, so that for each $i \in \{1,\dots,k_p(q)\}$ the map $\exp_p : U_i \to \tilde{V}$ is a diffeomorphism on some open set $\tilde{V} \subset N$ that contains q. We want to show that there is a neighborhood $V \subset M$ of q such that for every $x \in V$ and for any M-distance minimizing unit speed geodesic γ from p to x there is $i \in \{1,\dots,k_p(q)\}$ such that

$$\gamma(t) = \exp_p\left(t\frac{X}{\|X\|_g}\right), \quad \text{for some } X \in M_p \cap U_i.$$

Clearly this implies the inequality (4.17).

If the former is not true then there exist a sequence $q_k \in M$ that converges to q and $X_k \in M_p$ so that for each $k \in \mathbb{N}$ we have

- $\exp_p(X_k) = q_k$
- $\|X_k\|_g = d_M(p,q_k) \le \tau_{\text{exit}}(p, \frac{X_k}{\|X_k\|_g})$
- $\exp_p\left(t\frac{X_k}{\|X_k\|_g}\right)$ for $t \in [0, d_M(p,q_k)]$ is a unit speed distance minimizing geodesic from p to q_k.
- $X_k \notin U_1 \cup \ldots \cup U_{k_p(q)}$.

These imply that

$$\lim_{k\to\infty} \|X_k\|_g = \lim_{k\to\infty} d_M(p,q_k) = d_M(p,q).$$

Moreover, the sequence $X_k \in T_pM$ is contained in some compact subset K of T_pM. After passing to a sub-sequence we may assume $X_k \to X \in T_pM$ and the continuity of the exit time function on the non-trapping part of SM gives $\|X\|_g \le \tau_{\text{exit}}(p, \frac{X}{\|X\|_g})$. Thus $X \in M_p$ and by the continuity of $\exp_p$ we get $\exp_p(X) = q$, and $\|X\|_g = d_M(p,q)$.

Therefore $t \mapsto \exp_p\left(t\frac{X}{\|X\|_g}\right)$ is a M-distance minimizing geodesic from p to q and X must coincide with w_i for some $i \in \{1,\dots,k_p(q)\}$. Therefore $X_k \in U_i$ for large enough $k \in \mathbb{N}$. This contradicts the choice of X_k, and possibly after choosing a smaller $\tilde{V}$, we can set $V = M \cap \tilde{V}$. $\square$

Suppose now that $q \in T(p)$ is a typical cut point, and $V \subset M$ is a neighborhood of q as in

Lemma 67. Then by (4.17) it holds that

$$\mathrm{cut}(p) \cap V \subset T(p), \tag{4.18}$$

and $E_{p,q} = \{w_1, w_2\} \subset T_pN$ are the directions that give the two distance minimizing geodesics $\exp_p(t w_i)$, $t \in [0,1]$ from p to q. Let $U_1, U_2 \subset T_pN$ be the neighborhoods of w_1 and w_2 and $\tilde{V} \subset N$ a neighborhood of q as in the proof of Lemma 67. Finally we consider the distance vector fields Y_1, Y_2 related to p and U_1 and U_2. Since these vector fields do not vanish on $\tilde{V}$ the function

$$\rho : \tilde{V} \to \mathbb{R}, \quad \rho(x) = \|Y_1(x)\|_g - \|Y_2(x)\|_g,$$

is smooth. The following result is an adaptation of [44, Propositions 2.3 & 2.4].

Lemma 68. *Let Riemannian manifold (M, g) be as in Theorem 65 and $p \in M$. Let $q \in T(p) \subset M$ and define the closed manifold N as in Proposition 60. Let the neighborhood $\tilde{V} \subset N$ of q and function $\rho : \tilde{V} \to \mathbb{R}$ be as above. Then possibly after choosing a small enough $\tilde{V}$ we have*

$$\rho^{-1}\{0\} \cap M = \tilde{V} \cap cut(p). \tag{4.19}$$

Moreover, the set $\rho^{-1}\{0\}$ is a smooth hyper-surface of N whose tangent bundle is given by the orthogonal complement of the vector field $Y_1 - Y_2$.

Proof. We prove first the equation (4.19).

- Let $x \in \rho^{-1}\{0\} \cap M$. By the proof of Lemma 67 we can assume that a M-distance minimizing unit speed geodesic from p to x is given by $\exp_p(tX_1)$, $t \in [0,1]$, for some $X_1 \in M_p \cap U_1$. Also $x = \exp_p(X_2)$ for some $X_2 \in U_2$, but we do not know *a priori* if $\exp_p(tX_2) \in M$ for all $t \in [0,1]$ or equivalently if $X_2 \in M_p$. However, by the definition of the distance vector fields and the assumption $x \in \rho^{-1}\{0\}$ we have that

$$d_M(p, x) = \|X_1\|_g = \|Y_1(x)\|_g = \|Y_2(x)\|_g = \|X_2\|_g. \tag{4.20}$$

  Let $\hat{M}$ be as in Proposition 60. Thus we can assume that $\tilde{V} \subset \hat{M}$. Since $q \in T(p)$ there is $w_2 \in U_2$ so that $\exp_p(w_2) = q$ and $\exp_p(t w_2) \in M \subset \hat{M}$, for every $t \in [0,1]$. Since q is an interior point of $\hat{M}$ we can again choose smaller $\tilde{V}$ so that $\exp_p(tX) \in \hat{M}$, for every $t \in [0,1]$ and $X \in U_i$, for $i \in \{1,2\}$. Since $x \in M$ and $\exp_p(tX_2)$ is a geodesic of $\hat{M}$

that connects p to x having the length of $\|X_2\|_g$, the equation (4.20) and Proposition 60 imply that $\exp_p(tX_2) \in M$ for every $t \in [0,1]$. Therefore equation (4.20) gives $x \in \tilde{V} \cap \mathrm{cut}(p)$.

- Let $x \in \tilde{V} \cap \mathrm{cut}(p) \subset T(p)$. Thus there are exactly two distance minimizing geodesics of M from p to x. Since $x \in \tilde{V}$, it holds by the proof of Lemma 67 that one of these geodesics has the initial velocity in U_1 and the other in U_2. Therefore $\rho(x)$ is zero by the definition of the distance vector fields.

Then we prove that the set $\rho^{-1}\{0\}$ is a smooth hyper-surface whose tangent bundle is orthogonal to the vector field $Y_1 - Y_2$. By (4.16) we get

$$X\rho(x) = \frac{\langle X, Y_1(x)\rangle_g}{\|Y_1(x)\|_g} - \frac{\langle X, Y_2(x)\rangle_g}{\|Y_2(x)\|_g} = \frac{\langle X, Y_1(x) - Y_2(x)\rangle_g}{\|Y_1(x)\|_g}, \quad \text{for every } x \in \rho^{-1}\{0\}. \tag{4.21}$$

Moreover the vector field $Y_1 - Y_2$ does not vanish on $\tilde{V}$, since the geodesics related to these two vector fields are different. This implies that the differential of the map ρ does not vanish in $\tilde{V}$. Thus the set $\rho^{-1}\{0\}$ is a smooth hyper-surface of N, and by (4.21) its tangent bundle is given by those vectors that are orthogonal to $Y_1 - Y_2$. □

Now we consider the set of conjugate cut points $Q(p)$. First we define a function

$$\delta : S_pN \to \{0, 1, \ldots, n-1\}, \quad \delta(v) \text{ is the dimension of the kernel of } \mathrm{D}\exp_p \text{ at } \tau_{\mathrm{con}}(p,v)v.$$

If $\tau_{\mathrm{cut}}(p,v) = \infty$ we set $\delta(v) = 0$.

Lemma 69. *Let (N,g) be a closed Riemannian manifold. Let $p \in N$ and $v_0 \in S_pN$ be such that $\delta(v_0) = 1$. There exists a neighborhood $U \subset S_pN$ of v_0 such that $\delta(\cdot)$ is the constant function one in U.*

Before proving this lemma we recall one auxiliary result from linear algebra.

Lemma 70. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint bijective linear operator. Then the index $i(L)$ of L, the dimension of the largest vector subspace of $\mathbb{R}^n$ where L is negative definite, equals the amount of the negative eigenvalues of the operator L counted up to a multiplicity.*

Proof. Since L is self-adjoint and invertible, the spectral theorem says that L has $k \le n$ positive eigenvalues $\lambda_1, \ldots, \lambda_k > 0$ and $n-k$ negative eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$. Let $e_1, \ldots, e_n$

be the respective orthonormal set of eigenvectors. Since L is negative definite on the vector subspace $\operatorname{span}(e_{k+1},\dots,e_n)$ we must have that $i(L)\geq n-k$.

Suppose that $i(L)>n-k$ and let Q be an $i(L)$-dimensional vector subspace of $\mathbb{R}^n$ where the operator L is negative definite. We choose some vectors $(v_1,\dots,v_{i(L)})$ to constitute a basis of Q. Let P be the vector subspace of $\mathbb{R}^n$ spanned by the eigenvectors $e_1,\dots,e_k$. Since L is positive definite on P we must have that $P\cap Q=\{0\}$. Thus, the vectors $e_1,\dots,e_k,v_1,\dots,v_{i(L)}$ are linearly independent which is impossible since $k+i(L)>n$. Therefore $i(L)=n-k$ is the amount of negative eigenvalues of the operator L. □

Proof of Lemma 69. In this proof we adopt the definitions and results of [15, Chapter 11] appearing in the proof of the Morse index theorem. For each $v\in S_pM$ and $t>0$ we use the notation $\mathscr{V}(t,v)$ for the vector space of all piecewise smooth vector fields that are normal to the geodesic $\gamma_{p,v}$ in the interval $[0,t]$ and vanish at the endpoints. Then we define the function $i\colon[0,\infty)\times S_pN\to\mathbb{N}$, to be the index of the symmetric bilinear form

$$I_{t,v}(X,Y):=\int_0^t\langle \mathrm{D}_sX,\mathrm{D}_sY\rangle_g+\langle R(\dot\gamma_{p,v},X)\dot\gamma_{p,v},Y\rangle_g\,\mathrm{d}s,\quad X,Y\in\mathscr{V}(t,v).$$

Hence,

$$\begin{cases}\delta(v)>i(t,v)=0, & t\leq\tau_{\mathrm{con}}(p,v)\\ \delta(v)=i(t,v), & t\in(\tau_{\mathrm{con}}(p,v),\varepsilon(v))\\ \delta(v)<i(t,v), & t>\varepsilon(v)\end{cases}$$

where $\varepsilon(v)>0$ depends on $v\in S_pN$. Moreover, no $\gamma_{p,v}(t)$, for $t\in(\tau_{\mathrm{con}}(p,v),\varepsilon(v))$ is conjugate to p along $\gamma_{p,v}$. We choose $t\in(\tau_{\mathrm{con}}(p,v_0),\varepsilon(v_0))$. Thus Lemma 64, gives $\delta(v)\leq i(t,v)$ for $v\in S_pN$ close enough to v_0. Our aim is to find a neighborhood $U\subset S_pN$ of v_0 for which

$$1\leq\delta(v)\leq i(t,v)=i(t,v_0)=\delta(v_0)=1,\quad\text{for } v\in U. \tag{4.22}$$

Clearly this gives the result of the lemma.

The space $\mathscr{V}(t,v)$ can be written as a direct sum of two of its vector sub-spaces $\mathscr{V}_+(t,v)$ and $\mathscr{V}_-(t,v)$, defined so that index form $I_{t,v}$ is positive definite on $\mathscr{V}_+(t,v)$ and the space $\mathscr{V}_-(t,v)$ is finite dimensional. Moreover, these vector spaces are $I_{t,v}$ orthogonal. Thus the index of $I_{t,v}$ coincides with the index of its restriction on $\mathscr{V}_-(t,v)$. Since the dimension of $\mathscr{V}_-(t,v)$ is independent of a direction $v\in S_pN$, that is close to v_0, we can identify all the spaces $\mathcal{V}_{-}(t,v)$ with $\mathcal{V}_{-}(t,v_0)$ and consider the bilinear forms $I_{t,v}$ as a family of operators on the finite dimensional vector space $\mathcal{V}_{-}(t,v_0)$, depending continuously on the parameter $v \in S_p N$.

For each $v \in S_p N$, we consider the linear operator $L_{t,v} \colon \mathcal{V}_{-}(t,v) \to \mathcal{V}_{-}(t,v)$, corresponding to the bilinear form $I_{t,v}$. Since $\gamma_{p,v_0}(t)$ is not a conjugate point to p along γ_{p,v_0}, zero is not an eigenvalue of the linear operator L_{t,v_0}. Thus Lemma 70 implies that the operator L_{t,v_0} has $i(t,v_0)$ negative eigenvalues. Since the eigenvalues of the operator $L_{t,v}$ depend continuously on the initial direction $v \in S_p N$, that are near v_0, we can find a neighborhood $U \subset S_p N$ of v_0 such that the linear operator $L_{t,v}$ is invertible and has $i(t,v_0)$ negative eigenvalues for every $v \in U$. Hence, by Lemma 70 we have again that $i(t,v) = i(t,v_0)$ for $v \in U$. We have verified the equation (4.22). □

Lemma 71. *Let (N,g) be a closed Riemannian manifold, $p \in N$ and suppose that δ is constant in some open set $U \subset S_p M$. Then $\tau_{con}(p,\cdot)$ is smooth in U.*

Proof. If δ is zero in U then $\tau_{\mathrm{con}}(p,\cdot)$ is infinite and we are done. So we suppose that δ equals to $k \in \{1,\dots,n-1\}$ in U and get $\tau_{\mathrm{con}}(p,v) < \infty$ for every $v \in U$.

Let $\xi_1,\dots,\xi_{n-1}$ be a base of $T_{v_0} S_p M$ and use the formula

$$J_{v_0,\beta}(t) = \mathrm{D}\exp_p\Big|_{t v_0} t\xi_\beta, \quad \text{for } \beta \in \{1,\dots,n-1\},$$

from [38, Proposition 10.10], to define $(n-1)$-Jacobi fields $J_{v_0,1}(t),\dots,J_{v_0,n-1}(t)$ along the geodesic γ_{p,v_0}. They span the vector space of all Jacobi fields along $\gamma_{p,v_0}(t)$ that vanish at $t=0$ and are normal to $\dot\gamma_{p,v_0}(t)$. As $\delta(v_0)=k$ we may assume that $J_{v_0,\beta}(\tau_{\mathrm{con}}(p,v_0)) = 0$ for $\beta \in \{1,\dots,k\}$, implying $D_t J_{v_0,\beta}(\tau_{\mathrm{con}}(p,v_0)) \neq 0$ and $J_{v_0,\alpha}(\tau_{\mathrm{con}}(p,v_0)) \neq 0$ for $\beta \in \{1,\dots,k\}$ and $\alpha \in \{k+1,\dots,n-1\}$. Moreover, the vectors

$$D_t J_{v_0,1}(\tau_{\mathrm{con}}(p,v_0)),\dots,D_t J_{v_0,k}(\tau_{\mathrm{con}}(p,v_0)), J_{v_0,k+1}(\tau_{\mathrm{con}}(p,v_0)),\dots J_{v_0,n-1}(\tau_{\mathrm{con}}(p,v_0)) \tag{4.23}$$

are linearly independent due to [32, Proposition 2.5.8 (ii)] and the properties of the geodesic flow on SN presented in [45, Lemma 1.40].

Since Jacobi fields are solutions of the second order ODE they depend smoothly on the coefficients of the respective equation. In particular, after choosing smaller U if necessary, we can construct the family of Jacobi fields $J_{v,1}(t),\dots,J_{v,n-1}(t)$ along the geodesic $\gamma_{p,v}(t)$

that depend smoothly on $v \in U$, and span the vector space of all Jacobi fields along $\gamma_{p,v}(t)$ that vanishes at $t = 0$ and are normal to $\dot{\gamma}_{p,v}(t)$. Therefore the function

$$f : U \times [0, \tau_{\text{con}}(p, v_0) + 1] \to \mathbb{R}, \quad f(v,t) = \det(J_{v,1}(t), \dots, J_{v,n-1}(t)),$$

is smooth and vanishes at (v,t) if and only if $\gamma_{p,v}(t)$ is conjugate to p.

We choose a parallel frame $E_1(t), \dots, E_{n-1}(t)$ along $\gamma_{p,v_0}(t)$ that is orthogonal to $\dot{\gamma}_{p,v_0}(t)$. With respect to this frame we write

$$J_{v_0,\beta}(t) = j^\alpha_\beta(t) E_\alpha(t), \quad \text{for } \alpha, \beta \in 1, \dots, n-1,$$

for some some smooth functions $j^\alpha_\beta(t)$. From here we get

$$\frac{\partial^j}{\partial t^j} f(v_0, \tau_{\text{con}}(p, v_0)) = \frac{\partial^j}{\partial t^j} \left(\sum_\sigma \text{sign}(\sigma) j_1^{\sigma(1)}(t) j_2^{\sigma(2)}(t) \cdots j_{n-1}^{\sigma(n-1)}(t) \right) \Bigg|_{t = \tau_{\text{con}}(p, v_0)} = 0,$$

for every $j \in \{0, \dots, k-1\}$. Above, σ is a permutation of the set $\{1, \dots, n-1\}$. Moreover the covariant derivative of $J_{v_0,\beta}$ along γ_{p,v_0} is written as $D_t J_{v_0,\beta}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} j^\alpha_\beta(t) \right) E_\alpha(t)$.

If A is the square matrix whose column vectors are given in the formula (4.23) we have

$$\frac{\partial^k}{\partial t^k} f(v_0, \tau_{\text{con}}(p, v_0)) = \pm k! \det(A) \neq 0.$$

Since $\delta(\cdot)$ is constant k in the set U we have that $\frac{\partial^{k-1}}{\partial t^{k-1}} f(v, \tau_{\text{con}}(p, v)) = 0$ for every $v \in U$. Therefore the Implicit function theorem gives that the conjugate distance function is smooth in some neighborhood $V \subset U$ of v_0. Since $v_0 \in U$ was chosen arbitrarily the claim follows. □

Let $v_0 \in S_p N$ be such that $\tau_{\text{con}}(p, v_0) < \infty$. Then by lemmas 59 and 64 the functions $e_c(v) = \exp_p(\tau_{\text{cut}}(p, v)v) \in M$, and $e_q(v) = \exp_p(\tau_{\text{con}}(p, v)v) \in N$ are well defined and continuous on some neighborhood $U \subset S_p N$ of v_0. Moreover, we have that

$$Q(p) = \{e_q(v) \in M : \tau_{\text{cut}}(p, v) = \tau_{\text{con}}(p, v)\}. \tag{4.24}$$

The following result is an adaptation of [25, Lemma 2].

Proposition 72. *Let Riemannian manifold (M, g) be as in Theorem 65 and $p \in M$. The*

Hausdorff dimension of $Q(p)$ does not exceed $n-2$.

Proof. By (4.24) we can write the conjugate cut locus $Q(p)$ as a disjoint union of the sets

$$A_1 = \{e_q(v) \in M : \tau_{\text{cut}}(p,v) = \tau_{\text{con}}(p,v),\ \delta(v) = 1\}$$

and

$$A_2 = \{e_q(v) \in M : \tau_{\text{cut}}(p,v) = \tau_{\text{con}}(p,v),\ \delta(v) \geq 2\}.$$

To prove the claim of this proposition it suffices to show that

$$A_1 \subset \{e_q(v) \in N : \dim\big(\mathrm{D}e_q(T_v S_p M)\big) \leq n-2\}, \tag{4.25}$$

since clearly we have that

$$A_2 \subset \{\exp_p(w) \in N : w \in T_p N,\ \dim\big(\mathrm{D}\exp_p(T_w(T_p N))\big) \leq n-2\},$$

and therefore by the generalization of the classical Sard's theorem [51] the Hausdorff dimension of $Q(p) = A_1 \cup A_2$ is at most $n-2$.

We choose $v_0 \in S_p N$ such that $e_q(v_0) \in A_1$. By the properties of the Jacobi fields normal to γ_{p,v_0}, we can identify the kernel $\mathrm{D}\exp_p(\tau_{\text{con}}(p,v_0)v_0)$ with some vector sub-space of $T_{v_0} S_p M$.

Since $\dim T_{v_0} S_p M = n-1$ we can verify the inclusion (4.25) if we show that

$$\ker \mathrm{D}\exp_p(\tau_{\text{con}}(p,v_0)v_0) \subset \ker \mathrm{D}e_q(v_0). \tag{4.26}$$

Since $\delta(v_0) = 1$ we get by lemmas 69 and 71 that there exists a neighborhood $U \subset S_p N$ of v_0 where the conjugate distance $\tau_{\text{con}}(p,\cdot)$ and the map $e_q(v) = \exp_p(\tau_{\text{con}}(p,v)v)$ are smooth. Let $\xi \in T_{v_0} S_p M$ be in the kernel of the differential of the exponential map. Then by the chain and Leibniz rules we get

$$\mathrm{D}e_q(v_0)\xi = \dot{\gamma}_{p,v_0}(\tau_{\text{con}}(p,v_0))\mathrm{D}\tau_{\text{con}}(p,v_0)\xi.$$

Therefore $\mathrm{D}e_q(v_0)\xi = 0$ if and only if $\mathrm{D}\tau_{\text{con}}(p,v_0)\xi = 0$. So we suppose that $\mathrm{D}e_q(v_0)\xi \neq 0$.

Since $\delta(v) = 1$ for all $v \in U$, the conjugate distance function is smooth on U and therefore the set $\Sigma := \{\tau_{\text{con}}(p,v)v \in T_p N : v \in U\}$ is a smooth sub-manifold of dimension $n-1$.

Moreover the restriction of the exponential map on this sub-manifold is a constant rank map. Therefore it follows from the Rank theorem [37, Theorem 4.12] that the subset of T_pN, near $\tau_{\text{con}}(p,v_0)v_0$, where $\mathrm{D}\exp_p$ vanishes is diffeomorphic to a smooth sub-bundle of $TU \subset TS_pN$. Then we use the existence of the ODE theorem to choose a smooth curve $v(\cdot)\colon(-1,1)\to U\subset S_pN$ such that $v(0)=v_0$, $\dot v(0)=\xi$ and $\dot v(t)\in\ker \mathrm{D}\exp_p(\tau_{\text{con}}(p,v(t))v(t))$ for every $t\in(-1,1)$. Thus $c(t):=e_q(v(t))$ is a smooth curve in $\hat M$ that satisfies

$$\dot c(t)=\dot\gamma_{p,v(t)}(\tau_{\text{con}}(p,v(t)))\frac{d}{dt}(\tau_{\text{con}}(p,v(t))). \tag{4.27}$$

Since $\frac{d}{dt}(\tau_{\text{con}}(p,v(t_0)))=\mathrm{D}\tau_{\text{con}}(p,v_0)\xi$ we can assume that $\frac{d}{dt}(\tau_{\text{con}}(p,v(t)))>0$ on some interval $(-\varepsilon,\varepsilon)$ for $0<\varepsilon<1$. Thus by equation (4.27) and the Fundamental theorem of calculus we get that the length of $c(t)$ on $[-\varepsilon,0]$ is $\mathscr{L}(c)=\int_{-\varepsilon}^0\|\dot c(t)\|_g\mathrm{d}t=\int_{-\varepsilon}^0\frac{d}{dt}(\tau_{\text{con}}(p,v(t)))\mathrm{d}t=\tau_{\text{con}}(p,v_0)-\tau_{\text{con}}(p,v(-\varepsilon))$. Below we denote the length of $c(t)$ as $\mathscr{L}(c)$. From here by the assumption $\tau_{\text{con}}(p,v_0)=\tau_{\text{cut}}(p,v_0)$ and the triangle inequality we get

$$\begin{aligned}\tau_{\text{con}}(p,v(-\varepsilon)) &\ge d_{\hat M}(p,e_q(v(-\varepsilon)))\\ &\ge d_{\hat M}(p,e_q(v_0))-d_{\hat M}(e_q(v_0),e_q(v(-\varepsilon)))\\ &\ge d_{\hat M}(p,e_q(v_0))-\mathscr{L}(c)\\ &\ge \tau_{\text{con}}(p,v(-\varepsilon)),\end{aligned}$$

and the inequality above must hold as an equality. Therefore

$$d_{\hat M}(p,e_q(v(-\varepsilon)))+\mathscr{L}(c)=d_{\hat M}(p,e_q(v_0)),$$

and the curve $c(\cdot)\colon[-\varepsilon,0]\to\hat M$ is part of some distance minimizing geodesic γ of $\hat M$ from p to $e_q(v(0))$ that contains $e_q(v(-\varepsilon))$. Thus we have after some reparametrization $t=t(s)$ that

$$\gamma(s)=e_q(v(t(s)))=c(t(s))$$

for every $t(s)\in(-\varepsilon,0)$. By (4.27) we get that γ is a parallel to $\gamma_{p,v(t)}$ for every $t\in(-\varepsilon,0)$. This is not possible unless the geodesics $\gamma_{p,v(t)}$ are all the same for every $t\in(-\varepsilon,0)$. Hence $v(t)$ and $c(t)$ are constant curves. This leads to a contradiction. The inclusion (4.26) is confirmed and the proof is complete. □

We are ready to present the proof of Theorem 65.

Proof of Theorem 65. Let $p \in M$. In this proof we combine the observations made earlier in this section. The proofs of the four sub-claims are given below.

(1) By Lemma 68 we know that $T(p)$ is a smooth hyper-surface of M whose tangent space is normal to the vector field $\nu(q) = Y_1(q) - Y_2(q)$ for $q \in T(p)$. Since $Y_1(q) \neq Y_2(q)$ and $\|Y_1(q)\|_g = \|Y_2(q)\|_g$ we get from the Cauchy-Schwarz inequality that

$$\langle Y_1(q), \nu(q)\rangle = \|Y_1(q)\|_g^2 - \langle Y_1(q), Y_2(q)\rangle > 0$$
$$\langle Y_2(q), \nu(q)\rangle = -\|Y_2(q)\|_g^2 + \langle Y_1(q), Y_2(q)\rangle < 0.$$

Thus $Y_1(q)$ and $Y_2(q)$ hit $T(p)$ from different sides. If $q \in T(p) \cap \partial M$ and these surfaces are tangential to each other at q we arrive in a contradiction: Since $\nu(q)$ is normal to both $T(T(p))$ and $T\partial M$ we can without loss of generality assume that $Y_2(q)$ is inward pointing at q. This is not possible since the geodesic related to $Y_2(q)$, that connects p to q, is contained in M. Thus by equation (4.15) $Y_2(q)$ is also outward pointing which is not possible.

(2) By Proposition 72 we know that the Hausdorff dimension of the conjugate cut locus $Q(p)$ does not exceed $n-2$. If we can prove the same for the set $L(p)$ of a-typical cut points the claim (2) follows from Lemma 66.

Recall that $L(p) \subset (\mathrm{cut}(p) \setminus Q(p))$ is the set of points in M that can be connected to p with more than two distance minimizing geodesics of M. Let $q \in L(p)$ and define $k_p(q) \in \mathbb{N}$ to be the number of distance minimizing geodesics from p to q. Then we choose vectors $w_1, \ldots, w_{k_p(q)} \in M_p$ and their respective neighborhoods $U_i \in T_pN$ such that for each $i \in \{1, \ldots, k_p(q)\}$

$$\exp_p(w_i) = q, \quad \text{and } \exp_p \colon U_i \to \tilde{V}$$

is a diffeomorphism on some open set $\tilde{V} \subset N$. Let Y_i for $i \in \{1, \ldots, k_p(q)\}$ be the distance vector fields related to the U_i and p. Then we define a collection of smooth functions

$$\rho_{ij} \colon \tilde{V} \to \mathbb{R}, \quad \rho_{ij}(x) = \|Y_i(x)\|_g - \|Y_j(x)\|_g, \quad i, j \in \{1, \ldots, k_p(q)\}.$$

By the proof of Lemma 68 it holds that the sets $K_{ij} := \rho_{ij}^{-1}\{0\}$, for $i < j$ are smooth hyper-surfaces of N that contain q. Also by [44, Proposition 2.6] it holds that the sets

$$K_{i,j,k} := K_{ik} \cap K_{jk}, \quad \text{for } i < j < k$$

are smooth submanifolds of N of co-dimension two. Next we set $K(q) := \bigcup_{i<j<k} K_{i,j,k}$ and claim that

$$L(p) \cap \tilde{V} = K(q) \cap M. \tag{4.28}$$

Since the sets $K_{i,j,k}$ are smooth sub-manifolds of dimension $n-2$ their Hausdorff dimension is also $n-2$. Thus the equation (4.28) and Lemma 66 imply that Hausdorff dimension of $L(p)$ does not exceed $n-2$.

Finally we verify the equation (4.28). If $x \in L(p) \cap \tilde{V}$ it holds there are at least three distance minimizing geodesics of M connecting p to x. Thus there are $1 \le i < j < k \le k_p(q)$ so that

$$\|Y_i(x)\|_g = \|Y_j(x)\|_g = \|Y_k(x)\|_g = d_M(p,x),$$

which yields

$$\rho_{ik}(x) = \rho_{jk}(x) = 0, \quad \text{and } x \in K_{i,j,k} \subset K(q).$$

If $x \in K(q) \cap M$ then $x \in K_{i,j,k} \cap M$ for some $i < j < k$. Thus by the proof of Lemma 68 it holds that there are at least three distance minimizing geodesics of M connecting p to x. Therefore $x \in L(p) \cap \tilde{V}$.

(3) Since we can write the cut locus of p as a disjoint union $\text{cut}(p) = T(p) \cup L(p) \cup Q(p)$ the parts (1) and (2) in conjunction with Lemma 66 yield the claim of part (3).

(4) Since ∂M is a smooth hyper-surface of an n-dimensional Riemannian manifold M we have by part (1) that $T(p) \cap \partial M$ is a smooth sub-manifold of dimension $n-2$, thus it has the Hausdorff-dimension $n-2$. Also by part (3) we know that the Hausdorff dimension of $L(p) \cap Q(p)$ does not exceed $n-2$. We have proven that the Hausdorff dimension of the closed set $\text{cut}(p) \cap \partial M$ does not exceed $n-2$. Since the boundary of M has the Hausdorff-dimension $n-1$ it follows that $\partial M \setminus \text{cut}(p)$ is open and dense in ∂M. The density claim follows from the observation that by Lemma 66 the set $\text{cut}(p) \cap \partial M$ cannot contain any open subsets of ∂M as their Hausdorff dimension is $n-1$.

□

We are ready to prove Theorem 57.

Proof of Theorem 57. The proof follows from Proposition 63 and Theorem 65. □

4.3 Reconstruction of the Manifold

4.3.1 Geometry of the Measurement Region

In this section we consider only one Riemannian manifold (M, g) that satisfies the assumptions of Theorem 57 and whose partial travel time data (4.1) is known. Let $\nu(z)$ be the outward pointing unit normal vector field at $z \in \partial M$. The inward pointing bundle at the boundary is the set

$$\partial_{in} TM = \{(z, v) \in TM \mid z \in \partial M, \langle v, \nu(z)\rangle_g < 0\}.$$

We restrict our attention to the vectors that are inward pointing and of unit length: $\partial_{in} SM = \{(z, v) \in \partial_{in} TM \ : \ \|v\|_g = 1\}$. We emphasize that this set or its restriction on the open measurement region $\Gamma \subset \partial M$ is not *a priori* given by the data (4.1). Our first task is to recover a diffeomorphic copy of this set. We consider the orthogonal projection

$$h : \partial_{in} SM \to T\partial M, \qquad h(z, v) = v - \langle v, \nu(z)\rangle_g \, \nu(z), \tag{4.29}$$

and denote the set, that contains the image of h, as $P(\partial M) = \{(z, w) \in T\partial M \ : \ \|w\|_g < 1\}$. It is straightforward to show that the map h is a diffeomorphism onto $P(\partial M)$. For the convenience of the reader we provide the proof of this claim in the following lemma.

Lemma 73. *The mapping $h : \partial_{in} SM \to P(\partial M)$ is a diffeomorphism.*

Proof. Clearly the map h given in equation (4.29) is smooth. Let $(z, v) \in \partial_{in} SM$. Since $\langle \nu(z), h(z, v)\rangle_g = 0$, we have that $h(z, v)$ is tangential to ∂M at z. Moreover since $v \in \partial_{in} SM$ we have by a direct computation that $\|h(z, v)\|_g^2 < 1$. Thus $h(z, v) \in P(\partial M)$.

Again by a direct computation we see that the inverse function of h is given by the smooth map:

$$h^{-1} : P(\partial M) \to \partial_{in} SM, \qquad h^{-1}(z, w) = w - \sqrt{1 - \|w\|_g^2} \; \nu(z).$$

Therefore h is a diffeomorphism. $\square$

For the rest of this section we will be considering the vectors in $P(\partial M)$, and with a slight abuse of notation, each vector $(z,v)\in P(\partial M)$ represents an inward-pointing unit vector at z. In the next lemma we show that the data (4.1) determines the restriction of $P(\partial M)$ on Γ.

Lemma 74. *Let Riemannian manifold (M,g) be as in Theorem 57. The first fundamental form $g|_\Gamma$ of Γ and the set*

$$P(\Gamma)=\{(z,v)\in T\partial M:\ z\in\Gamma,\ \|v\|_g<1\}$$

can be recovered from the data (4.1).

Proof. Let $(z,v)\in T\Gamma$. We choose a smooth curve $c:(-1,1)\to\Gamma$ for which $c(0)=z$, and $\dot c(0)=v$. Since the boundary ∂M of M is strictly convex the inverse function of the exponential map $\exp_z$ is smooth and well defined near z on M. In addition, we have that

$$r_z(c(t))=d(z,c(t))=\|\exp_z^{-1}(c(t))\|_g.$$

We set $\tilde c(t)=\exp_z^{-1}(c(t))\in T_zM$. As the differential of the exponential map at the origin is an identity operator we get $\tilde c(0)=0$, and $\dot{\tilde c}(0)=v\in T_zM$. From here the continuity of the norm yields

$$\lim_{t\to0}\frac{r_z(c(t))}{|t|}=\lim_{t\to0}\left\|\frac{\tilde c(0)-\tilde c(t)}{t}\right\|_g=\left\|\dot{\tilde c}(0)\right\|_g=\|v\|_g. \tag{4.30}$$

By the data (4.1) and the choice of the path $c(t)\in\Gamma$ we know the left hand side of equation (4.30). Therefore we have recovered the length of an arbitrary vector $(z,v)\in T\Gamma$. Moreover, the set $P(\Gamma)$ is recovered.

Since we know the unit sphere $\{v\in T_z\partial M:\ \|v\|_g=1\}$ for each $z\in\Gamma$ the reconstruction of the first fundamental form of Γ can be carried out as explained in the next lemma. $\square$

Lemma 75. *Let (X,g) be a finite dimensional inner product space. Let $a>0$ and $S(a):=\{v\in X:\ \|v\|_g=a\}$. Then any open subset U of $S(a)$ determines the inner product g on X.*

Proof. This proof is the same as the one in [29, Lemma 3.33]. For the convenience of the reader we repeat it here. Choose a base $e_1,\dots,e_n$ of X. Then for every $v,w\in U$ we can write

$$g(v,w)=g_{ij}v^iw^j,\quad \text{where } g_{ij}=g(e_i,e_j),\ i,j\in\{1,\dots,n\}.$$

Thus it suffices to recover the matrix g_{ij}. Since $U \subset S(r)$ is open it holds that we know the open cone,

$$C(U) = \{tv \in X : v \in U : t > 0\}$$

and the smooth function

$$F : C(U) \to \mathbb{R}, \quad F(x) := \frac{1}{2}\|x\|_g^2 = \frac{1}{2}g_{ij}x^i x^j = \frac{1}{2}t^2 g_{ij}v^i v^j = \frac{1}{2}t^2 r^2.$$

Therefore g_{ij} is the Hessian of F.

□

Let $p_0 \in M$. By Theorem 57 we can find a boundary point $z_0 \in \Gamma$ and neighborhoods U_{p_0} and V_{p_0} for p_0 and z_0 respectively such that the distance function $d(\cdot,\cdot)$ is smooth in the product set $U_{p_0} \times V_{p_0}$. For each $z \in \Gamma \cap V_{p_0}$ we let γ_z be the unique distance minimizing unit speed geodesic from p_0 to z. If we decompose the velocity of the geodesic γ_z at $r_{p_0}(z)$ into its tangential and normal components to the boundary, then the tangential component coincides with the boundary gradient of the travel time function r_{p_0} at z. For this vector field we use the notation $\mathrm{grad}_{\partial M}\, r_{p_0}(z) \in P_z(\Gamma)$. Furthermore, by Lemma 74 we have recovered the metric tensor of the measurement domain $\Gamma \subset \partial M$. Thus we can compute $\mathrm{grad}_{\partial M}\, r_{p_0}(z)$ whenever the respective travel time function r_{p_0} is differentiable on Γ.

4.3.2 Topological Reconstruction

We first show that the data (4.1) separates the points in the manifold M.

Lemma 76. *Let (M,g) be as in Theorem 57. Let $\Gamma \subset \partial M$ be open and $p_1, p_2 \in M$ be such that $r_{p_1}(z) = r_{p_2}(z)$ for all $z \in \Gamma$, then $p_1 = p_2$.*

Proof. First we choose open and dense subsets $W_1, W_2 \subset \partial M$ for the points p_1 and p_2 as we have for the point p_0 in Theorem 57. Then we choose any point $z_0 \in W_{p_1} \cap W_{p_2} \cap \Gamma$, neighborhoods U_{p_1} of p_1, U_{p_2} of p_2 and V_{p_1}, V_{p_2} of z_0 as we have for p_0 in Theorem 57. Thus the distance function $d(\cdot,\cdot)$ is smooth in the product sets $U_{p_1} \times V$ and $U_{p_2} \times V$, where $V = V_{p_1} \cap V_{p_2}$ is an open neighborhood of z_0. Moreover for each $(p,z) \in U_{p_i} \times V$, $i \in \{1,2\}$ there exists a unique distance minimizing geodesic of M connecting p to z.

If γ_i is the distance minimizing geodesic from p_i to z_0 for $i = \{1,2\}$ then by the discussion preceding this lemma we have that $\mathrm{grad}_{\partial M} r_{p_i}(z_0)$ represents the tangential component of $\dot{\gamma}_i$ at $r_{p_i}(z_0)$. Since $r_{p_1} = r_{p_2}$ the tangential components of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are the same. Since the velocity vectors of $\dot{\gamma}_i$ at $r_{p_i}(z_0)$ have unit length, they must also coincide. We get

$$z_0 = \gamma_1(r_{p_1}(z_0)) = \gamma_2(r_{p_2}(z_0)) \quad \text{and} \quad \dot{\gamma}_1(r_{p_1}(z_0)) = \dot{\gamma}_2(r_{p_2}(z_0)).$$

Thus the geodesics γ_1 and γ_2 agree and we have $p_1 = p_2$. □

We are now ready to reconstruct the topological structure of (M,g) from the partial travel time data (4.1). Let $B(\Gamma)$ be the collection of all bounded functions $f : \Gamma \to \mathbb{R}$ and $\|\cdot\|_\infty$ the supremum norm of $B(\Gamma)$. Thus $(B(\Gamma), \|\cdot\|_\infty)$ is a Banach space. Since (M,g) is a compact Riemannian manifold each travel time function r_p, for $p \in M$, is bounded by the diameter of M, which is finite. Thus

$$\{r_p = d(p,\cdot) : \Gamma \to [0,\infty) \mid p \in M\} \subset B(\Gamma),$$

and the map

$$R : (M,g) \to (B(\Gamma), \|\cdot\|_\infty), \quad R(p) = r_p \tag{4.31}$$

is well defined.

Proposition 77. *Let Riemannian manifold (M,g) be as in Theorem 57. The map R as in* (4.31) *is a topological embedding.*

Proof. By Lemma 76, we know that the map R is injective. Let $x, y \in M$ and set $z \in \Gamma$. Using the triangle inequality we have

$$|r_x(z) - r_y(z)| = |d(x,z) - d(y,z)| \leq d(x,y).$$

Thus, $\|r_x - r_y\|_\infty \leq d(x,y)$, and R is a 1-Lipschitz function. Hence, it is continuous.

Let K be a closed set in M. Since M is a compact Hausdorff space the set K is compact. Since the image of a compact set under a continuous mapping is compact, it follows that $R(K)$ is closed. This makes R a closed map and thus a topological embedding. □

The topology on M is then the inherited topology from the metric space $(C(\Gamma), \|\cdot\|_\infty)$.

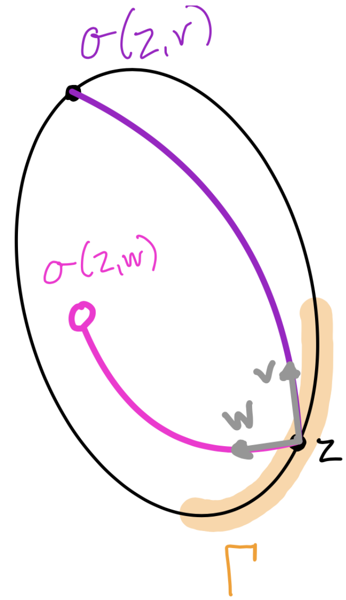

Figure 4.3 The set $\sigma(z,v)$ for $(z,v) \in P(\Gamma)$.

4.3.3 Boundary Determination

We recall that the data (4.1) only gives us the subset Γ of the boundary, and we do not know yet if the travel time function $r \in R(M)$ is related to an interior or a boundary point of M. In this subsection we will use the data (4.1) to determine the boundary of the unknown manifold M as a point set. However, due to Proposition 77 we may assume without loss of generality that the topology of M is known. Also the set $P(\Gamma)$, as in Lemma 74, is known to us.

Let $(z,v) \in P(\Gamma)$, and define the set,

$$\begin{aligned}\sigma(z,v) = \{p \in M \mid &\text{ the point } p \text{ has a neighborhood } U \subset M \text{ such that,} \\ &r_q : \Gamma \to \mathbb{R} \text{ is smooth near } z \text{ for every } q \in U, \\ &q \mapsto \operatorname{grad}_{\partial M} r_q(z) \text{ is continuous in } U, \\ &\operatorname{grad}_{\partial M} r_p(z) = -v\} \cup \{z\},\end{aligned} \tag{4.32}$$

where $\operatorname{grad}_{\partial M} r_p(z)$ is the boundary gradient of r_p at $z \in \Gamma$. We recall that by Proposition 77 we know the topology of M, and by Lemma 74 we know the geometry of Γ. These in conjunction with the data (4.1) imply that we can recover the set $\sigma(z,v)$ for every $(z,v) \in P(\Gamma)$. In the next lemma we generalize the result [35, Lemma 2.9] and relate $\sigma(z,v)$ to the maximal distance minimizing segment of the geodesic $\gamma_{z,v}$.

Lemma 78. *Let* $(z,v)\in P(\Gamma)$ *then* $\overline{\sigma(z,v)}=\gamma_{z,v}([0,\tau_{cut}(z,v)])$.

Proof. Let $t\in[0,\tau_{\mathrm{cut}}(z,v))$ and define $y:=\gamma_{z,v}(t)$. Thus y is not in the cut locus of z (see equation (4.11)). By Proposition 63 there exist neighborhoods $U,V\subset M$ of y and z respectively, having the property that the distance function $d(\cdot,\cdot)$ is smooth in the product set $U\times V$. Therefore the function $r_q(\cdot)=d(q,\cdot)|_\Gamma$ is smooth near z for any $q\in U$. Furthermore, $\mathrm{grad}_{\partial M}\, r_p(z)=-v$, and the function $p\mapsto \mathrm{grad}_{\partial M}\, r_p(z)$ is continuous in U. Therefore y is in $\sigma(z,v)$ and the inclusion $\gamma_{z,v}([0,\tau_{\mathrm{cut}}(z,v)))\subseteq\sigma(z,v)$ is true. This gives $\gamma_{z,v}([0,\tau_{\mathrm{cut}}(z,v)])\subseteq\overline{\sigma(z,v)}$.

Let $p\in\sigma(z,v)$, then $r_p(z)$ is smooth in a neighborhood of z and $\mathrm{grad}_{\partial M}\, r_p(z)=-v$. Thus $\gamma_{z,v}$ is the unique distance minimizing geodesic connecting z to p. Since the geodesic $\gamma_{z,v}$ is not distance minimizing beyond the interval $[0,\tau_{\mathrm{cut}}(z,v)]$ we have $p\in\gamma_{z,v}([0,\tau_{\mathrm{cut}}(z,v)])$ and therefore $\overline{\sigma(z,v)}\subset\gamma_{z,v}([0,\tau_{\mathrm{cut}}(z,v)])$. □

We set

$$T_{z,v}:=\sup_{p\in\sigma(z,v)} r_p(z)=\sup_{p\in\sigma(z,v)} d(p,z),\qquad \text{for } (z,v)\in P(\Gamma). \tag{4.33}$$

Notice that this number is determined entirely by the data (4.1), as opposed to $\tau_{\mathrm{cut}}(z,v)$ which requires our knowledge of when the geodesics were distance minimizing. By the following corollary, these two numbers are the same.

Corollary 79. *For any* $(z,v)\in P(\Gamma)$ *we have that* $T_{z,v}=\tau_{cut}(z,v)$.

Proof. Let $x\in\sigma(z,v)$. It follows from Lemma 78 that $d(x,z)\le\tau_{\mathrm{cut}}(z,v)$. By definition of $T_{z,v}$ we get $T_{z,v}\le\tau_{\mathrm{cut}}(z,v)$. Then we consider $t\in[0,\tau_{\mathrm{cut}}(z,v))$ and set $y=\gamma_{z,v}(t)$. By Lemma 78, we have $y\in\sigma(z,v)$. Thus, $t=d(y,z)\in[0,T_{z,v})$ making $[0,\tau_{\mathrm{cut}}(z,v))\subseteq[0,T_{z,v})$ and consequently, $\tau_{\mathrm{cut}}(z,v)\le T_{z,v}$. Therefore, we have verified that $T_{z,v}=\tau_{\mathrm{cut}}(z,v)$. □

We will use the sets $\sigma(z,v)$, for $(z,v)\in P(\Gamma)$ to determine the boundary ∂M of M. Since the topology of M is known by Proposition 77, we can determine the topology of these σ sets from the data. The next lemma shows if $\sigma(z,v)$ is closed then $\gamma_{z,v}(T_{z,v})$ is on the boundary of M.

Lemma 80. *Let* $(z,v)\in P(\Gamma)$*. If* $\sigma(z,v)$ *is closed then* $T_{z,v}=\tau_{exit}(z,v)$.

Proof. By the definition of $T_{z,v}$ we must have $T_{z,v} \le \tau_{\text{exit}}(z,v)$.

Suppose that $T_{z,v} < \tau_{\text{exit}}(z,v)$. From Corollary 79 then we also know

$$\tau_{\text{cut}}(z,v) = T_{z,v} < \tau_{\text{exit}}(z,v). \tag{4.34}$$

Let $p = \gamma_{z,v}(T_{z,v})$, and by Lemma 78 it holds that $p \in \overline{\sigma(z,v)} = \sigma(z,v)$. Since $p \in \text{cut}(z)$ we have by Lemma 58, that there either exists a second distance minimizing geodesic from z to p or p is a conjugate point to z along $\gamma_{z,v}$. In the first case let $w \in P(\Gamma)$ such that $\gamma_{z,w}$ is another unit-speed distance minimizing geodesic from z to p. We note that $T_{z,v} = \tau_{\text{cut}}(z,w)$.

Let U be a neighborhood of p as in (4.32). We consider a sequence $t_i \in [0, T_{z,v}]$, $i \in \mathbb{N}$ such that $t_i \to T_{z,v}$ as $i \to \infty$. Then for sufficiently large i the points $p_i = \gamma_{z,v}(t_i)$ and $q_i = \gamma_{z,w}(t_i)$ are in U and converge to p. By the continuity of the boundary gradient in U we have $\text{grad}_{\partial M} r_{p_i}(z) \to \text{grad}_{\partial M} r_p(z)$ and $\text{grad}_{\partial M} r_{q_i}(z) \to \text{grad}_{\partial M} r_p(z)$, when $i \to \infty$. However, by construction $\text{grad}_{\partial M} r_{p_i}(z) = -v$ while $\text{grad}_{\partial M} r_{q_i}(z) = -w$ for all $i \in \mathbb{N}$. Thus $\text{grad}_{\partial M} r_p(z)$ has multiple values, and r_p is not differentiable at z, contradicting that $p \in \sigma(z,v)$.

If the second case is valid, and since $p \in M^{int}$, we get by a similar proof as in [32, Theorem 2.1.12] that the exponential map $\exp_z$ is not a local injection at $T_{z,v} v \in T_z M$. From here [32, Theorem 2.1.14] implies that there is a sequence of points $(p_i)_{i=1}^{\infty}$ in M^{int} that converges to p and can be connected to z by at least two distance minimizing geodesics. By the same argument as in the previous case, r_{p_i} is not differentiable at z for any $i \in \mathbb{N}$, which contradicts the fact that $p \in \sigma(z,v)$. Thus inequality (4.34) cannot occur and we must have $T_{z,v} = \tau_{\text{exit}}(z,v)$. □

Lemma 81. *Let $p_0 \in \partial M$ and $z_0 \in \Gamma$, U_{p_0}, and V_{p_0} be as in Theorem 57. For every $p \in U_{p_0}$ we denote $\eta(p) = -\text{grad}_{\partial M} r_p(z_0)$. There exists a neighborhood $U'_{p_0} \subseteq U_{p_0}$ of p_0 such that for all $p \in U'_{p_0}$ we have that p is in the closed set $\sigma(z_0, \eta(p))$.*

Proof. By these assumptions, $d(\cdot,\cdot)$ in $U_{p_0} \times V_{p_0}$ is smooth. Define $v_0 = \eta(p_0)$ and $t_0 = \tau_{\text{exit}}(z_0, v_0)$, then $p_0 = \exp_{z_0}(t_0 v_0)$. Since z_0 was chosen to be a point outside the cut locus of p_0, these points are not conjugate to each other along the geodesic γ_{z_0,v_0} connecting them. Therefore the differential $\mathrm{D}\exp_{z_0}$ of the exponential map is invertible at $t_0 v_0 \in T_{z_0} M$. It follows from the Inverse function theorem that there exist respective neighborhoods $K \subseteq M$ of p_0 and $J \subseteq T_{z_0} M$ of $t_0 v_0$ such that $\exp_{z_0} : J \to K$ is a diffeomorphism.

Then we consider the open set

$$W = \left\{ \frac{\exp_{z_0}^{-1}(x)}{\|\exp_{z_0}^{-1}(x)\|_q} \; : \; x \in K \cap U_{p_0} \right\} \subset S_{z_0}M$$

that contains v_0. Since the exit time function τ_{exit} is smooth on W, there exists an open neighborhood $W' \subset W$ of v_0 in $S_{z_0}M$ such that

$$\exp_{z_0}(\tau_{\text{exit}}(z_0, v)v) \in U_{p_0} \cap \partial M, \quad \text{for every } v \in W'.$$

We define the cone

$$\Sigma = \left\{ t\, v \in T_{z_0}M : \; v \in W', \; t \in (0, \tau_{\text{exit}}(z_0, v)] \right\},$$

and the set

$$U'_{p_0} = \{\exp_{z_0}(w) \in M : \; w \in \Sigma \cap J\} \subset U_{p_0}.$$

Therefore U'_{p_0} is an open neighborhood of p_0 in M, and by its definition for each $p \in U'_{p_0}$ it holds that

$$r_p(z_0) = \|\exp_{z_0}^{-1}(p)\|_q \leq \tau_{\text{exit}}(z_0, \eta(p)),$$

and

$$q(p) := \gamma_{z_0,\eta(p)}(\tau_{\text{exit}}(z_0, \eta(p))) \in \partial M \cap U'_{p_0}.$$

Finally we get from the Proposition 57 that $z_0, p, q(p) \in \sigma(z_0, \eta(p))$. By Lemma 78 we know that the set $\sigma(z_0, \eta(p))$ is contained in the trace of the geodesic $\gamma_{z_0,\eta(p)}$. Since the boundary ∂M of M is strictly convex each geodesic can meet the boundary in at most two points. For the geodesic $\gamma_{z_0,\eta(p)}$ these points are z_0 and $q(p)$. Thus

$$\sigma(z_0, \eta(p)) = \gamma_{z,\eta(p)}([0, \tau_{\text{exit}}(z_0, \eta(p))]),$$

is closed. This ends the proof. □

Corollary 82. *Let $p_0 \in \partial M$, $z_0 \in \Gamma$ and U'_{p_0} be as in Lemma 81. If we denote $\eta(p) = -grad_{\partial M}\, r_p(z_0)$ then $T_{z_0,\eta(p)}$ is smooth for all $p \in U'_{p_0}$.*

Proof. Since the exit time function is smooth on those $(z, v) \in \partial_{in}SM$ that satisfy $\tau_{\text{exit}}(z, v) < \infty$ we only need to show that $T_{z,\eta(p)} = \tau_{\text{exit}}(z_0, \eta(p))$. This equation follows from lemmas 80 and 81. □

We are now ready to determine the boundary of M from the data (4.1).

Proposition 83. *Let (M,g) be as in Theorem 57 and $p_0 \in M$. Then $p_0 \in \partial M$ if and only if there exists $(z,v) \in P(\Gamma)$ such that $p_0 \in \sigma(z,v)$ and $r_{p_0}(z) = T_{z,v}$.*

Proof. If $p_0 \in \partial M$ then we get from Lemma 81 that there exists $(z_0, v) \in P(\Gamma)$ such that p_0 is in the closed set $\sigma(z_0, v)$. By Lemma 80 we have $T_{z_0,v} = \tau_{\text{exit}}(z_0, v)$. Firstly the strict convexity of ∂M implies that each geodesic has at most two boundary points. Secondly since $p_0 \neq z_0$ are both boundary points contained in $\sigma(z_0, v)$, which is a trace of a distance minimising geodesic, it follows that $T_{z_0,v} = r_{p_0}(z_0)$.

To show the reverse direction, let $(z,v) \in P(\Gamma)$ be such that $p_0 \in \sigma(z,v)$ and $T_{z,v} = r_{p_0}(z)$. Thus $\gamma_{z,v}([0, r_{p_0}(z)]) \subseteq \sigma(z,v)$, and it follows from Lemma 78 and Corollary 79 that $\sigma(z,v)$ is closed. By Lemma 80, the closedness of $\sigma(z,v)$ implies $T_{z,v} = \tau_{\text{exit}}(z,v)$. Thus, $r_{p_0}(z) = \tau_{\text{exit}}(z,v)$, making $p_0 \in \partial M$. $\square$

4.3.4 Local Coordinates

By Proposition 83 we have reconstructed the boundary ∂M of the smooth manifold M. In this section we use the partial travel time data (4.1) to construct two local coordinate systems for $p_0 \in M$. Since M has a boundary, we need different coordinates systems based on whether $p_0 \in M^{int}$ or $p_0 \in \partial M$.

Proposition 84. *Let (M,g) be as in Theorem 57. Let $p_0 \in M^{int}$, and choose $z_0 \in \Gamma$, U_{p_0}, and V_{p_0} as in Theorem 57. Let the map $\alpha : U_{p_0} \to P_{z_0}(\Gamma) \times \mathbb{R}$ be defined as*

$$\alpha(p) = (-grad_{\partial M}\, r_p(z_0), r_p(z_0)). \tag{4.35}$$

This map is a diffeomorphism onto its image $\alpha(U_{p_0}) \subset P_{z_0}(\Gamma) \times \mathbb{R}$.

Proof. Since the distance function $d(\cdot,\cdot)$ is smooth in $U_{p_0} \times V_{p_0}$ also the function α is smooth on U_{p_0}. By a direct computation we see that the inverse function of α, is given as,

$$\alpha^{-1}(v,t) = \exp_{z_0}\left(t\, h^{-1}(v)\right), \qquad \text{for } (v,t) \in P_{z_0}(\Gamma) \times \mathbb{R}.$$

where $h : \partial_{in} S_{z_0} M \to P_{z_0}(\Gamma)$, is the orthogonal projection given in (4.29). By the smoothness

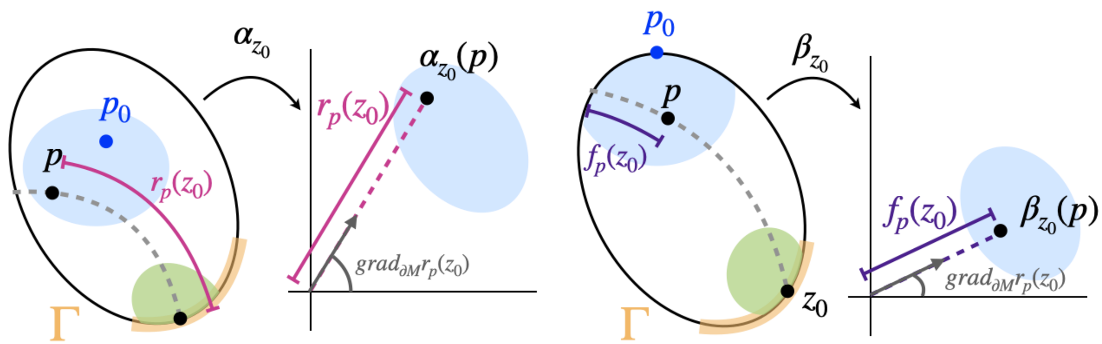

Figure 4.4 Local coordinates on M, which depend on whether p_0 is in M^{int} or ∂M.

of h^{-1} and the exponential map, it follows that α^{-1} is smooth. Thus, α is a diffeomorphism onto its image, which is open in $P_{z_0}(\Gamma) \times \mathbb{R}$. $\square$

In particular, the function α, in (4.35), gives a local coordinate system near the interior point p_0. In order to define a coordinate system for a point at the boundary we will adjust the last coordinate function of α to be a boundary defining function.

Proposition 85. *Let (M, g) be as in Theorem 57. Let $p_0 \in \partial M$ and choose $z_0 \in \Gamma$, and U'_{p_0} as in Lemma 81. Let $\eta(p) := -grad_{\partial M}\, r_p(z_0)$ and $\beta_{z_0} : U'_{p_0} \to P_{z_0}(\Gamma) \times [0, \infty)$ be defined as*

$$\beta(p) = (\eta(p), T_{z_0,\eta(p)} - r_p(z_0)). \tag{4.36}$$

This map is a diffeomorphism onto its image $\beta(U'_{p_0}) \subset P_{z_0}(\Gamma) \times [0, \infty)$.

Proof. Since the distance function $d(\cdot,\cdot)$ is smooth in $U'_{p_0} \times V_{p_0}$ and $p_0 \in \partial M$ we have by Corollary 82, that the map $T_{z_0,\eta(p)}$ is smooth for all $p \in U'_{p_0}$. Thus β is smooth in U'_{p_0}. Again by a direct computation we get that the inverse function of β is given as

$$\beta^{-1}(v, t) = \exp_{z_0}\left(\left(T_{z_0,v} - t\right) h^{-1}(v)\right) \qquad \text{for } (v, t) \in P_{z_0}(\Gamma) \times [0, \infty).$$

By the local invertibility of the exponential map $\exp_{z_0}$ at $r_{p_0}(z_0) h^{-1}(\eta(p_0)) \in T_{z_0}M$ and the equation $r_p(z_0) = \|\exp_{z_0}^{-1}(p)\|_g$ for $p \in U'_{p_0}$, the set $\eta(U'_{p_0}) \subset P_{z_0}(\Gamma)$ is open and the function $v \mapsto T_{z_0,v}$, in this set is smooth, making β^{-1} smooth. Thus, β is a diffeomorphism onto its image, which is open in $P_{z_0}(\Gamma) \times [0, \infty)$.

Finally by Proposition 80 we get that $T_{z_0,\eta(p)} - r_p(z_0) = 0$ if and only if $p \in U'_{p_0} \cap \partial M$. Thus this function defines the boundary. $\square$

Combining the results of Propositions 84 and 85, we know that for $p_0 \in M$, either the function α as in (4.35) or the function β as in (4.36), gives a smooth local coordinate system. Moreover these maps can be recovered fully from the data (4.1). As these two types of coordinate charts cover M the smooth structure on M is then the same as the maximal smooth atlas determined by these coordinate charts [38, Proposition 1.17].

4.3.5 Reconstruction of the Riemannian Metric

So far we recovered both the topological and smooth structures of the Riemannian manifold (M,g) from the data (4.1). In this section we recover the Riemannian metric g. We recall that by Lemma 74 we know the first fundamental form of Γ.

In order to recover the metric on M we consider the distance function

$$d(p,z) = r_p(z), \quad \text{for } (p,z) \in M \times \Gamma,$$

which we have recovered by Proposition 77. Let $p_0 \in M$. By Theorem 57 we can choose $z_0 \in \Gamma$ and neighborhoods U_{p_0} and V_{p_0} for p_0 and z_0 respectively such that the distance function $d(p,z)$ for $(p,z) \in U_{p_0} \times V_{p_0}$ is smooth. Thus the map

$$H_{p_0} \colon V_{p_0} \cap \Gamma \to T^*_{p_0}M, \quad H_{p_0}(z) = \mathrm{D}d(p_0,z) \tag{4.37}$$

is well defined and smooth. Here D stands for the differential of the distance function $d(p,z)$ with respect to the p variable in the open set $U_{p_0} \subset M$ and $T^*_{p_0}M$ is the cotangent space at p_0. As we have recovered the smooth structure of M we can find H_{p_0}.

For $z \in V_{p_0}$ the gradient $\text{grad}_p d(p,z)$ for $p \in U_{p_0}$ is the velocity of the distance minimizing unit speed geodesic from z to p (see for instance [38, theorems 6.31, 6.32]). In particular the map

$$\tilde{H}_{p_0} \colon (V_{p_0} \cap \Gamma) \ni z \to \text{grad}_p d(p_0,z) \in S_{p_0}M$$

is well defined and satisfies $\tilde{H}_{p_0}(z) = H_{p_0}(z)^\sharp$, where $\sharp\colon T^*_{p_0}M \to T_{p_0}M$ is the musical isomorphism, raising the indices, given in any local coordinates near p_0 as $(\xi^\sharp)^i = g^{ij}(p_0)\xi_j$. Note that the inverse of $\sharp$ is given by $\flat \colon T_{p_0}M \to T^*_{p_0}M$, that lowers the indices. Although we know the map H_{p_0}, we do not know its sister map $\tilde{H}_{p_0}$.

Lemma 86. *Let $p_0 \in M$. Let $z_0 \in \Gamma$, U_{p_0} and V_{p_0} be as in Theorem 57. Then the image of the* *map H_{p_0}, as in* (4.37)*, is contained the unit co-sphere*

$$S^*_{p_0}M := \{\xi \in T^*_{p_0}M : \|\xi\|_{g^{-1}} = 1\},$$

and has a nonempty interior.

Proof. Let $v \in T_{p_0}M$ such that $\exp_{p_0}(v) = z_0$. Since $\flat\colon T_{p_0}M \to T^*_{p_0}M$ is a linear isomorphism that preserves the inner product, the claim holds due to the local invertibility of the exponential map $\exp_p$ near v, the equality $d(p_0, z) = \|\exp^{-1}_{p_0}(z)\|_g$, which is true for all $z \in V_{p_0}$, and the continuity of the exit time function near $\frac{v}{\|v\|_g} \in S_{p_0}M$.

Since $\sharp$ is a linear isomorphism that preserves the inner product we have $[S^*_{p_0}M]^\sharp = S_{p_0}M$. Thus, if $\tilde{H}_{p_0}(V_{p_0}) \subseteq S_{p_0}(M)$ is open and nonempty then $H_{p_0}(V_{p_0}) = [\tilde{H}_{p_0}(V_{p_0})]^\flat$ is an open and nonempty subset of $S^*_{p_0}M$. So it suffices to show that $\tilde{H}_{p_0}(V_{z_0})$ is open and nonempty.

Since $d(p_0, z) = \|\exp^{-1}_{p_0}(z)\|_g$ for all $z \in V_{p_0}$, by the definition of $\tilde{H}$ given in (4.37), we have that

$$\tilde{H}_{p_0}(V_{p_0}) = \left\{ \operatorname{grad}_p d(p,z)\Big|_{p=p_0} : z \in V_{p_0} \cap \Gamma \right\} = \left\{ -\frac{\exp^{-1}_{p_0}(z)}{\|\exp^{-1}_{p_0}(z)\|_g} : z \in V_{p_0} \cap \Gamma \right\}.$$

Since the inverse of the exponential map $\exp^{-1}_{p_0}$ is a diffeomorphism from the neighborhood V_{p_0} of z_0 onto some neighborhood $W \subset T_{p_0}M$ of $\exp^{-1}_{p_0}(z_0)$ we see that the set $\tilde{A} := \left\{ -\frac{\exp^{-1}_{p_0}(z)}{\|\exp^{-1}_{p_0}(z)\|_g} : z \in V_{p_0} \right\}$ is open in $S_{p_0}M$.

In particular $-\frac{\exp^{-1}_{p_0}(z_0)}{\|\exp^{-1}_{p_0}(z_0)\|_g}$ is in $\tilde{A}$, and since $z_0 \in \Gamma$ we have that $\|\exp^{-1}_{p_0}(z_0)\|_g = \tau_{\text{exit}}\left(p_0, \frac{\exp^{-1}_{p_0}(z)}{\|\exp^{-1}_{p_0}(z)\|_g}\right)$. Thus by the smoothness of the exit time function the set W contains the image of the smooth function

$$f : A \to T_{p_0}M, \quad f(v) = -\tau_{\text{exit}}(p_0, v)v,$$

where $A \subset \tilde{A}$ is a neighborhood of $-\frac{\exp^{-1}_{p_0}(z_0)}{\|\exp^{-1}_{p_0}(z_0)\|_g}$. This implies that the set A is contained in $\tilde{H}_{p_0}(V_{z_0})$ as claimed. □

Finally Lemma 75 in conjunction with the previous lemma lets us recover the inverse metric $g^{-1}(p_0)$ and thus the metric g_{p_0}. This is formalized in the proposition below.

Proposition 87. *Let (M,g) be as in Theorem 57 and $p_0 \in M$. The data* (4.1) *determines the metric tensor g near p_0 in the local coordinates given in Propositions 84 and 85.*

Proof. Let $z_0 \in \Gamma$, U_{p_0} and V_{p_0} be as in Theorem 57. By Proposition 83 we can tell whether p_0 is an interior or a boundary point. Based on this we choose local coordinates of p_0 as in Proposition 84 or as in Proposition 85. Then we consider the function H_{p_0} given in the equation (4.37). By Lemma 86 we know that image of the function H_{p_0} contains an open subset of $S^*_{p_0}M$.

From here, by applying Lemma 75 we determine the inverse metric $g^{ij}(p_0)$ in the aforementioned coordinates. Finally taking the inverse of $g^{ij}(p_0)$ determines $g_{ij}(p_0)$. As this procedure can be done for any point $p \in M$, which is close enough to p_0, we have recovered the metric g near p_0 in the appropriate local coordinates. $\square$

4.4 The Proof of Theorem 55

Let Riemannian manifolds (M_1, g_1) and (M_2, g_2) be as in Theorem 55. We recall that the partial travel time data of these manifolds coincide in the sense of Definition 54. Let $(B(\Gamma_i), \|\cdot\|_\infty)$, for $i \in \{1,2\}$, be the Banach space of bounded real valued functions on Γ_i. We set a mapping

$$F : B(\Gamma_1) \to B(\Gamma_2), \qquad F(f) = f \circ \phi^{-1}, \tag{4.38}$$

where ϕ is the diffeomorphism from Γ_1 to Γ_2. By the triangle inequality we have that F is a metric isometry whose inverse mapping is given by $F^{-1}(h) = h \circ \phi$. Taking $R_i : (M_i, g_i) \to (B(\Gamma_i), \|\cdot\|_\infty)$, as in the equation (4.31), we have by the equation (4.2) in Definition 54 that

$$F(R_1(M_1)) = R_2(M_2).$$

Therefore we get from Proposition 77 that the map

$$\Psi : (M_1, g_1) \xrightarrow{R_1} (B(\Gamma_1), \|\cdot\|_\infty) \xrightarrow{F} (B(\Gamma_2), \|\cdot\|_\infty) \xrightarrow{R_2^{-1}} (M_2, g_2), \tag{4.39}$$

is a well defined homeomorphism, that satisfies the equation

$$d_2(\Psi(x), \phi(z)) = F(d_1(x, \cdot))(\phi(z)) = d_1(x, z), \qquad \text{for all } (x,z) \in M_1 \times \Gamma_1. \tag{4.40}$$

Here $d_i(\cdot,\cdot)$ is the distance function of (M_i, g_i). The goal of this section is to show that Ψ is a Riemannian isometry. In the following lemma we show first that ϕ preserves the Riemannian structure of the measurement regions.

Lemma 88. *Let Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *be as in Theorem 55. Then* $\Psi|_{\Gamma_1} = \phi$ *and* $\phi : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ *is a Riemannian isometry.*

Proof. Let z_1 be in Γ_1. From equation (4.40) we get $d_2(\Psi(z_1), \phi(z_1)) = 0$. Thus $\Psi(z_1) = \phi(z_1)$ and we have verified the first claim $\Psi|_{\Gamma_1} = \phi$. It follows from the proof of Lemma 74 and equation (4.40) that $\|\mathrm{D}\phi\, v\|_{g_2} = \|v\|_{g_1}$, for all $v \in T\Gamma_1$. Then the polarization identity implies that the differential $\mathrm{D}\phi$ of ϕ also preserves the first fundamental forms:

$$\langle \mathrm{D}\phi\, v_1, \mathrm{D}\phi\, v_2\rangle_{g_2} = \langle v_1, v_2\rangle_{g_1}, \quad \text{for all } v_1, v_2 \in T\Gamma_1,$$

making ϕ a Riemannian isometry. □

In particular we get from this lemma that $\mathrm{D}\phi(P(\Gamma_1)) = P(\Gamma_2)$. Next we show that the mapping Ψ takes the boundary of M_1 onto the boundary of M_2. In light of Proposition 83 we need to understand how this map carries over the sets $\sigma(z, v)$, as in (4.32). The following lemma gives an answer to this question.

Lemma 89. *Let Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *be as in Theorem 55. If* $(z_0, v) \in P(\Gamma_1)$ *then* $\Psi(\sigma(z_0, v)) = \sigma(\phi(z_0), \mathrm{D}\phi\, v)$*.*

Proof. Clearly we have that $\Psi(z_0) = \phi(z_0) \in \sigma(\phi(z_0), \mathrm{D}\phi\, v)$. So suppose that $p_0 \in \sigma(z_0, v) \setminus \{z_0\}$. Hence, by the same argument as in the proof of Lemma 80, we get that z_0 is not in the cut-locus of p_0. Thus by Proposition 63 we can choose a neighborhood $U \times V \subset M_1 \times M_1$ of (p_0, z_0) where the distance function $d_1(\cdot,\cdot)$ is smooth. Since the map Ψ is a homeomorphism the set $\Psi(U) \subset M_2$ is open, and we have by (4.40) that for each $q \in \Psi(U)$, the function $r_q(\cdot) = d_1(\Psi^{-1}(q), \phi^{-1}(\cdot))$ is smooth on the open set $\phi(V \cap \Gamma_1) \subset \Gamma_2$.

Since $\phi : \Gamma_1 \to \Gamma_2$ is a Riemannian isometry we have that

$$\mathrm{grad}_{\partial M_2} r_{\Psi(p)}(\phi(z_0)) = \mathrm{D}\phi(z_0)\mathrm{grad}_{\partial M_1} r_p(z_0), \quad \text{for } p \in U. \tag{4.41}$$

Here D stands for the differential, $\mathrm{grad}_{\partial M_1}$ for the boundary gradient of Γ_1 and $\mathrm{grad}_{\partial M_2}$ for that of Γ_2. Since the right hand side of equation (4.41) is continuous in p, the function

$$q = \Psi(p) \mapsto \mathrm{grad}_{\partial M_2} r_q(\phi(z_0))$$

is continuous in $\Psi(U)$. Finally

$$\mathrm{grad}_{\partial M_2} r_{\Psi(p_0)}(\phi(z_0)) = \mathrm{D}\phi(z_0)\mathrm{grad}_{\partial M_1} r_{p_0}(z_0) = -\mathrm{D}\phi(z_0)v$$

implies $\Psi(p_0) \in \sigma(\phi(z_0), \mathrm{D}\phi\, v)$.

On the other hand after reversing the roles of M_1 and M_2 we can use the same proof to show $\sigma(z_0, v) \supset \Psi^{-1}(\sigma(\phi(z_0), \mathrm{D}\phi\, v))$, implying $\Psi(\sigma(z_0, v)) = \sigma(\phi(z_0), \mathrm{D}\phi\, v)$. This ends the proof. □

Lemma 90. *Let Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *be as in Theorem 55. Then* $\Psi(\partial M_1) = \partial M_2$. *Moreover,* $\Psi(M_1^{int}) = M_2^{int}$.

Proof. Let $p \in \partial M_1$. Due to Proposition 83 there is a $(z, v) \in P(\Gamma_1)$ such that p is in the closed set $\sigma(z, v)$ and $r_{p_1}(z) = T_{z,v}$. Thus Lemma 89 gives $\Psi(\sigma(z, v)) = \sigma(\phi(z), \mathrm{D}\phi\, v)$, and since Ψ is a homeomorphism, also the set $\sigma(\phi(z), \mathrm{D}\phi\, v)$ is closed and contains $\Psi(p)$. Furthermore, by equation (4.40) we have that $r_{\Psi(q)}(\phi(z)) = r_q(z)$, for all $q \in \sigma(z, v)$. Therefore

$$T_{\phi(z),\mathrm{D}\phi\, v} = T_{z,v} = r_p(z) = r_{\Psi(p)}(\phi(z)).$$

From here Proposition 83 implies that $\Psi(p)$ is in ∂M_2. Thus $\Psi(\partial M_1) \subset \partial M_2$ and by using the same argument for Ψ^{-1} it follows that $\Psi(\partial M_1) = \partial M_2$. Since M_1^{int} and ∂M_1 are disjoint and Ψ is a bijection we also have that $\Psi(M_1^{int}) = M_2^{int}$. □

Lemma 91. *Let Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *be as in Theorem 55. The mapping* $\Psi : M_1 \to M_2$, *given in formula* (4.39), *is a diffeomorphism.*

Proof. Let $p_0 \in M_1$, and choose $W_{p_0} \subset \partial M_1$ as in Theorem 57. Since $\phi : \Gamma_1 \to \Gamma_2$ is a diffeomorphism, the set $\phi(W_{p_0} \cap \Gamma_1)$ is open and dense in Γ_2. Then for $\Psi(p_0) \in M_2$ we choose $W_{\Psi(p_0)} \subset \partial M_2$ as in Theorem 57 and consider the non-empty open set $W_{\Psi(p_0)} \cap \phi(W_{p_0} \cap \Gamma_1) \subset \Gamma_2$. We pick $z_0 \in W_{p_0} \cap \Gamma_1$ such that $\phi(z_0) \in W_{\Psi(p_0)} \cap \phi(W_{p_0} \cap \Gamma_1)$.

Let neighborhoods $U_{p_0} \subset M_1$ of p_0 and $V_{p_0} \subset M_1$ of z_0 be such that the distance function $d_1(\cdot, \cdot)$ is smooth in the product set $U_{p_0} \times V_{p_0}$. We also choose neighborhoods $U_{\Psi(p_0)} \subset M_2$ of

$\Psi(p_0)$ and $V_{\Psi(p_0)} \subset M_2$ of $\phi(z_0) = \Psi(z_0)$ to be such that the distance function $d_2(\cdot,\cdot)$ is smooth in the product set $U_{\Psi(p_0)} \times V_{\Psi(p_0)}$. Since $\Psi\colon M_1 \to M_2$ is a homeomorphism we may choose these four sets in such a way that they satisfy

$$\Psi(U_{p_0}) = U_{\Psi(p_0)}, \quad \text{and} \quad \Psi(V_{p_0}) = V_{\Psi(p_0)}.$$

By Lemma 90 we know that $\Psi(p_0) \in M_2^{int}$ if and only if $p_0 \in M_1^{int}$, and $\Psi(p_0) \in \partial M_2$ if and only if $p_0 \in \partial M_1$. Next we consider the interior and boundary cases separately.

Suppose first that p_0 is an interior point of M_1. The functions

$$U_{p_0} \ni p \mapsto \alpha_1(p) = (-\mathrm{grad}_{\partial M_1} r_p(z_0), r_p(z_0)) \in P_{z_0}(\Gamma_1) \times \mathbb{R},$$

and

$$U_{\Psi(p_0)} \ni q \mapsto \alpha_2(q) = (-\mathrm{grad}_{\partial M_2} r_q(\phi(z_0)), r_q(\phi(z_0))) \in P_{\phi(z_0)}(\Gamma_2) \times \mathbb{R},$$

as in Proposition 84, are smooth local coordinate maps of M_1 and M_2 respectively. Moreover, by the computations done in the proof of Lemma 89 we get for every $p \in U_{p_0}$ that

$$r_p(z_0) = r_{\Psi(p)}(\phi(z_0)), \quad \text{and} \quad \mathrm{D}\phi(z_0)\mathrm{grad}_{\partial M_1} r_p(z_0) = \mathrm{grad}_{\partial M_2} r_{\Psi(p)}(\phi(z_0)).$$

Therefore for any $(v,t) \in \alpha_1(U_{p_0})$ we have that

$$(\alpha_2 \circ \Psi \circ \alpha_1^{-1})(v,t) = (\mathrm{D}\phi(z_0)v, t).$$

Thus we have proven that the map $\alpha_2 \circ \Psi \circ \alpha_1^{-1} \colon \alpha_1(U_{p_0}) \to \alpha_2(U_{\Psi(p_0)})$ is smooth.

Then we let p_0 be a boundary point of M_1. Let $\eta_1(p) := -\mathrm{grad}_{\partial M_1} r_p(z_0)$ for $p \in U_{p_0}$ and choose $U'_{p_0} \subset U_{p_0}$ as in Lemma 81 to be such that the set $\sigma(z_0, \eta_1(p))$ is closed and the function $p \mapsto T_{z_0,\eta_1(p)}$ is smooth for every $p \in U'_{p_0}$. Let $U'_{\Psi(p_0)} := \Psi(U'_{p_0}) \subset U_{\Psi(p_0)}$ and denote $\eta_2(q) := -\mathrm{grad}_{\partial M_2} r_q(\phi(z_0))$ for $q \in U'_{\Psi(p_0)}$. Since we have that $\mathrm{D}\phi(z_0)\eta_1(p) = \eta_2(\Psi(p))$ it holds by Lemma 89 that the set $\sigma(\phi(z_0), \eta_2(\Psi(p))) = \Psi(\sigma(z_0, \eta_1(p)))$, is closed for every $p \in U'_{p_0}$, and thus the function $U'_{\Psi(p_0)} \ni q \to T_{\phi(z_0),\eta_2(q)}$ is smooth by Corollary 79. Moreover, we have $T_{z_0,\eta_1(p)} = T_{\phi(z_0),\eta_2(\Psi(p))}$ for every $p \in U'_{p_0}$.

Then we consider local coordinate maps

$$U'_{p_0} \ni p \mapsto \beta_1(p) = (\eta_1(p), T_{z_0,\eta_1(p)} - r_p(z_0)) \in P_{z_0}(\Gamma_1) \times [0,\infty),$$

of M_1 and

$$U'_{\Psi(p_0)} \ni q \mapsto \beta_2(q) = (\eta_2(q), T_{\phi(z_0),\eta_2(q)} - r_q(\phi(z_0))) \in P_{\phi(z_0)}(\Gamma_2) \times [0,\infty),$$

of M_2, as in Proposition 85. By the discussion above we have for any $(v,t) \in \beta_1(U'_{p_0})$ that

$$(\beta_2 \circ \Psi \circ \beta_1^{-1})(v,t) = (\mathrm{D}\phi(z_0)v, t),$$

which implies that the map $(\beta_2 \circ \Psi \circ \beta_1^{-1})\colon \beta_1(U'_{p_0}) \to \beta_2(U'_{\Psi(p_0)})$ is smooth.

By combining these two cases we have proved that for every $p_0 \in M$ a local representation of the map Ψ is smooth, making $\Psi\colon M_1 \to M_2$ smooth. Finally by an analogous argument for Ψ^{-1} we can show that this map is also smooth. Thus $\Psi\colon M_1 \to M_2$ is a diffeomorphism as claimed. $\square$

We are ready to present the proof of our main inverse problem:

Proof of Theorem 55. By Lemma 91 we know that the map $\Psi\colon M_1 \to M_2$ is a diffeomorphism. We define a metric tensor $\tilde{g}_2$ on M_1 as the pull back of the metric g_2 with respect to map Ψ. Thus it suffices to consider a smooth manifold $M = M_1$ with an open measurement region $\Gamma = \Gamma_1 \subset \partial M$ and two Riemannian metrics g_1 and $\tilde{g}_2$. Moreover ∂M is strictly convex with respect to both of these metrics.

Let $\tilde{d}_2(\cdot,\cdot)$ be the distance function of $\tilde{g}_2$. We note that due to equation (4.40) we have $d_1(p,z) = \tilde{d}_2(p,z)$, for all $(p,z) \in M \times \Gamma$. By Lemma 88 we get that $g_1(p) = \tilde{g}_2(p)$ for all $p \in \Gamma$. Let $p_0 \in M$. Thus the map H_{p_0} given by (4.37) is the same for both metrics. From here Lemma 86 and Proposition 87 imply that $g_1(p_0) = \tilde{g}_2(p_0)$. Therefore map Ψ is a Riemannian isometry as claimed. $\square$

CHAPTER

5

FUTURE RESEARCH QUESTIONS

A natural progression from the work presented in Chapter 4 is to consider the stability of the associated inverse problem. Recall that according to Hadamard [21], an inverse problem is considered stable if two data sets are ‘close’ then the reconstructed manifolds are ‘close’. Thus we will begin by introducing a notion of distance between manifolds.

5.1 Hausdorff & Gromov-Hausdorff Distances

Let (X,d) be a metric space and for a subset $S \subset X$ define the r**-neighborhood of** S to be

$$U_r(S) = \bigcup_{x \in S} B(x,r).$$

The **Hausdorff distance** between some subsets A and B of X is

$$d_H(A,B) = \inf_{r>0}\{A \subseteq U_r(B) \text{ and } B \subseteq U_r(A)\}.$$

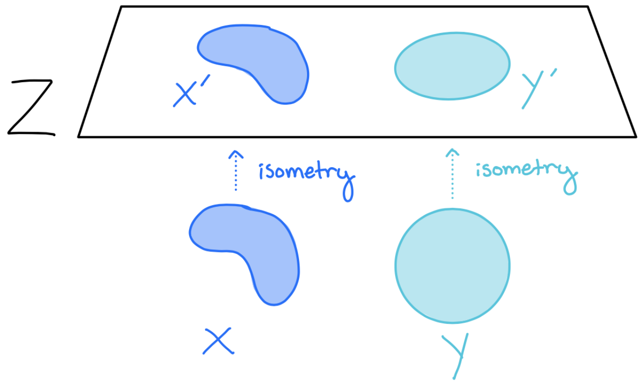

Figure 5.1 A metric space Z containing isometric copies of X and Y.

Example 92. *Let A be an open set in the metric space X. Then $d_H(A, \overline{A}) = 0$, so the Hausdorff distance cannot distinguish A from $\overline{A}$. However, if A and B are closed sets in X and $d_H(A, B) = 0$ then $A = B$. In particular, d_H provides a metric for the collection of all closed and bounded subsets of X [5].*

Let (X, d_x) and (Y, d_y) be compact metric spaces. Suppose (Z, d_Z) is a metric space with the property that it contains isometric embeddings of X and Y, denoted X' and Y' respectively. Such a metric space is illustrated in Figure 5.1. The collection of all metric spaces with this property will be denoted $\mathscr{Z}$. The **Gromov-Hausdorff distance** between X and Y, is defined as

$$d_{GH}(X, Y) = \inf_{Z \in \mathscr{Z}} \{d_H(X', Y') \text{ in } Z\}.$$

Example 93. *Let (X, d_x) and (Y, d_y) be compact metric spaces. If there exists an isometry between these metric spaces then $d_{GH}(X, Y) = 0$. This means that the Gromov-Hausdorff distance cannot distinguish between isometric manifolds.*

This exemplifies why d_{GH} is not a metric on the set of compact metric spaces. Instead, we will consider d_{GH} on the quotient space consisting of isometry classes of compact metric spaces.

Lemma 94. *Let $\mathscr{C}$ denote the set of compact metric spaces so that $\mathscr{C} / \sim$ is the set of isometry classes of compact metric spaces. The Gromov-Hausdorff distance is a metric on $\mathscr{C} / \sim$.*

Proof. Shown in [5, Theorem 7.3.30]. □

Thus define the metric space $(\mathscr{C}/\sim, d_{GH})$ to be the **Gromov-Hausdorff space**. The topology of this space, determined by the Gromov-Hausdorff distance, is called the **Gromov-Hausdorff topology**. It was shown in [49, pg. 296] that $(\mathscr{C}/\sim, d_{GH})$ is separable and complete. Moreover, a class of metric spaces $C \subset \mathscr{C}$ is said to be **pre-compact** if every sequence in C has a subsequence that is convergent in $(\mathscr{C}/\sim, d_{GH})$ [49, pg. 299].

5.2 Review of Some Stability Results

In an abstract sense, stability of an inverse problem can be defined as:

> Consider the data D_1, D_2 in metric space (X, d), where
>
> $$d(D_1, D_2) < \delta$$
>
> where the data has the associated manifolds (M_1, g_1) and (M_2, g_2). The inverse problem is considered **stable** if there exists a continuous function $\varepsilon : [0,\infty) \to [0,\infty)$ depending on δ such that $\delta \to 0$ implies $\varepsilon(\delta) \to 0$, and
>
> $$d_{GH}(M_1, M_2) < \varepsilon(\delta).$$

Example 95. *In [30] the authors study two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with boundary satisfy certain geometric bounds. The boundary distance data on the respective boundaries are denoted $R(M_i)$ where $R : x \mapsto r_x(\cdot)|_{\partial M_i}$.*

Let $S = \{z_i\}_{i=1}^N \subset \partial M$ be a finite δ-net for ∂M so that $d_H(S, \partial M) < \delta$, where d_H is the Hausdorff distance on ∂M. Then define $R_S(M)$ to be the restriction of $R(M)$ to S, so that $R_S : x \mapsto r_x(\cdot)|_S$. If $\hat{r}_p : S \to \mathbb{R}$ then the finite family $\hat{R}_S = \left\{ \left(\hat{r}_p(z_i)\right)_{i=1}^N \in \mathbb{R}^N \ : \ p = 1, ..., P \right\}$ is a δ-approximation to $R(M)$ if $d_H(\hat{R}_S, R_S(M)) < \delta$, where d_H is the Hausdorff distance on $\mathbb{R}^N$.

Taking $\hat{R}_s^i$ to be an δ-approximation of $R(M_i)$ then they show:

> *If $d_H(\hat{R}_s^1, \hat{R}_s^2) < \delta$ then there are uniform constants C_0 and δ_0 depending on the geometric bounds such that for $0 < \delta < \delta_0$ then $d_{GH}((M_2, g_2), (M_2, g_2)) < C_0 \delta^{1/18}$.*

A key idea in this proof is to use Toponogov's Theorem, which allows for the comparison of distances in the manifolds to distances in a model space of constant curvature. This connection to a model space is why there must be apriori geometric bounds on M_1 and M_2.

Additional stability results on manifolds using different types of data can be found in [17, 18] and [29, Section 4.4]. In the rest of this section we summarize the result presented by Ivanov in [27], since this paper studies the stability of a closely related inverse probem to our work. We note that their result does not provide explicit values of ε_δ in terms of δ, like Example 95. Thus, their techniques are promising for generalizing stability with a loose bound to the partial boundary distance data.

For a Riemannian manifold M, we denote by $\mathrm{diam}(M)$ the diameter, Sec_M is the sectional curvature, and inj_M is the injectivity radius. Define $\mathscr{M}(n, D, K, i_0)$ to be the class of all compact boundaryless Riemannian n-dimensional manifolds M with $\mathrm{diam}(M) \le D$, $|\mathrm{Sec}_M| \le K$, and $\mathrm{inj}_M \ge i_0$.

Lemma 96. *Let $n \ge 2$ and $K, D, i_0 \in (0, \infty)$ be given. The class of manifolds $\mathscr{M}(n, D, K, i_0)$ is pre-compact in the Gromov-Hausdorff topology.*

Proof. This result follows from [5, Theorem 10.7.2] and [49, Chapter 10]. They key idea is to show that there is a noncontracting map from M to the ball of radius D in a simply connected n-dimensional complete space of constant curvature K. Then for every sequence (M_i, g_i) in $\mathscr{M}(n, D, K, i_0)$ there exists a convergent subsequence $(M_{i_k} g_{i_k})$. □

For a Riemannian manifold $(M, g) \in \mathscr{M}(n, D, K, i_0)$ consider the nonempty open set $F \subset M$ to be the observation domain. The **distance difference data of** M is given by the map $D_F : (M, g) \to (C(F \times F), \|\cdot\|_\infty)$ where

$$D_F(x) = d(x, y) - d(x, z), \qquad x \in M, (y, z) \in F \times F.$$

Example 97 (Proposition 6.4 in [27])**.** *Let (M_1, g_1) and (M_2, g_2) belong to $\mathscr{M}(n, D, K, i_0)$. Assume that M_1 and M_2 share an open ball of radius $\rho_0 > 0$ denoted F, they induce the same topology and the same differential structure on F, and $g_1|_F = g_2|_F$.*

For $i = 1, 2$, let D_F^i denote the distance difference data of M_i and suppose that

$$d_H(D_F^1(M_1), D_F^2(M_2)) < \delta$$

where d_H is the Hausdorff distance in $C(F \times F)$. Then there exists an $\varepsilon > 0$ such that $d_{GH}(M_1, M_2) < \varepsilon$.

They key idea of the proof is to assume otherwise. It then uses the pre-compactness of

$\mathscr{M}(n, D, K, i_0)$ and a limiting argument to construct two sequences whose data are the same, but are not isometric. However, due to their earlier result (Theorem 1.3 in [27]) that is not possible.

5.3 Possible Directions

Ideally, we first want to show stability for the full boundary data case. This entails proving following:

Conjecture 98. *Let (M_1, g_1) and (M_2, g_2) be compact Riemannian manifolds with boundary such that $\phi : \partial M_1 \to \partial M_2$ is a diffeomorphism. For $i = 1, 2$ define the mappings $R_i : (M_i, g_i) \to (B(\partial M_i), \|\cdot\|_\infty)$ so that $R_i : x \mapsto r_x(\cdot)|_{\partial M_i}$ and $F : B(\partial M_1) \to B(\partial M_2)$ so that $F(f) = f \circ \phi^{-1}$. If $d_H(F \circ R_1(M_1), R_2(M_2)) < \delta$ where d_H is the Hausdorff distance in $B(\partial M_2)$ then there exists a continuous function $\varepsilon : [0, \infty) \to [0, \infty)$ depending on δ such that $\delta \to 0$ implies $\varepsilon(\delta) \to 0$, and $d_{GH}(M_1, M_2) < \varepsilon(\delta)$.*

One possible direction to prove this Conjecture is to use a limiting argument as in [27, Proposition 6.4]. This technique will likely require adding geometric bounds on the manifold or its boundary so that the manifolds are in a pre-compact space in the Gromov-Hausdorff sense. Currently we are unsure which pre-compact space encompasses these compact Riemannian manifolds with boundary so that the limiting argument works. Aside from $\mathscr{M}(n, D, K, i_0)$, some alternative pre-compact spaces in the Gromov-Hausdorff topology are discussed in [49, Chapter 10] and [5, Chapters 7 and 10].

In the event that we are able to prove Conjecture 98, and it becomes an interesting (i.e. non-trivial) result, the next step would be to generalize the method for the partial boundary data. However, it is not immediately clear how to properly formulate the corresponding conjecture. Specifically, we are not currently sure how to show when two data sets are 'close'.

Question 99. *Assume (M_i, g_i) are compact Riemannian manifolds with strictly convex boundary and $\Gamma_i \subset \partial M_i$ so that there is a diffeomorphism $\phi : \Gamma_1 \to \Gamma_2$. Define the maps $R_i : (M_i, g_i) \to (B(\Gamma_i), \|\cdot\|_\infty)$ so that $R_i : x \to r_x(\cdot)|_{\Gamma_i}$ and $F : B(\Gamma_1) \to B(\Gamma_2)$ so that $F(f) = f \circ \phi^{-1}$. Then what assumptions need to be on F or Γ_i so that $d_H(F \circ R_1(M_1), R_2(M_2)) < \delta$?*

Recall from (4.38) that if Γ_1 and Γ_2 are diffeomorphic and have coinciding data then $d_H(F \circ R_1(M_1), R_2(M_2)) = 0$. We might expect that if the Γ_i regions are from the same manifold then their data may be 'close', but even this assumption may not be enough. Consider the example of the catenoid where Γ_1 is the upper boundary circle and Γ_2 is the lower boundary circle so these two boundaries are diffeomorphic. However, there are many points p in the catenoid where $r_p(\cdot)|_{\Gamma_1}$ looks very different than $r_p(\cdot)|_{\Gamma_2}$, and so there is no guarantee the sets $F \circ R_1(M)$ and $R_2(M)$ will be close. Thus determining what makes two data sets 'close' in the space $B(\Gamma_2)$ will require more thinking.

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