Digital Signal Restoration Using Projection and Fuzzy Set Techniques

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April 1985

CCSP-TR-85/7

ABSTRACT

CIVANLAR, MEHMET REHA. Digital Signal Restoration Using Projection and Fuzzy Set Techniques.

(Under the direction of H.J. Trussell)

A new signal restoration method, which is capable of easily combining a large collection of a priori knowledge on the original signal and the degradation mechanism, is developed. The new method uses each piece of information for defining a fuzzy set restricting the set of acceptable solutions in the signal space. Using fuzzy sets gives considerable flexibility to the algorithm because, partially defined information can be modelled by fuzzy sets as well as exact knowledge.

The intersection of all fuzzy sets constructed for defining the solution is called the fuzzy feasibility set. The original signal is a member of the fuzzy feasibility set with a high membership value. Thus, the restoration problem is to find a member of the feasibility set with a large membership value. Any such member, being in accordance with the available information, is a non-rejectable solution. Projections onto convex sets and ordinary nonlinear optimi-

zation techniques can be used for finding a solution.

Ideally, the feasibility set must be a singleton containing the original signal. As the size of the feasibility set increases the chance of recovering the original signal decreases. Thus, the size of the feasibility set gives a quality measure for the solution.

The new method generated successful results in many signal restoration applications for which conventional restoration techniques have failed. Image coding and tomography are possible application areas for future work with the new technique.

BIOGRAPHY

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ACKNOWLEDGEMENTS

The author is grateful to his advisor Dr. H.J. Trussell for his guidance, encouragement and candid contribution at all stages of this research. His friendship made the task much more enjoyable.

He would also like to thank Dr. J. W. Silverstein for the interest shown and guidance extended to the author.

The author also wishes to thank his wife for her patience and support during this study.

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CHAPTER 1

INTRODUCTION

1.1. Background

1.1.1. Problem Definition and Noniterative Techniques

The signal restoration problem is to estimate the original form of a degraded and noise corrupted signal. It is a common problem of various fields in signal processing including image processing, speech processing, and system identification. Various techniques have been developed for the solution of the restoration problem under different degradation models and varying degrees of distortions [1],[2]. The current research is on finding new restoration techniques for extending the range of restorable distortions.

The most general distortion model can be written as:

$$\underline{g}=T[\underline{f}] \tag{1.1}$$

where; $\underline{g}$ is the degraded signal, $\underline{f}$ is the original signal, and T[.] is an operator. Throughout the thesis, all signals are assumed to be real and properly sampled, that is, there

is no aliasing. Thus, the signals are vectors in R^N, where N is the total number of samples.

The earliest approach to the restoration problem is inverse filtering [1], which can be described as finding the operator $T^{-1}[.]$ and applying it to the degraded signal. $T^{-1}[.]$ may be the actual inverse of $T[.]$, or a pseudo inverse such as the least squares inverse [1]. However, in most of the applications, inverse filtering is ill-conditioned [1],[3]. It amplifies the noise which is always present in practical applications.

The only information used by the inverse filter is the degradation operator. Better restoration methods are obtained through the use of additional knowledge about the distortion mechanism and the original signal. These methods include: the Wiener filter [1], which uses the covariances of the original signal and the contaminating noise; the constrained least squares method [4], which assumes a smooth original signal and needs the noise variance; and the stochastic estimation techniques [1],[5] which require probability density functions (pdf) of the signal and the noise. Obviously, the more information combined in a restoration algorithm, the better the restoration will be.

1.1.2. Iterative Techniques

A general formulation for the methods described in section 1.1.1 can be given as:

$$\hat{\underline{f}} = R[\underline{g}] \tag{1.2}$$

where, $\hat{\underline{f}}$ is the restored signal, and R is an operator derived from the degradation mechanism and a priori knowledge employed. These methods obtain the solution in one step. However, finding the operator R[.] becomes very difficult when the amount of a priori knowledge increases. Iterative techniques, which find the solution in more than one operation on the data, have a great flexibility in using various kinds of a priori information. They have become very popular in signal restoration because of this flexibility. A typical iteration equation can be written as:

$$\hat{\underline{f}}^{k+1} = O[\hat{\underline{f}}^{k}] \tag{1.3}$$

where, $\hat{\underline{f}}^{k}$ is the estimate for the original signal at k^{th} iteration and O[.] is an operator. The iterations are stopped when the estimate satisfies some convergence criterion.

In their survey paper Schafer et al. [6], formulate the constrained iterative restoration methods as finding the

fixed point of a transformation O[.] defined as:

$$O[.] = C[.] + \lambda \{ \underline{g} - T[C[.]] \} \qquad (1.4)$$

Any a priori knowledge about the original signal is expressed through the operator C[.]. This is done by constructing C[.] such that the original signal is a fixed point, i.e. $C[\underline{f}]=\underline{f}$. For example, if the original signal is known to be nonnegative, an appropriate operator will be a clipper zeroing the negative parts of its argument. The early applications of the iterative methods, Van Cittert [7] and Landweber [8] iterations, can be considered as special cases of this formulation with C[.]=I[.]. The nonzero scalar λ is to be used for controlling the rate of convergence. All other symbols have the same meaning as in equation 1.1. Several constraint operators for different kinds of a priori knowledge are studied in [6].

If the operator O[.] of equation 1.4 is a contraction then it has a unique fixed point and this fixed point is the original signal. This is because O[.] is constructed so that the original signal is a fixed point of it. For nonexpansive O[.]s the fixed point is not unique. However, if $\hat{\underline{f}}$ is a fixed point of O[.], either the residual signal which is defined by:

$$\underline{r} = \underline{g} - T[C[\hat{\underline{f}}]] \tag{1.5}$$

is null or $\hat{\underline{f}}$ does not satisfy the constraints. The second is not possible for the unique fixed point case, and for the nonexpansive case, a fixed point which does not satisfy the constraints can easily be discarded as a wrong solution. Thus, for all practical purposes, when the iterations converge the residual signal is zero. Clearly, for unconstrained iterations such as Landweber and Van Cittert the convergence point is inverse filter solution. As indicated above this solution is highly sensitive to noise.

In the constrained case, the ill-conditioning of the problem is not as severe as the straight forward inversion case. However, as indicated in [6], the algorithm is still sensitive to noise. This can be seen easily by including additive noise in the degradation model. For this case the degraded signal is:

$$\underline{g} = T[\ \underline{f}\ ]+\underline{n} \tag{1.6}$$

where $\underline{n}$ is the noise sequence. A solution, making the residual signal zero, is:

$$\underline{f}' = \underline{f} + (TC)^{-1}\ [\underline{n}] \tag{1.7}$$

The additional noise term may dominate the solution for many

cases.

A solution for this problem may be defining a new convergence criterion and stopping the iterations accordingly. In the work of Strand [9], the effects of truncation on the Landweber iteration are studied in detail. Trussell [10] uses the residual signal's sample variance for deciding on the convergence of the iterations. The iterations are stopped when the sample variance equals to the variance of the contaminating noise. This method has proven to be very successful.

1.2. Using Sets for Defining the Solution

1.2.1. Projection onto Convex Sets (POCS)

Generally, it is not easy to find a simple operator such that the set of its fixed points is equal to the set of signals satisfying an a priori information about the original signal. If complex operators are used, it will be very difficult to investigate the properties (such as contraction, non-expansivity, etc.) of their combinations. On the other hand, almost any kind of a priori knowledge can easily be modelled as a set in the signal space. Then, any signal in the intersection of these sets is consistent with all the available a priori information and acceptable as a solution.

In [11] these solutions were called "feasible solutions".

If all of the sets used in the description of the solution are closed and convex, a feasible solution can be found using the method of projections onto the closed convex sets (POCS). This method is presented by Bregman [12], and extended by Gubin et al.[13]. The methods for the extrapolation of bandlimited signals presented by Gerchberg [14], and Papoulis [15], although not stated explicitly, are the first applications of POCS in signal processing. In [16], Lent and Tuy used POCS in tomographic image reconstruction. In the works of Youla and Webb [17], and Sezan and Stark [18] a collection of closed and convex sets to be used in signal restoration applications are discussed. These sets include bandlimited signals, timelimited signals, positive signals, signals with positive Fourier transforms, etc..

A collection of sets describing the feasible solutions can be defined using the residual signal. Clearly, if the original signal is exactly recovered, the residual signal will be identical with the noise. The noise sequence is unknown; however, in most of the applications, many statistics of the noise are available. Thus, the sample statistics of the residual signal should be in accordance with the available noise statistics. The idea of using the residual signal's statistics for defining closed convex sets in

signal space can be used in almost all restoration problems. The POCS method makes it very easy to use various kinds of a priori knowledge in addition to the residual signal's statistics. This method has been successful both in one dimensional and in two dimensional applications [11].

1.2.2. Using Fuzzy Sets for Signal Restoration

Defining the feasible solutions using ordinary sets has a problem. Imprecise or vague information cannot be modelled easily. If the sets constructed using inaccurate information are too small, their intersection may be empty, and if they are too large their effect on the solution will be reduced. The convexity requirement is another factor restricting the information that can be used by the POCS technique. Application of a broader class of a priori information can be made possible by using fuzzy sets. In fact, among all the available methods for modelling a priori information, using fuzzy sets may be considered as the most general one.

In the case of ordinary sets, a signal is either an element of the sets or not. For fuzzy sets each signal has a "grade of membership" to each set and to the intersection set. In the fuzzy sets formulation of the problem, each of the sets is required to contain the original signal with a high grade of membership. Thus, the solution (not

necessarily unique) can be defined as the signal(s) with maximal grade of membership in the intersection set. If this maximal value is large, the solution is in accordance with the specified knowledge; otherwise the a priori specifications are inconsistent.

In POCS the size of the intersection set is a measure for the definition of the solution. If the size is large then there are many solutions satisfying the given knowledge, i.e. the problem is not well defined. In the case of fuzzy sets, the "energy" of the intersection set together with the membership value of the solution gives an improved measure for the definition of the solution. Obviously, if a solution is well defined, its quality will be high.

1.3. Review of the Research Methodology

In this dissertation, a new, general signal restoration method will be developed. This method is based on fuzzy set theory. It can be used in multi-dimensional signal processing applications. The new method is capable of generating high quality restorations in many cases for which conventional techniques have failed. An objective measure for the quality of a restoration will be presented also.

In the second chapter an outline of the POCS method is given. A survey of the necessary definitions and theorems from fuzzy set theory is presented in the Chapter 3. In Chapter 4, the formulation of the signal restoration problem using the fuzzy sets is given. The numerical techniques for finding the solution are discussed in Chapter 5. The definitions and the calculation techniques for the quality of restoration measures are given in Chapter 6. Chapter 7 presents the results obtained by the new method and a general discussion of its advantages and limitations. Possible areas for future work and general conclusions are discussed in Chapter 8.

CHAPTER 2

FEASIBLE SOLUTION IN SIGNAL RESTORATION

2.1. Introduction

In signal restoration, because of the noise and the null space of the degrading operator, it is not possible to recover the original signal with certainity. Therefore, the ultimate goal is to obtain the "best" estimate of the original signal. Conventional restoration methods employ different optimization criteria for selecting the "best" estimate and generally generate different solutions. The main reason for this difference is varying assumptions made by each method about the available a priori information.

In [11], a feasible solution of the restoration problem is defined to be a signal satisfying all the available a priori knowledge about the original signal. If the set of feasible solutions is sufficiently restrictive, any member of it is acceptable as the solution. Otherwise, the solution is not well defined, and further information about the original signal is needed.

Before defining any solution, it should be known how the data signal was formed. In [11] the standard linear model was used. However, it is possible to use more general

and complex models at the cost of additional computation time. The discrete linear signal formation model is given by:

$$\underline{g} = \underline{Hf} + \underline{n} \tag{2.1}$$

where, $\underline{g}$ is the recorded signal of length N; $\underline{f}$ is the original signal to be estimated of length N; $\underline{H}$ is the impulse response matrix (N*N), usually representing convolution; $\underline{n}$ is signal independent noise of length N.

In the most general case, only the data formation model is available, that is, the matrix $\underline{H}$ and some statistics of the noise are known. With this limited knowledge, the feasible solutions can be defined by using the residual signal which is given by:

$$\underline{r} = \underline{g} - \underline{H\hat{f}} \tag{2.2}$$

where $\underline{\hat{f}}$ is an estimate for the original signal. If $\underline{\hat{f}} = \underline{f}$ then the residual is identical with the noise. Thus, a feasible solution should generate a residual signal having statistics consistent with those of the noise. In the following sections, the method used for forcing the statistical constraints on the residual signal is described.

2.2. Residual Signal's Statistics

2.2.1. Guidelines for Selecting the Statistics to be Used

Generally, many statistics of the contaminating noise can be estimated. The most common statistics include : the mean value, the variance, the autocorrelation sequence and the power spectrum. Further, in many cases, the noise sequence is assumed to be Gaussian. Theoretically, all the available statistics should be used in defining the feasible solution. However, in practical applications, computational problems make a selection necessary. The following guideline proved to be useful in the selection process:

i. The statistics to be used should be easy to calculate, because during the iterations they are calculated many times.

ii. There should be an easy and objective way of modifying the data in order to change the statistics. This can be demonstrated through examples. If the histogram of the given data is to be compared with a given pdf; in the case of a discrepancy, it is not easy to identify the part of the data to be changed in order to achieve the required probability density. However, if the mean value is to be used, it can easily be changed by properly biasing the data.

iii. The set of statistics to be used should be nonredundant and sensitive to the relevant changes in data sequence. Otherwise, there will be unnecessary computations.

iv. In order to use the POCS technique, the set of signals satisfying the given statistics should be convex.

2.2.2. A Set of Statistics

The mean value, the variance, the power spectrum, and the maximum deviation from the mean are the statistics used in [11]. They are consistent with the guideline given above, and in many applications, these statistics of the noise sequence are assumed to be known [1],[4]. In the following sections, the definition of the feasible solutions for a Gaussian, zero mean, and white noise sequence with known variance σ_n^2 will be presented. These are common assumptions about the noise. However, for many other noise sequences with different properties, similar definitions can easily be constructed.

2.2.2.1. Mean Value

For a feasible solution, the sample mean of the residual signal is a zero mean, normal random variable with variance σ_n^2/N. The set of signals constructed using this

information is given by:

$$C_m = \{\hat{\underline{f}} \mid \sum_{i=1}^{N} ([\underline{g}]_i - [\underline{H}\hat{\underline{f}}]_i) \mid \leq \delta_m \} \qquad (2.3)$$

where subscript i selects the ith entry of the vectors g and Hf, and δ_m can easily be found from the standard normal distribution tables for a given confidence level. Clearly, C_m is convex.

2.2.2.2. Variance

Under the given assumptions, the sample variance of the residual of a feasible signal has a Chi square distribution. The set of feasible signals considering only the sample variance is approximated by:

$$C_v = \{\hat{\underline{f}} \mid \| \underline{g} - \underline{H}\hat{\underline{f}} \|^2 \leq \delta_v \} \qquad (2.4)$$

The constant δ_v may be found from Chi square distribution tables; however, in many cases the number of samples, N, is large enough for a Gaussian approximation. C_v is convex in this approximate form. For the exact case, the norm of the residual should have a lower limit, resulting in a nonconvex set. However, in practice this approximation does not cause a problem [11].

It should be noted that, the norm of the residual signal has been used as the constraint in the constrained least

squares [4] and the maximum power [19] restoration methods, and as a convergence criterion in iterative restoration techniques [10].

2.2.2.3. Power Spectrum

For Gaussian white noise the distribution of the periodogram has a Chi square distribution with two degrees of freedom [20]. Thus confidence limits for it can be found from standard tables. It is recognized that the periodogram is not a consistent estimator for the power spectrum; however, it is easy to calculate and modify. The convex set of signals defined using the periodogram is:

$$C_p = \{\hat{\underline{f}} \mid |G(k)-H(k)F(k)|^2 \le \delta_p \ , k = 1,2,\ldots,\frac{N}{2}-1 \} \quad (2.5)$$

where the upper case letters represents the DFT of their lower case counterparts. Note that to use the DFT, the matrix H is assumed to be circulant and H(k) represents the kth frequency coefficient of the DFT of its first row. The lower bounds of the periodogram coefficients are neglected for obtaining a convex set.

2.2.2.4. Maximum Deviation from Mean

The individual elements of the residual should have the same probability distribution as the noise. Thus, it is possible to construct a set of signals with acceptable

deviations from the mean value.

$$C_o = \{\hat{\underline{f}} \mid |[\underline{g}]_i - [\underline{H}\hat{\underline{f}}]_i| \le \delta_o \} \tag{2.6}$$

It should be noted that in fact C_o does not represent a single convex set but N convex sets; one for each point in the signal. It is possible to have a single set describing the combined effect of these sets, however, this was found unnecessary [11]. In calculating δ_o, the probability distribution of the maximum deviation from the mean value is considered. For the Gaussian case this is given by:

$$F(x) = K \left[\int_{-x}^{x} \exp[-\frac{1}{2} \frac{(\eta-\mu)^2}{\sigma^2}] \, d\eta \right]^N \tag{2.7}$$

2.3. Convex Sets Dependent on Signal Properties

There are many other constraints which can reasonably be applied to the deconvolution problem. These constraints depend upon the characteristics of the specific signal under consideration. Nonnegativity is one of the most important constraints especially for signals with many values near zero. The set defined by nonnegativity is obviously convex.

A list of useful convex sets can be found in [17]. Some of these are : bandlimited signals, support limited signals, signals whose Fourier transform is positive, and signals which have prescribed values on some region of support.

2.4. Obtaining the Feasible Solution

2.4.1. Projection Onto Convex Sets

In the preceding section, the constraints that can be imposed on a feasible solution are outlined. It remains to show how to produce a solution which satisfies all of these requirements. Individually, each of the constraints may be quite easy to satisfy. However, it is usually very difficult to satisfy all of the constraints simultaneously. The method used for this purpose is the method of successive projections for finding common points of convex sets. As summarized in Chapter 1, this method has been known for almost two decades and has been used in several signal processing applications.

The result of previous work on POCS which is of most interest here is stated as : let $C_0 = \bigcap_i C_i$ be nonempty, where C_i is a closed and convex set in a Hilbert space. Let P_i be the projection operator which projects a vector onto C_i. The iteration given by:

$$\hat{\underline{f}}^{k+1} = P_1 P_2 \ldots P_M \, [\hat{\underline{f}}^{k}] \tag{2.8}$$

converges weakly to a point in C_0. Clearly, convergence is strong for finite dimensional spaces.

This results states that a point in the intersection of the convex sets described in section 2.3.2 can be found using the successive projections. That point is in accordance with all the available a priori knowledge that is modelled as convex sets and thus it is a feasible solution.

2.4.2. Projection Operators

In order to find the projection operator onto a convex set C the following nonlinear, constrained minimization problem should be solved:

$$\min_{\underline{f}_p} \| \underline{f}_p - \underline{f}_0 \|^2 \quad \text{such that } \underline{f}_p \in C \tag{2.9}$$

where $\underline{f}_0$ is the signal to be projected onto C, and $\underline{f}_p$ is the projection. The derivations of the projection operators for the sets modelling the residual signal's statistics are given in [11]. The projections of a signal $\underline{f}_0$ with residual signal $\underline{r}_0$, are as follows:

2.4.2.1. Projection onto C_m

$$\underline{f}_p = \begin{cases} \underline{f}_0 + \dfrac{(\Sigma \underline{r}_{0i} - \delta_m)}{||\underline{h}_c||^2}\, \underline{h}_c & \Sigma \underline{r}_{0i} > \delta_m \\ \underline{f}_0 + \dfrac{(\Sigma \underline{r}_{0i} + \delta_m)}{||\underline{h}_c||^2}\, \underline{h}_c & \Sigma \underline{r}_{0i} < -\delta_m \\ \underline{f}_0 & \text{otherwise} \end{cases} \tag{2.10}$$

where $\underline{h}_c^t = (\Sigma[\underline{H}]_{i1}, \Sigma[\underline{H}]_{i2}, \ldots, \Sigma[\underline{H}]_{iN})$. In the case of circular convolution, h_c is a constant vector.

2.4.2.2. Projection onto C_v

If the norm of the residual signal is larger than δ_v, the projection of the signal is given by:

$$\underline{f}_p = \underline{f}_0 + (\underline{H}^t\underline{H} + 1/\lambda\ I)^{-1}\ \underline{H}^t(\underline{g}-\underline{H}\underline{f}_0) \tag{2.11}$$

where the constant λ is adjusted such that the norm of the projected residual is equal to δ_v. This operation is very similar to constrained least squares deconvolution as pointed out in [11].

2.4.2.3. Projection onto C_p

The projection for the kth discrete frequency is:

$$F_p(k) = \frac{1}{H(k)} \left[G(k) - \sqrt{\delta_p}\ \frac{G(k) - H(k)F_0(k)}{|G(k) - H(k)F_0(k)|} \right] \tag{2.12}$$

for a nonzero $H(k)$ and $|G(k) - H(k)F_0(k)|^2 > \delta_p$. If $H(k)$ is zero, there is nothing to be done. For this case the residual is the exact noise anyway.

2.4.2.4. Projection onto C_o

The projection onto this set is done by considering one point at a time. For the ith point of the signal the projection is:

$$\underline{f}_p = \begin{cases} \underline{f}_0 + \dfrac{(\underline{r}_{0i} - \delta_o)}{||\underline{h}_i||^2}\,\underline{h}_i & \underline{r}_i > \delta_o \\ \underline{f}_0 + \dfrac{(\underline{r}_{0i} + \delta_o)}{||\underline{h}_i||^2}\,\underline{h}_i & \underline{r}_i < -\delta_o \\ \underline{f}_0 & \text{otherwise} \end{cases} \tag{2.13}$$

where $\underline{h}_i$ is the column vector containing the ith row of the matrix $\underline{H}$.

2.5. Properties of the Solution

There are many factors which influence the solution that is obtained. Among the most important are the parameters of the signal formation model equation 2.1, the impulse response, $\underline{H}$, and the noise. The initial estimate $\underline{f}_0$ used to start the iterations, can be quite influential.

It is possible to relate all of the factors effecting the solution to the size of the intersection set. Without being specific about what is meant by size, intuitively, if the size of the intersection set is large, there are many feasible solutions and so, the chances of obtaining a good one are diminished.

Mathematically, the largest distance between any two points of a set can be defined as its size. It is very difficult to determine the size of the intersection set in this sense. However, upper bounds for the sizes of the component sets can be found in terms of the parameters of the model. Certainly, the size of the intersection set is related to that of the component sets, at least it will always be less than or equal to the smallest one.

2.5.1. Effect of Impulse Response

It is intuitive that an increase in the spatial or temporal extent of the impulse response should increase the size of the set of feasible solutions, that is, more vectors should be feasible. The quantitative effect of the extent of the impulse response on the size of the convex sets discussed earlier is studied in detail in [11]. As a summary, the extent of the impulse response is measured by the norm of the matrix $\underline{H}$. In [11], the matrix norm used was:

$$|| \underline{H} || = \max \{\sqrt{\lambda_i}\} \tag{2.14}$$

where λ_i is the ith eigenvalue of $\underline{H}^t\underline{H}$. This is the matrix norm induced by the Euclidean vector norm. If $\underline{H}$ represents circular convolution then these eigenvalues are magnitudes squared of the DFT coefficients of the impulse response.

In [11], it was shown that:

$$\max_{\underline{f}_1, \underline{f}_2 \in C_i} ||\underline{f}_1 - \underline{f}_2|| \leq 2\delta_i \; ||\underline{H}^{-1}|| \tag{2.15}$$

for the convex sets defined using the variance, the power spectrum and the maximum deviation from the mean value of the residual signal. For the convolution case $||\underline{H}^{-1}|| = \frac{1}{\min|H(k)|}$. Thus, the size of the sets depends on the maximum attenuation of the distortion operator. The set defined using the mean value has infinite extent.

In the case of a singular $\underline{H}$, the sizes of the sets defined by the residual statistics are unbounded. This is because of the null space of H, which is of infinite extent. For this case, the incorporation of other a priori knowledge will restrict the set of feasible solutions.

It should be noted that a set with unbounded size is not useless. In fact, most of the sets describing a priori knowledge about the signal, such as nonnegativity, have infinite extent, however they may severely limit the size of

the intersection set. Finally, the amplitude or the energy of a signal can always be restricted using a reasonable limit without loosing generality. The problem of estimating the proportion of the feasible region to this kind of a restricted region will be addressed in Chapter 6.

2.5.2. Effect of Noise

The noise in the linear degradation model, certainly, has a profound effect on the size of the sets. This is because all of the confidence limits, δ's, are functions of the noise. In the case of Gaussian white noise, these limits are proportional to the noise variance. From the bounds given in the previous section, it is seen that, as the severity of noise increases, so does the size of the feasible solution set.

2.5.3. Effect of Initial Estimate

In the case of a severe degradation without enough a priori knowledge, the set of feasible solutions can be very large or unbounded. Yet, the POCS algorithm is capable of producing reasonable solutions. However, the initial estimate becomes very important for this case.

In the successive projections method, each estimate is projected to the closest element in the next convex set. Intuitively, this means that the solution should be "close" to the initial estimate. In fact, if all of the sets are linear varieties, the solution is the closest point of their intersection to the initial estimate.

Although all the sets used in the restoration problem are not linear varieties, in [21], the solution was shown to be affected by the initial estimate. If the estimate starting the iterations is reasonable, the solution to which the method converges is also reasonable. In the example used in [21], the positions of peaks in a simulated x-ray fluorescence spectrum were determined by a restoration using a flat initial estimate; then the magnitudes of the peaks were refined by using an initial estimate which placed isolated peaks at the positions determined from the first restoration.

In the POCS technique, the initial estimate can be used for enforcing certain kind of a priori information such as smoothness or impulsiveness. Another way of using this kind of information is imposing a new set centered around a prototype signal. Such a constraint may be written as:

$$C_i = \{\underline{f} \mid \ ||\underline{f} - \underline{f}_p|| \leq \delta_i \} \qquad (2.16)$$

where $\underline{f}_p$ is a prototype signal which might be obtained by combining a priori information with the result of a preliminary restoration. The determination of the constant δ_i is subjective.

2.5.4. Effect of Other Constraints

Obviously, the solution gets better as more valid constraints are imposed on it. In general, a constraint which defines a small set in the signal space is effective. However, the effectiveness of a constraint set also depends on the characteristics of the original signal. In [11], simulated x-ray spectra and text images were used as examples. These signals have large regions near zero level. This makes the infinite extent nonnegativity constraint a very effective one.

2.6. Results

In order to demonstrate the POCS technique, a simulated x-ray fluorescence signal is used. These types of signals have isolated peaks and large regions near zero and they have been used in several previous studies [6], [10], [19]. The signal given in figure 2.1 is the original signal. The degraded signal is shown in figure 2.2. It is obtained from the original signal by convolving the original signal with a

Gaussian shaped impulse response with standard deviation 2.0, and adding zero mean white Gaussian noise with variance 0.001. The restorations obtained by modified inverse filtering and constrained least squares technique are given in figures 2.3 and 2.4 respectively. In figure 2.5, the restoration obtained by the POCS technique using the sets C_m, C_v, C_p, and C_o is demonstrated. This restoration, certainly, has a better resolution. However, the ringing in the regions where the signal is zero is disturbing. An obvious improvement is obtained through the use of the nonnegativity constraint. In the restoration displayed in figure 2.6 the original signal is almost exactly recovered.

The effects of noise and the extent of the impulse response are demonstrated in figures 2.7 and 2.8 respectively. As discussed earlier, the quality of restoration is higher for lower noise levels and narrow impulse responses.

The use of the initial estimate is demonstrated in figure 2.9. The degraded signal from which figure 2.9.b is obtained, has the same noise level with the one in figure 2.7.b. However, the initial estimate used in figure 2.7.b is the degraded signal. The initial estimate given in figure 2.9.a is constructed using the restoration in figure 2.7.b and the a priori knowledge about the impulsiveness of the original signal. Starting the iterations from this estimate

caused an improvement in the result obtained.

There is no restriction in the dimensionality of the signals to which the POCS technique can be applied. An example of the restoration of a text image is given in figure 2.10. The blurred and noise corrupted image is given in figure 2.10.a. figure 2.10.b shows the result without nonnegativity constraint, and the restoration obtained with nonnegativity is given in figure 2.10.c.

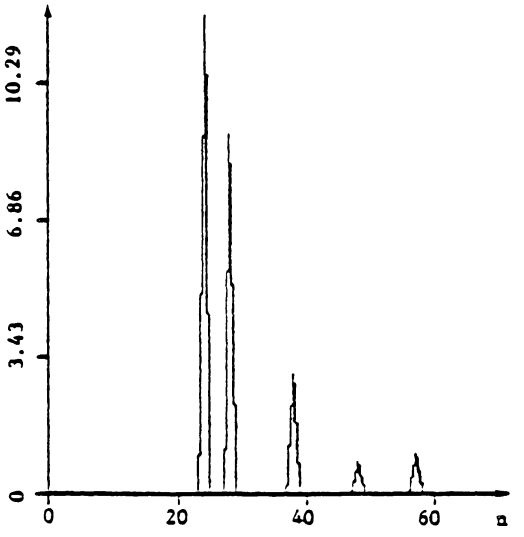

Figure 2.1 Original signal

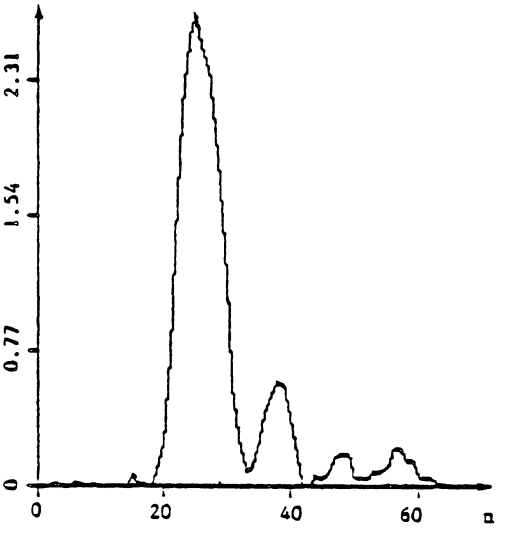

Figure 2.2 Degraded Signal

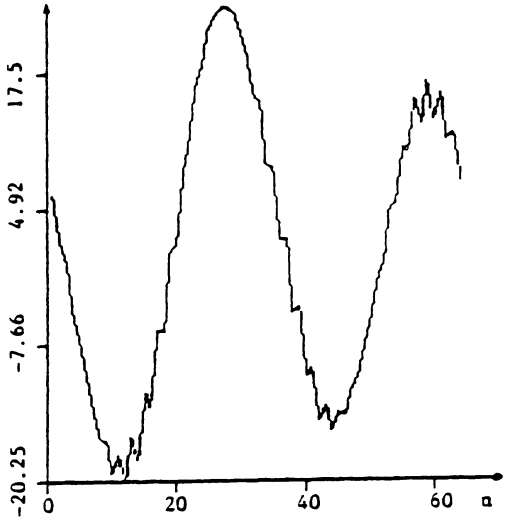

Figure 2.3 Restoration by Inverse Filter

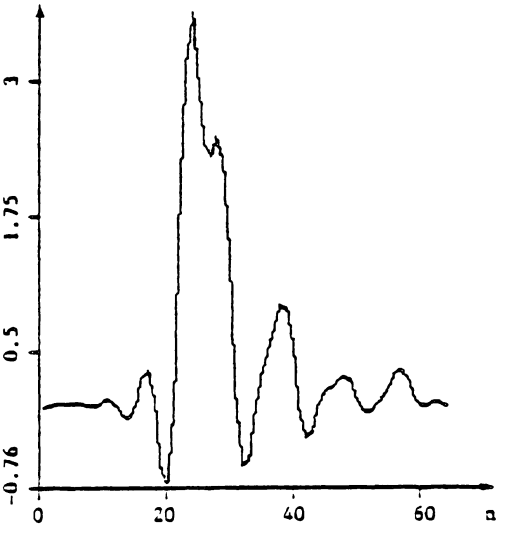

Figure 2.4 Restoration by Constrained Least Squares Method

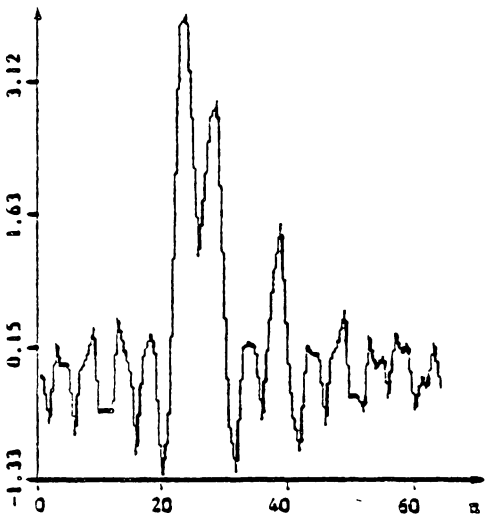

Figure 2.5 Restoration Using Residual Signal's Statistics

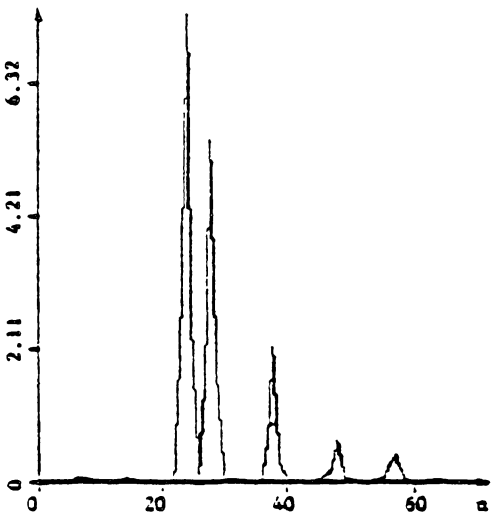

Figure 2.6 Restoration Using Residual Signal's Statistics and Nonnegativity

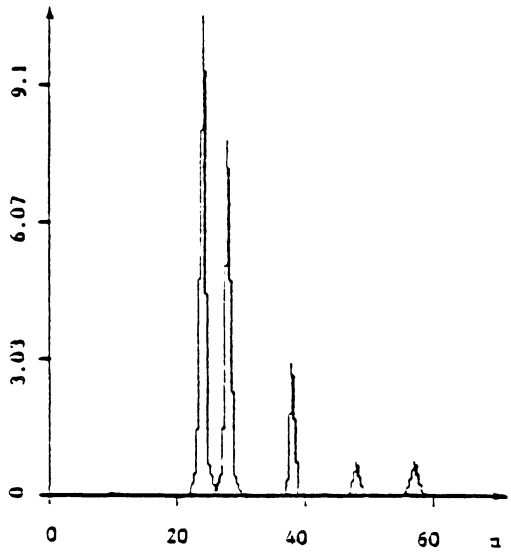

Figure 2.7.a Restoration when noise variance is 0.0001

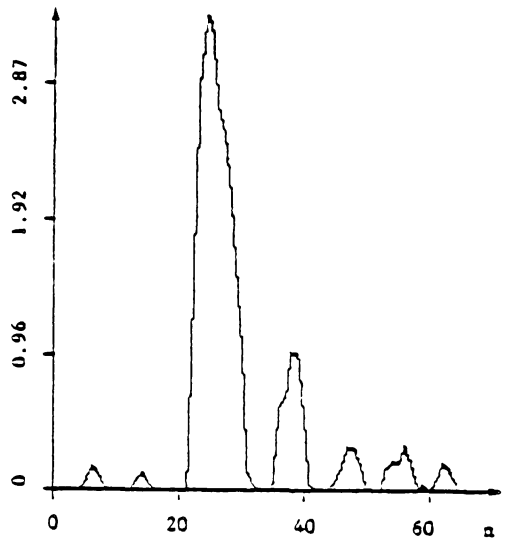

Figure 2.7.b Restoration when noise variance is 0.01

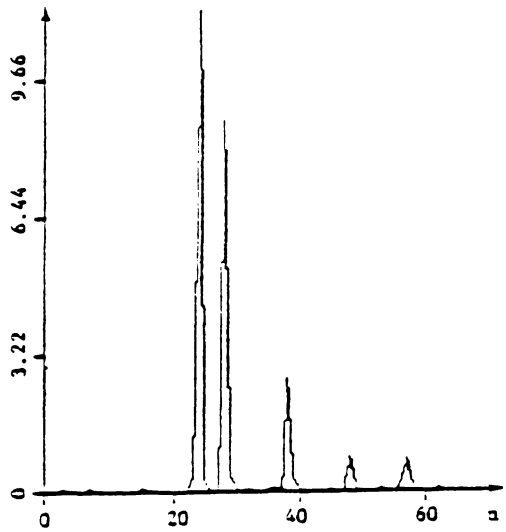

Figure 2.8.a Restoration when variance of psf is 1.

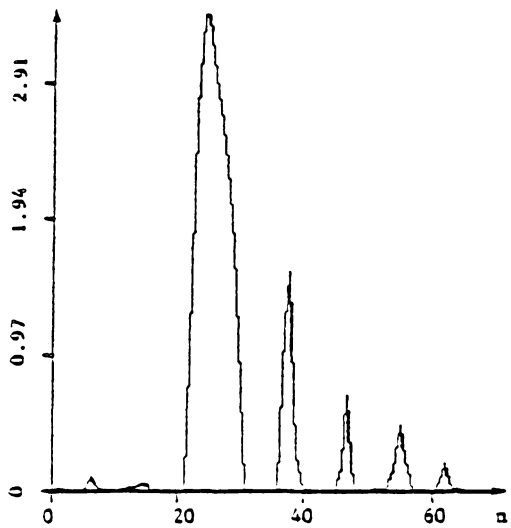

Figure 2.8.b Restoration when variance of psf is 8.

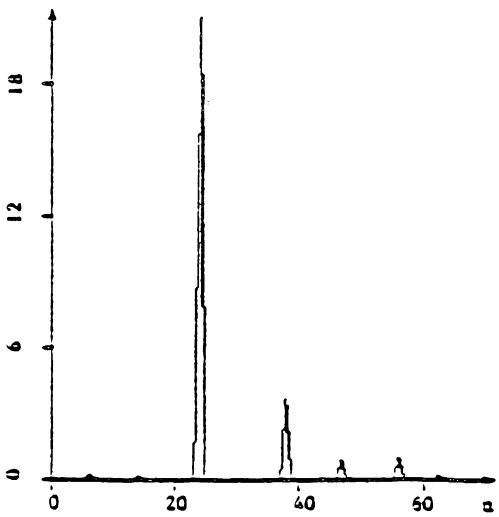

Figure 2.9.a Initial Estimate for the restoration.

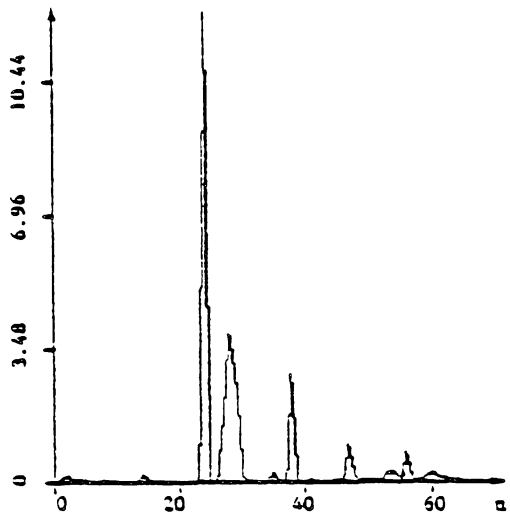

Figure 2.9.b Restoration Obtained using the initial estimate in Fig. 2.9.a

Figure 2.10.a. (Upper left) Degraded Signal, b. (Upper right) Restoration by using residual signal's statistics, c. (Lower) Restoration by using residual signal's statistics and nonnegativity.

2.7. Discussions

The described method is a recent formulation in signal restoration. It is capable of generating high quality restorations through the use of a large class of a priori information. The results can be obtained in reasonable number of iterations.

The idea of matching the statistics of the noise with those of the residual signal is very important in defining the solution. This is a general approach and can be used in finding objective constraints on the solutions of many estimation problems.

The POCS method can utilize any information that can be used to define a convex set of signals in the signal space. However, there may be information which cannot be modelled as a convex set of signals or even a set of signals. Imprecise information about the original signal is an example for this. Using fuzzy sets instead of the ordinary ones will extend the formulation capability of the method.

CHAPTER 3

A REVIEW OF THE FUZZY SET THEORY

3.1. Introduction

Many useful classifications of objects in the everyday life are vaguely defined. The set of men over 6 feet in height is well defined. A man is either a member of this set or not. However, it is not the same for the set of tall men. There is no fixed height separating tall men from the others. The transition is gradual. Still the classification of tall is useful. The tall men's stores cater to this vaguely defined set of people. It is the purpose of fuzzy set theory to give us a mathematical tool with which to manipulate this partially defined knowledge.

Imprecise information is common in signal restoration. Consider the following examples. Physical filters have transition regions, thus bandwidth of a filtered signal is not exact (unless we artificially assume so). For a lossless system the energy of the output signal, in the sense of mean squared signal value, is equal to that of the input signal, however because of noise we cannot estimate it accurately. The covariance structure of similar images are similar; however, they are not equal. If the original signal is known to

be "impulsive", we can judge a restoration using this knowledge; however, it is not easy to incorporate this knowledge in the restoration algorithm.

The requirements for the statistics of the residual signal, which are described in section 2.2.2, can be formulated better using fuzzy sets instead of ordinary sets of the POCS method. The following example may help in demonstrating this. Let the noise in the linear degradation model, equation 2.1, be zero mean, white, and normal, and let the sample mean of a residual signal be m . If the noise and the residual are the same, m is a normal random variable with zero mean and known variance σ^2. For the application of POCS technique, it is necessary to set limits for acceptable m 's. Practically, it is obvious that if $|m| = \sigma$, it is acceptable; and if $|m| = 100\sigma$, it is not. However, for $|m| = 2.5\sigma$ the decision is not easy. A limit for the convex set can be set using some level of confidence. Let that limit be 2σ corresponding to approximately 95% confidence. Under this limit, a residual signal with $m_1 = 1.9999\sigma$ will be considered acceptable, i.e. a member of the set, while another one with $m_2 = 2.00001\sigma$ will be rejected. However, the belief in the consistency of the data does not change abruptly. That is, there is no natural limit separating the acceptable and unacceptable m's, that transition occurs in a

region. Thus, C_m of section 2.2.2.1 has a vague boundary and a fuzzy version of it will be more appropriate.

The theory of fuzzy sets was founded by Zadeh [22] in 1964. There have been a large number of studies on this subject and it has been applied in many areas including automatic control and operations research. In [23] and [24] lists of references covering almost all of these works are included. This chapter summarizes the background material from fuzzy set theory which is used in the signal restoration problem.

3.2. Definitions

The characteristic (indicator) function $\mu_A : X \longrightarrow \{0,1\}$ of an ordinary (non fuzzy) set A in a universe X is defined as:

$$\mu_A(x) = \begin{cases} 1 & \text{iff} \quad x \in A \\ 0 & \text{iff} \quad x \notin A \end{cases} \tag{3.1}$$

where $\{0,1\}$ is called a valuation set. If the valuation set is the real interval $[0,1]$ then A is called a fuzzy set [24]. $\mu_A(x)$ is the grade of membership of x in A, and $\mu_A(.)$ is called the membership function of the fuzzy set A. A fuzzy set is completely specified by its membership func-

tion. Qualitatively, the membership function describes the strength of our belief that x is a member of A. If $\mu_A(x)=1$, it is certain that x is in A; and if $\mu_A(x)=0$, it is certain that x is not in A. Thus, a fuzzy set can be considered as a subset of X with vague boundaries.

Two fuzzy sets A and B are equal iff $\mu_A(x)=\mu_B(x)$ for all x in the universe X, and A is a subset of B iff $\mu_A(x)\leq\mu_B(x)$ for all x s in X. The set of x s for which $\mu_A(x) > 0$ is called the support of the fuzzy set A. The height of a fuzzy set is defined as:

$$\mathrm{hgt}(A) = \sup_{x\in X} \mu_A(x) \tag{3.2}$$

and finally, a fuzzy set is normalized if its height is unity.

The membership function is the key in fuzzy set theory. General techniques for finding membership functions have not been developed. A guideline for finding the membership functions for fuzzy sets to be used in signal restoration problem will be discussed in Chapter 4.

3.3. Fuzzy Set Algebra

In the scope of this dissertation, fuzzy sets are used to enforce several constraints in signal restoration, thus it is necessary to deal with several fuzzy sets. The

ordinary set theoretic operations, union, intersection and complement are extended to fuzzy sets. However, there is more than one set of definitions. The most common set of definitions is given as:

$$\begin{aligned} &\text{intersection} : \quad \mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x)) \\ &\text{union} : \quad \mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)) \\ &\text{complement} : \quad \mu_A(x) = 1 - \mu_A(x) \end{aligned} \tag{3.3}$$

It is noted that, under these definitions, the structure of the fuzzy subsets of the universe X is a pseudocomplemented distributive lattice, where max and min operators are the upper and lower bound operators of the lattice respectively [24].

A different set of definitions is the probabilisticlike operators. These are given as:

$$\begin{aligned} &\text{intersection} : \quad \mu_{A\otimes B}(x) = \mu_A(x)\mu_B(x) \\ &\text{union} : \quad \mu_{A\oplus B}(x) = \mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x) \end{aligned} \tag{3.4}$$

These operators generate a nondistributive lattice structure and they are interactive. The complementation given in equation 3.3 is not genuine for these definitions. The so called bold union and bold intersection generate a complemented nondistributive structure.

These operators are:

$$\begin{aligned} &\text{intersection} : \quad \mu_{A \sigma B}(x) = \max\,(0,\ \mu_A(x)+\mu_B(x)-1) \\ &\text{union} \quad : \quad \mu_{A \sigma B}(x) = \min\,(1,\ \mu_A(x)+\mu_B(x)) \end{aligned} \tag{3.5}$$

It should be noted that the intersection operators mentioned above are triangular norms [24]. A triangular norm is a function from [0,1]X[0,1] to [0,1] with the following properties :

i. $T(0,0)=0;\ T(a,1)=T(1,a)=a;$

ii. $T(a,b) \le T(c,d)$ whenever $a \le c$ and $c \le d$;

iii. $T(a,b)=T(b,a);$

iv. $T(T(a,b),c) = T(a,T(b,c))$

Furthermore, every triangular norm satisfies:

$$T_w(a,b) \le T(a,b) \le \min(a,b) \tag{3.6}$$

where

$$T_w(a,b) = \begin{cases} a & \text{if} \quad b=1 \\ b & \text{if} \quad a=1 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

The properties of triangular norms may be used for defining other intersection operators.

The best set of operators depends on the problem, and specifically what is meant by the words intersection, union and complement. The selection of the suitable set of operators for the restoration problem is discussed in the next chapter.

3.4. Measures for Fuzzy Sets

A mathematical way of "defining" an acceptable solution is restricting the set of solutions by using the available a priori information. If the definition is perfect, there should be a unique signal satisfying it. In other words, the intersection set should be a singleton. If the solution is not well defined, there will be many signals which can be accepted as the solution, and if the definitions are inconsistent, the result will have a low membership value. Thus, the "size" and the "height" of the intersection set can be used for measuring the quality of the definition. If the solution is well defined, clearly, its quality will be high.

Two measures, energy and entropy, have been proposed for fuzzy sets [25]. In [26], the energy measure is shown to be better suited for decision theoretic applications. The "size" of a fuzzy set, in the sense used here, is strongly related to the energy measure.

An energy measure for a fuzzy set A is defined as [26]:

$$e(A) = \int_X \varphi(\mu_A(x))\, dw(x) \tag{3.8}$$

where, φ is nondecreasing on [0,1]; $\varphi(x) = 0$ iff $x=0$; and w is a totally finite positive measure defined on a σ-algebra of subsets of X. (The finite subset X of the signal space can be constructed large enough to contain every possible solution without causing a loss of generality.) The main properties of this measure are :

I. e (A) is defined only for measurable fuzzy sets.

II. If $A_1 \subset A_2$ then $e(A_1) \leq e(A_2)$.

III. e (A)= 0 iff A is equivalent to empty set, or μ_A (x) vanishes almost everywhere.

IV. If φ is strictly increasing in [0,1], e (A) reaches its maximum value when A=X.

Two forms for this measure are given in equations 3.9 and 3.10:

$$e(A) = \frac{1}{hgt(A)} \int \mu_A(x)dx \tag{3.9}$$

$$e(A) = \frac{1}{hgt^2(A)} \int \mu_A^2(x)dx \tag{3.10}$$

Clearly, for a "small" set the energy measure is close to zero. However, in order to have a meaningful measure for

the size, the height and energy of the set under consideration should be specified together.

CHAPTER 4

APPLICATION OF FUZZY SET THEORY TO SIGNAL RESTORATION

4.1. Formulation Of The Problem

The signal restoration problem can be formulated as finding an element of the intersection of various sets describing a priori available properties of the original signal. As discussed in Chapter 2, this makes it possible to use a large variety of information in the restoration algorithm. A major problem encountered in this formulation is the modelling of partially defined or imprecise information which occurs in many practical applications. A solution of this problem is obtained by the use of fuzzy sets instead of ordinary sets.

To be useful in defining the solution, the ordinary set must contain the original signal. To guarantee this may require large sets, decreasing their effect on the solution. In the fuzzy sets formulation, this requirement is softened. Any fuzzy set which assigns a high membership value to the original signal can be added to the collection of the sets defining the solution. If all fuzzy sets used are expected to contain the original signal with a high membership value, then so is their intersection. Thus, a reasonable estimate

for the original signal is one of the members of the fuzzy intersection set with a high membership value.

There are three parts of the fuzzy sets formulation. The first one is finding an objective and methodical way for the construction of membership functions. Deciding on the intersection operator is the second part. Finally, a computational technique must be developed for finding a solution.

4.2. Membership Functions

The major problem with using fuzzy sets is defining the membership functions in an objective and systematic way. There is no general technique for the solution of this problem. In the literature, different techniques are suggested for various applications [24].

4.2.1. Background

Among the published methods for the construction of membership functions, the ones which are obviously applicable to the signal restoration problem are listed below.

4.2.1.1. Deformable Prototypes [27]

In this method, in order to construct the membership function, a prototype which is defined by n parameters is used. These parameters are adjusted for minimizing some distance measure between the prototype and the object for which

the membership value is to be calculated. The membership value is a function of the "dissimilarity" between the prototype and the object. In [27], the dissimilarity is defined as:

$$D(x) = \min_{p_1,\ldots,p_n} (m(x;p_1,\ldots,p_n) + w\delta(p_1,\ldots,p_n)) \quad (4.1)$$

where, m is the distance between the object x and the prototype; δ is a function describing the deformation energy, in other words the degree of distortion made on the prototype; and w is a weighting factor. A membership function for the fuzzy set described by the prototype may be given as:

$$\mu_p(x) = 1 - (D(x)/\sup D) \quad (4.2)$$

This method has been used for defining membership functions of fuzzy sets describing handwritten characters and certain classes of ECG signals [27]. In signal restoration, it is possible to use this technique for constructing fuzzy sets around prototype signals described in Chapter 2. Another important application is modelling of a signal which is known to be the output of a system defined by a set of parameters. As an example, consider a speech signal. The coefficients of a linear predictive coder can be used as the parameters of the prototype. The error made by coding the signal with specified number of coefficients is the distance

part of the dissimilarity. The deformation energy is related to the difference between the prototype's coefficients and the coefficients used for coding the signal. The fuzzy set constructed this way can be used for incorporating information such as "voiced speech", or "speech of a certain speaker" in a restoration algorithm.

4.2.1.2. Implicit Analytical Definition [28]

The main idea behind this method can be stated as : the marginal increase of a person's strength of belief that "x is A" is proportional to his belief that "x is A" and his belief that "x is not A". Mathematically this can be formulated as:

$$\frac{d\mu_A(x)}{dx} = k\mu_A(x)(1 - \mu_A(x)) \tag{4.3}$$

The solution of this differential equation is:

$$\mu_A(x) = \frac{1}{(1 + e^{a-bx})} \tag{4.4}$$

The parameters a and b are to be determined from other requirements. This method gives a justification for the shape of a membership function.

4.2.1.3. Use of Statistics

As in the case of finding probabilities, it is reasonable to rely on statistics in finding the membership func-

tions. This idea is applied in deriving membership functions for fuzzy sets to be used in linguistic applications [29], and in social sciences [30]. In [24], normalization of a histogram to make the highest ordinate equal to one is proposed for obtaining a possibility distribution. Considering the fuzzy sets defined on the real line, if a histogram or probability density function for the elements of a certain fuzzy set is available, assigning high membership values to those portions of the real line corresponding to high frequencies is reasonable. However, this does not justify using a function proportional to the histogram as a membership function. This problem is studied in the next section.

4.2.2. A Guideline for Constructing Membership Functions

4.2.2.1. Constructing Membership Functions Using Statistical Information

In constructing a fuzzy set defining the solution of a restoration problem, a parameter of the original signal or the degradation system, such as the bandlimit of the signal or a statistic of the contaminating noise, is used. As stated before, many times the exact value of the parameter is not available. However, information in the form of a his-

togram or a probability density function of the parameter can be obtained. The following conditions are found to be reasonable in constructing membership functions using this kind of information : let s be the parameter, such as the sample mean of the residual signal, to be used for the construction of a fuzzy constraint set, then;

I. $E\{\mu(\ s\ )|\ s$ is calculated for the original signal $\} \geq c$

where, c should be close to 1. Qualitatively, this requirement forces the average value of the membership of the original signal in this fuzzy set to be larger than c. It should be noted that, the original signal is not needed for the calculation of the required expected value. Using the sample mean example, this condition requires:

$$\int \mu(m)\ p(m \mid \hat{\underline{f}}=\underline{f})\ dm \geq c \qquad (4.5)$$

where m is the sample mean of the residual signal and the conditional pdf p is the density function of a zero mean normal random variable with variance σ^2 for the case described in section 3.1. If the proposed solution f is not equal to the original signal f, the pdf of the sample mean may not be p because the residual signal will not be equal to the noise unless the difference between the proposed solution and the original signal lies entirely in the null space of $\underline{H}$.

II. $0 \le \mu(x) \le 1$. Because, as far as the specific parameter used in defining the fuzzy set is concerned, there may be signals which are totally unacceptable as a solution ($\mu(x)=0$), signals which are indistinguishable from the solution ($\mu(x)=1$), and signals which are in between.

III. $\int \mu^2(x)\,dx$ should be minimized. This condition is required for obtaining a "selective" membership function, that is, the grade for signals which differ from the original signal should be as low as possible. The integral of the squared membership function is related to the size of a fuzzy set. Thus by minimizing it, the "smallest" set satisfying the other requirements can be obtained.

The optimal membership function defined by these conditions can be derived using constrained optimization techniques for infinite dimensional spaces [31]. This derivation is presented in Appendix 10.1. The optimal membership function is shown to be:

$$\mu(x) = \begin{cases} \lambda p(x) & \text{if} \quad \lambda p(x) < 1 \\ 1 & \text{if} \quad \lambda p(x) \ge 1 \end{cases} \tag{4.6}$$

where, $p(x)$ is the pdf or its estimate derived from the histogram of the parameter used for defining the fuzzy set, and the constant λ is to be solved from:

$$\lambda \int_{\lambda p(x)<1} p^2(\eta)d\eta + \int_{\lambda p(x)\geq 1} p(\eta)d\eta - c = 0 \qquad (4.7)$$

For standard normal pdf, the relation between the parameter λ and the confidence level is given in Appendix 10.1. For a specific pdf, λ can be solved from equation 4.7 using numeric root finding techniques, such as bisection or Newton-Raphson. A numeric integration is also necessary if a closed form for the indefinite integral of the pdf or its square does not exist.

It should be noted that, the optimal membership function depends on the size measure used in the third requirement. For example, if the integral of the membership function is used as the measure, the optimal solution is an ordinary set which is nothing but a classical confidence interval having the parameter "c" of the first condition as the confidence level. In other words, the sets similar to the ones in Chapter 2 are optimal if the integral of the membership function is minimized. This is discussed in Appendix 10.1. However, as explained in section 3.1, there is an abrupt change from nonmembership to full membership in these sets which is not realistic for many cases.

Clearly, it is not possible to prove the necessity or sufficiency of the proposed set of rules for constructing membership functions using statistical information. However,

they are reasonable requirements, and they generate reasonable membership functions.

4.2.2.2. Constructing Membership Functions When Statistics Are Not Available

If probabilistic knowledge is not available, the membership function should be defined using the available data so that its value for the original signal is close to one and it is as selective as possible. For example; if the original signal is known to be low pass filtered using a filter with a finite transition region, all signals bandlimited to the union of the passband and the transition band of this filter should have high membership in the set describing this information. However, lower membership values should be assigned to the signals which have their significant components only in the transition band.

Using a deformable prototype may help in constructing membership functions without using statistical data. Examples for this approach were given in section 4.2.1.1.

4.3. Fuzzy Sets For Signal Restoration Applications

In this section, membership functions for a collection of fuzzy sets which have potential in signal processing

applications will be presented. These membership functions are constructed following the guideline presented in sections 4.2.2.

4.3.1. Sets Derived From Residual Signal's Statistics

As in Chapter 2, in defining these sets, the underlying signal degradation model is assumed to be linear with additive zero mean, normal, and white noise with variance σ^2. Nonlinear degradation models with different noise characteristics can be modelled similarly at the cost of increased complexity.

4.3.1.1. Mean Value

The sample mean of a feasible residual signal is a normal random variable (r.v.) with zero mean and variance σ^2/N. Thus, a fuzzy set can be defined using the parameter

$$p_m = \frac{1}{\sigma\sqrt{N}} \Sigma \, [\underline{r}]_i \tag{4.8}$$

where r is the residual signal, and p_m is a standard normal r.v.. The membership function can easily be constructed for any confidence level, and is given as:

$$\mu_m(p_m) = \begin{cases} 1 & |p_m| \le a(c) \\ \frac{\lambda(c)}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}p_m^2} & \text{otherwise} \end{cases} \tag{4.9}$$

where; c is the confidence level , and a(c) and $\lambda(c)$ are the optimal parameters for normal pdf's, as described in Appendix 10.1.

4.3.1.2. Variance

The sum of the squares of a feasible residual signal divided by the noise variance has a chi square distribution with degrees of freedom N. However, for most of the applications the number of samples is large enough, for making a normal approximation. The parameter used in defining this set is:

$$p_v = \sqrt{2\,\frac{\Sigma[r]_i^2}{\sigma^2}} - \sqrt{2N-1} \tag{4.10}$$

The distribution of this parameter is approximately standard normal [32]. Thus, the membership function can be obtained from μ_m , by replacing p_m s with p_vs.

4.3.1.3. Power Spectrum

For discrete frequencies between 1 and (N/2 - 1), the magnitude squared DFT coefficients of a feasible residual signal have a chi square distribution with two degrees of freedom [20]. Thus, the pdf of the maximum magnitude square in these frequencies can be obtained as follows: let,

$$p_{pk} = \frac{2|R(k)|^2}{N\sigma^2} \tag{4.11}$$

where, R(k) is the kth discrete frequency component of the residual signal. The parameter p_{pk} has an exponential distribution with parameter 0.5 for $0 < k < N/2$. If p_{pm} denotes the maximum p_{pk}, then its probability distribution function can be calculated as:

$$\begin{aligned} F_{pm}(x) &= P\{p_{pm} \le x\} = P\{p_{pk} \le x, \forall k\} \\ &= \left[\int_0^x \frac{1}{2} e^{-\frac{\eta}{2}} d\eta\right]^{N'} = (1 - e^{-\frac{x}{2}})^{N'} \end{aligned} \tag{4.12}$$

where $N' = N/2 - 1$. The pdf can be obtained by differentiating equation 4.12 and is given by:

$$f_{pm}(x) = \frac{N'}{2}(1 - e^{-\frac{x}{2}})^{(N'-1)} e^{-\frac{x}{2}} \tag{4.13}$$

This is a unimodal pdf, hence corresponding membership function has the following form:

$$\mu_{pm}(x) = \begin{cases} 1 & \text{if} \quad a < x < b \\ \lambda f_{pm}(x) & \text{otherwise} \end{cases} \tag{4.14}$$

where the parameters a, b and λ are N dependent and to be found numerically. In Appendix 10.1, the values for the parameters a and b are tabulated for various confidence levels and numbers of samples.

4.3.1.4. Maximum Deviation From the Mean

Individual entries of the residual signal have normal distribution. The derivation of the pdf of the maximum value of the residual signal is similar to the derivation given in section 4.3.1.3., and it is given by:

$$f_o(x) = N\sqrt{\frac{2}{\pi}} \left[1 - 2Q(x)\right]^{(N-1)} e^{-\frac{x^2}{2}} \tag{4.15}$$

where Q(x) is the probability of having a standard normal random variable larger than x. The function $f_o(x)$ is a unimodal pdf, thus the form of the corresponding membership function is the same as μ_{pm} . The parameters of the membership function are dependent on N, and must be found numerically. A table of the a and b values is given in Appendix 10.1.

4.3.2. Fuzzy Sets Defined Using Signal's Statistics

In many applications, in addition to the noise statistics, various statistics of the original signal can be estimated also. There are restoration techniques based on this kind of knowledge. The power spectrum equalization technique [33] assumes the availability of the original signal's power spectrum. In the Wiener filter the covariance of the signal is used. Maximum a posteriori probability technique assumes the availability of a pdf for the original signal [34].

Obviously, fuzzy sets describing this information can easily be constructed. The membership functions derived for the sets defined by the residual signal's statistics can be used for the sets describing the same statistics of the original signal. Using the technique for deriving membership functions from statistical information, fuzzy sets defined by other statistics of the original signal can easily be constructed.

4.3.3. Fuzzy Sets Defined Using Signal Specifications

The following sets can be used for modelling various properties of signals which are encountered in many signal processing applications.

4.3.3.1. Energy of the Signal

In several applications, it is possible to estimate the energy of the original signal. In photography the average light intensity can be measured. Even if this measurement is not available, for many applications, a measurement done on a similar view can give a "close" value to the original one. Similarly, in speech processing, a knowledge about the average signal energy can be obtained by measuring the energy level of a similar speech signal. Imposing a bound on the maximum energy of the original signal is very useful in restricting the size of the feasibility set but, as can be seen from the examples above, generally, the upper bound of the energy is not known exactly. However, a fuzzy set can be used for introducing a soft upper bound. The membership function for this set may be of the form:

$$\mu_{eu}(\underline{f}) = e^{-\alpha_{eu}\underline{f}^t\underline{f}} \tag{4.16}$$

In addition to the upper bound, a soft lower bound can be imposed on the energy of the signal for restricting the size of the feasibility set. A soft lower bound can be obtained by considering, the attenuating nature of a general degradation system and and the energy of the output signal.

Another use of a lower bound on the energy is in defining impulsiveness which is a rather vague concept. In [19], maximizing the energy of the restored signal is shown to be useful in impulsive signal restoration. The reason for this is the relatively high autocorrelation of these signals at zero lag. This is shown in Appendix 10.2.

A soft lower bound on the energy of a signal can be modelled by a fuzzy set with a parametric membership function of the form:

$$\mu_{el}(\underline{f}) = 1 - e^{-\alpha_{el}\,\underline{f}^t\underline{f}} \tag{4.17}$$

It should be noted that this set cannot be approximated by an ordinary convex set.

4.3.3.2. Prototype Signals

It is possible to construct a fuzzy set around a signal which is expected to be similar to the original signal. There are two ways of obtaining a prototype signal. The first one is using the result of a prerestoration. In prerestoration, a conventional restoration technique may be used. It is observed that simple and fast techniques, such as thresholding for finite level signals, are capable of generating useful prototypes. The second method for finding

a prototype signal is using the available information about a property of the original signal. For example, if the signal is a text image, it is possible to estimate many letters using the meaning of the text.

A parametric membership function for a fuzzy set around a prototype signal can be:

$$\mu_{pr}(\underline{f}) = \exp(-\sum_{i=1}^{N} \alpha_i([\underline{f}]_i - [\underline{f}_p]_i)^2 \qquad (4.18)$$

where $\underline{f}_p$ is the prototype signal and α coefficients reflect the confidence in the prototype.

4.3.3.3. Smoothness

If the original signal is known to be smooth, a fuzzy set can be constructed using the approximate time or space derivative of the signal. In constrained least squares method [4], the original signal is assumed to be as smooth as possible. In the fuzzy set formulation, an adjustable weight can be assigned to the smoothness of the signal. A parametric membership function for this set may be:

$$\mu_s(\underline{f}) = e^{-\alpha_s \underline{f}^t \underline{D}^t \underline{D} \underline{f}} \qquad (4.19)$$

where $\underline{D}$ is a difference operator and α_s is for adjusting the required smoothness.

4.3.3.4. Finite Level Signals

There are many cases for which the original signal is known to have only a finite number of levels. An example for this can be given from industrial tomography. If the sample to be scanned is known to be made of certain kind of materials, then in the reconstructed image, there will be finite number of gray levels.

A membership function for a fuzzy set modelling the finite level signals may be:

$$\mu_{fl}(\underline{f}) = \prod_{i=1}^{N} \sum_{j=1}^{M} \beta_j \exp(-\alpha_j ([\underline{f}]_i - [\underline{flv}]_j)^2) \qquad (4.20)$$

where $\underline{flv}$ is a vector containing m possible finite levels, and α and β parameters can be adjusted using the relative amount of the number of levels.

4.3.3.5. Time, Amplitude and Bandlimits

Similar to the ordinary sets case, the knowledge about the time or space extent, the bandwith, and the amplitude range of the original signal can be used for defining fuzzy sets. However, fuzzy sets have the advantage of being able to model imprecise limits. The best example for this is the bandlimit of a lowpass filtered signal which is outlined in

section 4.2.2.2.

4.4. Selecting the Intersection Operator

A broad range of information about the solution of a restoration problem can be modelled using fuzzy sets. After the construction of these sets, the next step is finding the set of feasible solutions. In the case of ordinary sets, a feasible solution must be a member of all sets used in defining the solution, and any signal which is a member of all these sets must be a feasible solution. Therefore, the feasibility set is the largest common subset or the intersection of the sets defining the solution.

In the case of fuzzy sets, the largest common subset of a class of sets is their intersection defined using the minimum operator given in equation 3.3. In other words, straightforward extension of the ordinary sets formulation suggests using the minimum as the intersection operator. However, it is not necessary to construct the fuzzy sets formulation as an extension of the ordinary sets case, and, as outlined in section 3.3, there are many operators that can be used for defining the intersection of fuzzy sets.

One of the ways in finding a proper intersection operator for a specific application is defining requirements for

the intersection and then finding an operator satisfying them. In this formulation, the defining properties of the triangular norms, given in section 3.3, are taken as the smallest set of requirements for the intersection operator. This restricts the set of operators, but does not define a unique intersection. As indicated in section 3.3, most of the popular intersection operators including the minimum and the product are triangular norms.

In fact, if a feasible solution is defined as one which has a high grade of membership in the intersection set, the selection of a particular triangular norm will not have a major effect on the solution. However, the computation of the solution will be different for different intersection operators. This subject is addressed in Chapter 5. Another aspect of the problem for which the selection of the intersection operator plays an important role, is the calculation of the size of the feasibility set. The same classes of sets may have different sizes of intersection for different intersection operators. However, the meanings of these measures are different. This may be used to obtain a better definition for the shape of the feasibility set by comparing its sizes calculated using different intersection operators.

CHAPTER 5

COMPUTATIONAL METHODS

5.1. Using the POCS Method

5.1.1. Convex Fuzzy Sets

5.1.1.1. Definitions

The first step in extending the convex ordinary sets formulation is using convex fuzzy sets. A fuzzy set is said to be convex iff its membership function $\mu(x)$ is quasiconcave, that is, it satisfies:

$$\mu[\lambda x_1 + (1-\lambda)x_2] \geq \text{Min}\ [\mu(x_1),\mu(x_2)] \qquad (5.1)$$

for all x_1 and x_2 in the support of the fuzzy set, and all λ in [0,1] [22]. It is shown [22] that, an equivalent definition for convexity can be given using the α-level sets of a fuzzy set. An α-level set of a fuzzy set A with a membership function $\mu_A(x)$ is an ordinary set which is defined [22] as:

$$A_\alpha = \{x:\ \mu_A(x) \geq \alpha\ \} \qquad (5.2)$$

A fuzzy set is convex iff its α-level sets are convex for all α values in [0,1] [22].

The second form of the definition suggests that, if all fuzzy sets used in defining the solution of a signal restoration problem are convex, then their α-level sets form a collection of ordinary convex sets. A signal in the intersection of these level sets can be found using the POCS technique, and such a signal will have a grade of membership larger than or equal to α in every individual fuzzy set. Obviously, this signal's grade of membership in the fuzzy intersection set depends on the intersection operator. If the minimum operator is used for the intersection, the grade of membership of this signal in the intersection set is at least equal to α. For all other triangular norms, the grade will be smaller as indicated in section 3.3. However, if there exist signals with a specified grade of membership in the intersection set defined by a particular operator, it is possible to find one of them by using POCS and adjusting the parameter α according to the intersection operator.

5.1.1.2. Convex Fuzzy Sets For Signal Restoration

Obviously, the requirement for using convex fuzzy sets restricts the modelling capability of the fuzzy sets technique. However, as indicated in Chapter 2, many restoration problems can be successfully modelled using ordinary convex sets. Using convex fuzzy sets may be considered as an

intermediate method, extending the modelling capability of the ordinary POCS method.

Among the fuzzy sets defined in section 4.3.1, only the set constructed using the mean value of the residual signal is a convex fuzzy set. The fuzzy sets defined using the variance, the power spectrum, and the maximum deviation from the mean value of the residual signal are not convex. However, they can be approximated by using the smallest convex fuzzy sets which contain them. The experience obtained in applying the POCS technique shows that such an approximation will not have an adverse effect on the solution.

Most of the fuzzy sets defined using the signal specifications, which are given in section 4.3.3, are convex fuzzy sets. The exceptions are the fuzzy set defined using the minimum energy of a signal and the set modelling the finite level signals.

5.1.2. Successive Relaxation Technique

The convex fuzzy sets formulation introduces a successive relaxation algorithm for the POCS technique. As was discussed in [11], outliers can make the intersection of the ordinary sets empty. The proposed solution for this problem was enlarging the individual sets [11]. However, the amount

of the relaxation, and its relative effects on the confidence in the solution were not determined. The membership functions of the fuzzy sets give a relation between the sizes of the sets and the confidence in the solution obtained by using them. This makes it possible to adjust the sizes of the sets so that all sets describe the original signal with the same level of confidence. Additionally, the value of α will give a rating for the credibility of the solution for a certain relaxation level.

The successive relaxation algorithm can be stated as follows :

i. Set an initial value α_0 for α, set a value n_0 for maximum number of subiterations with α_0, set iteration counter i=0.

ii. Start POCS iterations. If a solution can be found in less than n_i subiterations, stop. Otherwise, reduce α_i. The main factors affecting the amount of this reduction are the previous α value, the previous number of iterations, cost of iterations, and maximum cost allocated for the problem.

iii. If α_i is less than a preset limit, then there is no solution worth obtaining with the allocated cost for the problem. Stop.

iv. Determine maximum number of subiterations for the $i+1^{th}$ iteration. The main factors to be considered in this are the

value of α_i, the previous number of iterations and the cost.

v. Use the last estimate as the initial estimate. Go to step ii.

The initial values of the parameters α and n are subjective. It is obvious that, in order to obtain a high quality restoration α must be close to one. However, when α is close to one, generally, the size of the intersection set is small, and finding a solution requires a larger number of iterations. This may require a compromise between the desired quality of the solution and the cost of finding it. Similarly, the amount of reduction in α and maximum number of subiterations with that value are decisions requiring compromises. Computational experience with using a collection of sets is valuable in determining these factors.

5.2. Using Optimization Techniques

In the most general case, the problem imposed by the fuzzy sets formulation is an unconstrained nonlinear optimization problem in the signal space. Mathematically, the set of solutions can be described as:

$$S^* = \{ \underline{f}^* : \underline{f}^* = \arg\sup \text{IO} [\mu_1(\underline{f}), \mu_2(\underline{f}), \ldots, \mu_m(\underline{f})] \} \quad (5.3)$$

where; IO is the intersection operator, and $\mu(.)$'s are the

membership functions of the fuzzy sets defining the solution. Ideally S^* is a singleton containing only the original signal. The relation between the size of the solution set and the quality of the solution is addressed in Chapter 6.

Finding an element of S^* is nothing but locating a maximum of a certain combination of nonlinear functions. However, it is well known that, no algorithm capable of solving every nonlinear optimization problem has been developed yet. For special problems, there are various techniques which are more efficient than other more general methods. Thus, in solving a signal restoration problem, which is formulated using fuzzy sets, the first step must be a search for a special technique suitable for the problem. Using POCS is an example for exploiting a special case of the problem namely, convexity of the individual sets.

In this section, general, iterative techniques for obtaining a solution will be outlined. The only assumption is the differentiability of the membership functions. Since all of the membership functions defined in section 4.3 are differentiable, nondifferentiable optimization techniques are not considered. The derivatives of some of the membership functions are given in Appendix 10.3.

It should be noted that, in an iterative optimization algorithm, the membership value of the estimate in the intersection set may be used as a convergence criterion. The iterations may be terminated when the membership value reaches a certain level which may be a constant or a function of the number of completed iterations. Because no assumptions are made about the convexity of the membership functions, any optimization algorithm may converge to a local suboptimal solution determined by the initial estimate. However, since the range of the membership function, and an acceptable level for its value is available, this will not cause a serious problem. If a specific algorithm converges to a point with a low membership value, it may be executed again using some other initial estimate.

5.2.1. Intersection Using Minimum Operator and Chebyshev Problem

As indicated in section 4.4, the straightforward extension of the ordinary POCS formulation requires the use of the minimum operator for the intersection of fuzzy sets. Using this operator the problem can be stated as:

$$\max_{\underline{f}} \ \mu_{min}(\underline{f}) \tag{5.4}$$

where, $\mu_{min}(\underline{f})$ is the value of the minimum membership

function in the set of membership functions defining the solution. This problem is known as Chebyshev problem [35]. Clearly, even if all of the membership functions are differentiable, $\mu_{min}(f)$ is not necessarily differentiable. Thus, conventional unconstrained optimization techniques cannot be applied to this problem. However, there are various solution techniques for the Chebyshev problem [35], [36], [37], [38]. In most of these techniques, the following equivalent formulation of the problem is preferred:

$$\begin{aligned} &\max z \\ &\text{s.t.} \\ &\mu_1(f) \geq z \\ &\mu_2(f) \geq z \\ &\ldots \\ &\mu_m(f) \geq z \end{aligned} \tag{5.5}$$

where, z is an additional scalar variable. Under the formulation given in equation 5.5, the problem becomes a nonlinear constrained optimization problem. Using Kuhn-Tucker theorem [31], a necessary condition for the optimal solution of the problem can be given as:

$$\sum_{i=1}^{m} \lambda_i \nabla\mu_i(f) = 0 \tag{5.6}$$

where; λ_i's are nonnegative constants whose sum is unity.

In [35], and [36] the solution method involves a linear programming subproblem. For this application, the number of samples, N, in the signal determines the dimension of this subproblem, which is, for many cases, too large to be practical with current state of technology. The algorithms described in [37] and [38] are essentially the same and based on finding an appropriate direction which gives a local increase to the function to be maximized. These algorithms were found to be applicable to the problem.

In order to present an overview of the algorithm, which is described in detail in [37], a certain amount of notation must be defined. Let $g_i(f)$ be the N X 1 gradient vector of the ith membership function evaluated for signal f. Associated with any signal f, let a set of indices be given by:

$$H(\underline{f},\delta) = \{i : \mu_i(\underline{f}) - \min_j[\mu_j(\underline{f})] \leq \delta \} \qquad (5.7)$$

where; $\delta \geq 0$. H(f,0) contains the indices of those membership functions which have the smallest values for the signal $\underline{f}$. Let $\underline{G}(\underline{f},\delta)$ denote an N X k matrix whose columns are the gradient vectors of those membership functions whose indices are included in the index set $H(f,\delta)$. Thus, k is the cardinality of H. An appropriate direction for modifying the signal $\underline{f}$ is shown to be:

$$\underline{s}(\underline{G}) = \underline{G}(\underline{G}^t\underline{G})^{-1}(1)/||\underline{G}(\underline{G}^t\underline{G})^{-1}(\underline{1})|| \qquad (5.8)$$

where; $(\underline{1})$ is a k X 1 vector whose elements are ones. If $\underline{s}$ is defined, then f can be updated using:

$$\underline{f} = \underline{f} + \tau\ \underline{s} \qquad (5.9)$$

where; τ is to be found by line search. Using these definitions, the main algorithm can be presented as follows :

0. Select an initial signal $\underline{f}$, and an arbitrary value δ. Set the iteration counter, i=0.

i. Determine the rank of the matrix $\underline{G}$. If it has full rank, go to v. If it has almost full rank, that is, if there is a submatrix of $\underline{G}$ which is obtained by deleting a single column of $\underline{G}$, and which is of full rank, go to iii. Otherwise, go to ii.

ii. Is there a null column in $\underline{G}$? If yes, terminate. Otherwise, do not update the solution, divide δ by two and go to vii.

iii. Find a vector $\underline{w}$ with positive sum of elements in the null space of $\underline{G}$. If all elements of w are nonnegative then go to iv. If it has a negative entry, construct a new index set from H by deleting the index of the membership function whose gradient is multiplied by the negative element of the vector $\underline{w}$. Using the new index set find the optimal direction

$\underline{s}$, and update the signal. Divide δ by two, go to vii.

iv. If δ is less than a preset limit γ, terminate. Otherwise divide δ by two, keep the signal unchanged, and go to vii.

v. Find the direction s. If for all i in the index set $\underline{g}_i^t\underline{s} > \delta$ holds, modify the signal using s, and go to vii. Otherwise go to vi.

vi. Construct a vector $\underline{w}$ using $\underline{w} = (\{\underline{1}\} + \underline{G}^t\underline{G})^{-1} (1)$ where; $\{\underline{1}\}$ and $(\underline{1})$ denote a matrix and a vector respectively, with all of their entries equal to one. Modify the index set H by dropping the index of the membership function corresponding to the minimum element of $\underline{w}$. Determine the direction $\underline{s}$ using the new index set. Modify the solution, divide δ by two, and go to vii.

vii. Increment the iteration counter i. If it is less than maximum number of iterations go to i, otherwise terminate.

This algorithm converges to the global optimum solution if all of the involved functions are pseudoconcave and the problem is nondegenerate, in the sense that, the $\underline{G}$ matrix corresponding to index set $H(\underline{f},0)$ is either full or almost full rank [37]. In the case of nonpseudoconcave functions, the conditions required for the natural termination of the algorithm, which are given in the second and the fourth steps, are nothing but the Kuhn-Tucker conditions given in

equation 5.5. Thus, the point of convergence will satisfy the necessary conditions for a local optimum. The algorithm, essentially being a hill climbing technique, will generate a better estimate than the initial signal, even if it is terminated at step vii.

The matrix which is inverted for determining the direction s, has a dimension equal to the cardinality of the index set which is at most equal to the number of the fuzzy sets. Any simple line search algorithm can be used for finding τ. A discussion of various line search techniques can be found in [39].

5.2.2. Formulations Using Other Intersection Operators

A differentiable membership function for the intersection set can be obtained by using forms of the intersection operator other than the minimum, equation 3.3.. There are various iterative algorithms for the solution of nonlinear optimization problems with differentiable cost functions $f(\underline{x})$ [40], and their convergence to a local optimum is guaranteed by the following conditions [40] :

i. $f(\underline{x})$ must be twice differentiable with bounded derivatives in the set

$$R(\underline{x}^0) = \{ \underline{x} : f(\underline{x}) \geq f(\underline{x}^0) \} \qquad (5.10)$$

where, $\underline{x}^0$ is the initial point.

ii. $R(\underline{x}^0)$ must be bounded.

iii. The solution $\underline{x}$ must be updated using $\underline{x}^{k+1} = \underline{x}^k + \lambda \underline{s}^k$ where, $\nabla f^t \underline{s} \geq 0$, and λ is to be found by one dimensional optimization.

iv. After some given number of iterations the projection of the direction vector s on the gradient must be positive.

Most of the conventional methods satisfy these conditions.

In this section, two different intersection operators which generate differentiable membership functions will be studied.

5.2.2.1. Product Form of the Intersection

In this formulation, the membership function of the intersection set is the product of the membership functions of the individual fuzzy sets. Since most of the membership functions are exponentials, taking the logarithm of the membership function of the intersection set simplifies this formulation. An important difference between the minimum and the product forms of the intersection operator is the interdependent nature of the latter. The membership of an element in the intersection set defined by the product

operator changes when its membership in any of the component sets change unless one of them is zero. For the minimum operator, any change in the membership value in a intersecting set will not affect the result provided that it is above the minimum membership value. Clearly, the membership value calculated by the product operator is always less than or equal to the one calculated by the minimum operator.

5.2.2.2. Formulation Using P-Norms

The main idea behind this formulation is the approximate calculation of the maximum element of a vector using its p-norm with large p values. In [41], the following formulation is proposed as a general form for the intersection operator:

$$\mu_{\cap_p}(f) = 1 - \min \{1, [\sum_{i=1}^{m} (1 - \mu_i(\underline{f}))^p]^{\frac{1}{p}} \} \qquad (5.11)$$

$\mu_{\cap_p}$ converges to minimum intersection operator as p approaches infinity. It is a triangular norm, and for p=1, it is the bold intersection operator defined in section 3.2. [41].

In using the p-norm form of the intersection operator for finding the solution of the restoration problem, it is possible to make simplifications. The most significant

simplification is removing the "min" operator from equation 5.11. This operator is effective only for very small values of the membership functions. Since a high membership value is required for the solution, removing this operator will not change the membership value of the solution. After this modification the problem can be stated as:

$$\max\ 1 - \left[\sum_{i=1}^{m} (1 - \mu_i(\underline{f}))^p \right]^{\frac{1}{p}} \tag{5.12}$$

which is equivalent to:

$$\min \sum_{i=1}^{m} (1 - \mu_i(\underline{f}))^p \tag{5.13}$$

The value of p must be adjusted considering the desired level of interdependence and the computational feasibility.

5.2.2.3. Solution Techniques

The main difficulty in finding the optimal solution is the high dimensionality of the problem. The large size of the Hessian matrix makes Newton's method impractical even if the membership functions are twice differentiable. For the same reason, quasi Newton or variable metric methods are not suitable.

An ordinary steepest ascent technique involves the smallest number of calculations and storage area per

iteration; but, its speed of convergence is quite slow. In order to have an algorithm with a faster convergence rate, a version of conjugate feasible directions techniques may be useful. Feasible direction techniques are iterative optimization algorithms. In each iteration of these algorithms, a "feasible" direction is calculated using a combination of the gradient of the cost function and the direction found in the previous iteration. The result of the iteration is found by a one dimensional optimization in this feasible direction. In [39], [40], and [42] the feasible direction algorithm developed by Polak is recommended for nonconvex problems with large dimensions. Without increasing the computational complexity, this method gives a superlinear convergence rate in a neighborhood of the solution where the cost function can be approximated by a quadratic. A detailed description of the method can be found in [39].

5.3. Similarities

In this chapter various methods for the solution of the fuzzy signal restoration problem were presented. The POCS method was shown to be applicable to convex fuzzy sets formulation. An algorithm which is proven to be convergent for pseudoconcave membership functions is presented for the case when the minimum operator is used as intersection. Finally,

ure was given in section 3.4. For ordinary sets, the energy measures given in equations 3.9 and 3.10 are nothing but the volumes of the sets. Clearly, the volume of a set is not directly related to the distance between its elements. This can be seen by comparing the rectangular and square sets shown in figure 6.1. However, a large volume implies large distances. In [26], the energy of a fuzzy set of alternatives is used as a measure for the difficulty in deciding on one of the alternatives. As indicated in [26], if a fuzzy set is used for defining a set of alternatives, the measure given in equation 3.10 gives a better description for the existence of a dominant alternative. This can be demonstrated by an example. Among the fuzzy sets A and B, whose membership functions are as depicted in figure 6.2, the definition of the original signal is, clearly, better in set B; but, when computed using equation 3.9, both sets have the same energy measure. On the other hand, under equation 3.10, the energy of the fuzzy set B is smaller.

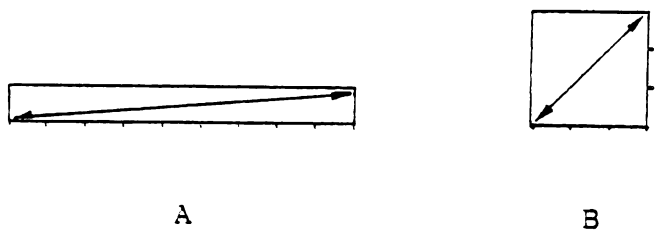

Figure 6.1 The areas of the sets A and B are equal to 9 units. However, the maximum distance between the elements of A is approximately 9.06, while the same distance for the set B is approximately 4.24.

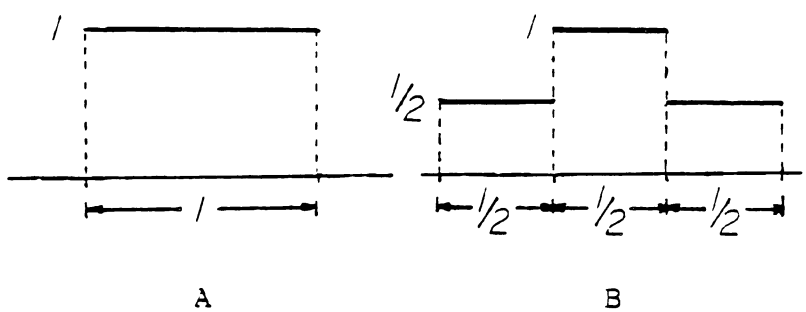

Figure 6.2 The area under the membership functions A and B are equal to 1 unit. The area under the squared membership function A is the same however, the area under the squared membership function B is 0.75.

6.1.3. Height of the Intersection Set

Following the arguments presented in the previous section, it may be concluded that a small intersection set is an indication of a well defined solution and high quality restoration. This conclusion is valid under the assumption of a consistent formulation. The intersection set may have a very small size but, this may be because of a low height, indicating an inconsistency in the incorporated information.

If a solution with a high membership value in a small sized intersection set cannot be found, the formulation of the problem must be checked for the source of an inconsistency. This may be done by removing some of the sets from the formulation and calculating the height of the new intersection set. In the following sections, a consistent formulation is assumed.

6.2. Computational Aspects

6.2.1. Infeasibility of an Analytical Calculation

A general analytical method for finding the size of the intersection of an arbitrary group of sets is not available. In fact, even for a very restrictive group of sets, the analytical calculation is quite difficult.

In [11], upper bounds for the sizes of the sets modelling the residual signal's statistics were derived. As noted in section 2.5.1, the smallest one of these bounds is an upper bound for the intersection set. However, in many applications, because of the null space of the degradation operator, these bounds are not defined. Even when some of the sets in the collection have known bounds, these are extremely large and of little value.

In a specific problem, it may be possible to find an analytical expression for the size of the intersection set. However, developing a general, analytical technique does not seem possible.

6.2.2. Monte Carlo Methods

6.2.2.1. Using Random Signals

The most straightforward way of finding the size of the intersection set is generating random signals and using them for the approximate calculation of the measures given in equations 6.1 and 3.10. The maximum distance between two elements of an α-level set of the fuzzy intersection set can be approximated by the largest distance between the random signals falling in this set. Probabilistic confidence levels on such an approximation can be calculated [43].

An estimate for the energy of an α-level set of the intersection set can be given by:

$$e(\cap_\alpha) = \frac{V}{NS} \sum_{i=1}^{M} \mu_N^2(\underline{f}_i) \tag{6.2}$$

where, NS is the total number of random signals used in the experiment, M is the number of random signals which fall into the α-level set, and V is the volume over which the signals are distributed uniformly.

There are two advantages of using random signals. The first one is the required analysis and coding effort is minimal, and the second one is the result can be obtained before solving the problem. Obviously, in order to find a limited region over which the random signals are to be generated, some information about the original signal is required. In the worst case, this region can be defined using a large enough energy bound on the original signal. However, if this region is too large, the high dimensionality of the signal space necessitates performing enormous number of experiments. In certain applications, the size of the region over which the Monte Carlo experiments are to be performed can be restricted using some of the sets defining the solution. For example, if the original signal is known to be positive, there is no reason for generating test sig-

nals with negative components.

6.2.2.2. Random Perturbations Around a Solution

If a solution is required regardless of the size of the intersection set, or if the region over which the Monte Carlo experiments are to be performed is too large for being computationally feasible, then a solution must be computed. After finding the solution, the size of the intersection set can be computed as a quality measure for the solution. Certainly, knowing a point with maximal membership in the intersection set helps a great deal in designing the experiment.

For finding the maximum distance between the elements, it is possible to search for the furthest element of the intersection set from the solution in randomly generated directions. If the POCS technique is used to obtain the solution, then it will be on the boundary of at least one of the intersecting α-level sets. For this case, instead of random directions the normals of the surfaces of the α-level sets can be used as search directions. However, the computation of the normal directions is not easy, and this method is not used in this dissertation.

Using randomly generated signals as initial signals, and finding the distances between the obtained restorations is another technique for computing the size. This technique is based on the results obtained in [21], which indicate the profound effect of the initial estimate on the solution when the size of the intersection set is large. The effect of the initial estimate decreases as the size gets smaller. However, solving the problem several times involves a large amount of computation.

To find the energy of the intersection set, an N dimensional sphere centered on the obtained solution can be used as an experimentation region. The size of the sphere can be adjusted during the experiment using a successive relaxation type algorithm. If most of the random signals in the sphere are inside the intersection set its radius is increased, if a very small number of the signals are inside the intersection, a smaller radius is used.

6.2.2.3. Using a Prototype for the Original Signal

As explained in the previous section, knowing a solution makes it easier to find the size of the intersection set. However, as indicated in section 6.1.1, if the size of the intersection set is available before solving the problem, it will be possible to determine the sufficiency of the

incorporated information for obtaining a reasonable solution.

The effectiveness of various types of information for restricting the set of solutions can be studied by computing the size reduction obtained by the inclusion of a new set in a collection of sets describing a signal. Removing ineffective sets from the formulation will reduce the amount of the computations required to obtain the solution. If experiments can be done on a specific group of sets to be used in restoring a particular group of signals, the most effective collection can be selected before making a restoration.

In many cases, a reasonable prototype signal "similar" to the original signal can be generated. As an example, lets consider a degraded image. Most of the time, the main objects in the picture, such as buildings, ships, men faces, text, etc., can be identified. After such an identification, it is possible to find a similar picture. Degrading this new picture with the same degradation, a new degraded picture can be obtained. Since the original of the new degraded picture is available, experiments can be done as if a solution is available. It is reasonable to assume that the size of the intersection set of this simulation is close to that of the original case.

Another application of using prototype signals might be in classifying the most effective sets for certain group of signals.

6.3. Results And Discussions

6.3.1. Results

In order to demonstrate the effects of variations in the parameters of a degradation system and the α level on the size of the intersection set, Monte Carlo experiments were performed on the linear degradation example described in Chapter 2. The original signal is the one given in figure 2.1. This signal is convolved with a Gaussian shaped impulse response, and zero mean, white noise is added to obtain a degraded signal. The severity of the degradation is varied by changing the spread of the impulse response and the noise variance. The sets used in the experiments are those defined by the residual signal's statistics and the nonnegativity.

Upper bounds for the sizes of the α-level sets of the fuzzy sets modelling the residual signal's statistics can be calculated using equation 2.15. However, as discussed in section 6.2.1, these bounds will be extremely large. For this example, when the variance of the normalized Gaussian

impulse response is four, the magnitudes of the DFT coefficients fall below 0.0001 after the 17th frequency coefficient. Thus, the main factor determining the size of the intersection set is the interaction between the sets used in the problem.

A Monte Carlo experiment was performed for estimating the energy measure of the α-level sets of the intersection set under varying degradation and α levels. In this experiment, uniform random signals are generated inside an N dimensional cube around the original signal and the energy measure defined in equation 3.10 is estimated using equation 6.2. The results are presented in table 6.1 where, 1000 signals, which are uniformly distributed inside a hypercube with noted side length are used for each experiment. The size of the hypercube is adjusted according to the severity of the degradation so that a reasonable number of points fall inside the intersection set. It should be noted that, changing the size of the hypercube while keeping the number of trials constant causes a change in the variance of the estimated energy. It is possible to estimate this variance and improve the experiment, however, in this dissertation, a detailed experiment design will not be included. The results obtained by the simple Monte Carlo experiment are consistent with the theoretical expectations and encouraging

for further studies on the subject.

It should be noted that, the size of the intersection set is quite small even under severe degradations. For example, when the variance of the contaminating noise is one, and the variance of the degrading function is nine, out of 1000 random signals uniformly distributed in an amplitude range ±12.5, only 174 satisfied the mean value constraint at membership level .95, 42 signals are in the set defined by outliers, and there are no signals in the other sets. This demonstrates the necessity for using an experimentation region around the solution or a prototype signal.

In order to estimate the maximum distance between the elements of the α-level sets of the intersection set, random signals are generated so that their DFT coefficients are nonzero at the frequencies for which the DFT coefficients of the degrading function are almost zero. If the nonnegativity constraint is not present, since the sets defined by the residual signal's statistics are insensitive to the signals in these frequencies, it is possible to obtain signals inside any α-level set which are extremely remote to the original signal. However, when the nonnegativity constraint is enforced, the size of the intersection set reduces significantly. As in the case of estimating energy, a restricted range is used for the frequency coefficients so that a

reasonable proportion of the generated 500 signals signals are contained in the intersection set. The results obtained in the Monte Carlo experiment for finding the maximum distance is presented in Table 6.2.

Table 6.1 The Energy of the Intersection Set

Noise Variance	Impulse Response's Variance	Conf. Level	side of the 64-cube	# of Pts. in the intersection	Energy
0.01	4.0	0.99	0.5	117	6.34e-21
0.01	4.0	0.10	0.5	309	10.52e-21
0.10	4.0	0.99	2.0	047	0.87e+18
0.10	4.0	0.10	2.0	166	1.79e+18
0.10	8.0	0.99	2.0	173	1.82e+18
0.10	8.0	0.10	2.0	196	2.12e+18

Table 6.2 Maximum Deviation from the Original Signal

Noise Variance	Impulse Response's Variance	Conf. Level	range of coeff. magnitude	# of Pts. in the intersection	Maximum Squared Distance
0.01	4.0	0.99	6.0	36	1.05
0.01	4.0	0.10	8.0	21	1.87
0.10	4.0	0.99	23.0	18	11.17
0.10	4.0	0.10	25.0	120	14.10
0.10	8.0	0.99	25.0	15	11.44
0.10	8.0	0.10	25.0	100	15.07

6.3.2. Discussions

The size of the intersection set is a valuable information on the quality of the restoration result. It is possible to obtain estimates for the size measure using Monte Carlo experiments. Special structures of the sets must be used in designing the experiments for reducing the computational burden. The estimate for the maximum distance between the elements of the intersection set seems to be more meaningful than the energy measure. However, the latter can be used for a relative comparison of different formulations.

The computation of the size measures using Monte Carlo experiments is long and costly. However, for the cases in which the collection of the fuzzy sets defining the solution is fixed, this drawback may be easier to overcome by performing experiments with prototype data and storing the results. For the general case, a value for the size of the intersection set may be obtained at a low confidence level by using a moderate size Monte Carlo experiment.

CHAPTER 7

RESULTS

7.1. Extension of the POCS Method Using Convex Fuzzy Sets

The POCS method can be used for finding an element in the intersection of α-level sets of a collection of convex fuzzy sets defining the original signal. In this section, the results obtained by this technique will be discussed.

7.1.1. Convex Approximations

The fuzzy sets defined using the residual signal's statistics, with the exception of the one using the mean value, are not convex. This is owing to the fact that, the variance, the maximum power spectrum and the maximum deviation from the mean value have lower and upper bounds. However, as in [11], it is reasonable to assume that, initially these statistics of the residual signal are more likely to be larger than the upper bounds and stay in the same region during the iterations. Under this assumption, the lower bounds can be neglected. This is equivalent to using the convex hull of the nonconvex α-level sets.

The convex approximations for the α-level sets of the fuzzy sets defined using residual signal's statistics have

the same form as the sets C_v, C_p, and, C_o defined in section 2.2.2 however, the δ parameters are α dependent. The relations between α and δ parameters are as follows:

$$
\begin{aligned}
\delta_m &= c1\ \sigma_n\ \sqrt{N} \\
\delta_v &= \sigma_n^2\ N\ \left(1 + \frac{c1^2}{2(N-1)} + \frac{c1\sqrt{2N-1}}{N-1}\right) \\
\delta_p &= \sigma_n^2\ N\ \left(\frac{b(c)}{2} - \ln\alpha\right) \\
\delta_o &= \sigma_n\ \sqrt{b^2(c) - 2\ \ln\alpha}
\end{aligned}
\qquad (7.1)
$$

where,

$$c1 = \sqrt{a^2(c) - 2\ \ln\alpha}$$

and a(c) and b(c) are the confidence level dependent parameters as defined in Appendix 10.1.

7.1.2. Restorations at Varying Membership Values

In order to observe the effect of the α level on the quality of the restoration and the computation time, the degraded signal shown in figure 2.2 is restored using three different levels of membership in the fuzzy sets defined by the residual signal's statistics and nonnegativity. The nonnegativity requirement is also made fuzzy using the following membership functions:

$$\mu_{nn}(\underline{f}_i) = \begin{cases} e^{\alpha_{nn}\underline{f}_i} & \underline{f}_i < 0 \\ 1 & \text{otherwise} \end{cases} \tag{7.2}$$

where, α_{nn} is adjusted for an acceptable negative value when the membership value of $\underline{f}_i$ is close to one.

The restoration shown in figure 7.1.a is obtained using 0.1 level sets in 64 iterations. In figure 7.1.b, the restoration at 0.5 level is displayed. This result is computed in 135 iterations. The result demonstrated in figure 7.1.c is obtained in 151 iterations and its membership in each set is larger than 0.99. As expected, the quality of the restoration gets higher and the amount of the computations for obtaining it increases as α gets larger.

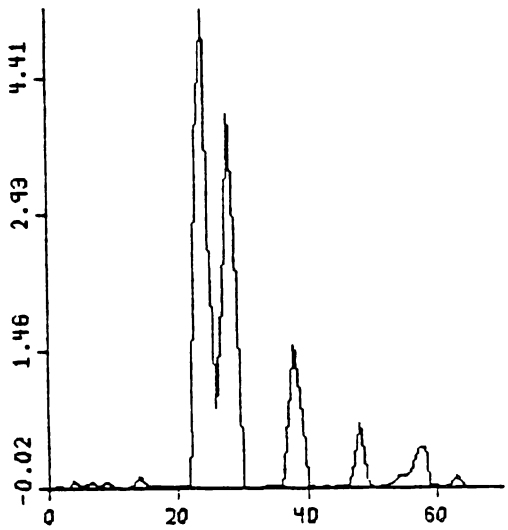

Figure 7.1.a Restoration when α = 0.1.

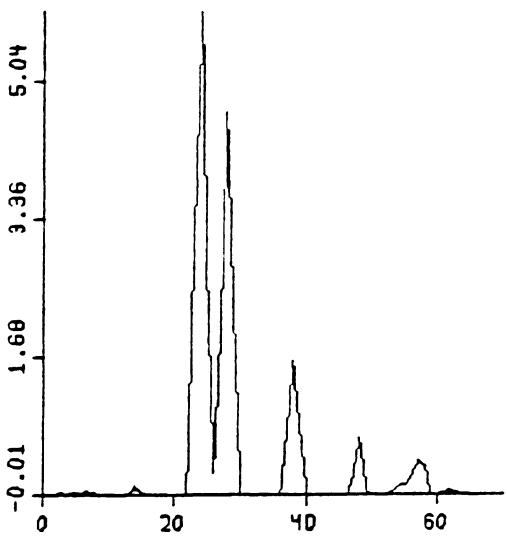

Figure 7.1.b Restoration when α = 0.5.

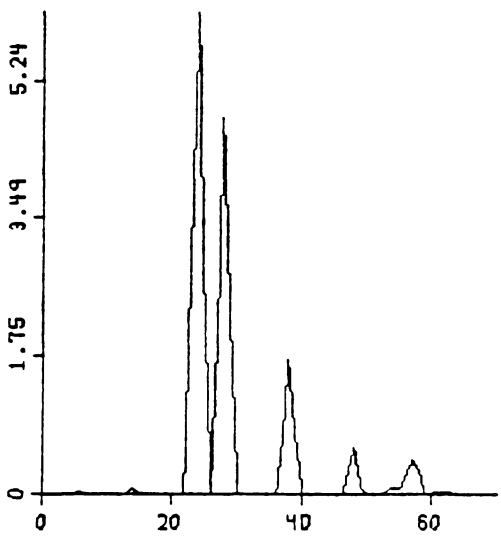

Figure 7.1.c Restoration when α = 0.99.

7.1.3. Sensitivity

Because of the logarithmic relation between α and δ values, the relative sensitivity of the results to changing α values decreases as α gets larger. For the example demonstrated in figure 7.1, to obtain the result at 0.89 level, 149 iterations are required. That is, only two iterations are gained by a 0.1 reduction in α. However, finding a solution, when α is 0.2, takes 99 iterations. Thus, the difference between the number of iterations for obtaining solutions at α levels 0.1 and 0.2 is 35.

There are many other factors effecting the sensitivity of the results to membership level. Number of samples, noise variance, and severity of the degradation are some of these factors. As can be seen from equation 7.1, the derivatives of the δ bounds of the sets with respect to α are functions of these factors. The effect of the noise variance on the sensitivity of the number of iterations to α can be demonstrated by an example. Figure 7.2 displays the restorations of the same degraded signal with different noise levels. The restoration shown in figure 7.2.a is obtained in 29 iterations when the noise variance is 0.002 and α is 0.5. At the same noise level, the solution when α is 0.99 is obtained in 34 iterations and displayed in figure 7.2.b.

When the noise variance is 0.0001, it takes 559 iterations to find the solution with membership value 0.5 which is demonstrated in figure 7.2.c. Figure 7.2.d displays the result when α is 0.99 and the noise variance is 0.001. This result is computed in 638 iterations.

It should be noted that, if the sensitivities of the sets used in a restoration problem varies significantly, the order of projections will affect the membership value of a result obtained after a certain number of iterations. Projecting onto more sensitive sets last may reduce the number of iterations required for obtaining a solution with a certain membership value.

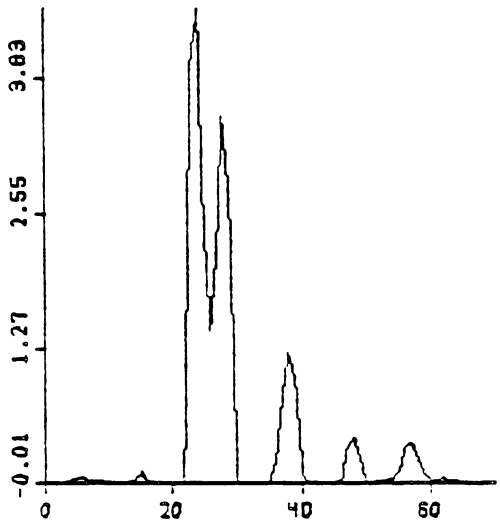

Figure 7.2.a Restoration when α = 0.5 and σ^2 = 0.002.

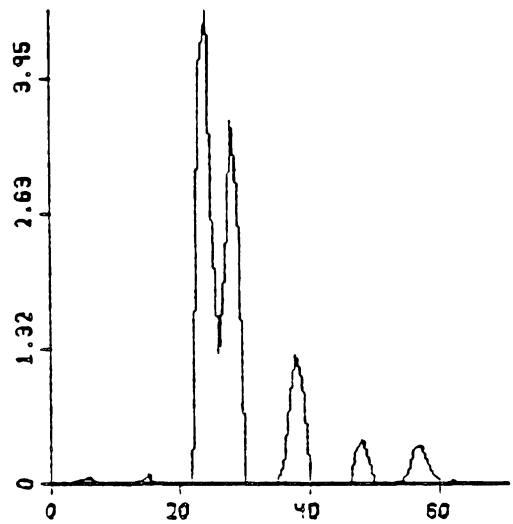

Figure 7.2.b Restoration when α = 0.99 and σ^2 = 0.002.

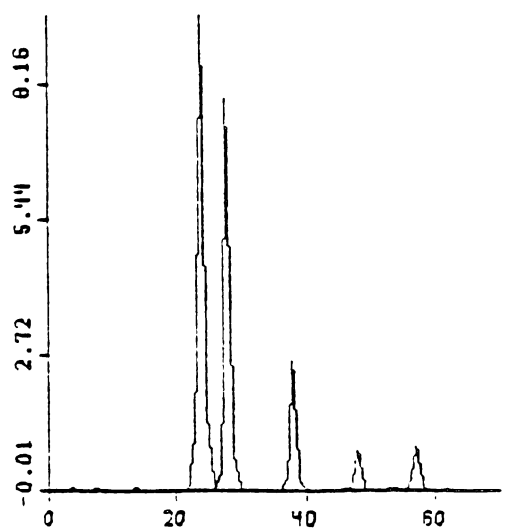

Figure 7.2.c Restoration when α = 0.5 and σ^2 = 0.0001.

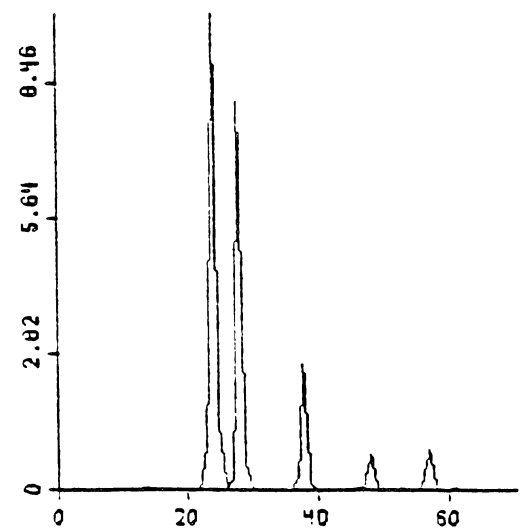

Figure 7.2.d Restoration when α = 0.99 and σ^2 = 0.0001

7.1.4. Successive Relaxation and Contraction Experiments

As discussed in section 5.1.2, a successive relaxation algorithm can be used in restoration using convex fuzzy sets. In cases for which the noise sequence has improbable sample statistics, or a degradation model with an erroneous parameter, the intersection set may be very small for high α values. This causes a large number of iterations. Reducing α for enlarging the sizes of the sets can be a solution for this problem. The membership level may also be reduced if a solution is required in a small number of iterations.

In figure 7.3, a result obtained by successive relaxation is displayed. For this example, the degraded signal is the same with the one used in obtaining the restorations displayed in figure 7.2.a and b. However, a spike with magnitude 0.1 is added to the 15th sample of the degraded signal. The restoration given in figure 7.3.a is obtained in 285 iterations when α is 0.99. In order to obtain the restoration in figure 7.3.b, a starting α value of 0.99 is used. After a 30 iterations with this value, the achieved membership level is 0.009. Then, α is reduced to 0.5 at which another 30 iterations are run. The achieved membership level in 60 iterations is 0.03. At this point, α is reduced to 0.1 and the result is obtained in 2 additional itera-

tions. The restoration displayed in figure 7.3.a is better then the one in figure 7.3.b, however, the latter may also be a sufficient restoration for many purposes. Thus, for this example, a 78% reduction in the computation time is achieved by using the relaxation technique and lowering the required quality. If instead of relaxing the α level, it is kept constant at 0.99 and the iterations are terminated at the 62th iteration, the result will not be much different than the ones shown in figure 7.3. However, there is no way of finding the sufficient number of iterations before the experiment.

Another way of approaching the outliers problem may be using a successive contraction. For this approach, a solution with a low membership value is computed first. Depending on the number of iterations for obtaining the solution with the low membership value, a decision can be made on running more iterations or terminating the process. For example, for the case described in the successive relaxation application, finding a solution at 0.1 level takes 40 iterations, while for the same signal without the added spike, 24 iterations are enough for obtaining a solution at the same level. Increasing the level to 0.2 takes another 120 iterations for the case with outlier while the second one reaches 0.99 level in 8 additional iterations. This demonstrates the

effect of outliers on the number of iterations. As in the successive relaxation case, deciding on the termination level involves subjective criteria.

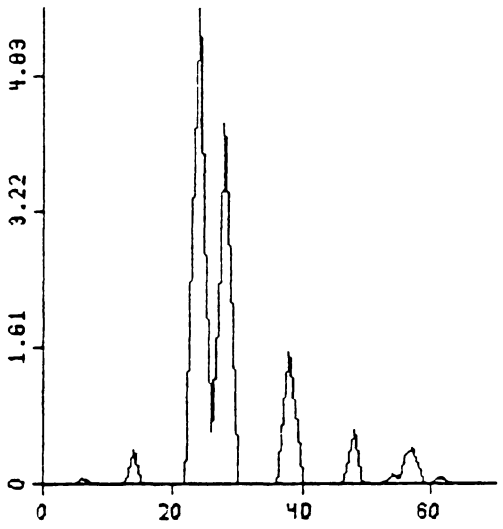

Figure 7.3.a Restoration obtained in 285 iterations.

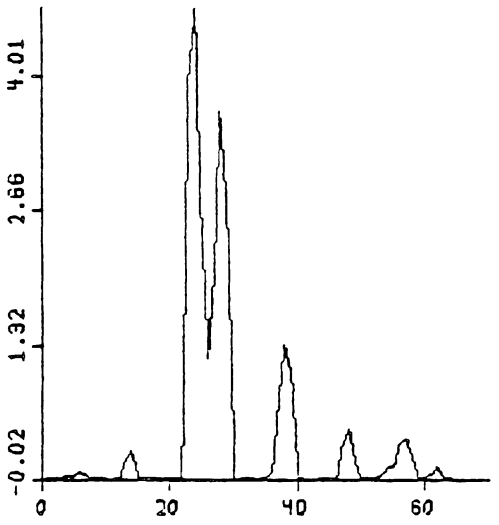

Figure 7.3.b Restoration obtained in 62 iterations with succ. relaxation.

7.1.5. Discussions

The α-level used in the restoration process affects the quality of the obtained result. The rate of change of quality with respect to α-level depends on the severity of the degradation. It has been observed that, acceptable solutions can be obtained in low α values for several degradation levels.

The effect of the α-level on the number of iterations is more pronounced. The sensitivity of the number of iterations to α-level also depends on the severity of the degradation. However, the relative change in the number of iterations corresponding to a fixed change in α value is almost constant for varying distortion levels. This can be seen in the example discussed in section 7.1.3, where a 0.49 change in α causes approximately 13% change in the number of iterations at two different noise levels. As discussed in section 7.1.3, another factor affecting the sensitivity of the number of iterations to changes in α is the magnitude of α.

Preliminary experiments with successive relaxation or contraction type of algorithms gave promising results for reducing the large number of iterations caused by outliers. A detailed study of these algorithms will be left for further research.

7.2. Results Obtained by Using Optimization Techniques

7.2.1. Motivations

As discussed in the previous section, if all fuzzy sets used for defining the solution are convex, the POCS technique can be used for solving the restoration problem. Essentially, the POCS technique may be considered as a special optimization algorithm, and as stated in Chapter 5, other nonlinear optimization techniques can also be used for finding a solution. There are two main motivations for investigating different optimization techniques. The first one is to find a more efficient algorithm than the POCS method for collections of convex fuzzy sets, and the second one is to be able to use nonconvex fuzzy sets which cannot be approximated by convex sets.

7.2.2. Algorithms for Convex Fuzzy Sets

If all fuzzy sets defining the solution are convex, and the intersection is given by the p-norm or the product rule, any nonlinear optimization algorithm can be used for finding the solution. If the intersection is defined by the minimum rule, the algorithm presented in section 5.2.1 may be used.

In order to investigate the performance of various algorithms, experiments were performed using the three forms of the intersection operator. A conjugate gradient type algorithm was utilized with the p-norm and the product intersections and the special algorithm given in section 5.2.1 was used with the minimum intersection. The experiments were performed on the restoration example given in section 7.1.

The results obtained by using different intersection operators and algorithms were similar to figures displayed in section 7.1, and need not be shown here. The number of iterations required for obtaining a solution with a given membership value is observed to be the smallest for the algorithm given in section 5.2.1. However, this does not result in a significant reduction in the computation time required by the POCS method because of extra computations performed in each iteration, such as calculation of the gradients, determination of the optimal direction, and line search.

It should be noted that, one of the techniques described above, or a completely different technique might be superior to others for a special collection of sets. Thus, the most suitable algorithm will depend on the particular collection of sets.

7.2.3. Results Obtained by Using Nonconvex Sets

In the formulation of the signal restoration problem using fuzzy sets, a priori knowledge which cannot be modelled as a convex set in the signal space may also be utilized. In the following subsections the improvement in the restoration quality obtained by the incorporation of this kind of information will be demonstrated through examples.

In the following restoration examples, the product form of the intersection operator is used. The p-norm intersection rule may also be used, however, it requires a good initial estimate to have nonzero gradients at the initial iterations. For computing the solution, a straightforward hill climbing algorithm with line search is utilized. The membership values of the restorations in the intersection set are larger than 0.99.

7.2.3.1. Energy Constraint

The fuzzy sets imposing upper and lower bounds on the energy of the restored signal were discused in section 4.3.3.1. In this example, the effect of the fuzzy set placing a soft lower bound on the energy will be demonstrated.

The degraded signal shown in figure 7.4.a is obtained from the same degradation system and the original signal used in section 2.6. However, the noise variance is increased to 0.01. In figure 7.4.b and 7.4.c, the restorations obtained by the constrained least squares [4] and the POCS methods are displayed respectively. In the POCS algorithm the fuzzy sets defined by the residual signal's statistics and positivity are used. Figure 7.4.d displays the restoration obtained when the collection of the fuzzy sets defining the solution includes an additional fuzzy set enforcing a soft lower bound on the energy of the signal. Considering the performance of the other techniques at this noise level, the superiority of the result is obvious. The three large impulses of the original signal are completely recovered, the remaining two impulses are not very clear, however, they are at the correct places.

In determining the membership function of the lower bound set, the energy of the degraded signal is used to locate the lowest value of the membership function. The energy of the degraded signal is less than that of the original signal because of the attenuating nature of the degrading system's impulse response. Determination of the non-rejectable energy level requires a priori knowledge about the energy of the original signal. In many

applications this kind of information is available. For example, in image restoration a prototype can be used for determining an acceptable energy level. In x-ray spectroscopy, an estimate for the composition of the test material can be made to derive an approximate bound on the expected energy of the resulting signal. It should be noted that the result is not very sensitive to the errors in the acceptable energy levels. In the demonstrated example, any signal with an energy higher than 180 units is assigned a membership value larger than 0.99, while the energy of the original signal is 235.65 units. The quality of the result does not change for a ±10% change around this energy level.

In figure 7.5 the degraded and restored signals corresponding to 0.5 noise variance are displayed. The resolution of the impulses are still excellent, however the locations of some of the impulses are different than the original signal by one sample. This indicates that, for higher noise levels than this one, the result may not be correct and more information about the original signal may be necessary.

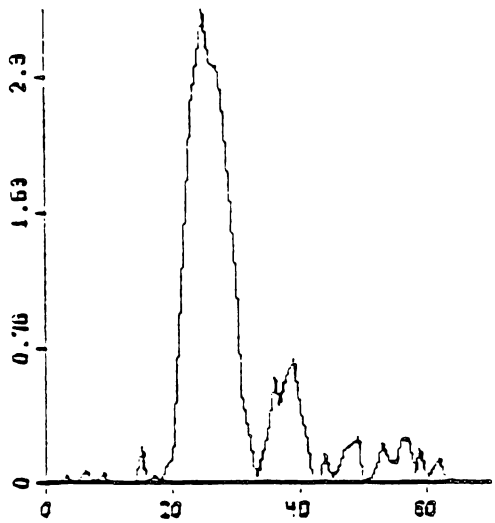

Figure 7.4.a Degraded Signal

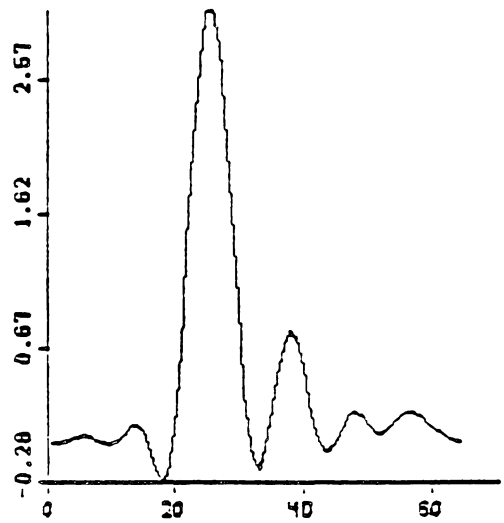

Figure 7.4.b Restoration by Constrained Least Squares Method.

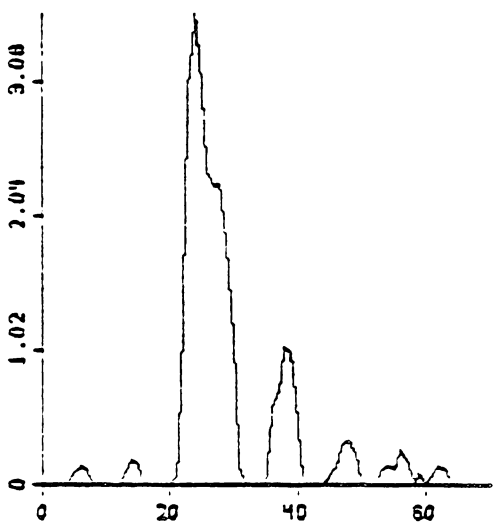

Figure 7.4.c Restoration by PCCS.

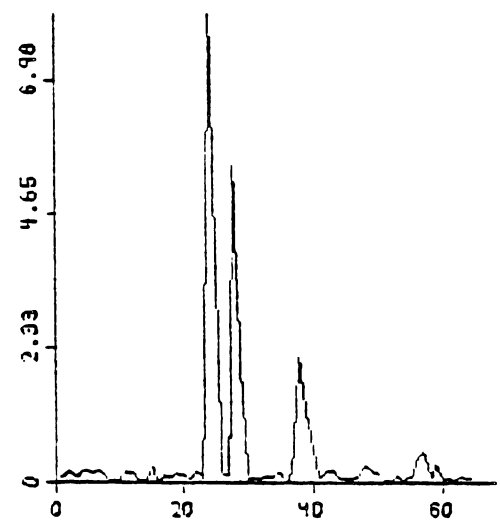

Figure 7.4.d Restoration by Imposing a Lower Bound on Energy of the Signal.

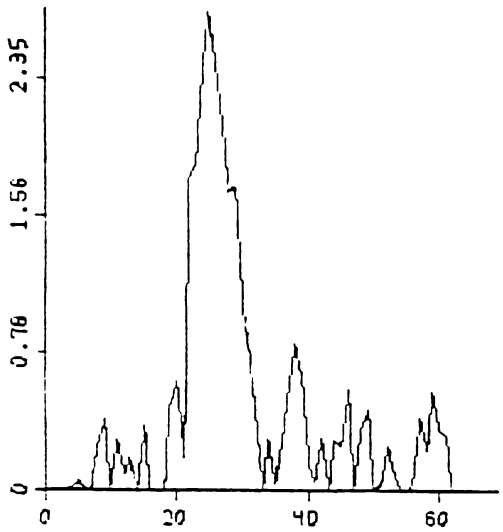

Figure 7.5.a Degraded Signal

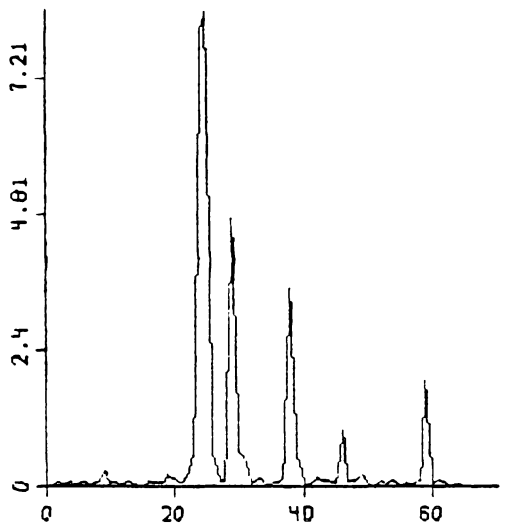

Figure 7.5.b Restoration

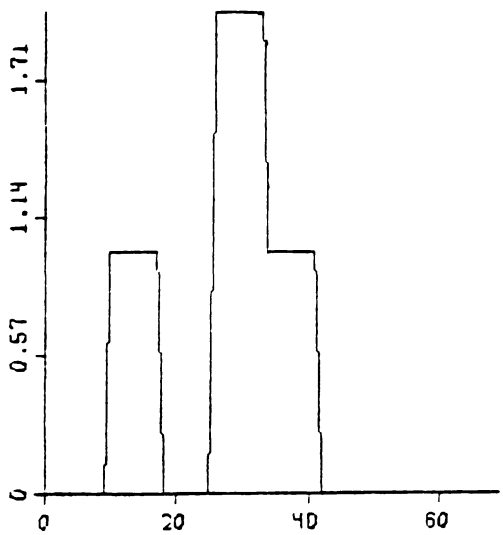

Figure 7.6.a Original Signal

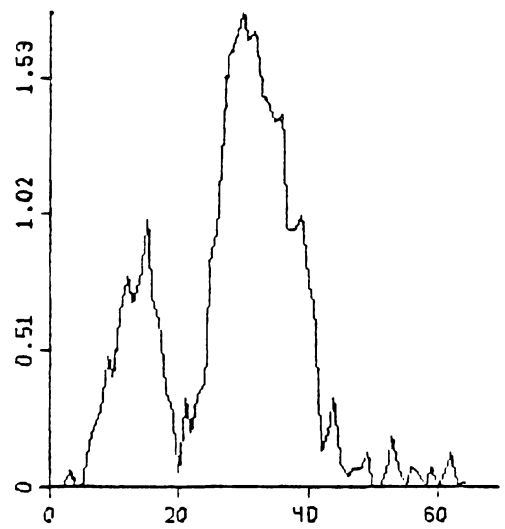

Figure 7.6.b Degraded Signal

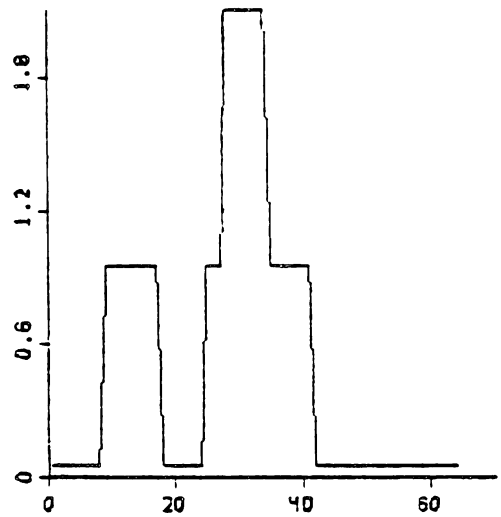

Figure 7.6.c Restored Signal

7.2.3.2. Known Ranges

In many applications, such as restoring text pictures or industrial tomography, the original signal is known to have finite levels. In such cases, the classical approach is thresholding a restoration of the signal obtained by a convenient restoration algorithm. That is, the closest available level is assigned to each sample of the restored signal. Usually, successful results can be obtained by this technique.

Obviously, in order to apply thresholding, the possible levels of the original signal must be known a priori. However, in many applications these levels are not known

exactly, but their ranges are available. An example for this was discussed in section 4.3.3.4. In such cases, thresholding cannot be applied, however, a fuzzy set modelling this property of the original signal can easily be constructed and included in the collection of sets describing the original signal.

In figure 7.6.a and 7.6.b a finite level signal and its degraded form are displayed respectively. The degrading impulse respone is of Gaussian shape with variance 8. Zero mean, white noise with variance 0.01 is added on the degraded signal. To obtain the restoration shown in figure 7.6.c, the fuzzy sets defined by the residual signal's statistics, positivity, and a fuzzy set modelling known ranges for signal values are utilized. The membership function of the last set is as given in equation 4.20, however, instead of the exact levels of the signal, 20% deviated values are used in constructing the membership function. The obtained restoration is reasonable at the stated degradation level.

7.2.4. Discussions

The signal restoration problem becomes a nonlinear optimization problem under the fuzzy sets formulation. Various algorithms for solving this problem are studied and

tested. For the examples used, no outstanding difference between the performances of these algorithms was observed. However, for a specific problem one algorithm might be superior to the others, and if a certain collection of sets are to be used many times, the most suitable algorithm must be determined.

The use of nonconvex fuzzy sets makes it possible to utilize a priori information which cannot be used otherwise. The use of energy resulted in an exceptionally good restoration. Utilization of known ranges produced a result comparable to thresholding.

CHAPTER 8

CONCLUSIONS AND SUGGESTIONS

8.1. Conclusions

In this dissertation a new signal restoration method is developed. This method is based on modelling all available information about the original signal and the degradation system using fuzzy sets in the signal space. The non-rejectable solutions of the restoration problem are members of the intersection of these sets with high membership values. Projections onto convex sets and nonlinear optimization techniques can be used for finding one of these elements. The size of the intersection set is related to the quality of the obtained solution and can be used as a measure for restoration quality.

The new method is capable of generating high quality restorations in cases for which the conventional techniques have failed. This is mainly because of the considerable flexibility of the method in incorporating a large collection of information to define the solution. Using fuzzy sets makes it possible to use partially defined information as well as exact information.

The developed method is a general and extendable technique. It can easily be adjusted according to the requirements of varying signal restoration problems. The quality of an obtained restoration can be improved by the addition of new sets to the existing collection.

8.2. Suggestions for Further Study

A natural extension for the new restoration technique is to define new fuzzy sets to model various properties of signals and degradation systems. The effectiveness of various fuzzy sets on restricting the size of the intersection set in different applications will also be a continuing research area.

The numerical algorithms for fast computation of the solution and special architectures for these algorithms may be investigated. For problems using a certain collection of fuzzy sets, it may be possible to obtain specialized solution techniques. An in depth study of the successive relaxation and contraction algorithms may prove useful in improving the computational efficiency of the projections onto α-level sets algorithm.

The size of the intersection set qualifies as an objective quality measure. Design of efficient and reliable Monte Carlo experiments to estimate the size measure may be

studied. Development of efficient techniques for the computation of the size of the intersection set will make this measure a convenient one for general applications.

In addition to the basic signal restoration applications, the new method can be used in tomography and in image coding. In industrial tomography, many times, there exist partial information about the construction of the scanned object. This information can be modelled using fuzzy sets and can be incorporated in the reconstruction algorithm. In transmitting or storing images, instead of the actual image, a collection of parameters describing the sets which in turn define the actual image may be transmitted or stored. In restoration applications, it has been observed that, a few sets may be adequate to approximate a signal well. If these sets can be defined by a small number of parameters, using these parameters instead of the image may reduce the required amount of stored or transmitted information considerably.

CHAPTER 9

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CHAPTER 10

APPENDICES

10.1. Finding the Optimal Membership Functions

10.1.1. Optimal Membership Function

Let X be the universe, and let P[.] map X into the reals. Assume that, for a certain collection of objects in X, the values of P has a known probability density function p(x). According to the conditions stated in section 4.2.2.1, the optimal membership function for a fuzzy set describing this certain class of objects can be found by solving the following problem:

$$\min_{\mu} \quad f(\mu) = \frac{1}{2}\int_{-\infty}^{\infty} \mu^2(x)dx$$

such that;

$$G(\mu) = c - E\{\mu\} = c - \int_{-\infty}^{\infty} \mu(x)p(x)dx \le 0, \tag{10.1}$$

and

$$\mu \in \Omega=\{\mu(x) \mid 0\le\mu(x)\le 1\}$$

where; $c < 1$. The 1/2 factor in the cost function is for notational simplicity.

The solution of the problem given in equation 10.1, is based on the following theorem:

Theorem [31,page 221] : Let X be a vector space, Z a normed space, Ω a convex subset of X, and P a closed positive cone with nonempty interior in Z. Let f be a real-valued convex functional on Ω. Let G be a convex and regular mapping from Ω into Z (i.e. there exist an $x_1 \in \Omega$ for which $G(x_1) < \vartheta$). Then, x^* is the optimal solution of the problem:

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } G(x) \leq \vartheta, \qquad x \in \Omega \end{aligned} \tag{10.2}$$

if and only if the Lagrangian defined by:

$$L(x,z^{\#}) = f(x) + \langle G(x), z^{\#} \rangle \tag{10.3}$$

where $z^{\#}$ is an element of the dual space of Z, possesses a saddle point at $(x^*, z^{\#*})$; i.e.

$$L(x^*,z^{\#}) \leq L(x^*,z^{\#*}) \leq L(x,z^{\#*}) \tag{10.4}$$

for all $x \in \Omega$, and $z^{\#} \geq \vartheta$.

For the problem given in equation 10.1, let X be the space of piecewise continuous functions, Z the real numbers, P nonnegative real numbers. Let f, G, and Ω be as given in equation 10.1. Clearly, Ω is a convex set, and G is a convex function. The regularity of G can easily be seen by using $\mu = 1$ as the x_1 of the theorem. The convexity of the functional f can be shown as follows: define F by

$$F = \lambda(1 - \lambda) \int_{-\infty}^{\infty} (\mu_1(x) - \mu_2(x))^2 dx \geq 0$$

for all λ in [0,1], and $\mu(.)$ in Ω. F can be reordered after adding and subtracting

$$(1 - \lambda)\int_{-\infty}^{\infty} \mu_2^2(x)dx$$

as:

$$F = \lambda\int_{-\infty}^{\infty} \mu_1^2(x)dx + (1 - \lambda)\int_{-\infty}^{\infty}\mu_2^2(x)dx - \int_{-\infty}^{\infty}\lambda^2 \mu_1^2(x)$$
$$+ 2\lambda(1 - \lambda)\mu_1^2\mu_2^2 + (1 - \lambda)^2\mu_2^2(x)dx \geq 0$$

which is equivalent to:

$$\lambda f(\mu_1) + (1 - \lambda)f(\mu_2) - f(\lambda\mu_1 + (1 - \lambda)\mu_2) \geq 0$$

The dual space of the reals is the reals. Thus, all conditions for the application of the theorem hold and the Lagrangian of the problem is:

$$L(\mu,\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \mu^2(x)dx + \lambda \{ c - \int_{-\infty}^{\infty} \mu(x)p(x)dx \} \quad (10.5)$$

where, $\lambda \geq 0$ is the Lagrange multiplier.

The Lagrangian is convex with respect to μ. Thus, the necessary and sufficient condition for a function μ^* to minimize L is [31] :

$$L'_{\mu^*}(\mu - \mu^*) \geq 0 \tag{10.6}$$

for all $\mu \in \Omega$, where; the quantity on the left of the inequality sign in equation 10.6 is the Gateaux derivative of $L(.,\lambda)$, calculated at μ^*, in the direction of $(\mu - \mu^*)$. The condition stated in equation 10.6 requires:

$$\int_{-\infty}^{\infty} (\mu^*(x) - \lambda p(x))(\mu(x) - \mu^*(x))dx \geq 0. \tag{10.7}$$

for all $\mu \in \Omega$. The optimal solution satisfying equation 10.7 is:

$$\mu^*(x) = \begin{cases} \lambda p(x) & \text{if} \quad \lambda p(x) \leq 1 \\ 1 & \text{otherwise} \end{cases} \tag{10.8}$$

Substituting this optimal value into equation 10.5, the following form for the Lagrangian is obtained:

$$L(\mu^*,\lambda) = \frac{1}{2}\int_{-\infty}^{\infty} I(\lambda p(x))(\lambda p(x)-1)^2 - \lambda^2 p^2(x)dx + \lambda c \tag{10.9}$$

where,

$$I(x) = \begin{cases} 0 & \text{if} \quad x \leq 1 \\ 1 & \text{otherwise} \end{cases} \tag{10.10}$$

It remains to find the maximizing λ value at the saddle point.

Lemma : The λ making:

$$C(\lambda) = \int_{-\infty}^{\infty} I(\lambda p(x))p(x)+[1-I(\lambda p(x))]\lambda p^2(x)\, dx-c \quad (10.11)$$

zero maximizes the Lagrangian given in equation 10.9.

Proof :

I. Existence and uniqueness :

By simple substitution, $C(0) = -c < 0$ can be obtained. Next, it will be shown that $C(\lambda)$ is a monotonically increasing function of λ.

Let $\lambda_2 > \lambda_1$, then

$$C(\lambda_2) - C(\lambda_1) =$$

$$\int_{R_1} p(x)(1-\lambda_1 p(x))dx + \int_{R_2} (\lambda_2 - \lambda_1)p^2(x)dx > 0$$

where,

$$R_1 = \{x \mid \frac{1}{\lambda_1} > p(x) \geq \frac{1}{\lambda_2}\}$$

and

$$R_2 = \{x \mid p(x) < \frac{1}{\lambda_1}\}$$

The continuity of $C(\lambda)$ for positive λ values can be shown as follows:

The absolute difference between the C values corresponding to two different λ values is:

$$D(\lambda_1,\lambda_2) = |C(\lambda_1) - C(\lambda_2)| =$$

$$\int_{R_1} p(x)(1 - \lambda_s p(x))dx - \varepsilon \int_{R_2} p^2(x)dx$$

where, $\lambda_s = \min\{\lambda_1,\lambda_2\}$, $\lambda_m = \max\{\lambda_1,\lambda_2\}$, $\varepsilon = \lambda_m - \lambda_s$,

$$R_1 = \{x|\ \frac{1}{\lambda_s} > p(x) \geq \frac{1}{\lambda_m}\}$$

and

$$R_2 = \{x|\ p(x) < \frac{1}{\lambda_m}\}$$

An upper bound for D can be obtained as follows:

$$D(\lambda_1,\lambda_2) \leq$$

$$(1 - \frac{\lambda_s}{\lambda_m}) \int_{R_1} p(x)dx + \frac{\varepsilon}{\lambda_m} \int_{R_2} p(x)dx$$

$$\leq \frac{2\ |\varepsilon|}{\lambda_m}$$

Thus, for any positive δ and λ_0, there exist a positive ε given by:

$$\varepsilon = \frac{\lambda_0\ \delta}{2}$$

for which, $D(\lambda_0,\lambda) < \delta$ whenever $|\lambda_0 - \lambda| < \varepsilon$.

Finally, since

$$\lim_{\lambda \to \infty} C(\lambda) = 1-c > 0$$

there exist a large enough value of λ for which $C(\lambda) \geq 0$.

Thus, $C(\lambda)$ must have a unique root λ^*.

II. Optimality of the λ^*:

Let $\lambda = \lambda^* + \eta$ be a Lagrange multiplier where, η is an arbitrary number and λ^* is the solution of equation 10.11. The difference between the Lagrangians corresponding to λ and λ^* is:

$$\begin{aligned} & L(\mu^*,\lambda) - L(\mu^*,\lambda^*) = \\ & \frac{1}{2}\int_{-\infty}^{\infty} I(\lambda^* p(x))(2\lambda^*\eta p^2(x) + \eta^2 p^2(x) - 2\eta p(x)) \\ & - 2\lambda^*\eta p^2(x) + \eta^2 p^2(x)\ [I(\lambda p(x)) - I(\lambda^* p(x))] \\ & (\lambda p(x) - 1)^2\, dx + \eta c \end{aligned} \tag{10.12}$$

Using equation 10.11, equation 10.12 can be simplified as:

$$\begin{aligned} & \frac{1}{2}\int_{-\infty}^{\infty} [I(\lambda p(x)) - I(\lambda^* p(x))](\lambda p(x) - 1)^2 \\ & - [1-I(\lambda^* p(x))]\ \eta^2 p^2(x)\, dx \end{aligned} \tag{10.13}$$

which is equivalent to:

$$\int_{R_1} (\lambda p(x) - 1)^2 - \eta^2 p^2(x)dx$$
$$- \int_{R_2} (\lambda p(x) - 1)^2 dx - \int_{R_3} \eta^2 p^2(x)dx \tag{10.14}$$

where,

$$R_1 = \{x \mid \lambda p(x)>1,\ \lambda^* p(x)<1\}$$

$$R_2 = \{x \mid \lambda p(x)<1, \lambda^* p(x)>1\}$$

$$R_3 = \{x \mid \lambda p(x)<1, \lambda^* p(x)<1\}$$

The expression given in equation 10.14 is always negative, because the last two terms are negative integrals of squared functions and the first term is equal to:

$$\int_{R_1} (\lambda^* p(x)-1)(\lambda p(x)-1+\eta p(x))dx$$

which has a negative integrand.

Thus λ^* maximizes the Lagrangian (QED).

10.1.2. Classical Confidence Intervals as Optimal Sets

If the cost functional $f(\mu)$ of equation 10.1 is replaced by:

$$\int_{-\infty}^{\infty} \mu(x)dx \tag{10.15}$$

which is another measure for the size of a fuzzy set, then the optimal membership function will be:

$$\mu^*(x) = \begin{cases} 0 & \text{if} \quad \lambda p(x) < 1 \\ 1 & \text{otherwise} \end{cases} \tag{10.16}$$

where λ is to be solved from:

$$\int_{\lambda p(x) > 1} p(x)dx = c$$

The set described by equation 10.16 is not fuzzy, and the probability of having one of the objects from the collection to be modelled in this set is equal to the confidence level.

10.1.3. Optimal Membership Functions For Specific PDFs

10.1.3.1. Gaussian Distribution

If the probability density function p(x) is :

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \tag{10.17}$$

then, the optimal membership function is determined by a single scalar "a" which is a function of the confidence level c defined in section 4.2.2.1. A sample membership function for standard normal distribution is illustrated in

figure 10.1. The plot given in figure 10.2 shows the values of the parameter a versus the confidence level c. The value of the optimal Lagrange multiplier can be calculated using:

$$\lambda^*(c) = \sqrt{2\pi}\, e^{\frac{a^2}{2}} \qquad (10.18)$$

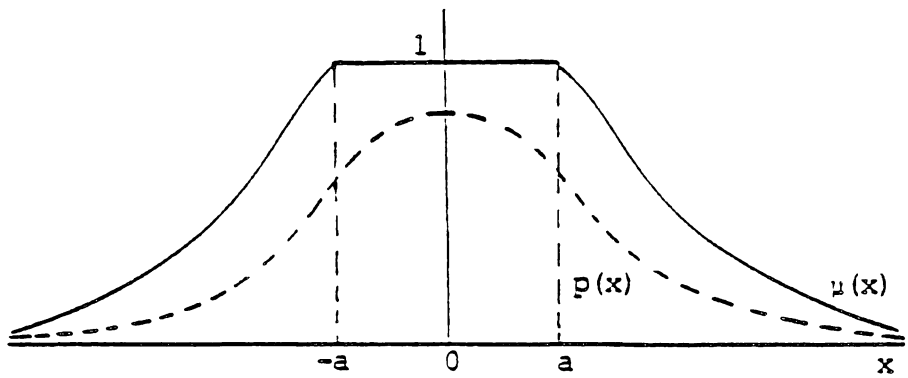

Figure 10.1 The membership function corresponding to the standard normal density.

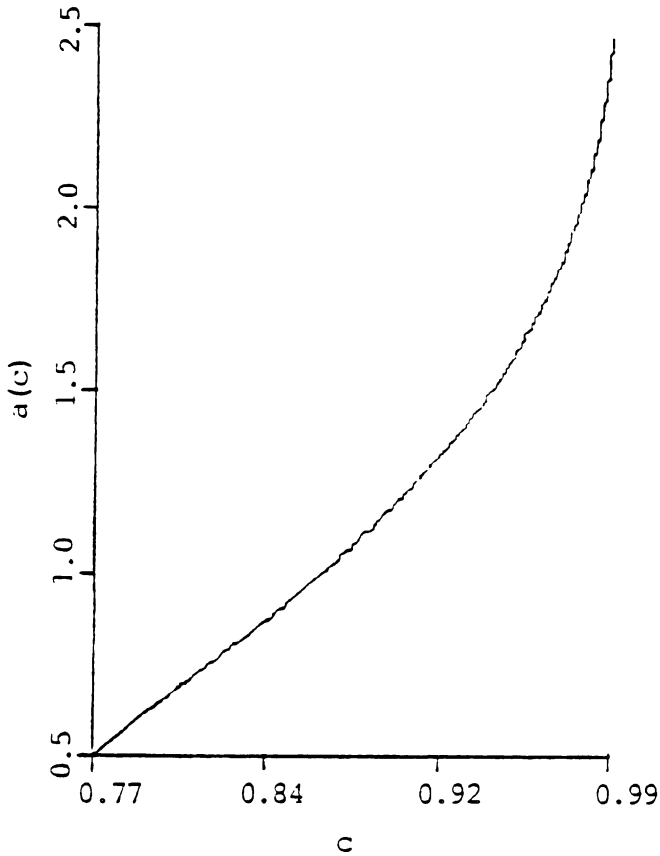

Figure 10.2 The parameter "a" of the membership function corresponding to the standard normal density versus confidence level.

3.2. PDFs for Power Spectrum and Maximum Deviation

The probability density functions for the power spectrum of the residual and maximum residual value were given in equation 4.13 and equation 4.15 respectively. As stated in Chapter 4, these are unimodal densities and the membership functions derived from them can be specified by two parameters, namely the starting value "a", and the final value "b" of the region in which the membership function equals to 1. The values of these parameters for various confidence levels, and number of samples are given in table 10.1.

Table 10.1 Optimal parameters for the membership functions of the sets described by the power spectrum and the maximum value of the residual signal.

		Power Spectrum Set			Max. Residual Set		
N	c	lambda	a	b	lambda	a	b
64	0.90	13	4.75	10.25	2.4	2.03	3.05
64	0.95	25	4.23	11.75	5.0	1.90	3.31
64	0.98	48	3.87	13.14	14.0	1.78	3.62
128	0.90	13	6.14	11.67	2.3	2.30	4.26
128	0.95	25	5.61	13.17	4.9	2.18	3.51
128	0.98	47	5.24	14.51	14.0	2.06	3.81
256	0.90	13	7.53	13.06	2.2	2.54	3.45
256	0.95	25	7.00	14.57	4.8	2.43	3.58
256	0.98	46	6.63	15.87	14.0	2.32	3.99
4096	0.90	13	13.08	18.62	1.8	3.38	4.13
4096	0.95	25	12.54	20.13	4.2	3.30	4.35
4096	0.98	45	12.17	21.38	13.5	3.21	4.62
16384	0.90	13	15.85	21.39	1.6	3.75	4.43
16384	0.95	25	15.31	22.90	3.5	3.67	4.62
16384	0.98	44	14.96	24.11	8.5	3.61	4.81
65576	0.90	13	18.63	24.17	1.2	4.11	4.66
65576	0.95	25	18.08	25.67	2.3	4.04	4.82
65576	0.98	40	17.79	26.68	7.0	3.96	5.00

10.2. Impulsive Signals

10.2.1. Defining an Impulsive Signal

In most of the common signal restoration techniques, the original signal is assumed to be smooth [1],[4]. This assumption helps in filtering the contaminating noise. However, if the original signal is impulsive, forcing the solution to be smooth causes unsuccessful restorations. An example for this can be seen in figure 2.4, where the restoration of an impulsive signal with the constrained least squares method is shown. Since in many applications, such as x-ray spectroscopy, the original signal is actually impulsive, a method suitable for this kind of signals is required.

In the fuzzy sets formulation, if the original signal is known to be not smooth, the set describing smooth signals is not used. However, if the original signal is known to be impulsive, this a priori knowledge can be used for improving the quality of the restoration.

The main properties of impulsive signals are large zero regions or regions of constant value, and a few, isolated nonzero samples. As stated in [11], nonnegativity constraint is very effective in the restoration of impulsive signals

owing to the existence of large zero regions. In order to use the other property to define a fuzzy set, it is assumed that γ out of N samples of the signal are nonzero, and the values of these nonzero samples are independent random variables with an identical pdf $f_p(x)$. The ratio $\alpha=\frac{\gamma}{N}$ will be called the impulse density.

10.2.2. Statistical Properties

The probability distribution function for the magnitude of a sample can be derived as:

$$\begin{aligned} F(X) &= P\{x \le X\} = P\{x \le X | x \text{ is nonzero}\}P\{x \text{ is nonzero}\} \\ &+ P\{x \le X | x \text{ is zero}\}P\{x \text{ is zero}\} \\ &= \alpha\, F_p(X) + (1-\alpha)\, u(X) \end{aligned} \tag{10.19}$$

where, $F_p(X)$ is the probability distribution function of the values of the nonzero samples, and $u(X)$ is the unit step function. The corresponding pdf is:

$$f(X) = \alpha\, f_p(X) + (1-\alpha)\, \delta(X) \tag{10.20}$$

where, $\delta(X)$ is Dirac's delta function. Obviously, all moments of the signal are equal to the corresponding moments of the nonzero values times the impulse density α. The autocorrelation of the signal is:

$$R_{xx}(k) = \begin{cases} \alpha\ E\{p^2\} & k=0 \\ \alpha\frac{(\gamma-1)}{(N-1)}E\{p^2\} & \text{otherwise} \end{cases} \tag{10.21}$$

where, $E\{p^2\}$ is the second moment of the nonzero values. Finally, the power spectrum is given by:

$$P(k) = \begin{cases} \gamma R_{xx}(0) & k=0 \\ \frac{N-\gamma}{N-1}\ R_{xx}(0) & 1 \le k \le N-1 \end{cases} \tag{10.22}$$

10.2.3. Modelling Impulsiveness

The statistical properties of the impulsive signal model can be used to define fuzzy sets in many ways. One of these is using the specific correlation structure. The autocorrelation in zeroth lag, which is the energy of the signal, is larger than all other correlations by a factor α^{-1}, which should be quite large. This fact can be used in the restoration algorithm by forcing the energy of the signal to be larger than a lower bound, while keeping the correlations at other lags small. In [19], maximization of the energy under bounded variance of the residual signal is found effective in impulsive signal restoration. This suggests that, if other constraints defining the signal are restrictive enough, the maximization of energy may be

sufficient. However, other sets defined by the properties such as the equality of the correlations for nonzero lags and the specific shape of the power spectrum may also be included in signal definition for further restricting the intersection set.

10.3. Gradients

In this appendix the derivatives of some of the residual signal's functions with respect to the estimated signal are given. For the linear signal degradation model, equation 2.1, the residual signal is defined as:

$$\underline{r} = \underline{g} - \underline{H}\hat{\underline{f}} \tag{10.23}$$

If the matrix H represents circular convolution, then its first row is enough in specifying the whole matrix. The signal whose samples are the elements of the first row of H will be called the point spread function (psf), and will be denoted by h(n). A signal, such as $\underline{r}$, has three representations, namely, vector form as in equation 10.23, which is denoted by a small letter, sequence form as h(n) and discrete Fourier transformed (DFT) form, which will be denoted by using a capital letter with explicit frequency dependence, as in R(k).

In the following equations, the notation $[]_i$ is used for denoting the ith element of a vector; and the subscripts .,i and i,. of a matrix are used for denoting i^{th} column vector and i^{th} row vector of the matrix respectively. For example, for the circular convolution case:

$$h(i) = [\underline{H}_{1,.}]_i = \frac{1}{N} \sum_{k=1}^{k=N} H(k) \exp\{\frac{2\pi}{N}(k-1)(i-1)\}$$
$$= \text{IDFT}\ \{H(k)\}$$

where, the notation IDFT() indicates the inverse DFT, and the result is either the vector or the sequence form of a signal, whichever is suitable to the equation.

10.3.1. Sum of Residual Elements

Sum of the residual signal's elements is used in defining the set based on the sample mean statistics. The gradient of this quantity is:

$$\nabla_{\underline{f}} \sum_{i=1}^{i=N} [\underline{r}]_i = - [\sum_{i=1}^{i=N} [\underline{H}_{.,1}]_i, \sum_{i=1}^{i=N} [\underline{H}_{.,2}]_i, \ldots, \sum_{i=1}^{i=N} [\underline{H}_{.,N}]_i]^t \quad (10.24)$$

If H represents circular convolution, then:

$$\nabla_{\underline{f}} \sum_{i=1}^{i=N} [\underline{r}]_i = - \sum_{i=1}^{i=N} [\underline{h}]_i \quad [\ 1,\ 1,\ \ldots\ ,\ 1\ ]^t \quad (10.25)$$

In many applications, the sum of the elements of the psf is unity. For this case, the gradient is an N dimensional vector of ones.

10.3.2. Sum of Squared Residual Elements

$$\nabla_{\underline{f}} \sum_{i=1}^{i=N} [\underline{r}]_i^2 = -2\, \underline{H}^t\, \underline{r} \qquad (10.26)$$

If H represents circular convolution, then the following form of equation 10.26 can be computed easily using FFT:

$$\nabla_{\underline{f}} \sum_{i=1}^{i=N} [\underline{r}]_i^2 = -2\ \mathrm{IDFT}\{H(k)^{*}\ R(k)\} \qquad (10.27)$$

10.3.3. Maximum Deviation from the Mean Value

The absolute value of the maximum deviation is not continuously differentiable. However, an approximation for it can be calculated by using the p-norm approximation for the Chebyshev's norm. The p-norm converges to The Chebyshev's norm for large p values. Using a sufficiently large p value, an approximate gradient for the maximum deviation can be obtained as follows:

$$\nabla_{\underline{f}}\ |[\underline{r}]_i|_{max} = \nabla_{\underline{f}} \left(\sum_{i=1}^{i=N} [\underline{r}]_i^p \right)^{\frac{1}{p}} = - \sum_{i=1}^{i=N} \left(\frac{[\underline{r}]_i}{rx}\right)^{(p-1)} \underline{H}_{i,.} \qquad (10.28)$$

where,

$$rx = \left(\sum_{i=1}^{i=N} [\underline{r}]_i^p \right)^{\frac{1}{p}} \tag{10.29}$$

and p is a large even integer. For the circular convolution case, $\underline{H}_{i,.}$ is the (i-1) times circularly rotated psf.

10.3.4. Maximum Power Spectrum Value

As in the maximum deviation case, only an approximate derivative can be found. If the periodogram is denoted by P(k), then:

$$\nabla_f P(k)_{max} = \nabla_f \left(\sum_{i=2}^{i=\frac{N}{2}} |R(i)|^{2p} \right)^{\frac{1}{p}} =$$

$$- 2 \sum_{i=2}^{i=\frac{N}{2}} \left(\frac{|R(i)|^2}{Pmx} \right)^{(p-1)} \tag{10.30}$$

$$Re\{R(i)\ H^*(i)\ [\ w^0,\ w^i,\ \ldots\ ,\ w^{(N-1)i}\ ]t\}$$

where,

$$Pmx = \sum_{i=2}^{i=\frac{N}{2}} \left(|R(i)|^{2p} \right)^{\frac{1}{p}}$$

$w = \exp(j\frac{2\pi}{N})$, and Re{} denotes the real part of the quantity inside the braces.

product and p-norm techniques are studied for the most general case. In this section, a similarity of the iterations generated by these different formulations will be demonstrated.

Assume that for a specific restoration problem, m fuzzy sets with pseudoconcave membership functions are used for the description of the problem. Let the estimate at the kth iteration be denoted by $\underline{f}^k$. In using the POCS technique with a specified α value, the equation for the kth iteration will be:

$$\underline{f}^{k+1} = P_1 P_2 \ldots P_m(\underline{f}^k) \tag{5.14}$$

where, P_i 's are the projectors. The projection of the signal $\underline{f}^{ki}$, which is inside the α-level set of the ith fuzzy set at k^{th} iteration, on the α-level set of the $i+1^{th}$ fuzzy set can be found by solving:

$$\begin{aligned} &\min \|\underline{f}^{ki} - \underline{f}^{k(i+1)}\|^2 \\ &\text{s.t.} \quad \mu_{i+1}(\underline{f}^{k(i+1)}) \geq \alpha \end{aligned} \tag{5.15}$$

If $\underline{f}^{ki}$ is already inside the α-level set of the $i+1^{th}$ fuzzy set, then its projection is itself. Otherwise, the Kuhn-Tucker conditions for the problem in equation 5.15 gives the following form for the solution:

$$\underline{f}^{(i+1)k} = \underline{f}^{ik} + \lambda_{(i+1)k} \nabla\mu_{i+1}(\underline{f}^{ik}) \tag{5.16}$$

where, $\lambda_{(i+1)k}$'s are constants. Thus, the equation giving the kth iteration of the POCS algorithm is:

$$\underline{f}^{k+1} = \underline{f}^{k} + \sum_{ik \in H_k} \lambda_{ik} \nabla\mu_{ik}(\underline{f}^{ik}) \tag{5.17}$$

where, H_k is the set of indices of the fuzzy sets which are projected on in the kth iteration.

If the same problem is solved using the minimum operator as the intersection and the technique outlined in section 5.2.1, the kth iteration will have the following form:

$$\underline{f}^{k+1} = \underline{f}^{k} + \tau \sum_{i \in H(\underline{f}^{k},\delta)} \lambda'_i \nabla\mu_i(\underline{f}^{k}) \tag{5.18}$$

where, τ is to be found with one dimensional optimization, $H(\underline{f}^{k},\delta)$ is the set of indices of those membership functions which are within δ distance to the minimum, and λ'_i constants are determined from equation 5.8.

If the product of membership functions is used for finding the membership function of the intersection set, the kth iteration of a steepest ascent algorithm used for maximizing the logarithm of the cost function will be:

$$\underline{f}^{k+1} = \underline{f}^{k} + \tau' \sum_{i=1}^{m} \frac{1}{\mu_i(\underline{f}_k)} \nabla\mu_i(\underline{f}^{k}) \tag{5.19}$$

where, τ' is to be found by one dimensional optimization. Similarly, if the p-norm is used with the same algorithm, the equation for the kth iteration is given by:

$$\underline{f}^{k+1} = \underline{f}^{k} + \tau' \sum_{i=1}^{m} p(1 - \mu_i(\underline{f}_k))^{p-1} \nabla\mu_i(\underline{f}^{k}) \qquad (5.20)$$

In all of these update equations, the estimate at the kth iteration is modified using a combination of the gradients of those membership functions which have small values for $\underline{f}^{k}$. In equation 5.18, this is explicitly stated, in equation 5.19 and equation 5.20, the gradients of the membership functions with smaller values will dominate the summations. In the projection technique, those fuzzy sets assigning high membership values to $\underline{f}^{k}$ will contain it, and will not take part in the projections. Thus, these techniques generate similar iterations. However, it should be noted that, all of them are not applicable unless all the membership functions involved are pseudoconcave and exponential.

CHAPTER 6

A MEASURE FOR THE QUALITY OF A RESTORATION

6.1. Size of the Intersection Set as a Quality Measure

6.1.1. Introduction

In the fuzzy sets formulation of the signal restoration problem, the main idea is using all available information for defining fuzzy sets to restrict the set of possible solutions. Obviously, all signals in the intersection of these fuzzy sets will be in accordance with the modelled information. Ideally, the intersection set must be a singleton containing only the original signal. For this ideal case, the available information uniquely defines the original signal. However, in practice, a complete definition of the original signal is almost never available and the intersection set has more than one element with above threshold membership values. An arbitrary one of these elements will be the solution obtained by the POCS algorithm or the nonlinear optimization techniques described in Chapter 5.

In a consistent formulation, the height of the intersection set is close to one. Thus, the elements with maximal membership in the intersection set are non-rejectable solutions. However, their "closeness" to the original signal

depends on the "size" of the intersection set. Heuristically, the size of the intersection set is a measure for the combined information about the original signal. If the size is small, the original signal is well defined; any signal in the intersection set is close to the original signal, and is a high quality restoration. However, the chance of obtaining a high quality restoration by selecting an arbitrary element of a large intersection set is small. Consequently, the size of the intersection set can be used as an indicator for the quality of the obtained solution. Obviously, formal definitions for the distance and the size are necessary for a quantitative study.

It should be noted that, the size of the intersection set can be computed before obtaining the solution. If the size can be computed more easily than a solution, the size measure may be found first and depending on the result a decision can be made on computing the solution.

6.1.2. Distance and Size Measures

6.1.2.1. Distance Between Two Elements of The Intersection Set

In section 2.5 and [11], the size of an ordinary set was defined to be:

$$\text{size}(\ C\ ) = \sup_{\underline{f}_1, \underline{f}_2 \in C} \| \underline{f}_1 - \underline{f}_2 \| \tag{6.1}$$

where, the Euclidean norm was used. This definition is certainly in a complete agreement with the size concept in this context. In fact, it is an upper bound for the Euclidean distance between the worst possible solution and the original signal.

An easy extension of this definition to fuzzy sets can be obtained by using the α-level sets of a fuzzy set. The α-level sets, as defined in section 5.1.1.1, are ordinary sets, and their sizes can be defined as in equation 6.1. The sizes of all α-level sets as a function of α conveys considerable information about a fuzzy set. In restoration applications, the rate of change of this function is particularly important because, it reflects the sensitivity of the results to the membership level.

6.1.2.2. The Energy of the Intersection Set

Measures for fuzzy sets have been studied by various researchers [25],[26]. The entropy and energy measures are results of these studies. The entropy of a fuzzy set is a measure for the degree of fuzziness, and is not related to the size. However, the energy of a fuzzy set is related to the volume of it. The formal definition of the energy meas-