A NEW INEQUALITY IN ABSTRACT L^p SPACES

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Theorem. Let (Ω,Σ,μ) be a finite measure space and let q be an extended positive real number. Then for every t in $[1,+\infty[$ and every real-valued function X in $L^q(\Omega,\Sigma,\mu)$ such that $X\neq 0$ a.e. the function

$$f_X\colon [0,q] \to \left[(\mu(\Omega))^{1-t},+\infty\right]$$

$$p \mapsto \frac{\int_\Omega |X|^{tp}\,d\mu}{\left(\int_\Omega |X|^p\,d\mu\right)^t}$$

is nondecreasing.

Proof. First, let X be a simple function

$$X=\sum_{i=1}^{n} x_i 1_{A_i} \quad \text{where} \quad \begin{cases} 0<x_1<x_2<\ldots<x_n \\ A_1,\ldots,A_n \text{ disjoint sets in } \Sigma \end{cases} \tag{1}$$

Let p be an arbitrary real number in $[0,q]$ and, for every i in $\{1,\ldots,n\}$, let $p_i=\mu(A_i)$. We have

$$f_X(p) = \frac{\sum_{i=1}^{n} p_i x_i^{tp}}{\left(\sum_{i=1}^{n} p_i x_i^p\right)^t} \tag{2}$$

We need to show that $df_X(p)/dp\geq 0$. Using (2) and the fact that $(\sum_{i=1}^{n} p_i x_i^p)^{t-1}>0$, this is equivalent to showing

$$(\sum_{i=1}^{n} p_i x_i^{tp}\ln(x_i))(\sum_{i=1}^{n} p_i x_i^p) - (\sum_{i=1}^{n} p_i x_i^{tp})(\sum_{i=1}^{n} p_i x_i^p\ln(x_i)) \geq 0 \tag{3}$$

If $n=1$ or $t=1$ the result is trivial. We shall therefore assume otherwise. Let i and j be any two integers in $\{1,\ldots,n\}$ such that $i<j$. Then $x_j>x_i$ and therefore $x_j^{(t-1)p}-x_i^{(t-1)p}>0$ and $\ln(x_j)>\ln(x_i)$. Thus

$$(x_j^{(t-1)p}-x_i^{(t-1)p})\ln(x_j) > (x_j^{(t-1)p}-x_i^{(t-1)p})\ln(x_i) \tag{4}$$

Hence

$$x_i^{(t-1)p}\ln(x_i) + x_j^{(t-1)p}\ln(x_j) > x_i^{(t-1)p}\ln(x_j) + x_j^{(t-1)p}\ln(x_i) \tag{5}$$

Upon multiplying through by $p_i p_j x_i^p x_j^p$ in (5) and summing up, we get

$$\sum_{1\le i<j\le n} (p_i x_i^{tp}\ln(x_i))(p_j x_j^p)+(p_i x_i^p)(p_j x_j^{tp}\ln(x_j)) >$$

$$\sum_{1\le i<j\le n} (p_i x_i^{tp})(p_j x_j^p\ln(x_j))+(p_i x_i^p\ln(x_i))(p_j x_j^{tp}) \tag{6}$$

Consider the identity

$$\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) = \sum_{i=1}^n a_i b_i + \sum_{1\le i<j\le n} a_i b_j + a_j b_i \tag{7}$$

Then, adding $\sum_{i=1}^n p_i^2 x_i^{(t+1)p}\ln(x_i)$ to each side in (6) yields

$$\left(\sum_{i=1}^n p_i x_i^{tp}\ln(x_i)\right)\left(\sum_{i=1}^n p_i x_i^p\right) > \left(\sum_{i=1}^n p_i x_i^{tp}\right)\left(\sum_{i=1}^n p_i x_i^p\ln(x_i)\right) \tag{8}$$

Which establishes (3). Hence the theorem holds for every nonnegative simple function. Now let X be an arbitrary function in $L^q(\Omega,\Sigma,\mu)$ such that $X\neq 0$ a.e. Then, there exists a nondecreasing sequence $\{X_n\}_{n\ge 0}$ of nonnegative simple functions converging a.e. to $|X|$ and

$$\int_\Omega |X|\,d\mu = \lim_{n\to+\infty} \int_\Omega X_n\,d\mu \tag{9}$$

It follows that

$$\int_\Omega |X|^p\,d\mu = \lim_{n\to+\infty} \int_\Omega X_n^p\,d\mu \quad \text{and} \quad \int_\Omega |X|^{tp}\,d\mu = \lim_{n\to+\infty} \int_\Omega X_n^{tp}\,d\mu \tag{10}$$

Therefore

$$\lim_{n\to+\infty} f_{X_n}(p) = \frac{\lim_{n\to+\infty} \int_\Omega X_n^{tp}\,d\mu}{\left(\lim_{n\to+\infty} \int_\Omega X_n^p\,d\mu\right)^t} = \frac{\int_\Omega |X|^{tp}\,d\mu}{\left(\int_\Omega |X|^p\,d\mu\right)^t} = f_X(p) \tag{11}$$

We conclude that since each term in the sequence $\{f_{X_n}\}_{n\ge 0}$ is a nondecreasing function of p, so is the limit f_X. $\square$

The Theorem leads at once to the following inequality.

Corollary 1. Let (Ω,Σ,μ) be a finite measure space, let $p \le q$ be any two extended positive real numbers, and let X be a measurable real-valued function. Then

$$(\forall t \in [1,+\infty[) \qquad \int_\Omega |X|^q \, d\mu \left(\int_\Omega |X|^{tp} \, d\mu \right)^{1/t} \le \int_\Omega |X|^p \, d\mu \left(\int_\Omega |X|^{tq} \, d\mu \right)^{1/t}$$

The application of the Theorem to probability theory is immediate.

Corollary 2. Let (Ω,Σ,P) be a probability space and let q be an extended positive real number. Then for every t in $[1,+\infty[$ and every random variable X in $L^q(\Omega,\Sigma,P)$ such that $X \neq 0$ a.s. the function

$$\begin{aligned} f_X \colon [0,q] &\to [1,+\infty] \\ p &\mapsto \frac{E|X|^{tp}}{E^t|X|^p} \end{aligned}$$

is nondecreasing.

The last Corollary is obtained by considering the case where $\Omega=\{1,\ldots,n\}$, Σ is the σ-algebra of all subsets of Ω, and μ is the counting measure.

Corollary 3. Let n be a positive integer and let $x=(x_1,\ldots,x_n)$ be a nonzero vector in $\mathbb{R}^n$. Then for every t in $[1,+\infty[$ the function

$$\begin{aligned} f_x \colon \mathbb{R}_+ &\to [n^{1-t},+\infty[\\ p &\mapsto \frac{\sum_{i=1}^{n} |x_i|^{tp}}{(\sum_{i=1}^{n} |x_i|^p)^t} \end{aligned}$$

is nondecreasing.

Notes

1. For measure theoretic notions, see: Paul R. Halmos, *Measure Theory,* New York: Van Nostrand, 1950.

2. For all $r<t$ in R_+^* a consequence of Hölder's inequality is

$$\Big(\int_\Omega |Y|^r \, d\mu\Big)^{1/r} \leq \Big(\int_\Omega |Y|^t \, d\mu\Big)^{1/t} (\mu(\Omega))^{1/r-1/t} \tag{12}$$

For $\mu(\Omega)<+\infty$, $|Y|=|X|^p$ ($p \in \mathrm{R}_+^*$), and $r=1$ it follows at once that $f_X(p) \geq (\mu(\Omega))^{1-t}$.

3. From (12), if (Ω,Σ,μ) is a finite measure space, then $L^q(\Omega,\Sigma,\mu) \subset L^p(\Omega,\Sigma,\mu)$ if $p \leq q$. Hence f_X is well defined (we never have ∞/∞).

4. From (8), f_X is increasing if X is a simple function, $n>1$, and $t>1$.

5. We only require that X be measurable in Corollary 1. Indeed, all the integrals in the inequality exist and, if any of them is $+\infty$, so is the rightmost one by (12) (the result then is trivial). The result is also trivial if $X=0$ a.e. In all the other cases, the result follows from the Theorem.

6. For $t=2$, the result of Corollary 2 was needed in P. L. Combettes' Ph.D. dissertation *(Set Theoretic Estimation in Digital Signal Processing,* Department of Electrical and Computer Engineering, NC State University. Raleigh, June 1989).

7. A more elegant proof of Corollary 2 has recently been communicated to the author by Professor O. Wesler.