An Approximate Analysis of a Bufferless N x N Synchronous Clos ATM Switch

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Abstract

We consider a synchronized bufferless Clos ATM switch with input cell processor queues. The arrival process to each input port of the switch is assumed to be bursty and it is modeled by an Interrupted Bernoulli Process. Each cell processor queue is modeled as an IBP/Geo/1/K queue. We present an exact analysis of the IBP/Geo/1 queue and IBP/Geo/1/K queue. These results are then used in an approximation algorithm for analyzing this ATM switch. The approximation algorithm was validated by carrying out extensive comparisons against simulation data. The approximate results have a better error when the squared coefficient of variation of the arrival process increases.
1 Introduction

The Asynchronous Transfer Mode (ATM) appears to be the most promising solution for broadband ISDN. ATM provides the means to transporting different types of highly bursty traffic such as voice, video and bulk files. It is based on the principle of packet switching, and all the packets (known as cells) have a fixed length of 53 bytes. Various types of architectures have been proposed for an ATM switch. Most of these architectures are based on multi-stage interconnection networks. The switching elements in a multi-stage interconnection network may be buffered or unbuffered. In the unbuffered case, there may be buffers at the input ports of the switch, or at the output ports, or at both input and output ports. There have been several performance studies of such switch architectures (see for instance Oie, Suda, Masayuki, and Miyahara [1], Shaikh, Schwartz, and Szymanski [2], Hluchyj and Karol [3]). For the performance analysis of switches with buffered switching elements see Yoon, Lee, and Liu [4], and the references within. Finally, we note that there is a different type of switch architecture based on the concept of the shared buffer. For further references, we refer the reader to Huang and Knauer [5], and Devault, Cochennec, and Servel [6].

In this paper, we consider a bufferless ATM switch with buffers at the input ports. The switch fabric is a Clos three-stage interconnection network. For this type of switch we construct an approximation algorithm under the assumption of bursty arrivals. The approximate results were extensively validated against simulation data. We now proceed to describe the switch under study in detail.

2 Model description

Let us consider an \( N \times N \) Clos cell switch, as shown in figure 1. There are three stages in the switch. At each stage, there are \( n \) or \( \sqrt{N} \) number of bufferless switching elements. Each switching element is an \( n \times n \) crossbar switch. For each input buffer, there is a cell processor to handle the incoming cells. The cell processor is responsible for determining the output port of a cell and then transmitting the cell through the switch fabric.

The operation of the Clos switch is synchronous. That means, at the beginning of a time slot all the busy cell processors will launch a cell through the switch. A cell will successfully get through the switch if it finds a free path to the sought output port. No contention takes place at the 1st stage of the Clos ATM switch. The cells in the switch will compete for the output lines at 2nd and the 3rd stages. In a switching element, if more than one input lines compete for the same output line, one of the input lines will get through. This input line is randomly selected. If a cell is blocked in the switch, it is assumed that the switching element will notify the cell processor. The transmission of a cell will be aborted. The cell processor will retransmit the cell at the beginning of the next time
The retransmissions will continue until the cell is successfully transmitted through the switch.

![Diagram of a synchronous $N \times N$ Clos ATM Switch.](image)

**Figure 1:** Synchronous $N \times N$ Clos ATM Switch.

The time required for a cell to pass through the switch fabric is equal to one slot time. It is assumed that each incoming line is slotted. The length of a slot is equal to a slot of the ATM switch. Each incoming line is not synchronized with the other incoming lines and with the switch. A cell that arrives at an idle cell processor in the middle of a slot of the ATM switch is not transmitted until the beginning of the next time slot of the ATM switch.

The arrival process to each input port of the ATM switch is modeled as a discrete time Interrupted Bernoulli Process (IBP).

\[
\begin{pmatrix}
\text{Active} & \text{Idle} \\
Active & \binom{p}{1-p} \\
Idle & \binom{1-q}{q}
\end{pmatrix}
\tag{1}
\]

An IBP process is governed by a two-state Markov chain. These states are: active state and idle state. If the process is in the idle state no arrivals occurs. If the process is in the
active state arrivals occur in a Bernoulli fashion. In particular, if the process is in the idle state, then in the next time slot it remains in the idle state with probability $q$ or it will change to the active state with probability $1-q$. Likewise, if it is in the active state, then in the next slot it will remain in the same state with probability $p$ or it will change to idle with probability $1-p$. Those transition rates are shown in (1). If the input process is in the active state, a slot will contain a cell with probability $\alpha$. Let $\bar{t}$ be the interarrival time of a cell and $\bar{t}_1$ be the time interval from a slot in the idle slot to the time of the next arrival. Then, $\bar{t}$ is equal to 1 with probability $p\alpha$ if the process is in the active state in the next time slot and the slot is filled with a cell. If the process is in the active state in the next time slot and the slot is empty, from the memoryless property, $\bar{t}$ is equal to $1 + \bar{t}_1$ with probability $p(1-\alpha)$. If the process changes to the idle state in the next time slot, $\bar{t}$ is equal to $1 + \bar{t}_1$ with probability $1-p$. $\bar{t}_1$ is obtained using similar arguments and we get

\[
\bar{t} = \begin{cases} 
1 & \text{, } p\alpha \\
1 + \bar{t} & \text{, } p(1-\alpha) \\
1 + \bar{t}_1 & \text{, } 1-p 
\end{cases} \tag{2}
\]

\[
\bar{t}_1 = \begin{cases} 
1 + \bar{t}_1 & \text{, } q \\
1 & \text{, } (1-q)\alpha \\
1 + \bar{t} & \text{, } (1-q)(1-\alpha) 
\end{cases} \tag{3}
\]

Hence,

\[
E\{z^{\bar{t}}\} = z p\alpha + z p(1-\alpha)E\{z^{\bar{t}}\} + z(1-p)E\{z^{\bar{t}_1}\} \tag{4}
\]

\[
E\{z^{\bar{t}_1}\} = z q E\{z^{\bar{t}_1}\} + z \alpha (1-q) + z(1-q)(1-\alpha)E\{z^{\bar{t}}\} \tag{5}
\]

Let $A(z)$ be the $z$-transform of the probability distribution of the interarrival time; i.e.

\[
A(z) = E\{z^{\bar{t}}\} = \frac{z\alpha[p + z(1-p-q)]}{(1-\alpha)(p + q - 1)z^2 - [q + p(1-\alpha)]z + 1} \tag{6}
\]

From this $z$-transform we can obtain the mean interarrival time $E(\bar{t})$ and the squared coefficient of variation of the interarrival time, $C^2$, as:

\[
E\{\bar{t}\} = \frac{(2-p-q)}{\alpha (1-q)} \tag{7}
\]

\[
C^2 = \frac{Var(\bar{t})}{[E(\bar{t})]^2} = 1 + \alpha \left( \frac{(1-p)(3-q)}{(2-p-q)^2} - 2 \right) + \alpha^2 \frac{(1-q)^2}{(2-p-q)^2} \tag{8}
\]
If $\alpha = 1$, $C^2$ is:

$$C^2 = \frac{(1-p)(p+q)}{(2-p-q)^2} \quad (9)$$

In this paper, we present an approximation method for analyzing the performance of the ATM switch described above. In section 3, we obtain the probability that a cell will be successfully transmitted when it is launched through the switch. In section 4, we analyze the buffer in front of a cell processor as an IBP/Geo/1 queue first with infinite capacity and then with finite capacity. We also give a simple approximation algorithm for analyzing the ATM switch. In section 5, the approximation results are compared against simulation for a $16 \times 16$ switch under a variety of traffic loads. Finally, conclusions are given in section 6.

**3 Probability of success for transmission through the Clos switch**

In this section we extend the probability of success analysis of a crossbar switch [7, 8] to the Clos ATM switch. Let us consider a crossbar switch of size $n \times n$. Assume that in each time slot a cell arrives at each input port independently with probability $p$ and the destination of each cell is randomly selected. The probability that an incoming cell will be successfully transmitted to its destination output port, $P_s$, is computed as follows:

$$P_s = \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{p}{n} \right)^k \left( 1 - \frac{p}{n} \right)^{n-k-1} \frac{1}{k+1}$$

Another way to compute the success probability is the following. In a switching element, the probability that all $n$ input lines will not select a specific output line is $(1 - \frac{p}{n})^n$. The probability that a particular output line is requested by any of the input lines is $1 - (1 - \frac{p}{n})^n$. Thus, the expected number of busy output lines is $n(1 - (1 - \frac{p}{n})^n)$ and the expected number of input lines is $n\rho$. The average probability that an input line will be successfully connected to the destined output line of the crossbar switch is calculated as:

$$P_s = \frac{\text{Expected number of busy output lines}}{\text{Expected number of busy input lines}}$$

$$= \frac{1}{\rho} \left\{ [1 - (1 - \frac{p}{n})^n] \right\} \quad (11)$$

We now proceed to calculate the probability of success for the entire switch.
3.1 Symmetric input case

Let $\rho$ be the utilization of each cell processor of the Clos ATM switch, or in another words, the probability that in each time slot a cell processor has a cell to transmit. Let $N$ be the total number of independent input lines of the Clos ATM switch. Assume that the destination of a cell is uniformly selected. Let $\rho_1$ be the average output line utilization of a switching element in the 1st stage of the Clos ATM switch. From the assumption that there is no blocking in the 1st stage of the switch, we get

$$\rho_1 = \rho$$  \hspace{1cm} (12)

Let $\rho_2$ and $\rho_3$ be the average output line utilizations of the switching element in the 2nd and 3rd stage of the switch given that each input line has $\rho_1, \rho_2$ probability of being busy. We have

$$\rho_i = 1 - \left(1 - \frac{\rho_{i-1}}{n}\right)^n \quad i = 2 \text{ and } 3$$  \hspace{1cm} (13)

The probability for a cell successfully passing through the Clos ATM switch, $1 - \sigma$, is computed as

$$1 - \sigma = \frac{\text{Average number of busy output lines at the 3rd stage}}{\text{Average number of busy input lines at the 1st stage}} = \frac{N \rho_3}{N \rho} = \frac{\rho_3}{\rho}$$  \hspace{1cm} (14)

3.2 Asymmetric input case

Let $\rho_i, i = 1 \cdots N$ be the utilization of the $i$th cell processor of the Clos ATM switch. Assume that in a time slot the $i$th input line of the ATM switch has a cell to transmit with probability $\rho_i$ and the input cells are equally likely destined to the $N$ outputs. The transmission at an input line is assumed to be independent of the other lines. Let $\rho_{i}^{[1]}, \rho_{i}^{[2]}, \rho_{i}^{[3]}$ be the $i$th output line utilizations of the switching elements in the 1st, the 2nd, and the 3rd stage of the Clos ATM switch. Following the assumption that there is no contention in the 1st stage of the switch, the average output line utilization in the 1st stage is computed as:

$$\rho_{kn+j}^{[1]} = \left(\sum_{(j-1)n+1}^{jn} \rho_i\right) / n \quad k = 0, \cdots, n - 1, \quad j = 1, 2, \cdots, n$$  \hspace{1cm} (15)

Since the output lines of the switching element in the previous stage of the switch are the input lines of the next stage and each switching element is an $n \times n$ crossbar switch, we
apply the analysis in crossbar switch to compute $\rho_i^{[2]}$ and $\rho_i^{[3]}, i = 1 \ldots N$. We get

$$\rho_{kn+j}^{[2]} = 1 - \prod_{(j-1)n+1}^{jn} (1 - \frac{\rho_i^{[1]}}{n}) \quad k = 0, \ldots, n - 1, \quad j = 1, 2, \ldots, n$$

and

$$\rho_{kn+j}^{[3]} = 1 - \prod_{(j-1)n+1}^{jn} (1 - \frac{\rho_i^{[2]}}{n}) \quad k = 0, \ldots, n - 1, \quad j = 1, 2, \ldots, n$$

The probability for a cell successfully passing through the Clos ATM switch, $1 - \sigma$, is computed as:

$$1 - \sigma = \frac{\text{Average number of busy output lines at the 3rd stage}}{\text{Average number of busy input lines at the 1st stage}}$$

$$= \frac{\sum_{i=1}^{N} \rho_i^{[3]}}{\sum_{i=1}^{N} \rho_i}$$

$$= \frac{\sum_{i=1}^{N} \rho_i^{[3]}}{\sum_{i=1}^{N} \rho_i}$$

(18)

4 Exact analysis of the IBP/Geo/1 and IBP/Geo/1/K queues

Let us consider a cell processor queue. It is obvious from the previous section that the total time it takes for the cell processor to transmit successfully a cell through the switch fabric is geometrically distributed with probability $\sigma$. Therefore, a cell processor queue can be modeled by a discrete IBP/Geo/1/K queue. Customers are served in this queue in a FIFO manner. The IBP/Geo/1 queue is a special case of the GI/Geo/1 queue which has been already analyzed (see Hunter[9], chapter 9). It can be also shown that the GI/Geo/1/K queue can be analyzed due to a duality with the Geo/GI/1/K queue. The dual of the GI/Geo/1/K queue is constructed by looking at the flow of holes through the queue. The GI distribution becomes the arrival distribution for the holes, and the Geo distribution becomes their service distribution. In this case, it can be shown that the queue-length distribution in a GI/Geo/1/K queue is equal to the distribution of the holes obtained by analyzing a Geo/GI/1/K queue. Now, the queue-length distribution in the latter queue is obtained by truncating the queue-length distribution in the Geo/GI/1 queue. We note that this approach requires that the resulting Geo/GI/1 queue is stable, which may not be the case in computer communication systems. For further references see Hunter, Neuts[10], Klimko and Neuts[11], Neuts and Klimko[12], and Heimann and Neuts[13]. Below, we give an exact analysis of the IBP/Geo/1 queue. We first consider the case where the cell processor queue is infinite and subsequently we analyze the case where the cell processor queue is finite.
We also note that these two discrete queues can be analyzed numerically under more general assumptions regarding the arrival process (such as using a Markov Modulated Bernoulli Process[14]). In this case, the rate matrix has a block tri-diagonal structure, and one can use the matrix geometric procedure or any other numerical procedure to solve the problem.

4.1 The IBP/Geo/1 queue

Figure 2: The evolution of the number of customers in the system.

In the analysis of the queueing system we can consider the successive arrival points as shown in figure 2. These arrival points are the imbedded Markov points of the queueing system. Our approach is to focus attention on arrival instants and the number of customers seen by an arriving customer. Let $\tilde{\ell}$ be the interarrival time and $\tilde{x}$ be the number of slots needed for a cell to be successfully transmitted to the output line. We define

\begin{align}
P[\tilde{\ell} = l] &= a_l, \quad l \geq 1 \quad \text{and} \\
\gamma_{[x = l]} &= (1 - \sigma)\sigma^{i-1}, \quad i \geq 1
\end{align}

where $a_l$ is the probability of an interarrival time of $l$ slots, and $\sigma$ is the probability of a cell to be blocked in the Clos ATM switch and retransmitted.

Let $\gamma_j$ be the steady state probability that an arriving cell finds $j$ customers in the system immediately prior to its arrival. For $j > 1$, see figure 2, $i + 1 - j$ customers must be served during an interarrival time. Thus, we get

\begin{align}
\gamma_j &= \sum_{i=j-1}^{\infty} \sum_{l=1}^{\infty} \gamma_i \left( \binom{l}{i+1-j} (1 - \sigma)^{i+1-j} \sigma^{i-(i+1-j)} a_l \right)
\end{align}

For this discrete system, a customer arriving to an idle system can not be served until the next time slot. Thus for $\gamma_1$ we get

\begin{align}
\gamma_1 &= \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \gamma_i \left( \binom{l}{i} (1 - \sigma)^{i} a_l \right) + \sum_{l=1}^{\infty} \gamma_0 \left( \binom{l-1}{0} (1 - \sigma)^{0} a_l \right)
\end{align}
From equation (21), we obtain the solution:

\[ \gamma_i = C\alpha^i, \quad i \geq 1. \]  

(23)

where

\[ \alpha = A(\sigma + \alpha(1 - \sigma)) \quad \text{and} \quad A(z) = \sum_{i=1}^{\infty} a_iz^i \]

Since \( \gamma_1 \) is equal to \( C\alpha \) we have using equation (22)

\[ C\alpha = CA(\sigma + \alpha(1 - \sigma)) - CA(\sigma) + \frac{\gamma_0}{\sigma}A(\sigma) \]

which gives us that

\[ C = \frac{\gamma_0}{\sigma}. \]  

(24)

From the normalization equation \( \gamma_0 + \sum_{i=1}^{\infty} \gamma_i = 1 \) it is found that

\[ C = \frac{1 - \alpha}{\sigma(1 - \alpha) + \alpha} \]

Consequently the probability that a customer finds an empty system is

\[ \gamma_0 = \frac{(1 - \alpha)\sigma}{\sigma + \alpha(1 - \sigma)} \]  

(25)

In general, the probability that an arriving customer finds \( j \) customers in the system, \( \gamma_j \), is computed as

\[ \gamma_j = \begin{cases} \frac{(1-\alpha)\sigma}{\sigma(1-\alpha)+\alpha}, & j = 0 \\ \frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\alpha^{j-1}, & j \geq 1 \end{cases} \]

(26)

4.1.1 Waiting time distribution

Let \( x_0 \) be the time interval between the arrival instant and the beginning of the next slot. It is the time that every cell, or customer, has to spend in the system. Let a slot time be the unit of time and \( w \) be the waiting time in the system. An arriving customer has to wait \( x_0 \) if the arrival finds either an empty system or only one customer in the system which finishes
service in the slot. We get

\[
\begin{align*}
  w &= \begin{cases} 
    x_0, & \gamma_0 + \gamma_1(1-\sigma) \\
    1 + x_0, & \gamma_1 \sigma(1-\sigma) + \gamma_2(1-\sigma)^2 \\
    2 + x_0, & \gamma_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sigma^2(1-\sigma) + \gamma_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sigma(1-\sigma)^2 + \gamma_3 \begin{pmatrix} 2 \\ 2 \end{pmatrix} (1-\sigma)^3 \\
    3 + x_0, & \gamma_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sigma^2(1-\sigma) + \gamma_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \sigma^2(1-\sigma)^2 + \gamma_3 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \sigma(1-\sigma)^3 + \gamma_4 \begin{pmatrix} 3 \\ 3 \end{pmatrix} (1-\sigma)^4 \\
    \vdots & \vdots \\
    \vdots & \vdots \\
  \end{cases}
\end{align*}
\]

From equations (26), the distribution of the waiting time is as follows:

\[
P[w = x_0 + j] = \begin{cases} 
  \gamma_0(1 + \frac{\alpha(1-\sigma)}{\sigma}), & j = 0 \\
  \alpha(1-\alpha)(1-\sigma)[\sigma + \alpha(1-\sigma)]^{j-1}, & j \geq 1 
\end{cases}
\]  

(27)

From the above equation, the average waiting time in the system, \(E(w)\), is calculated as:

\[
E(w) = x_0 + E(j) \\
= x_0 + \sum_{n=0}^{\infty} nP(j = n) \\
= x_0 + \frac{\alpha(1-\alpha)(1-\sigma)}{(1 - [\alpha + \alpha(1-\sigma)])^2}
\]  

(28)

4.2 The IBP/Geo/1/K queue

![IBP/Geo/1/K Queue Diagram](image)

*Figure 3: The finite buffer queue.*

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The queueing model under study is shown in figure 3. We assume for the moment that the probability of retransmission $\sigma$ is known.

Let $K$ be the system or buffer size. In this system, if a customer finds that the buffer is full, the customer will be lost. Let $\gamma_j$, $K \geq j \geq 0$, be the steady state probability that an arriving customer finds $j$ customers in the system. In order to obtain $\gamma_K$ we need to study the system at instants of arrivals. In steady state, we have

$$\gamma_K = \sum_{l=1}^{\infty} \gamma_{K-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (1 - \sigma)^0 \sigma^l a_l + \sum_{l=1}^{\infty} \gamma_K \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (1 - \sigma)^0 \sigma^l a_l$$

$$= A(\sigma)(\gamma_{K-1} + \gamma_K)$$

(29)

and

$$\gamma_{K-1} = \sum_{l=1}^{\infty} \gamma_{K-1} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) (1 - \sigma)^1 \sigma^{l-1} a_l + \sum_{l=1}^{\infty} \gamma_{K-2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (1 - \sigma)^0 \sigma^l a_l$$

$$+ \sum_{l=1}^{\infty} \gamma_K \left( \begin{array}{c} 1 \\ 1 \end{array} \right) (1 - \sigma)^1 \sigma^{l-1} a_l$$

$$= \gamma_{K-1}(1 - \sigma)A'(\sigma) + \gamma_{K-2}A(\sigma) + \gamma_K(1 - \sigma)A'(\sigma)$$

(30)

and

$$\gamma_{K-2} = \sum_{l=1}^{\infty} \gamma_{K-3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (1 - \sigma)^0 \sigma^l a_l + \sum_{l=1}^{\infty} \gamma_{K-2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) (1 - \sigma)^1 \sigma^{l-1} a_l$$

$$+ \sum_{l=1}^{\infty} \gamma_{K-1} \left( \begin{array}{c} 1 \\ 2 \end{array} \right) (1 - \sigma)^2 \sigma^{l-1} a_l + \sum_{l=1}^{\infty} \gamma_K \left( \begin{array}{c} 1 \\ 2 \end{array} \right) (1 - \sigma)^2 \sigma^{l-2} a_l$$

$$= \gamma_{K-3}A(\sigma) + \gamma_{K-2}(1 - \sigma)A'(\sigma) + \gamma_{K-1}\frac{(1 - \sigma)^2}{2!}A''(\sigma) + \gamma_K\frac{(1 - \sigma)^2}{2!}A''(\sigma)$$

(31)

and so on.

From the above equations, we get the relations between $\gamma_j$ and $\gamma_{j+1}$. We obtain

$$\gamma_{K-1} = \frac{1 - A(\sigma)}{A(\sigma)} \gamma_K = b_{K-1} \gamma_K$$

(32)

$$\gamma_{K-2} = \frac{1 - (1 - \sigma)A'(\sigma) - \frac{1}{b_{K-1}}(1 - \sigma)A'(\sigma)}{A(\sigma)} \gamma_{K-1} = b_{K-2} \gamma_{K-1}$$

(33)

$$\gamma_{K-3} = \frac{1 - (1 - \sigma)A'(\sigma) - \frac{1}{b_{K-2}}(1 - \sigma)^2 A''(\sigma) - \frac{1}{b_{K-1} b_{K-2}}(1 - \sigma)^2 A''(\sigma)}{A(\sigma)} \gamma_{K-2} = b_{K-3} \gamma_{K-2}$$

(34)

e tc.
In general the probability that an arriving customer finds \( j \) customers in the system, \( \gamma_j \), can be computed as follows;

for \( j = K - 1 \),

\[
\gamma_{K-1} = \frac{1 - A(\sigma)}{A(\sigma)} \gamma_K
\]  \hspace{1cm} \text{(35)}

for \( K - 1 > j > 0 \),

\[
\gamma_j = \left[ \frac{1 - (1-\sigma)A'(\sigma) - \sum_{i=2}^{K-1-j} \left( \prod_{l=1}^{i-1} \frac{1}{b_{j+l}} \right) \frac{1}{i!} (1-\sigma)^i A(i)(\sigma)}{\frac{A(\sigma)}{A(\sigma/\sigma)}} \right] \gamma_{j+1}
\]  \hspace{1cm} \text{(36)}

for \( j = 0 \),

\[
\gamma_0 = \left[ \frac{1 - (1-\sigma)A'(\sigma) - \sum_{i=2}^{K-1} \left( \prod_{l=1}^{i-1} \frac{1}{b_{i+l}} \right) \frac{1}{i!} (1-\sigma)^i A(i)(\sigma)}{\frac{A(\sigma)}{A(\sigma/\sigma)}} \right] \gamma_1
\]  \hspace{1cm} \text{(37)}

The probability of blocking in the system, \( \gamma_K \), is computed from the normalization equation:

\[
\sum_{i=0}^{K} \gamma_i = 1.
\]  \hspace{1cm} \text{(38)}

### 4.2.1 Computation of \( \frac{A^{(i)}(\sigma)}{i!} \)

From equation (6), we have

\[
A(z) = \frac{z\alpha[p + z(1 - p - q)]}{(1 - \alpha)(p + q - 1)z^2 - [q + p(1 - \alpha)]z + 1}
\]

\[
= \frac{az + bz + cz^2}{dz + ez + fz^2}
\]

Let us consider the function \( A(z) \) which is analytic in a neighborhood of a point \( z = \sigma \), \( 0 < \sigma < 1 \). Then we can expand \( A(z) \) in a Taylor series.

\[
A(z) = \sum_{m=0}^{\infty} \frac{A^{(m)}(\sigma)}{m!} (z - \sigma)^m
\]  \hspace{1cm} \text{(39)}
We can now compute \( \frac{A(i)(\sigma)}{i!} \), \( K \geq i \geq 0 \), as follows:

**step 0.** Set up initial values as: \( i = 0, a = a_1 = 0, b = b_1 = p\alpha, c = c_1 = \alpha(1 - p - q)d = d_1 = 1, e = e_1 = -p(1 - \alpha), f = f_1 = (1 - \alpha)(p + q - 1) \).

**step 1.** Compute \( A(\sigma) \) or \( A^{(0)}(\sigma) \) as

\[
A(\sigma) = \frac{a + b\sigma + c\sigma^2}{d + e\sigma + f\sigma^2}
\]

**step 2.** Let \( i = i + 1 \) and set coefficients: \( b_2 = b_1 - e_1 A^{(i-1)}(\sigma), c_2 = c_1 - f_1 A^{(i-1)}(\sigma) \)

**step 3.** Compute \( \frac{A(i)(\sigma)}{i!} \) as

\[
\frac{A(i)(\sigma)}{i!} = \frac{b_2 + 2c_2\sigma}{d_1 + e_1\sigma + f_1\sigma^2}
\]

**step 4.** Set coefficients : \( b_1 = c_2, c_1 = 0 \).

**step 5.** If \( i \leq K \) then go to step 2, else STOP.

4.2.2 Waiting time distribution.

Let \( x_0 \) be the time interval between the arrival instant and the beginning of the next slot and \( w \) be the waiting time in the system. Thus, we get

\[
w = \begin{cases}
  x_0, & \gamma_0 + \gamma_1(1 - \sigma) \\
  1 + x_0, & \gamma_1\sigma(1 - \sigma) + \gamma_2(1 - \sigma)^2 \\
  2 + x_0, & \gamma_1 \binom{2}{0} \sigma^2(1 - \sigma) + \gamma_2 \binom{2}{1} \sigma(1 - \sigma)^2 + \gamma_3 \binom{2}{2} (1 - \sigma)^3 \\
  3 + x_0, & \gamma_1 \binom{3}{0} \sigma^3(1 - \sigma) + \gamma_2 \binom{3}{1} \sigma^2(1 - \sigma)^2 + \cdots + \gamma_4 \binom{3}{3} (1 - \sigma)^4 \\
  \vdots \\
  K - 2 + x_0, & \gamma_1 \binom{K - 2}{0} \sigma^{K-2}(1 - \sigma) + \cdots + \gamma_{K-1} \binom{K - 2}{K - 2} (1 - \sigma)^{K-1}
\end{cases}
\]
In general, we have the waiting time distribution:

\[
P[w = x_0 + j] = \begin{cases} 
\gamma_0 + \gamma_1(1 - \sigma), & j = 0 \\
\sum_{n=1}^{j+1} \gamma_n \left( \begin{array}{c} j \\ n-1 \end{array} \right) \sigma^{j-n+1}(1 - \sigma)^{n-1}(1 - \sigma), & \text{for } K - 2 \geq j \geq 1. \\
\sum_{n=1}^{K-1} \gamma_n \left( \begin{array}{c} j \\ n-1 \end{array} \right) \sigma^{j-n+1}(1 - \sigma)^{n-1}(1 - \sigma), & \text{for } j > K - 2.
\end{cases}
\]

5 Approximate analysis of the switch

In this section, we describe a simple approximation algorithm for analyzing the ATM switch. We assume that each cell processor queue has a finite capacity. The approximation algorithm utilizes expression (18) for \( \sigma \), and the exact results for the IBP/Geo/1/K queue obtained in section 4.2. The algorithm is an iterative scheme involving the following steps:

**step 0.** Set up the initial value of the utilization of the \( i \)th cell processor, \( \rho_{i}^{(0)}, i = 1, 2 \cdots N \).

**step 1.** Compute the probability of retransmission of a cell due to contention in the Clos ATM switch, \( \sigma^{(0)} \).

**step 2.** Solve the IBP/Geo/1/K queueing system to find the probability of cell loss due to the fact that the buffer is full, \( \gamma_K \).

**step 3.** Compute the utilization of the cell processor, \( \rho_{i}^{(1)} \):

\[
\rho_{i}^{(1)} = \frac{\alpha(1 - q)}{2 - p - q(1 - \gamma_K)}, \quad i = 1, 2 \cdots N.
\]

**step 4.** Compute \( \sigma^{(1)} \) the probability of retransmission of a cell due to contention in the Clos ATM switch by using the new utilization of the cell processors, \( \rho_{i}^{(1)}, i = 1, \cdots, N \).

**step 5.** If \( |\sigma^{(1)} - \sigma^{(0)}| < \epsilon \) then STOP else set \( \sigma^{(0)} = \sigma^{(1)} \) and go to step 2.
6 Numerical Results

The approximation algorithm described in the previous section was employed to analyze a 16 x 16 bufferless Clos ATM switch. Each switching element was a 4 x 4 crossbar switch. The buffer size of each cell processor queue was set equal to 32. The arrival process to each cell processor was assumed to be an IBP with \( \alpha = 1 \). That is, during the busy period each slot contains a cell. The approximation results were compared against results obtained by simulation. The results are summarized in figure 6 to 6. Figures 6 to 6 are for a symmetric case, and figures 6 to 6 are for two asymmetric cases.

In the symmetric case, we assume the same arrival process to each input line. Each arriving cell chooses one of the destination output lines randomly. In figure 6, we plot the approximate and simulation results for the queue-length distribution of cell processor queue for two different values of \( C^2 \) the squared coefficient of variation of the interarrival time of the arrival process. The average arrival rate, \( \lambda \), was set equal to 0.4. In figure 6, we give the absolute errors calculated as (simulation - approximation) for the results given in figure 6. The log(base 10) of the approximation and simulation results for the blocking probability \( \gamma_K \) are given in figure 6 as a function of \( C^2 \). The absolute errors are given in figure 6.

In the two asymmetric cases considered here we assume that the average arrival rate to each input port \( i \) has a different value. In case 1, \( \lambda_i = 0.4, \lambda_{16} = 0.1 \), and the values of the remaining \( \lambda_i \)'s are uniformly distributed between 0.4 and 0.1. In case 2, \( \lambda_i = 0.4, i = 1, 2, \ldots, 8 \), and \( \lambda_i = 0.1, i = 9, 10, \ldots, 16 \). In figure 6, we plot the approximate and simulation results for the queue-length distribution of the cell processor queues corresponding to input line 1 (the most heavily used) and input line 16 (the least used). The value of \( C^2 \) was set to 20. The absolute errors are given in figure 6. The log(base 10) of the approximate and simulation results for the blocking probability for the same two cell processor queues are given in figure 6 as a function of \( C^2 \). The corresponding absolute errors are given in figure 6. Finally, figures 6 and 6 give results for the asymmetric case 2, assuming \( C^2 = 20 \) for line 1 and \( C^2 = 5 \) for line 16.

From the above results, we see that the approximation algorithm has a satisfactory accuracy. The absolute error of the blocking probabilities becomes large when the average utilization of the input line is small or the squared coefficient of variation \( C^2 \) of the interarrival time is small. The reason for this is that we have observed empirically that the variance of the number of busy input lines depends on \( C^2 \). It becomes bigger when \( C^2 \) is small, and smaller when \( C^2 \) is large. However, the way we calculate the probability of success, \( 1 - \sigma \), implies that the number of busy input lines has a Bernoulli distribution. This assumption becomes more justified when \( C^2 \) is large.
Figure 4: Queue length distribution for various values of $C^2$ (symmetric case).

Figure 5: Absolute errors for the results given in figure 4.
Figure 6: Approximation and simulation results for the blocking probability (symmetric case).

Figure 7: Absolute errors for the results given in figure 6.
Figure 8: Queue length distribution (asymmetric case 1, C**2=20).

Figure 9: Absolute errors for the results given in figure 8.
Figure 10: Approximation and simulation results for the blocking probability (asymmetric case 1).

Figure 11: Absolute errors for the results in figure 10.
Figure 12: Queue length distribution (asymmetric case 2).

Figure 13: Absolute errors for the results given in figure 12.
7 Conclusion

A synchronous $N \times N$ Clos ATM switch with queueing capability at the input ports was analyzed. The arrival process is modeled by an Interrupted Bernoulli Process. The probability of success for transmission through the Clos switch was obtained approximately under an independent assumption and assuming that arrivals are Bernoulli distributed. The cell processor queue was analyzed as an $IBP/Geo/1$ queue first with infinite capacities and then with finite capacities. Using the above results the ATM switch was then analyzed approximately. The approximation algorithm was extensively validated using simulation results. The algorithm has a satisfactory accuracy. The absolute error of the blocking probability increases when the average utilization of the input line is small or the squared coefficient of variation $C^2$ of the interarrival time is small.

References


