A NEW INEQUALITY IN ABSTRACT $L^p$ SPACES

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CCSP-TR-89/14
Theorem. Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $q$ be an extended positive real number. Then for every $t$ in $[1, +\infty]$ and every real-valued function $X$ in $L^q(\Omega, \Sigma, \mu)$ such that $X \neq 0$ a.e. the function

$$f_X: [0, q] \rightarrow [(\mu(\Omega))^{1-t} , +\infty]$$

$$p \mapsto \frac{\int \Omega |X|^p d\mu}{\left( \int \Omega |X|^p d\mu \right)^t}$$

is nondecreasing.

Proof. First, let $X$ be a simple function

$$X = \sum_{i=1}^n x_i 1_{A_i} \quad \text{where} \quad \left\{ \begin{array}{l} 0 < x_1 < x_2 < \ldots < x_n \\ A_1, \ldots, A_n \text{ disjoint sets in } \Sigma \end{array} \right.$$  \hspace{1cm} (1)

Let $p$ be an arbitrary real number in $[0, q]$ and, for every $i$ in $\{1, \ldots, n\}$, let $p_i = \mu(A_i)$. We have

$$f_X(p) = \frac{\sum_{i=1}^n p_i x_i^p}{\left( \sum_{i=1}^n p_i x_i^p \right)^t}$$  \hspace{1cm} (2)

We need to show that $df_X(p)/dp \geq 0$. Using (2) and the fact that $\left( \sum_{i=1}^n p_i x_i^p \right)^{t-1} > 0$, this is equivalent to showing

$$(\sum_{i=1}^n p_i x_i^p \ln(x_i)) (\sum_{i=1}^n p_i x_i^p) - (\sum_{i=1}^n p_i x_i^p)(\sum_{i=1}^n p_i x_i^p \ln(x_i)) \geq 0$$  \hspace{1cm} (3)

If $n = 1$ or $t = 1$ the result is trivial. We shall therefore assume otherwise. Let $i$ and $j$ be any two integers in $\{1, \ldots, n\}$ such that $i < j$. Then $x_j > x_i$ and therefore $x_j^{(t-1)p} - x_i^{(t-1)p} > 0$ and $\ln(x_j) > \ln(x_i)$. Thus

$$(x_j^{(t-1)p} - x_i^{(t-1)p}) \ln(x_j) > (x_j^{(t-1)p} - x_i^{(t-1)p}) \ln(x_i)$$  \hspace{1cm} (4)

Hence

$$x_i^{(t-1)p} \ln(x_i) + x_j^{(t-1)p} \ln(x_j) > x_i^{(t-1)p} \ln(x_j) + x_j^{(t-1)p} \ln(x_i)$$  \hspace{1cm} (5)
Upon multiplying through by $p_i p_j x_i^p x_j^p$ in (5) and summing up, we get

$$
\sum_{1 \leq i < j \leq n} (p_i x_i^p \ln(x_i))(p_j x_j^p) + (p_i x_i^p)(p_j x_j^p \ln(x_j)) >
$$

$$
\sum_{1 \leq i < j \leq n} (p_i x_i^p)(p_j x_j^p \ln(x_j)) + (p_i x_i^p \ln(x_i))(p_j x_j^p)
$$

(6)

Consider the identity

$$
(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} b_i) = \sum_{i=1}^{n} a_i b_i + \sum_{1 \leq i < j \leq n} a_i b_j + a_j b_i
$$

(7)

Then, adding $\sum_{i=1}^{n} p_i x_i^{t+1_\eta} \ln(x_i)$ to each side in (6) yields

$$
(\sum_{i=1}^{n} p_i x_i^{t_\eta} \ln(x_i))(\sum_{i=1}^{n} p_i x_i^p) > (\sum_{i=1}^{n} p_i x_i^{t_\eta})(\sum_{i=1}^{n} p_i x_i^p \ln(x_i))
$$

(8)

Which establishes (3). Hence the theorem holds for every nonnegative simple function. Now let $X$ be an arbitrary function in $L^q(\Omega, \Sigma, \mu)$ such that $X \neq 0$ a.e. Then, there exists a nondecreasing sequence $\{X_n\}_{n \geq 0}$ of nonnegative simple functions converging a.e. to $|X|$ and

$$
\int |X| d\mu = \lim_{n \to +\infty} \int X_n d\mu
$$

(9)

It follows that

$$
\int |X|^p d\mu = \lim_{n \to +\infty} \int X_n^p d\mu \quad \text{and} \quad \int |X|^p d\mu = \lim_{n \to +\infty} \int X_n^p d\mu
$$

(10)

Therefore

$$
\lim_{n \to +\infty} f_{X_n}(p) = \frac{\lim_{n \to +\infty} \int X_n^p d\mu}{\left(\lim_{n \to +\infty} \int X_n^p d\mu\right)^t} = \frac{\int |X|^p d\mu}{\left(\int |X|^p d\mu\right)^t} = f_X(p)
$$

(11)

We conclude that since each term in the sequence $\{f_{X_n}\}_{n \geq 0}$ is a nondecreasing function of $p$, so is the limit $f_X$. □
The Theorem leads at once to the following inequality.

**Corollary 1.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space, let $p \leq q$ be any two extended positive real numbers, and let $X$ be a measurable real-valued function. Then

$$\left( \forall t \in [1, +\infty[ \right) \quad \int_{\Omega} |X|^q \, d\mu \left( \frac{\int_{\Omega} |X|^p \, d\mu}{\int_{\Omega} |X|^q \, d\mu} \right)^{1/t} \leq \int_{\Omega} |X|^p \, d\mu \left( \frac{\int_{\Omega} |X|^q \, d\mu}{\int_{\Omega} |X|^p \, d\mu} \right)^{1/t}$$

The application of the Theorem to probability theory is immediate.

**Corollary 2.** Let $(\Omega, \Sigma, P)$ be a probability space and let $q$ be an extended positive real number. Then for every $t$ in $[1, +\infty[$ and every random variable $X$ in $L^q(\Omega, \Sigma, P)$ such that $X \neq 0$ a.s. the function

$$f_X: [0, q] \rightarrow [1, +\infty]$$

$$p \mapsto \frac{E|X|^p}{E^t|X|^p}$$

is nondecreasing.

The last Corollary is obtained by considering the case where $\Omega=\{1,\ldots,n\}$, $\Sigma$ is the $\sigma$-algebra of all subsets of $\Omega$, and $\mu$ is the counting measure.

**Corollary 3.** Let $n$ be a positive integer and let $x=(x_1,\ldots,x_n)$ be a nonzero vector in $\mathbb{R}^n$. Then for every $t$ in $[1, +\infty]$ the function

$$f_x: \mathbb{R}_+ \rightarrow [n^{-t}, +\infty[$$

$$p \mapsto \frac{\sum_{i=1}^{n} |x_i|^p}{(\sum_{i=1}^{n} |x_i|^p)^t}$$

is nondecreasing.

2. For all $r < t$ in $\mathbb{R}_+$ a consequence of Hölder’s inequality is

$$\left( \int_Y |Y|^r d\mu \right)^{1/r} \leq \left( \int_Y |Y|^t d\mu \right)^{1/t} (\mu(\Omega))^{1/r-1/t}$$

(12)

For $\mu(\Omega) < +\infty$, $|Y|^r = |X|^p$ ($p \in \mathbb{R}_+$), and $r = 1$ it follows at once that $f_X(p) \geq (\mu(\Omega))^{1-t}$.

3. From (12), if $(\Omega, \Sigma, \mu)$ is a finite measure space, then $L^q(\Omega, \Sigma, \mu) \subset L^p(\Omega, \Sigma, \mu)$ if $p \leq q$. Hence $f_X$ is well defined (we never have $\infty/\infty$).

4. From (8), $f_X$ is increasing if $X$ is a simple function, $n > 1$, and $t > 1$.

5. We only require that $X$ be measurable in Corollary 1. Indeed, all the integrals in the inequality exist and, if any of them is $+\infty$, so is the rightmost one by (12) (the result then is trivial). The result is also trivial if $X = 0$ a.e. In all the other cases, the result follows from the Theorem.

6. For $t=2$, the result of Corollary 2 was needed in P. L. Combettes’ Ph.D. dissertation (*Set Theoretic Estimation in Digital Signal Processing*, Department of Electrical and Computer Engineering, NC State University. Raleigh, June 1989).

7. A more elegant proof of Corollary 2 has recently been communicated to the author by Professor O. Wesler.