Computer Aided Analysis of Nonlinear Microwave Analog Circuits

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Abstract

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A new tool for the computer-aided analysis of nonlinear microwave analog circuits is reported. The technique uses generalized power series descriptions for the nonlinear elements and a spectral balance approach to operate entirely in the frequency domain. This allows circuits having large non-harmonically related input signals to be efficiently analyzed. To verify the analysis technique, a generalized power series based model is developed for a metal-semiconductor field-effect transistor (MESFET). A commercially available transistor is characterized and the appropriate model parameters extracted. This model is used with a computer implementation of the analysis to simulate a MESFET amplifier circuit. The results are compared to measured circuit performance for single and two-tone inputs showing good agreement. In addition the analysis technique is applied to calculate the effective impedance of the nonlinear circuit elements at large signal levels. The modeling capability is improved by considering nonlinear functions of two variables and a new MESFET model is developed. Algebraic formulas are developed for calculating the output of a generalized power series function of two variables having a multifrequency input thereby extending the analysis technique to this more general situation.
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Chapter 1

Introduction

1.1 Motivation for This Study

Microwave circuit designers have traditionally relied on the use of experimental modifications in order for their designs to meet the required specifications. This practice, however, is becoming increasingly impractical as circuit dimensions shrink and circuit complexity increases. This is particularly unwieldy for the design of microwave monolithic integrated circuits which cannot be easily modified once they are made and are too expensive to allow many design iterations. Thus there is an increasing interest in the computer-aided design of microwave circuits. While a variety of powerful design tools have been developed for linear circuits, few tools are available for the design of nonlinear analog circuits.

The objective of this study is the development of a tool suitable for the analysis of nonlinear microwave circuits. In particular, this study concentrates on analog circuits having sinusoidal excitation. The problems presented by these circuits differ significantly from those of low-frequency and of digital circuits and require new simulation strategies. The technique presented here is a frequency domain approach and as such is suited to the analysis of nonlinear circuits having multi-
frequency excitation.

The new analysis technique is verified through the simulation of an amplifier circuit whose active element is a metal-semiconductor field-effect transistor (MESFET). This device is chosen because it is an important component in microwave circuits and because it is difficult to model and thus provides a good test of the analysis technique. In the following section we provide an overview of the organization of this report.

1.2 Thesis Overview

Because of the large volume of research that has been done in the area of nonlinear circuit analysis, any study in this area requires a thorough literature review to place the work in proper context. This is done in Chapter 2 which presents a review of the nonlinear circuit analysis techniques that are most relevant to the simulation of microwave analog circuits. This includes a review of time domain methods, frequency domain methods, and hybrid methods. In each case the methods are evaluated with respect to the special characteristics of microwave analog circuits.

The next three chapters form the core of this thesis. Chapter 3 is devoted to the development of a new frequency domain nonlinear circuit analysis method. This technique uses generalized power series (power series having complex coefficients and time delays) descriptions for the nonlinear circuit elements in combination with a spectral balance method for circuit analysis. Generalized power
series descriptions are used to enable the output of a nonlinear element to be directly calculated in the frequency domain given the frequency domain input and the generalized power series coefficients. The spectral balance method finds the steady state solution by minimizing an error function based on the deviation from Kirchoff's current law at each frequency. The resulting analysis method is a true frequency domain technique that is capable of analyzing nonlinear circuits having large-signal multifrequency inputs.

In Chapter 4 a MESFET model is developed which is compatible with the analysis technique of Chapter 3. Two commercially available transistors are characterized and the model parameters extracted. These models are then used to verify the analysis technique in Chapter 5.

In Chapter 5 a MESFET amplifier circuit is simulated and the results compared to measured circuit performance. Comparisons are made for a single tone input and for two input signals. In addition, the effective impedance of each nonlinear element in the MESFET model is calculated.

The next two chapters present techniques for improving the modeling capability of generalized power series analysis. In Chapter 6 an improved MESFET model is developed that is valid over the entire range of useful operating points. This model uses power series descriptions that are functions of two variables. These descriptions are reduced to single variable power series at the desired operating point and used with the previously developed analysis technique.
In Chapter 7 a method is developed to include two variable power series as nonlinear elements in the analysis procedure. Algebraic formulas are developed for the frequency domain output of a two variable power series given the frequency domain input and the power series coefficients. Formulas are also developed for the derivative of the output with respect to the input.

Chapter 8 contains a summary, conclusions, and suggestions for further study.

1.3 Original Contributions

The original contributions reported here include:

i) the development and computer implementation of an analysis technique for nonlinear multi-terminal circuits using generalized power series and a spectral balance approach,

ii) the development of a MESFET model using generalized power series descriptions of the nonlinear elements,

iii) experimental verification of the generalized power series based analysis technique and the associated MESFET model,

iv) the development of a technique for calculating the effective impedance of a nonlinear element at large signal levels,

v) the development of a model for the dc characteristics of a MESFET using bivariate power series,
vi) the development of a global MESFET model using bivariate power series that is compatible with the existing simulator,

vii) the development of an algebraic formula for the output of a nonlinear system described by a bivariate generalized power series and having two multifrequency inputs, and

viii) the development of an algebraic formula for the derivative of the output of a nonlinear system described by a bivariate generalized power series with respect to the multifrequency input signals.
Chapter 2

Review of Nonlinear Circuit Analysis Techniques

2.1 Introduction

There is an ever-growing library of publications in the field of nonlinear circuit analysis. They range from abstract mathematics-intensive works to works analyzing a specific circuit and from the analysis of a complete system to the analysis of a single electronic device. The purpose of this chapter is to sample the available literature and to review the developments that are most relevant to the analysis of nonlinear microwave analog circuits. Even with such a focus it is beyond the scope of this chapter to discuss all of the work dealing with this area. Instead, the intent is to provide a review of the major techniques that are available.

When comparing analysis methods for a particular application, special characteristics of the circuits involved must be considered. For nonlinear microwave analog circuits these include:

i) microwave circuits are typically made up of only a few circuit elements,

ii) linear circuit elements outnumber nonlinear elements,

iii) the steady-state response is generally more important than the transient behavior,
iv) the circuits are driven by sums of sinusoids,

v) some circuit elements (e.g. transmission lines) are difficult to describe using time-domain expressions,

vi) the circuits frequently contain time constants that differ by orders of magnitude, and

vii) the frequencies involved in the solution may be widely separated.

In the review that follows, the analysis methods discussed are evaluated with respect to these characteristics.

Although the methods presented here for analyzing nonlinear circuits are varied, they are all based on solving the set of nonlinear differential equations resulting from application of three basic laws: Kirchoff's voltage law, Kirchoff's current law, and the constitutive relations (e.g. the element characteristics). The methods may be characterized by the way in which the nonlinear elements are treated and can be classified into one of three groups:

**time domain methods** Both the linear and nonlinear elements are analyzed in the time domain

**hybrid methods** The nonlinear elements are analyzed in the time domain and the linear elements are analyzed in the frequency domain

**frequency domain methods** The entire circuit is analyzed in the frequency domain.
Each of these groups will be discussed separately in the following sections.

2.2 Time Domain Methods

As mentioned previously, time domain methods analyze the nonlinear circuit by solving the nonlinear differential equations governing the circuit in the time domain. In this section we examine three such techniques: direct numerical integration of the state equations, associated discrete circuit modeling, and shooting methods.

2.2.1 Direct Integration of the State Equations

The most direct method for analyzing nonlinear circuits is numerical integration of the differential equations describing the network. By applying Kirchoff’s voltage and current laws and using the characteristic equations for the circuit elements (frequently using the tableau or modified nodal formulations), the state equations can often be written in normal form [1,2]

\[ \dot{X} = f(X, t) \] (2.2.1)

or more generally in the implicit form

\[ g(\dot{X}, X, t) = 0 \] (2.2.2)

where

\[ X = [X_1, X_2, X_3, \ldots, X_n] \]
is a set of voltages and currents. The characteristic equations for nonlinear resistors are of the form [1]

\[ p(v_R, i_R) = w_R. \]  \hspace{1cm} (2.2.3)

Charge and flux are used as additional variables to describe nonlinear capacitors and inductors respectively. Nonlinear capacitors are described by a pair of equations

\[ C(v_C, q_C) = 0 \]
\[ i_C = \frac{d}{dt} q_C \]  \hspace{1cm} (2.2.4)

and nonlinear inductors by a similar pair

\[ L(i_L, \phi_L) = 0 \]
\[ v_L = \frac{d}{dt} \phi_L. \]  \hspace{1cm} (2.2.5)

The solution is found by numerical integration of (2.2.2) typically using a predictor-corrector scheme [1,2,3]. In this approach, the nonlinear differential equations are converted into nonlinear algebraic equations through discretization of the time variable. This is often done using the backward difference formula which approximates the derivative at the next time step, \( \dot{X}(t_{n+1}) \), in terms of \( X(t_{n+1}) \) and the \( k \) past values \( X(t_n), X(t_{n-1}), \ldots, X(t_{n-k+1}) \) (the predictor). The resulting algebraic equations are solved iteratively often using Newton's method (the corrector). This is repeated until the solution is found for all time points \( t_n \). The resulting time domain solution can then be converted into a frequency domain
representation if required using a Fourier transform. This procedure is illustrated in Fig. 2.2.1.

While this method is quite general, it has some limitations when applied to microwave circuit analysis. The selection of an appropriate time step is one such problem. Microwave circuits typically have widely separated time constants resulting in a set of stiff state equations. The consequence is that a small time step must be chosen and a large number of iterations may be required to reach steady state [2], leading to excessive computation time. Similarly, it may be difficult to identify the steady-state solution when widely spaced frequencies are present.

2.2.2 Associated Discrete Circuit Modeling

A related method involves the use of associated discrete circuit models (also called companion models) [2] and is the technique used in the popular simulator SPICE [3,4]. This method is fundamentally the same as that just described in that the state equations are integrated numerically, however, the order of operations is changed. This approach begins by applying the time discretization step directly to the equations describing the circuit element characteristics. The nonlinear differential equations are thereby converted to nonlinear algebraic equations. Kirchoff's voltage and current laws are then applied to form a set of algebraic equations which are solved iteratively as before at each time point.

Converting the differential equations describing the element characteristics into
Figure 2.2.1: Flowchart for a nonlinear analysis method based on direct numerical integration of the state variable equations.
algebraic equations changes the network from a nonlinear dynamic circuit to a nonlinear resistive circuit, thus the method is called associated discrete circuit modeling. In effect the differential equations describing the capacitors and inductors are approximated by resistive circuits associated with the numerical integration algorithm. The term "associated" refers to the model's dependence upon the integration method while "discrete" refers to the model's dependence on the discrete time value.

The backward Euler algorithm for solving the differential equation

\[ \dot{v} = f(v) \]

with step size \( h \) is

\[ v_{n+1} = v_n + hf(v_{n+1}) \]

\[ = v_n + h\dot{v}_{n+1}. \quad (2.2.6) \]

For a linear capacitor,

\[ i(t) = C \frac{dv}{dt} = C \dot{v} \]

or

\[ \dot{v}_{n+1} = \frac{1}{C} \dot{i}_{n+1}. \quad (2.2.7) \]

Substituting (2.2.7) into (2.2.6) and rearranging, we find

\[ \dot{i}_{n+1} = \frac{C}{h} v_{n+1} - \frac{C}{h} v_n. \quad (2.2.8) \]

This equation can be modeled by a constant resistance in parallel with a current source that depends on the previous time step as shown in Fig. 2.2.2. A similar
Figure 2.2.2: Resistive circuit associated with the backward Euler algorithm for a linear capacitor and a discrete time step.
procedure is used to find the associated circuit for a linear inductor as well as for nonlinear capacitors and inductors. For the nonlinear elements the associated models will be nonlinear resistors.

Although this method suffers from the same problems as the direct numerical integration method, it has been successfully used to simulate nonlinear microwave circuits. In [5], for example, a microwave amplifier was simulated using SPICE. Valid results were obtained only after simulating the circuit for ten periods of the 2 GHz fundamental. More complicated circuits require simulation of as many as 30 periods in order to reach the steady-state solution [5]. In addition, a small time step must be used. The result is that a significant amount of computing power is required to solve a relatively simple problem. Despite these difficulties, this method and the direct integration method have the ability to calculate the transient or steady-state response for a complex nonlinear circuit.

2.2.3 Shooting Methods

In contrast the shooting methods attempt to find the steady-state solution without calculating the transient response. This is advantageous in situations that would require many iterations for the transient components to die out, if direct integration methods are used. It is assumed that the nonlinear circuit has a periodic solution and that the solution can be determined by finding an initial state such that there is no transient component.
As before, the problem is to solve the state equations (2.2.1), however, in this method the problem is converted into the two-point boundary value problem [6,7]

\[
X(0) = X(T)
\]
\[
X(T) = \int_0^T f(X, \tau) d\tau + X(0)
\]
(2.2.9)

where \( T \) is the period such that

\[
X(t) = X(t + T).
\]

This can be solved using the Newton's method iteration

\[
X^{k+1}(0) = X^k(0) - \left[ I - \frac{\partial X^k(T)}{\partial X^k(0)} \right]^{-1} [X^k(0) - X^k(T)]
\]
(2.2.10)

where the superscripts refer to iteration numbers and \( X^k(T) \) is found by integrating the circuit equations over one period from the initial state \( X^k(0) \).

To begin the analysis the period \( T \) is determined and the initial state \( X(0) \) is estimated. Using these values, the circuit equations are numerically integrated from \( t = 0 \) to \( t = T \) and the necessary derivatives calculated. Then, the estimate of the initial state is updated using the Newton iteration (2.2.10). This process is repeated until \( X(0) = X(T) \) is satisfied within a reasonable tolerance. This algorithm is illustrated in Fig. 2.2.3. The calculation of the derivatives required for the Newton iteration can be time consuming for large problems and an alternative optimization scheme has been developed [8].

Shooting methods are attractive for problems that have small periods. Unlike the direct integration methods, the circuit equations are only integrated over one
Figure 2.2.3: Algorithm for nonlinear circuit analysis using the shooting method.
period (per iteration). They are therefore more efficient provided that the initial state can be found in a number of iterations that is smaller than the number of periods that must be simulated before steady-state is reached in the direct methods. Unfortunately, shooting methods can only be applied to find periodic solutions. Also shooting methods become less attractive for cases where the circuit has a large period, for example when several nonharmonic signals are present.

### 2.3 Hybrid Methods

In contrast to the time domain methods just presented, the hybrid methods avoid numerical integration of the state equations. These methods directly calculate the steady-state response of the nonlinear circuit, and are generally referred to as harmonic balance methods. The term describes the solution algorithm, in which the assumed frequency components of the solution (the harmonics) are adjusted until the algebraic equations governing the circuit are “balanced”.

Hybrid methods are variations of Galerkin’s method [9,10] applied to nonlinear circuits. Galerkin’s method, first described in 1915, assumes a solution containing unknown coefficients [11]. The assumed solution is substituted into the governing equations. The unknown coefficients are then chosen to make the assumed solution satisfy the governing equations as accurately as possible.

When the assumed solution is a sum of sinusoids, this procedure has been referred to as harmonic balance. The name appears as early as 1937 in the work
of the Russian scientists Kryloff and Bogoliuboff (translated into English in 1943 [12]). More recently, the method has been developed and applied to nonlinear circuits by Baily [13] and Lindenlaub [14] in the 1960s. The modern version of the harmonic balance method was presented by Nakhla and Vlach in 1976 [15]. They reduce the number of variables to be optimized by partitioning the network into smaller subnetworks that are composed of either linear circuit elements or nonlinear elements. The linear subnetworks are solved in the frequency domain. Only the variables associated with the connection of the subnetworks need to be optimized. They called the resulting technique piecewise harmonic balance. In recent years, their method has been adopted and the adjective piecewise dropped. This basic version of the harmonic balance method is reviewed in the following section.

2.3.1 Harmonic Balance

Because many variations of the harmonic balance method have been developed, we first review the basic concepts and then present the most important modifications separately. The review that follows is based on the work of Nakhla and Vlach [15] and Gilmore [16], as well as the recent review paper by Kundert and Sangiovanni-Vincentelli [17].

We present the basic harmonic balance method by considering a single transistor circuit driven by two independent sources. For purposes of this example,
Figure 2.3.1: A single transistor circuit partitioned into linear and nonlinear subcircuits in order to be analyzed using the harmonic balance method.

the transistor is the only nonlinear element in the circuit and is viewed as a three terminal black box. Knowing the voltages at the three terminals, we can calculate the currents into the transistor. In general, the nonlinear elements must be representable by algebraic (memoryless) equations, and can be functions of voltage or current. Nonlinear capacitors and inductors are included by writing charge as a function of voltage and flux as a function of current respectively. Converting charge to current and flux to voltage is done in the frequency domain, where differentiation and integration with respect to time become algebraic operations.

The circuit to be analyzed is partitioned into linear and nonlinear subcircuits as shown in Fig. 2.3.1. The currents into the transistor terminals can be calculated
(noniteratively) in the time domain as

\[ i_1(t) = f(v_1, v_2, v_3) \]
\[ i_2(t) = g(v_1, v_2, v_3) \]
\[ i_3(t) = h(v_1, v_2, v_3) \]

(2.3.1)

where \( f, g, \) and \( h \) are nonlinear functions. The time domain waveforms can be transformed into the frequency domain (retaining \( N \) components), adopting the following notation for the currents into the nonlinear subcircuit

\[ i_1(t) \leftrightarrow I_1(\omega), \]

for the currents into the linear subcircuit

\[ \hat{i}_1(t) \leftrightarrow \hat{I}_1(\omega), \]

and for the voltages

\[ v_1(t) \leftrightarrow V_1(\omega). \]

The currents into the linear subcircuit are easily calculated directly in the frequency domain using the nodal admittance representation of the linear network

\[
\begin{pmatrix}
\hat{I}_1(\omega) \\
\hat{I}_2(\omega) \\
\hat{I}_3(\omega)
\end{pmatrix} =
\begin{pmatrix}
Y_{11}(\omega) & Y_{12}(\omega) & Y_{13}(\omega) & Y_{14}(\omega) & Y_{15}(\omega) \\
Y_{21}(\omega) & Y_{22}(\omega) & Y_{23}(\omega) & Y_{24}(\omega) & Y_{25}(\omega) \\
Y_{31}(\omega) & Y_{32}(\omega) & Y_{33}(\omega) & Y_{34}(\omega) & Y_{35}(\omega)
\end{pmatrix}
\begin{pmatrix}
V_1(\omega) \\
V_2(\omega) \\
V_3(\omega) \\
V_{S1}(\omega) \\
V_{S2}(\omega)
\end{pmatrix}
\] (2.3.2)
where $\omega$ is one of the $N$ frequency components being considered.

Satisfaction of Kirchoff's current law requires that the current into the nonlinear subcircuit equal the current into the linear subcircuit

$$ i(t) = \hat{i}(t) \quad \text{or} \quad \begin{pmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \end{pmatrix} = \begin{pmatrix} \hat{i}_1(t) \\ \hat{i}_2(t) \\ \hat{i}_3(t) \end{pmatrix}. \quad (2.3.3) $$

In this example, the principle of harmonic balance is simply the restatement of (2.3.3) in the frequency domain

$$ \begin{pmatrix} I_1(\omega_i) \\ I_2(\omega_i) \\ I_3(\omega_i) \end{pmatrix} = \begin{pmatrix} \hat{I}_1(\omega_i) \\ \hat{I}_2(\omega_i) \\ \hat{I}_3(\omega_i) \end{pmatrix} \quad (2.3.4) $$

for $i = 1, \ldots, N$, or using vector notation

$$ \mathbf{F}(\mathbf{V}) = \mathbf{I}(\mathbf{V}) - \mathbf{\hat{I}}(\mathbf{V}) = 0 \quad (2.3.5) $$

and is referred to as the determining equation [18]. When the nonlinear elements are voltage controlled and the linear subnetwork is represented by an admittance matrix (one for each frequency), the principle of harmonic balance is simply the frequency domain statement of Kirchoff's current law (e.g. the current into each node of the nonlinear subnetwork must equal the current into the corresponding node of the linear subnetwork at each frequency). In general, with voltage and current controlled elements present, Kirchoff's current and voltage laws are applied and a hybrid matrix used to represent the linear subnetwork.
The steady-state solution is found by varying the frequency domain voltages to satisfy (2.3.5) and (2.3.2) as accurately as possible. This algorithm is pictured in Fig. 2.3.2. All harmonic balance methods use this basic algorithm. There are, however, many different approaches for solving the determining equation (2.3.5) and for transforming the voltages and currents between the time and frequency domains. The following sections explore the most significant of these alternatives.

2.3.2 Solution of the Determining Equation

One area that differentiates the various harmonic balance methods is the solution of the determining equation (2.3.5). The two distinct approaches to this problem are: (a) to find the solution by minimizing an error function based on (2.3.5), and (b) to directly solve the system of nonlinear equations. The following sections describe these approaches and the various ways of implementing them. Under certain conditions, the two methods become very similar.

Solution via Minimization

One approach to the solution of the determining equation of harmonic balance is to minimize an objective function that is based on the deviation of $F$ from $0$ in (2.3.5) [15,17,19,20]. The particular choice of objective function influences the choice of numerical method used to find the solution and thereby affects the performance of the simulator. We examine several choices here.
Figure 2.3.2: Algorithm for analyzing a nonlinear circuit using the harmonic balance method.
The objective function used by Nakhla and Vlach [15] is based on a time domain statement of the determining equation

\[ f(t) = i(t) - \hat{i}(t) = 0 \quad 0 \leq t \leq T \]

where \( T \) is the period and \( f, i \) and \( \hat{i} \) are vectors. They formulate the scalar function \( E \) of the node voltage vector \( V \) as

\[ E(V) = \int_0^T f(\tau)^T f(\tau) d\tau. \quad (2.3.6) \]

A quasi-Newton algorithm is then used to find a vector of node voltage phasors to minimize \( E \). This approach complicates the problem needlessly by ignoring the structure of the frequency domain equations.

Recently, a technique has been proposed that uses the time domain determining equation but only attempts to satisfy it for discrete time samples [21]. This has been referred to as time-domain harmonic balance [21] and is being applied to the simulation of microwave circuits [22] although no results are currently available. Unfortunately, at this time there is not enough information available to evaluate this approach.

A similar objective function can be formulated in the frequency domain [17]

\[ E(V) = F(V)^\dagger F(V) \quad (2.3.7) \]

where \( F^\dagger \) denotes the conjugate transpose of \( F \) (\( F^\dagger = (F^T)^* \)), and \( F \) is defined in (2.3.5). The Gauss-Newton method can be used to find the minimum using an iteration procedure that is essentially Newton’s method applied to (2.3.5).
Contrary to the comments of [17], this is a very useful formulation and is closely related to the direct solution of (2.3.5) as a system of nonlinear equations. This relationship is explored in detail in Appendix A.

One benefit of solving the determining equation using minimization is that circuit parameters can be included with the unknowns and circuit performance can be optimized as part of the solution algorithm [19,20]. Unfortunately, this results in a less efficient algorithm because the number of unknowns is no longer equal to the number of terms in the objective function. Also the objective function will not, in general, have a zero residual. This requires a more complicated algorithm having less desirable convergence properties (see Appendix A).

**Direct Solution of the Nonlinear Equations**

An alternative approach to the solution of (2.3.5) is to directly solve it as a system of nonlinear equations. This can be accomplished in a variety of ways, several of which are discussed in the following.

Perhaps the simplest method of solution is that referred to as harmonic relaxation [17]. An example of this approach is the fixed point iteration used by Hicks and Khan [23]

\[ V^{k+1} = pY^{-1}I(V^k) + (1 - p)V^k \]  

(2.3.8)

where the superscripts are iteration numbers, \( Y \) is the admittance representation of the linear subnetwork, and \( 0 < p \leq 1 \) is a convergence parameter. This method
is very attractive in that it is simple and each iteration takes relatively little computing time. The disadvantage is that convergence can be slow and is dependent on the impedances of the linear and nonlinear subcircuits.

A more complicated method that has much better convergence properties is Newton's method. This method has been used in a number of harmonic balance programs including [17,24,25,26]. This technique finds the solution to (2.3.5) through the iteration

\[ \mathbf{V}^{k+1} = \mathbf{V}^k - \mathbf{J}(\mathbf{V}^k)^{-1}\mathbf{F}(\mathbf{V}^k) \]  

(2.3.9)

where \( \mathbf{J} \) is the Jacobian of \( \mathbf{F} \). Newton's method is a local method, having good convergence properties provided an appropriate initial estimate of the solution can be found. It has the disadvantage of requiring the calculation of the inverse of the Jacobian in addition to the function evaluation. Although significant computer time is expended in calculating \( \mathbf{J}^{-1} \), Newton's method is generally accepted as an efficient technique for solving a system of nonlinear equations.

Continuation or homotopy methods can be used to globalize Newton's method [27] as has been done in several harmonic balance simulators [24,28]. The basic approach is to avoid solving the nonlinear equations directly, but to solve a series of simpler problems leading up to the actual problem in small steps. This is done by introducing a parameter \( \eta \) (called the homotopy parameter) such that for \( \eta = \eta_0 \) the solution is known or can be readily determined whereas \( \eta = \eta_* \) corresponds to the more difficult problem to be solved. In the case of nonlinear circuit analysis,
an obvious choice for $\eta$ is the intensity of the driving sources. When the source intensity is zero ($\eta = \eta_0$) the solution is easily determined. The source intensity ($\eta$) is increased in steps and the solution from the previous step used to estimate the new solution. At each step, Newton's method is used to solve the resulting nonlinear equations. By taking small enough steps and using appropriate estimation strategies, the Newton iterations will converge rapidly. This is repeated until the desired source intensity is reached ($\eta = \eta_*$. This method is illustrated in Fig. 2.3.3.

2.3.3 Conversion Between the Time and Frequency Domains

In addition to the solution of the determining equation, the other major computational aspect of the harmonic balance algorithm is the conversion between the time and frequency domains. This conversion must be efficient since it is performed twice during each iteration of the nonlinear equation solver: once to convert the frequency domain voltages to time domain waveforms to be input to the nonlinear subnetwork, and once to convert the resulting time domain currents to the frequency domain. It must also be flexible enough to allow consideration of input signals consisting of a single frequency and its harmonics or of multiple frequencies that may not be related. Here we examine three such techniques.
Figure 2.3.3: An algorithm for solving the determining equation of harmonic balance using a continuation or homotopy method.
Discrete Fourier Transform

Perhaps the most obvious candidate for converting between the time and frequency domains is the discrete Fourier transform. To use this technique the time domain waveform must be sampled at a rate at least twice the highest frequency contained in the spectrum (the Nyquist rate). Taking $N$ time samples at an interval of $T_s$ corresponding to $t = \tau T_s$ for $\tau = 0, 1, 2, \ldots, N-1$ we can write the discrete Fourier transform $V(\nu)$ of the time samples $v(\tau)$ as

$$V(\nu) = N^{-1} \sum_{\tau} v(\tau) e^{-j2\pi(\nu/N)\tau} \quad (2.3.10)$$

yielding frequency components at $\omega = (2\pi\nu)/(NT_s)$ for $\nu = 0, 1, 2, \ldots, N/2$ [29]. Similarly we can write the inverse discrete Fourier transform as

$$v(\tau) = \sum_{\nu} V(\nu) e^{j2\pi(\nu/N)\tau}. \quad (2.3.11)$$

This technique is well established and efficient calculation procedures are available (e.g., fast Fourier transforms). It is, however, limited to signals that contain frequencies that are integer multiples of a single fundamental (in this case $\omega_0 = (2\pi)/(NT_s)$).

Nonlinear circuits having multi-tone excitations have been simulated using harmonic balance and discrete Fourier transforms [19,20,26,30] by choosing a fundamental frequency $((2\pi)/(NT_s))$ that is subharmonically related to all frequencies considered in the simulation. For example, with input frequencies at 10.0 GHz and 10.1 GHz, the sampling interval and the number of samples could be chosen
to make the fundamental frequency 100 MHz. In this way, the input frequencies and all resulting intermodulation products become harmonics of the 100 MHz fundamental. Unfortunately, this often requires a large number of time samples, reducing the efficiency of this approach.

An alternative is to use multidimensional Fourier transforms [31,32,33]. In general, an almost-periodic signal with periods $T_1, T_2, \ldots, T_N$ can be written using an expression of the form [34]

$$v(t) = \sum_m \sum_n \cdots \sum_q V_{mn\ldots q} \exp[j(m\omega_1 + n\omega_2 + \cdots + q\omega_N)t]. \quad (2.3.12)$$

For the two-tone case the series becomes

$$v(t) = \sum_m \sum_n V_{mn} \exp[j(m\omega_1 + n\omega_2)t]. \quad (2.3.13)$$

The basic idea is to consider $x = \omega_1 t$ and $y = \omega_2 t$ to be independent variables so that the voltage is a doubly periodic function of $x$ and $y$. Thus, a two-dimensional fast Fourier transform may be used to find the spectral components of the voltage and current waveforms [32]. This formulation is quite general and allows an arbitrary relationship between the two frequencies and has been used in conjunction with harmonic balance [31,32]. As this is a two-dimensional transform, a two-dimensional array of time samples is required. In [31] a circuit was simulated using a bidimensional discrete Fourier transform to convert between the frequency and time domains. To determine the desired 18 frequency components a $21 \times 21$ matrix was required (441 time samples). The same problem has been solved using
the almost periodic Fourier transform (described below) requiring only 100 time
samples [18]. A modified version of the almost periodic Fourier transform would
require approximately 61 samples [35]. This approach has also been used with
three input signals and a corresponding three-dimensional fast Fourier transform
[33]. The technique is computationally intensive, however, and a supercomputer is
used. While multidimensional Fourier transforms are useful in harmonic balance
simulations, more efficient techniques are available.

The Discrete Fourier Transform and Controlled Aliasing

Recently a technique, called modified harmonic balance, has been presented using
the discrete Fourier transform with a reduced number of time samples [16,36]. The
number of time samples is reduced by sampling at a rate below the Nyquist rate.
The resulting aliasing can be predicted and used to determine the original Fourier
coefficients.

Consider the simplified spectrum of Fig. 2.3.4 consisting of two input tones
at \( f_1 \) (\( d \)) and \( f_2 \) (\( e \)), the third order intermodulation frequencies \( 2f_1 - f_2 \) (\( c \)) and
\( 2f_2 - f_1 \) (\( f \)), the second order product at \( f_2 - f_1 \) (\( b \)), and dc (\( a \)). The signals
are separated by \( \Delta = f_2 - f_1 \) and the four highest frequencies are enclosed in
the band of width \( B \) beginning at \( f_1 - 2\Delta \). Additionally, the lower edge of the
band, \( B_0 \), is assumed to be an integer multiple of \( 2B \). If we sample the waveform
at the bandpass rate of \( T_s = 1/2B \) (significantly below the Nyquist rate) and
apply a discrete Fourier transform, the spectrum of Fig. 2.3.5 results. Because
Figure 2.3.4: Simple spectrum consisting of two input tones, one at $f_1$ and one at $f_2$, the third order intermodulation frequencies $2f_1 - f_2$ and $2f_2 - f_1$, the second order product at $f_2 - f_1$, and dc (after [16]).

Figure 2.3.5: Spectrum after sampling at the rate $T_0 = 1/2B$, resulting in aliasing (after [16]).
the sampling is done at such a slow rate, the band $B$ appears along with the original low frequency components due to aliasing. In this example, the resulting component at $\Delta$ is due to the original signal at $\Delta$ plus the original signal at $B_0 + \Delta$ (e.g. $b + c$). Only four components of the original signal can be determined ($a, d, e,$ and $f$). To determine the remaining components, the input signals are shifted down in frequency by $\Delta$ such that their relative spacing is preserved. The waveform is again sampled at a slow rate and a discrete Fourier transform applied with the result that the $c$ component overlaps with the $a$ component at $dc$, the $b$ component overlaps the $d$ component at $\Delta$, and the $e$ and $f$ components are at $2\Delta$ and $3\Delta$ respectively. Since $a$ and $d$ are known from the first transform, $b$ and $c$ can be found from subtraction. In general, this process can be repeated until all of the Fourier coefficients are found.

The modified harmonic balance method has been successfully applied to non-linear circuit analysis [16,36]. The advantages of this approach are that it is computationally efficient, requiring relatively few time samples and using little computer memory. The disadvantage is that because the signals must be shifted several times in frequency, only low-Q circuits with small frequency separations ($\Delta$) can be considered. If the circuit response is different at the shifted frequencies, errors are introduced into the calculated Fourier coefficients.
Almost Periodic Fourier Transform

The final method we review for converting between the time and frequency domains is the technique that has become known as the almost periodic Fourier transform. First developed by Chua and Ushida in 1981 [37] and used in connection with harmonic balance in 1984 [18], the method has recently been improved by reducing the number of time samples required [35].

To derive the transformation, a time domain waveform is assumed to be representable by a series of the form

\[
v(t) = a_0 + \sum_{k=1}^{N} (a_{2k-1} \cos(\omega_k t) + a_{2k} \sin(\omega_k t))
\]  

(2.3.14)

where the frequencies \(\omega_k\) are not, in general, harmonically related. Frequently, \(\omega_k\) can be expressed as a linear combination of two or more base frequencies as

\[
\omega_k = m\nu_1 + n\nu_2
\]  

(2.3.15)

where \(m\) and \(n\) are integers. The frequencies \(\nu_1\) and \(\nu_2\) need not be related and can even be incommensurable (in which case (2.3.14) is not periodic). In general, the series (2.3.14) is referred to as an almost periodic function [38]. We seek the coefficients \(a_k\) such that the series (2.3.14) approximates the time domain waveform \(v(t)\) best in the least squares sense.

Taking \(M\) samples of the time domain waveform, a linear system of \(M\) equations is developed from (2.3.14), with each equation corresponding to a discrete
value of $t$ denoted $t_i$

$$
\begin{bmatrix}
v(t_1) \\
v(t_2) \\
v(t_3) \\
\vdots \\
v(t_M)
\end{bmatrix} =
\begin{bmatrix}
1 & \cos(\omega_1 t_1) & \sin(\omega_1 t_1) & \cdots & \cos(\omega_N t_1) & \sin(\omega_N t_1) \\
1 & \cos(\omega_1 t_2) & \sin(\omega_1 t_2) & \cdots & \cos(\omega_N t_2) & \sin(\omega_N t_2) \\
1 & \cos(\omega_1 t_3) & \sin(\omega_1 t_3) & \cdots & \cos(\omega_N t_3) & \sin(\omega_N t_3) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & \cos(\omega_1 t_M) & \sin(\omega_1 t_M) & \cdots & \cos(\omega_N t_M) & \sin(\omega_N t_M)
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{2N-1} \\
a_{2N}
\end{bmatrix}
$$

or using matrix notation

$$v = \Gamma A. \quad (2.3.16)$$

The problem of finding the best vector $A$ in the least squares sense can be restated using this notation as

$$\text{given} \quad \Gamma \in \mathbb{R}^{M \times (2N+1)}, \quad M \geq (2N+1), \quad v \in \mathbb{R}^M$$

$$\text{find} \quad \min_{A \in \mathbb{R}^{2N+1}} \|\Gamma A - v\|_2. \quad (2.3.17)$$

This is the classic linear least squares problem and, provided $\Gamma^T \Gamma$ is nonsingular, has solution [39]

$$A = (\Gamma^T \Gamma)^{-1} \Gamma^T v. \quad (2.3.18)$$

Thus, we have defined a transform pair, with the frequency domain to time domain transform defined by (2.3.16) and the reverse transform defined by (2.3.18). It can be shown that, when the signal is periodic and the time samples are chosen correctly, this is equivalent to the conventional discrete Fourier transform [18].

The almost periodic Fourier transform has been successfully used in connection with harmonic balance simulations [18,35]. In the method of Ushida and Chua
[18], the time samples are equally spaced and the number of samples is greater than the number of coefficients to be determined \((M > 2N + 1)\). This leads to an overdetermined set of equations having the solution (2.3.18). While this is a valid approach, it is desirable from an efficiency standpoint to minimize the number of time samples required. Also, the matrix \(\Gamma^T \Gamma\) tends to be ill-conditioned and is difficult to invert accurately. A solution proposed by Sorkin, et al. [35] is to start with a large number of time samples, e.g. \(M = 2(2N + 1)\), and to choose a set of \(2N + 1\) samples so as to improve the conditioning of \(\Gamma\). This process is only performed once for a given set of frequencies. With \(M = 2N + 1\) the matrix \(\Gamma\) is square and (2.3.18) is simplified to

\[
A = \Gamma^{-1} v. \tag{2.3.19}
\]

This is a more efficient approach than (2.3.18) and uses a minimum number of time samples.

### 2.4 Frequency Domain Methods

The final category of nonlinear circuit analysis techniques is the frequency domain methods, methods that avoid explicit time domain calculations. This is accomplished through expansion of the input-output characteristics of the nonlinear elements. The expansion is chosen such that for certain inputs (typically a sum of sinusoids) the response can be calculated for each component of the expansion. The responses are then summed to give the total response of the nonlinear
element. In this section, we review three types of expansions: Volterra series, algebraic functional expansion, and power series.

2.4.1 Volterra Series Expansion

The oldest and most analytically developed frequency domain methods are the methods based on Volterra series. Around 1910 Volterra showed that every functional $G(x)$ continuous in the field of continuous functions can be represented by the expansion [40]

$$G(x) = \sum_{n=0}^{\infty} F_n(x)$$

(2.4.1)

where $F_n(x)$ is a regular homogeneous functional of the form

$$F_n(x) = \int_{a}^{b} \cdots \int_{a}^{b} h_n(\xi_1, \xi_2, \ldots, \xi_n)x(\xi_1)x(\xi_2)\cdots x(\xi_n)d\xi_1d\xi_2\cdots d\xi_n$$

(2.4.2)

and the functions $h_n(\xi_1, \xi_2, \ldots, \xi_n)$ are known as the $n$th order Volterra kernels. This series, called a Volterra series, can be viewed as a functional power series and the theorem as a generalization of the Weierstrass approximation theorem (which shows that a continuous function can be approximated by a polynomial).

In 1942 Wiener applied this type of functional series to the analysis of nonlinear systems [41]. He suggested that a weak nonlinearity could be represented with the first few terms of such a series. His ideas have subsequently been developed by many researchers including significant contributions by Bedrosian and Rice [42] and Bussgang, Ehrman, and Graham [43]. This review is based largely on those works as well as the more recent book by Weiner and Spina [44].
The Volterra series of (2.4.1) and (2.4.2) is a time domain representation. The frequency domain representation is much more useful and can be found by taking the n-fold Fourier transform of \( h_n \)

\[
H_n(f_1, f_2, \ldots, f_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) \\
\exp[-j2\pi(f_1\tau_1 + \cdots + f_n\tau_n)]d\tau_1 d\tau_2 \cdots d\tau_n
\]  

(2.4.3)

where \( H_n \) is called the nonlinear transfer function of order \( n \). The \( n \) th order kernel, \( h_n \), is called the nonlinear impulse response of order \( n \), and is found by the inverse Fourier transform

\[
h_n(\tau_1, \tau_2, \cdots, \tau_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(f_1, f_2, \ldots, f_n) \\
\exp[j2\pi(f_1\tau_1 + \cdots + f_n\tau_n)]df_1 df_2 \cdots df_n.
\]  

(2.4.4)

Given a time domain input-output relation \( y(t) = f(x(t)) \), we can put it in the form of the Volterra series (2.4.1)

\[
y(t) = \sum_{n=1}^{\infty} y_n(t)
\]  

(2.4.5)

where

\[
y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n)x(t - \tau_1) \cdots x(t - \tau_n)d\tau_1 \cdots d\tau_n
\]  

(2.4.6)

and \( x(t) \) is the input. Substituting (2.4.4) into (2.4.6) and noting that

\[
\int_{-\infty}^{\infty} \exp(j2\pi f_i \tau_i)x(t - \tau_i)d\tau_i = X(f_i)\exp(j2\pi f_i t)
\]  

(2.4.7)
where $X(f)$ is the Fourier transform of $x(t)$, we find

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(f_1, \ldots, f_n) \prod_{i=1}^{n} X(f_i) \exp(j2\pi f_i t) df_i. \quad (2.4.8)$$

This expresses the $n$th order terms of the output as a function of the input spectrum. The order of the terms refers to the fact that multiplication of the input by a constant $A$ results in multiplication of the $n$th order terms by $A^n$.

Taking the Fourier transform of (2.4.8) and using

$$\int_{-\infty}^{\infty} \exp[j2\pi(f_1 + \cdots + f_n)t] \exp(-j2\pi f t) dt = \delta(f - f_1 - \cdots - f_n) \quad (2.4.9)$$

where $\delta(\cdot)$ is the delta function, gives the output spectrum

$$Y_n(f) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(f_1, \ldots, f_n) \delta(f - f_1 - \cdots - f_n) \prod_{i=1}^{n} X(f_i) df_i. \quad (2.4.10)$$

We can then write a frequency domain series for the output

$$Y(f) = \sum_{n=1}^{\infty} Y_n(f) \quad (2.4.11)$$

in terms of the input spectrum and the nonlinear transfer functions. In general, a finite summation is used as illustrated in Fig. 2.4.1 which shows $N$ parallel blocks, each having a common input. The total response is found by summing the response due to each individual block.

This is a very general procedure that is applicable to an arbitrary input. In the analysis of microwave analog circuits the most common input is a sum of sinusoids

$$x(t) = \sum_{q=1}^{Q} |E_q| \cos(2\pi f_q t + \theta_q)$$

$$= \frac{1}{2} \sum_{q=-Q}^{Q} E_q \exp(j2\pi f_q t). \quad (2.4.12)$$
Figure 2.4.1: Illustration of nonlinear analysis using Volterra series.
Because this is a discrete spectrum (e.g. the Fourier series is a sum of delta functions) the integrals of (2.4.8) reduce to summations, and the $n$th order response can be written

$$y_n(t) = \frac{1}{2^n} \sum_{q_1=-Q}^{Q} \cdots \sum_{q_n=-Q}^{Q} E_{q_1} \cdots E_{q_n} H_n(f_{q_1}, \ldots, f_{q_n}) \exp[j2\pi(f_{q_1} + \cdots + f_{q_n})t]$$

(2.4.13)

or in the frequency domain

$$Y_n = \sum_{q_1=-Q}^{Q} \cdots \sum_{q_n=-Q}^{Q} H_n(f_{q_1}, \ldots, f_{q_n}) E_{q_1} \cdots E_{q_n}.$$  

(2.4.14)

The term $H_n(f_{q_1}, \ldots, f_{q_n}) E_{q_1} \cdots E_{q_n}$ is an $n$th order intermodulation product of frequency $(f_{q_1} + \cdots + f_{q_n})$. In this notation, $f_{-i} = -f_i$ and $E_{-i} = E_i^*$. To calculate a response of a certain order, all intermodulation products of that order are summed. Each intermodulation product is a product of the input phasors and the appropriate nonlinear transfer function. The nonlinear transfer functions (the $H_n$) do not depend on the input and are only determined by the nonlinearity.

Volterra series methods have been successfully applied to nonlinear circuit analysis by a number of researchers. They were first applied to transistor circuit simulation in 1967 [45] and have more recently been applied to microwave circuit simulation [46,47]. This is an attractive approach in that it can be applied to a wide variety of nonlinear systems and can handle arbitrary inputs. The disadvantage is that the nonlinear transfer functions, the $H_n$, are difficult to determine for high orders of $n$. Typically, determination of $H_n$ for $n > 3$ is impractical. Thus, only weak nonlinearities can be considered making Volterra series methods mainly
2.4.2 Algebraic Functional Expansion

In contrast to the approach used in Volterra series analysis, the method reviewed in this section uses an algebraic approach to nonlinear functional expansion based on noncommutative power series. A symbolic calculus has been developed which generalizes the operational calculus of Heaviside, used in linear system theory, to the domain of nonlinear systems. This method is related to Volterra series, but the analysis procedure is quite different and relies on symbolic computation.

The power series used in this technique are the functional power series developed by Fliess, et al. [48]. The power series are in terms of the symbols $x_0$ and $x_1$, where the symbol $x_0$ denotes integration with respect to time and the symbol $x_1$ the integration with respect to time after multiplying by the input $u$. The resulting power series is noncommutative because

$$\int_0^t d\tau_1 \int_0^{\tau_1} u(\tau_2) d\tau_2 \neq \int_0^t u(\tau_1) d\tau_1 \int_0^{\tau_2} d\tau_2$$  \hspace{1cm} (2.4.15)$$

which is written symbolically as

$$x_0 x_1 \neq x_1 x_0.$$  \hspace{1cm} (2.4.16)$$

A symbolic calculus based on this series has been developed by Lamnabhi [49,50] and applied to the analysis of nonlinear circuits. The review presented here is based on that work.
The Volterra expansion of \( y(t) \) with input \( u(t) \)
\[
y(t) = \int_0^t h_1(t, \tau_1)u(\tau_1)d\tau_1 + \int_0^t \int_0^{\tau_2} h_2(t, \tau_2, \tau_1)u(\tau_1)u(\tau_2)d\tau_1d\tau_2 + \cdots \tag{2.4.17}
\]
can be expressed using the notation just defined as
\[
g = g_1 + g_2 + \cdots + g_n + \cdots \tag{2.4.18}
\]
where
\[
g_n = \sum_{i_0, i_1, \ldots, i_n \geq 0} c_{i_0, \ldots, i_n} x_0^{i_0} x_1 x_0^{i_1} \cdots x_1 x_0^{i_n} \tag{2.4.19}
\]
and \( g \) is called the noncommutative generating power series associated with \( y(t) \).
In a similar way, the symbolic series representation can be found for a variety of nonlinear functions.

To analyze nonlinear circuits using this approach, the nonlinear differential equations describing the circuit are written using the symbolic notation. For example consider the equation
\[
\frac{dv}{dt} + \alpha v + \beta v^2 = i(t) \tag{2.4.20}
\]
where \( i(t) \) is the input and \( v(t) \) the unknown. Assuming a zero initial condition, we can write this in its integral form
\[
v(t) = \alpha \int_0^t v(\tau)d\tau + \beta \int_0^t v^2(\tau)d\tau = \int_0^t i(\tau)d\tau \tag{2.4.21}
\]
leading to the symbolic representation
\[
g + \alpha x_0 g + \beta x_0 g \circ g = x_1 \tag{2.4.22}
\]
where \( g \) is the generating power series associated with \( v(t) \). Rearranging, we obtain

\[
g = -\beta(1 + \alpha x_0)^{-1}x_0[g \ast g] + (1 + \alpha x_0)^{-1}.
\] (2.4.23)

The operation denoted \( \ast \) is called the shuffle product. Defining the set \( X = \{x_0, x_1, \ldots, x_m\} \) (called the alphabet) and the set \( X^* \) to be the set consisting of finite sequences of elements of \( X \), e.g. \( x_j \cdots x_{j_0} \) (called words), and \( 1 \) to be a neutral element of \( X \) (called the empty word), the shuffle product is defined

\[
1 \ast w = 1, \quad \forall w \in X^*, \quad w \ast 1 = 1 \ast w = w
\]

\[
\forall x_j, x_j' \in X, \quad \forall w, w' \in X^*
\]

\[
(x_j w) \ast (x_j' w') = x_j[w \ast (x_j' w')] + x_j'[(x_j w) \ast w'].
\]

(2.4.24)

This operation results in a mixing of the letters of the two words while retaining the order of the letters in each one. For example,

\[
x_0 x_1 \ast x_2 = x_0 x_1 x_2 + x_0 x_2 x_1 + x_2 x_0 x_1.
\]

(2.4.25)

Thus, (2.4.23) is totally an algebraic equation and can be solved using a fixed-point type iteration with the result

\[
g = g_1 + g_2 + \cdots + g_n + \cdots
\]

(2.4.26)

where

\[
g_1 = (1 + \alpha x_0)^{-1}x_1
\]

(2.4.27)
and $g_n$ consists of all terms of $-\beta(1 + \alpha x_0)^{-1} x_0 [g \circ g]$ depending on $g_1, g_2, \ldots, g_{n-1}$ and containing $n$-occurrences of $x_1$. The solution can be found using a digital computer and a symbolic mathematics package. To find the solution for a particular input, first express the input in the form

$$u(t) = \sum_{n \geq 0} u_n \frac{t^n}{n!} \quad (2.4.28)$$

from which the generating power series is found as

$$g_u = \sum_{n \geq 0} u_n x_0^n \quad (2.4.29)$$

which is called the Laplace-Borel transform. The transformed input is substituted into the algebraic equation for the response, the resulting equation is decomposed into partial fractions, and transformed back into the time domain giving a closed form solution. The transient or steady-state response can be found using this approach which is analogous to the Laplace transform method of linear systems analysis.

The advantage of this technique is that it can be applied to the same sort of problems Volterra series techniques can analyze, but a higher order approximation can be used since all the algebraic manipulations are performed on the computer. It has the disadvantage of requiring software to manipulate symbolic equations. This results in an approach that is inefficient and is difficult to integrate with existing computer-aided design tools.
2.4.3 Power Series Expansion

The final frequency domain nonlinear analysis methods we review are the power series methods. These methods expand the nonlinear input-output characteristics in conventional power series. As with the previous frequency domain methods, the motivation for this type of expansion is that the output components can be readily calculated when the input consists of a sum of sinusoids, allowing a time domain representation of the nonlinearity to be used while eliminating the need for transformation between the time and frequency domains. In contrast to the previously discussed frequency domain methods, power series methods are applicable to strongly nonlinear elements and are easily integrated into existing computer-aided design tools.

This approach is quite old and has been investigated by a number of researchers as the basis for hand calculations as well as for computer based simulations. One of the early contributors was Wass [51] who in 1948 developed a procedure for calculating the intermodulation products generated by a nonlinearity of the form

\[ y(t) = a_1v + a_2v^2 + \cdots + a_rv^r + \cdots \]  

(2.4.30)

with an input of the form

\[ v(t) = a \cos(2\pi At) + b \cos(2\pi Bt) + c \cos(2\pi Ct) + \cdots \]  

(2.4.31)

but, these results were not amenable to efficient computer implementation. An approach more suitable for computer calculation was simultaneously developed.
by Engel, et al. in 1967 [52] and by Sea in 1968 [53]. An improved calculation strategy was subsequently developed by Sea and Vacroux in 1969 [54]. In 1973 Heiter proposed using a modified power series having order dependent time delays to represent the nonlinearities [55]. This series is of the form

\[ y(t) = c_0 + c_1 v(t - \tau_1) + c_2 v^2(t - \tau_2) + c_3 v^3(t - \tau_3) + \cdots \]  

(2.4.32)

and is useful for modeling phase nonlinearities. Steer and Khan continued this development in 1983 with the addition of complex coefficients to the power series [56] resulting in the form

\[ y(t) = A \sum_{l=0}^{\infty} \left[ a_l \left\{ \sum_{k=1}^{N} b_k x_k(t - \tau_{k,l}) \right\}^l \right] \]  

(2.4.33)

(called a generalized power series) where \( y(t) \) is the output of the system; \( l \) is the order of the power series terms; \( a_l \) is a complex coefficient; \( \tau_{k,l} \) is a time delay that depends on both power series order and the index of the input frequency component; and \( b_k \) is a real coefficient. They developed an algebraic formula for the output components when the input is a sum of sinusoids. This approach is described in detail in the following chapter. Like the algebraic functional expansions, power series can be shown to be related to Volterra series [57].

These techniques have been applied in various forms to simulate nonlinear microwave circuits. Tucker used a third-order series with a two-tone input to investigate intermodulation distortion in transistor amplifiers [58,59]. By using a simple device model and a low-order power series, analytical formulas were developed for the intermodulation products. A similar procedure was used by Higgins
and Kuvas [60]. More recently computer implementations of the algebraic formulas of Steer and Khan have been used to simulate diode mixers [61,62] and IMPATT oscillators [63,64]. These simulations combine the power series formulas with an approach similar to the harmonic balance methods but do not require the transformation between the time and frequency domains. The following chapters describe the use of these techniques in simulating microwave circuits containing MESFETs.

2.5 Conclusion

This chapter has presented a review of techniques available for the analysis of nonlinear microwave analog circuits. The methods were grouped into three categories: time domain methods, hybrid methods, and frequency domain methods. The time domain methods are important for their capabilities of calculating the transient response. They are, however, inefficient for finding the steady-state response of microwave circuits, particularly for cases of multifrequency excitation. In contrast, the hybrid methods are only useful for calculating the steady-state response. These methods are efficient for analyzing nonlinear circuits excited by a single tone and its harmonics. Research is in progress to improve their ability to analyze circuits having multifrequency excitation. The frequency domain methods are ideally suited to the analysis of multifrequency excitation. They trade the ability to handle arbitrary nonlinear characteristics for the ability to handle
arbitrary input-output spectra. These techniques should have an advantage in simulating circuits such as mixers and effects such as intermodulation provided the nonlinear devices can be appropriately modeled. In the following chapters, we discuss in detail how generalized power series can be used to simulate nonlinear analog circuits and how this technique can be applied to the analysis of microwave MESFET circuits.
Chapter 3

Development of Generalized Power Series Analysis

3.1 Introduction

This chapter presents the heart of this thesis: the development of the generalized power series technique for the analysis of nonlinear analog circuits. The method uses generalized power series descriptions of the nonlinear elements and an analysis algorithm similar to harmonic balance. Unlike the traditional harmonic balance approaches, this technique does not require any explicit time domain calculations. Unlike other frequency domain methods, this technique is applicable to the large signal analysis of strongly nonlinear circuits and can be efficiently implemented on a digital computer. As with other frequency domain methods, it is ideally suited to the analysis of nonlinear analog circuits with multifrequency excitation.

Modified power series descriptions (having time delays and complex coefficients) of the nonlinear elements are used so we term the method *generalized power series analysis* (GPSA). Earlier developments of generalized power series analysis are described in [55,56,58] and applications reported for the simulation of diode mixers [61,62] and IMPATT oscillators [63,64]. Here we extend these concepts
to the simulation of complex multi-terminal nonlinear circuits. In the following sections, we review the basic properties of generalized power series, discuss the evaluation of the formulas for the output, present formulas for the derivatives of the output, show how generalized power series descriptions can be incorporated into a harmonic-balance-type algorithm, and discuss the computer implementation of these ideas.

3.2 Basic Properties of Generalized Power Series

One of the major components of any harmonic-balance-type simulator is a facility for calculating the frequency domain output of the nonlinear elements given the frequency domain input. This becomes particularly troublesome when the frequencies involved are not harmonically related. This is the motivation for using power series descriptions for the nonlinear elements, e.g. a power series whose input is an arbitrary sum of sinusoids has an output that is also a sum of sinusoids. Furthermore, this output can be calculated from algebraic formulas based on the coefficients of the power series. This section presents these formulas which effectively transform the time domain descriptions of the nonlinear elements (the power series) into frequency domain descriptions (the algebraic formulas).

Here, the output $y(t)$ of a nonlinear element having an $N$ component multifrequency input

$$x(t) = \sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} |X_k| \cos(\omega_k t + \phi_k)$$

(3.2.1)
is described by the generalized power series

\[ y(t) = A \sum_{\ell=0}^{\infty} \left[ a_{\ell} \left\{ \sum_{k=1}^{N} b_k x_k(t - \tau_{k,\ell}) \right\} \right] \tag{3.2.2} \]

where \( y(t) \) is the output of the system; \( \ell \) is the order of the power series terms; \( a_{\ell} \) is a complex coefficient; \( \tau_{k,\ell} \) is a time delay that depends on both power series order and the index of the input frequency component; and \( b_k \) is a real coefficient. Using complex coefficients and time delays enables a broad class of nonlinear circuits and systems to be described [55,56,58,61,63,64]. Note that \( |X_k| \) is the peak magnitude of an input sinusoid so that a dc input component has \( \omega_k = 0 \) and \( \phi_k = 0 \) or \( \pi \) radians. Rewriting the input using phasor notation,

\[ x_k(t - \tau_{k,\ell}) = |X_k| \cos(\omega_k t + \phi_k - \omega_k \tau_{k,\ell}) \]

\[ = \frac{1}{2} X_k \Gamma_{k,\ell} e^{j\omega_k t} + \frac{1}{2} X_k^* \Gamma_{k,\ell}^* e^{-j\omega_k t} \]

where \( X_k \) is the phasor of \( x_k \) and

\[ \Gamma_{k,\ell} = \exp(-j\omega_k \tau_{k,\ell}). \]

Using the multinomial expansion theorem, the power series of (3.2.2) can be expanded and terms collected according to frequency. As a result, the phasor component of the output, \( Y_q \), corresponding to the radian frequency \( \omega_q \), can be expressed as a sum of intermodulation products (various powers of \( X_k \) multiplied together) as given in [56]

\[ Y_q = \sum_{n=0}^{\infty} \sum_{|n_1| + \cdots + |n_N| = n} U_q \tag{3.2.3} \]
where \( \omega_q = \sum_{k=1}^{N} n_k \omega_k \), a set of \( n_k \)'s defines an intermodulation product (called an IPD), and \( n \) is the order of intermodulation. The second summation is over all possible combinations of \( n_1, \ldots, n_N \) such that \( |n_1| + \cdots + |n_N| = n \). The summations are therefore over the infinite number of intermodulation products (the \( U_q \)'s) yielding the \( q \) th output component \( (Y_q) \). When a nonlinear circuit is excited by a finite number of sinusoids, an infinite number of frequency components are present. In order to analyze such a problem numerically, the number of frequency components considered in the analysis must be truncated. Here we consider \( N \) frequency components. Each intermodulation product in (3.2.3) is given by \([56]\)

\[
U_q = \text{Re} \{ A e_n T \} \omega_q \tag{3.2.4}
\]

where

\[
T = \sum_{\alpha=0}^{\infty} \sum_{\eta_1, \ldots, \eta_N} \left\{ \left( \frac{(n + 2\alpha)!}{2^{(n+2\alpha)}} \right) a_{n+2\alpha} R_{n+2\alpha} \Phi \right\} \tag{3.2.5}
\]

and

\[
\Phi = \prod_{k=1}^{N} \frac{(X_k^t)^{n_k} |X_k|^{2s_k} b_k^{(n_k+2s_k)}}{s_k!(n_k+s_k)!} \tag{3.2.6}
\]

In these expressions \( X_k \) is the phasor of \( x_k \),

\[
X_k^t = \begin{cases} 
X_k & n_k \geq 0, \\
X_k^* & n_k < 0
\end{cases}, \tag{3.2.7}
\]

\[
R_{n+2\alpha} = \exp(-j \sum_{k=1}^{N} n_k \omega_k \tau_{k,n+2\alpha}), \tag{3.2.8}
\]
\[ \varepsilon_n = \begin{cases} 
1 & n = 0 \\
2 & n \neq 0 \end{cases}, \quad (3.2.9) \]

and \( \text{Re}\{\omega_q\} \) is defined such that for \( \omega_q \neq 0 \) it is ignored and for \( \omega_q = 0 \) the real part of the expression in braces is taken. The formula expressed by (3.2.3)-(3.2.9) essentially turns a time domain description (the generalized power series) into a frequency domain description. GPSA has the advantage of retaining the time domain description of the nonlinearities but requiring no explicit time domain calculations in order to find the frequency domain representation for the output. The formula is considerably simpler for nonlinear components that can be described by conventional power series [56].

### 3.3 Evaluating the Generalized Power Series Expressions

As shown in (3.2.3), calculating the frequency domain representation of the output of a nonlinear element involves the summation of the appropriate intermodulation products. For example, consider a two tone input

\[ x(t) = |X_1| \cos(\omega_1 t + \phi_1) + |X_2| \cos(\omega_2 t + \phi_2) \]

which can be written using phasor notation as

\[ x(t) = \frac{1}{2} \left[ X_1 e^{j\omega_1 t} + X_1^* e^{-j\omega_1 t} + X_2 e^{j\omega_2 t} + X_2^* e^{-j\omega_2 t} \right]. \]
The first three powers of $z$ can be easily expanded manually, e.g. expanding $z^2$ gives

$$x^2(t) = \left(\frac{1}{2}\right)^2 \left[ X_1^2 e^{j2\omega_1 t} + 2X_1 X_1^* + 2X_1 X_2 e^{j(\omega_1 + \omega_2)t} + 2X_1 X_2^* e^{j(\omega_1 - \omega_2)t} \right] + \left( X_1^* \right)^2 e^{-j2\omega_1 t} + 2X_1 X_2 e^{j(\omega_2 - \omega_1)t} + 2X_1 X_2^* e^{-j(\omega_1 + \omega_2)t} + X_2^2 e^{j2\omega_2 t} + 2X_2 X_2^* + (X_2^*)^2 e^{-j2\omega_2 t} \right], \quad (3.3.1)$$

and similarly expanding $z^3$ yields

$$x^3(t) = \left(\frac{1}{2}\right)^3 \left[ X_1^3 e^{j3\omega_1 t} + 3X_1^2 X_1^* e^{j\omega_1 t} + 3X_1^2 X_2 e^{j(2\omega_1 + \omega_2)t} + 3X_1^2 X_2^* e^{j(2\omega_1 - \omega_2)t} \right] + 3X_1 (X_1^*)^2 e^{-j\omega_1 t} + 6X_1 X_1^* X_2 e^{j\omega_2 t} + 6X_1 X_1^* X_2^* e^{-j\omega_2 t} + 3X_1 X_2^2 e^{j(\omega_1 + 2\omega_2)t} + 6X_1 X_2 X_2^* e^{j\omega_1 t} + 3(X_2^*)^2 e^{-j3\omega_1 t} + 3(X_1^*)^2 X_2 e^{j(\omega_2 - 2\omega_1)t} + 3(X_1^*)^2 X_2^* e^{-j(2\omega_1 + \omega_2)t} + 3X_1 X_2^2 e^{j(2\omega_2 - \omega_1)t} + 3X_1 (X_2^*)^2 e^{-j(\omega_1 + 2\omega_2)t} + X_2^3 e^{j3\omega_2 t} + 6X_1 X_2^2 X_2 e^{-j\omega_1 t} + 3X_2^2 X_2^* e^{j3\omega_2 t} + 3X_2 (X_2^*)^2 e^{-j\omega_2 t} + (X_2^*)^3 e^{-j3\omega_2 t} \right], \quad (3.3.2)$$

so that the output of the cubic equation

$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t)$$

can be calculated for the two tone input. Table 3.3.1 lists the various intermodulation products resulting from $x$, $x^2$, and $x^3$ and groups them according to frequency. (Only the terms for the positive frequencies are listed.) To calculate the output at a specific frequency, the appropriate terms are summed as in (3.2.3). For example,
Table 3.3.1: The intermodulation products resulting from $x$, $x^2$, and $x^3$ where $x$ is a two tone signal, showing only the positive frequencies.

<table>
<thead>
<tr>
<th>Intermodulation Product</th>
<th>Frequency</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}X_1X_1^*$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{1}{2}X_2X_2^*$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{1}{2}X_1$</td>
<td>$\omega_1$</td>
<td>1</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1^3X_1^*$</td>
<td>$\omega_1$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^36X_1X_2X_2^*$</td>
<td>$\omega_1$</td>
<td>3</td>
</tr>
<tr>
<td>$\frac{1}{2}X_2$</td>
<td>$\omega_2$</td>
<td>1</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_2^2X_2^*$</td>
<td>$\omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^36X_1X_1^*X_2$</td>
<td>$\omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^2X_1^2$</td>
<td>$2\omega_1$</td>
<td>2</td>
</tr>
<tr>
<td>$(\frac{1}{2})^2X_2^2$</td>
<td>$2\omega_2$</td>
<td>2</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1^3$</td>
<td>$3\omega_1$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_2^3$</td>
<td>$3\omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$\frac{1}{2}X_1X_2$</td>
<td>$\omega_1 + \omega_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{1}{2}X_1X_2^*$</td>
<td>$\omega_1 - \omega_2$</td>
<td>2</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1^2X_2$</td>
<td>$2\omega_1 + \omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1^2X_2^*$</td>
<td>$2\omega_1 - \omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1X_2^2$</td>
<td>$\omega_1 + 2\omega_2$</td>
<td>3</td>
</tr>
<tr>
<td>$(\frac{1}{2})^3X_1^*X_2^2$</td>
<td>$2\omega_2 - \omega_1$</td>
<td>3</td>
</tr>
</tbody>
</table>
the phasor output at \( \omega_1 \) is given by the sum of three intermodulation products

\[
Y_{\omega_1} = a_1 \left( \frac{1}{2} \right) X_1 + 3a_3 \left( \frac{1}{2} \right)^3 X_1^2 X_1^* + 6a_3 \left( \frac{1}{2} \right)^3 X_1 X_2 X_2^*.
\]  

(3.3.3)

The algebraic formulas presented in the preceding section generalize this procedure for \( N \) input frequencies and for the more complicated generalized power series descriptions of the nonlinear elements.

When a sum of sinusoids is input to a nonlinear element additional frequency components are generated. In our previous example there are components of the output due to the \( z^2 \) term at \( 2\omega_1, 2\omega_2, (\omega_1 + \omega_2), (\omega_2 - \omega_1) \), and dc in addition to the original frequencies \( \omega_1 \) and \( \omega_2 \). (There are also components at the corresponding negative frequencies.) When these components are input to the nonlinear element even more frequencies are generated. A similar situation occurs for the other terms in the power series. In order to make the analysis tractable, the number of frequency components considered must be limited. For the two tone input just discussed, the frequencies generated are integer combinations of the two inputs, e.g. \( \omega = m\omega_1 + p\omega_2 \) where \( m \) and \( p \) are integers. One way of limiting the number of frequencies is to consider only the combinations of \( m \) and \( p \) such that

\[
| m | + | p | \leq n_{\text{max}}
\]

assuming that all products of order greater than \( n_{\text{max}} \) are negligible. This approach is illustrated in Fig. 3.3.1. For the more general case in which \( N \) components are
Figure 3.3.1: Possible combinations of two input frequencies including fifth order products.
considered as inputs, a set of integers, denoted $n_k$ are used to specify the frequency,

$$\omega = \sum_{k=1}^{N} n_k \omega_k$$

where the order of intermodulation is given by

$$n = \sum_{k=1}^{N} |n_k| .$$

This set of integers ($n_k$) is called the intermodulation product description (IPD). IPD's up to a maximum order $n_{\text{max}}$ are predetermined and stored in a database although in the evaluation of the algebraic formula, all intermodulation products of the same order are calculated and added to the total response for that frequency component until the desired fractional accuracy is obtained.

To further illustrate the concept of intermodulation products and IPD's, consider the simple spectrum for a mixer circuit shown in Fig. 3.3.2. This spectrum retains only those components integral to the operation of the mixer: $f_2$ is the LO; $f_3$ is the RF; $f_4$ is dc; and $f_1 = f_2 - f_3$ is the IF, all of which are considered as inputs when evaluating the algebraic formula. A partial listing of the intermodulation products is given in Table 3.3.2. For example, Table 3.3.2 lists the fourth order intermodulation product description $f_1 = 2f_1 - f_2 + f_3$ which yields a component at $f_1$ and corresponds to an intermodulation product of the form $X_1^2X_2X_3$. The evaluation of the algebraic formula for all IPD's is illustrated in Fig. 3.3.3. This shows that each intermodulation product is calculated independently and summed to give the output at a particular frequency.
Table 3.3.2: Partial listing of IPD’s when dc is not an input to the algebraic formula (after [61]).

<table>
<thead>
<tr>
<th>Output Frequency</th>
<th>n</th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$, IF</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-2</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>$f_2$, LO</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>-2</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>3</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>$f_3$, RF</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>2</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-3</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>$f_4$, dc</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>2</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>
3.4 Derivative Calculations for Generalized Power Series

Formulas can also be derived for calculating the derivatives of the output phasors with respect to the input phasors when the nonlinear elements are described using generalized power series. Partial derivatives of a nonlinear node current phasor with respect to the magnitude and phase of a node voltage phasor are obtained by differentiating the algebraic formula (3.2.3)-(3.2.9). Using the notation previously defined, the derivative of the phasor of the $q$ th component of the output of the nonlinearity with respect to the magnitude of the phasor of the $m$ th input
Figure 3.3.3: Illustration of the algebraic formula evaluation.
component, \( X_m = |X_m| e^{i\phi_m} \), is found from (3.2.3) through differentiation

\[
\frac{\partial Y_q}{\partial |X_m|} = \sum_{n=0}^{\infty} \sum_{|n_1|+\cdots+|n_N|=n} \frac{\partial U_q}{\partial |X_m|}
\]

(3.4.1)

where

\[
\frac{\partial U_q}{\partial |X_m|} = \text{Re} \left\{ A_\epsilon_n \frac{\partial T}{\partial |X_m|} \right\}_{\omega_q}
\]

and

\[
\frac{\partial T}{\partial |X_m|} = \sum_{\alpha=0}^{\infty} \sum_{\sum \alpha = \alpha} \left\{ \left( \frac{(n+2\alpha)!}{2^{n+2\alpha}} \right) a_{n+2\alpha} R_{n+2\alpha} \frac{\partial \Phi}{\partial |X_m|} \right\}.
\]

(3.4.2)

Rewriting (3.2.6)

\[
\Phi = \left( \frac{(X_m^1)^{|n_m|}}{s_m! |n_m| + s_m!} \right) \prod_{k=1}^{N} \frac{(X_k^1)^{|n_k|}}{s_k! |n_k| + s_k!}
\]

(3.4.3)

and noting that for \(|X_m| \neq 0\)

\[
\frac{\partial (X_m)^n}{\partial |X_m|} = \frac{\partial}{\partial |X_m|} \left( |X_m|^n e^{i\phi_m} \right)
\]

\[
= n |X_m|^{n-1} e^{i\phi_m}
\]

\[
= \frac{n}{|X_m|} (X_m)^n
\]

we find the derivative

\[
\frac{\partial \Phi}{\partial |X_m|} = \left( |n_m| + 2s_m \right) \left( \frac{\Phi}{|X_m|} \right)
\]

(3.4.4)
where it is assumed that $|X_m| \neq 0$. Similarly, the derivative of the phasor of the $q$ th component of the output of the nonlinearity with respect to the angle of the phasor of the $m$ th component of the input is found to be

$$\frac{\partial Y_q}{\partial \phi_m} = \sum_{n=0}^{\infty} \sum_{n_1, \ldots, n_N} \frac{\partial U_q}{\partial \phi_m}$$

where

$$\frac{\partial U_q}{\partial \phi_m} = \text{Re} \left\{ A_{\epsilon_n} \frac{\partial T}{\partial \phi_m} \right\}_{\omega_q}$$

and

$$\frac{\partial T}{\partial \phi_m} = \sum_{\alpha=0}^{\infty} \sum_{s_1, \ldots, s_N} \left\{ \left( \frac{(n + 2\alpha)!}{2^{n+2\alpha}} \right) a_{n+2\alpha} R_{n+2\alpha} \frac{\partial \Phi}{\partial \phi_m} \right\}.$$ 

Taking the derivative of (3.4.3) and noting that

$$\frac{\partial (X_m^n)}{\partial \phi_m} = \frac{\partial}{\partial \phi_m} \left( |X_m|^n e^{jn\phi_m} \right)$$

$$= jn |X_m|^n e^{jn\phi_m}$$

$$= jn (X_m)^n$$

we find

$$\frac{\partial \Phi}{\partial \phi_m} = jn_m \Phi.$$  

(3.4.8)

Calculation of the partial derivatives is computationally inexpensive as many of the terms are precalculated in the evaluation of an intermodulation product. The
Following section will show how these formulas can be incorporated in a harmonic-balance-type algorithm to analyze nonlinear circuits.

### 3.5 Circuit Simulation Using Generalized Power Series

In this section we develop a technique for the simulation of nonlinear analog circuits using the generalized power series formulas just presented. This technique is similar to the harmonic balance methods discussed in the previous chapter in that the steady state solution is found by solving the system of nonlinear equations resulting from the frequency domain statement of Kirchoff's current law. The method presented here uses voltage dependent generalized power series descriptions of the nonlinear elements and a nodal admittance matrix description of the linear elements. We show how the determining equation is derived and present an algorithm for implementing the analysis technique on a digital computer.

The circuit to be analyzed is arranged as shown in Fig. 3.5.1, and is divided into linear and nonlinear subcircuits. The linear subcircuit has $Q$ nodes; $P$ of which are common to the nonlinear subcircuit, and $M$ of which are connected to independent voltage sources. The nonlinear subcircuit has $P$ nodes and is composed of nonlinear elements each of which is characterized by a generalized power series. Here we consider $N$ frequency components, so that at the $p$th node the instantaneous current into the linear subcircuit is

$$
i_p(t) = \sum_{q=1}^{N} \text{Re}[\hat{I}_p(\omega_q)e^{i\omega_q t}]$$

(3.5.1)
where $\hat{I}(\omega_q)$ is a phasor current at radian frequency $\omega_q$. Similarly, the current into the nonlinear subcircuit at the $p$ th node is

$$i_p(t) = \sum_{q=1}^{N} \text{Re}[I_p(\omega_q)e^{j\omega_q t}]$$ (3.5.2)

and the voltage at the $p$ th node is

$$v_p(t) = \sum_{q=1}^{N} \text{Re}[V_p(\omega_q)e^{j\omega_q t}].$$ (3.5.3)

To satisfy Kirchoff's current law,

$$i_p(t) + i_p(t) = 0$$ (3.5.4)

for all $p$ from 1 to $P$, e.g. at all nodes of the nonlinear subcircuit. We can rewrite this in the frequency domain as

$$F(V) = I + \hat{I} = 0$$ (3.5.5)
where \( \hat{I} \) is the real valued vector of length \( 2NP \) consisting of the real and imaginary components of the phasor currents into the linear subcircuit at all frequencies

\[
\hat{I} = \begin{bmatrix}
\hat{I}_1^R(\omega_1) & \hat{I}_1^I(\omega_1) & \cdots & \hat{I}_P^R(\omega_1) & \hat{I}_P^I(\omega_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\hat{I}_1^R(\omega_2) & \hat{I}_1^I(\omega_2) & \cdots & \hat{I}_P^R(\omega_2) & \hat{I}_P^I(\omega_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\hat{I}_1^R(\omega_N) & \hat{I}_1^I(\omega_N) & \cdots & \hat{I}_P^R(\omega_N) & \hat{I}_P^I(\omega_N)
\end{bmatrix}^T,
\tag{3.5.6}
\]

\( I \) is the corresponding vector of phasor currents into the nonlinear subcircuit

\[
I = \begin{bmatrix}
I_1^R(\omega_1) & I_1^I(\omega_1) & \cdots & I_P^R(\omega_1) & I_P^I(\omega_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
I_1^R(\omega_2) & I_1^I(\omega_2) & \cdots & I_P^R(\omega_2) & I_P^I(\omega_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
I_1^R(\omega_N) & I_1^I(\omega_N) & \cdots & I_P^R(\omega_N) & I_P^I(\omega_N)
\end{bmatrix}^T,
\tag{3.5.7}
\]

and \( V \) is the vector of node voltage phasors

\[
V = \begin{bmatrix}
V_1^R(\omega_1) & V_1^I(\omega_1) & \cdots & V_P^R(\omega_1) & V_P^I(\omega_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
V_1^R(\omega_2) & V_1^I(\omega_2) & \cdots & V_P^R(\omega_2) & V_P^I(\omega_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
V_1^R(\omega_N) & V_1^I(\omega_N) & \cdots & V_P^R(\omega_N) & V_P^I(\omega_N)
\end{bmatrix}^T.
\tag{3.5.8}
\]

In these expressions, \( X_i^R(\omega_q) = \text{Re}[X_i(\omega_q)] \) and \( X_i^I(\omega_q) = \text{Im}[X_i(\omega_q)] \), where \( X = I, \hat{I}, \) and \( V \). Thus \( F \) has components of the form \( I_i^R(\omega_q) + \hat{I}_i^R(\omega_q) \) and \( I_i^I(\omega_q) + \hat{I}_i^I(\omega_q) \).

The steady state solution is found by solving the system of nonlinear equations represented by (3.5.5), called the determining equation [18]. As with the harmonic balance methods, we can solve (3.5.5) directly as a system of nonlinear equations, or we can solve it as a minimization problem. A popular technique for solving such a nonlinear system is Newton's method (also called the Newton-Raphson method) [39]. This method seeks to find the vector of node voltages to satisfy (3.5.5) using the iterative procedure

\[
V^{i+1} = V^i - J^{-1}(V^i)F(V^i)
\tag{3.5.9}
\]
where the superscripts are iteration indexes. The matrix $J$ is the Jacobian of $F$ whose elements are given by

$$
[J(V^i)]_{j,k} = \frac{\partial F_j(V^i)}{\partial V^i_k} = \frac{\partial I_j(V^i)}{\partial V^i_k} + \frac{\partial \hat{I}_j(V^i)}{\partial V^i_k}. 
$$

(3.5.10)

This method can also be used to solve (3.5.5) as a minimization problem if we consider the function to be minimized as

$$
E(V) = \frac{1}{2} F^T F.
$$

(3.5.11)

In this case, it is referred to as the Gauss-Newton method [39] (refer to Appendix A for more details).

To use this technique, we must be able to calculate the function values $F$ as well as their derivatives. Calculation of the function values requires calculation of the real and imaginary components of the phasor currents into the linear and nonlinear subcircuits. The current into the linear subcircuit is found through multiplication of the node voltage vector by the nodal admittance matrix of the linear subcircuit (see Appendix B), e.g.

$$
\hat{I} = YV.
$$

(3.5.12)

The current into the nonlinear subcircuit is found by evaluating the algebraic formula (3.2.3)-(3.2.9) for each nonlinear element. (These calculations require conversion of the node voltages into a polar representation and a corresponding conversion of the phasor currents into rectangular form.)
The calculation of the Jacobian requires partial derivatives of the real and imaginary components of the phasor currents with respect to the real and imaginary components of the node voltage phasors for all nodes (1 to \(P\)) and all frequency components (1 to \(N\)). For the linear subcircuit, this information is contained in the nodal admittance matrix. In particular,

\[
\frac{\partial I_j}{\partial V_k} = [Y]_{j,k}. \tag{3.5.13}
\]

The derivatives for the nonlinear subcircuit are calculated for each element using the algebraic formula (3.4.1)-(3.4.8). While this formula calculates derivatives with respect to polar quantities, they can be easily converted to derivatives with respect to real and imaginary variables using the chain rule

\[
\frac{\partial I(\omega_q)}{\partial V^R(\omega_q)} = \left( \frac{\partial I(\omega_q)}{\partial |V(\omega_q)|} \right) \left( \frac{\partial |V(\omega_q)|}{\partial V^R(\omega_q)} \right) + \left( \frac{\partial I(\omega_q)}{\partial I(\omega_q)} \right) \left( \frac{\partial I(\omega_q)}{\partial V^I(\omega_q)} \right) \tag{3.5.14}
\]

and

\[
\frac{\partial I(\omega_q)}{\partial V^I(\omega_q)} = \left( \frac{\partial I(\omega_q)}{\partial |V(\omega_q)|} \right) \left( \frac{\partial |V(\omega_q)|}{\partial V^I(\omega_q)} \right) + \left( \frac{\partial I(\omega_q)}{\partial I(\omega_q)} \right) \left( \frac{\partial I(\omega_q)}{\partial V^I(\omega_q)} \right) \tag{3.5.15}
\]

along with

\[
\frac{\partial I^R(\omega_q)}{\partial V^R(\omega_q)} = \text{Re} \left[ \frac{\partial I(\omega_q)}{\partial V^R(\omega_q)} \right] \tag{3.5.16}
\]

\[
\frac{\partial I^I(\omega_q)}{\partial V^R(\omega_q)} = \text{Im} \left[ \frac{\partial I(\omega_q)}{\partial V^R(\omega_q)} \right] \tag{3.5.17}
\]

\[
\frac{\partial I^R(\omega_q)}{\partial V^I(\omega_q)} = \text{Re} \left[ \frac{\partial I(\omega_q)}{\partial V^I(\omega_q)} \right] \tag{3.5.18}
\]

\[
\frac{\partial I^I(\omega_q)}{\partial V^I(\omega_q)} = \text{Re} \left[ \frac{\partial I(\omega_q)}{\partial V^I(\omega_q)} \right]. \tag{3.5.19}
\]
This form of the derivatives is required in the implementation of GPSA. The calculation of the derivatives of voltage magnitude and phase with respect to real and imaginary components is discussed in Appendix C.

An algorithm for implementing this analysis procedure is shown in Fig. 3.5.2. The analysis proceeds by calculating the nodal admittance representation of the linear subcircuit and making an initial estimate of the node voltage phasors. This estimate is used to calculate the function values ($F$), the error function ($E$), and the Jacobian ($J$). The voltage estimate is then updated using the iteration (3.5.9). The Newton's method iteration is continued until the error function is reduced to a small value, as compared to the square of a typical current in the circuit.

The major difference between this technique and the standard harmonic balance methods is the calculation of the frequency domain currents from the frequency domain voltages. This difference is illustrated in Fig. 3.5.3. In the standard harmonic balance method the frequency domain voltages are transformed into time domain voltages. The time domain voltages are then applied to the non-linear elements to calculate time domain currents. The time domain currents are then transformed into the frequency domain. In generalized power series analysis, the frequency domain currents are calculated directly from the frequency domain voltages using the algebraic formula (3.2.3)-(3.2.9) and the generalized power series coefficients. We refer to generalized power series analysis as a spectral balance technique as the method is similar to harmonic balance without the restriction of
INPUT CIRCUIT PARAMETERS

CALCULATE Y-MATRIX FOR LINEAR SUBCIRCUIT

ESTIMATE SOLUTION

EVALUATE ERROR AND JACOBIAN

IS ERROR SMALL ENOUGH?

YES

OUTPUT VARIABLES

NO

UPDATE VALUES USING NEWTON'S METHOD

Figure 3.5.2: An algorithm for analyzing a nonlinear circuit using generalized power series analysis and a minimization approach.
harmonically related frequencies. In the next section, we describe a computer implementation of this technique which we have named **FREDA** for **FREquency Domain Analysis**.

### 3.6 Implementation of GPSA – FREDA

We have developed a nonlinear circuit simulator based on generalized power series analysis named **FREDA**, whose flowchart is shown in Fig. 3.6.1. This program uses Newton’s method, globalized using a continuation, or homotopy method to solve the determining equation. In this approach, the power level of the sources is used as the homotopy parameter. For small input powers, the circuit behaves almost linearly and can be solved easily from an arbitrary initial estimate of the node voltage phasors. The input power can then be increased and this solution appropriately scaled and used as the new initial estimate of the node voltages. With the power increment properly chosen, the Newton’s method iterations will converge rapidly, and the process can be repeated until the desired power level is reached. In many cases, such as amplifier simulations, this effort provides a sweep of the output power as a function of input power.

**FREDA** is designed to simulate nonlinear analog circuits containing a single MESFET. The circuit to be simulated is separated into linear and nonlinear subcircuits, with the nonlinear subcircuit consisting of a model for the nonlinear behavior of the transistor (to be discussed later). The linear subcircuit can be
Figure 3.5.3: A comparison of the standard harmonic balance technique and the generalized power series analysis approach to the calculation of frequency domain currents from frequency domain voltages.
Figure 3.6.1: Flowchart for the generalized power series based program FREDA, for the simulation of nonlinear analog circuits containing a single MESFET.
arbitrarily specified in a format similar to that used in commercially available simulators such as SPICE [4] and TOUCHSTONE [65]. This format consists of the element type, its value, the nodes to which it is attached, and a name to be used as a label.

After entering the descriptions of the linear and nonlinear subcircuits, information is input defining the independent sources, their power levels, and the frequencies to be considered. Based on the frequency information a table of intermodulation products is developed. An initial estimate of the node voltage phasors (at a low power level) is then used to calculate the currents in the linear and nonlinear subcircuits, the magnitude of the error function, and the Newton step. The voltage estimate is updated using (3.5.5) until the error is reduced below a predetermined value (based on a typical current flowing in the circuit). The solution is output and the power level of the source increased. The previous solution is scaled and used as the new initial estimate. This is repeated until the final power level is reached.

3.7 Conclusion

In this chapter we have presented the basic properties of generalized power series and shown how they can be used in a harmonic-balance type algorithm to find the steady-state response of nonlinear analog circuits. We have also discussed FREDA, a program based on this algorithm, which is designed to simulate analog
circuits containing a MESFET as the nonlinear element. In subsequent chapters, we will discuss the nonlinear subcircuit used to model the MESFET, compare simulated results to measured circuit behavior, and comment on the performance of FREDA in comparison to other simulators.
Chapter 4

MESFET Modeling: Model Selection, Device Characterization, and Parameter Extraction

4.1 Introduction

In this chapter we present a MESFET model that is compatible with the generalized power series analysis just described. The development of an appropriate model involves three major steps: model selection, device characterization, and parameter extraction. In the model selection step, a model is chosen that is capable of representing a certain type of device, for example a bipolar junction transistor or a metal semiconductor field effect transistor (MESFET). This general model is then applied to a particular device, here an Avantek AT-8250 GaAs MESFET, by adjusting the variables in the model. These variables are determined in the characterization and parameter extraction steps. In the characterization step the particular device is measured under a variety of operating conditions. This information is then used in the parameter extraction step to determine the variables needed for the model.

In the following sections we describe the model selected for the MESFET and discuss the use of generalized power series to describe the nonlinear elements. We
also discuss how a particular transistor is characterized and how the appropriate parameters are extracted for the model. To illustrate this process two examples will be presented, a device that has been extensively described in the literature and a device that we have characterized and tested.

4.2 Model Selection

There are two basic types of MESFET models, physical and empirical. Traditionally physical models have been used by device designers while empirical models have been used by circuit designers. In this section we give a brief overview of the two approaches and discuss the empirical model used with generalized power series analysis.

In the physical approach the transistor is represented by the system of nonlinear differential equations resulting from consideration of the laws of physics and the construction of the device. Models based on this approach can be subdivided into numerical and analytic techniques. The numerical techniques discretize the time and space variables and use finite difference methods to convert the system of nonlinear differential equations into a system of nonlinear algebraic equations. The resulting equations are then solved iteratively. This approach has been used by a number of researchers and is useful for understanding the physical operation of the device [66,67,68]. There have been several efforts to incorporate a numerical physics based model into a circuit simulator [68,69], but unfortunately these mod-
els are too inefficient to be useful for circuit design. As an example, one numerical physics based transistor model used in conjunction with a time domain circuit simulator had to run overnight on a VAX 11/780 to reach a steady state solution when it was embedded in a simple external circuit [69].

The analytic techniques attempt to solve the system of nonlinear differential equations more efficiently than can the numerical techniques while still retaining the dependence upon the physical device construction. This is accomplished by making a number of simplifying assumptions so that some of the equations can be solved analytically [70,71,72]. Analytic models have been successfully integrated into circuit simulators [72], and represent a good compromise between tools useful only to device designers (e.g. numerical physics based models) and tools useful only to circuit designers (e.g. empirical models).

In contrast, the empirical models do not attempt to solve the system of nonlinear differential equations at all. Instead the device is modeled with an equivalent circuit. The circuit is chosen to reproduce the device behavior as measured under a variety of operating conditions. This approach requires an existing device and cannot be used as a tool for designing devices. (Alternatively, an empirical model could be developed from the circuit characteristics of a physics based model, in which case an existing device would not be required.) But because a circuit model is used this approach is computationally efficient and can be easily included in circuit simulators [25,73,74,75]. This is the approach used here and described in
the following.

The equivalent circuit used in FREDA to model the nonlinear behavior of a MESFET is shown in Fig. 4.2.1. This is a commonly accepted model and has been used successfully in a number of microwave circuit applications [25,74,75,76,77]. In general all of the elements are nonlinear functions of the gate-source and drain-source voltages, enabling a wide variety of effects to be simulated. In FREDA the drain-source current is modeled by the transconductance ($G_m$) which is a function of the voltage across $C_{gs}$, and the drain-source resistance ($R_{ds}$) which is a function of the voltage from drain to source. The gate junction is modeled by the nonlinear gate-source capacitance ($C_{gs}$) and the gate-source resistance ($R_{gs}$) which allows forward conduction through the gate to be considered. Both elements are functions of the voltage across $C_{gs}$. The resistance, $R_i$, models the charging time of the junction and is also a function of the voltage across $C_{gs}$. Gate-drain breakdown is modeled by the resistive element $R_{gd}$ which is a function of the voltage from gate to drain. The gate-drain capacitance ($C_{gd}$) is also a function of this voltage, while the drain-source capacitance ($C_{ds}$) is a function of the voltage from drain to source.

It should be noted that this is a lumped-element model. As the operating frequency increases and the device dimensions become appreciable fractions of the wavelength, the distributed nature of the device becomes significant. For typical FETs this will occur at frequencies of tens of gigahertz [78]. If transistors are
Figure 4.2.1: The circuit used in **FREDA** to model the nonlinear behavior of a MESFET. Each circuit element is a nonlinear function of voltage.
to be simulated at these frequencies, the lumped-element model presented here will not be sufficient. Another effect that should be considered at high microwave frequencies is the formation of a dipole region in the channel near the drain. This can be modeled by an additional nonlinear capacitance connected to the drain terminal and between $C_{gs}$ and $R_i$ [73,79] and has been shown to be a limiting factor in high frequency operation [80]. This could be easily added to the model used here if operation at high frequencies is desired.

While the model of Fig. 4.2.1 is capable of modeling the nonlinear behavior of the transistor, additional circuit elements are required to form the complete model for a packaged device as shown in Fig. 4.2.2. In generalized power series analysis, the circuit elements of Fig. 4.2.1 will be part of the nonlinear subcircuit while the additional elements of Fig. 4.2.2 will be part of the linear subcircuit. In the program FREDA the topology of the nonlinear subcircuit is fixed while the linear subcircuit can be specified arbitrarily. Thus the linear circuit used to model the parasitics associated with the packaged transistor can be easily changed as needed for different devices.

Adapting the model of Fig. 4.2.1 to generalized power series analysis requires that each of the nonlinear circuit elements be described using generalized power series. In particular, FREDA requires a power series for the current through each element as a function of a voltage in the circuit. For the resistive elements a series
Figure 4.2.2: A complete circuit model for a packaged MESFET.
of the form

\[ i(t) = a_0 + a_1 v(t) + a_2 v^2(t) + \cdots + a_p v^p(t) \quad (4.2.1) \]

is used where \( i(t) \) is the time domain current through the nonlinear element and \( v(t) \) is the time domain voltage across the element. The transconductance element is similarly represented with the addition of the time delay \( \tau \) (to model the finite transit time of the carriers)

\[ i_{gm}(t) = g_m v(t - \tau) + g_m v^2(t - \tau) + \cdots + g_m v^p(t - \tau) \quad (4.2.2) \]

where the voltage \( v \) is the voltage across the gate-source capacitance \( C_{gs} \). The nonlinear capacitors are described by a series for capacitance in terms of voltage

\[ C(v) = C_0 + C_1 v(t) + C_2 v^2(t) + \cdots + C_p v^p(t) \quad (4.2.3) \]

where the voltage \( v \) is the voltage across the capacitor. This is converted into an equation for charge as a function of voltage by integrating with respect to voltage

\[ q(v) = \int C(v) dv. \quad (4.2.4) \]

As the capacitance is given by a power series, this integration is readily performed to yield

\[ q(v) = C_0 v(t) + \frac{1}{2} C_1 v^2(t) + \frac{1}{3} C_2 v^3(t) + \cdots + \frac{1}{p+1} C_p v^{p+1}(t) + q_c \quad (4.2.5) \]

where \( q_c \) is a constant. The current through the nonlinear capacitor is then found by taking the derivative of the charge with respect to time

\[ i(t) = \frac{dq}{dt}. \quad (4.2.6) \]
If $I_k$ is the phasor current through the capacitor and $Q_k$ the phasor representation of the charge at frequency $\omega_k$ then $I_k = j\omega_k Q_k$. This is the approach used in FREDA, e.g. the algebraic formulas of Chapter 3 are used to calculate the various frequency components of the charge which are then converted to frequency components of current through multiplication by $j\omega$. The following section details the procedure used to characterize a transistor so that the parameters used in the model can be found.

### 4.3 Device Characterization

In this section we discuss the procedure used to characterize a transistor. Where appropriate, example data is presented for an Avantek AT-8250 gallium arsenide MESFET. This transistor is a low noise, medium power device designed for use in the 2 to 8 GHz frequency range [81] and was selected arbitrarily. It is packaged in a 70 mil stripline package, has a 0.5 $\mu$m gate length, and a total gate width of 500 $\mu$m. The device behavior was measured under a number of operating conditions including dc and small-signal ac excitation.

#### 4.3.1 DC Measurements

Perhaps the most commonly available performance data for a transistor are the dc characteristics. For MESFETs, the drain-source current as a function of drain-source and gate-source voltages is the most important dc behavior. This data is
presented in Fig. 4.3.1 and Fig. 4.3.2 for the Avantek AT-8250. The drain-source current as a function of the drain-source voltage is shown in Fig. 4.3.1 for several different values of gate-source voltage. In Fig. 4.3.2 the data is presented as a function of the gate-source voltage for several different values of drain-source voltage. These measurements were taken using a Hewlett Packard (HP) 4145 automated semiconductor parameter analyzer with the transistor in a 70 mil Avantek test fixture and the dc signals applied through broadband bias tees. The current was measured as the drain-source voltage was varied from 0 V to 5.0 V in 0.2 V steps, and as the gate-source voltage ranged from -1.0 V to 0.5 V in 0.1 V steps.

These measurements provide information about the behavior of the nonlinear transconductance element, $G_m$, and the nonlinear drain-source resistance, $R_{ds}$, in the model of Fig. 4.2.1.

In addition to the standard drain-source current characteristics, dc measurements were made to determine the parasitic resistances $R_s$, $R_d$, and $R_g$ in the model of Fig. 4.2.2. This procedure was described by Fukui [82] and requires two types of measurements. The first type is a measurement of the forward biased gate current as a function of the gate voltage for several different ground connections. In particular, measurements are made firstly with the source and drain both connected to ground, then with only the source connected, and finally with only the drain connected to ground. The results of these measurements are shown in Fig. 4.3.3. This data was taken manually using a digital voltmeter and
Figure 4.3.1: The measured dc drain-source current as a function of the drain-source voltage for an Avantek AT-8250 GaAs MESFET. The gate-source voltage is stepped in 0.1 V increments from -0.7 V to 0.5 V.
Figure 4.3.2: The measured dc drain-source current as a function of the gate-source voltage for an Avantek AT-8250 GaAs MESFET. The drain-source voltage is stepped from 0.2 V to 0.8 V in 0.2 V increments, and from 1.0 V to 5.0 V in 1.0 V increments.
a digital ammeter, varying the gate bias so that the gate current ranged from 0 to 50 mA. Again the device was placed in an Avantek test fixture and the bias applied through broadband bias tees. The second type of measurement is the drain-source current as a function of the gate-source voltage while the drain-source voltage is fixed at 0.05 V. The results of this measurement are shown in Fig. 4.3.4. This data was also taken manually with the gate-source varying from -0.9 V to 0.5 V in 0.1 V increments. The interpretation of these results is explained in the upcoming section on parameter extraction.

4.3.2 AC Measurements

In addition to the dc measurements, a series of small-signal ac measurements were made. In particular the scattering parameters were measured at 201 different frequencies in the range 45 MHz to 18 GHz. The S-parameters were measured at a number of different dc bias points, including all combinations of: \( V_{ds}=0, 0.3, 0.5, 0.7 \) V; \( V_{ds}=1.0 \) V to \( V_{ds}=5.0 \) V in 0.5 V steps; \( V_{gs}=-2.0, -1.5, -1.0 \) V; and \( V_{gs}=-0.8 \) V to \( V_{gs}=0.1 \) V in 0.1 V steps. Although it is impractical to show all of this data here, Fig. 4.3.5 shows \( S_{11} \) for three different gate-source voltages with a constant drain-source voltage. Similarly, Fig. 4.3.6 shows \( S_{22} \) for four different drain-source voltages with a constant gate-source voltage. In both figures the frequency varies from 45 MHz to 12.6 GHz. (Above this frequency the S-parameters display an undesired resonant-type behavior that is difficult to model using the equivalent
Figure 4.3.5: The measured $S_{11}$ from 45 MHz to 12.6 GHz for an Avantek AT-8250 GaAs MESFET. The drain-source voltage is 3.0 V and the three curves represent gate-source voltages of $-0.8$ V, $-0.6$ V, and $-0.4$ V in order from larger to smaller radius. The frequencies shown are in GHz.
circuits considered here, so this data is omitted.) The magnitude of $S_{21}$ is shown in Fig. 4.3.7 for a number of different gate-source voltages. All of the S-parameter data was taken using an HP 8510A automated network analyzer. The transistor was placed in an HP 85041A test fixture with a 70 mil insert. An HP 8340B synthesized sweeper was used as the ac signal source and dc bias was applied through the HP 8515A S-parameter test set. The transistor measurements were de-embedded using the HP 85014A active device measurement software. In the following section we will discuss the determination of the model parameters based on these measurements.

4.4 Parameter Extraction

In this section we discuss the interpretation of the measurements just described and the determination of the element values for the circuit model of the transistor. The approach used here is the one described by Rauscher and Willing [83] and requires a number of assumptions to be made. In particular, the nonlinear elements are assumed to be time invariant with the only time dependence due to the time dependence of the voltages in the circuit. In addition, it is assumed that the large-signal behavior of the nonlinear elements can be determined from small-signal measurements. This technique uses the device behavior under rf-perturbed static excitation to determine the large-signal dynamic behavior and is called the quasi-static approach [83]. These assumptions are justified on the basis of their use with
Figure 4.3.6: The measured $S_{22}$ from 45 MHz to 12.6 GHz for an Avantek AT-8250 GaAs MESFET. The gate-source voltage is $-0.2$ V and the four curves represent drain-source voltages of 1.5 V, 1.0 V, 0.7 V, and 0.5 V in order from larger to smaller radius. The frequencies shown are in GHz.
other simulation techniques, the results of which have been experimentally verified at frequencies through X band [73,83]. In the following we discuss the use of the dc and ac measurements and the development of the power series descriptions for the nonlinear elements.

4.4.1 Interpretation of DC Measurements

The dc measurements discussed above may be used to determine values for the transconductance and the drain-source conductance as a function of bias as well as values for the parasitic resistances in series with the gate, source, and drain. The transconductance is defined as the change in drain-source current produced by a change in gate-source voltage [84] and can be written

\[ G_m = \frac{\partial I_{ds}}{\partial V_{gs}} \]  

(4.4.1)

where the drain-source voltage is held constant. The transconductance may be calculated from the data for \( I_{ds} \) as a function of \( V_{ds} \) and \( V_{gs} \) by numerically differentiating with respect to \( V_{gs} \) for a fixed value of \( V_{ds} \). The data shown in Fig. 4.3.2 was used to calculate the dc transconductance and the results are shown in Fig. 4.4.1. The derivatives were found using a three point forward difference formula for the two points closest to the lower endpoint, a three point backward difference formula for the two points closest to the upper endpoint, and a four point central difference formula for the remaining points [85]. The values shown in Fig. 4.4.1 represent the transconductance seen at the terminals of the transistor. The intrin-
Figure 4.4.1: The dc transconductance seen at the device terminals as a function of the gate-source voltage for an Avantek AT-8250 GaAs MESFET as calculated from the measured dc characteristics. The drain-source voltage is stepped from 0.2 V to 0.8 V in 0.2 V increments and from 1.0 V to 5.0 V in 1.0 V increments.
sic transconductance (e.g. the element labeled $G_m$ in Fig. 4.2.1) is related to this value through the expression

$$G_{m(terminal)} = \frac{G_m}{1 + R_s G_m}$$

(4.4.2)

which accounts for the feedback caused by the parasitic resistance $R_s$ [84].

In a similar manner, the drain-source conductance is defined as the change in drain-source current produced by a change in drain-source voltage and is written

$$G_{ds} = \frac{\partial I_{ds}}{\partial V_{ds}}$$

(4.4.3)

where the gate-source voltage is held constant [84]. This can be found from the characteristics of Fig. 4.3.1 by differentiating with respect to $V_{ds}$ for a fixed value of $V_{gs}$. Taking the derivatives as before, the resulting conductance values are shown in Fig. 4.4.2. Again, the values shown are for the conductance seen at the device terminals. The intrinsic conductance can be found using the expression

$$G_{ds(terminal)} = \frac{G_{ds}}{1 + (R_s + R_d)G_{ds}}$$

(4.4.4)

which removes the effect of the parasitic resistances $R_s$ and $R_d$ [84].

The parasitic resistance values are found using the procedure described by Fukui [82]. To determine the values for $R_s$, $R_d$, and $R_g$, the gate built-in voltage, $V_b$, is found by plotting the logarithm of the forward biased gate current as a function of the gate voltage with the source and drain connected to ground. This is shown in Fig. 4.4.3 and uses the data from Fig. 4.3.3. From the linear part of the curve two gate voltages are chosen, $V_{g(m)}$ and $V_{g(m-1)}$ corresponding to gate
Figure 4.4.2: The dc output conductance seen at the device terminals as a function of the drain-source voltage for an Avantek AT-8250 GaAs MESFET as calculated from the measured dc characteristics. The gate-source voltage is stepped in 0.2 V increments from −0.7 V to 0.3 V.
currents $I_{g(m)} = 10^m$ A and $I_{g(m-1)} = 10^{m-1}$ A. In this case $m$ is chosen to be -4, corresponding to

\[
\begin{align*}
I_{g(m)} &= 10^{-4} \text{ A} \\
V_{g(m)} &= 0.409 \text{ V} \\
I_{g(m-1)} &= 10^{-5} \text{ A} \\
V_{g(m-1)} &= 0.337 \text{ V}.
\end{align*}
\]

The ideality parameter $n$ is then given by [82]

\[ n = 16.8[V_{g(m)} - V_{g(m-1)}] \quad (4.4.5) \]

and in this case yields $n = 1.21$. The built-in voltage follows from

\[ V_b = 0.768 - 0.06 \log \left[ \frac{10^y}{L_g Z} \right] \quad (4.4.6) \]

where

\[ y = 12 + m - \frac{1}{1 - (V_{g(m-1)}/V_{g(m)})}, \quad (4.4.7) \]

$L_G$ is the gate length in $\mu$m, and $Z$ is the total gate width in mm [82]. For this transistor, $y = 2.319$ and $V_b = 0.593$ V. Next the pinch-off voltage, $V_p$, is found from the measurement of drain-source current as a function of gate-source voltage at a constant drain-source voltage of 0.05 V, shown in Fig. 4.3.4. An estimate is made for the pinch-off voltage based on an extrapolation of the curve in Fig. 4.3.4 to $I_{ds} = 0$. The current values are converted into resistance values through

\[ R_{ds} = \frac{V_{ds}}{I_{ds}} \]
Figure 4.4.3: The measured forward biased gate current as a function of gate voltage with the source and drain connected to ground for the Avantek AT-8250 GaAs MESFET.
and displayed as a function of the parameter $X$ defined as [82]

$$X = \frac{1}{1 - \sqrt{(V_b - V_{gs})/(V_b + V_p)}}.$$

If the resulting plot is not a straight line, the value of the pinch-off voltage is changed and the process repeated until the resistance values are a linear function of $X$. This process is illustrated in Fig. 4.4.4 which shows the results for three different choices of $V_p$. A pinch-off voltage of 0.25 V is seen to be the best choice. The point where the line intercepts the $R_{ds}$ axis is taken to be the value for $R_s + R_d$. In this case, with $V_p = 0.25$ V, we find $R_d + R_s = 7.75$ Ω. Taking the inverse of the slope of the $I_g - V_g$ characteristic (shown in Fig. 4.3.3) with the source connected to ground and subtracting the inverse of the slope with the drain connected to ground gives a value for $R_s - R_d$. (The slopes are evaluated in the 40 - 50 mA region.) Values for $R_d$ and $R_s$ can then be separated giving $R_s = 2.4$ Ω and $R_d = 5.3$ Ω. The parasitic gate resistance is found by taking the inverse of the slope of the $I_g - V_g$ characteristic with the source and drain connected to ground and subtracting the contribution of the paralleled $R_s$ and $R_d$. The result for this device is $R_g = 2.9$ Ω.

### 4.4.2 Interpretation of AC Measurements

The remainder of the elements required for the transistor model are found from the ac measurements. The parasitic reactive elements can be found from the S-parameter measurements made with $V_{ds} = 0.0$ V. With zero drain-source voltage,
Figure 4.4.4: The resistance $R_{ds} = V_{ds}/I_{ds}$ as a function of the parameter $X = \left[1 - \sqrt{(V_b - V_{gs})/(V_b + V_p)}\right]^{-1}$ for three different values of $V_p$, (a) corresponds to $V_p = 0.35$ V, (b) corresponds to $V_p = 0.25$ V, and (c) corresponds to $V_p = 0.15$ V. The lines are shown for reference purposes.
the model for the transistor can be simplified from the circuit shown in Fig. 4.2.2, for example, to the one shown in Fig. 4.4.5. This model is simpler in that the elements used to describe the active region of the transistor are replaced by an \( R - C \) network containing fewer unknown parameters [86]. The element values are found by optimizing the circuit to match the measured S-parameters using a linear circuit analysis program such as TOUCHSTONE [65]. The values of \( R \) and \( C \) are also optimized but are not used in the nonlinear simulations.

The nonlinear element values are found by optimizing the circuit model (such as the one shown in Fig. 4.2.2) to match the measured S-parameters. For each combination of \( V_{ds} \) and \( V_{gs} \) the previously determined linear element values are used and the values of the nonlinear elements optimized. This results in a table of nonlinear element values as a function of \( V_{ds} \) and \( V_{gs} \).

### 4.4.3 Developing the Power Series Descriptions

**FREDA** requires the nonlinear elements to be described by generalized power series. The power series are found in order to best reproduce the table of nonlinear element values. The procedure used here is to find a power series that fits the data best in the least squares sense over the desired operating range. This is described in detail in Appendix D.

Power series are fit directly to the optimized values for capacitance as a function of voltage. For the resistive elements, however, power series are fit to the optimized
Figure 4.4.5: The circuit used to model the transistor with zero drain-source voltage.
values of conductance as a function of voltage. These series are then integrated to give power series for current as a function of voltage. For example, the power series for the drain-source conductance as a function of the drain-source voltage is converted into a power series for the current through the $G_{ds}$ (or $R_{ds}$) element as a function of voltage by integrating with respect to the drain-source voltage, e.g.

$$I_{gds}(V_{ds}) = \int G_{ds}(V_{ds})dV_{ds}. \quad (4.4.8)$$

This integration is readily performed since the conductance is described by a power series. The resulting constant of integration can be determined from the dc current characteristics. The parameter extraction process is illustrated in detail in the following section which presents two modeling examples.

### 4.5 Modeling Examples

In this section we present two examples to demonstrate the modeling procedure. The first is a model based on a device that has been extensively characterized in the literature and the second is a device that we have characterized.

#### 4.5.1 A Model for the GaAs MESFET Characterized by Willing, et al. [73]

Although the literature contains a number of reports on transistor characterization and simulation, few publications present the details necessary to reconstruct the
device model. An exception is the paper by Willing, et al. [73] in which a model for a GaAs MESFET is described in detail and experimental data presented for the transistor with small-signal and large-signal excitation. This data is used here to develop a model suitable for use with generalized power series. The device characterized in [73] is a packaged Texas Instruments GaAs MESFET having a gate length of 1.7 μm and a gate width of 600 μm.

The circuit chosen to model the transistor is shown in Fig. 4.5.1 and contains linear and nonlinear elements used to model the behavior of the device as well as linear elements to model the package parasitics. This model is based on the circuit used in [73], but for simplicity contains only three nonlinear elements: the transconductance \( G_m \), the gate-source capacitance \( C_{gs} \), and the drain-source resistance \( R_{ds} \). The values of \( C_{gd}, C_{gs}, R_{ds}, R_i \), and \( G_m \) are specified in [73] for a gate-source voltage of -2.0 V and a drain-source voltage of 6.0 V (the chosen operating point). Using these values, the remaining element values in the model of Fig. 4.5.1 were found by optimizing this circuit to match the S-parameters presented in [73] (using TOUCHSTONE [65] for the optimization). These S-parameters were measured over the range 2-10 GHz at the previously mentioned bias point. The element values found in the optimization are presented in Table 4.5.1. The S-parameters of the optimized model are compared to the measured S-parameters in Fig. 4.5.2 and Fig. 4.5.3. The model accurately predicts \( S_{11}, S_{22}, \) and \( S_{21} \). The values for \( S_{12} \) do not match the measured values, but the
Figure 4.5.1: The circuit used with generalized power series analysis to model the MESFET characterized by Willing, et al. [73].
Table 4.5.1: The optimized linear element values for the circuit used to model the MESFET characterized by Willing, et al. [73].

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_a$</td>
<td>0.035 pF</td>
</tr>
<tr>
<td>$L_a$</td>
<td>0.39 nH</td>
</tr>
<tr>
<td>$C_b$</td>
<td>0.3 pF</td>
</tr>
<tr>
<td>$R_a$</td>
<td>0.17 Ω</td>
</tr>
<tr>
<td>$C_c$</td>
<td>0.0006 pF</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.011 Ω</td>
</tr>
<tr>
<td>$L_b$</td>
<td>0.18 nH</td>
</tr>
<tr>
<td>$R_d$</td>
<td>2.7 Ω</td>
</tr>
<tr>
<td>$L_c$</td>
<td>0.24 nH</td>
</tr>
<tr>
<td>$C_t$</td>
<td>0.06 pF</td>
</tr>
<tr>
<td>$R_i$</td>
<td>17 Ω</td>
</tr>
<tr>
<td>$C_{ds}$</td>
<td>0.12 pF</td>
</tr>
<tr>
<td>$C_{gd}$</td>
<td>0.05 pF</td>
</tr>
<tr>
<td>$\tau$</td>
<td>6.6 pS</td>
</tr>
</tbody>
</table>

magnitude of $S_{12}$ is small and so this discrepancy should not present a problem in the large-signal simulations.

In [73] S-parameters were measured at a number of different dc biases. The circuit model was then optimized to reproduce the measured S-parameters at each bias resulting in a series of figures showing the bias dependence of each nonlinear element. These figures are used here to find the power series descriptions for the nonlinear elements. The voltages controlling the nonlinear elements are assumed to be the same as the terminal voltages, due to the small values of the parasitic resistances $R_s$, $R_d$, and $R_g$.

The power series for the transconductance is found using the data presented
Figure 4.5.2: The measured $S_{11}$ and $S_{22}$ for the MESFET characterized in [73] compared to the S-parameters of the optimized circuit model. The measured results are shown as points over the frequency range 2-10 GHz. The frequencies shown are in GHz, and the points are at spacings of 2 GHz.
Figure 4.5.3: The measured $S_{21}$ and $S_{12}$ for the MESFET characterized in [73] compared to the S-parameters of the optimized circuit model. The measured results are shown as points over the frequency range 2-10 GHz. The outer radius corresponds to a value of 3.0 for $S_{21}$ and 0.1 for $S_{12}$. The frequencies shown are in GHz, and the points are at spacings of 2 GHz.
in Fig. 4 (b) of [73]. A power series is fit using the least squares method described in Appendix D to the data for the transconductance as a function of gate-source voltage with a constant drain-source voltage of 6.0 V. The transconductance calculated from the resulting power series is shown in Fig. 4.5.4 and compared to the optimized values from [73]. As explained previously, the power series for the transconductance is converted into a power series for current by integrating, e.g.

\[ I_{gm}(V_{gs}) = \int G_m(V_{gs}) dV_{gs}. \]

The constant of integration is chosen so that when this expression is added to the power series for the current through the drain-source conductance, the correct value of drain-source current is calculated at the operating point. In this example, the constant is chosen so that the current through the transconductance element is zero at a gate-source voltage of -2.0 V. The drain-source current calculated using this power series is shown in Fig. 4.5.5 along with the measured values of dc current from Fig. 1 of [73] shown for comparison. The current at -2.0 V is nonzero due to the contribution from the nonlinear drain-source resistance element.

The power series for the gate-source capacitance is found by considering the data presented in Fig. 4 (a) of [73]. In particular a power series is fit using the least squares method to the data for the gate-source capacitance as a function of gate-source voltage for a constant drain-source voltage of 6.0 V. The data calculated from the resulting power series is shown in Fig. 4.5.6 and compared to the optimized values from [73].
Figure 4.5.4: The transconductance as a function of gate-source voltage as calculated from the power series representation. The points are the optimized values from [73] for a constant drain-source voltage of 6.0 V.
Figure 4.5.5: The drain-source current as a function of gate-source voltage for a constant drain-source voltage of 6.0 V as calculated from the power series for the current through the transconductance element. Shown for comparison are the dc current measurements from [73], shown here as points.
Figure 4.5.6: The gate-source capacitance as calculated from its power series description compared to the optimized values from [73] (shown as points). The drain-source voltage is 6.0 V.
Values for the drain-source resistance, \( R_{ds} \), are provided in Fig. 4 (e) of [73] as a function of bias. These values are converted to conductances, e.g. \( G_{ds} = 1/R_{ds} \), and a power series fit to the values of conductance as a function of drain-source voltage for a constant gate-source voltage of -2.0 V. The conductance values calculated from this series are shown in Fig. 4.5.7 and compared to the optimized values from [73]. As with the transconductance element, the power series for the drain-source conductance is converted into a power series for current. In this case the integration is with respect to the drain-source voltage

\[ I_{rd}(V_{ds}) = \int G_{ds} dV_{ds}. \]

The constant of integration is chosen so that the total drain-source current \( I_{ds} = I_{gm} + I_{rd} \) is equal to the measured dc value at \( V_{ds} = 6.0 \) V and \( V_{gs} = -2.0 \) V. The drain-source current calculated from the resulting series is shown in Fig. 4.5.8 as a function of drain-source voltage. The measured values of current are shown for comparison.

It should be noted that the calculated drain-source current does not match the measured dc current as a function of drain-source voltage. This is because the element describing the change in \( I_{ds} \) with respect to \( V_{ds} \), \( G_{ds} \), is determined from the high frequency behavior of the transistor. For GaAs MESFETs, the drain-source conductance is known to increase with frequency [87,88]. The high frequency behavior is considered more important here because the goal of this work is to predict the nonlinear behavior at microwave frequencies.
Figure 4.5.7: The drain-source conductance as a function of drain-source voltage for a constant gate-source voltage of -2.0 V. The values calculated from the power series fit are shown compared to the optimized values from [73] (shown here as points).
Figure 4.5.8: The drain-source current as a function of drain-source voltage for a constant gate-source voltage of -2.0 V. The values calculated from the power series expression are shown compared to the measured values of current from [73] (shown here as points).
Table 4.5.2: The power series coefficients used in the model developed for the MESFET characterized in [73].

<table>
<thead>
<tr>
<th>Order</th>
<th>( I_{gm} ) (A)</th>
<th>( I_{rds} ) (A)</th>
<th>( C_{gs} ) (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.074618</td>
<td>-0.0411638</td>
<td>0.72 \times 10^{-12}</td>
</tr>
<tr>
<td>1</td>
<td>0.039559</td>
<td>0.1356875</td>
<td>0.183 \times 10^{-12}</td>
</tr>
<tr>
<td>2</td>
<td>0.101898 \times 10^{-2}</td>
<td>-0.088446</td>
<td>0.0317 \times 10^{-12}</td>
</tr>
<tr>
<td>3</td>
<td>-0.18746 \times 10^{-3}</td>
<td>0.0322235</td>
<td>-0.333 \times 10^{-14}</td>
</tr>
<tr>
<td>4</td>
<td>-0.178092 \times 10^{-3}</td>
<td>-0.68416 \times 10^{-2}</td>
<td>-0.17 \times 10^{-14}</td>
</tr>
<tr>
<td>5</td>
<td>-0.69647 \times 10^{-4}</td>
<td>0.844562 \times 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.764807 \times 10^{-5}</td>
<td>-0.56194 \times 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.275233 \times 10^{-6}</td>
<td>0.155754 \times 10^{-5}</td>
<td></td>
</tr>
</tbody>
</table>

The power series coefficients for the nonlinear elements are summarized in Table 4.5.2. Included are power series for the current through the transconductance element, for the current through the drain-source resistance element, and for the gate-source capacitance. As discussed earlier, power series descriptions for nonlinear capacitors are input to FREDA as series for capacitance and converted in the program to descriptions for current as a function of voltage. Large-signal simulations using this model will be presented in the following chapter and compared to measured results from [73].

4.5.2 A Model for the Avantek AT-8250 GaAs MESFET

Because more information was needed about the transistor's behavior (under both small and large-signal conditions) than was available in the literature, an Avantek
Table 4.5.3: The optimized linear element values for the circuit used to model the Avantek AT-8250 GaAs MESFET.

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{g1}$</td>
<td>0.0463 pF</td>
</tr>
<tr>
<td>$L_g$</td>
<td>0.6475 nH</td>
</tr>
<tr>
<td>$C_{g2}$</td>
<td>0.195 pF</td>
</tr>
<tr>
<td>$R_g$</td>
<td>2.9 Ω</td>
</tr>
<tr>
<td>$R_s$</td>
<td>2.4 Ω</td>
</tr>
<tr>
<td>$L_s$</td>
<td>0.00323 nH</td>
</tr>
<tr>
<td>$R_d$</td>
<td>5.3 Ω</td>
</tr>
<tr>
<td>$L_d$</td>
<td>0.6228 nH</td>
</tr>
<tr>
<td>$C_{d2}$</td>
<td>0.00094 pF</td>
</tr>
<tr>
<td>$C_d$</td>
<td>0.06936 pF</td>
</tr>
<tr>
<td>$R_i$</td>
<td>10 Ω</td>
</tr>
<tr>
<td>$\tau$</td>
<td>6.56 pS</td>
</tr>
</tbody>
</table>

AT-8250 GaAs MESFET was characterized and tested locally. In this section we discuss the model developed for this transistor. This device is a low noise medium power transistor designed for use in the 2 to 8 GHz frequency region and was arbitrarily selected. The circuit chosen to model this transistor is shown in Fig. 4.2.2. Five elements are considered to be nonlinear: $C_{gs}$, $C_{gd}$, $C_{ds}$, $G_{ds}$, and $G_m$. Values for the parasitic resistances $R_s$, $R_d$, and $R_g$ are found using the Fukui procedure [82] described above. The remaining linear elements are found by optimizing the circuit shown in Fig. 4.4.5 to match the measured S-parameters with $V_{ds} = 0.0V$ over the frequency range 45 MHz to 9 GHz (the useful range of the device). The resulting element values are listed in Table 4.5.3. The measured S-parameters are compared to the S-parameters calculated from the optimized
circuit in Fig. 4.5.9 and Fig. 4.5.10. In Fig. 4.5.9 values are shown for $S_{11}$ and $S_{22}$, while Fig. 4.5.10 shows $S_{12}$, (The values of $S_{21}$ are close to those for $S_{12}$ and are not shown.)

The nonlinear element values are found by optimizing the circuit of Fig. 4.2.1 to match the measured S-parameters over the frequency range 45 MHz to 9 GHz for a number of different bias conditions. The linear element values are fixed at the values presented in Table 4.5.3. There is too much data to present here, but an example of the results is shown in Fig. 4.5.11 and Fig. 4.5.12. The measured S-parameters are compared to the calculated S-parameters for $V_{gs} = -0.1$ V and $V_{ds} = 3.0$ V (the chosen operating point).

A power series was fit to the optimized transconductance values as a function of gate-source voltage, for $V_{ds} = 3.0$ V. The values of gate-source voltage at the terminals of the device were corrected to account for the voltage drop across $R_s$, so that the transconductance was found as a function of the intrinsic gate-source voltage (e.g. the voltage across $C_{gs}$). The result is shown in Fig. 4.5.13 with the optimized values compared to the power series expression. The power series for the transconductance was integrated with respect to gate-source voltage to give a power series for the component of drain-source current due to $G_m$. A constant was added so that the current through $G_m$ was zero at $V_{gs} = -0.1$ V. The resulting drain-source current as a function of gate-source voltage is shown in Fig. 4.5.14 and compared to the dc characteristics for $V_{ds} = 3.0$ V. (In this figure,
Figure 4.5.9: The measured $S_{11}$ and $S_{22}$ (points) compared to the S-parameters calculated from the optimized circuit model (lines) for the Avantek AT-8250, with $V_{ds} = 0.0$ V, over the frequency range 45 MHz to 9 GHz. The frequencies shown are in GHz, and the points are spaced every 0.72 GHz for $S_{22}$ and 0.18 GHz for $S_{11}$. 
Figure 4.5.10: The measured $S_{12}$ (points) compared to the calculated $S_{12}$ from the optimized circuit model for the Avantek AT-8250, with $V_{ds} = 0.0$ V, over the frequency range 45 MHz to 9 GHz. The value corresponding to the outer radius is 0.4. The frequencies shown are in GHz, and the points are spaced every 0.18 GHz.
Figure 4.5.11: The measured S-parameters (points) compared to the values calculated from the optimized circuit model (lines) for the Avantek AT-8250, with $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V, over the frequency range 45 MHz to 9 GHz. The frequencies shown are in GHz, and the points are spaced every 0.18 GHz for $S_{11}$ and 0.36 GHz for $S_{22}$. 
Figure 4.5.12: The measured S-parameters (points) compared to the values calculated from the optimized circuit model (lines) for the Avantek AT-8250, with $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V, over the frequency range 45 MHz to 9 GHz. The value corresponding to the outer radius is 6.0 for $S_{21}$ and 0.15 for $S_{12}$. The frequencies shown are in GHz, and the points are spaced every 0.36 GHz.
Figure 4.5.13: The optimized values of transconductance (points) compared to the power series description as a function of the intrinsic gate-source voltage for the Avantek AT-8250.
the current through $R_{ds}$ at $V_{ds} = 3.0$ V has been added to the values calculated from the power series.) The power series does not exactly match the measured dc characteristics because the power series for the transconductance was found from the ac measurements, which in this case yield different values for $G_m$ than those calculated from the dc measurements.

A power series was fit to the optimized values of drain-source conductance as a function of drain-source voltage for $V_{gs} = -0.1$ V. The values of drain-source voltage were corrected for the voltage drops across $R_d$ and $R_s$ and the power series found as a function of the intrinsic drain-source voltage (e.g. the voltage across $R_{ds}$). The power series is compared to the optimized conductance values in Fig. 4.5.15. The power series for $G_{ds}$ was integrated with respect to the drain-source voltage to find a power series for the component of drain-source current due to $R_{ds}$. A constant term was added to make $I_{ds}$ equal to the measured dc value at $V_{ds} = 3.0$ V. The current calculated from this power series is compared to the measured dc drain-source current as a function of drain-source voltage in Fig. 4.5.16. Again, the dc characteristics are not exactly reproduced due to the use of ac conductance values. It has been proposed to add a series $R - C$ element (linear) connected in parallel with $R_{ds}$ to account for the drain-source conductance change with respect to frequency [87]. This allows for a change in conductance but only allows for a constant difference in the dc and ac conductance values with respect to drain-source voltage. For this device, the difference in dc and ac
Figure 4.5.14: The measured dc drain-source current (points) for $V_{ds} = 3.0$ V compared to the power series representation for the current through $G_m$ as a function of the terminal gate-source voltage for the Avantek AT-8250.
Figure 4.5.15: The optimized values of drain-source conductance (points) compared to the power series description as a function of the intrinsic drain-source voltage for $V_{gs} = -0.1$ V for the Avantek AT-8250.
conductances is not a constant and this technique is not used. Instead, a single power series description is used to model the conductance for the entire frequency range considered in the simulations.

A power series was fit to the optimized values of gate-source capacitance as a function of the intrinsic gate-source voltage for \( V_{ds} = 3.0 \) V. The power series description is compared to the optimized values in Fig. 4.5.17. Similarly, a power series was fit to the optimized values of drain-source capacitance as a function of the intrinsic drain-source voltage for \( V_{gs} = -0.1 \) V. The optimized values are compared to the power series description in Fig. 4.5.18. Finally, a power series was fit to the optimized values of gate-drain capacitance as a function of the intrinsic drain-gate voltage (e.g. the voltage across \( C_{gd} \)) for \( V_{gs} = -0.1 \) V. The power series description is compared to the optimized values in Fig. 4.5.19.

The power series descriptions are summarized in Table 4.5.4. The current through the transconductance \( (I_{gm}) \) and the gate-source capacitance \( (C_{gs}) \) are functions of the intrinsic gate-source voltage. The current through the gate-drain resistance \( (I_{rds}) \) and the drain-source capacitance \( (C_{ds}) \) are functions of the intrinsic drain-source voltage. The gate-drain capacitance \( (C_{gd}) \) is a function of the intrinsic drain-gate voltage. The highest order used is sixth, for the two resistive elements, and in all cases the power series accurately fit the optimized element values.
Figure 4.5.16: The measured dc drain-source current (points) for $V_{gs} = -0.1$ V compared to the power series representation for the current through $G_{ds}$ as a function of the terminal drain-source voltage for the Avantek AT-8250.
Figure 4.5.17: The optimized values of gate-source capacitance (points) compared to the power series description as a function of the intrinsic gate-source voltage for $V_{ds} = 3.0$ V for the Avantek AT-8250.
Figure 4.5.18: The optimized values of drain-source capacitance (points) compared to the power series description as a function of the intrinsic drain-source voltage for $V_{gs} = -0.1$ V for the Avantek AT-8250.
Figure 4.5.19: The optimized values of gate-drain capacitance (points) compared to the power series description as a function of the intrinsic drain-gate voltage for $V_{gs} = -0.1$ V for the Avantek AT-8250.
Table 4.5.4: The power series coefficients used to model the Avantek AT-8250 GaAs MESFET.

<table>
<thead>
<tr>
<th>Order</th>
<th>$I_{gm}$ (A)</th>
<th>$C_{ds}$ (pF)</th>
<th>$I_{rs}$ (A)</th>
<th>$C_{gs}$ (pF)</th>
<th>$C_{gd}$ (pF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.01146</td>
<td>0.286</td>
<td>-0.83186 $\times 10^{-2}$</td>
<td>0.765918</td>
<td>0.34697</td>
</tr>
<tr>
<td>1</td>
<td>0.08886</td>
<td>-0.022345</td>
<td>0.036662</td>
<td>0.94329</td>
<td>-0.33135</td>
</tr>
<tr>
<td>2</td>
<td>0.049871</td>
<td>0.0043288</td>
<td>-0.035092</td>
<td>0.564949</td>
<td>0.1576</td>
</tr>
<tr>
<td>3</td>
<td>-0.030178</td>
<td>-0.0003038</td>
<td>0.01963</td>
<td>0.480278</td>
<td>-0.032687</td>
</tr>
<tr>
<td>4</td>
<td>-0.011395</td>
<td>-0.58022 $\times 10^{-2}$</td>
<td>0.29057</td>
<td>0.0024389</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.024355</td>
<td>0.85425 $\times 10^{-3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.0109178</td>
<td>-0.493237 $\times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.6 Conclusion

In this chapter we have discussed the development of a MESFET model suitable for use with generalized power series analysis. We have discussed the selection of an empirical circuit-type model, the measurements required to characterize a device, and the process used to extract the parameters for the model. This was illustrated by the development of models for two different transistors. In the following chapter these models will be used to predict the large-signal behavior of the transistors in an amplifier circuit and the simulated results compared to experimental performance.
Chapter 5

Experimental Verification of Generalized Power Series Analysis

5.1 Introduction

In this chapter the nonlinear circuit analysis technique developed in Chapter 3 and the MESFET model developed in Chapter 4 are verified through the comparison of simulated and experimental large-signal MESFET amplifier performance. Both the model for the device characterized by Willing, et al. [73] and the model for the Avantek device are used along with FREDA to predict the performance of an amplifier circuit and the results compared to measurements. For the Willing device results are compared for a single-tone input, while for the Avantek transistor results for single and two-tone inputs are compared. As a by-product of the simulations, the effective impedance of each nonlinear element in the MESFET model is found as a function of input power. These results are presented for the Avantek device. Finally, the performance of the simulator is discussed.
5.2 Results Using the Model of the Willing [73] Device

5.2.1 Introduction

In the following sections the generalized power series model developed for the MESFET characterized by Willing et al. [73] is used with FREDA to predict the large-signal behavior of a common source amplifier circuit. The results of the simulation are compared to the measured results presented in [73] and the agreement discussed.

5.2.2 Results for a Single-Tone Input

The circuit that is simulated is shown in Fig. 5.2.1 and is a MESFET connected in a common source amplifier configuration. This circuit is the one described in [73] and is simply the transistor in a 50.Ω system. This circuit was chosen to concentrate on the behavior of the transistor and to insure that the source and load impedances would be well behaved over a wide frequency range. The transistor is biased at $V_{gs} = -2.0$ V and $V_{ds} = 6.0$ V and it is assumed that the bias network is perfect, i.e. that the inductors are sufficiently large to totally block the rf and that the capacitors are sufficiently large to be invisible to the rf signal. The transistor is modeled using the generalized power series MESFET model described in the previous chapter. Four frequencies are considered in the simulations: dc, the fundamental, and two harmonics. (The inclusion of additional
frequencies did not significantly effect the results.)

The results of the simulation are shown in Fig. 5.2.2 and are compared to the measured results from [73]. A single tone was input at 2 GHz and the input power swept from -5 dBm to 20 dBm. The resulting power output in the fundamental (a) and in the second harmonic (b) are shown. The fundamental response is accurately predicted while the simulated second harmonic power is approximately 4 dB less than the measured response at low input powers. As the input power increases above 12 dBm the shape of the simulated second harmonic deviates from the shape of the measured response.

The gain is shown in Fig. 5.2.3 as a function of input power and the simulated results compared to the measured results from [73]. The small signal gain was measured to be 7.7 dB as compared to the simulated result of 8.1 dB. At 20 dBm input power the gain was compressed by approximately 4 dB. Again, it can be seen that the fundamental response is accurately predicted by the simulations.

If the gate-source bias used in the simulation is changed from $V_{gs} = -2.0 \text{ V}$ to $V_{gs} = -2.4 \text{ V}$, the predicted behavior more closely matches the measured response as shown in Fig. 5.2.4. In this case, the small signal gain is 7.8 dB which is closer to the measured value of 7.7 dB, and the simulated second harmonic power is within 0.3 dB of the measured value at low levels of input power. This behavior is readily explained by slight inaccuracies in the device characterization and modeling steps.
Figure 5.2.1: The common source amplifier circuit used in the simulations.
Figure 5.2.2: The results of the single tone test for an amplifier using the transistor characterized by Willing, et al. [73]. The curves are the simulated results for the power output at the fundamental frequency of 2 GHz (a) and the second harmonic (b). The points are measured results from [73].
Figure 5.2.3: The gain of the amplifier using the transistor characterized by Willing, et al. [73] as a function of input power at 2 GHz. The curve is the simulated result and the points are measured values from [73]. The transistor is biased at $V_{gs} = -2.0$ V and $V_{ds} = 6.0$ V.
Figure 5.2.4: The results of the single tone test for an amplifier using the transistor characterized by Willing, et al. [73] with a different bias used in the simulations. The curves are the simulated results for the power output at the fundamental frequency of 2 GHz (a) and the second harmonic (b).
5.2.3 Discussion

Overall, the simulated results are in agreement with the measured response, particularly for the lower levels of input power. There are, however, several explanations for the deviation of the second harmonic as the input power is increased. One is that the model used here is based on bias dependent element values presented in [73] for a slightly different circuit topology. The nonlinear MESFET model used in FREDA has a fixed topology and does not include all of the elements used in [73]. Another explanation is that the data presented in [73] does not cover a large enough voltage range, so that the behavior of the transconductance, for example, is not known for large gate-source voltages. Knowledge of this behavior is important for accurate simulations at large signal levels. It should also be noted that the simulations presented in [73] for the second harmonic (using the same characterization information) are different from the measured values by 4–5 dB which is comparable to the discrepancy between our simulations and the experimental results.

5.3 Results Using the Model of the Avantek AT-8250

5.3.1 Introduction

In order to completely verify the simulation technique, more information was needed about the transistor's behavior than was available in the literature. Consequ-
sequently, an Avantek AT-8250 GaAs MESFET was characterized and tested locally. In Chapter 4 the characterization procedure is described and a model developed for the device. In the following sections, we describe the experimental setup used to make large-signal measurements of the transistor’s behavior in a common source amplifier configuration. We then compare the measured performance to simulations using *FREDA* for a single-tone input and for a two-tone input and discuss the agreement.

### 5.3.2 Experimental Setup

The experimental setup used to measure the large-signal performance of the transistor in a common source amplifier configuration is shown in Fig. 5.3.1. The transistor is placed in a 70 mil Avantek test fixture and bias applied through broadband bias tees. Two rf signal sources are used: one an HP8690A with an HP8692B 2–4 GHz plug-in which was phase locked using an EIP 578 source locking frequency counter, and the other an HP8620C with a HP86222A 0.01–2.4 GHz solid-state plug-in (not phase locked). The sources were combined using a 2–8 GHz power divider and the output passed through a low-pass filter to reduce the level of any harmonics that might be present. The input power level was computer controlled using an HP9826 computer, a digital to analog converter, and an HP8732A PIN modulator. The power was measured using an HP8566A spectrum analyzer which was also under computer control. For the single tone test, the phase locked
Figure 5.3.1: The experimental setup used to measure the large-signal performance of the Avantek AT-8250 in a common source amplifier configuration.
HP8690A was used and the power divider eliminated. A calibrated measurement was made by removing the device and connecting the input to the device directly to the spectrum analyzer. The voltage to the PIN modulator was varied and the corresponding power level noted. The device was then inserted and the power swept over the chosen range by varying the voltage to the PIN modulator as before. The corresponding output power at all of the frequencies of interest is read from the spectrum analyzer by the computer. The PIN modulator has a dynamic range of approximately 40 dB while the maximum power was determined by the output level controls on the rf sources and by the manual attenuator included in the circuit.

5.3.3 Preliminary Results for a Single-Tone Input

The circuit used in the simulations is the one shown in Fig. 5.2.1 and the model used is the one developed in the previous chapter for the Avantek transistor. The results of the simulation with a single-tone input are shown in Fig. 5.3.2 and compared with the measured performance. (The simulations include four frequencies: dc, the fundamental, and two harmonics. As before, the inclusion of additional frequencies is seen to have little effect.) The fundamental, second harmonic, and third harmonic are shown for a fundamental frequency of 3 GHz and a bias of $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V. While the predictions are good, the shape of the second harmonic curve is not accurate above 7 dBm input power, and
both of the predicted harmonic power levels are greater than the measured values.

By changing the bias, the power output in the fundamental is more accurately predicted as seen in Fig. 5.3.3. This is a plot of the simulated gain as a function of input power for a fundamental frequency of 3 GHz and a bias of $V_{ds} = 3.0$ V and $V_{gs} = -0.24$ V. Included for comparison are the experimental results corresponding to $V_{gs} = -0.1$ V, as shown before. The small signal gain is 10 dB and an input power of 10 dBm corresponds to approximately 4 dB of gain compression. The following section presents a modified model of this device, developed to improve the predictions.

### 5.3.4 Development of an Improved Model

It was theorized that the predicted harmonic power levels are higher than the measured values due to inaccuracies in the device model as reflected by the comparison of the simulated and measured values of $S_{22}$ shown in Fig. 4.5.11. In this section an improved model is developed by re-optimizing the circuit of Fig. 4.2.2 to match the measured S-parameters, with increased emphasis placed on matching $S_{22}$. The optimized results presented in Chapter 4 were obtained using the model function in TOUCHSTONE [65]. This feature attempts to match two sets of S-parameter data (e.g. measured results and modeled results) by changing the values of a set of circuit parameters. Unfortunately, each S-parameter is treated
Figure 5.3.2: The results of the single tone test for the Avantek AT-8250. Shown are the simulated (curves) and measured values (points) for power output at the fundamental (a), the second harmonic (b), and the third harmonic (c) as a function of input power.
Figure 5.3.3: The gain of the common source amplifier using the Avantek AT-8250. The simulated results (curve) with a bias of $V_{gs} = -0.24$ V are compared with measured values (points) for a bias of $V_{gs} = -0.1$ V. In both cases $V_{ds} = 3.0$ V.
Table 5.3.1: The linear element values used in the improved model for the Avantek AT-8250 GaAs MESFET.

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{g1}$</td>
<td>0.1386 pF</td>
</tr>
<tr>
<td>$L_g$</td>
<td>0.69414 nH</td>
</tr>
<tr>
<td>$C_{g2}$</td>
<td>0.30707 pF</td>
</tr>
<tr>
<td>$R_g$</td>
<td>2.9 Ω</td>
</tr>
<tr>
<td>$R_s$</td>
<td>2.4 Ω</td>
</tr>
<tr>
<td>$L_s$</td>
<td>0.00323 nH</td>
</tr>
<tr>
<td>$R_d$</td>
<td>5.3 Ω</td>
</tr>
<tr>
<td>$L_d$</td>
<td>0.41143 nH</td>
</tr>
<tr>
<td>$C_{d2}$</td>
<td>0.09012 pF</td>
</tr>
<tr>
<td>$C_d$</td>
<td>0.00341 pF</td>
</tr>
<tr>
<td>$R_i$</td>
<td>10 Ω</td>
</tr>
<tr>
<td>$\tau$</td>
<td>6.56 pS</td>
</tr>
</tbody>
</table>

equally. For the data used in the transistor optimizations, the magnitude of $S_{22}$ is generally smaller than $S_{11}$ and much smaller than $S_{21}$. The result is that $S_{21}$ and $S_{11}$ are accurately fit at the expense of $S_{22}$. This can be corrected by avoiding the model function and instead directly specifying goals for the optimization and weights for each S-parameter. This is the approach used in the following.

The new model was developed using the parasitic resistance values found previously along with the nonlinear expressions found for $C_{ds}$ and $C_{gd}$. The remaining linear circuit elements were determined by re-optimizing the model to match the S-parameters measured at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V. The resulting linear element values are summarized in Table 5.3.1.

The S-parameters calculated from the new model are compared to the measured
S-parameters (at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V) in Fig. 5.3.4 and Fig. 5.3.5. The values of $S_{11}$, $S_{21}$, and $S_{22}$ in particular are seen to be in good agreement throughout the 45 MHz to 9 GHz frequency range shown in the figures. As desired, the values of $S_{22}$ calculated from the new model fit the measured response more accurately than the results presented earlier, in Fig. 4.5.11. The remaining S-parameters are similar to those calculated from the model of Chapter 4.

With the linear element values fixed, the values of the nonlinear elements were found as before by re-optimizing the model of Fig. 4.2.2 to match the measured S-parameters at a number of different bias conditions. Power series were then fit to match the optimized values as a function of bias. The optimized values of transconductance ($G_m$) are shown in Fig. 5.3.6 and compared to the values calculated from the new power series expression for a constant drain source voltage of $V_{ds} = 3.0$ V. The new power series is found so that the values of transconductance decrease as $V_{gs}$ is increased towards 0.5 V as suggested by the dc characteristics shown in Fig. 4.4.1. (The S-parameters were not available at this bias.) This power series is converted into a power series for current by integrating with respect to gate-source voltage as before. The constant of integration is chosen so that the current through the transconductance element is zero at the operating point ($V_{gs} = -0.1$ V). This value is chosen for convenience, the drain-source current at the operating point will be determined by the power series for the current through the drain-source conductance element. A power series was fit to the op-
Figure 5.3.4: The values of $S_{11}$ and $S_{22}$ as calculated from the improved model for the Avantek AT-8250 compared to the measured values from 45 MHz to 9 GHz for $V_{gs} = -0.1$ V and $V_{ds} = 3.0$ V. The frequencies shown are in GHz, and the points are spaced every 0.18 GHz for $S_{11}$ and every 0.36 GHz for $S_{22}$. 
Figure 5.3.5: The values of $S_{12}$ and $S_{21}$ as calculated from the improved model for the Avantek AT-8250 compared to the measured values from 45 MHz to 9 GHz for $V_{gs} = -0.1$ V and $V_{ds} = 3.0$ V. The frequencies shown are in GHz and the points are spaced every 0.36 GHz. The outer radius is 0.15 for $S_{12}$ and 6.0 for $S_{21}$. 
Figure 5.3.6: The optimized values of transconductance (points) for the improved model of the Avantek AT-8250 compared to values calculated from the new power series description as a function of the intrinsic gate-source voltage.
Table 5.3.2: The power series coefficients used in the improved model for the Avantek AT-8250 GaAs MESFET.

<table>
<thead>
<tr>
<th>Order</th>
<th>( I_{gm} ) (A)</th>
<th>( C_{ds} ) (pF)</th>
<th>( I_{rd,s} ) (A)</th>
<th>( C_{gs} ) (pF)</th>
<th>( C_{gd} ) (pF)</th>
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<td>0</td>
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<td>0.286</td>
<td>0.0113</td>
<td>0.62039</td>
<td>0.34697</td>
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<tr>
<td>1</td>
<td>0.094677</td>
<td>-0.022345</td>
<td>0.037698</td>
<td>0.792475</td>
<td>-0.33135</td>
</tr>
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<td>0.050134</td>
<td>0.0043288</td>
<td>-0.035092</td>
<td>-0.02648</td>
<td>0.1576</td>
</tr>
<tr>
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<td>-0.044482</td>
<td>-0.0003038</td>
<td>0.01963</td>
<td>-0.22036</td>
<td>-0.032687</td>
</tr>
<tr>
<td>4</td>
<td>-0.03577</td>
<td>-0.58022 \times 10^{-2}</td>
<td></td>
<td>0.0024389</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0032192</td>
<td>0.85425 \times 10^{-3}</td>
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</tr>
<tr>
<td>6</td>
<td>0.0032238</td>
<td>-0.493237 \times 10^{-4}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Optimized values of \( C_{gs} \) as a function of gate-source voltage for a constant \( V_{ds} = 3.0 \) V. The results are shown in Fig. 5.3.7, where the optimized values are compared to values calculated using the new power series.

The power series found previously for the drain-source conductance is made to fit the optimized values of \( G_{ds} \) for the improved model by modifying the constant term. The values of \( G_{ds} \) calculated from the resulting power series are shown in Fig. 5.3.8 and compared to the optimized values for the improved model. This power series was converted into a power series for current by integrating with respect to drain-source voltage. The constant of integration was chosen so that the current through \( G_{ds} \) is equal to the measured dc value of \( I_{ds} \) at the operating point. The power series coefficients used in the improved model are summarized in Table 5.3.2.
Figure 5.3.7: The optimized values of gate-source capacitance (points) for the improved model of the Avantek AT-8250 compared to values calculated from the new power series description as a function of the intrinsic gate-source voltage.
5.3.5 Results for a Single-Tone Input

The improved model was used to simulate the common source amplifier circuit discussed above with a single-tone input at 3 GHz and with $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V. The results of this simulation are compared with the measured performance in Fig. 5.3.9. The predicted harmonic power levels are closer to the measured values than those predicted using the previous model, shown in Fig. 5.3.2. The predictions at large input powers are also improved. In particular the new simulations show the dip in the second harmonic power and accurately predict the point at which the second harmonic power is equal to the third harmonic power.

This circuit was also simulated with a single tone input at 2.35 GHz and the results compared to measured performance in Fig. 5.3.10. Again, the simulation is seen to accurately predict the circuit performance.

5.3.6 Results for a Two-Tone Input

Figure 5.3.11 shows the results of an intermodulation test for the common source amplifier circuit using the Avantek transistor. The simulated results are shown along with the measured response for two input signals of equal amplitude at 2.35 GHz and 2.4 GHz with $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V. The power output at 2.35 GHz is shown along with the power output in the third order intermodulation product at 2.3 GHz as a function of input power. Good agreement is seen. The power output at the second harmonic frequency of 4.7 GHz is shown in Fig. 5.3.12
Figure 5.3.9: The results of the single tone test for the Avantek AT-8250 using the improved model and a 3 GHz fundamental. Shown are the simulated (curves) and measured values (points) for power output at the fundamental (a), the second harmonic (b), and the third harmonic (c) as a function of input power.
Figure 5.3.10: The results of the single tone test for the Avantek AT-8250 using the improved model and a 2.35 GHz fundamental. Shown are the simulated (curves) and measured values (points) for power output at the fundamental (a), the second harmonic (b), and the third harmonic (c) as a function of input power.
along with the power output in the 2.35 GHz fundamental. Again the simulated results are in agreement with the measured response. In these simulations, five frequencies were considered: dc, 2.35 GHz, 2.4 GHz, 4.7 GHz, and 2.3 GHz.

5.3.7 Discussion

The excellent agreement between the simulated results and the measured circuit performance verifies the use of generalized power series analysis and the associated MESFET model. The small differences between the simulated and measured values are attributable to a combination of the following: measurement inaccuracies, deviation of the measurement system from 50 $\Omega$, and the use of descriptions for the nonlinear elements that depend only on a single variable. (The limitations resulting from the consideration of nonlinear functions of a single variable is addressed in the following chapter.)

5.4 Effective Impedance of the Nonlinear Elements

In addition to calculating output powers at all the frequencies considered, FREDA provides information about the behavior of the nonlinear elements. This information can be used to calculate the effective value of the elements at the fundamental frequency as a function of input power. When the steady state solution has been found, FREDA provides the currents through the nonlinear elements and the voltages across the elements at all frequencies used in the simulation. Dividing
Figure 5.3.11: The results of the two tone test for the Avantek AT-8250 using the improved model and two equal amplitude signals input at 2.35 GHz and 2.4 GHz. Shown are the simulated (curves) and measured values (points) for the power output at 2.35 GHz (a) and at the third order intermodulation product at 2.3 GHz (b).
Figure 5.3.12: The results of the two tone test for the Avantek AT-8250 using the improved model and two equal amplitude signals input at 2.35 GHz and 2.4 GHz. Shown are the simulated (curves) and measured values (points) for the power output at 2.35 GHz (a) and at the second harmonic frequency of 4.7 GHz (b).
the fundamental component of the voltage across an element by the fundamental component of the current through the element gives a quantity which can be regarded as the effective impedance seen at the fundamental frequency. The calculated impedances can then be interpreted as resistors, capacitors, or inductors as required.

Effective element values, as seen at the fundamental, were calculated for each nonlinear component of the Avantek AT-8250 model with a single tone input at 3 GHz using the calculation procedure described above. In Fig. 5.4.1 the effective value of the transconductance \((G_m)\) as a function of input power is plotted and is seen to decrease by 26 percent as the input power increases from -15 dBm to 10 dBm. The effective value of the time delay \((\tau)\) associated with the transconductance is shown in Fig. 5.4.2 as a function of input power. The delay increases from a small signal value of 6.56 pS to a peak of 9.97 pS (corresponding to 7.5 dBm input power) and then decreases again as the input power is increased. The delay is not considered to be a nonlinear element in the MESFET model used in FREDA, however the phase of the current through the nonlinear transconductance changes as the input power is increased (due to the nonlinearity of \(G_m\)) and can be interpreted as a nonlinear delay.

Similar effects are noticed in the large signal behavior of the gate-source capacitance \((C_{gs})\). As the power input to the transistor is increased, the current through the nonlinear capacitor begins to have a component that is in phase with
Figure 5.4.1: The effective value of the transconductance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at $V_{ds} = 3.0 \text{ V}$ and $V_{gs} = -0.1 \text{ V}$. 
Figure 5.4.2: The effective value of the time delay associated with the transconductance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V.
the voltage across the capacitor. This is a parametric effect and can be modeled at the fundamental as a resistance in parallel with a capacitance. This resistance represents the power converted from the fundamental to another frequency. Figure 5.4.3 shows the values of capacitance while Fig. 5.4.4 shows the values of the parallel resistance needed to predict the behavior of $C_g^*$ as a function of input power. The effective capacitance has a dip of 13 percent (corresponding to the peak in the delay) while the effective resistance decreases by a factor of 140.

The effective value of gate-drain capacitance is shown in Fig. 5.4.5. It is seen to increase by 52 percent as the power level is increased. There is an effective resistance in parallel with this capacitance also, and is seen to decrease by a factor of 300. The value of this resistance, however, is large enough in comparison to the other elements to be insignificant and so is not shown here.

The effective value of drain-source conductance is shown in Fig. 5.4.6. The conductance increases by 17 percent over the range of input power shown. The effective values of drain-source capacitance remain essentially constant over the range of input power considered and are not shown.

This information may be used to design matching networks for large-signal amplifiers or mixers or to design oscillator circuits. The effective impedance presented by the input and by the output of the transistor can be found from the effective impedance values of the individual elements. This information may also be used to calculate the ‘large-signal S-parameters’ of the transistor as a function
Figure 5.4.3: The effective value of the gate-source capacitance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V.
Figure 5.4.4: The effective value of the gate-source resistance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at \( V_{ds} = 3.0 \) V and \( V_{gs} = -0.1 \) V.
Figure 5.4.5: The effective value of the gate-drain capacitance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V.
Figure 5.4.6: The effective value of the drain-source conductance at the fundamental frequency for the Avantek AT-8250 in a common source amplifier circuit. A single tone was input at 3 GHz and the transistor biased at $V_{ds} = 3.0$ V and $V_{gs} = -0.1$ V.
of input power. Large-signal S-parameters are an attempt to extend the linear circuit concept of S-parameters to the large-signal nonlinear regime. They have been used to design amplifiers [89] and oscillators [90,91] and are typically found through a rather complicated measurement procedure [92]. Alternatively, a technique has been recently proposed to calculate large-signal S-parameters using the harmonic balance method [93], but is more involved than the method proposed here.

5.5 Comments on Simulator Performance

As explained in Chapter 3, FREDA uses a Newton's method iteration scheme globalized with a continuation method to find the steady-state solution for a nonlinear circuit. This technique converges rapidly, typically in less than 10 iterations and frequently less than 5 iterations are required. The iterations continue until the error function is reduced to an acceptable level. The error function is based on the deviation from Kirchoff's current law and thus the amount of error that can be tolerated depends on the magnitude of typical currents in the circuit.

Here we consider the Euclidean norm of the vector error function, defined in (3.5.5), divided by the dc drain-source current to be the normalized error. Good results are obtained when this normalized error is reduced below $10^{-4}$, corresponding to a 0.01 percent fractional accuracy. Figure 5.5.1 shows the normalized error as a function of the iteration number for several points in the intermodulation
simulation. Curve (a) shows the error for the simulation with -10 dBm input power. The initial estimate of the node voltage phasors is arbitrary for the ac voltages while the dc voltages are set to the dc bias point. The solution from this simulation is then scaled and used as the initial estimate for the -5 dBm input power case. The error for this simulation is shown as curve (b). This procedure is repeated for 0 dBm and for 5 dBm input power with the results shown in curves (c) and (d) respectively.

It can be seen from Fig. 5.5.1 that convergence is rapid for low values of input power and where the initial estimate of node voltages is close to the solution. Indeed if the nonlinear elements are actually linear, convergence is obtained after one or two iterations. The method is very efficient for weakly nonlinear circuits or when the applied signals are small. Computational effort increases only when the nonlinearity is significant.

5.6 Conclusion

This chapter has presented the results of nonlinear circuit simulations using the generalized power series based program FREDA developed in Chapter 3 and the MESFET model developed in Chapter 5. The results were compared with measured circuit performance for single-tone inputs and for two-tone inputs with good agreement in both cases, thereby validating the analysis procedure. In addition, a method was presented for calculating the effective impedance of the nonlinear
Figure 5.5.1: The normalized error as a function of iteration number for several points in the intermodulation simulation. Curve (a) is for -10 dBm input power, (b) corresponds to -5 dBm, (c) to 0 dBm, and (d) to 5 dBm of input power.
elements as a function of input power and results presented showing the behavior of the nonlinear elements in the MESFET model. The convergence properties of FREDA were also examined.
Chapter 6

Development of A Global MESFET Model

6.1 Introduction

In the previous chapter, the generalized power series based MESFET model is used to simulate an amplifier circuit and the results compared to measured circuit performance. The simulations are in agreement with the measurements, justifying the use of the model developed in Chapter 4. This model, however, is only designed to simulate the behavior of a transistor around a specific operating point (for the Avantek device $V_{gs} = -0.1 \text{ V}$ and $V_{ds} = 3.0 \text{ V}$), and thus may be described as a local model. While this approach was shown in Chapter 5 to accurately predict the circuit performance at the chosen operating point, it is inconvenient to develop such a model for every possible bias point. In this chapter we develop a new MESFET model that is valid over the range of useful bias points, and thus is termed a global model. This approach extends the techniques developed previously through the use of power series that are functions of two variables.

This chapter consists of three parts. In the first part, a new model is presented for the dc characteristics of a MESFET. These ideas are extended in the second part where a new model for the high-frequency behavior of a MESFET is
developed. Finally, we discuss a technique for incorporating the new model into FREDA.

6.2 A New dc MESFET Model

6.2.1 Background

In the following sections we present a new model for the dc characteristics of a MESFET. This model uses a power series that is a function of gate-source and drain-source voltage to approximate the drain-source current and may be used with any computer-aided analysis technique that can accommodate functions of two variables. Here we present a brief review of available dc models, followed by a description of our new model, and a discussion of it's relative merit.

A major component of any empirical MESFET model is an analytical expression for the drain-source current as a function of the terminal voltages. There are a number of different expressions currently in use, although many of these use the hyperbolic tangent function (proposed by Van Tuyl and Liechti [94]) to approximate the shape of the characteristics as a function of drain-source voltage. An example of this approach is the model presented in [95] which uses the expression

$$I_{ds} = I_{dss} \left(1 - \frac{V_{gs}}{V_{p0} + \gamma V_{ds}}\right)^2 \tanh \left(\frac{\alpha V_{ds}}{V_{gs} - V_{p0} - \gamma V_{ds}}\right). \quad (6.2.1)$$

This expression is a function of gate-source and drain-source voltage and contains four parameters that can be varied depending on the particular transistor being
simulated. Another expression that uses this basic form is the one developed by Curtice [69]. This model has become popular in the microwave community and uses the expression

$$I_{ds} = (A_0 + A_1 V_1 + A_2 V_1^2 + A_3 V_1^3) \tanh(\gamma V_{ds})$$  \hspace{1cm} (6.2.2)

where

$$V_1 = V_{gs}[1 + \beta(V_{ds0} - V_{ds})]. \hspace{1cm} (6.2.3)$$

This is a more complex expression than (6.2.1) and contains seven parameters. Recently, a new model has been proposed [96] which does not use the hyperbolic tangent. Instead, the expression

$$I_{ds} = \frac{\beta(V_{gs} - V_T)^2}{1 + b(V_{gs} - V_T)} \left\{ 1 - \left( 1 - \frac{\alpha V_{ds}}{3} \right)^3 \right\} (1 + \lambda V_{ds})$$  \hspace{1cm} (6.2.4)

for $0 < V_{ds} < 3/\alpha$, and

$$I_{ds} = \frac{\beta(V_{gs} - V_T)^2}{1 + b(V_{gs} - V_T)} (1 + \lambda V_{ds})$$  \hspace{1cm} (6.2.5)

for $V_{ds} \geq 3/\alpha$ is used. In this model, the hyperbolic tangent is replaced by a third order polynomial in $V_{ds}$ for small values of drain-source voltage and by unity in the saturation region.

While each of these models provides a useful approximation for many devices, the variety of behavior that can be modeled is limited. The behavior as a function of drain-source voltage, for example, is dominated by the hyperbolic tangent function for the expression of (6.2.1) and (6.2.2) while (6.2.5) has strictly a linear
dependence on $V_{ds}$ in the saturation region. The Avantek AT-8250 GaAs MESFET characterized in Chapter 4 exhibits a more complicated behavior (see Fig. 4.3.1). In particular, the current does not have a simple linear shape in the saturation region. In the following section a new model is presented that is capable of accurately modeling this type of behavior. Before presenting the new model, the modeling techniques used in previous chapters are briefly reviewed.

6.2.2 A Model Based on Bivariate Power Series

In the model presented in Chapter 4 all of the nonlinear elements are described using power series that are functions of a single variable. This requires that the drain-source current be represented as the sum of two nonlinear terms: one to describe the behavior as a function of the gate-source voltage (the transconductance), and one to describe the behavior as a function of the drain-source voltage (the drain-source conductance). The result is an expression of the form

$$I_{ds} = I_{gm}(V_{gs}) + I_{gds}(V_{ds})$$

which we have been modeling as

$$I_{ds} = \sum_{n=0}^{P} \left( a_n V_{gs}^n + b_n V_{ds}^n \right).$$

To see the limitations of this approach, a model was developed to fit the measured dc characteristics of the Avantek AT-8250 GaAs MESFET. Using the data for $I_{ds}$ with $V_{gs} = -0.1 \, \text{V}$, a single variable power series was found to describe the
Table 6.2.1: The power series coefficients used to model the dc behavior of the Avantek AT-8250 using a sum of two single variable power series of the form (6.2.7).

<table>
<thead>
<tr>
<th>Order</th>
<th>Element</th>
<th>$I_{gm}$ (A)</th>
<th>$I_{rd}$ (A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0.0052755</td>
<td>0.0006486</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.0553126</td>
<td>0.0304521</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.0235932</td>
<td>-0.026741</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>-0.019615</td>
<td>0.0110322</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.0038974</td>
<td>-0.002057</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.0140191</td>
<td>0.0001420</td>
</tr>
</tbody>
</table>

drain-source current as a function of drain-source voltage. Similarly, the data for $I_{ds}$ with $V_{ds} = 3.0$ V was used to fit a power series as a function of gate-source voltage. The coefficients found are shown in Table 6.2.1 and include terms of up to fifth order. The complete model for $I_{ds}$ is the sum of these two series as in (6.2.7). This model was used to calculate the current for a number of combinations of $V_{ds}$ and $V_{gs}$. The results are compared to the measured data in Fig. 6.2.1. While this model accurately describes the behavior in a region around the point at which the description was developed, the fit becomes inaccurate for low drain-source voltages and for gate-source voltages approaching pinch-off.

A more general representation for the drain-source current is a function of two variables, e.g.

$$I_{ds} = I(V_{gs}, V_{ds})$$

(6.2.8)
as described in the previous section. The natural extension of the power series
Figure 6.2.1: The dc current calculated from the sum of the two single variable power series (shown as the solid curves) compared to the measured values (shown as dashed curves) for the Avantek AT-8250 GaAs MESFET. The gate-source voltage is varied from -0.4 V to 0.5 V in 0.1 V increments.
model to this case is a power series in two variables such as

$$I_{ds} = \sum_{n=0}^{P} \sum_{m=0}^{Q} d_{nm} V_{gs}^n V_{ds}^m$$  \hspace{1cm} (6.2.9)

which may be described as a bivariate power series and is the approach used in this chapter. This is a more general expression than the single variable power series and can be reduced to (6.2.7) if the only nonzero values of $d_{nm}$ are:

$$d_{00} = a_0 + b_0$$

$$d_{n0} = a_n$$

$$d_{0n} = b_n.$$

The coefficients used in the bivariate power series are found in the same manner as discussed previously for the single variable power series. Despite the large number to be found, the problem of determining the coefficients is a linear one and they can be found using a noniterative technique (described in Appendix E). Finding the coefficients with single variable power series involves fitting a curve to data, while with bivariate power series the process involves fitting a surface to data.

A bivariate power series was fit to the measured dc data for the Avantek AT-8250. This series was found by considering 338 data points: all combinations of $V_{gs}$ ranging from -0.7 to 0.5 V in 0.1 V increments, and $V_{ds}$ ranging from 0.0 to 5.0 V in 0.2 V increments. The resulting series contains terms of up to third order in $V_{gs}$ and up to sixth order in $V_{ds}$. Table 6.2.2 summarizes the coefficients and is
Table 6.2.2: The bivariate power series coefficients for the model of the dc drain-source current for the Avantek AT-8250 GaAs MESFET. The resulting series has units of Amperes.

<table>
<thead>
<tr>
<th>Order of $V_{ds}$ Term</th>
<th>Order of $V_{gs}$ Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$0.2276 \times 10^{-3}$</td>
</tr>
<tr>
<td>1</td>
<td>0.04752</td>
</tr>
<tr>
<td>2</td>
<td>-0.05000</td>
</tr>
<tr>
<td>3</td>
<td>0.026661</td>
</tr>
<tr>
<td>4</td>
<td>$-0.7393 \times 10^{-2}$</td>
</tr>
<tr>
<td>5</td>
<td>$0.1029 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$-0.5690 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

explained in the following.

The bivariate power series described in Table 6.2.2 is a sum of terms that correspond to all combinations of $V_{gs}^n$ and $V_{ds}^m$ multiplied together where, in this example, $0 \leq n \leq 3$ and $0 \leq m \leq 6$. Thus, there is a term of the form $A V_{gs}^2 V_{ds}^4$ whose coefficient $A$ is listed in Table 6.2.2 as 0.02046. In this example, there are 28 coefficients.

The current calculated from the bivariate power series expression is compared to the measured dc data in Fig. 6.2.2 and Fig. 6.2.3. In Fig. 6.2.2 the data is shown as a function of drain-source voltage for several different gate-source voltages, while Fig. 6.2.3 shows the data as a function of gate-source voltage for several different drain-source voltages. In both cases, excellent agreement is seen over the entire range of voltages considered.
Figure 6.2.2: The dc current calculated from the bivariate power series (shown as the curves) compared to the measured values (shown as crosses) for the Avantek AT-8250 GaAs MESFET. The gate-source voltage is varied from -0.7 V to 0.5 V in 0.1 V increments.
Figure 6.2.3: The dc current calculated from the bivariate power series (shown as the curves) compared to the measured values (shown as crosses) for the Avantek AT-8250 GaAs MESFET. The drain-source voltage is varied from 0.2 V to 0.8 V in 0.2 V increments, and from 1.0 V to 5.0 V in 1.0 V increments.
6.2.3 Discussion

The model just presented for the dc characteristics of a MESFET has several advantages over the methods reviewed here. In particular, the power series model is more general than the others. The other models use a fixed expression having only a few parameters that can be altered. In contrast, the power series model uses a relatively large number of coefficients and can therefore be adapted to a wider range of characteristics. Complex behavior as a function of drain-source voltage, for example, is easily modeled. Also, the power series model can be extended to cover negative drain-source voltages. Although many coefficients are needed, they are easily determined using a noniterative procedure, in contrast to the other models which must be found using nonlinear optimization. The disadvantage of this approach is that a significant amount of dc characterization information is required to find a good power series fit. This is not a problem, however, as dc measurements are relatively simple to obtain. In summary, the bivariate power series model presented here should be of use with any nonlinear circuit analysis technique that requires a dc MESFET model.
6.3 A New High-Frequency MESFET Model

6.3.1 Background

In the previous sections we saw that the use of single variable power series was limited to describing the dc characteristics of a MESFET in a localized region around the operating point. In order to accurately describe the behavior over a large range of gate-source and drain-source voltage, a bivariate power series model was developed. The following sections show that this is also the case for the model of the high-frequency behavior of a MESFET. In particular, the circuit elements used to model the device are not simple functions of a single variable, but are functions of both $V_{ds}$ and $V_{gs}$. The sections that follow present a new high-frequency MESFET model using the concepts developed for the dc model just presented.

Just as analytical expressions for the dc current characteristics are needed, empirical MESFET models need analytical expressions for all of the nonlinear circuit elements. As an example, an expression developed for the gate-source capacitance used in conjunction with the program SPICE is [5]

$$C_{gs} = C_{gs0} + \lambda_{gs1}(\epsilon WL/t_c) \left( \frac{V_{gs} - V_{TO}}{\phi_G - V_{TO}} \right)^{m_{gs}} f_c(V_{ds}) \quad (6.3.1)$$

where

$$f_c = 1 + \lambda_{gs2} \left[ \frac{\phi_G - V_{TO}}{V_{gs} - V_{TO}} \right]^{1/m_{gs}} (V_{ds} - V_{dsat}) \quad (6.3.2)$$
for \( V_{ds} \geq V_{d_{\text{sat}}} \) and

\[
f_c = (\lambda_{gs2} V_{d_{\text{sat}}} + \beta_{gs} - 1) \left[ \frac{V_{ds}}{V_{d_{\text{sat}}}} \right]^2 + (2 - \lambda_{gs2} V_{d_{\text{sat}}} - 2\beta_{gs}) \left[ \frac{V_{ds}}{V_{d_{\text{sat}}}} \right] + \beta_{gs} \quad (6.3.3)
\]

for \( V_{ds} < V_{d_{\text{sat}}} \). This is a complex expression, dependent on \( V_{gs} \) and \( V_{ds} \) and containing five empirical parameters. Similarly, expressions have been developed for the gate-drain capacitance, such as [5]

\[
C_{gd} = \frac{C_{gdi}}{(1 - (\lambda_{gdi} V_{gs} - V_{ds})/(\lambda_{gdi} \phi_G))^{m_{gd}}(1 - \lambda_{gdi2} V_{gs}) + C_{gdo}} \quad (6.3.4)
\]

which is also a function of both \( V_{ds} \) and \( V_{gs} \) and contains five parameters.

Accurate modeling over a wide range of bias voltage requires that, as in these expressions, the nonlinear elements be functions of both gate-source and drain-source voltage. In the following section we develop such expressions for the nonlinear elements in the MESFET model using bivariate power series.

### 6.3.2 A Model Based on Bivariate Power Series

The goal of this section is to develop a MESFET model that is valid over a large range of gate-source and drain-source voltage. The idea is to retain the basic topology of the model developed in Chapter 4 (shown in Fig. 4.2.1) while allowing the nonlinear elements to be functions of both \( V_{ds} \) and \( V_{gs} \). This is in contrast to the approach used in Chapter 4 where each nonlinear element was assumed to be a function of a single voltage variable. Here the elements are described using bivariate power series which are found to fit the optimized element values found
in Chapter 4 to match the measured S-parameters. In each case approximately 100 optimized values are considered, corresponding to gate-source voltages ranging from -1.0 to 0.1 V and to drain-source voltages ranging from 0.3 to 5.0 V. These terminal voltages were corrected to account for the voltage dropped across the parasitic resistances so that the bivariate power series expressions were found as functions of the intrinsic voltage variables. The linear circuit elements in the MESFET model are the same ones used in Chapter 4, and the complete model is the one shown in Fig. 4.2.2.

Based on the optimized element values, four circuit elements need to be described as functions of both \( V_{ds} \) and \( V_{gs} \): the gate-source capacitance, the gate-drain capacitance, the transconductance, and the drain-source conductance. The remaining elements are accurately described with functions of a single variable, and new expressions for these elements will not be developed here.

First, a bivariate power series expression was developed for the gate-source capacitance using the technique described in Appendix E. The coefficients are summarized in Table 6.3.1. The values calculated from this expression are compared to the optimized values in Fig. 6.3.1 as a function of drain-source voltage and in Fig. 6.3.2 as a function of gate-source voltage. In both cases the bivariate power series accurately fit the optimized values. The improvement of this model over the single variable power series model can be seen in Fig. 6.3.2. In the model of Chapter 4, the gate-source capacitance is not a function of the drain-source
Table 6.3.1: The bivariate power series coefficients used to describe the gate-source capacitance for the model of the Avantek AT-8250. The values calculated from this series are in pF.

<table>
<thead>
<tr>
<th>Order of $V_{gs}$ Term</th>
<th>Order of $V_{ds}$ Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.4879</td>
</tr>
<tr>
<td>1</td>
<td>0.3121</td>
</tr>
<tr>
<td>2</td>
<td>4.2488</td>
</tr>
<tr>
<td>3</td>
<td>17.2211</td>
</tr>
<tr>
<td>4</td>
<td>22.1835</td>
</tr>
<tr>
<td>5</td>
<td>9.2139</td>
</tr>
</tbody>
</table>

Voltage and the curves shown in Fig. 6.3.2 would all have a zero slope. As can be seen in the figure, the optimized values do not lie on horizontal lines, indicating the improvement obtained with the bivariate power series.

Similarly, a bivariate power series expression was developed for the gate-drain capacitance and the coefficients summarized in Table 6.3.2. The values calculated from this expression are compared to the optimized values in Fig. 6.3.3 as a function of drain-source voltage and in Fig. 6.3.4 as a function of gate-source voltage. Again the bivariate power series accurately models the optimized values. The model of Chapter 4 was chosen to match the optimized values as a function of drain-gate voltage for $V_{gs} = -0.1$ V and cannot predict the behavior shown in Fig. 6.3.3 and Fig. 6.3.4.

As discussed in Chapter 4, the transconductance and drain-source conductance elements may be chosen based either on the dc or on the high-frequency behav-
Figure 6.3.1: The gate-source capacitance as a function of intrinsic drain-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for gate-source voltages of -1.0, -0.8, -0.6, -0.4, -0.2, -0.1, 0.0, and 0.1 V (from the lower curve to the upper curve respectively).
Figure 6.3.2: The gate-source capacitance as a function of intrinsic gate-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for drain-source voltages of 1.0 V to 5.0 V in 1.0 V steps (from the lower to the upper curve respectively).
Figure 6.3.3: The gate-drain capacitance as a function of intrinsic drain-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression. The solid curve is for $V_{gs} = -1.0 \text{ V}$, the smaller dashed curve is for $V_{gs} = -0.4 \text{ V}$, and the large dashed curve is for $V_{gs} = 0.1 \text{ V}$.
Figure 6.3.4: The gate-drain capacitance as a function of intrinsic gate-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for drain-source voltages of 0.3, 0.5, 0.7, 1.0, 2.0, 3.0, and 5.0 V, with the largest values corresponding to $V_{ds} = 0.3$ V.
Table 6.3.2: The bivariate power series coefficients used to model the gate-drain capacitance in the model of the Avantek AT-8250. The values calculated using this expression are in pF.

<table>
<thead>
<tr>
<th>Order of $V_{ds}$ Term</th>
<th>Order of $V_{gs}$ Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.6548</td>
</tr>
<tr>
<td>1</td>
<td>-1.5205</td>
</tr>
<tr>
<td>2</td>
<td>1.6737</td>
</tr>
<tr>
<td>3</td>
<td>-0.9333</td>
</tr>
<tr>
<td>4</td>
<td>0.2765</td>
</tr>
<tr>
<td>5</td>
<td>-0.04149</td>
</tr>
<tr>
<td>6</td>
<td>0.002476</td>
</tr>
</tbody>
</table>

ior. Here, as in Chapter 4, the model is developed based on the high-frequency behavior. Using the optimized values, a bivariate power series expression was developed for the transconductance and the coefficients summarized in Table 6.3.3. The values calculated from this expression are compared to the optimized values in Fig. 6.3.5 as a function of drain-source voltage and in Fig. 6.3.6 as a function of gate-source voltage, showing good agreement in both cases.

Finally, a bivariate power series expression was developed for the drain-source conductance. The coefficients are summarized in Table 6.3.4. The values calculated from this expression are compared to the optimized values in Fig. 6.3.7 as a function of drain-source voltage and in Fig. 6.3.8 as a function of gate-source voltage, showing good agreement.
Table 6.3.3: The bivariate power series coefficients used to model the transconductance of the Avantek AT-8250. The values calculated from this series are in millisiemens.

<table>
<thead>
<tr>
<th>Order of $V_{ds}$ Term</th>
<th>Order of $V_{gs}$ Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1.231</td>
</tr>
<tr>
<td>1</td>
<td>295.7</td>
</tr>
<tr>
<td>2</td>
<td>-390.0</td>
</tr>
<tr>
<td>3</td>
<td>246.8</td>
</tr>
<tr>
<td>4</td>
<td>-79.82</td>
</tr>
<tr>
<td>5</td>
<td>12.73</td>
</tr>
<tr>
<td>6</td>
<td>-0.794</td>
</tr>
</tbody>
</table>

Table 6.3.4: The bivariate power series coefficients used to model the drain-source conductance in the model of the Avantek AT-8250. The values calculated from this series are in millisiemens.

<table>
<thead>
<tr>
<th>Order of $V_{ds}$ Term</th>
<th>Order of $V_{gs}$ Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>78.42</td>
</tr>
<tr>
<td>1</td>
<td>-218.79</td>
</tr>
<tr>
<td>2</td>
<td>261.35</td>
</tr>
<tr>
<td>3</td>
<td>-154.45</td>
</tr>
<tr>
<td>4</td>
<td>47.57</td>
</tr>
<tr>
<td>5</td>
<td>-7.31</td>
</tr>
<tr>
<td>6</td>
<td>0.442</td>
</tr>
</tbody>
</table>
Figure 6.3.5: The transconductance as a function of intrinsic drain-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for extrinsic gate-source voltages of -1.0, -0.8, -0.6, -0.4, -0.2, -0.1, 0.0, and 0.1 V.
Figure 6.3.6: The transconductance as a function of intrinsic gate-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for extrinsic drain-source voltages of 1.0, 2.0, 3.0, 4.0, and 5.0 V.
Figure 6.3.7: The drain-source conductance as a function of intrinsic drain-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for extrinsic gate-source voltages of -0.6,-0.4,-0.2,-0.1, and 0.0 V.
Figure 6.3.8: The drain-source conductance as a function of intrinsic gate-source voltage. Optimized values (points) are compared to the values calculated from the bivariate power series expression for extrinsic drain-source voltages of 0.3, 0.5, 0.7, 1.5, and 2.5 V.
6.3.3 Discussion

We have presented a new MESFET model that accurately predicts the behavior of the device over a wide range of gate-source and drain-source voltage. This model uses the circuit topology developed in Chapter 4 and allows the nonlinear elements to be functions of two variables. The parasitic element values developed earlier are retained along with the nonlinear expression for \( C_{ds} \). Bivariate power series expressions were developed for \( C_{gs}, C_{gd}, G_m, \) and \( G_{ds} \). This model may be used with any circuit simulator that requires an empirical MESFET model and that can accommodate functions of two variables. In the next section, we present a technique for incorporating the new model into **FREDA**.

6.4 Incorporating the New MESFET Model into **FREDA**

The nonlinear circuit simulator **FREDA**, developed in Chapter 3, is based on the use of generalized power series functions of a single variable to describe the nonlinear elements. The MESFET model just developed can be incorporated in **FREDA**, however, by reducing the bivariate power series expressions into single variable power series. This is done by simply substituting into the expression the value of one variable at the operating point. All terms of similar power in the remaining variable are then summed, resulting in a power series function of a single variable. As examples, consider the transconductance and the gate-source capacitance. Collapsing these series about \( V_{ds} = 3.0 \text{ V} \) (e.g. substituting \( V_{ds} = 3.0 \)
Table 6.4.1: The single variable power series coefficients obtained by substituting $V_{ds} = 3.0$ V into the bivariate power series expressions for $G_m$ and $C_{gs}$.

<table>
<thead>
<tr>
<th>Order</th>
<th>$G_m$ (siemens)</th>
<th>$C_{gs}$ (pF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.09127</td>
<td>0.79458</td>
</tr>
<tr>
<td>1</td>
<td>0.08590</td>
<td>0.86989</td>
</tr>
<tr>
<td>2</td>
<td>-0.19786</td>
<td>-0.74821</td>
</tr>
<tr>
<td>3</td>
<td>-0.27121</td>
<td>-3.04968</td>
</tr>
<tr>
<td>4</td>
<td>-0.07819</td>
<td>-3.35611</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>-1.13277</td>
</tr>
</tbody>
</table>

V into the bivariate power series expressions found in the previous section, we get power series that are functions of $V_{gs}$ only. The resulting coefficients are shown in Table 6.4.1. The values calculated using these expressions are compared to the optimized values (for $V_{ds} = 3.0$ V) and to the values calculated using the single variable power series developed in Chapter 4 in Fig. 6.4.1 and Fig. 6.4.2. Fig. 6.4.1 compares the results for the gate-source capacitance while Fig. 6.4.2 compares the results for the transconductance. In both cases, the collapsed bivariate power series accurately model the optimized values and are similar to the expressions developed in Chapter 4. The drain-source conductance and the gate-drain conductance may be similarly collapsed into single variable series.

As explained in Chapter 4, FREDA requires that the resistive elements be described using power series for the current through the element as a function of voltage. The collapsed bivariate power series for the transconductance is integrated as explained in Chapter 4 to obtain a single variable power series for current.
Figure 6.4.1: Comparison of the expressions for the gate-source capacitance used in the model of the Avantek AT-8250. The points are optimized values, the dashed curve is the single variable power series expression, and the solid curve is the collapsed bivariate power series expression corresponding to $V_{ds} = 3.0$ V.
Figure 6.4.2: Comparison of the expressions for the transconductance used in the model of the Avantek AT-8250. The points are optimized values, the dashed curve is the single variable power series expression, and the solid curve is the collapsed bivariate power series expression corresponding to $V_{ds} = 3.0$ V.
This process is easily automated and incorporated into FREDA. The result is a MESFET model using single variable power series descriptions whose coefficients depend on the operating point: a global model.

6.5 Conclusion

In this chapter we have presented a technique for extending the modeling capabilities of generalized power series analysis. This approach extends the previously developed ideas through the use of power series functions of two variables. This was used to develop new models for the dc and high-frequency behavior of a MESFET. The new models are valid over the range of useful operating points and may be incorporated into any nonlinear circuit simulator requiring an empirical MESFET model. This is effectively a table based approach using a single interpolating function (the bivariate power series).

It was also shown that the new model may be used with FREDA to provide a single MESFET model that is valid at all operating points. This was accomplished by reducing the bivariate power series to single variable power series at the operating point. The following chapter continues this extension of generalized power series analysis through the development of formulas for the output of bivariate power series with multifrequency excitation.
Chapter 7

Extending Generalized Power Series Analysis to Functions of Two Variables

7.1 Introduction

In the previous chapter the use of bivariate power series was introduced and a method developed for incorporating these expressions into generalized power series analysis. This method, however, requires that the bivariate power series be collapsed into a single variable power series at the operating point. The purpose of this chapter is to provide the theoretical background necessary to completely include bivariate power series as functions of two variables in generalized power series analysis.

For bivariate power series to be completely included in an analysis procedure similar to the one developed in Chapter 3, formulas must be available to calculate the output of a bivariate power series given the two multifrequency inputs and the power series coefficients. Also, formulas for the derivatives of the output with respect to the inputs are required. In the following sections, these formulas are derived. We also investigate the relationship of the new formulas to the ones presented in Chapter 3 for single variable power series, and discuss the results.
7.2 Development of Algebraic Formulas

The following sections present the derivation of the algebraic formulas for the output of a bivariate power series having two multifrequency inputs, and for the derivatives of the output with respect to the inputs.

7.2.1 Formulas for the Output

In this section we derive an algebraic formula for the output components of a nonlinearity which can be described by a power series in two variables having complex coefficients and frequency-dependent time delays, when the inputs are sums of sinusoids. This represents a generalization of the formulas presented in [56] for the case of a power series in one variable. The derivations are similar and important differences are noted.

Here we describe the output $y(t)$ of a nonlinear element having the two multifrequency inputs (each having $N$ components)

\[
x(t) = \sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} |X_k| \cos(\omega_k t + \phi_k)
\]

(7.2.1)

and

\[
z(t) = \sum_{k=1}^{N} z_k(t) = \sum_{k=1}^{N} |Z_k| \cos(\omega_k t + \theta_k)
\]

(7.2.2)

by the bivariate generalized power series

\[
y(t) = \sum_{\sigma=0}^{\infty} \sum_{\rho=0}^{\infty} a_{\sigma,\rho} f(\sigma, x) g(\rho, z)
\]

(7.2.3)
with

\[
f(\sigma, x) = \left( \sum_{k=1}^{N} b_k x_k(t - \tau_{k,\sigma}) \right)^\sigma
\]

(7.2.4)

and

\[
g(\rho, z) = \left( \sum_{k=1}^{N} d_k z_k(t - \lambda_{k,\rho}) \right)^\rho.
\]

(7.2.5)

In these expressions, \( a_{\sigma,\rho} \) is a complex coefficient, \( b_k \) and \( d_k \) are real, and \( \tau_{k,\sigma} \) and \( \lambda_{k,\rho} \) are time delays that depend on both power series order and the index of the input frequency components. As in Chapter 3, we rewrite the input components using phasor notation. For the \( x \) input,

\[
x_k(t - \tau_{k,\sigma}) = |X_k| \cos(\omega_k t + \phi_k - \omega_k \tau_{k,\sigma})
\]

\[
= \frac{1}{2} X_k \Gamma_{k,\sigma} e^{j\omega_k t} + \frac{1}{2} X_k^* \Gamma_{k,\sigma}^* e^{-j\omega_k t}
\]

(7.2.6)

where \( X_k \) is the phasor of \( x_k \) and

\[
\Gamma_{k,\sigma} = e^{-j\omega_k \tau_{k,\sigma}}.
\]

For the other set of inputs, we write

\[
z_k(t - \lambda_{k,\rho}) = |Z_k| \cos(\omega_k t + \theta_k - \omega_k \lambda_{k,\rho})
\]

\[
= \frac{1}{2} Z_k \Upsilon_{k,\rho} e^{j\omega_k t} + \frac{1}{2} Z_k^* \Upsilon_{k,\rho}^* e^{-j\omega_k t}
\]

(7.2.7)

where \( Z_k \) is the phasor of \( z_k \) and

\[
\Upsilon_{k,\rho} = e^{-j\omega_k \lambda_{k,\rho}}.
\]
Using the multinomial expansion theorem [97], we expand (7.2.4) into a sum of products expression:

$$f(\sigma, x) = \sum_{l_1, \ldots, l_N, m_1, \ldots, m_N} \left\{ \left[ \exp \left( j \sum_{k=1}^{N} (l_k - m_k) \omega_k t \right) \right] \sigma! \right\}$$

$$\times \prod_{k=1}^{N} \left( \frac{\left( \frac{1}{2} b_k \right)^{l_k} + m_k (X_k)^{l_k} (X_k^*)^{m_k} (\Gamma_{k,\sigma})^{l_k} (\Gamma_{k,\sigma}^*)^{m_k}}{l_k! \ m_k!} \right) \right\}$$

(7.2.8)

where the summation is over all combinations of the integers $l_1, \ldots, l_N, m_1, \ldots, m_n$ such that $\sum_{k=1}^{N} l_k + m_k = \sigma$. Expanding (7.2.5), we get the similar expression

$$g(\rho, z) = \sum_{i_1, \ldots, i_N, j_1, \ldots, j_N} \left\{ \left[ \exp \left( j \sum_{k=1}^{N} (i_k - j_k) \omega_k t \right) \right] \rho! \right\}$$

$$\times \prod_{k=1}^{N} \left( \frac{\left( \frac{1}{2} d_k \right)^{i_k} + j_k (Z_k)^{i_k} (Z_k^*)^{j_k} (\Upsilon_{k,\rho})^{i_k} (\Upsilon_{k,\rho}^*)^{j_k}}{i_k! \ j_k!} \right) \right\}.$$ 

(7.2.9)

Thus the product of $f$ and $g$ becomes

$$f(\sigma, x) g(\rho, z) = \sum_{l_1, \ldots, l_N, m_1, \ldots, m_N} \left\{ \left[ \exp \left( j \sum_{k=1}^{N} (l_k + i_k - m_k - j_k) \omega_k t \right) \right] \sigma! \rho! \Psi \right\}$$

where

$$\Psi = \prod_{k=1}^{N} \left( \frac{\left( \frac{1}{2} b_k \right)^{l_k} + m_k (\frac{1}{2} d_k)^{i_k} + j_k (X_k)^{l_k} (X_k^*)^{m_k} (Z_k)^{i_k} (Z_k^*)^{j_k} (\Gamma_{k,\sigma})^{l_k} (\Gamma_{k,\sigma}^*)^{m_k} (\Upsilon_{k,\rho})^{i_k} (\Upsilon_{k,\rho}^*)^{j_k}}{l_k! \ m_k! \ i_k! \ j_k!} \right)$$
and the above summation is over all combinations of the nonnegative integers \( l, m, i, \) and \( j \) such that \( \sum_{k=1}^{N} l_k + m_k = \sigma \) and \( \sum_{k=1}^{N} i_k + j_k = \rho \). Just as for the single variable power series, the frequency of each component is given by

\[
\omega = \sum_{k=1}^{N} n_k \omega_k
\]

where \( n_k \) is a set of integers (IPD) that fixes the difference of \((l_k + i_k)\) and \((m_k + j_k)\) according to

\[
n_k = l_k + i_k - m_k - j_k
\]

and as before, the intermodulation order is given by \( n \), where

\[
n = \sum_{k=1}^{N} |n_k|.
\]

(It should be noted that the derivation presented here uses the same set of intermodulation product descriptions developed for the single variable power series in Chapter 3.)

Depending on the sign of \( n_k \), we can deduce the relative magnitude of \( l_k + i_k \) and \( m_k + j_k \), in particular

\[
(l_k + i_k) > (m_k + j_k) \quad n_k > 0
\]

\[
(l_k + i_k) < (m_k + j_k) \quad n_k < 0
\]

\[
(l_k + i_k) = (m_k + j_k) \quad n_k = 0
\]

so that by letting \((p_k + q_k)\) equal the larger of \((l_k + i_k)\) and \((m_k + j_k)\), and \((r_k + s_k)\) equal the smaller, we find

\[
p_k + q_k - r_k - s_k = |n_k|
\]
where \( p_k, q_k, r_k, s_k \geq 0 \).

For a given set of \( n_k \)'s corresponding to one intermodulation product description (IPD), the relevant components of \( f(\sigma, x)g(\rho, z) \) can be written as the sum of two terms (for \( n \neq 0 \))

\[
\left( \frac{1}{2} C \right) e^{j\omega t} + \left( \frac{1}{2} C \right)^{*} e^{-j\omega t} = \left( \frac{1}{2} U'_{Q} \right) e^{j\omega t} + \left( \frac{1}{2} U'_{Q} \right)^{*} e^{-j\omega t} \quad \omega_q \neq 0
\]

\[
= U'_{Q} \quad \omega_q = 0
\]  

(7.2.11)

where

\[
C = 2 \sum_{p_1, \ldots, p_N, r_1, \ldots, r_N} \sum_{q_1, \ldots, q_N, s_1, \ldots, s_N} \left\{ \left( \frac{1}{2} \right)^{\sigma + \rho} \frac{\sigma! \rho! \Phi}{p_k! r_k! q_k! s_k!} \right\}
\]  

(7.2.12)

\[
\Phi = \prod_{k=1}^{N} \frac{(b_k)^{p_k+r_k}(d_k)^{q_k+s_k}(X_k^{\dagger})^{p_k}(Z_k^{\dagger})^{r_k}(Z_k)^{s_k}(\Gamma_{k,\sigma})^{p_k}(\Gamma_{k,\rho})^{r_k}(\Upsilon_{k,\sigma})^{q_k}(\Upsilon_{k,\rho})^{s_k}}{p_k! r_k! q_k! s_k!}
\]  

(7.2.13)

Here we define

\[
X_k^{\dagger} = \begin{cases} 
X_k & n_k \geq 0 \\
X_k^* & n_k < 0
\end{cases}
\]

and

\[
X_k^\dagger = \begin{cases} 
X_k^* & n_k \geq 0 \\
X_k & n_k < 0
\end{cases}
\]
Note that $U'_q$ is the contribution to $f(\sigma, x) g(\rho, y)$ of one IPD. The two terms in (7.2.11) occur as for $n \neq 0$, $p_k$ and $q_k$ replace two sets of $i_k, j_k, l_k$, and $m_k$, one set corresponding to $(l_k + i_k) > (m_k + j_k)$ resulting in the $\text{e}^{j\omega t}$ term and a set corresponding to $(l_k + i_k) < (m_k + j_k)$ resulting in the $\text{e}^{-j\omega t}$ term. For $n = 0$ there is only one set corresponding to $(l_k + i_k) = (m_k + j_k)$. Thus, the $U'_q$ expression is one-half that in (7.2.14) for the case $n = 0$. For (7.2.14) to hold we make the restriction that no IPD be equal to the negative of another IPD. If $U_q$ is the component of $Y$ due to a single intermodulation product then

$$U_q = \sum_{\sigma=0}^{\infty} \sum_{\rho=0}^{\infty} a_{\sigma, \rho} U'_q.$$  
(7.2.15)

Using the Neumann factor, $\epsilon_n$ ($\epsilon_n = 1, n = 0; \epsilon_n = 2, n \neq 0$),

$$U_q = \text{Re} \left\{ \epsilon_n T \right\}_{\omega_q}$$  
(7.2.16)

where $\text{Re} \left\{ \right\}_{\omega_q}$ is defined such that it is ignored for $\omega_q \neq 0$ but for $\omega_q = 0$ the real part of the expression in brackets is taken. In (7.2.16)

$$T = \sum_{\alpha=0}^{\infty} \sum_{p_1, \ldots, p_N, r_1, \ldots, r_N}^{q_1, \ldots, q_N, \sigma_1, \ldots, \sigma_N} \left\{ \frac{\sigma! \rho! a_{\sigma, \rho}}{2(n+2\alpha)} \right\}$$  
(7.2.17)
and $\Phi$ is given by (7.2.13). The phasor of the $\omega_q$ component of the output $y(t)$ is then given by

$$Y_q = \sum_{n=0}^{\infty} \sum_{n_1, \ldots, n_N} U_q, \quad \text{if } |n_1| + \cdots + |n_N| = n$$

We have thus derived an algebraic formula for the output of a bivariate power series having two multifrequency inputs analogous to the formula presented in Chapter 3 for a single variable power series.

### 7.2.2 Formulas for the Derivatives

In addition to algebraic formulas for calculating the frequency domain representation of the output, a circuit simulator requires derivatives of the output with respect to the input components. In particular, formulas are needed for the derivative of $Y_q$, the phasor component of the output $Y$ at the frequency $\omega_q$, with respect to the magnitude and angle of the $m$th component of the input phasors, $X$ and $Z$, e.g.,

$$\frac{\partial Y_q}{\partial |X_m|}, \quad \frac{\partial Y_q}{\partial |Z_m|}, \quad \frac{\partial Y_q}{\partial \phi_m}, \quad \text{and,} \quad \frac{\partial Y_q}{\partial \theta_m}.$$

To calculate the derivatives with respect to magnitude, differentiate (7.2.18) to obtain

$$\frac{\partial Y_q}{\partial |X_m|} = \sum_{n=0}^{\infty} \sum_{n_1, \ldots, n_N} \frac{\partial U_q}{\partial |X_m|}, \quad \text{if } |n_1| + \cdots + |n_N| = n$$

(7.2.19)
where

\[ \frac{\partial U_q}{\partial |X_m|} = \text{Re} \left\{ \epsilon_n \frac{\partial T}{\partial |X_m|} \right\}_{\omega_q} \]

and

\[ \frac{\partial T}{\partial |X_m|} = \sum_{\alpha=0}^{\infty} \sum_{p_0, \ldots, p_N, r_1, \ldots, r_N} \left\{ \left( \frac{\sigma! \rho! a_{\sigma, \rho}}{2^{n+2\alpha}} \right) \frac{\partial \Phi}{\partial |X_m|} \right\}. \]

Differentiating (7.2.13) we find that for \( |X_m| \neq 0 \)

\[ \frac{\partial \Phi}{\partial |X_m|} = (p_m + r_m) \left( \frac{\Phi}{|X_m|} \right). \]  

(7.2.20)

The derivative with respect to \( |Z_m| \) is found in a similar manner, with

\[ \frac{\partial \Phi}{\partial |Z_m|} = (q_m + s_m) \left( \frac{\Phi}{|Z_m|} \right). \]

The derivative with respect to the angle of the \( m \) th component of the input \( x \) is found to be

\[ \frac{\partial Y_q}{\partial \phi_m} = \sum_{n=0}^{\infty} \sum_{n_1, \ldots, n_N} \frac{\partial U_q}{\partial \phi_m} \quad |n_1| + \cdots + |n_N| = n \]

where

\[ \frac{\partial U_q}{\partial \phi_m} = \text{Re} \left\{ \epsilon_n \frac{\partial T}{\partial \phi_m} \right\}_{\omega_q} \]
and

\[
\frac{\partial T}{\partial \phi_m} = \sum_{\alpha=0}^{\infty} \sum_{p_1 \cdots p_N, r_1 \cdots r_N \atop q_1 \cdots q_N, s_1 \cdots s_N} \left\{ \frac{(\sigma! \rho! a_{\sigma, \rho})}{2^{(n+2\alpha)}} \frac{\partial \Phi}{\partial \phi_m} \right\}.
\]

Rewriting the expression for \(X_k^j\) we find that if \(n_k \neq 0\)

\[
(X_k^j)^{p_k} = |X_k|^{p_k} \exp \left( j\phi_k \frac{p_k n_k}{|n_k|} \right)
\]

and

\[
(X_k^j)^{r_k} = |X_k|^{r_k} \exp \left( -j\phi_k \frac{r_k n_k}{|n_k|} \right).
\]

Thus, we obtain the derivative

\[
\frac{\partial \Phi}{\partial \phi_m} = \begin{cases} 
(j \frac{n_m}{|n_m|})(p_m - r_m) \Phi & n_m \neq 0 \\
0 & n_m = 0.
\end{cases} \tag{7.2.21}
\]

The derivative with respect to the angle of the \(m\) th component of the input \(z\) is found similarly to be

\[
\frac{\partial \Phi}{\partial \theta_m} = \begin{cases} 
(j \frac{n_m}{|n_m|})(q_m - s_m) \Phi & n_m \neq 0 \\
0 & n_m = 0.
\end{cases} \tag{7.2.22}
\]

Thus we have derived algebraic formulas for the derivatives of the output of a bivariate power series with respect to the multifrequency inputs.
7.3 Relationship to Formulas for Single Variable Series

In the following sections, we examine the relationship of the formulas just derived to the formulas presented in Chapter 3 for single variable power series.

7.3.1 Formulas for the Output

In this section we compare the formulas just derived for the output of a bivariate power series and the formulas presented in Chapter 3 for the single variable power series. We can consider the special case of a single variable generalized power series in $x$ by substituting

$$a_{\sigma, \rho} = \begin{cases} a_{\sigma} & \rho = 0 \\ 0 & \rho \neq 0 \end{cases}$$  \hspace{1cm} (7.3.1)

into (7.2.3), resulting in the single variable expression for the output $y$

$$y(t) = \sum_{\sigma=0}^{\infty} a_{\sigma} f(\sigma, x).$$  \hspace{1cm} (7.3.2)

With $\rho = 0$,

$$\sigma + \rho = \sigma = n + 2\alpha.$$

With $q_k$ and $s_k \geq 0$, $\rho = \sum_k (q_k + s_k)$, and $\rho = 0$, we find that $q_k$ and $s_k$ are zero, so that

$$p_k - r_k = |n_k|$$

or, solving for $p_k$

$$p_k = |n_k| + r_k.$$
This means that

\[
\sum_{k=1}^{N} (p_k + r_k) = \sum_{k=1}^{N} (|n_k| + 2r_k) = n + 2 \sum_{k=1}^{N} r_k.
\]

But, from the relationship between \(p_k, r_k,\) and \(\sigma\)

\[
\sum_{k=1}^{N} (p_k + r_k) = \sigma = n + 2\alpha.
\]

Therefore,

\[
\sum_{k=1}^{N} r_k = \alpha.
\]

Now we can eliminate \(p_k\) and reduce the summations in the output formula to a single index by rewriting the product

\[
(X_k^\dagger)^{p_k} (X_k^\dagger)^{r_k}
\]

as

\[
(X_k^\dagger)^{|n_k| + r_k} (X_k^\dagger)^{r_k} = (X_k^\dagger)^{|n_k|} |X_k|^{2r_k}
\]

using the fact that \(X_k^\dagger\) is the complex conjugate of \(X_k^\dagger\). Similarly, we use the fact that \(|\Gamma_{k,\sigma}| = 1\) to write

\[
(\Gamma_{k,\sigma}^\dagger)^{p_k} (\Gamma_{k,\sigma}^\dagger)^{r_k} = (\Gamma_{k,\sigma}^\dagger)^{|n_k|}.
\]

Furthermore, note that

\[
\Gamma_{k,\sigma}^\dagger = \begin{cases} 
\exp(-j\frac{n_k}{|n_k|}\omega_k\tau_{k,\sigma}) & n \neq 0 \\
1 & n = 0
\end{cases}
\]
so that

\[(\Gamma_{k,\sigma}^\dagger)^{|n_k|} = \exp(-jn_k\omega_k\tau_{k,\sigma})\]

and

\[
\prod_{k=1}^{N} (\Gamma_{k,\sigma}^\dagger)^{|n_k|} = \exp\left(-j\sum_{k=1}^{N} n_k\omega_k\tau_{k,\sigma}\right) = R_{\sigma} = R_{n+2\alpha}
\]

as defined earlier. The expression for \(T\) is now greatly simplified and reduced to the sum over the single index \(r_k\)

\[
T = \sum_{\alpha=0}^{\infty} \sum_{r_1,\ldots,r_N} \left\{ \left(\frac{(n+2\alpha)!}{2(n+2\alpha)}\right) a_{n+2\alpha} R_{n+2\alpha} \Phi \right\}
\]

(7.3.3)

where

\[
\Phi = \prod_{k=1}^{N} \frac{(X_k^\dagger)^{|n_k|} |X_k|^{2r_k} (b_k)^{|n_k|+2r_k}}{r_k! (|n_k|+r_k)!}.
\]

(7.3.4)

This is equivalent to the expression derived in [56] for a single variable generalized power series and presented in Chapter 3 as (3.2.3)-(3.2.9).

### 7.3.2 Formulas for the Derivatives

In this section we compare the formulas just derived for the derivatives of the output of a bivariate power series with respect to the inputs with the formulas presented in Chapter 3. Again we consider the special case of a single variable generalized power series in \(x\) so that as before,

\[
p_k = |n_k| + r_k.
\]
Using the expression for the derivative of $\Phi$ with respect to the magnitude of the $m$th input component as given by (7.2.20) we find

$$\frac{\partial \Phi}{\partial |X_m|} = (p_m + r_m) \left( \frac{\Phi}{|X_m|} \right)$$

$$= (|n_m| + 2r_m) \left( \frac{\Phi}{|X_m|} \right).$$

As before, the expressions reduce to summations over a single index such that

$$\frac{\partial Y_q}{\partial |X_m|} = \sum_{n=0}^{\infty} \sum_{n_1 \cdots n_N} \frac{\partial U_q}{\partial |X_m|}$$

where

$$\frac{\partial U_q}{\partial |X_m|} = \text{Re} \left\{ \epsilon_n \frac{\partial T}{\partial |X_m|} \right\} \omega_q$$

and

$$\frac{\partial T}{\partial |X_m|} = \sum_{\alpha=0}^{\infty} \sum_{r_1 \cdots r_N} \left\{ \left( \frac{(n + 2\alpha)!}{2(n + 2\alpha)} \right) a_{n+2\alpha} R_{n+2\alpha} \frac{\partial \Phi}{\partial |X_m|} \right\}$$

where $\Phi$ is given by (7.3.4). This is equivalent to the expression presented in (3.4.2) for single variable power series.

In a similar way, the derivative of $\Phi$ with respect to the phase of the $m$th input component, $\phi_m$ as given by (7.2.21) becomes

$$\frac{\partial \Phi}{\partial \phi_m} = j \frac{n_m}{|n_m|} (p_m - r_m) \Phi \quad (n_m \neq 0)$$

$$= j \frac{n_m}{|n_m|} (|n_m| + r_m - r_m) \Phi$$

$$= j n_m \Phi. \quad (7.3.7)$$
Using

\[ \frac{\partial Y_q}{\partial \phi_m} = \sum_{n=0}^{\infty} \sum_{n_1, \ldots, n_N}^{n_1 + \ldots + n_N = n} \frac{\partial U_q}{\partial \phi_m} \]  

(7.3.8)

where

\[ \frac{\partial U_q}{\partial \phi_m} = \text{Re} \left\{ \epsilon_n \frac{\partial T}{\partial \phi_m} \right\} \omega_q \]

and

\[ \frac{\partial T}{\partial \phi_m} = \sum_{\alpha=0}^{\infty} \sum_{\tau_1, \ldots, \tau_N} \left\{ \left( \frac{(n + 2\alpha)!}{2^{(n+2\alpha)}} \right) a_{n+2\alpha} R_{n+2\alpha} \frac{\partial \Phi}{\partial \phi_m} \right\} \]

(7.3.9)

and the reduced expression for \( \Phi \), (7.3.4), we obtain the expression presented in (3.4.7) for single variable power series.

7.4 Discussion

The formulas derived here are qualitatively similar to the ones presented in Chapter 3 for the single variable power series with the most obvious difference being the more complicated summations, as in (7.2.17). These complications arise from the consideration of terms such as

\[ (X_k)^{l_k} (X_k^*)^{m_k} (Z_k)^{i_k} (Z_k^*)^{j_k} \]

which have a number of combinations of the integers \( l, m, i, j \) that correspond to a given output frequency. In contrast, the single variable case involves terms of the form

\[ (X_k)^{l_k} (X_k^*)^{m_k} \]
for which there is a more limited number of combinations that yield a given output frequency. The single variable case can be simplified requiring only one set of integers to be retained, while the two variable case requires that all four integers specifying the powers of $X, X^*, Z,$ and $Z^*$ be retained. Consider the example presented in Chapter 3 for the intermodulation product description $f_1 = 2f_1 - f_2 + f_3$ yielding an output component at $f_1$ and corresponding to the intermodulation product $X_1^2X_2^*X_3$. For the two variable case, the same intermodulation product description corresponds to the following intermodulation products:

$$X_1^2X_2^*X_3$$
$$X_1^2X_2^*Z_3$$
$$X_1^2Z_2^*Z_3$$
$$X_1Z_1Z_2^*X_3$$

along with a number of others.

### 7.5 Conclusion

In this chapter we have derived algebraic formulas for the output of a nonlinearity described by a bivariate generalized power series having multifrequency inputs. We have also derived formulas for the derivatives of the output components with respect to the input components and discussed the relationship of these formulas to the formulas previously derived for single variable power series. These formulas enhance the capabilities of generalized power series analysis by allowing nonlinear...
functions of two variables to be considered in a general way.
Chapter 8

Conclusion

8.1 Summary

The objective of this study was the development of a new computer-aided analysis tool for nonlinear microwave analog circuits. This work consists of two parts: the development of the basic analysis technique, and improvements to the basic technique.

In the first part, a new computer-aided analysis tool was developed. The method uses generalized power series (power series having complex coefficients and time delays) descriptions for the nonlinear circuit elements in combination with a spectral balance method for circuit analysis. The resulting analysis method is a true frequency domain technique that is suited to the analysis of nonlinear circuits having large-signal multifrequency excitation. In order to verify the analysis technique, a generalized power series based MESFET model was developed. Transistors were characterized and the model parameters extracted. These models were used to simulate common source MESFET amplifiers and the results compared to measured circuit performance. Comparisons were made for single and two tone inputs showing excellent agreement, thereby validating the analysis procedure. In
addition, a method for calculating the effective impedance of each nonlinear circuit element in the MESFET model was presented.

In the second portion of this work, techniques were presented for enhancing the modeling capabilities of generalized power series analysis. This was done through the use of power series functions of two variables called bivariate power series. These expressions were used to develop new models for the dc and high-frequency behavior of a MESFET. The new models are valid over a wide range of operating points and, effectively, globalize the models developed earlier. A method was also presented for incorporating these ideas into generalized power series analysis by reducing the bivariate power series expressions to single variable power series at the operating point. Finally, the theoretical background was presented for completely including functions of two variables in generalized power series analysis.

8.2 Discussion

It is interesting to compare the performance of the generalized power series approach developed here to other simulation strategies. Unfortunately, this is a difficult task since the information is not available in the literature, and it is difficult to have access to all of the techniques. Recently, however, a second generation generalized power series analysis has been written [98] that includes, as an option, the standard harmonic balance routine. This program, called **FREDA2**, is an improved version of the tool developed here. It has improved convergence proper-
ties, is user friendly, and is not limited to a fixed nonlinear subcircuit topology. It also has an option that allows the circuit to be analyzed using a harmonic balance approach [99] utilizing the almost periodic transform along with a near orthogonal time point selection algorithm, as in [100]. **FREDA2** has been used to compare the two techniques with regard to execution speed and memory requirements [101].

Two circuits are considered in [101]: a class C amplifier using a bipolar transistor, and a MESFET amplifier with three incommensurable input frequencies. For the class C amplifier with a single input tone, the harmonic balance method is much faster and uses less memory than the generalized power series method. Because this circuit is highly nonlinear and the frequencies are harmonically related, a large table of intermodulation products must be used with generalized power series analysis. This results in a less efficient simulation.

The simulation of the MESFET amplifier is quite different. In this case, the signals are not harmonically related, if fact they are incommensurable. In this case, **GPSA** is more efficient than harmonic balance with respect to accuracy, computing time, and memory requirements. A large number of frequencies must be included in the harmonic balance method before the correct solution is found. In GPSA, only a relatively few frequencies need to be considered to reach the solution. For example, Fig. 8.2.1 shows the power at one of the intermodulation frequencies for the MESFET amplifier simulation as a function of the number of frequencies in the analysis. The harmonic balance method requires that 26 frequencies be considered
before the correct solution is found while GPSA requires only four frequencies to reach the same solution.

Thus, we see that generalized power series analysis, as developed here is an important tool for the computer-aided analysis of nonlinear analog circuits. It has distinct advantages over other simulation strategies and is most suited to analyzing nonlinear circuits having multifrequency excitation.

8.3 Suggestions for Further Study

There are two areas for further study that were identified in the course of this work:

i) continued development of the analysis method, and

ii) improved device modeling.

Included in the area of continued development of the analysis method is an efficient computer implementation of the formulas developed for bivariate power series. As developed, these formulas do not look amenable to efficient implementation and work is needed in this area. The relationship between the bivariate power series formulas and Volterra series should also be investigated. Volterra series are traditionally presented as functions of a single variable, however, they could also be extended to include functions of two variables. If extended to this case, they should be related to bivariate power series just as conventional Volterra...
Figure 8.2.1: The simulated power output at one of the IF frequencies for a MESFET amplifier with three incommensurable inputs. Shown are the values calculated using generalized power series analysis compared to the values calculated using harmonic balance as a function of the number of frequencies considered in the simulations (after [101]).
series are related to single variable power series [57]. There is also a need for
the simulation of complex nonlinear systems. The techniques used in this work
should be applicable to these situations also. Work has already begun in other
areas, including improvement in the iterative solution of the determining equation
and in simulating oscillator circuits using generalized power series [98]. Further
improvements along these lines could involve simulating the noise performance of
oscillator and amplifier circuits.

In the category of improved device modeling much work can be done. Included
is the development of models for different devices as well as improved MESFET
models. Also the process of device characterization and parameter extraction
deserves attention. Currently, this is an extremely slow and laborious process.
Another area deserving attention is the behavior of the nonlinear elements in the
MESFET model as a function of frequency. As mentioned in the text, the dc
behavior is different from the rf behavior and there is currently no good model for
this effect.
References


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Appendix A

Solving the Determining Equation of Harmonic Balance

In this appendix we investigate the relationship between two methods for solving the determining equation associated with the harmonic balance technique for nonlinear circuit analysis. One method finds the solution by minimizing an error function while the other solves a related system of nonlinear equations. In general, these methods are different and lead to separate solution algorithms having different properties. For this particular application, however, these techniques are very similar, a fact which has been misrepresented in a recent publication [17]. After a review of the basic theory of these two methods, we show how, when using the harmonic balance technique, they result in the same algorithm.

The general problem of solving a system of nonlinear equations can be formulated as:

\[
given \ F : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ find \ x_0 \in \mathbb{R}^n \ such \ that \ F(x_0) = 0 \quad (A.1)
\]

where it is assumed that \( F \) is continuously differentiable. An important technique for solving (A.1) is Newton's method. This is an iterative procedure, consisting of
the solution of

\[ J(x_k)s_k = -F(x_k) \]

\[ x_{k+1} = x_k + s_k \]  (A.2)

at each iteration \( k \) [39]. This is solution is frequently written as

\[ x_{k+1} = x_k - J^{-1}(x_k)F(x_k) \]  (A.3)

where \( J \) is the Jacobian of \( F \). This is a very powerful technique having good convergence properties provided an appropriate initial estimate of \( x_* \) can be made.

In contrast, the general problem of minimizing a nonlinear function can be formulated as:

\[ \min_{x \in \mathbb{R}^n} f : \mathbb{R}^n \to \mathbb{R} \]  (A.4)

where \( f \) is assumed to be twice continuously differentiable. Applying Newton’s method to this problem results in an algorithm consisting of the solution of

\[ \nabla^2 f(x_k)s_k = -\nabla f(x_k) \]

\[ x_{k+1} = x_k + s_k \]  (A.5)

at each iteration \( k \) [39] where \( H \) is the Hessian (\( \nabla^2 f \)). This solution is frequently written as

\[ x_{k+1} = x_k - H^{-1}\nabla f(x_k). \]  (A.6)

This algorithm results from applying Newton’s method to the system of nonlinear equations

\[ \nabla f(x) = 0 \]
and thus has similar convergence properties to that of (A.3), with certain conditions on $H$, but requires the calculation of the gradient as well as the Hessian matrix. There are a variety of quasi-Newton methods available that do not require explicit calculation of the Hessian but have different convergence properties. In summary, the problem of minimization is more involved than the solution of nonlinear equations.

A special case of the general minimization problem is the nonlinear least squares problem, which can be stated as:

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2} \mathbf{R}(x)^T \mathbf{R}(x) = \frac{1}{2} \sum_{i=1}^{m} r_i(x)^2$$  \hspace{1cm} (A.7)

where the residual function

$$\mathbf{R} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is nonlinear in $x$ and $r_i(x)$ is the $i$th component of $\mathbf{R}(x)$. Typically $m > n$ and the problem is viewed as an overdetermined system of equations. This problem can be solved using Newton's method as just presented, requiring the gradient and Hessian, or in some cases, the Gauss-Newton method may be used. This approach to the nonlinear least squares problem uses the iteration:

$$x_{k+1} = x_k - (J(x_k)^T J(x_k))^{-1} J(x_k)^T \mathbf{R}(x_k)$$  \hspace{1cm} (A.8)

where $J$ is the jacobian of $\mathbf{R}$ [39]. For zero residual problems ($\mathbf{R}(x_*) = 0$) this method has properties similar to that of Newton's method applied to a system of nonlinear equations. In fact, if the number of unknowns is equal to the number of
components in the residual function, \( n = m \), the Jacobian is square and, provided it is nonsingular, (A.8) can be reduced to the form of (A.3). When \( n = m \) the nonlinear least squares problem is very closely related to the solution of a system of nonlinear equations.

Nonlinear circuit simulation using harmonic balance requires the solution of the determining equation resulting from application of Kirchoff's current law in the frequency domain. This can often be written in the form

\[
F(V) = 0 \quad (A.9)
\]

where \( F \) is a complex valued vector function of \( V \), a complex valued vector of voltage phasors. As noted in [17] this can be solved using a minimization approach with the function to be minimized

\[
g(V) = F^\dagger F \quad (A.10)
\]

where \( F^\dagger \) denotes the conjugate transpose of \( F \) and \( g \) is a real function. This can be rewritten as

\[
g = \sum_{i=1}^{m} F_i^* F_i
\]

\[
= \sum_{i=1}^{m} (\text{Re}(F_i)^2 + \text{Im}(F_i)^2) \quad (A.11)
\]

where \( F_i^* \) denotes the complex conjugate of the \( i \)th component of \( F \). Each component of \( F \) corresponds to the sum of currents at one node and at one frequency and there is a corresponding component of the vector of node voltage phasors, \( V \). The
number of unknowns is equal to the number of squared terms to be minimized.

Using the notation of [17], an equivalent real valued function can be formed

\[ \bar{F} = [\text{Re}(F) \quad \text{Im}(F)]^T \]

with the result that

\[ F^\dagger F = \bar{F}^T \bar{F}. \]

The function to be minimized can be put in the form of (A.7)

\[ g = \frac{1}{2} \bar{F}^T \bar{F} \tag{A.12} \]

and the minimum can be found using the Gauss-Newton iteration which reduces to (A.3) since the number of unknowns equals the number of components of \( F \).

The solution algorithm is identical to that obtained by applying Newton's method directly to the system of equations (A.9). Intuitively, this assertion is reasonable assuming that there is a solution such that \( g(V_0) = 0 \). A sum of real valued functions that are squared can be equal to zero only if each function itself is equal to zero. The system of equations resulting from application of harmonic balance can thus be viewed as either a minimization problem or as a system of nonlinear equations and solved using the same algorithm.
Appendix B

Representation of the Linear Subcircuit

In the generalized power series based program FREDA (described in Chapter 3), the linear subcircuit is described by its nodal admittance matrix representation. Such a representation is found systematically from a list of circuit elements specifying the element type (e.g., capacitor, resistor, etc.), its value (e.g., C, R, etc.), and the nodes to which it is connected (provided that the elements have an admittance representation) [1]. For a linear subcircuit having Q nodes, the total current entering each node at radian frequency \( \omega_q \) is given by

$$
\begin{bmatrix}
\hat{I}_1(\omega_q) \\
\hat{I}_2(\omega_q) \\
\vdots \\
\hat{I}_Q(\omega_q)
\end{bmatrix} =
\begin{bmatrix}
Y_{11}(\omega_q) & Y_{12}(\omega_q) & \cdots & Y_{1Q}(\omega_q) \\
Y_{21}(\omega_q) & Y_{22}(\omega_q) & \cdots & Y_{2Q}(\omega_q) \\
\vdots & \vdots & \ddots & \vdots \\
Y_{Q1}(\omega_q) & Y_{Q2}(\omega_q) & \cdots & Y_{QQ}(\omega_q)
\end{bmatrix}
\begin{bmatrix}
V_1(\omega_q) \\
V_2(\omega_q) \\
\vdots \\
V_Q(\omega_q)
\end{bmatrix}
$$

(B.1)

where the subscripts refer to node numbers. In Chapter 3 the determining equation is solved by converting the complex valued vectors into real valued vectors, using the notation

$$
\hat{I}_1(\omega_q) = \hat{I}_1^R(\omega_q) + j\hat{I}_1^I(\omega_q)
$$
where $I_1^R$ and $I_1^I$ are real. Defining the admittance and voltage in a similar way, the equation

$$\hat{I} = YV$$

becomes

$$\hat{I}^R + j\hat{I}^I = (Y^R + jY^I)(V^R + jV^I)$$

from which we find

$$\hat{I}^R = Y^R V^R - Y^I V^I$$

and

$$\hat{I}^I = Y^I V^R + Y^R V^I.$$ 

If we define

$$\tilde{Y}_{ij}(\omega_q) = \begin{bmatrix} Y_{ij}^R(\omega_q) & -Y_{ij}^I(\omega_q) \\ Y_{ij}^I(\omega_q) & Y_{ij}^R(\omega_q) \end{bmatrix} \quad (B.2)$$

we can rewrite (B.1) as

$$\begin{bmatrix} \hat{I}_1^R(\omega_q) \\ \hat{I}_1^I(\omega_q) \\ \hat{I}_2^R(\omega_q) \\ \hat{I}_2^I(\omega_q) \\ \vdots \\ \hat{I}_Q^R(\omega_q) \\ \hat{I}_Q^I(\omega_q) \end{bmatrix} = \begin{bmatrix} \tilde{Y}_{11}(\omega_q) & \tilde{Y}_{12}(\omega_q) & \cdots & \tilde{Y}_{1Q}(\omega_q) \\ \tilde{Y}_{21}(\omega_q) & \tilde{Y}_{22}(\omega_q) & \cdots & \tilde{Y}_{2Q}(\omega_q) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{Y}_{Q1}(\omega_q) & \tilde{Y}_{Q2}(\omega_q) & \cdots & \tilde{Y}_{QQ}(\omega_q) \end{bmatrix} \begin{bmatrix} V_1^R(\omega_q) \\ V_1^I(\omega_q) \\ V_2^R(\omega_q) \\ V_2^I(\omega_q) \\ \vdots \\ V_Q^R(\omega_q) \\ V_Q^I(\omega_q) \end{bmatrix} \quad (B.3)$$
or, more concisely,

\[ \hat{I}(\omega_q) = \hat{Y}(\omega_q) \hat{V}(\omega_q). \]

We consider \( N \) frequencies simultaneously by again rewriting (B.1)

\[
\begin{bmatrix}
\hat{I}_1^R(\omega_1) \\
\hat{I}_1^I(\omega_1) \\
\vdots \\
\hat{I}_Q^R(\omega_1) \\
\hat{I}_Q^I(\omega_1) \\
\hat{I}_1^R(\omega_2) \\
\vdots \\
\hat{I}_Q^R(\omega_2) \\
\hat{I}_Q^I(\omega_2) \\
\vdots \\
\hat{I}_Q^R(\omega_N) \\
\hat{I}_Q^I(\omega_N)
\end{bmatrix}
\begin{bmatrix}
\hat{Y}(\omega_1) & 0 & \cdots & 0 \\
0 & \hat{Y}(\omega_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{Y}(\omega_N)
\end{bmatrix}
\begin{bmatrix}
V_1^R(\omega_1) \\
V_1^I(\omega_1) \\
\vdots \\
V_Q^R(\omega_1) \\
V_Q^I(\omega_1) \\
V_1^R(\omega_2) \\
\vdots \\
V_Q^R(\omega_2) \\
V_1^I(\omega_2) \\
\vdots \\
V_Q^I(\omega_N) \\
V_Q^I(\omega_N)
\end{bmatrix}.
\]  

(B.4)

We can reduce the number of variables by eliminating the dependent voltages associated with nodes internal to the linear subcircuit. Thus, the nodal admittance matrix originally \((2NQ) \times (2NQ)\) can be reduced to \(2N(P + M) \times 2N(P + M)\), where there are \( P \) nodes in common with the nonlinear subcircuit and \( M \) nodes connected to independent sources. Furthermore, the only currents of interest are those associated with the \( P \) nodes in common with the nonlinear subcircuit. Eliminating the rows associated with the \( M \) independent sources leaves a reduced nodal admittance matrix of dimension \(2NP \times 2N(P + M)\). Unfortunately, reducing
the nodal admittance matrix destroys its block diagonal form. The advantage is that the number of variables to be optimized is greatly reduced, resulting in an improvement in speed and efficiency. The resulting equation for the node currents expressed in the notation of Chapter 3 is

\[ \hat{I} = YV. \]  (B.5)
Appendix C

Derivative Calculations for a Nonlinear Element

In this appendix we derive the equations needed to calculate the derivatives of current with respect to the real and imaginary components of voltage for a nonlinear element. These derivatives are required for the Jacobian used in the Newton's method iteration. In Chapter 3, formulas are provided for calculating the derivatives of current with respect to the voltage magnitude and phase. By calculating derivatives of voltage magnitude and phase with respect to the real and imaginary components of the voltage phasor and using the chain rule, the desired quantities can be found.

Given the phasor current through a nonlinear element as a function of the difference of two node voltage phasors:

\[ I = f(V_1 - V_2) \quad \text{(C.1)} \]

we seek the derivatives of the real and imaginary components of \( I \) with respect to the real and imaginary components of \( V_1 \) and \( V_2 \), e.g.,

\[ \frac{\partial \text{Re}(I)}{\partial \text{Re}(V_1)} \frac{\partial \text{Re}(I)}{\partial \text{Re}(V_2)} \frac{\partial \text{Re}(I)}{\partial \text{Im}(V_1)} \frac{\partial \text{Re}(I)}{\partial \text{Im}(V_2)} \frac{\partial \text{Im}(I)}{\partial \text{Re}(V_1)} \frac{\partial \text{Im}(I)}{\partial \text{Re}(V_2)} \frac{\partial \text{Im}(I)}{\partial \text{Im}(V_1)} \frac{\partial \text{Im}(I)}{\partial \text{Im}(V_2)} \]

In practice only four derivatives need to be calculated since

\[ \frac{\partial \text{Re}(I)}{\partial \text{Re}(V_1)} = \text{Re} \left( \frac{\partial I}{\partial \text{Re}(V_1)} \right) \]
and

$$\frac{\partial \text{Im}(I)}{\partial \text{Re}(V_1)} = \text{Im} \left( \frac{\partial I}{\partial \text{Re}(V_1)} \right).$$

Making the following definitions:

$$g = |V_1 - V_2| \quad h = \angle(V_1 - V_2)$$

$$V_1 = a + jb \quad V_2 = c + jd$$

where $\angle(V_1 - V_2)$ is the phase of $(V_1 - V_2)$ and $a, b, c,$ and $d$ are real, we can rewrite the current as

$$I = f(g(a, b, c, d), h(a, b, c, d)) \quad \text{(C.2)}$$

where $g$ and $h$ are defined as

$$g(a, b, c, d) = \left((a - c)^2 + (b - d)^2\right)^{\frac{1}{2}} \quad \text{(C.3)}$$

$$h(a, b, c, d) = \tan^{-1}\left(\frac{b - d}{a - c}\right). \quad \text{(C.4)}$$

In Chapter 3 formulas are provided for calculating $\partial I/\partial g$ and $\partial I/\partial h$ when the nonlinear function $f$ is described by a generalized power series. Here we find the derivatives with respect to the real and imaginary components of $V_1$ and $V_2$.

From the chain rule, the derivative with respect to $\text{Re}(V_1)$ is

$$\frac{\partial I}{\partial \text{Re}(V_1)} = \frac{\partial I}{\partial a} = \frac{\partial I}{\partial g} \frac{\partial g}{\partial a} + \frac{\partial I}{\partial h} \frac{\partial h}{\partial a} \quad \text{(C.5)}$$

and it remains for $\partial g/\partial a$ and $\partial h/\partial a$ to be calculated. Taking the derivative of (C.3) with respect to $a$

$$\frac{\partial g}{\partial a} = \frac{1}{2} \left[(a - c)^2 + (b - d)^2\right]^{-\frac{1}{2}} 2(a - c)$$
Similarly, taking the derivative of (C.4) with respect to $a$

\[
\frac{\partial h}{\partial a} = \frac{1}{1 + \left[\frac{(b - d)}{(a - c)}\right]^2} \frac{1}{(a - c)^2} (b - d) (-1) \frac{1}{(a - c)^2}
\]

\[
= \frac{-(b - d)}{(a - c)^2 + (b - d)^2}
\]

\[
= - \frac{\text{Im}(V_1 - V_2)}{|V_1 - V_2|^2}.
\]

The chain rule is again used to find the derivative with respect to $\text{Im}(V_1)$

\[
\frac{\partial I}{\partial \text{Im}(V_1)} = \frac{\partial I}{\partial b} \frac{\partial b}{\partial g} + \frac{\partial I}{\partial h} \frac{\partial h}{\partial b}
\]

and we need to find $\partial g/\partial b$ and $\partial h/\partial b$. Taking the derivative of (C.3) with respect to $b$

\[
\frac{\partial g}{\partial b} = \frac{1}{2} \left[ \frac{(a - c)^2 + (b - d)^2}{2(b - d)} \right]^{-\frac{1}{2}} 2(b - d)
\]

\[
= \frac{\text{Im}(V_1 - V_2)}{|V_1 - V_2|}
\]

and taking the derivative of (C.4) with respect to $b$

\[
\frac{\partial h}{\partial b} = \frac{a - c}{1 + \left[\frac{(b - d)}{(a - c)}\right]^2}
\]

\[
= \frac{\text{Re}(V_1 - V_2)}{|V_1 - V_2|^2}.
\]

To find the derivative with respect to $\text{Re}(V_2)$, we proceed as before

\[
\frac{\partial I}{\partial \text{Re}(V_2)} = \frac{\partial I}{\partial c} = \frac{\partial I}{\partial g} \frac{\partial g}{\partial c} + \frac{\partial I}{\partial h} \frac{\partial h}{\partial c}
\]
where $\partial g / \partial c$ and $\partial h / \partial c$ are unknown. Taking the appropriate derivatives, we find

$$\frac{\partial g}{\partial c} = -\frac{1}{2} \left[ (a - c)^2 + (b - d)^2 \right]^{-\frac{1}{2}} 2(a - c)$$

$$= -\frac{\text{Re}(V_1 - V_2)}{|V_1 - V_2|} \quad (C.12)$$

and

$$\frac{\partial h}{\partial c} = \frac{b - d}{1 + [(b - d)/(a - c)]^2} \frac{1}{(a - c)^2}$$

$$= \frac{\text{Im}(V_1 - V_2)}{|V_1 - V_2|^2}. \quad (C.13)$$

Finally, the derivative with respect to $\text{Im}(V_2)$ is given by

$$\frac{\partial I}{\partial \text{Im}(V_2)} = \frac{\partial I}{\partial d} = \frac{\partial I}{\partial g} \frac{\partial g}{\partial d} + \frac{\partial I}{\partial h} \frac{\partial h}{\partial d} \quad (C.14)$$

where

$$\frac{\partial g}{\partial d} = -\frac{1}{2} \left[ (a - c)^2 + (b - d)^2 \right]^{-\frac{1}{2}} 2(b - d)$$

$$= -\frac{\text{Im}(V_1 - V_2)}{|V_1 - V_2|} \quad (C.15)$$

and

$$\frac{\partial h}{\partial d} = -\frac{a - c}{1 + [(b - d)/(a - c)]^2}$$

$$= -\frac{\text{Re}(V_1 - V_2)}{|V_1 - V_2|^2}. \quad (C.16)$$

It should be noted that these derivatives are only valid if the magnitude of $V_1 - V_2$ is nonzero.
This appendix has presented formulas for calculating the derivatives of the real and imaginary components of current with respect to the real and imaginary components of voltage for a nonlinear circuit element. These derivatives are required for the generalized power series analysis technique presented in Chapter 3.
The parameter extraction step of the MESFET modeling procedure described in Chapter 4 involves the determination of the coefficients in the power series descriptions of the nonlinear elements. These coefficients are chosen so that the power series most closely reproduce the measured data. In other words, given the $M$ data points $(x_i, y_i)$ we try to find a $p$th order power series of the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_p x^p \quad (D.1)$$

so that $y(x_i)$ is suitably close to $y_i$ for all data points. The data, $y_i$, can be values for capacitance, conductance, or current, while the independent variable $x$ is the controlling voltage. A commonly used measure of the closeness of the approximation is the least-squares error

$$E = \sum_{i=1}^{M} [y(x_i) - y_i]^2. \quad (D.2)$$

This measure can be justified on a statistical basis and results in a simple method for calculating the coefficients [102,103].

To find the power series coefficients using the method of least squares, substitute the data into (D.1) forming a set of $M$ equations, which can be written
as

\[
\begin{bmatrix}
  y(x_1) \\
  y(x_2) \\
  \vdots \\
  y(x_M)
\end{bmatrix} =
\begin{bmatrix}
  1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^p \\
  1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^p \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & x_M & x_M^2 & x_M^3 & \cdots & x_M^p
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_p
\end{bmatrix}
\]  
(D.3)

or in matrix notation

\[
Y = XA,
\]  
(D.4)

where the matrix \(X\) is a Vandermonde matrix [102]. Writing the data as a vector

\[
\tilde{Y} = \begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_M
\end{bmatrix}
\]

we can calculate the least-squares error using vector notation and the Euclidean norm

\[
E = (\|Y - \tilde{Y}\|_2)^2.
\]  
(D.5)

Minimizing \(\sqrt{E}\) yields the same coefficients as minimizing \(E\) and allows the problem to be simply stated: find the vector \(A\) so that \(\|Y - \tilde{Y}\|_2 = \|XA - \tilde{Y}\|_2\) is minimized. This is the classic linear least-squares problem whose solution can be found by solving the system

\[
(X^TX)A = X^T\tilde{Y}
\]  
(D.6)
called the normal equations [39]. If the matrix \((X^T X)\) is nonsingular the unique solution is

\[
A = (X^T X)^{-1} X^T \hat{Y}.
\]

(D.7)

The coefficients of the power series can thus be found directly (non-iteratively) using the least-squares technique because the output of a power series depends linearly on its coefficients.

Unfortunately, the matrix

\[
X^T X = \begin{bmatrix}
M & \sum_{i=1}^{M} x_i & \sum_{i=1}^{M} x_i^2 & \cdots & \sum_{i=1}^{M} x_i^p \\
\sum_{i=1}^{M} x_i & \sum_{i=1}^{M} x_i^2 & \sum_{i=1}^{M} x_i^3 & \cdots & \sum_{i=1}^{M} x_i^{p+1} \\
\sum_{i=1}^{M} x_i^2 & \sum_{i=1}^{M} x_i^3 & \sum_{i=1}^{M} x_i^4 & \cdots & \sum_{i=1}^{M} x_i^{p+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{M} x_i^p & \sum_{i=1}^{M} x_i^{p+1} & \sum_{i=1}^{M} x_i^{p+2} & \cdots & \sum_{i=1}^{M} x_i^{2p}
\end{bmatrix}
\]

(D.8)

tends to be ill-conditioned making the direct solution via (D.7) impractical in many cases. In cases where the direct method fails, there are several methods that factor \(X\) to find a solution without forming \((X^T X)\). The simpler of these methods is the QR decomposition which factors \(X\) into the product \(X = QR\) where \(Q\) is an \(M \times M\) orthogonal matrix and \(R\) is an \(M \times (p + 1)\) upper triangular matrix.

The unique solution to the linear least-squares problem can then be written

\[
A = R_u^{-1}(X^T \hat{Y})_u
\]

(D.9)

where \(R_u\) is the top \(n\) rows of \(R\) and \((X^T \hat{Y})_u\) is the first \(n\) elements of \((X^T \hat{Y})\) [39]. This assumes that \(X\) has full column rank. In the event that \(X\) does not have full
column rank, there is no unique solution to the linear least-squares problem [39]. A unique solution exists if the problem is further restricted to the one possible solution having smallest Euclidean norm. This solution can be found using the singular value decomposition where $X$ is factored into the product $X = UDV^T$ where $U$ is an $M \times M$ orthogonal matrix, $V$ is an $(p+1) \times (p+1)$ orthogonal matrix, and $D$ is an $M \times (p+1)$ diagonal matrix whose elements are the nonnegative square roots of the eigenvalues of $X^TX$ (for $M \geq (p + 1)$). The solution is then written

$$A = VD^+U^TY$$

(D.10)

where

$$D^+ = \begin{cases} 
  d_{ii}^+ & = & \begin{cases} 
    1/d_{ii} & d_{ii} > 0 \\
    0 & d_{ii} = 0 
  \end{cases} \\
  d_{ij}^+ & = & 0 \quad i \neq j
\end{cases}$$

and the elements $d_{ii}$ (the diagonal elements of $D$) are called the singular values of $X$ [39]. Routines are available for performing both of these decompositions in the LINPACK subroutine library [104]. In the application presented here, the QR decomposition represents a good compromise between the other two methods. It is more accurate than the direct solution approach and is computationally more efficient than the singular value decomposition.

A common occurrence when fitting power series to data is that the series will match the data but will have an oscillatory behavior between the data points. If the values of the independent variable $x_i$ can be arbitrarily chosen, the oscillatory
behavior will be minimized by clustering the values near the endpoints of the interval as opposed to having the $x_i$ equally spaced [102]. Another possibility is to numerically differentiate the data with respect to the independent variable and fit a power series to the derivative. This series can then be integrated to yield a power series that models the original data. The order of the power series is another variable that can be adjusted to achieve a better fit. Typically a low order power series is tried initially and the order increased until an acceptable fit is obtained. The order chosen is often a compromise since the order of the power series used in the MESFET model is linked to the efficiency of the nonlinear circuit simulator.
Appendix E

Surface Fitting with Power Series

The problem of determining the coefficients for a bivariate power series is solved in the same manner as for the single variable power series. As before, we are given $M$ data points and we try to find a power series to reproduce the data. In this case the data is of the form $(x_i, z_i, y_i)$ and we consider power series of the form

$$y(x, z) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{p+1} x^p + a_{p+2} z + a_{p+3} x z + \cdots$$

$$+ a_{p+4} x^2 z + \cdots + a_{2(p+1)} x^{2p} + a_{2p+3} z^2 + a_{2p+4} z^2 x + \cdots$$

$$+ a_{2p+6} x^2 z^2 + \cdots + a_{3(p+1)} z^2 x^p + \cdots + a_{(q+1)(p+1)} z^q x^p$$  (E.1)

which contain terms of $p$ th order in $X$ and $q$ th order in $z$, and has $((q+1)(p+1))$ coefficients. As before we choose to find the power series so that the approximation $y(x_i, z_i)$ is close to the measured data $y_i$ in the least squares sense, e.g. we seek to minimize

$$E = \sum_{i=1}^{M} [y(x_i, z_i) - y_i]^2.$$  (E.2)

To find the coefficients, substitute the data into (E.1) and form a set of $M$ equations. If we write

$$X_i = \begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 & \cdots & x_i^p \end{bmatrix},$$  (E.3)
then this set of equations can be written

\[
\begin{bmatrix}
  y(x_1) \\
  y(x_2) \\
  \vdots \\
  y(x_M)
\end{bmatrix}
= \begin{bmatrix}
  X_1 & z_1X_1 & z_1^2X_1 & \cdots & z_1^qX_1 \\
  X_2 & z_2X_2 & z_2^2X_2 & \cdots & z_2^qX_2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  X_M & z_MX_M & z_M^2X_M & \cdots & z_M^qX_M
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_N
\end{bmatrix}
\]  

(E.4)

where \( N = (p + 1) \cdot (q + 1) \), or in matrix notation

\[
Y = \Gamma A.
\]  

(E.5)

This equation is solved for \( A \) using the \( QR \) decomposition as explained in Appendix D. As is the case for single variable power series, finding the coefficients of a bivariate power series is equivalent to solving a system of linear equations.
Figure 4.3.3: The measured dc gate current as a function of gate voltage for an Avantek AT-8250 GaAs MESFET. Curve (a) is for the source and drain connected to ground, (b) is for the source connected to ground and the drain floating, and (c) is for the drain connected to ground and the source floating.
Figure 4.3.4: The measured dc drain-source current as a function of the gate-source voltage for an Avantek AT-8250 GaAs MESFET. The drain-source voltage is 0.05 V.