EQUIVALENCIES BETWEEN OPEN AND CLOSED QUEUEING NETWORKS WITH FINITE BUFFERS

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Abstract

Equivalencies between open and closed models of queueing networks with finite buffers are investigated. Interarrival times in open networks and the processing times are assumed to be exponentially distributed. Using such equivalencies bounds on the throughput of open networks are established.

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1. Introduction

Queueing network models can be classified as open and closed queueing networks. In particular, in open models, customers enter the network from outside and depart after receiving service at a number of service stations. In closed models, there is a fixed number of customers circulating in the network at all times. Hence, no arrivals to and no departures from the network are allowed.

In this paper, we consider both open and closed queueing networks with blocking. A closed queueing network with blocking (hereafter referred to as CQN-B) is assumed to consist of N nodes and K customers. Each node consists of a finite single queue served by a server with an exponentially distributed service time with rate $\mu_i$; $B_i$ is the capacity of node $i$ including the service space in front of server $i$, $i=1,\ldots,N$. A customer at node $i$ will join node $j$ with probability $p_{ij}$, $i,j=1,\ldots,N$. Customers at each node are served in a FIFO manner. Open queueing networks with blocking (hereafter referred to as OQN-B) studied here have the same set of parameters as the closed queueing networks. In addition, we assumed that a customer at node $i$ leaves the network with probability $p_{i0}$. Customers arrive at node $i$ in a Poisson fashion with rate $\lambda_i$. Furthermore, both CQN-B and OQN-B considered here are assumed to be deadlock free networks. A CQN-B is deadlock free if the number of customers in the network is less than the total capacity of each cycle in the network (cf. Akyildiz and Kundu [1]). An OQN-B is deadlock free if there is no cycle in the network, i.e. if we envisage the topology of such a queueing network as being depicted by a graph, then this
An important feature of networks with finite buffers is that a server may become blocked when the capacity of the destination node is full. Various types of blocking mechanisms that arose out of studying various real life systems have been considered in the literature. Onvural and Perros [3] have classified the most commonly used blocking mechanisms as follows:

**TYPE 1 Blocking Mechanism:** A customer upon completion of its service at node $i$ attempts to enter destination node $j$. If node $j$ at that moment is full, the customer is forced to wait in front of server $i$ until there is a departure from the destination node $j$. The server remains blocked for this period of time and it can not serve any other customers waiting in its queue. This blocking mechanism has been used to model systems such as production systems and disk I/O subsystems.

**TYPE 2 Blocking Mechanism:** A customer at node $i$ declares its destination node $j$ before it starts its service. If node $j$ is full, the $ith$ server becomes blocked, i.e. it can not serve its customer. When a departure occurs from destination node $j$, the $ith$ server becomes unblocked and the customer begins receiving service. This blocking mechanism has been used to model systems such as production systems and telecommunication systems. Depending on whether the blocking customer is allowed to occupy the position in front of the server when the server is blocked, this blocking mechanism is divided into the following two sub-categories.

**TYPE 2.1:** Position in front of the server can not be occupied when the server
is blocked.

**TYPE 2.2:** Position in front of the server can be occupied when the server is blocked.

**TYPE 3 Blocking Mechanism:** A customer upon service completion at server $i$ attempts to join destination node $j$. If node $j$ at that moment is full, the customer receives another service at node $i$. This is repeated until the customer completes a service at server $i$ at a moment that the destination node is not full. Within this category of blocking mechanism, the following two sub-categories have been studied in the literature.

**TYPE 3.1:** Once the customer's destination is determined it cannot be altered. This blocking mechanism arose in modeling telecommunication systems.

**TYPE 3.2:** A destination node is chosen at each service completion independently of the destination node chosen the previous time. This type of blocking is associated with reversible queueing networks with blocking.

Comparisons among these blocking mechanisms can be found in Onvural and Perros [3] and Onvural [2].

A bibliography of papers which contain exact, approximate and numerical results related to queueing networks with blocking is compiled by Perros [5]. A survey of closed queueing networks with finite buffers is given in Onvural [2]. Finally, two-node queueing networks (open or closed) are surveyed in Perros [6].

In this paper, we study the relationship between open and closed queueing networks with finite buffers. In particular, we establish how an OQN-B can be
converted to an equivalent CQN-B and vice versa. Such conversions are common in the analysis of queueing networks. For instance, it is a known fact that an open queueing network can be analyzed as a closed queueing network with an additional node representing the arrival process in the open queueing network. The two networks are equivalent if there are enough customers in the closed network so that the additional node is never empty. In the following section, we show that in the case of blocking, this requirement is not necessary. In section 3, we used the obtained equivalencies in order to establish bounds on the throughput of OQN-B. Finally, the conclusions are given in section 4.

2. Relations Between Open and Closed Queueing Networks

In this section, we will study the relationships between open and closed queueing networks with finite buffers. The equivalencies between such networks are first established for the case where, in the OQN-B, arrivals occur at only one node. Then, the case of multiple arrivals are studied in section 2.2. Applicability of the results to the case of multiple classes of customers are discussed in section 2.3.

2.1. The Case of a Single Arrival Stream

Let us consider an exponential open network with finite buffer capacities. Arrivals are assumed to occur only at one particular node. Interarrival time between arrivals are exponentially distributed with mean $1/\lambda$. 
The following procedure will be used to construct a CQN-B from a given OQN-B.

Algorithm 1

[0] Consider an N-node OQN-B with parameters $\mu_i$, $B_i$, $p_{ij}$, $i,j=1,...,N$, and $\lambda$. We have $B_i < \infty$, $i=1,...,N$. Without loss of generality assume that arrivals occur only at node 1. Let $S$ be the set of nodes where customers depart from the network, i.e. $S = \{l \mid p_{l0} > 0\}$.

[1] Construct a closed queueing network with $N+1$ nodes and with parameters $\mu_i$, $B_i$, and $p_{ij}$, $i=0,...,N$. Let $K$ be the number of customers in the network. Furthermore, assume that $B_0 = \infty$, $B_i = B_i$, $\mu_0 = \lambda$, $\mu_i = \mu_i$, $p_{ij} = p_{ij}$, $i,j=1,...,N$; $p_{01} = 1$ and $p_{l0} = p_{l0}$ for all $l \in S$.

To construct a CQN-B from a given OQN-B we add a node (node 0) to the open network with parameters $B_0 = \infty$ and $\mu_0 = \lambda$. In the OQN-B, arrivals occur only at node 1. Then we have $p_{01} = 1$ in the CQN-B. Departures in OQN-B occur from node $l$ with probability $p_{l0}$, $l \in S$. Hence, we have $p_{l0} = p_{l0}$, $l \in S$ in the CQN-B.

We note that the CQN-B constructed using algorithm 1 for a given OQN-B is deadlock free for any number of customers in it if the OQN-B is deadlock free.

Next, we will present results pertaining to equivalencies between OQN-B and CQN-B constructed using algorithm 1. Two such networks are equivalent if they have the same rate matrix. These results may be proved in two steps: 1) show that both networks have the same number of states, and 2) define a one-to-one mapping from the states of one network to the states of the other network such that transitions into and out of two equivalent states are the same, thus showing that both networks have the same rate matrix.
Lemma 1: Consider an OQN-B and a CQN-B constructed using algorithm 1, both under type 1 blocking mechanism. Let \( K = \sum_{i=1}^{N} B_i \). In the OQN-B, without loss of generality, let all the arrivals occur at node 1. Also, let \( B_1^* = B_1 - 1 \) in the CQN-B while keeping all other parameters the same. Then, two networks have the same rate matrix.

Proof: Consider a deadlock-free OQN-B under type 1 blocking with parameters \( \mu_i, B_i \) and \( p_{ij}, i,j=1,...,N \), where \( N \) is the number of nodes in the network. Without loss of generality, let all the arrivals occur only at node 1 with rate \( \lambda \). We assume that there is a path from node 1 to every other node in the network.

Node 1 cannot block a node since there cannot be directed cycles in deadlock-free networks. Let \((n_1,n)\) be the state of the network where \( n_1 \) is the number of customers at node 1 and \( n \) is the state of nodes 2 to \( N \). We have \( 0 \leq n_1 \leq B_1 \). For arbitrary topologies, \( n \) includes information on which nodes are blocked, which are the blocking nodes and in what order they are blocked, in addition to the distribution of customers at each node.

Now, consider the CQN-B constructed from this OQN-B using algorithm 1 and let \( K = \sum_{i=1}^{N} B_i \). Furthermore, let \((n_0,n_1,n)\) denote the state of this network. We have \( 1 \leq n_0 \leq K, 0 \leq n_1 \leq B_1 - 1 \) and \( n \) is the state of nodes 2 to \( N \). We note that the capacity of node 1 in the CQN-B is \( B_1 - 1 \) where \( B_1 \) is the capacity of node 1 in the OQN-B. A customer at node 0 is subject to blocking, i.e. if a customer upon service completion at node 0 finds node 1 full then it is blocked. Let \((n_0,B_1,n)\)
denote a state where node 1 is full, there are \( n_0 \) (\( \geq 1 \)) customers at node 0 and customer at node 0 is blocked by node 1. We note that both networks have the same number of states. Let us now define a mapping between the states of the OQN-B and the states of the CQN-B as follows:

\[
\begin{cases}
(n_1, n) = (n_0 = K - n_1 - n', n_1, n) & 0 \leq n_1 \leq B_1 - 1 \\
(B_1, n) = (n_0 = K - n_1 - n', B_1, n) & n_1 \leq B_1
\end{cases}
\]

(1)

where \( n' \) is the total number of customers at nodes 2 to N. The transitions between nodes 1 to N are the same, seeing that there is no difference between nodes 1 to N in both networks. In OQN-B, a customer will depart from node i with probability \( p_{i0} \). This transition corresponds to a customer joining node 0 in the CQN-B. In both cases, departure of a customer can not be blocked. Finally, arrivals occur at node 1 when \( 0 \leq n_1 \leq B_1 - 1 \) in both networks. In particular, when an arrival occurs at a time node 1 is full, it is lost in the OQN-B while when node 0 is blocked by node 1, server 0 can not serve any customer waiting in its queue. Hence, transitions into and out of equivalent states (defined by (1)) are the same. Therefore, the two networks have the same rate matrix. \( \square \)

**Lemma 2:** Consider a deadlock free OQN-B and a CQN-B constructed using algorithm 1, both under type 3.2 blocking mechanism. Let \( K = \sum_{i=1}^{N} B_i \). Then, the two networks have the same rate matrix. The same result holds for type 2.1, 2.2 and 3.1 blocking mechanisms.
Proof: The proof is similar to the one presented above for Lemma 1. Let \( \mathbf{n} \) denote the state of OQN-B under type 3.2 blocking and \( (n_0, \mathbf{a}) \) be the state of the CQN-B constructed using Algorithm 1. Then, the equivalencies of states can be summarized as follows:

\[
\mathbf{n} = (K - \sum_{i=1}^{N} n_i, \mathbf{a})
\]  

(2)

Clearly, both networks have the same number of states. Furthermore, it can be similarly shown that transitions into and out of equivalent states are the same. Hence, they have the same rate matrix. \( \Box \)

We will now discuss the following problem. Given a CQN-B, is it possible to find an equivalent OQN-B such that both networks have the same rate matrix?

Let us first consider a CQN-B with finite capacities and \( K \) customers, i.e. \( K > B_i, \ i=1,...,N \). Then, there can not be an equivalent OQN-B, seeing that departures from an open network can not be blocked while in this CQN-B departures from each node are subject to blocking.

Let us now assume that there is exactly one node with infinite capacity in a CQN-B and without loss of generality let it be node 1, i.e. \( B_1 = \infty \) and \( B_i < K, \ i=2,...,N \). If \( K < \sum_{i=2}^{N} B_i, \ ( N > 2 ) \), then there does not exist an equivalent OQN-B to this CQN-B. This is because, the number of states in these two networks can not be the same, i.e. there is a state in an open network in which all nodes are full which is not possible in such CQN-B. (As a special case, if \( N=2, B_1 = \infty, B_2 < \infty \) and \( K < B_2 \) then this is a non-blocking network and node 2 behaves like an
M/M/1/K queue where \( \mu_1, \mu_2 \) is the arrival and service rates of this queue respectively.)

Now, we will show that Algorithm 1 can be used to generate an equivalent OQN-B for a class of CQN-B. Let us consider a CQN-B with \( B_1 = \infty, B_i < \infty, i = 2, \ldots, N \) and \( p_{1l} = 1 \) for a given \( l \). Without loss of generality, let \( l = 2 \), i.e. \( p_{12} = 1 \), and \( p_{1j} = 0, j = 1, \ldots, N; j \neq 2 \). Furthermore, assume that deadlocks are not allowed. Now, let \( K \geq \sum_{i=2}^{N} B_i + 1 \) and consider the network under type 1 blocking mechanism. Then, there is an equivalent OQN-B to this CQN-B which can be constructed by applying first step 1 and then step 0 of algorithm 1 with the buffer capacity of node 2 is increased by one in the OQN-B. For CQN-B under type 2 or 3 blocking mechanisms, a similar result holds without any changes in the buffer capacities if \( K \geq \sum_{i=2}^{N} B_i \). Finally, we note that these last two results are in fact equivalent to the result presented in Lemmas 1 and 2.

2.2. The Case of Multiple Arrival Streams

In this section, we will consider deadlock free exponential queueing networks with multiple arrival streams. In an OQN-B, let \( A \) be the set of nodes in which arrivals occur. For each node \( i, i \in A \), the time between arrivals are assumed to be distributed exponentially with rate \( \lambda_i \). To find an equivalent closed queueing network to this OQN-B, we will add a node (node 0) to the OQN-B with parameters \( B_0 = \infty \) and \( \mu_0 = \lambda = \sum_{i \in A} \lambda_i \). We have \( p_{0i} = \lambda_i / \lambda, i \in A \). The total number of custo-
mers, K, in the CQN-B is equal to $\sum_{i=1}^{N} B_i$. Departures in the OQN-B are handled the same way as in the case of single arrival streams. Furthermore, node 0 in the CQN-B is subject to type 3.2 blocking independent of the blocking mechanism used in the given OQN-B. Hence, we may have two different blocking mechanisms used in the equivalent CQN-B. In the OQN-B, arrivals occurring to a full node are lost. In the equivalent CQN-B, a customer at node 0, upon service completion, attempts to join node $i$, $i \in A$. If node $i$ is full then it receives another service at node 0. Such transitions cancel out in the rate matrix. The destination chosen the next time is independent of the previous choices. We note that arrivals occur to node $i$ with rate $\lambda_i$, $i \in A$ as in the OQN-B. The following algorithm, similar to Algorithm 1, summarizes this procedure.

**Algorithm 2**

0. Consider an N-node OQN-B with parameters $\mu_i$, $B_i$, $p_{ij}$, $i,j=1,...,N$. We have $B_i < \infty$, $i=1,...,N$. Let $A$ be the set of nodes to which arrivals occur. The interarrival times are assumed to be distributed exponentially with rate $\lambda_i$. Let $S$ be the set of nodes where customers depart from the network, i.e. $S=\{l \mid p_{l0} > 0\}$.

1. Construct a closed queueing network with $N+1$ nodes and with parameters $\mu_i^*$, $B_i^*$, and $p_{ij}^*$, $i,j=0,...,N$. Let $K$ be the number of customers in the network. Furthermore, assume that $B_0^*=\infty$, $B_i^*=B_i$, $\mu_0^* = \sum_{i \in A} \lambda_i$, $\mu_i^* = \mu_i$, $p_{ij}^*=p_{ij}$, $i,j=1,...,N$; $p_{0k}^* = \lambda_k / \lambda$, $k \in A$ and $p_{l0}^* = p_{l0}$ for all $l \in S$.

The CQN-B constructed using Algorithm 2 have the same rate matrix as the given OQN-B if $K \geq \sum_{i=1}^{N} B_i$. The following lemma may be proved similar to the previous two lemmas.
Lemma 3: Consider a deadlock free OQN-B under type i blocking, i=1, 2.1, 2.2, 3.1 or 3.2, and a CQN-B constructed using algorithm 2. Let node 0 be subject to type 3.2 blocking while all other nodes are subject to type i blocking as in the OQN-B. Furthermore, let $K = \sum_{i=1}^{N} B_i$. Then, the two networks have the same rate matrix.

2.3. OQN-B with Multiple Classes of Customers

In this section, we consider OQN-B with R classes of customers. The parameters of the network are class dependent buffer capacities, routing probabilities and service rates. To the best of our knowledge, no approximation algorithms have been reported in the literature for such networks.

Algorithms 1 and 2 can be easily modified to construct an equivalent CQN-B to a given OQN-B with multiple classes of customers and with a single stream of arrivals. In this case, there is one source node in the CQN-B for each class. With this modification, Lemmas 1 and 3 can be easily extended to include multiple classes of customers. Also, in the case of multiple arrival streams for each class, there is one source node per class and the service rate of these source nodes are equal to the sum of the arrival rates for each class. All source nodes are subject to type 3.2 blocking, independent of the blocking mechanism used in the given OQN-B. Then, Lemma 3 is applicable to OQN-B with multiple arrival streams and with multiple classes of customers.
3. Upper and Lower Bounds of the Throughput of OQN-B

In this section, we use the equivalencies presented above to establish bounds on the throughput of OQN-B. Recent results on this problem have been reported by Shanthikumar and Jafari [7] and Van Dijk and Lamond [9].

Consider a CQN-B with exactly one node with infinite buffer capacity and let $\lambda(K)$ be its throughput when there are $K$ customers in it. For a moment, consider the same network with infinite buffer capacities and let $\beta(K)$ be its throughput. Clearly, $\lambda(K) \leq \beta(K)$. Furthermore, a CQN-B has a product form solution with $K = \min_i (B_i) + 1$ customers under type 1 blocking (cf. Onvural and Perros [4]), and with $K = \min_i (B_i)$ customers under type 2 and 3 blocking. It can be shown that throughput is a non-decreasing function of the number of customers in a CQN-B in which there is exactly one node with infinite buffer capacity (cf. Onvural and Perros [4] and Shanthikumar and Yao [8]). Now, consider an OQN-B and its equivalent CQN-B. Since they have the same rate matrix, the throughput of both networks is the same. Let $\lambda$ be the throughput of an OQN-B. Then:

$$
\begin{align*}
\beta(\min(B_i) + 1) & \leq \lambda = \lambda(\sum_{i=1}^{N} B_i) \leq \beta(\sum_{i=1}^{N} B_i) & \text{in type 1 blocking} \\
\beta(\min(B_i)) & \leq \lambda = \lambda(\sum_{i=1}^{N} B_i) \leq \beta(\sum_{i=1}^{N} B_i) & \text{otherwise}
\end{align*}
$$

In the table below, we compare the tightness of these bounds with the examples given in [7] and [9]. These examples were given for two stage tandem networks.
under type 2.2 blocking. We note that our upper bound is approximately the same as Van Dijk and Lamond's, and it is much tighter than Shanthikumar and Jafari's. Our lower bound is approximately the same or tighter than Van Dijk and Lamond's but not as tight as Shanthikumar and Jafari's. However, our approach is much simpler than these two methods. We only need to calculate the normalization constants of the equivalent CQN-B for \( K=1,\ldots, \sum_{i=1}^{N} B_i \). Then the required throughputs can be easily calculated from the normalization constants. Furthermore, our approach is readily applicable to arbitrary topologies and different types of blocking mechanisms.

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VD-L UB-LB : Upper and Lower Bounds by Van Dijk and Lamond [9]
O-P UB-LB : Upper and Lower Bounds by Onvural and Perros
Let us now consider the following four node network under type 1 blocking. The first three examples given below are related to triangle configurations. Example 4 is related to four-node configuration. Exact throughput in these examples were calculated numerically.

An Arbitrary Configuration with Four Nodes.

**Example 1:** \((B_1, B_2, B_3) = (4,3,2), (\mu_1, \mu_2, \mu_3) = (2,1,2), \lambda = 1.5, p_{10} = 0, p_{12} = 0.5, p_{13} = 0.5, p_{20} = 0, p_{23} = 1\)

Lower Bound = 0.9154; Exact Throughput = 1.202; Upper Bound = 1.3411

**Example 2:** \((B_1, B_2, B_3) = (5,4,3), (\mu_1, \mu_2, \mu_3) = (1,0.5,1), \lambda = 8, p_{10} = 0.2, p_{12} = 0.4, p_{13} = 0.4, p_{20} = 0.3, p_{23} = 0.7\)

Lower Bound = 0.783; Exact Throughput = 0.8608; Upper Bound = 0.9705

**Example 3:** \((B_1, B_2, B_3) = (5,1,2), (\mu_1, \mu_2, \mu_3) = (2,2,2), \lambda = 3, p_{10} = 0.2, p_{12} = 0.4, p_{13} = 0.4, p_{20} = 0.3, p_{23} = 0.7\)

Lower Bound = 1.1413; Exact Throughput = 1.6137; Upper Bound = 1.88205
Example 4: \((B_1, B_2, B_3, B_4) = (2, 2, 2, 2), (\mu_1, \mu_2, \mu_3, \mu_4) = (1, 1, 1, 1), \lambda = 5, p_{10} = 0.05, p_{12} = 0.35, p_{13} = 0.3, p_{14} = 0.3, p_{20} = 0.05, p_{23} = 0.05, p_{24} = 0.9, p_{30} = 0.05, p_{34} = 0.95\)

Lower Bound = 0.69; Exact Throughput = 0.789; Upper Bound = 0.904

Finally, we note that the approach presented above may not be readily used to establish lower and upper bounds of the throughput of OQN-B with multiple classes of customers.

4. Conclusions

We have answered the question as to when an open queueing network with finite buffers is equivalent to a closed queueing network and what are the parameters of the equivalent network. In the case of OQN-B under type 1 blocking and with single stream of arrivals, it is found that the buffer capacity of the node in which arrivals occur should be reduced by one in the equivalent CQN-B. Also, it suffices to analyze the equivalent CQN-B with \(K\) customers in it, where \(K\) is the sum of all the buffers in the OQN-B. Similar results was shown to hold in the case of multiple arrivals. However, in this case, nodes of the equivalent CQN-B may be subject to different types of blocking, i.e. additional node is always subject to type 3.2 blocking while all the other nodes in the equivalent CQN-B are subject to the same blocking as in the given OQN-B.
With these equivalencies, results presented in the literature for OQN-B can be applied to equivalent CQN-B and vice versa. In particular, consider an OQN-B with a single stream of arrivals. Since the throughput of such networks are not known, approximations developed in the literature for such networks are iterative. Furthermore, these algorithms have not been proved to be convergent. Its throughput could be found using an algorithm for the throughput of its equivalent CQN-B eliminating the need for iterations.

We also presented bounds on the throughput of OQN-B. Such bounds are very easy to calculate and are comparable with the existing bounding algorithms.
REFERENCES


