A NOTE ON THE COMPATIBILITY OF DISTRIBUTION FUNCTIONS

by

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A class of random variables is generally defined in either of two ways. One, as a class of measurable functions over a space with a probability measure, the other, by defining each element in terms of the properties it shares with the other members of the class - these properties being characterized by distribution functions. The latter development usually takes this form: associated with each finite subset $t_1, \ldots, t_n$ of some set $T$ we have a distribution function $F_{t_1, \ldots, t_n}$. This system of distribution must satisfy the well known consistency relations.

Let the integer $k$ be fixed. Suppose that associated with each finite subset $t_1, \ldots, t_n$ of $T$, with $n \leq k$, we have a distribution function $F_{t_1, \ldots, t_n}$ and that this system satisfies the consistency relations. Call such a system a $k$-fold system of distribution functions.

Can a $k$-fold system of distribution functions always be extended to define a class of random variables?

This question merits some consideration. Most of the convergence criteria for sequences of random variables are determined by the 2-fold distribution functions, as is the covariance function of a stochastic process. One may seek an example of a stochastic process with a particular property determined by the $k$-fold system of distribution functions, and having specified a satisfactory $k$-fold system, may ask whether a stochastic process exists with this $k$-fold system.

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The question also appears to have some significance in the axiomatic development of probability in terms of random experiments. We refer particularly to Cramer, *Mathematical Methods of Statistics*, paragraph 14.2. An affirmative answer to the above questions would seem necessary for the appropriateness of Axiom 3 of that paragraph.

The following example shows that, at least for 2-fold systems an extension is not always possible.

Let \( x_1, x_2, x_3 \) be any three elements from \( \{x_t\} \), \( t \in T \). Let \( a \neq b \) be two real numbers. Then the relations

\[
P \left( \sum x_i = a, x_j = b \right) = P \left( \sum x_i = b, x_j = a \right) = \frac{1}{2}, \quad i = 1, 2, 3;
\]

\( j = 1, 2, 3; \ i \neq j \), uniquely determines a consistent 2-fold system of distribution functions. But no distribution function for \( x_1, x_2, x_3 \) exists which is consistent with this system. For any such distribution function will be completely specified by the eight values

\[
P \left( \sum x_1 = c, x_2 = d, x_3 = e \right)
\]

where \( c, d, \) and \( e \) are each equal to \( a \) or \( b \). Since at least two of them, say \( c \) and \( d \), are equal, we have, by the consistency requirement,

\[
P \left( \sum x_1 = c, x_2 = c, x_3 = e \right) \leq P \left( \sum x_1 = c, x_2 = c \right) = 0
\]

The question of the compatibility of distribution functions must thus be confined to specific \( k \)-fold systems and specific values of \( k \).