

**The Use of Time Domain Information
to
Improve Transform Coding**

by

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December 1984

CCSP-TR-84/16

Revised

April 1985

1. Abstract

In time-varying image coding as well as in some still-frame image coding, pixel domain information can often be transmitted with relatively little bandwidth requirement. This is the case when there are large blocks of homogeneous intensity or, as in time-varying images, blocks of unchanged pixels. This information can be combined with transform coding techniques to produce images with reduced mean square error at the receiver.

This paper presents two methods of utilizing the pixel domain information given the constraint of fixed size transforms in the coding system. The first uses the method of successive projections to find an image which satisfies the transform and pixel domain constraints. The second allows known pixel information to behave as "don't cares" in the transform coder and utilizes the method of successive projections for two nonintersecting convex sets to find the optimal transform coefficients.

2. Introduction

Transform coding offers a simple and robust method of transmitting time-varying images (TVI) with reduces bandwidth requirements. Although it is usually not advantageous to also transmit single pixel value information, it is often the case that much time domain information can be transmitted with little additional bandwidth. This situation exists when there are blocks of unchanged information or blocks of uniform intensity values. If an image is to be transform coded in fixed sized blocks, segments of the pixel domain

information indicated above, referred to as the region of support (ROS) may extend into many of these fixed blocks. The ROS can then be used to reduce the mean square error in the received signal or to maintain a given error with decreased bandwidth. Further, by combining pixel and frequency domain information, it may be possible to attain better subjective quality.

In this paper two methods are presented which combine information in the pixel and frequency domains. It is assumed that these methods are to be employed on a system which uses transform coding on fixed sized blocks; therefore, the procedures are designed to utilize the existing transform hardware or software on the system. It should be noted here that these techniques, although originally formulated with image sequences in mind, are applicable to one dimensional signals since the problems are formulated in the vector space R^n . Further, each method may be incorporated into zonal or threshold coding systems.

3. Projection onto Convex Sets

By utilizing the method of successive projections [1], a signal can be found at the receiver which has the proper pixel domain information as indicated by the ROS as well as the transmitted transform coefficients. Assuming that the signals to be transmitted are vectors in R^n , all vectors with the same ROS describe a convex set in R^n as follows:

$$S_R = \{x \in R^n \mid x_i = k_i, i \in I_R\} \quad (1)$$

Where x_i is the i 'th component of x , k_i is a constant, and I_R is an

index set which indicates the known pixel values, the ROS. Since the set S_R is closed and convex, the constrained minimization problem, minimize $\|x - y\|$ subject to $y \in S_R$ has a unique solution. This solution is the projection of x onto S_R indicated by $y = P_R(x)$ where

$$y_i = \begin{cases} x_i, & i \text{ not } \in I_R \\ k_i, & i \in I_R \end{cases} \quad (2)$$

The transmitted transform coefficients also form a closed convex set, S_T , in a similar fashion.

$$S_T = \{x \in R^n \mid (Fx)_i = c_i, i \in I_T\} \quad (3)$$

Where F is a transform, e.g. DFT or DCT, and I_T is the index set indicating the transmitted coefficients. The projection onto S_T , $y = P_T(x)$, is given by

$$(Fy)_i = \begin{cases} (Fx)_i, & i \text{ not } \in I_T \\ c_i, & i \in I_T \end{cases} \quad (4)$$

Then $y = F^*y$ where F^* is the inverse transform.

By the successive projections theorem, the iterative procedure,

$$x_{k+1} = P_R P_T(x_k), \quad x_0 \in R^n \quad (5)$$

will converge to a vector in the intersection of S_T and S_R . Although the fixed point of the procedure is only guaranteed to be in the intersection, since S_R and S_T are subspaces, the fixed point also satisfies the minimization problem given by:

$$\text{minimize } \|x_0 - x\|$$

$$\text{subject to } x \in S_{\mathbb{R}} \cap S_{\mathbb{T}}$$

(6)

This technique may take many iterations to converge to the intersection, but it gives an improved solution after each iteration, and experiments show five to ten iterations to be a good stopping point in many cases. The convergence rate and error can also be affected by the initial estimate, order of projection and limited changes in the projection operator. This technique can easily incorporate new convex sets by adding their projections to the iterative scheme described by (5).

Use of the method of successive projections defined in this section can be compared with an optimal solution in the least squares sense by viewing the transform coefficients and the ROS together as basis vectors for \mathbb{R}^n . The ROS essentially defines coefficients for standard basis vectors for pixels in the ROS. A standard basis vector, e_i , contains a one in the i 'th position and zeroes elsewhere. The transform coefficients which are to be sent are normally given by $\langle x, \varphi_i \rangle$ where x is the desired signal, φ_i is a transform basis vector, and \langle, \rangle is the inner product. For the optimum solution, the set of φ_i for which a coefficient is to be transmitted should be extended from the e_i , $i \in I_{\mathbb{R}}$, via Gram-Schmidt orthogonalization to a new set ψ_i , and then $\langle x, \psi_i \rangle$ should be transmitted. The ROS vectors and the transform bases chosen should be linearly independent, however. If they are not, the system will still operate successfully since the extension of a linearly dependent φ will simply be the zero vector. Alternatively, the coeffi-

coefficients could be found by solving the normal equations.

Neither of these procedures, Gram-Schmidt or solving the normal equations, is easily incorporated into an existing transform coding system. The projection method, on the other hand, can give an improved estimate using the existing system and an additional frame buffer containing the ROS. This could be a previous-frame buffer as may be needed in a motion compensation system.

A drawback with the techniques described in the previous section is that they don't take full advantage of the known ROS. That is, both methods attempt to minimize the error between the received signal and the entire desired signal regardless of the fact that the ROS pixels will be assigned their exact values at the receiver. Further, iteration is required by the receiver and transmitter. The next section introduces an algorithm which assigns optimum values to the ROS pixels by considering them "don't cares", a term borrowed from digital logic design, and will alleviate the problems just mentioned.

4. Using Don't Cares in the Region of Support

Since components of the signal to be transmitted which are in the ROS will be assigned their exact values at the receiver, they become free variables or "don't care" elements when performing the transform at the transmitter. Therefore, the transmitter can assign values to these components so that when the transform is performed and only a subset of the coefficients are transmitted, the signal recovered by the receiver has the error minimized only in the non-

ROS components. It should be noted at this time that this algorithm is for use with transform coders which operate on fixed size blocks. If a variable number of noncontiguous pixels are allowed to be transformed coded, then the ROS pixels can be left out since they simply constitute unneeded constraints in a system of linear equations.

This reasoning may be stated as a constrained minimization problem as follows:

$$\begin{aligned} & \text{minimize } ||A(Hx - x)|| \\ & \text{subject to } Ax = b \end{aligned} \tag{7}$$

Where H is a map from R^n to R^n , which transforms, filters, then inverse transforms. Hx is actually the received signal. For example, $H = F*LF$, where F and F^* are the transform and its inverse, and L is a low pass filter. A is a map from R^n to R^k , $k < n$, which forms a vector whose components are those not in the ROS. That is, A is a selection operator which alters the cost function to only measure the error in the non-ROS components. The b is a constant vector containing the non-ROS values. The constraint equation forces the the non-ROS components of the solution to be correct while the ROS are allowed to vary. (Note that the solution is not the received signal, but the signal to transform coded at the transmitter.) This constrained problem does not appear to have an easy solution, but a simpler model which leads to a better solution is given next.

On closer examination, there is no need to force the solution to retain exact values in the non-ROS components, only that the received signal has minimum error in this region. This may be

stated as the following unconstrained minimization problem:

$$\text{minimize } ||AHx - b|| \quad (8)$$

Where A,H, and b are as before. This also involves solving a generalized inverse type of problem; however, an iterative solution can be obtained by applying the method of successive projections for non-intersecting convex sets. The theorem which is applied and a proof are given in the appendix. The theorem was first presented and proven by Cheney and Goldstein [6]. A less formal, and hopefully more informative, description follows.

Assume a transform coder is available which transmits m out of n coefficients. Without loss of generality, it can be assumed that these are the m low pass coefficients and that the receiver will assign zeros to the untransmitted coefficients. Define a set, C_F , to be all $x \in R^n$ such that the m low pass transform coefficients are bounded in magnitude, and the m-n remaining coefficients have assumed values, typically zero. C_F is a closed bounded convex set. The bounds for the m transmitted coefficients can be any two real numbers reasonable for the application. The m-n coefficients need not be zero, but the values which they are assumed to have must be known by the transmitter and the receiver.

Define another set, C_p , to be all $x \in R^n$ such that x has correct values in the non-ROS components, and that the remaining components are bounded. C_p is also closed, bounded and convex, and the bounds for the ROS components can be any real numbers reasonable for the application.

Any received signal which is a member C_P would be acceptable since the ROS will be assigned exact values. Further, any signal in C_F can be transmitted. The problem, then, is to find a signal in C_F which is closest to C_P . This idea can be more readily visualized by referring to figure 1.

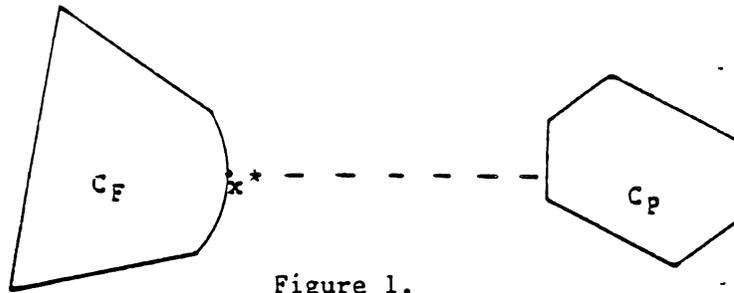


Figure 1.

The signal x^* in figure 1 can be found by using the method of successive projections as described in the appendix. That is, let P_F and P_P be the projectors onto C_F and C_P , respectively. Then the iterative procedure,

$$x_{k+1} = P_F P_P(x_k)$$

will converge to a vector in C_F closest to C_P (x^* in the case of figure 1). This fixed point actually minimizes the function

$$f(x) = ||x - P_P(x)||, \text{ for all } x \in C_F$$

It should be noted that x^* may not be unique and $C_F \cap C_P$ need not be empty. Also, in the appendix, x_0 is assumed to be in C_F , however, any $x_0 \in \mathbb{R}^n$ will suffice.

5. Discussion

Preliminary tests utilizing line-by-line transform coding and the non-intersecting set method give results which vary from little

error reduction to 50% reduction per line. This improvement was obtained with ten or fewer iterations per line. Further testing is in progress.

The application described in this paper is not the only one for which the technique is viable. For example, when transmitting positive signals, zero valued pixels could be allowed to take on any value less than or equal to zero. The receiver would then assign zero to negative pixels.

Another possibility is to allow various segments of an image to take on any pixel value within a range if that particular segment need not be transmitted with as much accuracy as other segments. This allows the decision to be implemented in the pixel domain much as high motion areas may be transmitted with fewer frequency coefficients. This case arises since human observers require less resolution in these areas.

In general, the non-intersecting method may be used whenever the transmittable signals and the acceptable signals form closed convex sets. The paper shows that the algorithm can produce good results in relatively few steps and can be implemented with a parallel architecture by operating concurrently on separate blocks.

Additional research which compares the use of alternating projections for intersecting sets with this new technique is still needed. Criteria for deciding which performs better in various situations and under what conditions are not yet defined. There are cases, however, when both methods do not apply, such as when the only the transmitter has knowledge of which pixels are allowed to be

don't cares. This discussion has dealt with only two nonintersecting sets. There may be usefulness for more than two sets, and this idea is mentioned in the appendix.

6. Appendix

The method of successive projections is used to locate a vector in the intersection of closed convex sets [1]. This appendix deals with using successive projections for closed bounded convex sets whose intersection is empty. The theorem presented shows that the fixed point reached minimizes the distance between the set for which the fixed point is determined and one other nonintersecting set. Before the basic theorem is given, a number necessary lemmas are presented. Some of the proofs or similar ones can be found in Youla's work [5] which provides a good reference for this paper. The main theorem and its proof is given in [6].

Let C_1, C_2, \dots, C_m be closed bounded convex subsets of R^n , and P_1, P_2, \dots, P_m be the projectors onto these sets. That is, given $x \in R^n$ and $P_i: R^n \rightarrow C_i$ (P_i maps R^n into C_i), $P_i x$ minimizes $\|x-y\|$ for all $y \in C_i$. This minimum is guaranteed to exist [2].

Lemma 0. If P is the projector onto the closed convex set C , then the following statements are true.

$$\|x-Px\| \leq \|x-y\| \text{ for all } y \in C \text{ and,}$$

$$\langle x-Px, y-Px \rangle \leq 0 \text{ for all } y \in C$$

Proof: [2].

Lemma 1. Let P be a projector onto a closed convex set C , then P is nonexpansive. That is,

$$\|Px - Py\| \leq \|x - y\| \text{ for all } x, y \in \mathbb{R}^n$$

Proof: By lemma 0 we can write the following [2]:

$$\langle x - Px, Py - Px \rangle \leq 0$$

$$\langle y - Py, Px - Py \rangle \leq 0$$

Add the above two

$$\langle x - Px, Py - Px \rangle + \langle y - Py, Px - Py \rangle \leq 0$$

Rearrange terms

$$\langle x - Px - y + Py, Py - Px \rangle \leq 0$$

$$\langle Py - Px, Py - Px \rangle + \langle x - y, Py - Px \rangle \leq 0$$

$$\|Py - Px\|^2 \leq \langle x - y, Py - Px \rangle$$

Applying Schwartz's inequality

$$\|Py - Px\|^2 \leq \|x - y\| \|Py - Px\|$$

Then, if $\|Py - Px\| \neq 0$

$$\|Py - Px\| \leq \|x - y\|$$

If $\|Py - Px\| = 0$, then the inequality in the lemma clearly holds.

It is now shown that the composition of a finite number of convex set projectors is nonexpansive. Let $T = P_m P_{m-1} \dots P_1$, P_i as before, then $T: C_m \rightarrow C_m$. An initial vector in C_m can always be found by applying P_m to any vector in \mathbb{R}^n .

Lemma 2. T_m is nonexpansive.

Proof (by induction): T_1 is nonexpansive by lemma 1. Assume T_{m-1} is nonexpansive.

$$\begin{aligned} \|T_m(x) - T_m(y)\| &= \|P_m T_{m-1}(x) - P_m T_{m-1}(y)\| \\ &\leq \|T_{m-1}(x) - T_{m-1}(y)\| \\ &\leq \|x - y\| \end{aligned}$$

This was gotten by applying lemma 1 and the inductive hypothesis.

Lemma 3. If C_m is a nonempty closed bounded convex set, then T_m , as defined above has at least one fixed point.

Proof: [5].

Lemma 4. The sequence $\{x_n\}$, where $x_{n+1} = T_m(x_n)$, converges strongly to x^* .

Proof: $T_m: C_m \rightarrow C_m$ and C_m is a closed and bounded subset of a finite dimensional Hilbert space. Therefore $\{x_n\}$ has a strongly convergent subsequence [3]. But $\{x_n\} \rightarrow x^*$ strongly if and only if $\{x_n\}$ has a subsequence which converges strongly [5].

The question still remains as to the significance of the fixed point reached. Further, since there may be more than one fixed point, do all the fixed points have the same significance?

Define the functional $f: R^n \rightarrow R$ as follows:

$$f(x) = \sum_{i=1}^{m-1} \|x - P_i x\|^2, \quad P_i \text{ as before}$$

So f defines a kind of total distance from x to the $m-1$ convex sets $C_{m-1}, C_{m-2}, \dots, C_1$. If f is minimized for all $x \in C_m$, then, if x^* is this minimizing vector, $f(x^*)$ defines the minimum total distance from C_m to the remaining $m-1$ sets in the least squares sense given by the expression for $f(x)$. As presented in the main body of this paper, this was the exact problem, for $m=2$, which needed to be solved. If the intersection of the C_i is not empty, then $f(x^*)=0$, and an x^* can be found by the method of successive projections. It is reasonable to hope, then, that if the intersection of the C_i is empty, the fixed point reached by T_m minimizes f . This will be shown for two nonintersecting sets by the final theorem of this

appendix and is the direction taken now. In fact, the theorem is not true for $m > 2$ which can be shown by counterexample. In general this does not mean that the fixed point found is insignificant, and indeed may be a minimum under certain conditions and projection orders. The following lemmas will be given for any m , however, for the sake of generality.

Lemma 5. The function f defined above is convex. That is

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad 0 < \alpha < 1$$

Proof:

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= \sum_{i=1}^{m-1} \|\alpha x + (1-\alpha)y - P_i(\alpha x + (1-\alpha)y)\| \\ &\text{since } C_i \text{ is convex, } [\alpha P_i(x) + (1-\alpha)P_i(y)] \in C_i \\ &\text{we apply lemma 0} \\ &\leq \sum_{i=1}^{m-1} \|\alpha x + (1-\alpha)y - [\alpha P_i(x) + (1-\alpha)P_i(y)]\| \\ &= \sum_{i=1}^{m-1} \|\alpha(x - P_i(x)) + (1-\alpha)(y - P_i(y))\| \\ &\leq \sum_{i=1}^{m-1} \alpha \|x - P_i(x)\| + \sum_{i=1}^{m-1} (1-\alpha) \|y - P_i(y)\| \\ &= \alpha f(x) + (1-\alpha)f(y) \end{aligned}$$

Lemma 6. If x^* minimizes f , then x^* is a global minimum.

Proof: [2]

Lemma 7. The function f is continuous.

Proof: P_i is continuous which can be seen by the nonexpansive property of P_i . Since the norm is continuous [2], f is a continuous function of a continuous function and is therefore itself continuous.

Lemma 8. The function f achieves a minimum on C_m .

Proof: [2].

Theorem (for $m=2$). Let C_1 and C_2 be closed bounded convex subsets of a finite dimensional Hilbert space and $P_1: H \rightarrow C_1$, $P_2: H \rightarrow C_2$ be the respective projectors onto these sets. Define $f: C_1 \rightarrow \mathbb{R}$ as follows:

$$f(x) = \|x - P_1(x)\|^2$$

The vector x^* minimizes the functional f if and only if $x^* = T_2 x^* = P_2 P_1 x^*$. That is, if and only if x^* is a fixed point for T_2 .

Proof:

I. Show $x^* = P_2 P_1 x^*$ implies $f(x^*)$ is minimum.

Let y be any member of C_2 , and, for ease of notation, $x \equiv x^*$. By lemma 0 we get the following two expressions:

$$\langle x - P_1 x, P_1 y - P_1 x \rangle \leq 0$$

$$\langle P_1 x - P_2 P_1 x, y - P_2 P_1 x \rangle = \langle P_1 x - x, y - x \rangle \leq 0$$

add the above two expressions

$$\langle x - P_1 x, P_1 y - P_1 x \rangle + \langle P_1 x - x, y - x \rangle \leq 0$$

$$\langle x - P_1 x, P_1 y \rangle + \langle x - P_1 x, -P_1 x \rangle + \langle P_1 x - x, y \rangle + \langle P_1 x - x, -x \rangle \leq 0$$

$$\langle x - P_1 x, x - P_1 x \rangle + \langle x - P_1 x, P_1 y - y \rangle \leq 0$$

$$\|x - P_1 x\|^2 - \langle x - P_1 x, y - P_1 y \rangle \leq 0$$

by Schwartz's inequality

$$\|x - P_1 x\|^2 \leq \|x - P_1 x\| \cdot \|y - P_1 y\|$$

$$\|x - P_1 x\| \leq \|y - P_1 y\|$$

Therefore $x = x^*$ is a minimizer.

II. Show x^* minimizes $f(x)$ " implies " $x^* = P_2 P_1 x^*$.

$$\begin{aligned} \|P_2 P_1 x^* - P_1 P_2 P_1 x^*\| &\leq \|P_2 P_1 x^* - P_1 x^*\|, \text{ by lemma 0} \\ &\leq \|x^* - P_1 x^*\|, \text{ by lemma 0} \end{aligned}$$

If strict less than holds, then x^* is not minimum. Therefore, equality must hold, and, by the uniqueness of the projection, it can be concluded that $x^* = P_2 P_1 x^*$

7. References

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