Finite Word Length Analysis of the LMS Adaptive Algorithm

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Abstract

The roundoff error noise power is derived as a function of the loop gain in the Least Mean Square (LMS) adaptive algorithm. A statistical approach using vector and matrix operations is used which lends itself to the roundoff error analysis of other adaptive filter algorithms. The results show that for small loop gain the roundoff noise increases inversely to the loop gain. For large loop gains the error power is proportional to loop gain. These resutls agree with previously published results [4,5].

Introduction

The Least Mean Square (LMS) algorithm is widely used in adaptive filters in applications ranging from adaptive equalization, echo cancellation, noise cancellation to adaptive sidelobe cancellation in an antenna array [1]. Currently, echo cancellers based on the LMS algorithm have been built using VLSI and LSI for real time applications [2,3]. However, until recently the finite wordlength implementation issues of the algorithm had not been studied. Two papers treated this problem and derived results showing the roundoff noise power as a function of loop gain [4,5]. The results, which show that roundoff noise increase inversely with loop gain for small values of loop gain, underline the importance of analytical results in the finite word-length implementation of algorithms. Simulations results, while useful, cannot be generalized and are often computationally expensive. Furthermore, analytical results lead to design issues and point to trade offs in implementation.

In this paper, a straightforward statistical approach will be used to derive the roundoff noise power for the LMS algorithm using vector and matrix operations. This general approach can be used to derive analytical results for other adaptive algorithms as well. The results obtained agree with those presented in [4] and [5]. In fact, the approach used here is similar to [4] in some aspects. In this paper we are interested in a methodology for the theoretical analysis of finite wordlength effects in adaptive algorithms.

1. Derivation of Roundoff Noise Power as a Function of Loop Gain

Consider the LMS adaptive algorithm described by the following coefficient update equation:

$$\underline{a}(n) = \underline{a}(n-1) + \alpha\, \underline{x}(n)\, e(n) \tag{1}$$

where

$$e(n) = y(n) - \hat{y}(n) \tag{2}$$

and

$$\hat{y}(n) = \underline{x}^T(n)\underline{a}(n-1) \tag{3}$$

In the above equations, $\underline{x}(n)$ is the vector containing delayed elements of the scalar input to the system $x(n)$, $y(n)$ is the scalar output of the system, and $\underline{a}(n)$ are the state estimates of the system at each iteration n. The loop gain is denoted by α. For example, $\underline{a}(n)$ may be the estimate of the truncated impulse response of the system.

An implementation of the algorithm is shown in figure 1. For a finite word length implementation of the algorithm, roundoff or truncation errors occur after multiplications. In a fixed point implementation, these errors can be modelled as additive noise after an infinite precision multiplication. The FW implementation is illustrated in figure 3. Figure 4 illustrates the additive nature of roundoff and truncation errors and also shows their density distributions [6]. From figure 3, the FW LMS update equations become

$$\underline{a}(n) = \underline{a}(n-1) + \alpha\underline{x}(n)e(n) + \underline{\mu}(n) \tag{4}$$

$$\hat{y}(n) = \underline{a}^T(n-1)\,\underline{x}(n) + \sum_{i=0}^{N-1} \varepsilon_i(n) \tag{5}$$

where

$$\underline{\mu}(n) = [\mu_0(n)\ \mu_1(n)\ \ldots\ \mu_{N-1}(n)]^T \tag{6}$$

For roundoff arithmetic

$$E[\mu_i(n)\mu_i(m)] = \sigma_\mu^2\ \delta mn \tag{7}$$

$$E[\varepsilon_i(n)\varepsilon_i(m)] = \sigma_\varepsilon^2\ \delta mn \tag{8}$$

$$E[\mu_i(n)] = 0 \tag{9}$$

$$E[\varepsilon_i(n)] = 0 \tag{10}$$

Furthermore, the error sequences $\mu_i(n)$ and $\varepsilon_i(n)$ are uncorrelated with the data and estimated coefficients. The quantities σ_μ^2 and σ_ε^2 depends on the register length and scaling factors used in the actual implementation. Their values will be derived later as a function of register length and scaling factors.

We will assume that the process y(n) is generated by

$$y(n) = \underline{a}^{*T}\underline{x}(n) \quad . \tag{11}$$

The vector $\underline{a}^*$ is to be estimated from the sequence x(n) and y(n) by the LMS algorithm (4). The estimate error in the finite word length implementation becomes using (5) and (11),

$$e'(n) = \underline{a}^{*T}\underline{x}(n) - \underline{a}^T(n-1)\,\underline{x}(n) - \sum_{i=0}^{N-1} \varepsilon_i(n) \tag{12}$$

where the prime denotes the FWL implementation. Substituting (12) into (4) we obtain

$$\underline{a}(n) = \underline{a}(n-1) + \alpha\underline{x}(n)\underline{x}^T(n)\underline{a}^* - \alpha\underline{x}(n)\underline{x}^T(n)\underline{a}(n-1) - \alpha \sum_{i=0}^{N-1} \varepsilon_i(n)\ \underline{x}(n) + \underline{\mu}(n)\ . \tag{13}$$

If we change variables such that

$$\underline{b}(n) = \underline{a}(n) - \underline{a}^* \tag{14}$$

Then substracting a* from both sides of (13) we can write,

$$\underline{b}(n) = \underline{b}(n-1) - \alpha\underline{x}(n)\underline{x}^T(n)\underline{b}(n-1) - \alpha\underline{x}(n) \sum_{i=0}^{N-1} \varepsilon_i(n) + \underline{\mu}(n) \tag{15}$$

which can be written,

$$\underline{b}(n) = [I-\alpha\underline{x}(n)\underline{x}^T(n)]\underline{b}(n-1) + \underline{w}(n) \tag{16}$$

where we have made the substitution

$$\underline{w}(n) = \underline{\mu}(n) - \alpha\underline{x}(n) \sum_{i=0}^{N-1} \varepsilon_i(n) \quad . \tag{17}$$

Now, if at n=0, the initital condition of the state estimate is b(0) then from (16) we can derive the exact relationships,

$$\underline{b}(n) = \prod_{i=1}^{n} [I-\alpha\underline{x}(i)\underline{x}^T(i)]\underline{b}(0) + \sum_{i=1}^{n} \prod_{j=i+1}^{n} [I-\alpha\underline{x}(j)\underline{x}^T(j)]\underline{w}(i) \;. \tag{18}$$

The term

$$\psi(n) = \sum_{i=1}^{n} \prod_{j=i+1}^{n} [I-\alpha\underline{x}(j)\underline{x}^T(j)]\underline{w}(i) \tag{19}$$

represents the deviation of the estimate vector b(n) in the finite wordlength implementation from the infinite precision estimate represented by the first term in the RHS of (18).

Thus $\underline{b}(n)=\underline{b}_0(n)+\psi(n)$ from which using (14) $\psi(n)=\underline{a}(n)-\underline{a}_0(n)$ where $a_0(n)$ is the infinite precision vector estimate. We are interested in the matrix

$$\Psi = E\{\psi(n)\psi^T(n)\} \tag{20}$$

from which we can obtain the variance of the steady state error of the finite wordlength coefficients. Taking the transpose of (19) we obtain,

$$\psi^T(n) = \sum_{i=1}^{n} \underline{w}^T(i) \prod_{j=i+1}^{n} [I-\alpha\underline{x}(j)\underline{x}^T(j)] \quad . \tag{21}$$

Thus,

$$\psi(n)\psi^T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=i+1}^{n} [I-\alpha\underline{x}(k)\underline{x}^T(k)]\underline{w}(i)\underline{w}^T(j) \prod_{m=j+1}^{n} [I-\alpha\underline{x}(m)\underline{x}^T(m)]. \tag{22}$$

Prior to taking the expectation of (22) we calculate,

$$R_x = E\{\underline{x}(k)\underline{x}^T(k)\} \tag{25}$$

where R_x is the autocorrelation matrix of the input data and

$$R_w = E\{\underline{w}(k)\underline{w}^T(k)\} \tag{26}$$

If x(i) is a sequence of zero mean uncorrelated random variables with variance $\sigma_x{}^2$ then

$$R_x = \sigma_x^2 I \tag{26}$$

and since $\mu_i(n)$ and $\varepsilon_i(n)$ are uncorrelated with x(n) we obtain from (17)

$$R_w = \sigma_\mu^2 I + \alpha^2 \sigma_x^2 N \sigma_\varepsilon^2 I \tag{27}$$

where

$$\sigma_\mu^2 = E\{\mu_i^2(n)\} \tag{28}$$

$$\sigma_\varepsilon^2 = E\{\varepsilon_i^2(n)\} \tag{29}$$

Define

$$\sigma_w^2 = \sigma_\mu^2 + \alpha^2 \sigma_x^2 N \sigma_\varepsilon{}^2 \tag{30}$$

Then $R_W = \sigma_w^2 I$

Furthermore,

$$E\{\underline{w}(i)\underline{w}^T(j)\} = \phi \text{ for } i \neq j$$

Also, we observe from (17) that $\underline{w}(n-1)$ is independent of $\underline{x}(n)$. Therefore, taking the expectation of both sides of (22) we obtain

$$\Psi = \sigma_w^2 \sum_{i-1}^{n} \prod_{k=i+1}^{n} E\{[I-\alpha\underline{x}(j)\underline{x}^T(j)][I-\alpha\underline{x}(j)\underline{x}^T(j)]\}$$

where the terms $i \neq j$ drop out based on (30).

$$\Psi = \sigma_w^2 \sum_{i-1}^{n} \prod_{k=i+1}^{n} E[I-\alpha\underline{x}(j)\underline{x}^T(j)-\alpha\underline{x}(j)\underline{x}^T(j)+\alpha^2\underline{x}(j)\underline{x}^T(j)x(j)\underline{x}^T(j)]$$

To evaluate the fourth moments

$$E\{\underline{x}(k)\underline{x}^T(k)\underline{x}(k)\underline{x}^T(k)\}$$

we note that, in general, if z_1, z_2, z_3, z_4 are real zero mean Gaussian random variables, then [7]

$$E\{z_1z_2z_3z_4\} = E[z_1z_2]E[z_3z_4] + E[z_1z_3]\ E[z_2z_4] + E[z_1z_4]E[z_2z_3]$$

In [8] this result is extended from scalars to vectors and the fourth moments are found to be

$$E\{\underline{x}(k)\underline{x}^T(k)A\underline{x}(k)\underline{x}^T(k)\} = 2R_xAR_x + R_xTr\{R_xA\}$$

where A is a positive definite matrix and $R_x = E\{\underline{x}(k)\underline{x}^T(k)\}$.

Substituting A=I and $R_x=\sigma^2{}_xI$ we obtain

$$E\{\underline{x}(k)\underline{x}^T(k)\underline{x}(k)\underline{x}^T(k)\} = 2\sigma^4{}_xI + \sigma^4{}_xNI$$

Thus

$$\Psi = \sigma_w^2 \sum_{i=1}^{n} \prod_{k=i+1}^{n} [I-2\alpha R_x+\alpha^2 R_x \text{Trace } R_x+2\alpha^2\sigma_x^4]$$

$$\Psi = \sigma_w^2 \sum_{i-1}^{n} (1-2\alpha\sigma_x^2 + \alpha^2 N\sigma_x^4 + 2\alpha^2\sigma_x^4)^{n-i}I$$

Let k=n-i. Then changing the summation variables leads to

$$\sigma_w^2 \sum_{k=n-1}^{0} (\bullet)^k I$$

Now, since $|1-2\alpha\sigma_x^2+\alpha^2(N+2)\sigma_x^4|<1$ for stable values of α we can use the identity,

$$\frac{1}{1-\varepsilon} = 1+\varepsilon+\varepsilon^2+\ldots |\varepsilon|<1$$

to replace the summation series by

$$\sum_{k=0}^{n-1} (1-2\alpha\sigma_x^2 \ \alpha^2 N\sigma_x^4)^k = (2\alpha\sigma_x^2-\alpha^2 N\sigma_x^4)^{-1} \quad n \to \infty \tag{31}$$

The above procedure implies a condition for stability namely,

$$\alpha < \frac{2}{\sigma_x^2(N+2)}$$

Substituting into (31) and replacing $\sigma^2{}_w$ gives

$$\Psi = (\sigma_\mu^2 + \alpha^2 N \sigma_x^2 \sigma_\varepsilon^2) \cdot [2\alpha\sigma_x^2 - \alpha^2 (N+2)\sigma_x^4]^{-1} I \quad (32)$$

or, finally,

$$\Psi = \left[\frac{\sigma_\mu^2}{\sigma_x^2} \cdot \frac{1}{\alpha} + \alpha N \sigma_\varepsilon^2 \right] [2 - \alpha(N+2)\sigma_x^2]^{-1} I \quad (33)$$

The above analysis is valid for rounding arithmetic. For truncations, the truncation error is not independent of the multiplication result and hence the derivation is not as straightforward.

Now,

$$e(n) = y(n) - \hat{y}(n)$$

$$e'(n) = y(n) - \hat{y}'(n)$$

$$\zeta(n) = \hat{y}'(n) - \hat{y}(n) = \underline{a}^T(n)\underline{x}(n) + \sum_{i=0}^{N-1} \varepsilon_i(n) - \underline{a}^T(n)\underline{x}(n)$$

$$\zeta(n) = \underline{\psi}^T(n)\underline{x}(n) + + \sum_{i=0}^{N-1} \varepsilon_i(n)$$

where we have used the definition of $\psi(n)$, i.e.,

$$\psi(n) = \underline{a}(n) - \underline{a}'(n).$$

We now proceed to derive the variance of the error between the FWL implementation prediction error $e'(n)$ and the infinite precision error $e(n)$.

We define,

$$\zeta(n) = e(n) - e'(n) \quad (34)$$

Now,

$$\sigma_\zeta^2 = E[\zeta^2(n)] = E\{\underline{x}^T(n)\underline{\psi}(n)\underline{\psi}^T(n)\underline{x}(n)\} + N\sigma_\varepsilon^2 \quad (35)$$

$$= \text{Trace } E[\underline{x}(n)\underline{x}^T(n)\underline{\psi}(n)\underline{\psi}^T(n)] + N\sigma_\varepsilon^2 \quad (36)$$

$$= \text{Trace } [R_x\Psi] + N\sigma_\varepsilon^2 \quad (37)$$

$$= N_{\sigma_x^2}\left[\frac{\sigma_\mu^2}{\sigma_x^2} \cdot \frac{1}{\alpha} + \alpha N\sigma_\varepsilon^2\right] \cdot [2 - \alpha(N+2)\sigma_x^2]^{-1} + N\sigma_\varepsilon^2 \quad (38)$$

Finally

$$\sigma_\zeta^2 = [N\sigma_\mu^2 \cdot \frac{1}{\alpha} + \alpha N^2\sigma_x^2\sigma_\varepsilon^2] \cdot [2-\alpha(N+2)\sigma_x^2]^{-1} + N\sigma_\varepsilon^2 \qquad (39)$$

Note that if rounding of the prediction $\hat{y}(n)$ occures after accumulation, the result becomes

$$\sigma_\zeta^2 = \frac{1}{2} [N\sigma_\mu^2 \cdot \frac{1}{\alpha} + \alpha N\sigma_x^2 \sigma_\varepsilon^2] \cdot [2-\alpha(N+2)\sigma_x^2]^{-1} + \sigma_\varepsilon^2 \qquad (40)$$

For α small we have

$$\sigma^2 = \frac{1}{2} N\sigma_\mu^2 \cdot \frac{1}{\alpha} + \sigma_\varepsilon^2 \qquad (41)$$

2. Simulation Results

The LMS algorithm was simulated in an echo cancellation application where a 9th order FIR impulse response was estimated. Two cases were studied. For the first case the roundoff error in the convolution equation ε was made small in order to analyse the effects of the gain α on the error introduced by roundoff quantization. In particular we are interested in checking the result

$$\sigma_\zeta^2 = \frac{N\sigma_\mu^2}{\alpha}$$

where we have neglected σ_ε^2 by making it very small. The results are as follows (5000 iterations, white gaussian input $\sigma_x^2 = 1.0$ ε=20 bits)

α	μ(bits)	σ_ζ^2
0.1	8	3.186×10^{-5}
0.05	8	1.467×10^{-5}
0.01	8	3.842×10^{-4}

The results clearly indicate that the mean error squared due to roundoff increases inversely to the loop gain as predicted.

In the second case, σ_μ^2 was made very small (μ=24 bits) while α was varied. The results were as follows (5000 iterations white gaussian noise σ_x^2=1.0, μ=24 bits)

α	ε(bits)	$\sigma_\zeta{}^2$
0.1	8	3.449×10^{-5}
0.05	8	1.322×10^{-5}
0.025	8	1.084×10^{-5}

In this case the error decreases as the loop gain is decreased. For α large, the mean error squared is proportional to α.

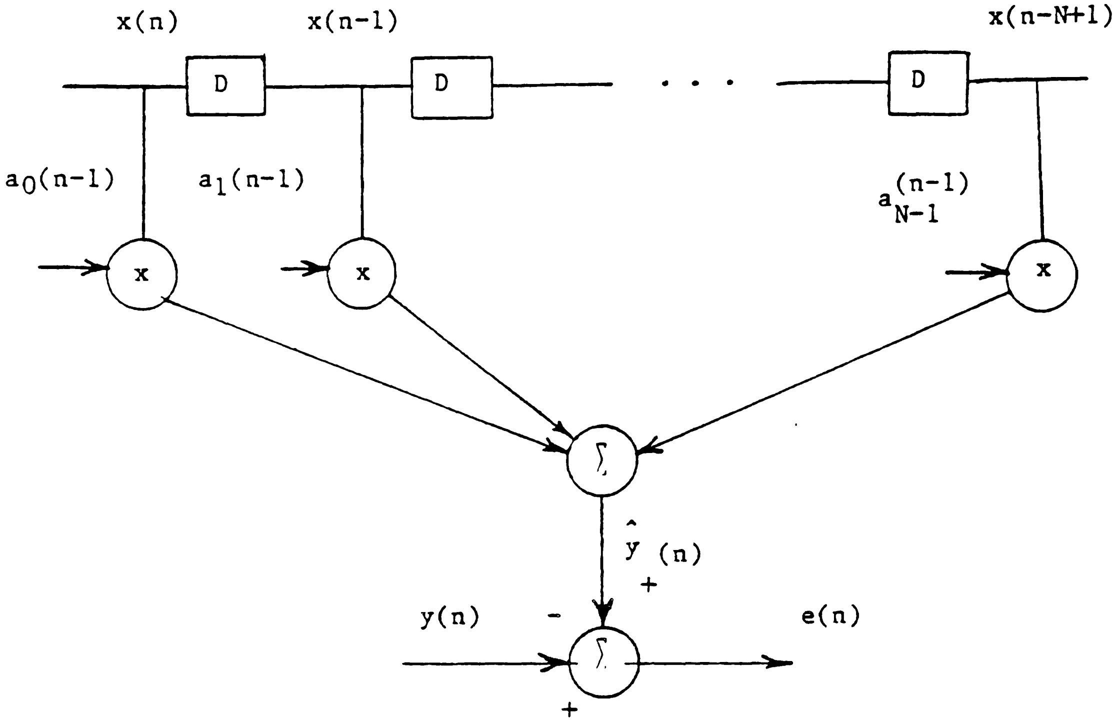


Figure 1. LMS Algorithm Structure

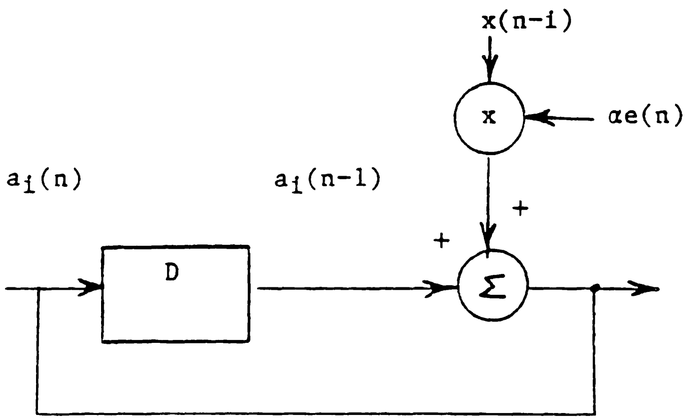


Figure 2. Coefficient Update Structure

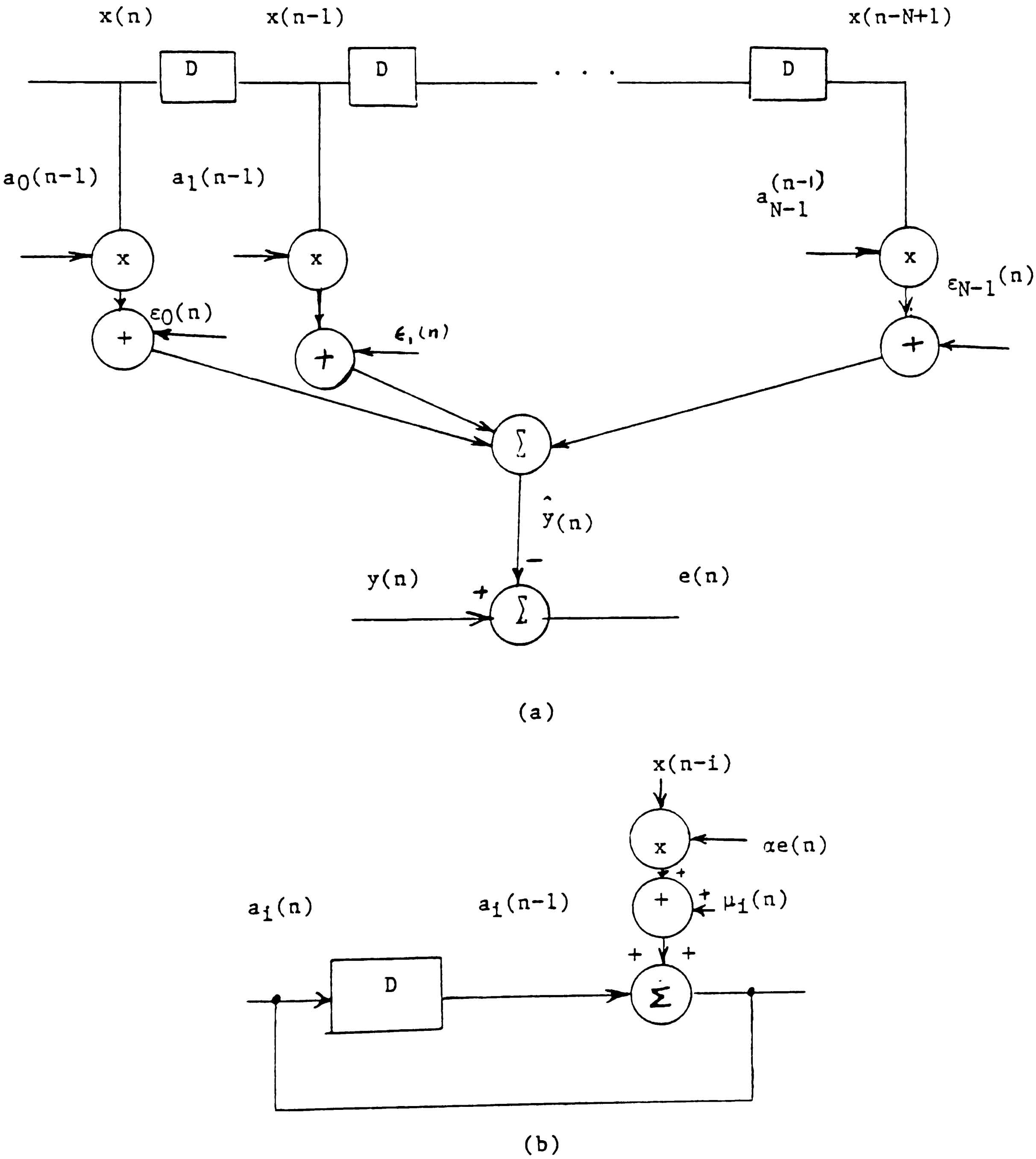
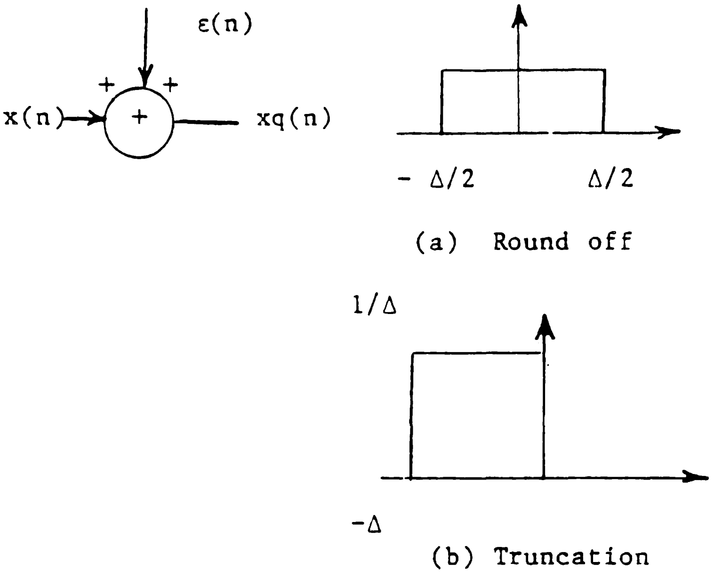


Figure 3. Finite Word Length LMS Algorithm Structure.



(a) Round off:

$$E[\varepsilon(n)] = 0$$

$$E[\varepsilon(n)\ (m)] = \frac{\Delta^2}{12}\ \delta mn$$

$$E[x(n)\varepsilon(n)] = 0$$

(b) Truncation:

$$E[\varepsilon(n)] = \Delta/2$$

$$E[\varepsilon(n)\varepsilon(m)] = \frac{\Delta^2}{12}\ \delta mn$$

$$E[x(n)\varepsilon(n)] = \Delta/2\ E(x(n)]$$

Figure 4: Fixed Point Roundoff and Truncation Error Models

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