ON A METHOD OF CONSTRUCTION OF GROUP DIVISIBLE DESIGNS

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This article presents a rather elementary method of constructing partially balanced incomplete block designs of group divisible type (GDPBIB designs, in short); some of the resulting designs seem to be new.

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Contents

1. A method of constructing GDPBIB designs and its properties

Let us take $v = mn$ treatments, $\{0,1,\ldots,mn-1\}$, m and n being positive integers, with the following association scheme of group divisible type:

$$
(1.1)\qquad
\begin{array}{llll}
\underline{0 \quad m \quad 2m \ldots\ldots\ldots\ldots (n-1)m} & \text{(1-st group)} \\
\underline{1 \quad m+1 \quad 2m+1 \ldots\ldots\ldots (n-1)m+1} & \text{(2-nd group)} \\
\underline{2 \quad m+2 \quad 2m+2 \ldots\ldots\ldots (n-1)m+2} & \text{(3-rd group)} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\
\underline{m-1 \quad 2m-1 \quad 3m-1 \ldots\ldots\ldots mn-1} & \text{(m-th group).}
\end{array}
$$

We want to construct a GDPBIB design with the parameters

(1.2) $\quad v = mn,\ k,\ b,\ r, \lambda_1, \lambda_2,\ m,\ n,$

for which it hold that

(1.3) $\quad v\,r = b\,k \quad \text{and} \quad \lambda_1 n_1 + \lambda_2 n_2 = r(k-1)\ ,$

where $n_1 = n-1$ and $n_2 = n(m-1)$.

Let us take any block of size k, consisting of k distinct treatments

$$B = \{x_1,\ldots,x_k\}$$

and let $\{y_1,\ldots,y_k\}$ be a set of integers corresponding to the block B given above in such a way that $y_i = j$ if x_i belongs to the $(j+1)$-th group of the association scheme (1.1), $i = 1,\ldots,k$; $j = 0,1,\ldots,m-1$. Furthermore, let $(a_o,a_1,\ldots,a_{m-1})$ be an ordered m-tuple consisting of the multiplicities of y_i 's in the set of integers, $\{0,1,\ldots,m-1\}$, i.e., a_j is the number of 'j' appearing in the set $\{y_1,\ldots,y_k\}$, $j = 0,1,\ldots,m-1$. It hold then that

(1.4) $\quad 0 \leqq a_j \leqq n \ (j = 0,1,\ldots,m-1) \text{ and } \sum_{j=0}^{m-1} a_j = k\ .$

Thus, for any given block of size k, there corresponds an ordered m-tuple; we shall denote this correspondence by a mapping f defined over the set of all blocks of size k :

$$f(\{x_1,\ldots,x_k\}) = (a_o,\ldots,a_{m-1})\ .$$

This correspondence is of course a many-to-one correspondence.

Now, let us consider a set of ordered m-tuples satisfying the conditions (1.4):

$$(1.5) \qquad S = \{(a_{po}, a_{p1}, \ldots, a_{pm-1})\} \; (p = 1, \ldots, s) \; ,$$

where

$$(1.6) \qquad 0 \leqq a_{pi} \leqq n \; (i=0,1,\ldots,m-1;\; p=1,\ldots,s) \text{ and } \sum_{i=0}^{m-1} a_{pi} = k \; , \quad (p = 1,\ldots,s) \; .$$

Let $C(S)$ be the set of all those blocks which are mapped onto the set S , i.e.,

$$C(S) = f^{-1}(S).$$

It is clear that $C(S)$ is a disjoint union of s subsets given by $C_p \equiv f^{-1}((a_{po},\ldots,a_{pm-1}))$, $p=1,\ldots,s$.

Since the cardinality of the set C_p is equal to

$$b_p(S) \equiv \prod_{i=o}^{m-1} \binom{n}{a_{pi}} \; , \; p=1,\ldots,s \; ,$$

the number of blocks in $C(S)$ is given by

$$(1.7) \qquad b(S) = \sum_{p=1}^{s} b_p(S) \; .$$

For each $(a_{po},\ldots,a_{pm-1})$ in S , the number of blocks in C_p which contain any given treatment belonging to the $(i+1)$-th group is given by

$$\binom{n}{a_{po}} \cdots \binom{n}{a_{pi-1}} \binom{n-1}{a_{pi}-1} \binom{n}{a_{pi+1}} \cdots \binom{n}{a_{pm-1}} = \frac{a_{pi}}{n} \cdot b_p(S)$$

from which it follows that the number of blocks in $C(S)$ which contain any given treatment belonging to the $(i+1)$-th group is equal to

$$(1.8) \qquad r_i(S) = \sum_{p=1}^{s} a_{pi} b_p(S)/n \; , \; i = 0,1,\ldots,m-1$$

The number of blocks belonging to $C(S)$ in which any two treatments in the $(i+1)$-th group occur together is given by

$$(1.9) \qquad \lambda_{1i}(S) = \sum_{p=1}^{s} a_{pi}(a_{pi}-1) b_p(S)/n(n-1) \; , \; i = 0,1,\ldots,m-1,$$

and the number of blocks in $C(S)$ in which any two treatments, one from the $(i+1)$-th group and the other from the $(j+1)$=th group, occur together, is given by

$$(1.10) \qquad \lambda_{2i,j}(S) = \sum_{p=1}^{s} a_{pi} a_{pj} b_p(S)/n^2 \ , \ i \neq j; \ i,j=0,1,\ldots,m-1 \ .$$

We shall call the set S given by (1.5) a selecting set for a GDPBIB design if the corresponding set of blocks, C(S), forms a GDPBIB design. Then, we have the following

Theorem 1. A necessary condition for S given by (1.5) to be a selecting set for a GDPBIB design with the parameters (1 2) is given by the conditions

$$(1.11) \qquad b(S) = b$$

$$(1.12) \qquad r_i(S) = r \ , \ i = 0,1,\ldots,m-1 \ ,$$

$$(1.13) \qquad \lambda_{1i}(S) = \lambda_1 \ , \ i = 0,1,\ldots,m-1 \ ,$$

and

$$(1.14) \qquad \lambda_{2i,j}(S) = \lambda_2, \ i \neq j; \ i,j = 0,1,\ldots,m-1 \ .$$

Conversely, if there exists a set S given by (1.5) satisfying the conditions (1.12), (1.13) and (1.14) for some set of positive integers, r, λ_1 and λ_2 , then S is a selecting set for a GDPBIB design with the parameters (1.2) with b = b(S).

Proof It is clear that the conditions are necessary.

To prove the sufficiency, it suffices to show that the conditions (1.3) are satisfied.

Summing up both sides of (1.12) with respect to i and using (1.6) and (1.11), we have $mr = kb/n$, from which it follows that $vr = bk$.

Since $n_1 = n-1$ and $n_2 = n(m-1)$, it follows from (1.13) and (1.14) that

$$\begin{aligned} \lambda_1 n_1 + \lambda_2 n_2 &= (n-1) \sum_{i=o}^{m-1} \lambda_{1i}(S)/m + n(m-1) \sum_{\substack{i,j=o \\ i \neq j}}^{m-1} \lambda_{2i,j}(S)/m(m-1) \\ &= (k^2 b - kb)/mn \\ &= r(k-1) \ , \end{aligned}$$

which proves the second equality of (1.3).

Corollary 1. In the special case when $k = m$,

$$S = \{ (1,\dots,1) \}$$

is a selecting set for a semi-regular GD design with parameters

(1.15) $\quad v - mn,\ k = m,\ b = n^m,\ r = n^{m-1},\ m,\ n, \lambda_1 = 0, \lambda_2 = n^{m-2}$

Bose, Shrikhande and Bhattacharya [1] showed that a semi-regular GD design with parameters

(1.16) $\quad v = mn,\ k = m,\ b = n^3,\ r = n^2,\ m, n, \lambda_1 = 0\ \lambda_2 = n,$

n being a power of a prime, is constructed from the finite projective geometry $PG(3,n)$. The design with the parameters (1.15) is the n^{m-3}-plicate of the design with the parameters (1.16) if n is a prime power.

Corollary 2. The set

$$S = \{ (k,0,\dots,0),\ (0,k,0,\dots,0), \dots, (0,\dots,0,k) \}$$

is a selecting set for a GDPBIB design with parameters

(1.17) $\quad v = mn,\ k,\ b = m\binom{n}{k},\ r = m\binom{n-1}{k-1},\ m,\ n, \lambda_1 = m\binom{n-2}{k-2}, \lambda_2 = 0$,

which is obtained from a BIB design with parameters

$$v = n,\ k,\ b = \binom{n}{k},\ r = \binom{n-1}{k-1},\ \lambda = \binom{n-2}{k-2}. \qquad ([4])$$

Corollary 3. Suppose that $n = 2$ and $k = 2t$ for some positive integer t. If each n-tuple of S is a permutation of the m-tuple $(\overbrace{2,2,\dots,2}^{t},0,\dots,0)$, then S is a selecting set for a singular GDPBIB design.

The following theorem states a sort of additive property of our method, the proof of which is easy and omitted.

Theorem 2. Let S' and S'' be two mutually disjoint selecting sets for GD designs with parameters

(1.18) $\quad v = mn,\ k,\ b',\ r',\ m,\ n, \lambda_1',\ \lambda_2'$

and

(1.19) $\quad v = mn,\ k,\ b'',\ r'',\ m,\ n, \lambda_1'', \lambda_2''$,

respectively. Then, the union, $S = S' \cup S''$, is a selecting set for a GDPBIB design with parameters

(1.20) $v = mn,\ k,\ b = b' + b'',\ r = r' + r'',\ m,\ n, \lambda_1 = \lambda_1' + \lambda_1'', \lambda_2 = \lambda_2' + \lambda_2''$.

Conversely, if $S = S' \cup S''$ and $S' \cap S'' = \phi$, and if S and S' are the selecting sets for GD designs (1.20) and (1.18) respectively, then S" is a selecting set for a GD design with parameters (1.19).

Now, in the final place, we state without proof the following

Theorem 3. Let

$$S = \{ (a_{po}, \ldots, a_{pm-1}) \} \quad (p = 1, \ldots, s) ,$$

be a selecting set for a GD design with the parameters (1.2) , and put

$$\bar{a}_{pi} = n - a_{pi} \text{ for all } i \text{ and } p . \text{ Then}$$

$$\bar{S} = \{ (\bar{a}_{po}, \ldots, \bar{a}_{pm-1}) \} \quad (p = 1, \ldots, s)$$

is a selecting set for the complementary design of (1.2).

2. Selecting sets.

In order to construct a GD design by the method stated in the preceding section for given values of v, m, n and k, one has to find a suitable selecting set, such that the number of replications, and hence of blocks, is as small as possible. This gives rise to some combinatorial problems. In the present section, we shall investigate these problems in some detail, though any solution has not yet been found for them.

Let us consider a set of m-tuples consisting of non-negative integers not exceeding n:

$$S = \{ (a_{po}, a_{p1}, \ldots, a_{pm-1}) \} \quad (p = 1, \ldots, s) .$$

We arrange the s m-tuples in S in an s x n rectangular array in such a way that s rows of the array are the s m-tuples in S :

$$
(2.1) \qquad S \qquad \begin{array}{cccc} a_{1o} & a_{11} & \cdots & a_{1m-1} \\ a_{2o} & a_{21} & \cdots & a_{2m-1} \\ & \cdots\cdots & & \\ a_{s0} & a_{s1} & \cdots & a_{sm-1} \end{array}
$$

Then, Theorem 1 assures us that S is a selecting set for some GD design if the following conditions are satisfied.

$$
(2.2) \begin{cases} \text{(i)} & \sum_{i=o}^{m-1} a_{pi} = k \ , \ p = 1,\ldots,s \\ \text{(ii)} & \sum_{p=1}^{s} a_{pi}\, b_p = c_1 \ , \ i = 0,1,\ldots,m-1 \ , \\ \text{(iii)} & \sum_{p=1}^{s} a_{pi}^2\, b_p = c_2 \ , \ i = 0,1,\ldots,m-1 \ , \\ \text{(iv)} & \sum_{p=1}^{s} a_{pi}\, a_{pj}\, b_p = c_3 \ , \ i \neq j;\ i,j = 0,1,\ldots,m-1, \end{cases}
$$

where c_1, c_2 and c_3 are constants and $b_p = \prod_{i=o}^{m-1} \binom{n}{a_{pi}}$, p= 1,...,s.

It is difficult in general to find a selecting set satisfying the conditions (2.2). Hence, we confine our attention to the case where the rows of the array (1.1) are obtained by permutations of a given m-tuple, $(a_o, a_1, \ldots, a_{m-1})$ say, for which $\sum_{i=o}^{m-1} a_i = k$. In this case, the values of b_p 's are the same, and one may seek for the array 2.1 under the conditions

$$
(2.3) \begin{cases} \text{(i)} & \sum_{p=1}^{s} a_{pi} = c'_1, \ i=o,1,\ldots m-1 \\ \text{(ii)} & \sum_{p=1}^{s} a_{pi}^2 = c'_2, \ i=o,1,\ldots m-1 \\ \text{(iii)} & \sum_{p=1}^{s} a_{pi}\, a_{pj} = c'_3 \ , \ i \neq j;\ i,\ j = 0,1,\ldots,m-1 \ , \end{cases}
$$

where c'_1, c'_2 and c'_3 are constants. This type of array is useful for the construction of GD designs, as will be seen in the following section.

One of the easiest ways of obtaining arrays satisfying (2.3) is as follows: Take all distinct permutations of a given m-tuple $(a_o, a_1, \ldots, a_{m-1})$ to form the rows of the array (2.1) . If the given m-tuple contains q distinct integers $d_1, \ldots, d_q$, with respective multiplicities $\psi_1, \ldots \psi_q$, then the number of rows in the array is given by

$$s = m! \; / \prod_{u=1}^{q} (\psi_u !) \; .$$

This number, however, appears too large for our purpose; it would be desirable to get a smaller value of s, for which any effective method has not yet been found.

The parameters of the design in this case are given by

$$v = mn, \; k = \sum_{u=1}^{q} \psi_u d_u, \; b = s \sum_{u=1}^{q} \binom{n}{d_u}^{\psi_u}, \; r = b \sum_{u=1}^{q} \psi_u d_u \,/\, mn,$$

(2.4)

$$\lambda_1 = b \sum_{u=1}^{q} \psi_u d_u(d_u - 1)/mn(n-1) \; ,$$

$$\lambda_2 = b(\sum_{u=1}^{q} \psi_u (\psi_u - 1)d_u^2 + 2 \sum_{\substack{u,u'=1 \\ u<u'}}^{q} \psi_u \psi_{u'} d_u d_{u'})/n^2 m(m-1).$$

Another easy way of obtaining arrays satisfying (2.3) is to use the incidence matrices of known BIB designs: Let us consider the case where the m-tuple $(a_o, a_1, \ldots, a_{m-1})$ consists of two distinct integers, d_1, d_2 , with respective multiplicities ψ_1, ψ_2. Take the transpose of the incidence matrix of a BIB design with parameters

$$v^* = m, \; k^* = \psi_1, \; b^*, \; r^*, \; \lambda^* \; ,$$

assuming that this design exists:

$$N' = \begin{bmatrix} n_{11} & n_{21} & \cdots & n_{m1} \\ n_{12} & n_{22} & \cdots & n_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ n_{1b^*} & & \cdots & n_{mb^*} \end{bmatrix}$$

We change the elements n_{ij} for ψ_1 if $n_{ij} = 1$ and for ψ_2 if $n_{ij} = 0$ to get a matrix or an array corresponding to (2.3):

$$
\begin{array}{cccc}
a_{10} & a_{11} & \cdots & a_{1m-1} \\
a_{2o} & a_{21} & \cdots & a_{2m-1} \\
 & \cdots\cdots\cdots & & \\
a_{b^*o} & a_{b^*1} & & a_{b^*m-1}
\end{array} \tag{2.5}
$$

Then, each row of this array is a permutation of the m-tuple $(a_o, a_1, \ldots, a_{m-1})$ and contains ψ_1 'd_1' and ψ_2 'd_2' . On the other hand, each column of this array contains r^* 'd_1' and b^*-r^* 'd_2' and among the b^* unordered pairs of elements, $\{(a_{pi}, a_{pj})\}$ $(p = 1,\ldots,b^*), i \neq j$, the unordered pairs (d_1, d_1), (d_1, d_2) and (d_2, d_2) appear λ^*, $2(r^*-\lambda^*)$ and $b^*-2r^*+\lambda^*$ times respectively. Hence the condition (2.3) is satisfied, where $s = b^*$.

The parameters of the resulting design are

$$
\begin{aligned}
& v = mn,\ k=\psi_1 d_1+\psi_2 d_2,\ b = b^* \sum_{u=1}^{2} \binom{n}{d_u}^{\psi_u},\ r = b(r^*d_1+(b^*-r^*)d_2)/nb^* , \\
& \lambda_1 = b\ (r^*d_1(d_1-1)+(b^*-r^*)d_2(d_2-1))\ /n(n-1)b^* , \\
& \lambda_2 = b\ (\lambda^* d_1^2+2(r^*-\lambda^*)d_1d_2+(b^*-2r^*+\lambda^*)d_2^2)/n^2b^*
\end{aligned} \tag{2.6}
$$

As for the other types of selecting sets we have not succeeded to get any suitable method of finding them. In the following section, some of the examples will exhibit the technique of finding suitable selecting set by an intuitive method.

In the final place of this section, it should be remarked that there might be a way of reducing the number of replications or of blocks of the resulting GD design: If we can reduce the number of inverse images of each m-tuple in S with respect to the mapping f, then the number of blocks of the resulting design is reduced. This would concern to a king of decomposability of the mapping f, which has also not investigated yet.

3. Some of the GD designs which can be constructed by our method.

In this section, we construct some of the regular GDPBIB designs by using our method; most of the singular and semi-regular GD designs obtained by applying the method mentioned in Section 1 have been already solved $\lfloor 2 \rfloor$. The designs in the examples below have $k \leqq 10$ and $r \leqq 10$, and they have not listed in $\lfloor 3 \rfloor$.

Example 1.

(3.1) $v = 6,\ k = 3,\ b = 18,\ r = 9, \lambda_1 = 3, \lambda_2 = 4,\ m = 2,\ n = 3,\ n_1 = 2,\ n_2 = 3.$

This design is obtained by using the selecting set

$$S: \quad \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$$

and the association scheme of the treatments and the blocks of the design are given by

$$\begin{matrix} 0 & 2 & 4 \\ \hline 1 & 3 & 5 \\ \hline \end{matrix} \qquad \begin{matrix} 0&0&0&0&0&0&2&2&2&1&1&1&1&1&1&3&3&3 \\ 2&2&2&4&4&4&4&4&4&3&3&3&5&5&5&5&5&5 \\ 1&3&5&1&3&5&1&3&5&0&2&4&0&2&4&0&2&4 \end{matrix} .$$

Example 2.

(3.2) $v = 8,\ k = 4,\ b = 16,\ r = 8, \lambda_1 = 4, \lambda_2 = 3,\ m = 2,\ n = 4,\ n_1 = 3,\ n_2 = 4$

This design is obtained in the following way: Consider first the case $m = 4$ and $n = 2$, and take the following set

$$S; \quad \begin{matrix} 2&0&1&1 \\ 1&1&2&0 \\ 0&1&1&2 \\ 1&2&0&1 \end{matrix}$$

Then, it is clear that this set cannot be a selecting set for any GD design with $v = 8$, $k = 4$, $m = 4$ and $n = 2$. We have however, the following values for the parameter (1.7), (1.8), (1.9) and (1.10):

$$b = 16, \quad r(0) = r(1) = r(2) = r(3) = 8\ ,$$
$$\lambda_1(0) = \lambda_1(1) = \lambda_1(2) = \lambda_1(3) = 4\ ,$$
$$\lambda_2(0,1)=\lambda_2(0,3)=\lambda_2(1,2)=\lambda_2(2,3)=3\ ,$$
$$\lambda_2(0,2)=\lambda_2(1,3)=4\ ,$$

where we have put $r_i(S)=r(i), \lambda_{1i}(S)=\lambda_1(i), \lambda_{2ij}(S)=\lambda_2(i,j)$, $i,j=0,1,\ldots,m-1$.

Comparing two association schemes corresponding to the cases $m = 4$, $n = 2$ and $m = 2$, $n = 4$:

$$\underset{(m=4,n=2)}{\begin{array}{cc} 0 & 4 \\ \hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline \end{array}} \quad \text{and} \quad \underset{(m=2,n=4)}{\begin{array}{cccc} 0 & 2 & 4 & 6 \\ \hline 1 & 3 & 5 & 7 \\ \hline \end{array}}\ ,$$

one can see easily that the above values of parameters give those of GDPBIB design with parameters (3.2), that is, the set S given above is a selecting set for the GD design (3.2).

The 16 blocks of the design are given by the columns of the following scheme:

$$\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 \\
2 & 2 & 3 & 6 & 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 & 1 & 1 & 5 & 5 \\
3 & 7 & 6 & 7 & 3 & 7 & 3 & 7 & 1 & 5 & 1 & 5 & 2 & 6 & 2 & 6
\end{array}$$

<u>Example 3</u>.

(3.3) $v = 8,\ k = 3,\ b = 24,\ r = 9, \lambda_1 = 6, \lambda_2 = 2,\ m = 4,\ n = 2,\ n_1 = 1,\ n_2 = 6.$

This design is obtained by taking the set

$$S = \{(2\ 1\ 0\ 0) \rightarrow\} \text{ (the set of all permutations)}$$

and the association scheme and the 24 blocks of the design are given by

$$\begin{array}{cc} 0 & 4 \\ \hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline \end{array} \qquad \begin{array}{cccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\
1 & 5 & 2 & 6 & 3 & 7 & 0 & 4 & 2 & 6 & 3 & 7 & 0 & 4 & 1 & 5 & 3 & 7 & 0 & 4 & 1 & 5 & 2 & 6
\end{array}$$

<u>Example 4</u>.

(3.4) $v = 10,\ k = 3,\ b = 20,\ r = 6, \lambda_1 = 4, \lambda_2 = 1,\ m = 5,\ n = 2,\ n_1 = 1,\ n_2 = 8.$

Firstly, let us take the selecting set

$$S = \{(2,1,0,0,0) \rightarrow\}$$

It is then easy to see that this selecting set results a GD design with parameters

$$v = 10,\ k = 3,\ b = 40,\ r = 12, \lambda_1 = 8, \lambda_2 = 2,\ m = 5,\ n = 2,$$

which is the duplicate of a GD design with parameters (3.4), if it exists. Hence, we try to find 10 tuples from the set S which form a selecting set for the design (3.4), for which it is easily noticed that we may seek for an arrangement like (2.1) such that each pair of columns contains the pair (unordered) (2,1) as its row exactly once and each column contains '2' and '1' exactly twice for each. Such an arrangement is given by

$$S': \quad \begin{array}{ccccc} 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \end{array}$$

This set generates the design (3.4), and the association scheme and blocks are given by

$$\begin{array}{cc} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{array} \qquad \begin{array}{cccccccccccccccccccc} 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4 \\ 5&5&5&5&6&6&6&6&7&7&7&7&8&8&8&8&9&9&9&9 \\ 1&6&2&7&2&7&3&8&3&8&4&9&0&5&4&9&0&5&1&6 \end{array}$$

<u>Example 5</u>.

(3.5) $v = 12,\ k = 6,\ b = 18,\ r = 9, \lambda_1 = 7, \lambda_2 = 3,\ m = 3,\ n = 4,\ n_1 = 3,\ n_2 = 8$

This design is obtained by using the selecting set

$$S: \quad \begin{array}{ccc} 4 & 0 & 2 \\ 2 & 4 & 0 \\ 0 & 2 & 4 \end{array} \quad ,$$

and the association scheme and blocks are given by

	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
0 3 6 9	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5
1 4 7 10	6	6	6	6	6	6	7	7	7	7	7	7	8	8	8	8	8	8
2 5 8 11	9	9	9	9	9	9	10	10	10	10	10	10	11	11	11	11	11	11
	2	2	2	5	5	8	0	0	0	3	3	6	1	1	1	4	4	7
	5	8	11	8	11	11	3	6	9	6	9	9	4	7	10	7	10	10

Example 6.

(3.6) $v = 14,\ k = 3,\ b = 42,\ r = 9, \lambda_1 = 6, \lambda_2 = 1,\ m = 7,\ n = 2,\ n_1 = 1,\ n_2 = 12$

By a similar investigation to that of Example 4, we have a generating set for this design:

$$S: \quad \begin{array}{ccccccc}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 \\
1 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 2
\end{array}$$

0 7	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
1 8	7	7	7	7	7	7	8	8	8	8	8	8	9	9	9	9	9	9
2 9	1	8	2	9	3	10	2	9	3	10	4	11	3	10	4	11	5	12
3 10																		
4 11	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5
5 12	10	10	10	10	10	10	11	11	11	11	11	11	12	12	12	12	12	12
6 13	4	11	5	12	6	13	5	12	6	13	0	7	6	13	0	7	1	8
	6	6	6	6	6	6												
	13	13	13	13	13	13												
	0	7	1	8	2	9												

Note that the above selecting set is obtained from the transpose of the incidence matrix of a BIB design with parameters

$$v^* = 7,\ k^* = 2,\ b^* = 21,\ r^* = 6, \lambda^* = 1 \quad ,$$

by changing one-half of the '1' in each column and in each row for '2'. Similar fact will be seen for the generating set S' given in Example 4.

Example 7.

(3.7) $v = 15,\ k = 4,\ b = 30,\ r = 8, \lambda_1 = 6, \lambda_2 = 1,\ m = 5,\ n = 3,\ n_1 = 2,\ n_2 = 12$

From the incidence matrix of a BIB design with parameters

$$v^* = 5,\ k^* = 2,\ b^* = 10,\ r^* = 4, \lambda^* = 1,$$

we easily have a selecting set for the above GD design (3.7):

$$S; \quad \begin{array}{ccccc} 3 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 \end{array}$$

The association scheme and the blocks are given by

0	5	10
1	6	11
2	7	12
3	8	13
4	9	14

0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
5	5	5	5	5	5	6	6	6	6	6	6	7	7	7	7	7	7
10	10	10	10	10	10	11	11	11	11	11	11	12	12	12	12	12	12
1	6	11	2	7	12	2	7	12	3	8	13	3	8	13	4	9	14

3	3	3	3	3	3	4	4	4	4	4	4
8	8	8	8	8	8	9	9	9	9	9	9
13	13	13	13	13	13	14	14	14	14	14	14
4	9	14	0	5	10	0	5	10	1	6	11

Example 8.

(3.8) $v = 15,\ k = 5,\ b = 30,\ r = 10, \lambda_1 = 8, \lambda_2 = 2,\ m = 5,\ n = 3,\ n_1 = 2,\ n_2 = 12$

A selecting set for this design is obtained by changing '1' for '2' in the selecting set S given in Example 7:

$$S: \quad \begin{array}{ccccc} 3 & 2 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 3 & 2 \\ 2 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 3 \end{array}$$

The association scheme and the blocks are

0	5	10
1	6	11
2	7	12
3	8	13
4	9	14

0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
5	5	5	5	5	5	6	6	6	6	6	6	7	7	7	7	7	7
10	10	10	10	10	10	11	11	11	11	11	11	12	12	12	12	12	12
1	1	6	2	2	7	2	2	7	3	3	8	3	3	8	4	4	9
6	11	11	7	12	12	7	12	12	8	13	13	8	13	13	9	14	14

3	3	3	3	3	3	4	4	4	4	4	4
8	8	8	8	8	8	9	9	9	9	9	9
13	13	13	13	13	13	14	14	14	14	14	14
4	4	9	0	0	5	0	0	5	1	1	6

Example 9.

(3.9) $v = 18,\ k = 5,\ b = 36,\ r = 10, \lambda_1 = 8, \lambda_2 = 2,\ m = 9,\ n = 2,\ n_1 = 1,\ n_2 = 16$

Firstly, let us check the selecting set

$$S = \{ (2,2,1,0,0,0,0,0,0) \rightarrow \} \ .$$

This set contains 252 tuples and generates a GD design with parameters

$$v = 18,\ k = 5,\ b = 504,\ r = 140,\ \lambda_1 = 112, \lambda_2 = 28 \ ,$$

which is the 14-plicate of the design (3.9) provided that the latter exists

Since 252 is divided by 14, 252 = 14 x 18, there might be a subset of S, consisting of 18 tuples in S, which generates the design (3.9). If so, each column of the arrangements (2.1) of these 18 tuples must contain 4 '2' and 2 '1', as will be enumerated from the values of r and λ_1 given by (3.9).

Now, let us take any two columns, the $(i + 1)$-th and the $(j + 1)$-th, and let x and y be the numbers of pairs (2,2) and (2,1) (unordered) among their 18 rows, respectively. Then, the number of incidences of any two treatments, one from the $(i + 1)$-th group and the other from the $(j + 1)$-th, is given by

$$\lambda_2(i,j) = (4x + 2y)2/4 = 2x + y.$$

Since this must be equal to 2 for any pair (i,j) $(i \neq j)$, the only allowable cases are $x = 1,\ y = 0$, and $x = 0,\ y = 2$.

Hence one may seek for such an arrangement of 18 distinct permutations of (2,2,1,0 0 0 0 0 0) that (i) each column contains 4 '2' and 2'1', and (ii) any two columns contain the pair (2,2) exactly once and the pairs (2,0), (1,0) or (0,0) elsewhere, or the pair (2,1) exactly twice and the pairs (2,0), (1,0) or (0,0) elsewhere, among their 18 rows.

Such an arrangement is given by

$$
S: \quad \begin{matrix}
2&2&0&0&0&1&0&0&0\\
2&0&2&0&0&0&1&0&0\\
2&0&0&2&0&0&0&1&0\\
2&0&0&0&2&0&0&0&1\\
0&2&2&0&0&0&0&0&1\\
0&2&0&2&0&0&1&0&0\\
0&2&0&0&2&0&0&1&0\\
0&0&2&0&2&1&0&0&0\\
0&0&2&1&0&0&0&2&0\\
0&0&1&2&0&0&0&0&2\\
0&0&0&2&1&2&0&0&0\\
0&0&0&1&2&0&2&0&0\\
1&0&0&0&0&2&0&2&0\\
0&1&0&0&0&2&0&0&2\\
0&0&1&0&0&2&2&0&0\\
1&0&0&0&0&0&2&0&2\\
0&1&0&0&0&0&2&2&0\\
0&0&0&0&1&0&0&2&2
\end{matrix}
$$

The association scheme and the blocks are given by

0 9	0	0	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2
1 10	9	9	9	9	9	9	9	9	10	10	10	10	10	10	11	11	11	11
2 11	1	1	2	2	3	3	4	4	2	2	3	3	4	4	4	4	7	7
3 12	10	10	11	11	12	12	13	13	11	11	12	12	13	13	13	13	16	16
4 13	5	14	6	15	7	16	8	17	8	17	6	15	7	16	5	14	3	12
5 14																		
6 15																		
7 6	3	3	3	3	4	4	5	5	5	5	5	5	6	6	6	6	7	7
8 17	12	12	12	12	13	13	14	14	14	14	14	14	15	15	15	15	16	16
	8	8	5	5	6	6	7	7	8	8	6	6	8	8	7	7	8	8
	17	17	14	14	15	15	16	16	17	17	15	15	17	17	16	16	17	17
	2	11	4	13	2	11	0	9	1	10	2	11	0	9	1	10	4	13

Example 10.

(3.10) $v = 18,\ k = 9,\ b = 18,\ r = 9, \lambda_1 = 8, \lambda_2 = 4,\ m = 9,\ n = 2,\ n_1 = 1,\ n_2 = 16$

By a similar investigation as in Example 9, it is not so difficult to find out a selecting set for this design:

$$
S: \quad \begin{array}{ccccccccc}
2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 2 \\
2 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 2 \\
1 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 & 2 & 0 & 1 & 2 \\
0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 1 \\
0 & 0 & 2 & 1 & 2 & 0 & 2 & 0 & 2 \\
0 & 0 & 1 & 2 & 2 & 2 & 0 & 2 & 0
\end{array}
$$

Association scheme and blocks are

0 9	0	0	0	0	0	0	0	0	1	1	1	1	1	1	2	2	3	3
1 10	9	9	9	9	9	9	9	9	10	10	10	10	10	10	11	11	12	12
2 11	1	1	4	4	2	2	3	3	4	4	2	2	3	3	4	4	4	4
3 12	10	10	13	13	11	11	12	12	13	13	11	11	12	12	13	13	13	13
4 13	2	2	7	7	6	6	5	5	5	5	5	5	6	6	6	6	5	5
5 14	11	11	16	16	15	15	14	14	14	14	14	14	15	15	15	15	14	14
6 15	3	3	8	8	7	7	8	8	6	6	8	8	7	7	8	8	7	7
7 16	12	12	17	17	16	16	17	17	15	15	17	17	16	16	17	17	16	16
8 17	4	13	1	10	5	14	6	15	0	9	7	16	8	17	3	12	2	11

Example 11.

(3.11) $v = 20,\ k = 7,\ b = 20,\ r = 7, \lambda_1 = 6, \lambda_2 = 2,\ m = 10,\ n = 2,\ n_1 = 1,\ n_2 = 18$

This design is generated by the selecting set

$$
S: \quad \begin{array}{cccccccccc}
2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 \\
0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\
0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2
\end{array}
$$

The association scheme and the blocks are given by

0 10	0	0	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	3	3
1 11	10	10	10	10	10	10	11	11	11	11	12	12	12	12	13	13	13	13	13	13
2 12	1	1	4	4	5	5	4	4	6	6	5	5	7	7	4	4	5	5	8	8
3 13	11	11	14	14	15	15	14	14	16	16	15	15	17	17	14	14	15	15	18	18
4 14	2	2	7	7	6	6	9	9	8	8	8	8	9	9	6	6	7	7	9	9
5 15	12	12	17	17	16	16	19	19	18	18	18	18	19	19	16	16	17	17	19	19
6 16	3	13	8	18	9	19	5	15	7	17	4	14	6	16	2	12	1	11	0	10
7 17																				
8 18																				
9 19																				

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