A NOTE ON THE COMPATIBILITY OF DISTRIBUTION FUNCTIONS

by

George Marsaglia

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By George Marsaglia.
Montana State University and University of North Carolina

A class of random variables is generally defined in either of two ways. One, as a class of measurable functions over a space with a probability measure, the other, by defining each element in terms of the properties it shares with the other members of the class - these properties being characterized by distribution functions. The latter development usually takes this form: associated with each finite subset $t_1, \ldots, t_n$ of some set T we have a distribution function $F_{t_1, \ldots, t_n}$. This system of distribution must satisfy the well known consistency relations.

Let the integer k be fixed. Suppose that associated with each finite subset $t_1, \ldots, t_n$ of T, with $n \leq k$, we have a distribution function $F_{t_1, \ldots, t_n}$, and that this system satisfies the consistency relations. Call such a system a k-fold system of distribution functions.

Can a k-fold system of distribution functions always be extended to define a class of random variables?

This question merits some consideration. Most of the convergence criteria for sequences of random variables are determined by the 2-fold distribution functions, as is the covariance function of a stochastic process. One may seek an example of a stochastic process with a particular property determined by the k-fold system of distribution functions, and having specified a satisfactory k-fold system, may ask whether a stochastic process exists with this k-fold system.

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The question also appears to have some significance in the axiomatic development of probability in terms of random experiments. We refer particularly to Cramer, Mathematical Methods of Statistics, paragraph 14.2 An affirmative answer to the above questions would seem necessary for the appropriateness of Axiom 3 of that paragraph.

The following example shows that, at least for 2-fold systems an extension is not always possible.

Let x_1, x_2, x_3 be any three elements from $\{x_t\}$, $t \in T$. Let $a \neq b$ be two real numbers. Then the relations

$$P[x_i = a,\ x_j = b] = P[x_i = b,\ x_j = a] = \tfrac{1}{2}, \quad i = 1, 2, 3;$$

$j = 1, 2, 3;\ i \neq j,$ uniquely determines a consistent 2-fold system of distribution functions. But no distribution function for x_1, x_2, x_3 exists which is consistent with this system. For any such distribution function will be completely specified by the eight values

$$P[x_1 = c,\ x_2 = d,\ x_3 = e]$$

where c, d, and e are each equal to a or b. Since at least two of them, say c and d, are equal, we have, by the consistency requirement,

$$P[x_1 = c,\ x_2 = c,\ x_3 = e] \leq P[x_1 = c,\ x_2 = c] = 0$$

The question of the compatibility of distribution functions must thus be confined to specific k-fold systems and specific values of k.