

**A PERFORMABILITY ANALYSIS OF
A RECONFIGURABLE DUPLICATION SYSTEM**

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CCSP-TR-86/9

April 1986

Abstract

An N duplicate-server system, where each server consists of two reconfigurable duplicated units which are subject to breakdowns, is described. This system is analyzed using the generating function technique and also using numerical techniques. The generating function technique is difficult to generalize for $N \geq 2$. The numerical method is applicable for any value of N . The response time of the N duplicate-server system is compared with that of the multiple server system.

1. Introduction

In recent years, the interest in fault-tolerant computing systems has been increased because many computing systems require very high reliability and/or availability. For example, ultra-high reliability is the most important factor when designing an aircraft flight-control system and high availability is imposed on many commercial database systems (e.g., airline reservation systems).

There are several techniques to increase reliability and/or availability, such as fault-detection, masking redundancy, and dynamic redundancy. Several dynamic redundancy techniques utilize a combination of fault-detection, fault-masking, and reconfiguration. Among dynamic redundancy techniques, reconfigurable duplication and graceful degradation are of interest.

Reconfigurable duplication employs hardware comparators as shown in Figure 1. In this figure, two duplicated units are functioning in parallel, but only one of the duplicated units is connected to the system's output. When a fault is detected by a mismatch, there are several methods of determining the faulty copy and switching it out. Four methods are discussed in Siewiorek and Swarz [26], which are to run a diagnostic program when a mismatch occurs, to include self-checking capabilities on each module, to use a watchdog timer, and to use an outside arbiter.

Graceful degradation techniques use the redundant hardware as part of the system's normal resources at all times. For example, consider a gracefully degrading system with N identical processors which are subject to breakdowns. If the system is fault-free, then all processors are active and are able to process jobs concurrently. Each processor is self-testing, and in the presence of a single faulty processor the system is able to recover (with a specified coverage) to an $(N-1)$ -processor configuration, provided that $N \geq 2$. In this configuration, the system

behaves the same as a fault-free version of the system with $N-1$ processors. When only a single processor remains fault-free, fault recovery is no longer possible. Failure to recover from a processor fault results in a total loss of processing capability (system is down).

In studies of fault-tolerant systems, it is assumed that the coverage factor is known. In [26], two measurements of coverage are discussed. The first, called general coverage, is more qualitative. Usually, general coverage specifies the class of failures that are detectable, and may include failure detection percentage for different classes of failures. The second form of coverage is more explicit. It is the probability (denoted by c) that a failure (any failure) is detected. c could be determined from the general coverage specifications by using the average of the coverages for all possible classes of failures. In practice, c is difficult to obtain and indeed may not be known. In many instances, simplifying assumptions could allow us to estimate the coverage.

Since a gracefully degrading system is designed so that continued (though degraded) processing is possible in the presence of failures, performance varies widely over time. So, gracefully degrading systems cannot be modeled by a separation of reliability/availability measures from performance measures. In other words, a combined performance/reliability measure is necessary in order to evaluate gracefully degrading systems.

Such performance/reliability measures have been obtained by many authors using various approaches. Relevant studies include, for example, Beaudry [3] (performance-related reliability measures), Gay and Ketelsen [5] (performance evaluation of degradable systems), Mine and Hitayama [15] (job-related reliability), Castillo and Siewiorek [4] (performance-reliability models for computing systems), Osaki and Nishio [23] (reliability of information), Huslende [6] (combined performance-reliability evaluation for degradable systems), Meyer [13, 14] (performability), and Kulkarni, Nicola, and Trivedi [11, 12] (completion time of a job).

The performance/reliability of reconfigurable duplication systems has not been adequately studied. In view of this, we seek to develop analytic and numerical procedures for analyzing such systems. In this report, we study the performance/reliability analysis of fault-tolerant systems which use dynamic redundancy techniques involving reconfiguration. In section 2, we describe the system studied in this paper consisting of N duplicated servers. In section 3, we discuss how our model can be analyzed using existing solution methods. In the subsequent section, we compare the N duplicate-server system with a gracefully degrading system. Conclusions and extensions are given in section 5.

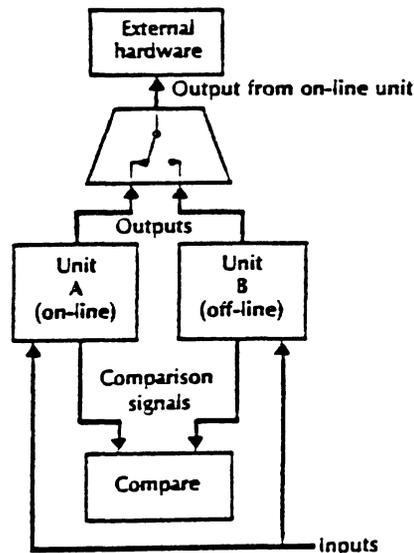


Figure 1. Reconfigurable Duplication

2. The N duplicate-server System

Let us first consider a gracefully degrading system consisting of N identical servers in parallel. The servers are subject to breakdowns. Failed servers are repaired immediately by one of N repair crews and are brought back to an operative state. A breakdown occurring while a job is being served results in preemption of the job. That is, the interrupted job is not lost but it joins the head of the queue, and it has to be restarted (preemptive-restart) whenever a server becomes available.

This model was analyzed by Mitrani and Avi-Itzhak [16], Meyer [14], Gay and Ketelsen [5], Munarin [18], Nakamura and Osaki [19, 24], Neuts and Lucantoni [20, 21], and Nicola, Kulkarni, and Trivedi [22]. In some of these papers [16, 20, 22] the exact solution with perfect coverage ($c = 1$) was obtained. The remaining papers [5, 14, 18, 19] were concerned with approximate solutions.

In this paper, we apply dynamic redundancy techniques involving reconfigurable duplication to the above-mentioned gracefully degrading system. This means that each of the N identical servers consists of two reconfigurable duplicated units as shown Figure 2. We call this new system *N duplicate-server system*.

The service mechanism of this system can be described as follows:

Service Mechanism

Any functioning unit of a server may become inoperative due to breakdown. In this case, service continues uninterrupted on its duplicate unit. The breakdown unit is switched out and it becomes operative again after it has been repaired. During the time that the breakdown unit is being repaired, the other duplicate unit may also break down. In this case, the server is completely broken down. The total number of servers is reduced by one. The interrupted job is not lost but it joins the head of the queue and it is restarted (preemptive restart) whenever a server

becomes available. The server becomes operative when one of its two duplicate units is repaired.

For this system, any repaired unit is assumed to be perfect. It is assumed that there is an infinite waiting room and that the service discipline is first-come-first-served (FCFS). We also assume that all relevant random variables such as the inter-arrival time, the service time, the time to failure, and the repair time of the server have exponential distribution with parameters λ , μ , p , and r , respectively.

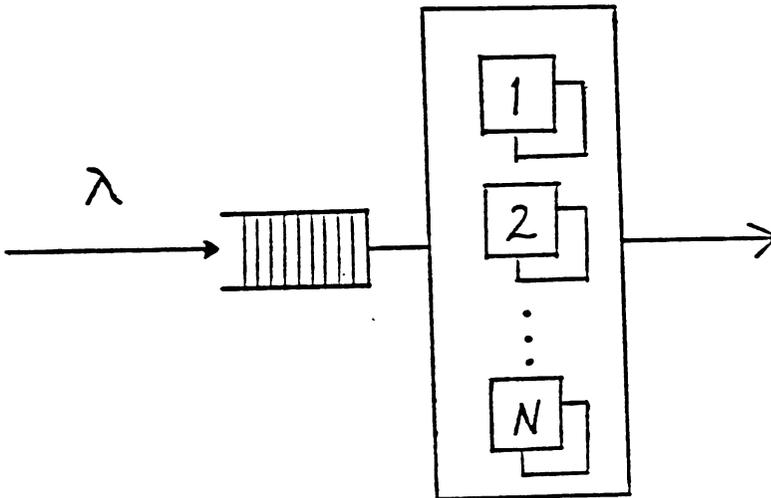


Figure 2. N Duplicate-server System

3. Solutions

We analyze the N duplicate-server system using the generating function technique and also using numerical techniques. The generating function technique is difficult to generalize for $N \geq 2$. The numerical method is applicable for any value of N . Below, we obtain the solution to the system using the generating function approach for a single duplicate-server system. In section 3.2 we describe how the matrix-geometric solution can be employed to analyze numerically N duplicate-server systems.

3.1. Generating Function Method

For simplicity, we only consider the case of perfect coverage ($c=1$). We denote the steady-state probability of the system being in state (s,n) by $P_{s,n}$, for $n = 0,1,2,\dots$, and $s = 0,1,2$. The steady-state balance equations of the system can be obtained easily from Figure 3:

$$\begin{aligned}
 (\lambda + 2p)P_{0,0} &= \mu P_{0,1} + rP_{1,0} \\
 (\lambda + \mu + 2p)P_{0,n} &= \lambda P_{0,n-1} + \mu P_{0,n+1} + rP_{1,n}, \quad \text{for } n \geq 1, \\
 (\lambda + p + r)P_{1,0} &= \mu P_{1,1} + 2pP_{0,0} + 2rP_{2,0} \\
 (\lambda + \mu + p + r)P_{1,n} &= \lambda P_{1,n-1} + \mu P_{1,n+1} + 2pP_{0,n} + 2rP_{2,n}, \quad \text{for } n \geq 1, \\
 (\lambda + 2r)P_{2,0} &= pP_{1,0} \\
 (\lambda + 2r)P_{2,n} &= \lambda P_{2,n-1} + pP_{1,n}, \quad \text{for } n \geq 1.
 \end{aligned} \tag{1}$$

Let the generating function $G_s(z)$, $|z| \leq 1$, $s=0,1,2$, be defined by

$$G_s(z) = \sum_{n=0}^{\infty} z^n P_{s,n}. \tag{2}$$

By Eq. (2), $G_s(1)$ is the steady-state probability that there are $(2-s)$ operative units.

Multiplying both sides of Eqs. (1) by z^n and summing over all n , we obtain the following equations:

$$\begin{aligned}
& [\lambda z(1-z) + 2pz - \mu(1-z)]G_0(z) - rzG_1(z) = -\mu(1-z)P_{0,0} \\
& -2pzG_0(z) + [\lambda z(1-z) + (p+r)z - \mu(1-z)]G_1(z) - 2rzG_2(z) = -\mu(1-z)P_{1,0} \\
& -pG_1(z) + [\lambda(1-z) + 2r]G_2(z) = 0.
\end{aligned} \tag{3}$$

Equation (3) can be written in matrix form,

$$A(z)g(z) = (1-z)b(z), \tag{4}$$

where

$$A(z) = \begin{bmatrix} \lambda z(1-z) + 2pz - \mu(1-z) & -rz & 0 \\ -2pz & \lambda z(1-z) + (p+r)z - \mu(1-z) & -2rz \\ 0 & -p & \lambda(1-z) + 2r \end{bmatrix},$$

$$g(z) = \begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix},$$

$$\text{and } b(z) = \begin{bmatrix} -\mu P_{0,0} \\ -\mu P_{1,0} \\ 0 \end{bmatrix}.$$

It is easy to show that $|A(z)|=0$ at $z=1$. We can write

$$|A(z)| = (1-z)Q(z), \tag{5}$$

where $Q(z)$ is a polynomial of degree 4 in z since $|A(z)|$ is a polynomial of degree 5 in z .

Equation (4) may now be rewritten in the form

$$Q(z)G_s(z) = |A_s(z)|, \quad s=0,1,2 \tag{6}$$

where $A_s(z)$ is obtained by replacing the $(s+1)$ st column of $A(z)$ with $b(z)$.

Let $G(z)$ denote the generating function of steady-state number of jobs in the system, which is given by

$$G(z) = \sum_{s=0}^2 G_s(z). \tag{7}$$

We have two unknown probabilities $P_{0,0}$ and $P_{1,0}$. From Eqs. (5) and (6), we have

$$P_{0,0} + P_{1,0} = \frac{(2p+r)r\mu - (p+r)^2\lambda}{\mu(p+r)^2}. \tag{8}$$

To find one more additional equation relating $P_{0,0}$ and $P_{1,0}$ we note from Eq. (6) that $|A_s(z_0)|$ must equal to zero whenever $Q(z_0)=0$, $0 < z_0 < 1$. It can be shown that $Q(z)$ has only one real

root in the interval (0,1). This statement can be proved using similar arguments as in Mitrani and Avi-Itzhak [16]. For the two-server case, they have obtained an explicit solution for z_0 . Unfortunately, in our case an explicit solution expression for the root z_0 is difficult to obtain. We use numerical methods to determine z_0 instead.

Given z_0 , we have from $|A_2(z_0)|=0$ that

$$P_{0,0} = -\frac{1}{2pz_0}(\lambda z_0(1-z_0)+2pz_0-\mu(1-z_0))P_{1,0}. \quad (9)$$

Substituting (9) into (8), we have

$$P_{1,0} = \frac{2pz_0}{(1-z_0)(\mu-\lambda z_0)} \cdot \frac{(2p+r)r\mu-(p+r)^2\lambda}{\mu(p+r)^2}, \quad (10)$$

$$P_{0,0} = \frac{(2p+r)r\mu-(p+r)^2\lambda}{\mu(p+r)^2} - P_{1,0}.$$

The expected number of jobs in the system, $E(N)$, can be computed by

$$E(N) = \frac{dG(z)}{dz} \Big|_{z=1}$$

$$= \frac{d}{dz}(G_0(z)+G_1(z)+G_2(z)) \Big|_{z=1} \quad (11)$$

$$= \frac{d}{dz} \left[\frac{|A_0(z)|+|A_1(z)|+|A_2(z)|}{Q(z)} \right] \Big|_{z=1}.$$

The steady-state probability of the number of jobs in the system can be obtained as follows [25]

:

$G(z)$ can be expressed in the following form,

$$G(z) = \frac{S_0 - S_1z + S_2z^2 - S_3z^3}{1 - z(T_1 - T_2z + T_3z^2 - T_4z^3)}, \quad (12)$$

where $S_i, i=0,1,2,3$, and $T_j, j=1,2,3,4$, are represented in terms of parameters λ, μ, p, r , and z_0 . If $|z(T_1 - T_2z + T_3z^2 - T_4z^3)| < 1$, $G(z)$ can be rewritten into

$$G(z) = (S_0 - S_1z + S_2z^2 - S_3z^3) \sum_{k=0}^{\infty} z^k (T_1 - T_2z + T_3z^2 - T_4z^3)^k. \quad (13)$$

Now, using the multinomial expansion formula, we can write:

$$\begin{aligned}
& \sum_{k=0}^{\infty} z^k (T_1 - T_2 z + T_3 z^2 - T_4 z^3)^k \\
&= \sum_{k=0}^{\infty} \sum_{n_1, n_2, n_3, n_4} \frac{k!}{n_1! n_2! n_3! n_4!} (-1)^{n_2+n_4} T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4} z^{n_1+2n_2+3n_3+4n_4} \\
&= \sum_{k=0}^{\infty} A_k z^k,
\end{aligned} \tag{14}$$

where

$$A_k = \sum_{n_1, n_2, n_3, n_4} \frac{k!}{n_1! n_2! n_3! n_4!} (-1)^{n_2+n_4} T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4},$$

and the nonnegative integers n_1, n_2, n_3, n_4 in the above summation take values such that $n_1 + 2n_2 + 3n_3 + 4n_4 = k$.

Thus, Eq. (13) can be rewritten as follows:

$$\begin{aligned}
G(z) &= S_0 A_0 + (S_0 A_1 - S_1 A_0)z + (S_0 A_2 - S_1 A_1 + S_2 A_0)z^2 \\
&+ \cdots + (S_0 A_n - S_1 A_{n-1} + S_2 A_{n-2} - S_3 A_{n-3})z^n + \cdots
\end{aligned} \tag{15}$$

Comparing (15) with (7), we have:

$$\begin{aligned}
P_0 &= P_{0,0} + P_{1,0} + P_{2,0} = S_0 A_0 \\
P_1 &= P_{0,1} + P_{1,1} + P_{2,1} = S_0 A_1 - S_1 A_0 \\
P_2 &= P_{0,2} + P_{1,2} + P_{2,2} = S_0 A_2 - S_1 A_1 + S_2 A_0 \\
&\cdot \\
&\cdot \\
&\cdot \\
P_n &= P_{0,n} + P_{1,n} + P_{2,n} \\
&= S_0 A_n - S_1 A_{n-1} + S_2 A_{n-2} - S_3 A_{n-3}, \quad n \geq 3.
\end{aligned} \tag{16}$$

From P_n , we can obtain $P_{s,n}$, $s=0,1,2$, by using $G_0(1)$, $G_1(1)$, and $G_2(1)$:

$$P_{s,n} = G_s(1) \cdot P_n \quad \text{for } s=0,1,2. \tag{17}$$

The quantities A_n , $n=0,1,2,\dots$, can be easily obtained recursively:

$$\begin{aligned}
A_0 &= 1, \\
A_1 &= T_1, \\
A_2 &= A_1 T_1 - A_0 T_2, \\
A_3 &= A_2 T_1 - A_1 T_2 + A_0 T_3, \\
A_n &= A_{n-1} T_1 - A_{n-2} T_2 + A_{n-3} T_3 - A_{n-4} T_4, \text{ for } n \geq 4.
\end{aligned} \tag{18}$$

We know from Eq. (2) that $G_s(1)$, $s=0,1,2$, is the probability of being (2-s) operative units in the system. From this fact we can obtain the availability, A , of this system,

$$\begin{aligned}
A &= G_0(1) + G_1(1) \\
&= \frac{r^2}{(p+r)^2} + \frac{2pr}{(p+r)^2} \\
&= \frac{r^2 + 2pr}{(p+r)^2}.
\end{aligned} \tag{19}$$

It may be shown that the stability condition for one duplicate-server system is

$$\frac{\lambda}{\mu} < \frac{r^2 + 2pr}{(p+r)^2}. \tag{20}$$

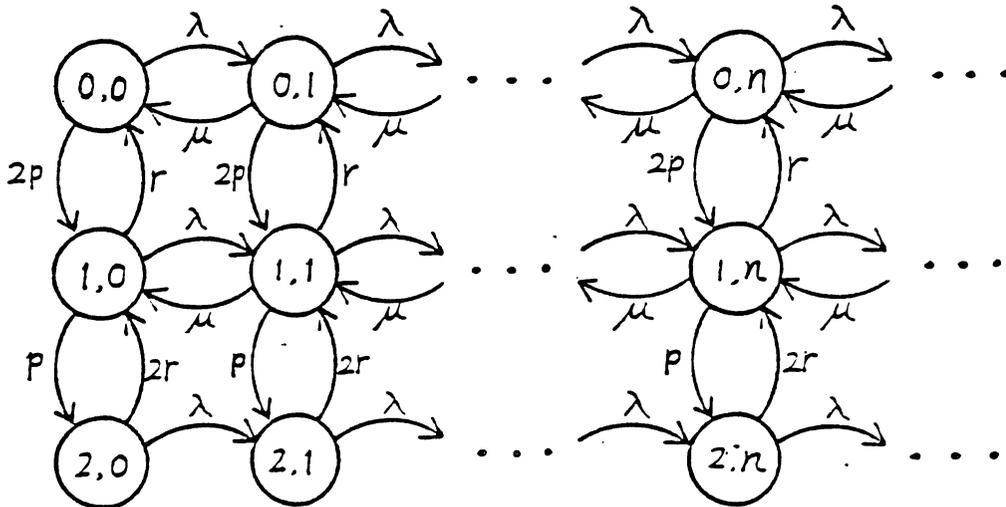


Figure 3. Transition Diagram for a Single Duplicate-server System with Perfect Coverage ($c = 1$)

3.2. Matrix-geometric Solution

An N duplicate-server system can be modeled by a discrete state continuous time Markov process. We define the states of the system (k_2, k_1, n) , where n denotes the number of jobs in the system, k_2 is the number of servers whose duplicate units are both operative, and k_1 is the number of servers with only one operative unit, for $n=0,1,\dots$, $k_2=0,1,\dots,N$, $k_1=0,1,\dots,N$, and $0 \leq k_2 + k_1 \leq N$. Note that the total number of operative servers in the system is equal to $k_2 + k_1$.

The numerical procedure requires the generation of all possible states and the generation of the rate matrix Q , which contains all the possible transitions from one state to all the other states. Each row of Q corresponds to a specific state and each element of each row (except a diagonal element) is a transition rate, which is non-negative. The diagonal elements of Q are strictly negative. The sum of all the elements of each row must equal to zero.

After the rate matrix Q is obtained, the stationary probability vector $\underline{x} = (x_0, x_1, \dots)$, if it exists, can be found by

$$\begin{aligned} \underline{x}Q &= 0 \\ \sum_{i=0}^{\infty} x_i \underline{e} &= 1, \end{aligned} \tag{21}$$

where x_i , $i=0,1,2,\dots$, are $(N+1)(N+2)/2$ vectors and column vector \underline{e} has all elements equal to one. The above system of linear equation $\underline{x}Q = 0$ is solved by using the matrix-geometric method [21]. This is because the rate matrix Q has a special block tri-diagonal structure, as shown in Figure 4.

In the stable case, there exists a non-negative matrix R with spectral radius less than one, such that

$$x_i = x_{N-1} R^{i-N+1}, \quad \text{for } i \geq N-1, N \geq 2. \tag{22}$$

The vectors x_0, x_1, \dots, x_{N-1} are uniquely determined by

the stability condition is given by

$$\pi A_2 \mathbf{e} > \pi A_0 \mathbf{e} . \quad (26)$$

From the above inequality, the general condition of stability for the N duplicate-server system can be obtained as follows:

In the Appendix, we see that the condition of stability for the 1 duplicate-server system is

$$\frac{\lambda}{\mu} < 1 - \left(\frac{p}{p+r} \right)^2 .$$

For the 2 duplicate-server system, the row vector π , $A_2 \mathbf{e}$, and $A_0 \mathbf{e}$ are given by

$$A_2 \mathbf{e} = (2\mu \ 2\mu \ \mu \ 2\mu \ \mu \ 0)^T ,$$

$$A_0 \mathbf{e} = (\lambda \ \lambda \ \lambda \ \lambda \ \lambda \ \lambda)^T ,$$

and

(k_2, k_1)	(2,0)	(1,1)	(1,0)	(0,2)	(0,1)	(0,0)
π_{k_2, k_1}	$\frac{r^4}{D_1}$	$\frac{4pr^3}{D_1}$	$\frac{2p^2r^2}{D_1}$	$\frac{4p^2r^2}{D_1}$	$\frac{4pr^3}{D_1}$	$\frac{p^4}{D_1}$

$$(D_1 = (p + r)^4) .$$

Then we have

$$\begin{aligned} \pi A_2 \mathbf{e} &= \frac{2\mu}{D_1} \left[r^4 + 4r^3p + 5r^2p^2 + 2rp^3 \right] \\ &= \frac{2\mu}{D_1} \left[D_1 - r^2p^2 - 2rp^3 - p^4 \right] \\ &= 2\mu \left[1 - \frac{p^2(p+r)^2}{D_1} \right] \\ &= 2\mu \left[1 - \left(\frac{p}{p+r} \right)^2 \right] , \end{aligned}$$

and $\pi A_0 \mathbf{e} = \lambda$. The condition of stability for the 2 duplicate-server system is

$$\frac{\lambda}{2\mu} < 1 - \left(\frac{p}{p+r} \right)^2 .$$

The following vectors are the row vector π , $A_2 \mathbf{e}$, and $A_0 \mathbf{e}$ for the 3 duplicate-server system:

$$A_2 \mathbf{e} = (3\mu \ 3\mu \ 2\mu \ 3\mu \ 2\mu \ \mu \ 3\mu \ 2\mu \ \mu \ 0)^T, \\ A_0 \mathbf{e} = (\lambda \ \lambda \ \lambda)^T,$$

and

(k_2, k_1)	(3,0)	(2,1)	(2,0)	(1,2)	(1,1)	(1,0)	(0,3)	(0,2)	(0,1)	(0,0)
π_{k_2, k_1}	$\frac{r^6}{D_2}$	$\frac{6r^5p}{D_2}$	$\frac{3r^4p^2}{D_2}$	$\frac{12r^4p^2}{D_2}$	$\frac{12r^3p^3}{D_2}$	$\frac{3r^2p^4}{D_2}$	$\frac{8r^3p^3}{D_2}$	$\frac{12r^2p^4}{D_2}$	$\frac{6rp^5}{D_2}$	$\frac{p^6}{D_2}$

$$(D_2 = (p+r)^6).$$

Then,

$$\begin{aligned} \pi A_2 \mathbf{e} &= \frac{3\mu}{D_2} \left[r^6 + 6r^5p + 14r^4p^2 + 16r^3p^3 + 9r^2p^4 + 2rp^5 \right] \\ &= \frac{3\mu}{D_2} \left[D_2 - p^2(r^4 + 4r^3p + 6r^2p^2 + 4rp^3 + p^4) \right] \\ &= \frac{3\mu}{D_2} \left[D_2 - p^2(r+p)^4 \right] \\ &= 3\mu \left[1 - \left(\frac{p}{p+r} \right)^2 \right] \end{aligned}$$

and $\pi A_0 \mathbf{e} = \lambda$. From inequality (26) we have the condition of stability for the 3 duplicate-server system, i.e.,

$$\frac{\lambda}{3\mu} < 1 - \left(\frac{p}{p+r} \right)^2.$$

Now, the condition of stability for the N duplicate-server system is as follows:

$$\frac{\lambda}{N\mu} < 1 - \left(\frac{p}{p+r} \right)^2. \quad (27)$$

The right-hand-side of the above inequality is the percentage of time a duplicate server is available. The quantity on the left-hand-side is the percentage of time that the duplicate server is busy. Therefore, in order to have a stable system the above inequality has to hold. The availability of the N duplicate-server system can be obtained easily:

$$Availability = 1 - \pi_{0,0}$$

$$= 1 - \left(\frac{p}{p+r} \right)^{2N}, \quad (28)$$

where $\pi_{0,0}$ denotes the probability that the system is down.

4. Results

In this section, we compare N duplicate-server systems with multiple server systems (or otherwise known as gracefully degrading systems) in which the system response time as a measure of performance is used.

$N=1(D)$ and $N=2(D)$ denote the system with one and two duplicated servers, respectively. The multiple server system with two and four servers are denoted by $N=2(M)$ and $N=4(M)$, respectively. For all these systems, the values of parameters used for comparisons are as follows:

service time ; $1/\mu = 1/20/\text{hr}$,

arrival rate ; $\lambda = .1\mu, .3\mu, .5\mu, .7\mu, .9\mu$,

failure rate ; $p = .006/\text{hr}, .012/\text{hr}$,

repair rate ; $1/r = 24/\text{hrs}$.

Figure 5-10 summarize the results. These results are obtained by using the matrix-geometric solution method.

The results show that the coverage factor c affects the response time. Since the coverage factor can be interpreted as the probability that a failure is detected, the coverage factor of $N=2(D)$ system could be higher than that of $N=4(M)$ system. As we see in the Figure 8-10, $N=2(D)$ system does not have advantages over $N=4(M)$ system for the same values of coverage. But $N=2(D)$ system with $c = .999$ and $c = .99$ has smaller response time than $N=4(M)$ system with $c = .99$ and $c = .97$, respectively, for some values of λ/μ . However, comparing

$N=1(D)$ with $N=2(M)$ system, the $N=1(D)$ system does not seem to have that much of an advantage over the $N=2(M)$ system as the $N=2(D)$ system has (see Figure 5-7).

In general, we conclude that the N duplicate-server system might have smaller response time than the $2N$ multiple server systems for some values of λ/μ , if it has a coverage higher than that of the $2N$ multiple server system. This needs further investigation.

5. Conclusions and Extensions

We have studied a fault-tolerant computing system, N duplicate-server system, where each server consists of two reconfigurable duplicated units which are subject to breakdowns. The analysis of N duplicate-server system has been done using the generating function technique and also using the matrix-geometric solution method. We have compared the response time of this system with that of the multiple server system.

This analysis have been carried out for a single node consisting of a single queue served by N duplicate-server. We will extend the analysis of networks of such nodes.

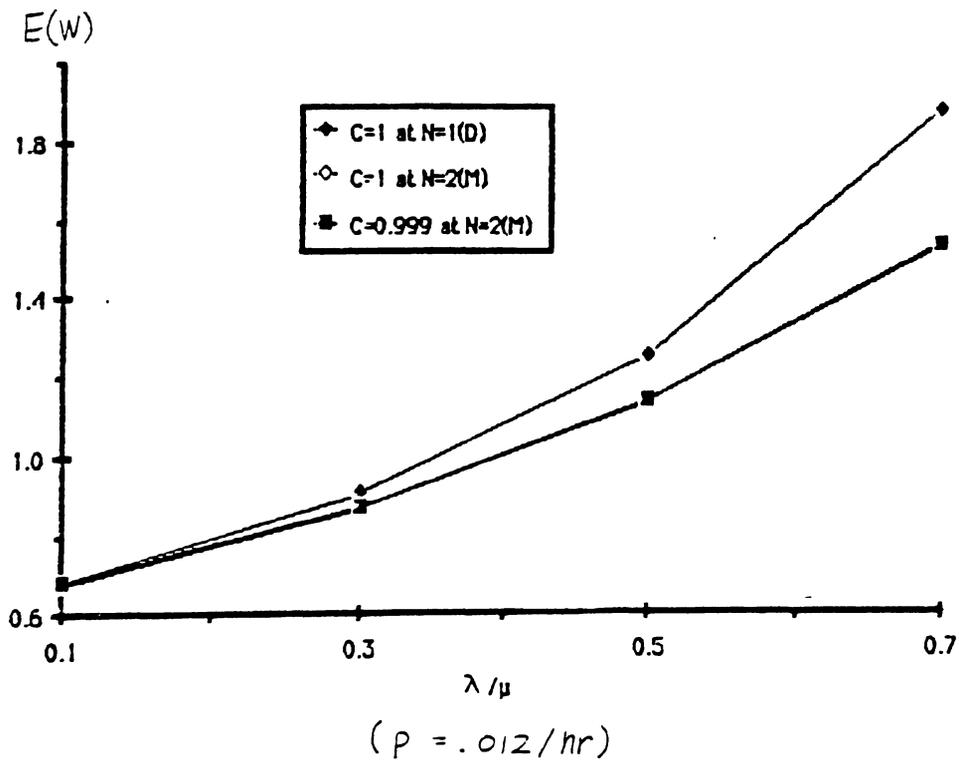
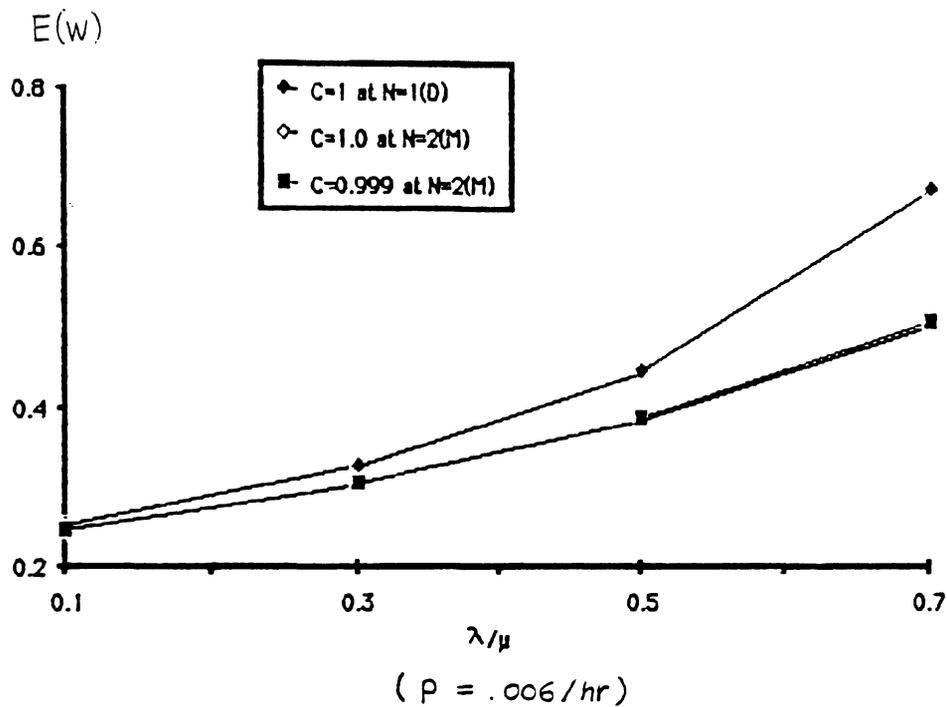


Figure 5. Response Time of N=1(D) and N=2(M)

($c = 1.$ and $.999$)

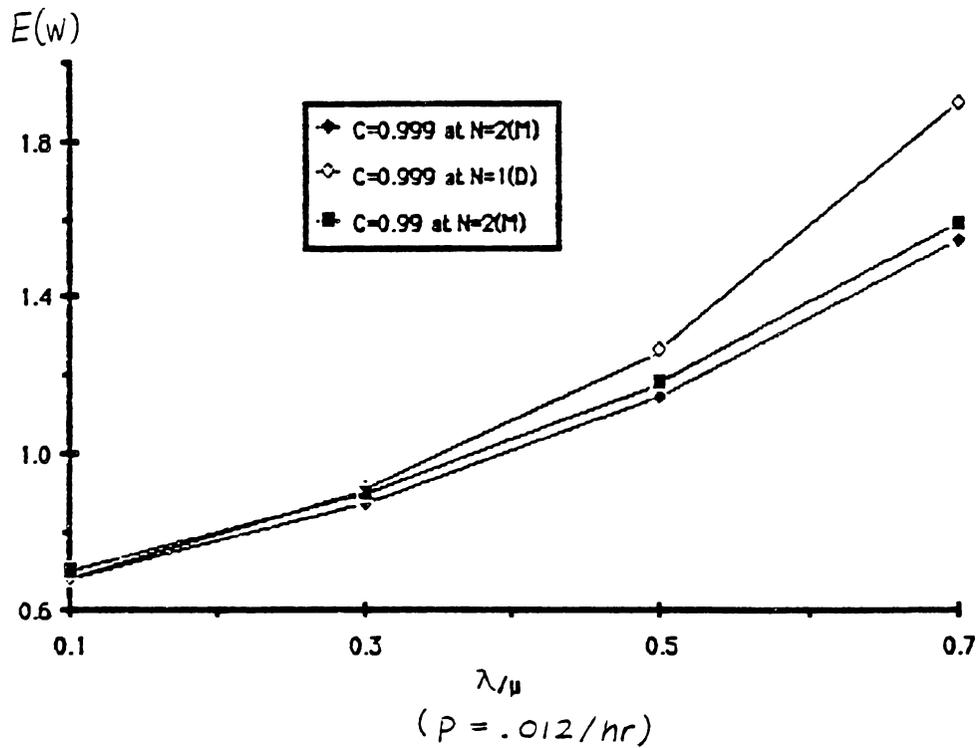
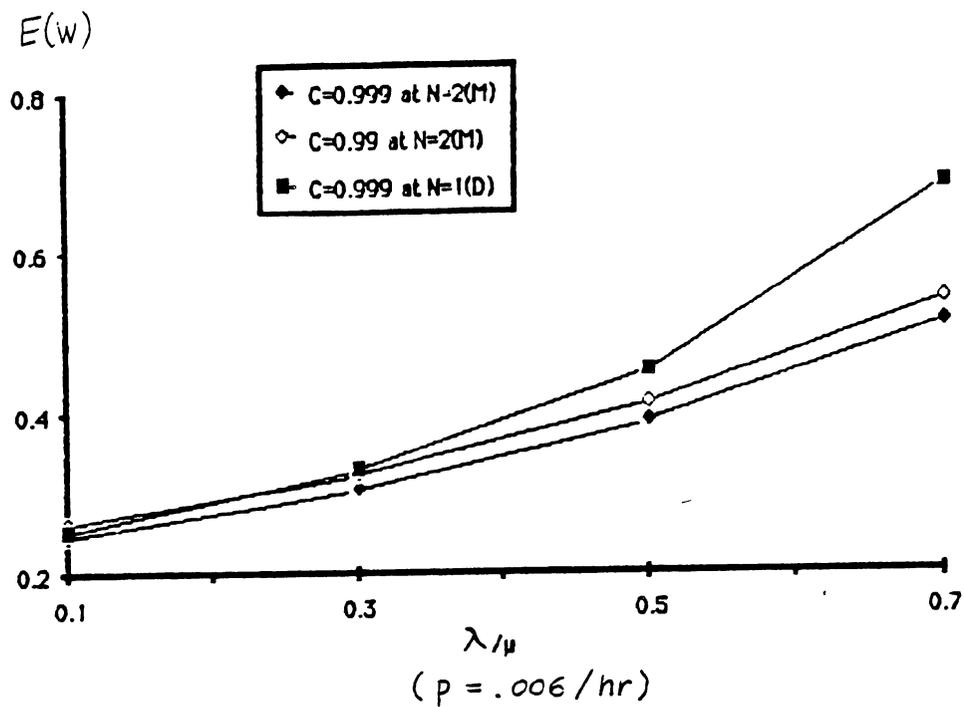


Figure 6. Response Time of N=1(D) and N=2(M)

($c = .999$ and $.99$)

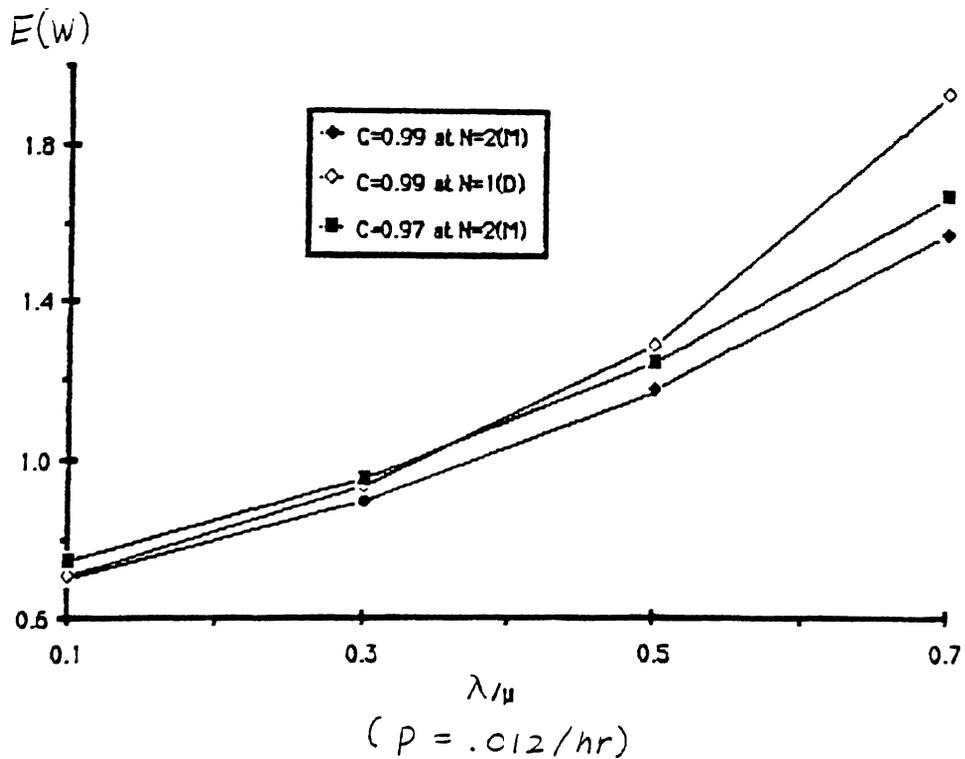
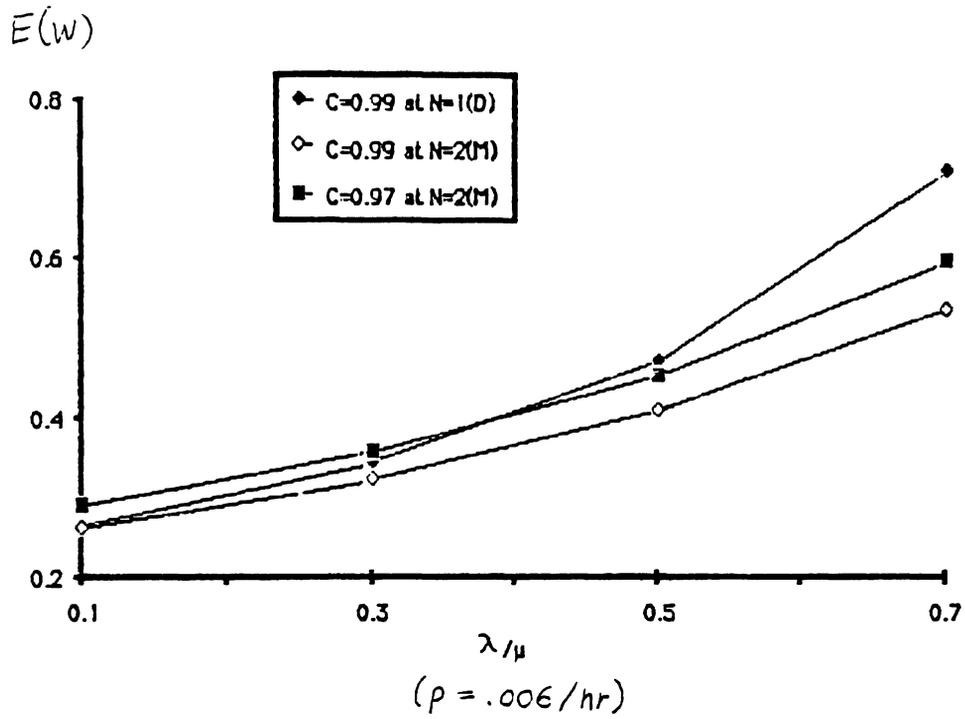


Figure 7. Response Time of N=1(D) and N=2(M)

(c = .99 and .97)

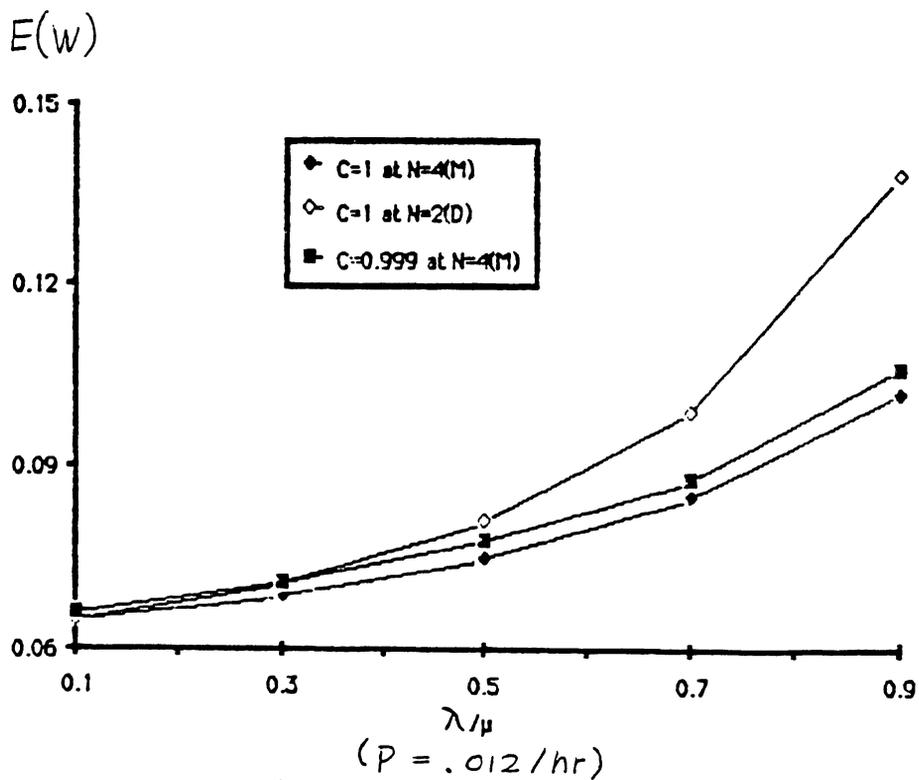
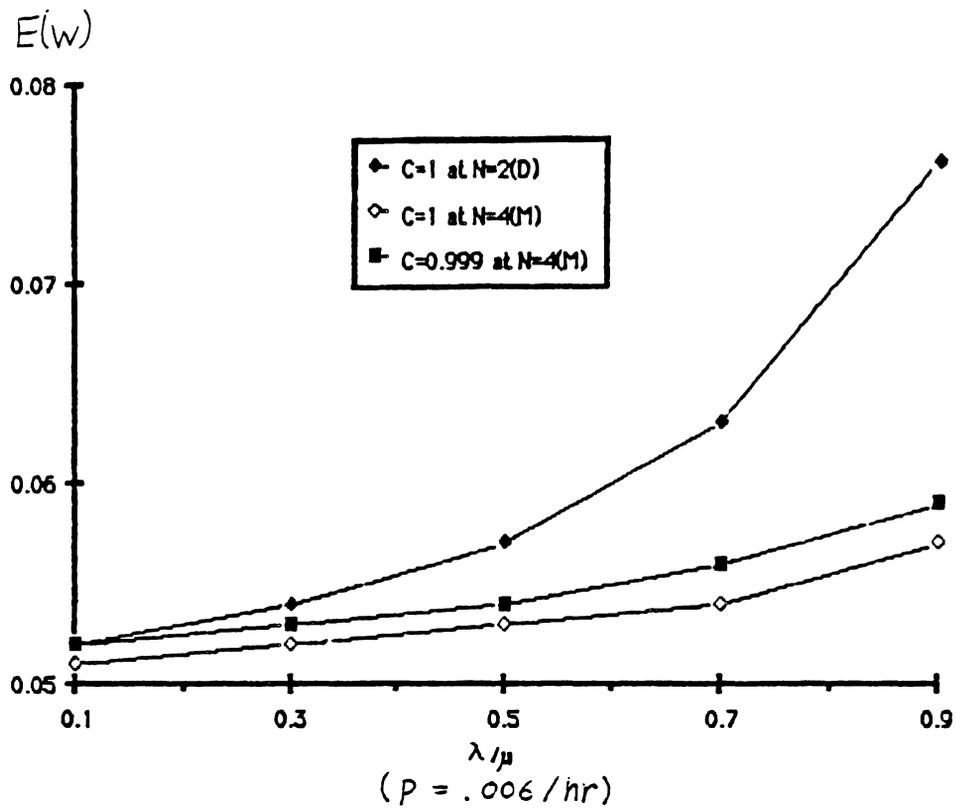


Figure 8. Response Time of N=2(D) and N=4(M)

(c = 1. and .999)

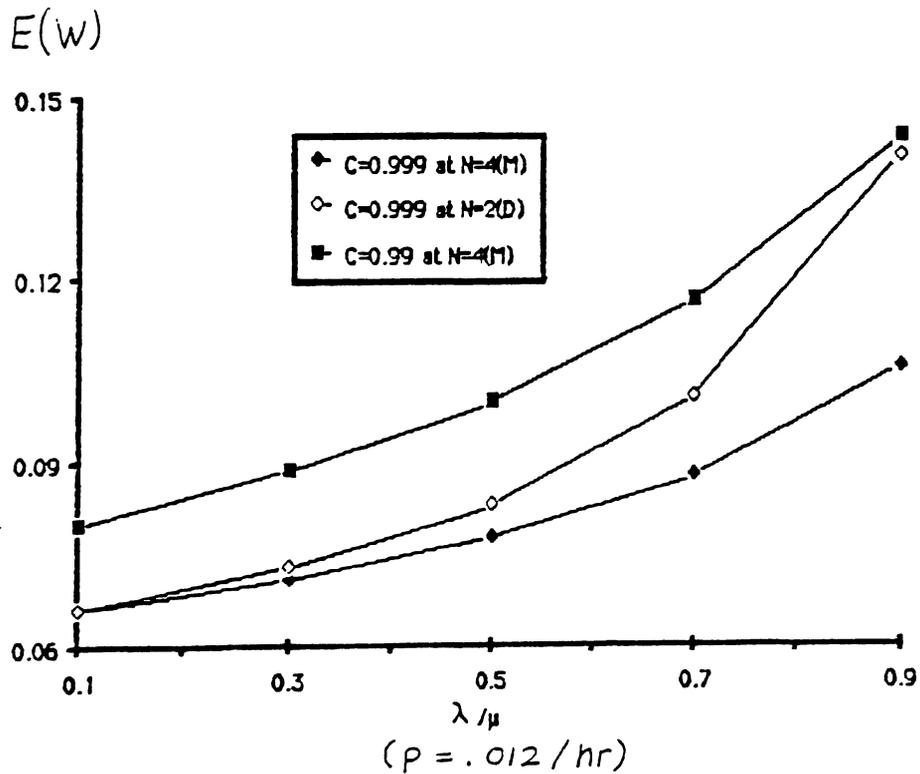
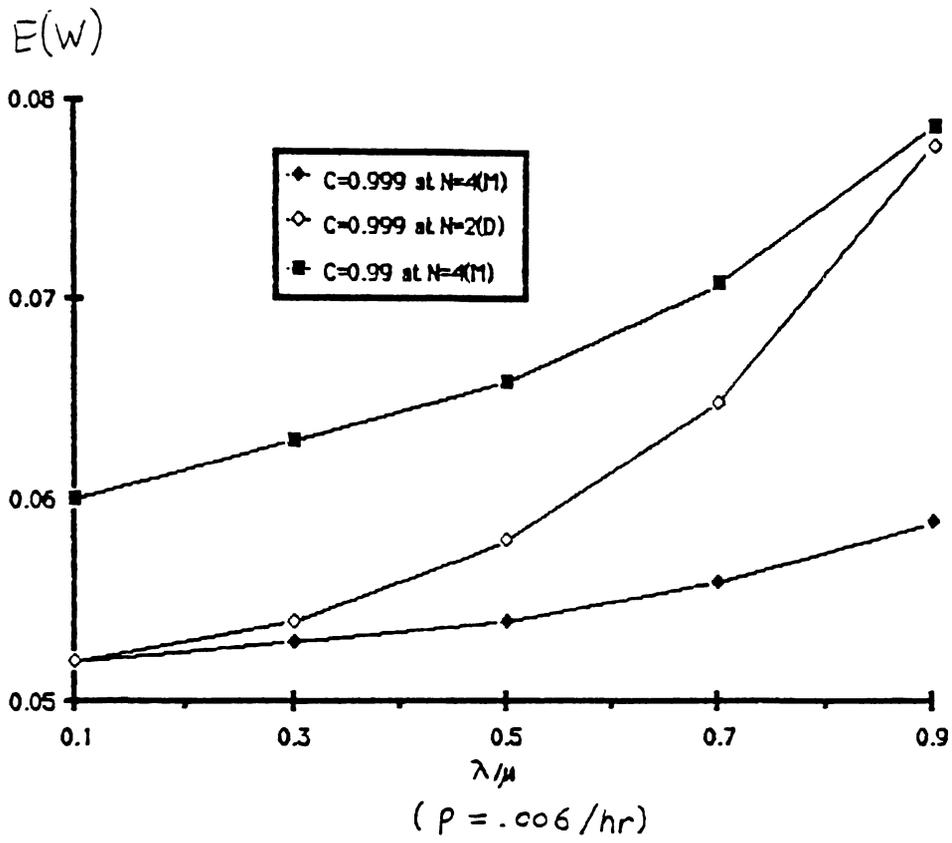


Figure 9. Response Time of N=2(D) and N=4(M)

($c = .999$ and $.99$)

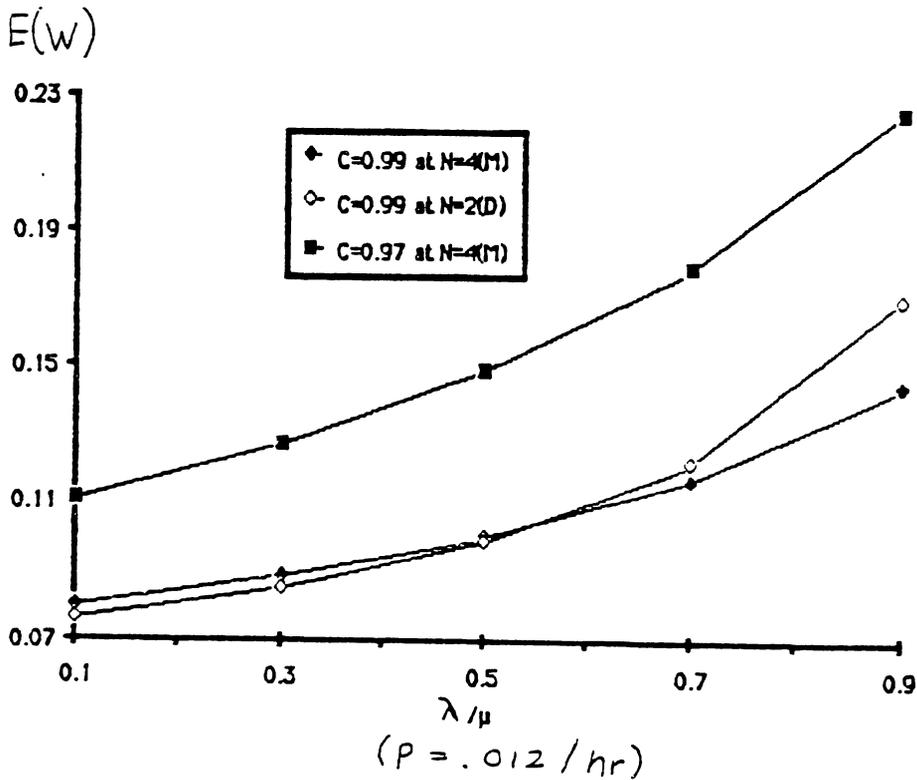
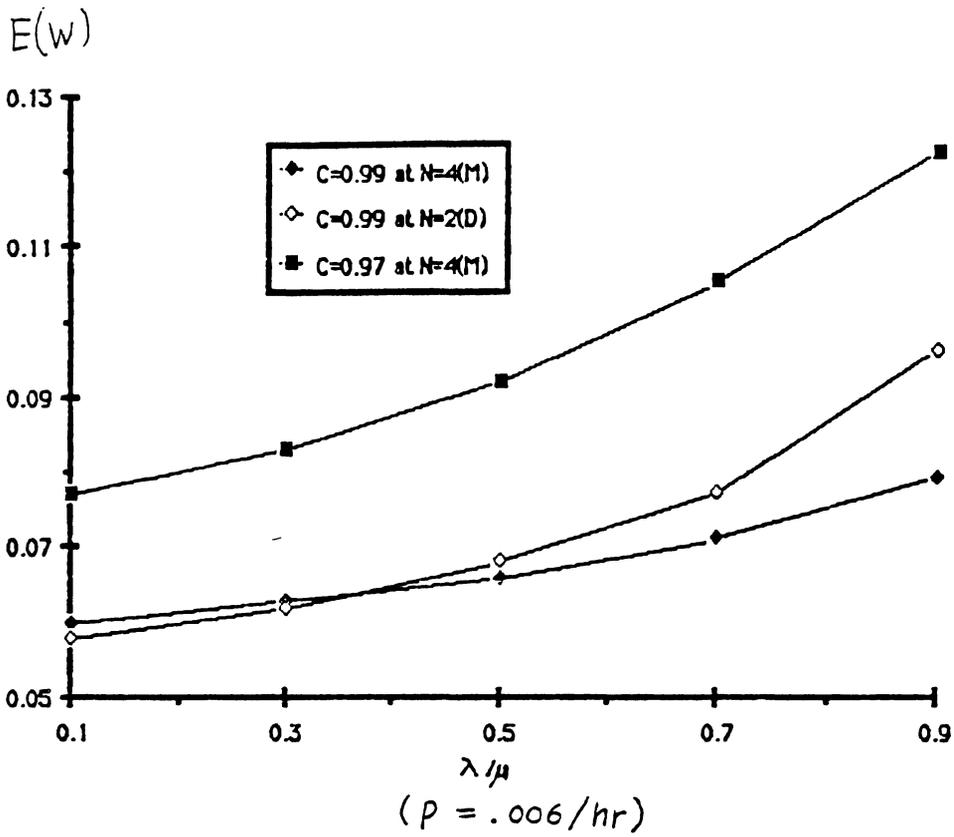


Figure 10. Response Time of N=2(D) and N=4(M)

($c = .99$ and $.97$)

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Appendix

Let us consider an example in order to understand the matrix-geometric method more clearly. Consider a system with only one duplicated server, hereafter referred to as an $N=1(D)$ system. For the $N=1(D)$ system, the states can be changed into the simple form as follows:

$$(k_2=1, k_1=0, n) \rightarrow (0, n),$$

$$(k_2=0, k_1=1, n) \rightarrow (1, n),$$

$$(k_2=0, k_1=0, n) \rightarrow (2, n),$$

where (s, n) means that n is the number of jobs in the system and s , $s=0,1,2$, is the number of failed units in the system.

We draw the state transition diagram for the $N=1(D)$ system in Figure 11. c is the coverage factor defined as the conditional probability of successful recovery upon the occurrence of a failure. The corresponding rate matrix Q is shown in Figure 12.

We determine the solution to the components of π in terms of the parameters as:

$$\begin{aligned}\pi_0 &= \frac{r^2}{(p+r)^2} \\ \pi_1 &= \frac{2pr}{(p+r)^2} \\ \pi_2 &= \frac{p^2}{(p+r)^2}.\end{aligned}\tag{29}$$

The availability of the $N=1(D)$ system is given by

$$\begin{aligned}\pi_0 + \pi_1 &= \frac{r^2 + 2pr}{(p+r)^2} \\ &= 1 - \left(\frac{p}{p+r} \right)^2.\end{aligned}\tag{30}$$

From the inequality (26), the stability condition can be obtained:

$$\frac{\lambda}{\mu} < 1 - \left(\frac{p}{p+r} \right)^2.\tag{31}$$

This condition can be explained intuitively. For a system to be stable, the mean arrival rate must be less than the total available service rate.

By Neuts' theorem, there exists a non-negative matrix R with spectral radius less than one such that

$$x_i = x_1 R^{i-1} \quad \text{for } i \geq 1, \quad (32)$$

and R can be obtained by the recursive equations (28). x_0 and x_1 are uniquely determined by

$$\begin{aligned} x_0 A_{00} + x_1 A_2 &= 0 \\ x_0 \mathbf{e} + x_1 (I - R)^{-1} \mathbf{e} &= 1. \end{aligned} \quad (33)$$

Using Eqs. (32) and (33), we can compute the probability of the number of jobs in the system.

Let $E(N)$ and $E(W)$ denote the expected number of jobs in the system and the mean response time, respectively. By definition, we have

$$\begin{aligned} E(N) &= \sum_{n=0}^{\infty} n x_n \mathbf{e} \\ &= x_1 (I - R)^{-2} \mathbf{e}. \end{aligned} \quad (34)$$

Applying Little's law, then we have

$$E(W) = E(N)/\lambda. \quad (35)$$

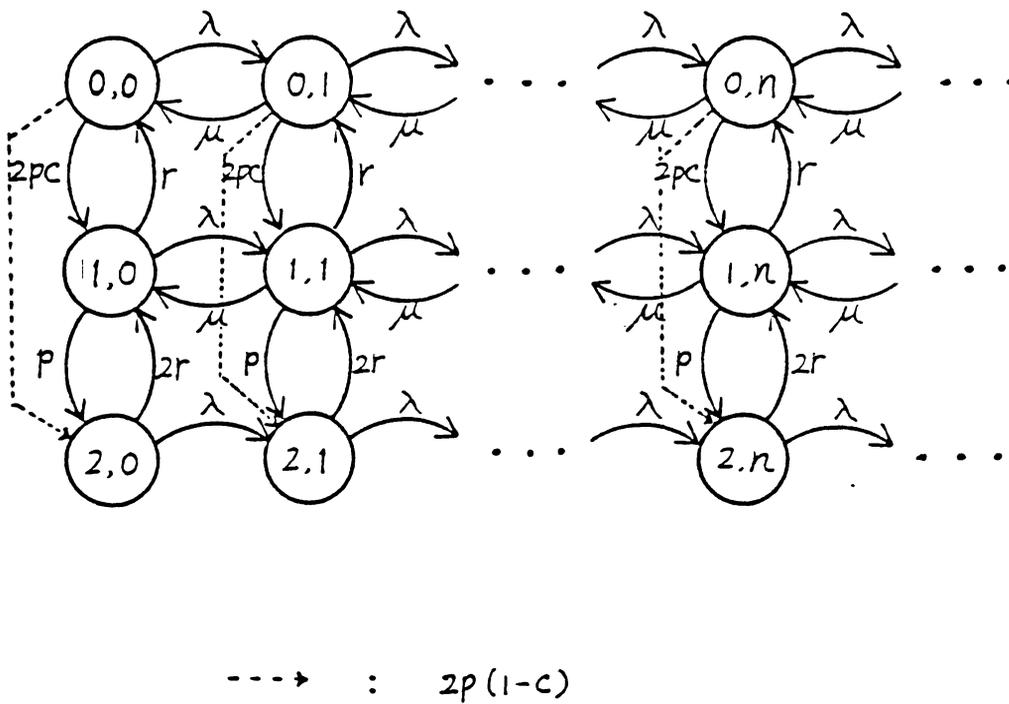


Figure 11. Transition Diagram for a Single Duplicate-server System
with Imperfect Coverage ($c \leq 1$)

$$Q = \begin{bmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix},$$

where

$$A_{00} = \begin{bmatrix} -2p - \lambda & 2pc & 2p(1-c) \\ r & -p - r - \lambda & p \\ 0 & 2r & -2r - \lambda \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$A_1 = A_{00} + A_2.$$

Figure 12. Rate Matrix Q for N=1(D) System