SOME POISSON-TYPE LIMIT THEOREMS FOR SEQUENCES
OF DEPENDENT RARE EVENTS, WITH APPLICATIONS
by
Richard M. Meyer
University of North Carolina
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DEPARTMENT OF STATISTICS
University of North Carolina
Chapel Hill, N. C.
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INTRODUCTION AND SUMMARY

The main objective of this dissertation is to develop and apply generalizations of some well-known results, valid for sequences of independent events (or random variables) to sequences of dependent events (or random variables). In particular we consider some generalizations of such results as the Poisson limit of the binomial, the Poisson limit for Poisson-Bernoulli trials, and a Borel-Cantelli lemma to the case of dependent events (or random variables). These generalizations turn out to have many interesting and diverse applications. We consider two applications; one relates to a probability model for the quantum-biophysics of vision (essentially the initiation of periods when k or more servers are busy in an M/G/\infty or G/M/\infty queueing system), the other relates to the problem of upcrossings of a high level by certain discrete time stochastic processes.

In Chapter 1 we develop some general theorems for sequences of certain types of dependent events. In particular, we extend the Poisson limit theorem (for sequences of independent and identical binomial events) to f(n)-dependent and certain strongly mixing sequences of events. Furthermore, the Poisson limit for Poisson-Bernoulli trials is also extended to such sequences of events. A dependent analogue of a Borel-Cantelli lemma is used in the process of establishing the above result. The motivation for many of these results is the work of Watson [19] and Loynes [12] both of which deal specifically with discrete time
stationary stochastic processes. By using a multivariate analogue of
the Bonferroni Inequalities, also developed in this dissertation, we
establish multivariate analogues of the Watson and Loynes results, and
these have applications in certain investigations concerning asymptotic
independence of groups of dependent events.

Chapter 2 deals with a very specific probability model for the
quantum-biophysics of vision considered, for example, by Ikeda [10].
This work, in fact, originally motivated a closer analysis of \( f(n) \)-
dependent events and this lead to the work of Chapter 1. This chapter
is essentially an application of Corollary 1.1.3.1 (i.e. essentially a
Poisson-type result for dependent events). The work provides an
approach to the asymptotic behavior of the model not before considered,
and it provides some new and interesting results.

In Chapter 3 we apply the general results of Chapter 1 to obtain
some new and interesting results on the asymptotic properties of up-
crossings of a high level by certain discrete time stochastic processes.
The results, valid for a broad class of discrete-time processes, are
similar to those obtained by Cramer and Leadbetter [4] for certain con-
tinuous time normal stationary processes.
CHAPTER I

POISSON-TYPE LIMIT THEOREMS FOR SEQUENCES OF m-DEPENDENT,
f(n)-DEPENDENT AND STRONGLY MIXING 'RARE' EVENTS

1.0 Summary

In this chapter we establish several Poisson-type limit theorems for 'rare' events which are not necessarily independent. These results are generalizations of well-known (see Feller [5], von Mises [13], and Hodges and LeCam [7]) Poisson-type limit theorems for sequences of independent binomial (multinomial) events. Specific results are given for certain sequences of m-dependent f(n)-dependent and strongly mixing 'rare' events. For example, under mild assumptions on the probabilities of f(n)-dependent 'rare' events, it is shown that the probability that exactly k among n of them occur is asymptotically Poisson as n → ∞. The results obtained will be used to obtain useful generalizations of a result due to Watson [19] dealing with the asymptotic distribution of extreme values of certain m-dependent strictly stationary stochastic processes. Results related to those obtained by Loynes [12] concerning strongly mixing processes will also be given.

1.1 Introduction and preliminary results

There has been considerable interest in extending the validity of certain limit results, known to hold under certain 'ideal' conditions, to situations where these 'ideal' conditions are not satisfied. In this chapter we shall, in particular, be concerned with extending the validity
of certain Poisson-type limit results for 'rare' events to 'non-ideal' situations which allow for some dependence.

In a sense, such results serve to establish the robustness of the asymptotic result under deviations from 'ideal' conditions. Of course, the ultimate goal of such investigations is to find necessary and sufficient conditions under which the limit result of interest is valid. Needless to say, such a task is generally very difficult. Frequently, too, it is impossible to evaluate the 'closeness' of the asymptotic result to the finite situation.

The following investigation was originally suggested by a theorem of Watson [19] concerning the limiting behavior of extreme values for certain m-dependent strictly stationary stochastic processes. We shall begin by stating Watson's original theorem. First we state two basic definitions.

**Definition 1.1.1** A sequence of p-dimensional random vectors 
\[ \{X_i, i = 1, 2, \ldots\} \] is termed strictly stationary if (i) for every pair of non-negative integers \( n \) and \( h \), and (ii) for every choice \( i_1, i_2, \ldots, i_n \) of \( n \) distinct indices, and (iii) for every choice \( S_1, S_2, \ldots, S_n \) of \( n \) p-dimensional (Borel) sets, we have that

\[
\begin{align*}
    P(\bigcap_{j=1}^{n} X_{i_j} \in S_j) &= P(\bigcap_{j=1}^{n} X_{i_j+h} \in S_j).
\end{align*}
\]

**Definition 1.1.2** A sequence of p-dimensional random vectors 
\[ \{X_i, i = 1, 2, \ldots\} \] is termed \( f(n) \)-dependent (\( f(n) \) a non-negative integer \( \leq n \)) provided the following condition is satisfied: \( s+r > f(n) \) implies that the two sets of random vectors \( \{X_1, X_2, \ldots, X_r\} \) and \( \{X_s, X_{s+1}, \ldots, X_n\} \) are independent.

**Definition 1.1.2** of \( f(n) \)-dependence, introduced by Hoeffding and
Robbins [8], implies that any number of blocks of successive terms from 
\( \{X_i, \ i=1,2,\ldots \} \) are independent whenever the largest subscript is not
larger than \( n \) and successive blocks are separated by more than \( f(n) \)
indices. An important special case is when \( f(n) = m \) for all \( n \) (\( n \geq m \)).
In such a case the sequence \( \{X_i (i=1,2,\ldots)\} \) is termed \( m \)-dependent.

We now state the theorem due to Watson [19].

Theorem 1.1.1 Let \( \{X_i (i=1,2,\ldots)\} \) be an \( m \)-dependent strictly stationary
sequence of random variables which are unbounded above, and which have
the property that

\[
\lim_{c \to \infty} \frac{1}{P(X_i > c)} \max_{0 < |i-j| \leq m} P(X_i > c, X_j > c) = 0.
\]

Then if \( \xi = n P(X_i > c_n(\xi)) \) for \( \xi > 0 \) fixed,

\[
\lim_{n \to \infty} P[X_i \leq c_n(\xi) (i=1,2,\ldots,n)] = e^{-\xi}.
\]

This result, already a generalization of a well-known result for
sequences of iid random variables, may be expanded in several directions.
First, Watson's theorem may be generalized so as to include \( m \)-dependent
vector-valued strictly stationary sequences, along with 'extremal sets'
more general than half-infinite intervals. Second, the limiting proba-
bilities of events such as \( X_i > c_n(\xi) \) for exactly \( k \) among \( i=1,2,\ldots,n \)
may be derived. In addition, the frequency with which the \( X_i \)'s fall in
the cells of a multichotomfy rather than a dichotomy may be considered.
Finally, more general types of dependence may be considered, and the
requirement of stationary may be relaxed.

A closer examination of the above situation suggests that a fruit-
ful approach is to abandon reference to a stochastic process altogether,
and instead deal only with events. Such an approach does, in fact, lead to general results about Poisson-type limit theorems, of which the Watson theorem and its generalizations become special cases. We shall have need of the analogue of Definition 1.1.2 for events. For completeness we state it here.

**Definition 1.1.3** A sequence of events \( \{E_i(i=2,2,\ldots)\} \) is termed \( f(n) \)-dependent provided the following condition is satisfied: \( s-r > f(n) \) implies that the two collections of events \( \{E_1,\ldots,E_r\} \) and \( \{E_s,\ldots,E_n\} \) are mutually independent.

The remarks following Definition 1.1.2 also specialize to the case at hand.

In order to provide a proper setting for treating Poisson-type limit theorems for 'rare' events, for each \( n \) we introduce a basic probability space \( (\Omega_n,\mathcal{A}_n,\mathbb{P}^n) \) and sequence of events \( \{A_i^n(i=1,2,\ldots)\} \) therein. For notational convenience we shall suppress the index \( n \) on the probability measure \( \mathbb{P}^n \), and we shall, for example, write \( \mathbb{P}(A_i^n) \) instead of the (proper) expression \( \mathbb{P}^n(A_i^n) \). There may or may not be a relationship between the various probability spaces. It is felt that no confusion will arise from our notational simplification.

It is well-known that if, for each \( n \), the events \( \{A_i^n(i=1,\ldots,n)\} \) are independent and \( \mathbb{P}(A_i^n) = \xi/n \ (\xi > 0; i=1,2,\ldots,n) \), then for any non-negative integer \( k \), \( \mathbb{P}\{\text{exactly } k \text{ among } A_i^n \ (i=1,\ldots,n) \text{ occur}\} \rightarrow e^{-\xi} \xi^k/k! \) as \( n \rightarrow \infty \). This is a simple example of the Poisson limit of the binomial distribution. It is not clear, however, how robust such a result is. That is, how much and what kind of dependence can exist amongst the events \( \{A_i^n(i=1,2,\ldots,n)\} \) and still have the same Poisson
limit result? It will be seen that the following theorem, concerning
\( f(n) \)-dependent events, gives a sufficient condition. A generalization
of Watson's theorem is a special case. Our proof is patterned after
Watson [19].

**Theorem 1.1.2** For each \( n(n \geq m) \) let \( \{A_i^n(i=1,2,\ldots,n)\} \) be a sequence of
\( m \)-dependent events such that for some \( \xi > 0 \) we have, as \( n \to \infty \),

\[
(1.1.4) \quad a_n = \max_{j=1,\ldots,n} |P(A_j^n)-\xi/n| = o(1/n)
\]

and

\[
(1.1.5) \quad b_n = \max_{i,j=1,2,\ldots,n} P(A_i^n|A_j^n) = o(1).
\]

Then, as \( n \to \infty \),

\[
(1.1.6) \quad P(\text{exactly } k \text{ amongst } A_i^n(i=1,\ldots,n) \text{ occur}) \to e^{-\xi k/k!}.
\]

**Proof.** Let \( S_r^n = \sum_{i=1}^{r} P(A_i^n \ldots A_i^n) \), the summation extending over all
\( r \)-tuples \( (i_1,\ldots,i_r) \) such that \( 1 \leq i_1 < \ldots < i_r \leq n \). We first consider
the case \( k=0 \). By the Bonferroni Inequalities (see, for example, Feller
[5] p. 100) we have for any even integer \( \ell, 0 \leq \ell \leq n \),

\[
(1.1.7) \quad b^n_{\ell,0} \leq P(\bigcap_{i=1}^{n} A_i^n) \leq u^n_{\ell,0},
\]

where

\[
(1.1.8) \quad b^n_{\ell,0} = 1 - S_1^n + S_2^n - \ldots + (-1)^{\ell-1} S^n_{\ell-1}
\]

and

\[
(1.1.9) \quad u^n_{\ell,0} = 1 - S_1^n + S_2^n - \ldots + (-1)^{\ell} S^n_{\ell}.
\]

We have from (1.1.4) that \( S_1^n = \sum_{i=1}^{n} P(A_i^n) \to \xi \) as \( n \to \infty \) since,

\[
0 \leq |S_1^n - \xi| + \sum_{i=1}^{n} |P(A_i^n) - \xi| \leq \sum_{i=1}^{n} |P(A_i^n) - \xi/n| \leq n a_n \to 0.
\]
Now we consider \( S_2^n \). By \( m \)-dependence we have

\[
(1.1.10) \quad S_2^n = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(A_i^n A_j^n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{m'} P(A_i^n A_j^n) + \sum_{i=1}^{n-m-1} \sum_{j=m'+1}^{n} P(A_i^n) P(A_j^n),
\]

where \( m' = \min(n, i+m) \). From (1.1.5) we have that as \( n \to \infty \),

\[
(1.1.11) \quad \sum_{i=1}^{n-1} \sum_{j=i+1}^{m'} P(A_i^n A_j^n) \leq nm \left[ 1 + \frac{m+1}{2n} \right] \max_{0 < |i-j| \leq m} P(A_i^n A_j^n) \sim \xi m \left[ 1 + \frac{m+1}{2n} \right] b_n \to 0.
\]

Further, the uniformity of condition (1.1.4) guarantees that as \( n \to \infty \),

\[
(1.1.12) \quad \sum_{i=1}^{n-m-1} \sum_{j=m'+1}^{n} P(A_i^n) P(A_j^n) \sim \frac{1}{2} (n-m)(n-m-1) \left[ \frac{n^2}{n^2 + o(1/n^2)} \right] \to \frac{\xi^2}{2!}.
\]

The general term \( S_r^n \) \((r \geq 2)\) contains \( \binom{n}{r} \) terms. Let \( C_j^n \) \((j=0,1,\ldots,r)\) denote that subset of these terms in which exactly \( j \) of the \( A_i^n \)'s appearing differ in subscript by more than \( m \) from their nearest neighbor.

(Assume \( n \) is large enough for all cases to be possible, and observe that \( C_r^n = C_{r-1}^n \).) Letting \( N_j^n \) denote the number of elements in \( C_j^n \), it is known (see Watson [19]) that \( N_j^n = 0(n^{j+1}) \) for \( j < r-1 \) and \( N_r^n = N_{r-1}^n \sim n^{r}/r! \) as \( n \to \infty \). Using \( m \)-dependence and (1.1.4) it then follows that the contribution to \( S_r^n \) of the elements in \( C_j^n \) \((j < r-1)\) is bounded by

\[
(1.1.13) \quad K \cdot n^j \left[ \frac{\xi}{n} + o(1/n) \right]^j b_n,
\]

where \( K \) is a constant independent of \( n \). From (1.1.5) the expression in (1.1.13) tends to zero as \( n \to \infty \). However, the contribution to \( S_r^n \) of the elements in \( C_r^n \) is, asymptotically,
\[(1.1.14) \quad \frac{n^r}{r!} \left[ \frac{\xi}{n} + o\left(\frac{1}{n}\right) \right]^r \rightarrow \frac{\xi^r}{r!}. \]

Thus, for \( r \geq 2 \), \( S_n^r \rightarrow \frac{\xi^r}{r!} \) as \( n \rightarrow \infty \), and so for every even integer \( \ell \), we have established that

\[(1.1.15) \quad \sum_{j=0}^{\ell-1} \frac{(-\xi)^j}{j!} \leq \liminf_{n \rightarrow \infty} P\left( \bigcap_{i=1}^{n} A_i^n \right) \leq \limsup_{n \rightarrow \infty} P\left( \bigcap_{i=1}^{n} A_i^n \right) \leq \sum_{j=0}^{\ell} \frac{(-\xi)^j}{j!}. \]

Since \( \ell \) is arbitrary,

\[(1.1.16) \quad \lim_{n \rightarrow \infty} P\left( \bigcap_{i=1}^{n} A_i^n \right) = e^{-\xi}. \]

For \( k > 0 \), again using the Bonferroni inequalities, it follows that for any even integer \( \ell \) such that \( k + \ell \leq n \),

\[(1.1.17) \quad B_{\ell,k}^n \leq P\{ \text{exactly } k \text{ amongst } A_i^n (i = 1, \ldots, n) \text{ occur} \} \leq u_{\ell,k}^n, \]

where

\[(1.1.18) \quad B_{\ell,k}^n = S_k^n - \binom{k+1}{k} S_{k+1}^n + \ldots + (-1)^{\ell-1} \binom{k+\ell-1}{k} S_{k+\ell-1}^n, \]

and

\[(1.1.19) \quad u_{\ell,k}^n = S_k^n - \binom{k+1}{k} S_{k+1}^n + \ldots + (-1)^{\ell} \binom{k+\ell}{k} S_{k+\ell}^n. \]

We have already shown that \( S_{k+j}^n \rightarrow \frac{\xi^{k+j}}{(k+j)!} \) as \( n \rightarrow \infty \), and so

\[(1.1.20) \quad \lim_{n \rightarrow \infty} B_{\ell,k}^n = \sum_{j=0}^{\ell-1} \binom{k+j}{k} (-1)^j \frac{\xi^{k+j}}{(k+j)!} = \frac{\xi^k}{k!} \left[ \sum_{j=0}^{\ell-1} \frac{(-\xi)^j}{j!} \right]. \]

and
\[
\lim_{n \to \infty} \sum_{k=0}^{\ell} \binom{k+j}{k} (-1)^j \frac{\xi^{k+j}}{(k+j)!} = \frac{\xi^k}{k!} \left[ \sum_{j=0}^{\ell} \frac{(-\xi)^j}{j!} \right].
\]

Thus, since \(\xi\) is arbitrary,

\[
P\text{(exactly } k\text{ amongst } A^i_k(i=1,\ldots,n)\text{ occur)} \to e^{-\xi} \frac{\xi^k}{k!} \text{ as } n \to \infty.
\]

With this the proof is completed.

As one might suspect, the conditions (1.1.4) and (1.1.5) can be still further weakened and yet have (1.1.6) remain valid. We shall pursue this point later. For the moment, we shall concentrate on some immediate consequences of this particular (robust) version of the Poisson limit to the binomial.

First, it is clear that Watson's theorem is a special case of Theorem 1.1.2. If \(\{x_i (i=1,2,\ldots)\}\) is an \(m\)-dependent strictly stationary sequence of random variables, unbounded above, define

\[
A^n_j = \{x_j > c_n(\xi)\} \text{ where } c_n(\xi) \text{ is chosen so that } \frac{\xi}{n} = P(x_j > c_n(\xi)), \quad (\xi > 0).
\]

If the hypotheses of Watson's Theorem 1.1.1 are satisfied, the sets \(\{A^n_j(j=1,2,\ldots,n)\}\) satisfy the conditions of Theorem 1.1.2. In fact, we state two vector-valued versions (one stationary, one not necessarily stationary) of Watson's original result. They will be of use in Chapter II.

**Corollary 1.1.2.1** Let \(\{Y_i (i=1,2,\ldots)\}\) be a strictly stationary sequence of \(p\)-dimensional, \(m\)-dependent random vectors. Further, let

\[
S_n = S_n(\xi)(\xi > 0, n = 1,2,\ldots) \text{ be a sequence of } p\text{-dimensional (Borel)}
\]

sets such that as \(n \to \infty\)

\[
(1.1.21) \quad P(Y_i \in S_n) = \frac{\xi}{n}
\]
and

\[ (1.1.22) \quad \max_{0 < |i-j| \leq m} P(Y_i \in S_n \mid Y_j \in S_n) \rightarrow 0. \]

Then \( P(Y_i \in S_n \text{ for exactly } k \text{ amongst } i = 1, 2, \ldots, n) \rightarrow e^{-\xi \xi/k!} \) as \( n \rightarrow \infty \).

Note that Theorem 1.1.2 applies to sequences of \( m \)-dependent events all of whose elements need not have equal probability. Condition (1.1.4) requires only that the differences amongst the probabilities of the events in the sequence be uniformly \( o(1/n) \) as \( n \rightarrow \infty \). Thus, Corollary 1.1.2 has an analogue for sequences of \( p \)-dimensional \( m \)-dependent random vectors that are not necessarily stationary. For completeness we state this analogue.

**Corollary 1.1.2.2** Let \( \{Y_i (i=1, 2, \ldots)\} \) be a sequence of \( p \)-dimensional, \( m \)-dependent random vectors. Let \( S_n = S_n(\xi) \) \( (\xi > 0; n=1, 2, \ldots) \) be a sequence of \( p \)-dimensional (Borel) sets such that as \( n \rightarrow \infty \).

\[ (1.1.23) \quad a_n' = \max_{i=1, \ldots, n} \left| P(Y_i \in S_n) - \frac{\xi}{n} \right| = o(1/n) \]

and

\[ (1.1.24) \quad b_n' = \max_{0 < |i-j| \leq m} P(Y_i \in S_n \mid Y_j \in S_n) = o(1). \]

Then, as \( n \rightarrow \infty \),

\[ P(Y_i \in S_n \text{ for exactly } k \text{ amongst } i=1, 2, \ldots, n) \rightarrow e^{-\xi \xi/k!}. \]

The case where \( \{A_i^n (i=1, 2, \ldots)\} \) are mutually independent suggests that Theorem 1.1.2 may be valid under certain weaker assumptions. In particular, the following result will be of use in later applications.

**Theorem 1.1.3** For each \( n \) \( (n \geq m) \) let \( \{A_i^n (i=1, 2, \ldots, n)\} \) be a sequence of \( m \)-dependent events such that for some \( \xi > 0 \), as \( n \rightarrow \infty \).
\[ a_n^i = \max_{j=1, \ldots, N} |P(A_j^n) - \xi/n| = o(1/n), \]

\[ b_n^i = \max_{0 < |i-j| \leq m} P(A_i^n|A_j^n) = o(1), \]

and

\[ N \sim n. \]

Then, as \( n \to \infty \),

\[ P\{\text{exactly } k \text{ amongst } A_i^n (i=1, \ldots, n) \text{ occur} \} \to e^{-\xi \xi^k/k!}. \]

**Proof.** It is easily seen that the proof of Theorem 1.1.2 carries through when \( n \) is replaced by \( N = n+o(n) \) in the appropriate places.

It will be of use in Chapter II to have the following related result.

**Corollary 1.1.3.1** Let \( (Y_i (i=1,2,\ldots)) \) be a stationary sequence of \( p \)-dimensional \( m \)-dependent random vectors. Further, let \( S_n = S_n(\xi) \) \((\xi > 0; n=1,2,\ldots)\) be a sequence of \( p \)-dimensional (Borel) sets such that as \( n \to \infty \),

\[ 0 < P(Y_i \in S_n) = \xi/n + o(1/n), \]

\[ b_n^i = \max_{0 < |i-j| \leq m} P(Y_i \in S_n|Y_j \in S_n) = o(1), \]

and

\[ N \sim n. \]

Then, as \( n \to \infty \)

\[ P\{Y_i \in S_n \text{ for exactly } k \text{ amongst } i=1,2,\ldots, N \} \to e^{-\xi \xi^k/k!}. \]

The following result elaborates the asymptotic Poisson process character of the occurrences of events of type \( A_i^n \). It is of interest in later applications, and also in suggesting future results in the
f(n)-dependence and strong-mixing cases.

Theorem 1.1.4 For each n let \( \{A^n_i : i=1,2,...\} \) be a sequence of m-dependent events satisfying the following conditions. For some \( \xi > 0 \) and as \( n \to \infty \),

\[
(1.1.33) \quad a_n = \max_{i=1,2,...} |P(A^n_i) - \xi/n| = o(1/n),
\]
and

\[
(1.1.34) \quad b_n = \max_{0 < |i-j| \leq m \atop i,j=1,2,...} P(A^n_i | A^n_j) = o(1).
\]

Further, let \( s \) and \( t \) (\( 0 \leq s < t \)) be any two real numbers. Then, as \( n \to \infty \),

\[
P^k_n(s,t) = P(\text{exactly } k \text{ amongst } A^{[sn]+1}_j, A^{[sn]+2}_j, ..., A^{[tn]}_j \text{ occur}) \to e^{-\xi(t-s)\left[\frac{\xi(t-s)}{k}\right]^k/k!}.
\]

Proof. For simplicity we shall treat \( sn \) and \( tn \) as integers. This avoids certain notational difficulties, and the theorem is valid whether or not this is the case. Let \( N = (t-s)n \), and \( B^n_j = A^{sn+j}_j \); \( j = 1,2,...,N \). Now using (1.1.33) we have

\[
(1.1.35) \quad N a_N = N \max_{j=1,...,N} \left| P(B^n_j) - \frac{\xi(t-s)}{N} \right| \leq N \max_{j=1,...,N} \left| P(A^n_j) - \frac{\xi/n}{(t-s)nax_n} \right| \to 0 \text{ as } n \to \infty.
\]

Furthermore, as \( n \to \infty \) (\( N \to \infty \)),

\[
b_N = \max_{0 < |i-j| \leq m \atop i,j=1,...,N} P(B^n_i | B^n_j) \leq \max_{0 < |i-j| \leq m \atop i,j=1,2,...} P(A^n_i | A^n_j) \to 0
\]

by (1.1.34). Thus, via Theorem 1.1.2 the desired result is established.
We note that under assumptions of Theorem 1.1.4, $P_n^k(0, t/\xi) \to e^{-t} t^k/k!$. Thus, judged in terms of 'time units' of length $n/\xi$, the waiting time until the $k$-th occurrence of an event of the type $A_i^n$ has a distribution function $F_k^n(t)$ satisfying the relationship

$$P[\text{Time to } k\text{-th event } > t] = 1 - F_k^n(t) = \sum_{j=0}^{k-1} P_j^n(0, t/s).$$

Since the right hand side of (1.1.36) tends to a limit as $n \to \infty$, $F_k^n(t)$ has a limit, say $F_k(t)$, for every fixed $k \geq 1$. Thus, in terms of 'time units' of length $n/\xi$, the distribution function $F_k(t)$ of the (asymptotic) waiting time until the $k$-th occurrence of an event of type $A_i^n$ is that of a random variable with a Gamma $(k)$ distribution, viz.

$$F_k(t) = 1 - (1+t/1!+t^2/2! + \ldots + t^{k-1}/(k-1)!) e^{-t}, \ t > 0.$$

Theorem 1.1.4 could have been stated in slightly more general terms. For example, if $a_{ij} = (a_{1j}, a_{2j})$, $j = 1, 2, \ldots, r$, are $r$ disjoint bounded intervals of $[0, \infty)$ and $\theta = \sum_{j=1}^r (a_{2j} - a_{1j})$, then under the assumptions of Theorem 1.1.4 the limiting probability that exactly $k$ amongst $A_i^n$ with subscripts $i$ in $\bigcup_{j=1}^n [a_{1j}+1, \ldots, a_{2j}]$ occur is $e^{-\theta \xi (a_{2j})^k/k!}$. We refer ahead to Corollary 1.3.1.2 for a more refined result.

1.2 A Poisson-Bernoulli-type limit result for $f(n)$-dependent events

In the preceding Section, the Poisson limit of the binomial was extended to include certain sequences of $m$-dependent events. These extensions permitted the individual events in the sequence to have differing probabilities, yet the differences were required to be, in a sense, uniformly $o(1/n)$ as $n \to \infty$.

In the present section we show that by examining the structure of
f(n)-dependent events more closely, an analogue of the Poisson-
Bernoulli limit theorem (see Feller [5] p. 264, Hodges and LeCam [7])
can be proved for certain sequences of f(n)-dependent events. Two
versions of such a result will be stated. In addition, an extension of
the results of the preceding section to sequences of f(n)-dependent
events is available upon specializing the results.

Von Mises [13] proved that if for each \( n \) \( \{ A^n_i; i=1,2,\ldots \} \) is a
sequence of mutually independent events such that as \( n \to \infty \),
\[
a_n = \sum_{i=1}^{n} P(A^n_i) = a < \infty \quad \text{and} \quad \max_{1 \leq i \leq n} P(A^n_i) \to 0, \]
then \( P(\text{exactly } k \text{ amongst } A^n_1(1,\ldots,n) \text{ occur}) \to e^{-a} a^k / k! \) as \( n \to \infty \). Hodges and LeCam [7]
proved a more powerful result, still for independent events, that
provided a bound on the difference between the limit value and the
actual finite state value of the above probability.

The work of this section provides for f(n)-dependent events a
result intermediate to the Von Mises and Hodges and LeCam results.
Unfortunately, no bound is available for the difference between the
limiting probability and its value at any finite stage.

**Theorem 1.2.1** For each \( n \), let \( \{ A^n_i(i=1,2,\ldots,n) \} \) be a sequence of
f(n)-dependent events. Let \( p^n_i = P(A^n_i) \) \( (i=1,2,\ldots,n) \) and define
\[
a_n = \sum_{i=1}^{n} p^n_i. \quad \text{Suppose that as } n \to \infty
\]
\[
(1.2.1) \quad \rho_n = \max_{1 \leq i \leq n} p^n_i = o(1/n),
\]
\[
(1.2.2) \quad f(n) = o(n),
\]
(1.2.3) \[ c_n^{(1)} = \max_{I^n_1} P(A_{i_1}^n | A_{i_2}^n ) = o(1/f(n)) \] and \[ c_n^{(r)} = \max_{I^n_{r+1}} P(A_{i_1}^n | A_{i_2}^n \ldots A_{i_{r+1}}^n ) = o(1/f(n)) \] where \( I^n_r, r=1,2,\ldots \) is the set \( \{(i_1,\ldots,i_{r+1})| 1 \leq i_1 < \ldots < i_{r+1} \leq n; i_{j+1}-i_j \leq f(n); i_j = 1,2,\ldots,n; j=1,\ldots,r\} \).

Then, as \( n \to \infty \),

(1.2.4) \[ P(\text{exactly } k \text{ amongst } A_{i_1}^n(i=1,\ldots,n) \text{ occur}) \sim e^{-\alpha_n} \alpha_n^k/k!. \]

**Proof.** We prove the theorem for the special case where \( \alpha_n \equiv a < \infty \); \( n = 1,2,\ldots \). It will be seen that the proof carries through with appropriate modifications, for the general case, since (1.2.1) implies that \( \{\alpha_n\} \) is a bounded sequence.

First we prove the theorem for the case \( k = 0 \). Let

(1.2.5) \[ P_k^n = P(\text{exactly } k \text{ amongst } A_{i_1}^n(i=1,2,\ldots,n) \text{ occur}), \]

and

(1.2.6) \[ S_r^n = \sum_{J_r} P(A_{i_1} \ldots A_{i_r}), \text{ where } J_r = \{(i_1,\ldots,i_r)| 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}. \]

Using the Bonferroni inequalities we have for any even integer \( \ell \), \( 0 \leq \ell \leq n \),

(1.2.7) \[ 1 - S_1^n + S_2^n - \ldots - S_{\ell-1}^n \leq P_0^n \leq 1 - S_1^n + S_2^n - \ldots + S_\ell^n. \]

We propose to show that \( S_r^n \to a^r/r! \) as \( n \to \infty \). Certainly this is true for \( r=1 \) since \( S_1^n = \sum_{i=1}^n P_i^n \equiv a \). Consider now \( S_2^n \). By definition
\( S_2^n = \sum_{1 \leq i < j \leq n} P(A_i^n A_j^n) = (\Sigma + \Sigma) P(A_1^n A_j^n) \),
where

\[ N = \{(i, j) | 1 \leq i < j \leq n, \quad j - i \leq f(n)\} \text{ and } F = \{(i, j) | 1 \leq i < j \leq n, \quad j - i > f(n)\}. \]

Now, \( \Sigma P(A_i^n A_j^n) = \Sigma P(A_i^n | A_j^n) P(A_j^n) \leq \rho_n \Sigma P(A_i^n | A_j^n) \leq n \rho_n f(n) c_n^1 \to 0 \)
as \( n \to \infty \) via (1.2.1) and (1.2.3). Furthermore, from \( f(n) \)-dependence,

\[ \Sigma P(A_i^n A_j^n) = \Sigma P_i^n P_j^n = \frac{a^2}{2} - \frac{1}{2} \sum_{i=1}^{n} (p_i^n)^2 - \Sigma p_i^n P_j^n. \]
Now using (1.2.1) and (1.2.2), \( 1/2 \sum_{i=1}^{n} (p_i^n)^2 \leq 1/2 \rho_n a \to 0 \) while \( \Sigma p_i^n P_j^n \leq n f(n) \rho_n^2 \to 0 \)
as \( n \to \infty \). Therefore, \( S_2^n \to a^2/2! \) as \( n \to \infty \).

We now consider \( S_r^n, r > 2 \). Recall,

\[ S_r^n = \sum_{\mathcal{C}} P(A_{i_1} \ldots A_{i_r}) = (\Sigma + \Sigma) P(A_{i_1} \ldots A_{i_r}), \text{ where} \]

\[ \mathcal{C} = \{(i_1, \ldots, i_r) | 1 \leq i_1 < \ldots < i_r \leq n\}, \]

\[ F = \mathcal{C} \setminus \{(i_1, \ldots, i_r) | i_r - i_r - 1 > f(n), \ldots, i_2 - i_1 > f(n)\}, \text{ and} \]

\[ N = \mathcal{C} \setminus F. \]

We shall show that as \( n \to \infty \), (a) \( \Sigma P(A_{i_1} \ldots A_{i_r}) \to a^r/r! \) and

(b) \( \Sigma P(A_{i_1} \ldots A_{i_r}) \to 0 \). We first prove (a). Note that via \( f(n) \)-dependence,

\[ \Sigma P(A_{i_1} \ldots A_{i_r}) = \sum_{\mathcal{C}} P_{i_1}^n \ldots P_{i_r}^n = (\Sigma - \Sigma) P_{i_1}^n \ldots P_{i_r}^n, \]

and via symmetry,

\[ \sum_{\mathcal{C}} P_{i_1}^n \ldots P_{i_r}^n = \frac{1}{r!} (a^r - \sum_{k=2}^{r} D_k P_{i_1}^n \ldots P_{i_r}^n), \]

where \( D_k = \{(i_1, \ldots, i_r) | 1 \leq i_1 \leq i_2 \leq \ldots \leq i_r \leq n \text{ and exactly } k \text{ of} \)

the $i_j$'s are equal). For fixed $k$, $(2 \leq k \leq r)$, and some constant $K$

independent of $n$,

$$\sum_{\Omega_n}^{K(n_0)_{r-k}} \sum_{j=1}^{n} (p_{j_1}^n)^k \leq K n^{-k} r^{-l} a \to 0$$

as $n \to \infty$ via (1.2.1). Thus to establish (a) it remains only to show

that $\sum_{N}^{p_{i_1}^n \ldots p_{i_r}^n} \to 0$ as $n \to \infty$. To do this, we introduce the following

notation. Let $\Lambda = \{\lambda_1, \ldots, \lambda_{r-1} \mid \lambda_i = 0, 1; i=1, \ldots, r-1; \Sigma_1 < r-1\}$. For

each $\lambda = (\lambda_1, \ldots, \lambda_{r-1}) \in \Lambda$ let $N_{\lambda}$ be the following subset of $N$:

$$N_{\lambda} = N \cap \{(\lambda_1, \ldots, \lambda_{r-1}) \mid \lambda_j = 1(0); j = 2, \ldots, r\}.$$

Clearly, $N = \bigcup_{\lambda \in \Lambda} N_{\lambda}$, and this is a (disjoint) union of $2^{r-1}-1$ terms.

We shall show that for each $\lambda \in \Lambda$, as $n \to \infty$

$$\sum_{N_{\lambda}}^{p_{i_1}^n \ldots p_{i_r}^n} \to 0.$$

By symmetry it will suffice to show that for fixed but arbitrary $k$,

$0 \leq k < r-1$,

$$\sum_{N_{\lambda^k}}^{p_{i_1}^n \ldots p_{i_r}^n} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$\lambda^k = (1, \ldots, 1, 0, \ldots, 0).$$

Now, assuming $n$ is large enough, we have

$$\sum_{N_{\lambda^k}}^{p_{i_1}^n \ldots p_{i_r}^n} = \sum_{i_r = \ell_r}^{u_r} \sum_{i_{r-1} = \ell_{r-1}}^{u_{r-1}} \ldots$$

$$\sum_{i_1 = \ell_1}^{u_1} \sum_{i_2 = \ell_2}^{u_2} \ldots$$

$$\sum_{i_{r-1} = \ell_{r-1}}^{u_{r-1}} \sum_{i_r = \ell_r}^{u_r}.$$
where
\[
\begin{align*}
\ell_1 &= 1 \\
\ell_2 &= (f(n)+1) + 1 \\
&\vdots \\
\ell_{k-1} &= (k-1)(f(n)+1) + 1 \\
\ell_k &= \max\{(k-1)(f(n)+1) + 2, i_{k-1} = f(n)\} \\
\ell_{k+1} &= \max\{(k-1)(f(n)+1) + 3, i_{k+1} = f(n)\} \\
&\vdots \\
\ell_r &= \max\{(k-1)(f(n)+1) + r+1-k, i_{r-1} = f(n)\} \\
\ell_r &= (k-1)(f(n)+1) + r+2-k
\end{align*}
\]

Thus,
\[
\Sigma p_{i_1}^n \cdots p_{i_r}^n \leq \sum_{\ell=1}^k \sum_{i_{r-1} = \ell_{r-1}}^{\ell_r} \sum_{i_k = \ell_k}^{\ell_{k+1}} \cdots \sum_{i_1 = \ell_1}^{\ell_2} \sum_{a_k} \cdot
\]

Now, a geometric argument (analogous to the case \(r=2\)) can be used to show that
\[
\Sigma p_{i_1}^n \cdots p_{i_r}^n \leq \sum_{\ell=1}^k \sum_{i_{r-1} = \ell_{r-1}}^{\ell_r} \sum_{i_k = \ell_k}^{\ell_{k+1}} \cdots \sum_{i_1 = \ell_1}^{\ell_2} \sum_{a_k} \cdot
\]

and thus via (1.2.19), (1.2.1) and (1.2.2) we have, as \(n \to \infty\),
\[
(1.2.21) \quad \Sigma p_{i_1}^n \cdots p_{i_r}^n \leq a_k n \rho_n [f(n)]^{r-k} \to 0
\]

With this, we have established (a).
To establish (b) we again make use of the above decomposition of
the set \( N \) and write

\[(1.2.22) \sum_{N_\lambda'} \sum_{\lambda \in \Lambda} P(A^n_1 \ldots A^n_r) = \sum_{N_\lambda'} \sum_{\lambda \in \Lambda} P(A^n_1 \ldots A^n_r).\]

We shall show that for each fixed \( \lambda \in \Lambda \), \( \sum_{N_\lambda'} P(A^n_1 \ldots A^n_r) \to 0 \) as \( n \to \infty \). An appeal to symmetry again shows that it is sufficient to
prove that for fixed but arbitrary \( k \), \( 0 \leq k < r-1 \),

\[(1.2.23) \sum_{N_\lambda'} P(A^n_1 \ldots A^n_r) \to 0 \text{ as } n \to \infty,\]

where \( \lambda' = (1, \ldots, 1, 0, \ldots, 0) \).

By \( f(n) \)-dependence we have

\[(1.2.24) \sum_{N_\lambda'} P(A^n_1 \ldots A^n_r) = \sum_{i_n=\ell_n}^{u_n} \sum_{i_r=\ell_r}^{u_r} \prod_{i=1}^{u_r} P(A^n_{i}) P(A^n_{i_1} \ldots A^n_{i_k}) \sum_{i_2=\ell_2}^{u_2} \sum_{i_1=\ell_1}^{u_1} P(A^n_{i_1} \ldots A^n_{i_{k+1}}) \sum_{i_r=\ell_r}^{u_r} \sum_{i_{r-1}=\ell_{r-1}}^{u_{r-1}} \sum_{i_{k+1}=\ell_{k+1}}^{u_{k+1}} P(A^n_{i_{k+1}} \ldots A^n_{i_r}) \leq a^k \sum_{i_r=\ell_r}^{u_r} \sum_{i_{r-1}=\ell_{r-1}}^{u_{r-1}} \sum_{i_{k+1}=\ell_{k+1}}^{u_{k+1}} P(A^n_{i_{k+1}} \ldots A^n_{i_r}) \prod_{i=1}^{u_r} P(A^n_{i}) P(A^n_{i_1} \ldots A^n_{i_k}) \prod_{i=1}^{u_2} P(A^n_{i_1} \ldots A^n_{i_{k+1}}) \prod_{i=1}^{u_1} P(A^n_{i_1} \ldots A^n_{i_{k+1}}) \prod_{i=1}^{u_1} P(A^n_{i_1} \ldots A^n_{i_{k+1}}) \leq \prod_{j=1}^{r-k+1} \left[ f(n) C_n^{(j)} \right]. \]
Thus, by (1.2.25), (1.2.1) and (1.2.3) we have,

\[ \sum_{N_{\lambda}^{r-k-1}} P(A_1^n \ldots A_r^n) \leq a_n \prod_{\rho_n f(n)c(j)}^{k-1} \rightarrow 0 \]

as \( n \rightarrow \infty \).

With this, (b) is established.

From this point on, the arguments presented on the proof of Theorem 1.1.2 (from 1.1.15 onward) apply word for word. We may therefore consider the proof for \( k=0 \), and general \( k \), accomplished.

It will be seen that conditional probabilities in (1.2.3) can be 'much larger' than the unconditional probabilities, though they must tend to zero as \( n \rightarrow \infty \). Furthermore, setting \( f(n) \equiv m \) gives a Poisson-Bernoulli limit theorem for \( m \)-dependent events. However, a somewhat 'better' result can be obtained when \( f(n) \equiv m \) (\( n \geq m \)). In order to be complete we now state this theorem. It is related in form to Von Mises [13] theorem.

**Theorem 1.2.2** For each \( n \), let \( \{A_i^n(i=1,2,\ldots,n)\} \) be a sequence of \( m \)-dependent events. Let \( p_i^n = P(A_i^n) \) (\( i=1,2,\ldots,n \)) and define \( a_n = \sum_{i=1}^{n} p_i^n \).

Suppose that as \( n \rightarrow \infty \),

\[ \{a_n\} \text{ is a bounded sequence} \]

\[ \rho_n = \max_{1 \leq i \leq n} p_i^n = o(1), \]

and

\[ b_n = \max_{0 < |i-j| \leq m} P(A_i^n | A_j^n) = o(1). \]

Then,

\[ P(\text{exactly } k \text{ amongst } A_i^n(i=1,2,\ldots,n) \text{ occur}) \sim e^{-a_n k/k!}. \]
The proof of this theorem is similar to that of Theorem 1.2.1. It will be noted that the major difference between the present theorem and Theorem 1.2.1 specialized to \( f(n) = m \) is the order of \( \rho_n = \max_{1 \leq i \leq n} p_i^n \). In the present theorem we require only that \( \rho_n = o(1) \). Since this does not imply that \( \{a_n\} \) is bounded we need assumption (1.2.27).

Theorem 1.2.2 is to be compared with the Von Mises' theorem for independent events. That theorem (requiring that \( a_n \to a \) as \( n \to \infty \)) does not need (1.2.29) because of independence. The Hodges-LeCam theorem does not require \( \{a_n\} \) to be bounded, nor does it require (1.2.29) since the events considered are independent as well.

Some strengthening of Theorem 1.2.2 is possible if the sequence \( \{a_n\} \) properly diverges to \( +\infty \). We propose to show that if in the preceding theorem \( a_n \to \infty \) as \( n \to \infty \) then the conclusion (1.2.30) remains valid when the expression \( e^{a_n k/k!} \) is replaced by its limiting value of zero. In order to do this, we need the following lemma. It will be seen that it is in one sense an m-dependent analogue of a well-known Borel-Cantelli lemma (see Feller [5], p. 189).

Lemma 1.2.1  For each \( n \), \( (n \geq m) \), let \( \{A_i^n(i=1,2,\ldots,n)\} \) be a sequence of m-dependent events. Let \( p_i^n = P(A_i^n) \) \( (i=1,2,\ldots,n) \) \( (n \geq m) \), and \( a_n = \sum_{i=1}^{n} p_i^n \). If \( a_n \to \infty \) as \( n \to \infty \), then for any non-negative integer \( k \), as \( n \to \infty \),

(1.2.31)  \[ P(\text{exactly } k \text{ amongst } A_i^n(i=1,\ldots,n) \text{ occur}) \to 0. \]

Proof. We first prove the lemma for the case \( m = 0 \) (i.e., for independent events). Now by independence, as \( n \to \infty \),

(1.2.32)  \[ 0 \leq P(\text{no } A_i^n(i=1,\ldots,n) \text{ occurs}) = \sum_{i=1}^{n} (1-p_i^n)^n < e^{-a_n} = e^{-a_n} \to \text{ and} \]
Thus, \( P[\text{at least one } A^n_i(i=1,\ldots,n) \text{ occurs}] \rightarrow 1 \) as \( n \rightarrow \infty \).

For each \( n \geq 2 \) it is possible to divide the sequence

\[ [A^n_i(i=1,2,\ldots,n)] \]

into two subsequences, say \( [A^n_{\alpha i}(i=1,\ldots,N^1_1)] \) and

\[ [A^n_{\beta i}(i=1,2,\ldots,N^2_2)] \]

in such a manner that \( a^{(\alpha)}_n = \sum_{i=1}^{N^1_1} p^n_{\alpha i} \rightarrow \infty \) and

\[ a^{(\beta)}_n = \sum_{i=1}^{N^2_2} p^n_{\beta i} \rightarrow \infty \text{ as } n \rightarrow \infty. \]

Then, as above,

\[
P[\text{no } A^n_{\alpha i}(i=1,2,\ldots,N^1_1) \text{ occurs}] = \prod_{i=1}^{N^1_1} (1-p^n_{\alpha i}) < e^{-\sum_{i=1}^{N^1_1} p^n_{\alpha i}} = e^{-a^{(\alpha)}_n} \rightarrow 0, \text{ and}
\]

\[
P[\text{no } A^n_{\beta i}(i=1,\ldots,N^2_2) \text{ occurs}] = \prod_{i=1}^{N^2_2} (1-p^n_{\beta i}) < e^{-\sum_{i=1}^{N^2_2} p^n_{\beta i}} = e^{-a^{(\beta)}_n} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Thus, combining these two results, we have as \( n \rightarrow \infty \),

\[
(1.2.33) \quad P[\text{at least two } A^n_i(i=1,2,\ldots,n) \text{ occur}] \geq P[\text{at least one } A^n_{\alpha i}
\text{ and at least one } A^n_{\beta i} \text{ occur}]
= 1 - P[\text{no } A^n_{\alpha i} \text{ or no } A^n_{\beta i} \text{ occurs}]
\geq 1 - P[\text{no } A^n_{\alpha i} \text{ occurs}] - P[\text{no } A^n_{\beta i} \text{ occurs}]
\rightarrow 1.
\]

This procedure can be carried out inductively for any positive integer.

In particular \( P[\text{at least } (k+1) A^n_i(i=1,2,\ldots,n) \text{ occur}] \rightarrow 1 \) as \( n \rightarrow \infty \), so that \( P[\text{exactly } k \text{ amongst } A^n_i(i=1,2,\ldots,n) \text{ occur}] \rightarrow 0 \) as \( n \rightarrow \infty \) for any non-negative integer \( k \).

We now consider \( m > 0 \). Assuming that \( n \) is large enough, for each \( n \) extract from \( \{A^n_i(1=1,2,\ldots,n)\} \) a subsequence \( \{A^n_{\alpha i}(1=1,2,\ldots,N^n_i)\} \) with the following properties:
\[(1.2.34) \quad n_1 - n_{i-1} > m; \quad i=1,2,...,N_n \quad \text{and} \quad a_n = \sum_{i=1}^{N_n} p_{n_i}^{n} \to \infty \quad \text{as} \quad n \to \infty.\]

(A proof of the fact that such a subsequence can always be selected is contained in Lemma 1.2.2.) Obviously, \(N_n \to \infty\) as \(n \to \infty\). By \(m\)-dependence the events in the subsequence \(\{A_{n_i}^n (i=1,2,...,N_n)\}\) are mutually independent. We may therefore apply the results of the preceding case \((m=0)\) to them. Clearly for any non-negative integer \(\ell\),

\[(1.2.35) \quad P[\text{at least} \ \ell \ \text{amongst} \ A_{n_i}^n (i=1,2,...,N_n \ \text{occur})] \leq P[\text{at least} \ \ell \ \text{amongst} \ A_{i}^n (i=1,\ldots,n \ \text{occur})] \]

Moreover, from the preceding case \(m=0\), it can be deduced that

\[(1.2.36) \quad P[\text{at least} \ \ell \ \text{amongst} \ A_{n_i}^n (i=1,2,...,N_n \ \text{occur})] \to 1 \quad \text{as} \quad n \to \infty.\]

This, in conjunction with \((1.2.35)\), serves to establish the Lemma.

As mentioned in the proof of the preceding Lemma, we need the following result.

**Lemma 1.2.2**  For each \(n\), let \(\{p_{i}^{n}(i=1,2,...,n)\}\) be a set of real numbers such that \(0 \leq p_{i}^{n} \leq 1\) and \(a_n = \sum_{i=1}^{N_n} p_{i}^{n} \to \infty \) as \(n \to \infty\). It is always possible to extract from each \(\{p_{i}^{n}(i=1,2,...,n)\}\) a subset \(\{p_{i}^{n}(i=1,2,...,N_n)\}\) with the property that for some fixed positive integer \(m\), \(n_i - n_{i-1} > m\), \(i=1,2,...\) and \(a_n = \sum_{i=1}^{N_n} p_{n_i}^{n} \to \infty \) as \(n \to \infty\).

**Proof.** We give a constructive proof. For each \(n\), let

\[b_n = \max(a_1^{(1)},a_2^{(2)}) \quad \text{where} \quad a_1^{(1)} = \sum_{i=1}^{\lceil n/2 \rceil + 1} p_{2i-1}^{n}, a_2^{(2)} = \sum_{i=1}^{\lceil n/2 \rceil} p_{2i}^{n}, \quad \text{that is} \]

\[b_n \quad \text{is the maximum of the sums of the odd numbered terms and even numbered terms in} \ \{p_{i}^{n}(i=1,2,...,n)\}. \quad \text{If} \quad b_n = a_1^{(1)}, \text{let} \]


\[ C_n = \max \{b_n^{(11)}, b_n^{(12)}\} \] where \( b_n^{(11)} \) is the sum of the terms in \( \{p_i^n(i=1,\ldots,n)\} \) which have subscripts \( \equiv 1 \pmod{4} \) and \( b_n^{(12)} \) is the sum of terms in \( \{p_i^n(i=1,\ldots,n)\} \) which have subscripts \( \equiv 3 \pmod{4} \). If \( b_n = a_n^{(2)} \), let \( C_n = \max \{b_n^{(21)}, b_n^{(22)}\} \), where \( b_n^{(21)} \) is the sum of the terms in \( \{p_i^n(i=1,2,\ldots,n)\} \) which have subscripts \( \equiv 2 \pmod{4} \) and \( b_n^{(22)} \) is the sum of the terms in \( \{p_i^n(i=1,2,\ldots,n)\} \) which have subscripts \( \equiv 0 \) \( \pmod{4} \). This procedure is carried out a total of \( r \) times, where \( r \) is the smallest integer such that \( 2^r > m+1 \). By this construction we arrive at a subsequence \( \{p^n_{N_i}(i=1,2,\ldots,N_i)\} \) of \( \{p^n_i(i=1,2,\ldots,n)\} \) for each \( n \), with the property that \( N_i = N_{i-1} = 2^r - 1 > m; i=1,2,\ldots,N_i \), and 
\[
a_n^i = \sum_{i=1}^{N_i} p^n_{i} \rightarrow \infty \text{ as } n \rightarrow \infty.\]
With this the Lemma is established, and Theorem 1.2.2 is shown to remain valid when \( \{a_n^i\} \) properly diverges to \( +\infty \).

From the statement of Theorem 1.2.1 it is easily seen that an \( f(n) \)-dependence version of Theorem 1.1.2 is available. Since in many cases \( f(n) \)-dependence is a more realistic assumption than \( m \)-dependence, such a result is of interest.

**Theorem 1.2.3** For each \( n \), let \( \{A^n_i(i=1,2,\ldots,n)\} \) be a sequence of \( f(n) \)-dependent events. Suppose that for some \( \xi > 0 \) we have, as \( n \rightarrow \infty \),

\[
(1.2.33) \quad a_n^i = \max_{j=1,\ldots,n} |P(A^n_j) - \xi/n| = o(1/n)
\]

\[
(1.2.34) \quad f(n) = o(n),
\]

and

\[
(1.2.35) \quad c_n^{(1)} = \max_{i=1,2} P(A^n_i | A^n_{i+1}) = o(1/f(n))
\]

and

\[
(1.2.36) \quad c_n^{(r)} = \max_{i=1,2,\ldots,\ldots,\ldots} P(A^n_{i_1} | A^n_{i_2} \ldots A^n_{i_{r+1}}) = o(1/f(n)), \quad r \geq 2,
\]
where \( i_1^n = (i_1, i_2, \ldots, i_{r+1} | 1 \leq i_1 < \ldots < i_{n+1} \leq n; i_{j+1} - i_j \leq f(n); j = 1, 2, \ldots, n \}. \) Then as \( n \to \infty \)

(1.2.37) \[ P(\text{exactly } k \text{ amongst } A_{i_1}^n; i=1, 2, \ldots, n \text{ occur}) \to e^{-\frac{n}{k}}. \]

Proof. We note that (1.2.33) implies that \( a_n = \sum_{i=1}^{n} P(A_i^n) \to \xi \) as \( n \to \infty. \) Certainly \( \max_{j=1, \ldots, n} P(A_j^n) = O(1/n). \) Thus the conditions of Theorem 1.2.1 are satisfied and (1.2.37) follows directly.

We remark that under assumptions such as (1.2.34) and (1.2.35), \( f(n) \)-dependent versions of the results of Section 1.1 are almost immediately available. It will be necessary to state these extensions directly.

1.3 A Multivariate Analogue of Theorem 1.1.2

As mentioned in Section 1.1, Theorem 1.1.2 is essentially a generalization (to \( m \)-dependent events) of the Poisson Limit theorem for sequences of independent binomial trials. A logical extension of Theorem 1.1.2 is, therefore, to an \( m \)-dependent equivalent of sequences of independent multinomial trials. Some care must be taken in stating such a theorem, and hence it may at first appear a bit cumbersome. However, some applications of this theorem prove quite simple and interesting. We shall attempt to state the theorem in the most general form practical. Following the pattern of the previous sections of this chapter, we shall defer a discussion of some specific applications to later chapters. As before for each \( n \) we postulate the existence of a basic probability space \( (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \) in which the collection of sequences \( \{A_{i_{j}}^n (j=1, \ldots, n)\} (i=1, 2, \ldots, r) \) discussed in the following theorems are event sequences. For notational convenience we shall suppress the index \( n \) on the probability measure \( n. \)
We state a basic form of the multivariate analogue of Theorem 1.1.2 as

**Theorem 1.3.1.** For each \( n \) let \( \{A_{1j}^n(j=1,2,\ldots,n)\}, \{A_{2j}^n(j=1,2,\ldots,n)\}, \ldots,\{A_{nj}^n(j=1,2,\ldots,n)\} \) be \( r \) sequences of mutually \( m \)-dependent events. That is, events \( A_{1j_1}^n \) and \( A_{1j_2}^n \) are independent provided only that \( |j_2 - j_1| > m \), regardless of the values of \( i_1 \) and \( i_2 \). Suppose that for some \( \delta > 0; i=1,2,\ldots,r \) we have, as \( n \to \infty \),

\[
(1.3.1) \quad \alpha_n(i) = \max_{1 \leq j \leq n} |P(A_{1j}^n) - \frac{\delta_j}{n}| = o\left(\frac{1}{n}\right); \quad i=1,2,\ldots,r
\]

and

\[
(1.3.2) \quad \beta_n = \max_S P(A_{1j_1}^n | A_{1j_2}^n) = o(1),
\]

where

\( S \) is the set defined as \( S = \{(i_1,j_1), (i_2,j_2) | (i_1,j_1) \neq (i_2,j_2), \)

\( |j_2 - j_1| \leq m, i_1,i_2 = 1,\ldots,r; j_1,j_2 = 1,\ldots,n \}. \) Then, for any choice of \( r \) non-negative integers \( m_1, m_2, \ldots, m_r \) we have as \( n \to \infty \)

\[
(1.3.3) \quad P(\bigcap_{i=1}^r \{\text{exactly } m_i \text{ amongst } A_{1j}^n(j=1,2,\ldots,n) \text{ occur}\})
\]

\[
\to \prod_{i=1}^r \left(1 - \frac{\delta_i}{m_i!}\right). \]

**Proof.** We prove the theorem for the case \( r = 2 \). The changes necessary to establish the result for general \( r \) are straightforward and mainly notational. Let \( P_{[m_1,m_2]}^n \) denote the probability that exactly \( m_1 \) events amongst \( \{A_{1j}^n(j=1,2,\ldots,n)\} \) occur and exactly \( m_2 \) events amongst \( \{A_{2j}^n(j=1,2,\ldots,n)\} \) occur. Further, for \( 0 \leq k \leq n, 0 \leq h \leq n \) define the quantity

\[
(1.3.4) \quad S_{k,h}^n = \sum_{C_n^k} P(A_{1i_1}^n \ldots A_{1i_k}^n A_{2j_1}^n \ldots A_{2j_h}^n), \quad \text{where}
\]

\( C_n^k \) is the set consisting of all distinct \((k+h)\)-tuples of integers of
the form \((i_1, \ldots, i_k; j_1, \ldots, j_h); 1 \leq i_1 < \ldots < i_k \leq n; 1 \leq j_1 < \ldots < j_h \leq n.\)

It is known (see, for example, Fréchet [6]) that

\[
C^n_{m_1, m_2} = \sum_{k=m_1}^{n} \sum_{h=m_2}^{n} (-1)^{k-h}(m_1 + m_2) \binom{n}{k} \binom{k}{h} s^n_{k, h},
\]

where \(s^n_{k, h}\) is understood to be zero if \(k + h > n.\) We now proceed to evaluate the asymptotic value of (1.3.5) as \(n \to \infty\) by evaluating the asymptotic behavior of (1.3.4) as \(n \to \infty.\) Consider the terms contributing to the sum \(s^n_{k, h}\). Let \(C^n_{s, t} (s=0, 1, \ldots, k; t=0, 1, \ldots, h)\) denote the subset of terms in \(s^n_{k, h}\) in which exactly \(S\) events of type \(\{A^n_{j, 1}(j=1, 2, \ldots, n)\}\) and exactly \(T\) events of type \(\{B^n_{j, 2}(j=1, 2, \ldots, n)\}\) appearing differ in (second) subscript by more than \(m\) from their nearest neighbor of either type (assuming \(n\) is sufficiently large for all cases to be possible). Let \(N^n_{s, t}\) denote the number of elements in the class \(C^n_{s, t}\) (and note that \(C^n_{k-1, h} \equiv C^n_{k, h-1} \equiv C^n_{k, h}\)). As in Theorem 1.1.2 it can be shown that \(N^n_{s, t} = 0(n^{s+t+1})\) for \(s+t < k+h-1\) and \(N^n_{s, t} \sim n^{k+h}/k!h!\) as \(n \to \infty.\) Therefore, from mutual \(m\)-dependence and (1.3.1) it follows that the contribution to \(s^n_{k, h}\) of terms in \(C^n_{s, t}\) \((s+t < k+h-1)\) is bounded by

\[
K_{s, t} n^{s+t} [\xi_1^{1/n+O(1/n)}]^S [\xi_2^{1/n+O(1/n)}]^T b_n,
\]

where \(K_{s, t}\) is a constant independent of \(n.\) The expression in (1.3.6) tends to zero as \(n \to \infty\) by (1.3.2). However, the contribution to \(s^n_{k, h}\) of terms in \(C^n_{s, t}\) is, via mutual \(m\)-dependence and (1.3.1), asymptotically

\[
n^{k+h}/k!h! [\xi_1^{1/n+O(1/n)}]^k [\xi_2^{1/n+O(1/n)}]^h \to \xi^k_1 \xi^h_2/k!h!.
\]

Thus \(s^n_{k, h} \to \xi^k_1 \xi^h_2/k!h!\) as \(n \to \infty.\)
Using a multivariate analogue of the Bonferroni Inequalities (see Appendix C), we have for any non-negative integer \( \ell, \ell \leq \lfloor n/2 \rfloor \),

\[
(1.3.8) \quad P_{m_1 m_2}^{n} \leq \sum_{t = m_1 + m_2}^{m_1 + m_2 + 2\ell} (-1)^{i+j} \sum_{m_1 \leq i \leq \ell + m_1, m_2 \leq j \leq \ell + m_2} \binom{i}{m_1} \binom{j}{m_2} S_{i,j}^{n}.
\]

and

\[
(1.3.9) \quad P_{m_1 m_2}^{n} \geq \sum_{t = m_1 + m_2}^{m_1 + m_2 + 2\ell + 1} (-1)^{i+j} \sum_{m_1 \leq i \leq \ell + 1 + m_1, m_2 \leq j \leq \ell + 1 + m_2} \binom{i}{m_1} \binom{j}{m_2} S_{i,j}^{n}.
\]

Thus, as \( n \to \infty \), we have by (1.3.7) and (1.3.8),

\[
(1.3.10) \quad \limsup_{n \to \infty} P_{m_1 m_2}^{n} \leq \sum_{t = m_1 + m_2}^{m_1 + m_2 + 2\ell} (-1)^{i+j} \sum_{m_1 \leq i \leq \ell + m_1, m_2 \leq j \leq \ell + m_2} \binom{i}{m_1} \binom{j}{m_2} S_{i,j}^{n}.
\]

\[
= \frac{\ell_1 \ell_2}{m_1 m_2} \frac{2\ell}{m_1 m_2 !} \left( \sum_{t = 0}^{m_1 + m_2} (-1)^{i+j} \frac{(-\ell_2)^{i} (-\ell_2)^{j}}{i!j!} \right),
\]

and similarly by (1.3.7) and (1.3.9),

\[
(1.3.11) \quad \liminf_{n \to \infty} P_{m_1 m_2}^{n} \geq \frac{\ell_1 \ell_2}{m_1 m_2} \frac{2\ell + 1}{m_1 m_2 !} \left( \sum_{t = 0}^{m_1 + m_2} (-1)^{i+j} \frac{(-\ell_2)^{i} (-\ell_2)^{j}}{i!j!} \right).
\]

Since \( \ell \) is arbitrary, and the expressions on the right hand side of (1.3.10) and (1.3.11) both tend to a common limit (as \( \ell \to \infty \)), we
conclude that $P_{[m_1, m_2]}^n$ tends to this common limit as $n \to \infty$. Hence, as $n \to \infty$,

$$P_{[m_1, m_2]}^n \to (e^{-\delta_1 m_1 / m_1!}) (e^{-\delta_2 m_2 / m_2!}).$$

(1.3.12)

With this the proof of the theorem is complete.

We shall have use for several results that follow directly as corollaries from the above theorem. Rather than postponing these until a later chapter, we state them here.

**Corollary 1.3.1.1** For each $n$, let $\{A_{ij}^n (j=1,2,\ldots,N_i^1)\}_{i=1,2,\ldots,r}$ be $r$ sequences of mutually m-dependent events. Let $N_i^1$, $\xi_i^1 > 0$ and $0 \leq C_1^{(i)} \leq C_2^{(i)} \leq 1$, $i=1,2,\ldots,r$ be such that as $n \to \infty$,

$$a_n^{(i)} = \max_{1 \leq k \leq N_i^1} |P(A_{ijk}^n) - \xi_i^1/n| = o(1/n); i=1,2,\ldots,r,$$

(1.3.13)

$$b_n^{(i)} = \max_{S', j_1, j_2} P(A_{i_1,j_1,\ldots,i_r,j_r}^n | A_{i_1,j_1,\ldots,i_r,j_r}^n) = o(1),$$

(1.3.14)

where $S' = \{(i_1, j_1), (i_2, j_2) | (i_1, j_1) \neq (i_2, j_2), |j_2 - j_1| \leq m; j_1=1,2,\ldots,N_i^1; j_2=1,2,\ldots,N_i^2; i_1, i_2=1,2,\ldots,r\}$, and

$$N_i^1 \sim \lambda_i n, i=1,2,\ldots,r.$$

(1.3.15)

Then, for each choice of $r$ non-negative integers $m_1, m_2, \ldots, m_r$ we have that as $n \to \infty$,

$$P \left( \bigcap_{i=1}^r \{\text{exactly } m_i \text{ amongst } A_{ij}^n \text{ with } j = [C_1^{(i)} N_i^1]+1, \ldots, [C_2^{(i)} N_i^1] \text{ occur} \} \right) \to \prod_{i=1}^r \left( e^{-\delta_1 m_i / m_i!} \right),$$

(1.3.16)

where $\delta_1 = \lambda_i^{\xi_i^1} (C_2^{(i)} - C_1^{(i)})$, $i=1,2,\ldots,r$.

The statement of the corollary looks complicated because it is
intended to serve an omnibus purpose. With $\lambda_i = 1$, $i=1,2,\ldots,r$, the corollary is seen to be closely related to Theorem 1.1.4. In fact, an easy generalization of the proof of Theorem 1.1.4 suffices to establish the present corollary under this special condition. With $C_1^{(i)} = 0$, $C_2^{(i)} = 1$, $i=1,\ldots,r$, the present corollary is seen to be a multivariate analogue of Theorem 1.1.3. In fact it amounts to a slightly more general version of Theorem 1.1.3 when specialized to the univariate case. In any event, the result is proved easily by choosing $N = \max_{i=1,\ldots,r} N_i$ and reducing this case to the previous case. Combining the two cases produces no difficulties, and the corollary follows.

As remarked in section 1.1, the results of Theorem 1.1.4 can be refined. We are now in a position to provide that refinement. An easy application of Corollary 1.3.1.1 yields the following result.

**Corollary 1.3.1.2** For each $n$, let $\{A^n_i(i=1,2,\ldots)\}$ be a sequence of $m$-dependent events such that for some $\xi > 0$, as $n \to \infty$,

\[
(1.3.16) \quad a_n = \max_{j=1,2,\ldots} |P(A^n_i) - \xi/n| = o(1/n),
\]

and

\[
(1.3.17) \quad b_n = \max_{i,j=1,2,\ldots} P(A^n_i|A^n_j) = o(1).
\]

Further, let $\delta_j = (\delta_{1j},\delta_{2j})$; $j=1,2,\ldots,r$ be $r$ bounded, non-overlapping intervals of $(0,\infty)$ (either closed, open or semi-open), and let $m_j$; $j=1,2,\ldots,r$ be $r$ non-negative integers. Then, as $n \to \infty$,

\[
(1.3.18) \quad P(\Pi \{\text{exactly } m_j \text{ amongst } A^n_i \text{ with } i = [\delta_{1j}n]+1,\ldots,[\delta_{2j}n] \text{ occur}) \to \prod_{j=1}^{r} \left( e^{\delta_{1j}(\delta_{2j}^{-1/m_j})} \right) ,
\]
where \( \delta_j = \delta_{2j-1} \); \( j = 1, 2, \ldots, r \).

**Proof.** This corollary follows from Corollary 1.3.1.1. Define

\[ A_{ij}^n = A_{[n \delta_{1i}]}^n + \delta_{ij}; \quad j = 1, 2, \ldots, [\delta_{1i}] = N_i; \quad i = 1, 2, \ldots, r. \]

It will be seen that the sets, so defined, satisfy the conditions of Corollary 1.3.1.1 with \( \xi_i = \xi \) and \( \lambda_i = \delta_i \); \( i = 1, 2, \ldots, r \). It will be seen that the other conditions of corollary 1.3.1.1 are satisfied and that Corollary 1.3.1.2 is therefore immediate.

At this point it is convenient to summarize some of the results of the preceding sections. In order to do this, we introduce the following definition.

**Definition 1.3.1.** Let \( \{ U_n(t); 0 \leq t \leq T \} \quad (T \leq \infty; \ n = 1, 2, \ldots) \) be a sequence of counting processes defined on the interval \([0, T]\). For any real \( s, t \) \( (0 \leq s < t \leq T) \) let \( U_n(s, t) = U_n(t) - U_n(s) \) designate the number of 'counts' in the interval \([s, t]\) for the process \( \{ U_n(t); 0 \leq t \leq T \} \). The sequence \( \{ U_n(t); 0 \leq t \leq T \} \) \( (n = 1, 2, \ldots) \) is said to converge weakly to a Poisson process of intensity \( \xi \) on the interval \([0, T]\) provided the following conditions are satisfied: (i) for any real \( s, t, h \)

\( (0 \leq s < t < t+h \leq T) \) and non-negative integer \( k \),

\[ P[U_n(s, t) = k] \rightarrow e^{-\xi(t-s)}[\xi(t-s)]^k/k! \text{ and } P[U_n(s, t) = k] \rightarrow P[U_n(s+h, t+h) = k] \rightarrow 0 \text{ as } n \rightarrow \infty, \]

and (ii) for any real numbers \( 0 \leq s < t \leq v < w \leq T \) and pair of non-negative integers \( k, \ell \),

\[ P[U_n(s, t) = k, U_n(v, w) = \ell] \rightarrow [e^{-\xi(t-s)}[\xi(t-s)]^k/k!][e^{-\xi(w-v)}[\xi(w-v)]^\ell/\ell!] \text{ as } n \rightarrow \infty. \]

In other words, Definition 1.3.1 requires that over the interval \([0, T]\) the process \( U_n(t) \) has, asymptotically, stationary independent increments and Poisson marginals.

The above definition now enables us to summarize many of the
preceding results in a single theorem concerning an asymptotic Poisson process.

**Theorem 1.3.2.** For each $n$ ($n \geq m$) let $\{A_i^n (i=1,2,\ldots,n)\}$ be a sequence of $m$-dependent events such that for some $\xi < 0$ we have, as $n \to \infty$,

\[(1.3.19) \quad a_n = \max_{j=1,2,\ldots,n} |P(A_j^n) - \xi/n| = o(1/n),\]

and

\[(1.3.20) \quad b_n = \max_{0 < |i-j| \leq m} P(A_i^n A_j^n) = o(1).\]

Let $U_n(t)$ ($0 \leq t \leq 1; n=1,2,\ldots$) be the number of events $A_i^n$ that occur with $1 \leq i \leq \lfloor nt \rfloor$. Then, as $n \to \infty$, the sequence of counting processes $\{U_n(t); 0 \leq t \leq 1\}$ ($n=1,2,\ldots$) converges weakly over $[0,1]$ to a Poisson process of intensity $\xi$.

This theorem follows from Theorems 1.1.2 and 1.3.1. It will prove quite convenient for the purposes of future applications to have results summarized in the above form. A similar summary form should be possible for results involving sequences of $f(n)$-dependent events, both stationary and non-stationary. However, we shall not do this here.

**1.4 Some generalizations of a result due to Loynes [12]**

As was pointed out in Sections 1.1 and 1.2, the Poisson limit for independent binomial events can be extended to $m$-dependent, and in general $f(n)$-dependent, events. Furthermore, 'multivariate' Poisson limit results are also available for classes of such dependent events. It is natural to inquire whether the Poisson limit result can be established for sequences of events which are dependent in a more extensive manner than $m$- or $f(n)$-dependent events. In particular, we...
would like to establish such a Poisson limit result (as found in Theorem 1.1.2) for sequences of events in which each event conceivably depends upon each other event (in some specified manner). Naturally, one would expect that such dependence would have to 'die off' quite rapidly.

In this section we present a result quite closely related to a theorem given by Loynes [12] which dealt with the maximum of a stationary stochastic process. We shall state Loynes' result, and then state and prove a related result involving sequences of events rather than stochastic process. It will be seen that such an approach is quite fruitful.

First, we shall need the following definition of uniform (strong) mixing.

**Definition 1.4.1** A discrete-time stochastic process \( \{x_i (i=1,2,\ldots)\} \) is termed uniformly (strongly) mixing (with mixing function \( g \)) if whenever \( A \in \mathcal{B}(x_1, \ldots, x_m) \) and \( B \in \mathcal{B}(x_{m+k+1}, \ldots) \) for some \( m \), then

\[
|P(AB) - P(A)P(B)| < g(k).
\]

Here \( \mathcal{B}(\ldots) \) denotes the Borel field of events generated by the indicated random variables, and \( g(k) \) is a function tending to zero as \( k \to \infty \).

We now state Loynes' theorem

**Theorem 1.4.1** (Loynes) Let \( \{x_i (i=1,\ldots)\} \) be a uniformly mixing strictly stationary stochastic process with mixing function \( g \). Let \( c_n = c_n(\xi) \) (\( \xi > 0; \ n=1,2,\ldots \)) satisfy the relation

\[
P(x_i > c_n) \leq \xi/n \leq P(x_i \geq c_n).
\]

If there are sequences of integers \( \{p_m\} \) and \( \{q_m\} \), \( m=1,2,\ldots \) satisfying the following conditions, as \( m \to \infty \),

\[
(1.4.1) \quad mg(q_m) \to 0, \ q_m/p_m \to 0, \ p_{m+1}/p_m \to 1,
\]
with the property that (writing \( p = p_m, q = q_m, t = m(p+q) \))

\[
\sum_{i=1}^{p-1} \frac{(p-1)/p}{P(x_i > c_t, x_{i+1} > c_t) / P(x_i > c_t)} \to 0
\]

as \( m \to \infty \), then as \( n \to \infty \)

\[
P(x_i \leq c_n(\xi)(i=1, \ldots, n)) \to e^{-\xi}.
\]

It can be shown that Watson's original theorem follows from Loynes' theorem by choosing the \( g_m \) constant, independent of \( m \). What is not clear is that with a slight strengthening of the hypothesis of Theorem 1.4.1 one can prove (in a different setting) a Poisson limit theorem for events that are uniformly (strongly) mixing. To be precise, we specialize Definition 1.4.1 to sequences of events.

**Definition 1.4.2** A sequence of events \( \{A_i(i=1,2,\ldots)\} \) is termed uniformly (strongly) mixing (with mixing function \( g \)) if whenever \( |i-j| \leq k \), then \( |P(A_iA_j) - P(A_i)P(A_j)| \leq g(k) \), where \( g(k) \to 0 \) as \( k \to \infty \).

Accordingly, an event in a uniformly mixing sequence of events may not be independent of any other event in the sequence. However, the dependence (judged by comparing joint and marginal probabilities 'dies off' according to the function \( g(k) \)) as \( k \to \infty \), we now proceed to formulate an analogue of Loynes' result in an 'event-sequence' setting rather than a stochastic process setting. The extension will be applicable to stochastic processes that are not necessarily stationary, although the non-stationarity must be 'uniformly small' in a sense made precise by (1.4.4) and (1.4.8) in the theorem. By addition of a stronger uniform mixing condition (we essentially require 'exponential' decrease in dependence; see (1.4.5)), we are able to obtain a 'full-Poisson' result, whereas Loynes' essentially obtained only the first term in the Poisson
distribution. We now state and prove this result. The cumbersome form of the hypothesis is necessary to admit 'non-stationary' applications.

**Theorem 1.4.1** For each \( n \) let \( \{A^n_i(\text{i}=1,2,\ldots,n)\} \) be a uniformly mixing sequence of events (with mixing function \( g \)). Suppose that for some \( \xi > 0 \),

\[
(1.4.4) \quad a_n = \max_{1 \leq i \leq n} |P(A^n_i) - \xi/n| = o(1/n) \text{ as } n \to \infty.
\]

Suppose further that there are sequences of integers \( \{p_m\} \) and \( \{q_m\} \) \((m=1,2,\ldots)\) such that as \( m \to \infty \),

\[
(1.4.5) \quad m^{-r}g(q_m) \to 0 \text{ for any fixed } r > 0,
\]

\[
(1.4.6) \quad q_m/p_m \to 0, \quad p_{m+1}/p_m \to 1,
\]

and writing \( p = p_m, q = q_m, t = m(p+q) \),

\[
(1.4.7) \quad \sum_{k=0}^{m-1} \sum_{j<k} P(A^t_iA^t_j) \to 0 \text{ and } \sum_{j<k} P(A^t_iA^t_j) \to 0,
\]

where \( \ell = k(p+q), u = \ell + p \). Finally suppose that if \( E_p^t \) is an event defined in terms of \( A^t_1,\ldots,A^t_p \) and \( P(E_p^t) \sim \xi/m \) as \( m \to \infty \), then

\[
(1.4.8) \quad \max_{1 \leq i \leq m} \left| P\{T^{(p+q)}(i-1)(E_p^t) \} - \xi/m \right| = o(1/m),
\]

where \( T^j(E) \) denotes the translate to the right (along the \( \{A^n_i\} \) sequence) by \( j \) of \( E \). Then, as \( n \to \infty \),

\[
(1.4.9) \quad P[\text{exactly } k \text{ amongst } A^n_i(\text{i}=1,\ldots,n) \text{ occur}] \to e^{-\xi^k/k!}.
\]

**Proof.** For fixed \( m \), partition the positive integers into consecutive blocks of size \( p_m \) and \( q_m \) alternately, beginning with the initial block \((1,2,\ldots,p_m)\) of size \( p_m \). Let \( P_m(q_m) \) denote those positive integers
falling into \( p_m(q_m) \) blocks. Supressing the index \( m \) (as in (1.4.1)), define \( \mathcal{E}_k^t \) as the event "exactly \( k \) amongst \( A_i^t \); \( i = 1, 2, \ldots, t \) occur." 

We first show that, asymptotically, if an event \( A_i^t \) occurs, then \( i \in p_n \). Thus, events \( A_i^t \) with \( i \in Q_m \) can be neglected in the limit. Then, we show that if \( k \) amongst \( A_i^t \); \( i = 1, \ldots, t \) occur, then, asymptotically, all \( k \) indices lie in separate \( p_m \)-blocks. Finally, we use the Bonferroni inequalities to obtain the asymptotic probability of the event \( \mathcal{G}_k^t \) defined as "exactly \( k \) amongst \( A_i^t \); \( i = 1, 2, \ldots, t \) occur and each \( A_i^t \) lies in a separate \( p_m \)-block." Write, say,

\[
(1.4.10) \quad P(\mathcal{E}_k^t) = P(\mathcal{G}_k^t) + P(\mathcal{G}_k^t) - P(\mathcal{G}_k^t),
\]

where

\[
(1.4.11) \quad \mathcal{G}_k^t \text{ is the event "exactly } k \text{ amongst } A_i^t (i = 1, 2, \ldots, t) \text{ occur and all } i \in p_n, \text{ and } \]

\[
\mathcal{G}_k^t \text{ is the event "exactly } k \text{ amongst } A_i^t (i = 1, 2, \ldots, t) \text{ occur and some } i \in Q_m."
\]

Now from (1.4.4), as \( m \to \infty \), we have

\[
(1.4.12) \quad P(\mathcal{G}_k^t) \leq \sum_{i \in Q_m, i < t} P(A_i^t) \sim mq(t/t) \to 0.
\]

Thus \( P(\mathcal{E}_k^t) \) and \( P(\mathcal{G}_k^t) \) are identical in the limit. Next write, say,

\[
(1.4.13) \quad P(\mathcal{G}_k^t) = P(\mathcal{G}_k^t) + P(\mathcal{G}_k^t),
\]

where

\[
(1.4.14) \quad \mathcal{G}_k^t \text{ is the event "} \mathcal{G}_k^t \text{ occurs and all } A_i^t \text{'s occurring have indices in separate } p_m \text{-blocks,"}
\]
and

\[ Z^t_k \] is the event "\[ \mathcal{B}^t_k \] occurs and some \( A^t_i \)'s occurring have indices in the same \( p_m \)-block."

Via (1.4.7) we have, as \( m \to \infty \),

\[
P(\mathcal{Z}^t_k) \leq \sum_{k=0}^{m-1} \sum_{l<i<j \leq u} P(A^t_i A^t_j) \to 0.
\]

Thus, \( P(\mathcal{E}^t_k) \) and \( P(\mathcal{G}^t_k) \) are identical in the limit.

We now proceed to evaluate the asymptotic value of \( P(\mathcal{G}^t_k) \). For this purpose, we define an event \( \mathcal{B}^t_{k,h} \) for which \( P(\mathcal{B}^t_{k,h}) \sim P(\mathcal{G}^t_k) \) as \( m \to \infty \) as follows:

\[
(1.4.16) \quad \mathcal{B}^t_{k,h} \text{ is the event } "\text{exactly } k \text{ amongst } G^t_i (i=1,2,\ldots,m) \text{ occur}," \text{ where}\n
(1.4.17) \quad G^t_i \text{ is the event } "\text{exactly one } A^t_j (j=(i-1)(p+q)+1, \ldots, (i-1)(p+q)+p) \text{ occurs}".

Using Bonferroni's inequalities we have, for any even integer \( \ell, \ell+k \leq m, \)

\[
(1.4.18) \quad \mathcal{L}^t_{k,\ell} \leq P(\mathcal{B}^t_{k,h}) \leq \mathcal{U}^t_{k,\ell}, \quad \text{where}\n
(1.4.19) \quad \mathcal{L}^t_{k,\ell} = S^t_k - \binom{k+1}{k} S^t_{k+1} + \cdots + (-1)^{\ell-1} \binom{k+\ell-1}{k} S^t_{k+\ell-1},

(1.4.20) \quad \mathcal{U}^t_{k,\ell} = S^t_k - \binom{k+1}{k} S^t_{k+1} + \cdots + (-1)^{\ell} \binom{k+\ell}{k} S^t_{k+\ell}, \text{ with}\n
(1.4.21) \quad S^t_r = \sum_{1 \leq i_1 < \ldots < i_r \leq m} P(G^t_{i_1} \ldots G^t_{i_r}).

Via the uniform mixing property of \( \{A^n_i, i=1,2,\ldots\} \) we have, for fixed \( i_1, \ldots, i_r, \)

\[
(1.4.22) \quad | P(G^t_{i_1} \ldots G^t_{i_r}) - \prod_{j=1}^{r} P(G^t_{i_j}) | \leq r g(q_m).\]
We now make use of assumption (1.4.8). Consider the event $G^t_1$. It depends only upon the events $A^t_i (i=1, \ldots, p)$. Furthermore, each event $G^t_{1,j}$ is a translate of $G^t_1$ by an amount $(i,j-1)(p+q)$. Using Bonferroni's inequalities again, we have

\begin{equation}
T^t_1 - T^t_2 \leq P(G^t_1) \leq T^t_1,
\end{equation}

where

\begin{equation}
T^t_1 = \sum_{j=1}^{p} P(A^t_j) \quad \text{and} \quad T^t_2 = \sum_{1 \leq i < j \leq p} P(A^t_i A^t_j).
\end{equation}

Now from (1.4.4) and (1.4.7) we have, as $m \to \infty$,

\begin{equation}
mT^t_1 \sim mp(\xi/t) \to \xi \quad \text{and} \quad mT^t_2 \to 0.
\end{equation}

Therefore, combining (1.4.25) with (1.4.23) yields

\begin{equation}
P(G^t_1) \sim \frac{\xi}{m} \quad \text{as} \quad m \to \infty
\end{equation}

and so, by assumption (1.4.8)

\begin{equation}
m \max_{j=1,2,\ldots,r} |P(G^t_{1,j}) - \frac{\xi}{m}| = o(1/m).
\end{equation}

The above step now allows us to write (1.4.22) in the form

\begin{equation}
|P(G^t_{1,1} \ldots G^t_{1,r}) - [\frac{\xi}{m} + o(1/m)]^r| \leq r g(q_m),
\end{equation}

and so from (1.4.21),

\begin{equation}
|S^t_r - (m_r)[\frac{\xi}{m} + o(1/m)]^r| \leq r \left(\frac{m}{r}\right) g(q_m).
\end{equation}

It is then obvious that as $m \to \infty$, (1.4.5) and (1.4.29) together yield the desired conclusion, namely that $S^t_r \to \frac{\xi^r}{r!}$ as $m \to \infty$. With this we have established that for any even integer $\ell$, as $m \to \infty$, \ldots
\[
\begin{align*}
(1.4.30) \quad \ell_t^{k,\ell} & \Rightarrow \sum_{j=0}^{\ell-1} \binom{k+j}{k} (-1)^j \frac{k+j}{(k+j)!} = \frac{k}{\ell!} \left[ \sum_{j=0}^{\ell-1} \frac{(-\xi)^j}{j!} \right] \\
\ell_t^{k,\ell} & \Rightarrow \sum_{j=0}^{\ell} \binom{k+j}{k} (-1)^j \frac{k+j}{(k+j)!} = \frac{k}{\ell!} \left[ \sum_{j=0}^{\ell} \frac{(-\xi)^j}{j!} \right],
\end{align*}
\]

which together with previous remarks serve to establish that
\[
(1.4.31) \quad P(\ell_t^{k}) \sim P(\ell_t^{k}) \Rightarrow e^{-\xi \ell_t^{k}}/k!.
\]

We have shown that $P(\ell_t^{k})$ converges along a sequence $t=m(p+q)$ as $m \to \infty$. However, since an arbitrary integer $n$ lies between two consecutive values of $t$ and property (1.4.6) holds, it is easily shown that (1.4.31) implies (1.4.9). With this the proof is completed.

For completeness we state the following immediate corollary of Theorem 1.4.1. It will be of use in the sequel, and will be seen to be a vector-valued analogue of Loynes' original result. It will also illustrate how a stationarily assumption simplifies the conditions of Theorem 1.4.1.

**Corollary 1.4.1.1** Let \( \{Y_i (i=1,2,\ldots)\} \) be a uniformly mixing strictly stationary $p$-dimensional vector-valued stochastic process (with mixing function $g$). Suppose that $S_n = S_n(\xi)$ ($\xi > 0$; $n=1,2,\ldots$) is a sequence of $p$-dimensional (Borel) sets such that $P(Y_1 \in S_n) = \xi/n + o(1/n)$. If there are sequences of integers \( \{p_m\} \) and \( \{q_m\} \) (in the notation of 1.4.1), satisfying (1.4.5) and (1.4.6) as $m \to \infty$, and such that
\[
(1.4.32) \quad \sum_{i=1}^{p-1} [(p-1)/p] P(Y_1 \in S_t, Y_{i+1} \in S_t) / P(Y_1 \in S_t) \to 0,
\]
then we have, as $n \to \infty$,
\[ P(Y_i \in S_n \text{ for exactly } k \text{ amongst } i=1,2,...,n) \rightarrow e^{-s/s}^k/k! \]

**Proof.** If we define events \( A^n_i = (Y_i \in S_n); i=1,2,...,n \), it is easily seen that the stationarity hypothesis on \( \{Y_i; i=1,2,...\} \) assures that these events satisfy the conditions of Theorem 1.4.1.

As mentioned above, this corollary illustrates the simplifications brought about in the statement of Theorem 1.4.1 when the sequence of events \( \{A^n_i; i=1,2,...\} \) can be assumed stationary (by a stationary sequence of events we mean a sequence of events whose characteristic functions satisfy Definition 1.1.1). In this stationary case, both parts of condition (1.4.7) are the same, \( P(A^n_i = P(A^n_j), i,j=1,2,...; \) and condition (1.4.8) is clearly satisfied. Furthermore, to stress the relationships between Theorems 1.4.1 and 1.1.2, we can rewrite condition (1.4.7) (for the stationary case) in terms of conditional probabilities; it can be written as

\[ \frac{1}{p} \sum_{\lambda=1}^{p-1} (p-1) P[A^t_{i+1} | A^t_i] = o(1) \text{ as } m \rightarrow \infty \text{ (} p_m \rightarrow \infty). \]

It is to be noticed that the expression in (1.4.34) is actually a weighted mean of probabilities of events conditioned on a fixed event, more weight given to probabilities of events 'close' to the conditioning event, less weight given to probabilities of events 'far' from the conditioning event.

Of course, the question immediately arises as to whether Theorem 1.4.1 has a multivariate analogue, similar to Theorem 1.3.1. The answer is affirmative, but to state such a theorem in the same generality as Theorem 1.4.1 would prove very cumbersome. As an alternative, we state a multivariate analogue of Theorem 1.4.1 only for the
'stationary' case. This case is, in part, suggested with a future application in mind, and it is to be understood that a more general 'non-stationary' version could be stated.

**Theorem 1.4.2** For each \( n \) let \( \{A_{ij}^n; j=1,2,\ldots\} \); \( i=1,2,\ldots,r \) be \( r \) strongly mixing sequences of events (with common mixing function \( g \)) that are also jointly stationary (in the sense that if \( E \) is an event defined in terms of some collection of the above events, \( A_{ij}^n \), then \( P(E) = P(T(E)) \), where \( T(E) \) is the translate event in which each \( A_{ij}^n \) is replaced by \( A_{i,j+1}^n \)). Let \( \xi_i > 0; i=1,2,\ldots,r \) be such that

\[
(1.4.35) \quad P(A_{ij}^n) \sim \frac{\xi_i}{n}; i=1,\ldots,r.
\]

Suppose further that there are sequences of integers \( \{p_m\} \) and \( \{q_m\} \) such that, as \( m \to \infty \),

\[
(1.4.36) \quad m^r g(q_m) \to 0 \text{ for } r > 0,
\]

\[
(1.4.37) \quad q_m/p_m \to 0 \text{ and } p_{m+1}/p_m \to 1
\]

and writing \( p=p_m \), \( q=q_m \), \( t=m(p+q) \)

\[
(1.4.38) \quad \frac{1}{p} \sum_{j=1}^{p-1} (p-j) P(A_{i_1}^t, A_{i_2}^{t,j+1}) = t (i_1, i_2) \to 0; \quad i_1, i_2 = 1, 2, \ldots, r.
\]

Then, for any choice of \( r \) non-negative integers \( m_1, m_2, \ldots, m_r \), as \( n \to \infty \),

\[
(1.4.39) \quad P( \bigcap_{i=1}^r \{ \text{exactly } m_i \text{ amongst } A_{ij}^n; j=1,2,\ldots,n \text{ occur} \} \to \prod_{i=1}^r (e^{-\xi_i/m_i}) \).}

\]
Proof. Assume without loss of generality that equality holds in (1.4.35). We prove the theorem for the special case \( r=2 \). As in Theorem 1.4.1, the integers are partitioned into two sets, \( P_m \) and \( Q_m \), and \( E_{k,h}^t \) is defined as the event "exactly \( k \) events amongst \( \{ A_{ij}^t \}; j=1,\ldots,t \) and exactly \( h \) events amongst \( \{ A_{2j}^t \}; j=1,2,\ldots,t \) occur." As before, we intend to show that in the limit events \( \{ A_{ij}^t \}; j=1,2,\ldots,t; i=1,2 \) with \( j \in Q_m \) do not occur. Furthermore, we show that, asymptotically, if \( k \) events amongst \( \{ A_{1j}^t \}; j=1,\ldots,t \) and \( h \) events amongst \( \{ A_{2j}^t \}; j=1,\ldots,t \) occur, then all the \( j \)-indices lie in separate \( P_m \)-blocks. Finally, we evaluate the limit value of \( P(E_{k,h}^t) \), where \( E_{k,h}^t \) is the event "exactly \( k \) events amongst \( \{ A_{1j}^t \}; j=1,2,\ldots,t \) and exactly \( h \) events amongst \( \{ A_{2j}^t \}; j=1,2,\ldots,t \) occur and all \( j \)-indices lie in separate \( P_m \)-blocks."

Let

\[
(1.4.40) \quad P(E_{k,h}^t) = P(B_{k,h}^t) + P(C_{k,h}^t), \text{ say}
\]

where \( B_{k,h}^t \) and \( C_{k,h}^t \) are the events defined, respectively, as "\( E_{k,h}^t \) occurs and all \( j \)-indices lie in \( P_m \)" and "\( E_{k,h}^t \) occurs and some \( j \)-indices lie in \( Q_m \)." Now from (1.4.35), (1.4.37), and mutual stationarity it follows that

\[
(1.4.41) \quad P(C_{k,h}^t) \leq \sum_{j \in Q_m, j \leq t_m} \sum_{j \in Q_m, j \leq t_m} P(A_{1j}^t \cup A_{2j}^t) \leq m q \left( \frac{s_1}{t} + \frac{s_2}{t} \right) \to 0 \quad \text{as } m \to \infty.
\]

Thus \( P(E_{k,h}^t) \) and \( P(B_{k,h}^t) \) are the same in the limit. Now,

\[
(1.4.42) \quad P(B_{k,h}^t) = P(D_{k,h}^t) + P(F_{k,h}^t), \text{ say},
\]

where \( D_{k,h}^t \) and \( F_{k,h}^t \) are the events defined, respectively, as "\( D_{k,h}^t \) occurs and all of the \( \{ A_{ij}^t \}; j=1,2,\ldots,t; i=1,2 \) occurring have \( j \)-indices}
in separate $p_m$-blocks" and "$B_{k,h}^t$ occurs and some of the $\{ A_{ij}^t; j=1, \ldots, t; i=1,2 \}$ occurring have $j$-indices in the same $p_m$-blocks."

Now via (1.4.37), (1.4.38), and joint stationarity it follows that

\[(1.4.43) \quad P(\theta_{k,h}^t) \leq \frac{m}{p} \sum_{j=2}^{p} \left[ P(A_{11}^t A_{1j}^t) + P(A_{12}^t A_{2j}^t) + P(A_{21}^t A_{2j}^t) \right] + P(A_{11}^t A_{1j}^t) \]

\[= \frac{m}{p} \left\{ \delta_1 [I_m(1,1) + I_m(1,2)] + \delta_2 [I_m(2,1) + I_m(2,2)] \right\} \]

\[\rightarrow 0 \text{ as } m \rightarrow \infty.\]

Thus, $P(\theta_{k,h}^t)$ and $P(\theta_{k,h}^t)$ are the same in the limit. To evaluate the latter quantity, we define the following events. Let $G_{i}^t; i=1,2,\ldots,m$ be the event "exactly one amongst $A_{1j}^t; j=(i-1)(p+q)+1,\ldots,(i-1)(p+q)+p$ occurs" and let $H_{i}^t; i=1,2,\ldots,m$ be the event "exactly one amongst $A_{2j}^t; j=(i-1)(p+q)+1,\ldots,(i-1)(p+q)+p$ occurs." From arguments presented above, it is clear that $P(\theta_{k,h}^t)$ is asymptotically the same as $P(\gamma_{k,h}^t)$ where $\gamma_{k,h}^t$ is the event "exactly $k$ amongst $G_{i}^t; i=1,2,\ldots,m$ and exactly $h$ amongst $H_{i}^t; i=1,2,\ldots,m$, occur, but no $G_{i}^t H_{i}^t; i=1,2,\ldots,m$ occurs."

Now, using a multivariate analogue of Bonferroni's inequalities (see Appendix C) we have for any integer \( \ell, 0 \leq \ell \leq \lfloor m/2 \rfloor \),

\[(1.4.44) \quad L_{k,h}^t \leq P(\gamma_{k,h}^t) \leq U_{k,h}^t,\]

where

\[(1.4.45) \quad L_{k,h}^t = \sum_{t=k+h}^{k+h+2\ell+1} \sum_{i+j=t}^{(i)} (-1)^{t-(k+h)} \binom{i}{k} \binom{j}{h} S_{i,j}^t, \]

\[(1.4.46) \quad U_{k,h}^t = \sum_{t=k+h}^{k+h+2\ell} \sum_{i+j=t}^{(i)} (-1)^{t-(k+h)} \binom{i}{k} \binom{j}{h} S_{i,j}^t, \]
with

\[(1.4.47) \quad s_{i,j}^t = \Sigma \left\{ p(G_{\alpha_1}^t \ldots G_{\alpha_i}^t H_{\beta_1}^t \ldots H_{\beta_j}^t) \right\} \]

de C is defined as \(C = \{(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j) | 1 \leq \alpha_1 < \ldots < \alpha_i \leq m; 1 \leq \beta_1 < \ldots < \beta_j \leq m; \alpha_r \neq \beta_s r=1,\ldots,i; s=1,\ldots,j\} \). Now by the strong mixing property and joint stationarity it follows that for fixed \((\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j)\)

\[(1.4.48) \quad |p(G_{\alpha_1}^t \ldots C_{\alpha_i}^t H_{\beta_1}^t \ldots H_{\beta_j}^t) - p_j(G_{\alpha_1}^t) p_j(H_{\beta_1}^t)| < i_j g(q_m),\]

and so, by (1.4.36),

\[(1.4.49) \quad |s_{i,j}^t - (m) (m^{-i}) p_j(G_{\alpha_1}^t) p_j(H_{\beta_1}^t)| < i_j (m) (m^{-i}) g(q_m) \rightarrow 0 \]

as \(m \rightarrow \infty\).

Using Bonferroni's inequalities,

\[(1.4.50) \quad t_{11}^t - t_{12}^t \leq p(G_{1}^t) \leq t_{11}^t, t_{21}^t - t_{22}^t \leq p(H_{1}^t) \leq t_{21}^t,\]

where

\[(1.4.51) \quad t_{11}^t = \Sigma_{j=1}^{p} p(A_{i,j}^t) \text{ and } t_{12}^t = \Sigma_{1 \leq r < s \leq p} p(A_{i,\alpha_r}^t A_{i,\beta_s}^t), i=1,2.\]

Now clearly, from (1.4.35) and (1.4.38),

\[(1.4.52) \quad m t_{11}^t = mp(\xi_1/t) \rightarrow \xi_1 \text{ and } m t_{12}^t = mp t \xi_1 \imath(t, i) \rightarrow 0 \]

as \(m \rightarrow \infty\),

and so we have as \(m \rightarrow \infty\),

\[(1.4.53) \quad p(G_{1}^t) \sim \xi_1/m \text{ and } p(H_{1}^t) \sim \xi_2/m.\]

Combining (1.4.53) with (1.4.49), we have, via assumption (1.4.36),
(1.4.54) \[ S_{i,j}^t \sim \binom{m}{ij} \left( \frac{\xi_1}{m} \right)^i \left( \frac{\xi_2}{m} \right)^j \rightarrow \frac{\xi_1^i \xi_2^j}{i! j!} \], as \( m \rightarrow \infty \).

With this, we have established that as \( m \rightarrow \infty \)

(1.4.55) \[ L_{k,h}^t \rightarrow \frac{k^h}{k! h!} \sum_{t=0}^{2\ell+1} \sum_{0<i<\ell+1, 0<j<\ell+1} (-\xi_1)^i (-\xi_2)^j i! j! \text{ and } U_{k,h}^t \rightarrow \frac{k^h}{k! h!} \sum_{t=0}^{2\ell} \sum_{0<i<\ell} (-\xi_1)^i (-\xi_2)^j i! j! . \]

Since \( \ell \) is arbitrary, and both \( L_{k,h}^t, U_{k,h}^t \) tend to a common limit as \( \ell \rightarrow \infty \), we conclude that \( P(U_{k,h}^t) \) tends to this common limit as well.

Thus, combining the above results we finally conclude that as \( m \rightarrow \infty \),

(1.4.56) \[ P(U_{k,h}^t) \sim P(U_{k,h}^t) \rightarrow (e^{-\xi_1 k/h})(e^{-\xi_2 h/h}). \]

Since an arbitrary integer \( n \) lies between two consecutive values of \( t \), and property (1.4.37) holds, (1.4.56) implies the desired result, thereby concluding the proof.

It is clear how the mutual stationarity assumptions simplified the proof of the above theorem as compared with Theorem 1.4.1. Particular simplification occurs at (1.4.41), (1.4.43) and specially (1.4.47) through (1.4.54). Now this assumption of mutual stationarity of the event sequences \( \{A_{i,j}^n; j=1,2,\ldots,n\}; i=1,2,\ldots,r \) is satisfied in a particular interesting class of situations related to stationary processes.

To illustrate this point we state a multivariate analogue (in terms of stochastic processes) of Loynes' original result. This version will find an application in Chapter III.
Corollary 1.4.2.1 Let \( \{Y_i(i=1, \ldots)\} \) be strongly mixing, strictly stationary \( p \)-dimensional vector-valued stochastic process, with mixing function \( g \). Further, let \( S_{in}(\xi_1), \ldots, S_{rn}(\xi_r) \), \( (\xi_i > 0; i=1, \ldots, r; n=1,2,\ldots) \) be \( r \) sequences of \( p \)-dimensional (Borel) sets such that

\[
P(Y_i \in S_{in}(\xi_i)) \sim \frac{\xi_i}{n} \text{ as } n \to \infty, \ i=1, \ldots, r.
\]

Suppose also that there exist sequences of integers \( \{p_m\}, \{q_m\} \) satisfying (1.4.5) and (1.4.6) of Theorem 1.4.1. Finally, defining

\[
A_{ij}^n = \{Y_j \in S_{in}(\xi_i)\} \ (i=1,2,\ldots,r; j=1,2,\ldots),
\]

suppose these events satisfy condition (1.4.38) of Theorem 1.4.2. Then, for any choice of \( r \) non-negative integers \( m_1, m_2, \ldots, m_r \), we have as \( n \to \infty \)

\[
P(\bigcap_{i=1}^r \{Y_j \in S_{in}(\xi_i)\} \text{ for exactly } m_i \text{ amongst } (j=1,2,\ldots,n)) \to \prod_{i=1}^r \left( e^{-\frac{\xi_i}{m_i}} \right).
\]

Proof. The only conditions that need verification are the mutual stationarity and common strong mixing conditions for the event sequences \( \{A_{ij}^n(j=1,2,\ldots)\}; i=1,2,\ldots,r \) as defined above. But, as mentioned in the introduction to this corollary, the strong mixing and strict stationarity properties of the underlying stochastic process \( \{Y_i(i=1,2,\ldots)\} \) are precisely sufficient to insure that the corresponding conditions are satisfied for the event sequences.

It will prove useful to have an explicit statement of some 'asymptotic independence' results that can be obtained directly from Theorem 1.4.2. The results we have in mind are to be analogous to Theorem 1.1.4, Corollary 1.3.1.1 and Corollary 1.3.1.2. We begin with the following result. For convenience, these results will be stated
Theorem 1.4.3  For each n, let \( A_i^n (i=1,2,...) \) be a stationary uniformly mixing sequence of events (with mixing function \( g \)). Suppose that for some \( \xi > 0 \), \( P(A_i^n) \sim \xi/n \) as \( n \to \infty \). Further, suppose that there exist sequences of integers \( \{p_m\} \) and \( \{q_m\} \) satisfying (1.4.5) and (1.4.6). Then provided \( \frac{1}{p} \sum_{j=2}^{p} (p-1) P(A_i^t | A_j^t) \to 0 \) as \( m \to \infty \) and \( N/n \to \lambda \) as \( n \to \infty \), we have for \( 0 \leq s < t < \infty \) \( P(\text{exactly } k \text{ amongst } A_i^n (i=[sN]+1,...,[tN]) \text{ occur}) \to e^{-\delta s^k/k!} \), where \( \delta = \lambda \xi (t-s) \).

The multivariate analogue of the above theorem is now presented.

Theorem 1.4.4  For each n, let \( A_i^{n,j} (j=1,2,...,r \text{ be } r \text{ mutually stationary and mixing sequences of events, with common mixing function } g \). Let \( \xi_i > 0; i=1,2,...,r \) be such that \( P(A_i^n) \sim \xi_i/n; i=1,2,...,r \) as \( n \to \infty \). Further, suppose that there exist sequences of integers \( \{p_m\} \) and \( \{q_m\} \) such that (1.4.36), (1.4.37), and (1.4.38) are satisfied. Then for \( 0 \leq s_i < t_i < \infty , N_i/n \sim \lambda_i \) (as \( n \to \infty \) \( i=1,2,...,r \)) and for any choice of \( r \) non-negative integers \( m_1,m_2,...,m_r \), we have as \( n \to \infty \),

\[
P(\bigcap_{i=1}^{r} \{\text{exactly } m_i \text{ amongst } A_i^{n,j} (j=[s_i N_i]+1,...,[t_i N_i])\}) \to \prod_{i=1}^{r} (e^{-\delta_i s_i^m/m_i !}) .
\]

where \( \delta_i = \xi_i \lambda_i (t_i-s_i) ; i=1,2,...,r \).

Proof. For \( \lambda_i = 1; s_i=0, t_i=1; i=1,2,...,r \) this is precisely the same as Theorem 1.4.2. The proof of the present theorem is merely an adaptation of the proof of that theorem and will not be presented here.

Finally we are in a position to state an interesting theorem explicitly concerning asymptotic independence of the number of events
occurring in disjoint intervals. It will have an interesting application in Chapter III.

**Theorem 1.4.5** For each \( n \), let \( \{ A_i^n (i=1,2,\ldots) \} \) be a stationary, strongly mixing sequence of events (with mixing function \( g \)). Suppose \( P(A_i^n) \sim \xi_i/n \) as \( n \to \infty \) and that there exists sequences \( \{ p_m \} \) and \( \{ q_m \} \) so that the condition of Theorem 1.4.3 are satisfied. Further, let \( S_i = (S_{i1}, S_{i2}) \); \( i=1,2,\ldots,r \) be \( r \) bounded, non-overlapping subintervals of \((0,\infty)\) (either closed, open, or semi-closed) and let \( m_1, m_2, \ldots, m_r \) be \( r \) non-negative integers. Then, as \( n \to \infty \)

\[
(1.4.59) \quad P\left( \bigcap_{i=1}^r \{ \text{exactly } m_i \text{ amongst } A_i^n; j=[nS_{i1}]^j+1, \ldots, [nS_{i2}] \text{ occur} \} \right) \to \prod_{i=1}^r \left( e^{\xi_i S_i} \frac{S_i^{m_i}}{m_i!} \right),
\]

where \( S_i = S_{i2} - S_{i1} \); \( i=1,2,\ldots,r \).

**Proof.** This result follows from Theorem 1.4.4. Define \( A_{ij}^n = A_i^n [nS_{i1}]^j; j=1,2,\ldots,[\xi_i n] = N_i; i=1,2,\ldots,r \). It can be shown that the sets so defined satisfy the conditions of Theorem 1.4.4 with \( \xi_i^* = \xi_i \) and \( \lambda_i^* = \delta_i \); \( i=1,\ldots,r \). The other conditions of the theorem are also satisfied, and the result is established.

For completeness we now state a 'multivariate waiting time' result.

**Theorem 1.4.6** For each \( n \), let \( \{ A_{ij}^n; j=1,2,\ldots; i=1,2,\ldots,r \} \) be \( r \) sequences of jointly stationary strongly mixing (function \( g \)). Sequences of events. Suppose that as \( n \to \infty \), \( P(A_{ij}^n) \sim \xi_i^*/n \); \( \xi_i^*>0; i=1,2,\ldots,r \), and that there exist sequences of integers \( \{ p_m \} \) and \( \{ q_m \} \) such that conditions (1.4.36), (1.4.37) and (1.4.38) of Theorem 1.4.2 are satisfied. Then if for \( r \) non-negative integers \( m_1, m_2, \ldots, m_r \) and \( r \) real numbers
\[ t_i; 0 \leq t_i < \omega; \ i=1,2,\ldots,r, \ F(m_1,\ldots,m_r; t_1,\ldots,t_r) = P( \bigcap_{i=1}^{r} \{ \text{waiting time for } m_i \text{th occurrence of } A_{ij}^n; j=1,2,\ldots \text{ greater than } \lceil nt_i / \xi_i \rceil \} ) \]
\[ \rightarrow \prod_{i=1}^{r} (1 - \sum_{j=0}^{\infty} e^{-t_i / j}) \text{ as } n \rightarrow \infty. \]

**Proof.** This is, of course, a simple application of Theorem 1.4.5.

Again, as at the end of Section 1.3, we can summarize some of the preceding section in the form of a theorem concerning weak convergence to a Poisson process. We now state such a theorem for stationary sequences of events.

**Theorem 1.4.7** For each \( n \) let \( \{ A_i^n; i=1,2,\ldots,n \} \) be a uniformly mixing stationary sequence of events (with mixing function \( g \)). Suppose that for some \( \xi > 0 \), \( P(A_i^n) \sim \xi / n \) as \( n \rightarrow \infty \), and that there exist sequences of integers \( \{ p_m \} \) and \( \{ q_m \} \) such that the condition of Theorem 1.4.3 are satisfied. Let \( U_n(t) \) \( (0 \leq t \leq 1; \ n=1,2,\ldots) \) be the number of events \( A_i^n \) that occur with \( 1 \leq i \leq \lceil nt \rceil \). Then, as \( n \rightarrow \infty \), the sequence of counting process \( \{ U_n(t); 0 \leq t \leq 1 \} \) \((n=1,2,\ldots)\) converges weakly over [0,1] to a Poisson process of intensity \( \xi \).

This theorem follows from the stationary version of Theorem 1.4.1 and Theorem 1.4.2. Of course, a more general non-stationary version is possible. However, we shall not pursue such a result here.
CHAPTER II

ASYMPTOTIC STRUCTURE OF A QUANTUM BIOPHYSICS OF VISION MODEL

2.0 Summary

As an application of the results of the preceding chapter, we consider a certain stochastic model for the quantum biophysics of vision. The limiting behavior of this model has been considered by Ikeda [10], Isii [11], Yamamoto et al. [20], Bouman and Van Der Velden [2] (and in a different context by van Elteren et al. [18]) and others. In this chapter we describe and extend the known limiting results for this model, and expose some of its rich limiting structure.

2.1 Introduction

We first introduce the stochastic model for the quantum biophysics of vision that will be examined in this chapter. One considers light quanta being absorbed by (or, arriving at) the retina of the eye according to a positive renewal process \( \{y_i; i=1,2,\ldots\} \), that is, a process for which the 'interarrival' times \( \{x_i = y_i - y_{i-1}; i=1,2,\ldots\} \) \((y_0 = 0)\) form a sequence of independent and identically distributed random variables for which \( P(x_i = 0; i=1,2,\ldots) \). Let us assume that the \( i \)th particle (quanta), which arrives at time \( y_i \), has a (random) lifetime \( \tau_i; i=1,2,\ldots \). A \( k \)th \((k \geq 1)\) order visual response is defined to occur if at least \( k \) particles are simultaneously 'alive' for some interval of time. Of interest is the limiting behavior of the probability of m \( k \)th order
visual response in the time interval \((0,t]\) as \(t \to \infty\) in some manner.

In the model above, let \(X(t)\) represent the number of particles simultaneously 'alive' at time \(t\). One way of representing \(X(t)\) is as follows:

\[
X(t) = \sum_{n=1}^{N(t)} f(t-x_n, x_n), \quad 0 < t < \infty,
\]

where \(N(t)\) is the number of particles arriving before or at time \(t\), and \(f(x,y)\) is a function defined for \(y > 0\) as

\[
f(x,y) =\begin{cases} 
1, & \text{if } 0 \leq x \leq y \\
0, & \text{if } x < 0 \text{ or } x > y.
\end{cases}
\]

A typical realization of the process \(\{X(t), 0 < t < \infty\}\) might, in part, appear as follows:

![Graph showing a typical realization of \(X(t)\)](image)

FIGURE 2.1.1

Ikeda [10], Isii [11], Yamamoto et al. [20], Bouman and Van Der Velden [2] and others have considered certain limiting behavior of such a process in relation to a problem of visual response in certain experimental research in quantum-biophysics of vision. It is easily seen that a \(k\)th order visual response begins at time \(t'\) if and only if at time \(t'\)
the process $X(t)$ attains (or exceeds) the level $X=k$ from below. The response lasts until the process first drops below the level $X=k$. Of specific interest is the behavior of $P(m \text{ kth order visual responses in } (0,t)]$ as $t \to \infty$ in some suitable manner (later to be specified). For convenience, this will be referred to as the 'Limit Problem' (for a non-random time interval).

Of course, the quantum-biophysics of vision model is not the only situation where the process $X(t)$ and the 'Limit Problem' might arise. The process has, in fact, been considered in a different setting by van Elteren et al. [18]. There, the authors consider a model of the form as $X(t)$ for the 'thickness' at time $t$ of a strand of fiber being woven from individual filaments. The $i$th filament, entering the strand at time $y_i$, has a (random) length $\tau_i$.

The simplest form of the 'Limit Problem' is the case where the

$(y_i; i=1,2,\ldots)$ form a Poisson process of intensity $\mu$ (i.e. where the random variables $z_i = \mu(y_i - y_{i-1}); i=1,2,\ldots$, with $y_0 = 0$, are independent and identically distributed -- iid -- exponential random variables with unit mean), and the lifetimes $(\tau_i; i=1,2,\ldots)$ are identical positive constants, $(\tau_i = \tau > 0; i=1,2,\ldots)$. In this case it has been shown (see, reference 5 in [10]) that for $t \to \infty$ with $\mu^k t \to \lambda$ ($\lambda$ a positive constant)

$P^k_m(t) = P(\text{exactly } m \text{ kth order visual responses in } (0,t])$

$\to e^{-\xi} \xi^m / m! ,$

where $\xi = \lambda \tau^{k-1}/(k-1)!$. For convenience, we refer to (2.1.3) as the 'Poisson Limit Result' (for non-random $t$, $(y_i)$ Poisson ($\mu$), $(\tau_i)$ constant and equal).
The Limit Problem for iid random lifetimes has been studied. With \( \alpha = \mathcal{E}_\tau < \infty \) replacing \( \tau \) in the above definition of \( \xi \), the Poisson Limit Result has been established for various special cases using different techniques. With the \( \{\tau_i\} \) iid exponential random variables (independent of \( \{y_i\} \) with mean \( \alpha \)), van Elteren et al. [18] and Yamamoto [20] have, for example, established (2.1.3); with the \( \{\tau_i\} \) iid positive random variables (independent of \( \{y_i\} \) with finite second moment, Ikeda [9] established (2.1.3); with the \( \{\tau_i\} \) iid positive random variables (independent of \( \{y_i\} \) with a finite moment of specified order \( \delta \), \( 1 < \delta < 2 \), Isii [11] established (2.1.3); finally, under the sole assumption that the \( \{\tau_i\} \) are positive random variables (independent of \( \{y_i\} \) with a finite first moment \( \alpha = \mathcal{E}_\tau \)), Ikeda [10] established (2.1.3).

It is felt that the diverse techniques used to establish the various cases of Poisson Limit Result, (2.1.3), have in a sense hidden much of the rich limiting structure of the process \( X(t) \) defined by (2.1.1). Furthermore, these diverse techniques all rely heavily on the 'nice' properties of Poisson arrivals. We will pursue a different type of approach. Using the results of Chapter 1 (in particular, the generalization of Watson's theorem found in Corollary 1.1.2.1) all previously known cases of the Poisson Limit Result will be established. In the process, the limiting structure of the process \( X(t) \) is nicely exposed, and the effects of the increasingly more general assumptions on the (moments of) \( \{\tau_i\} \) become apparent. For example, the moment assumptions on the \( \{\tau_i\} \) will be seen to be reflected in the limiting behavior of the lifetimes of individual particles measured in terms of the number of subsequent particles they survive. An interesting boundary case will be presented.
The results of Chapter 1 will also be used to consider the Limit Problem for some new situations. In particular, it will be shown that an analogue of the Limit Result is valid if the \( \{y_i; i=1,2,\ldots\} \) form a positive iid renewal process while the lifetimes are iid (and independent of \( \{y_i\} \)) exponential random variables.

First, however, we shall discuss the relationship between a certain aspect of our method of approach (non-random \( n \)), and the approaches previously taken (non-random \( t \)).

2.2 The Limit Problem for non-random \( n \) vs. non-random \( t \)

In the Limit Problem, one might consider both the limiting behavior of

\[
q_m^k(n) = P(\text{exactly } m \text{ kth order visual responses in first } n \text{ arrivals}) \text{ as } n \to \infty,
\]

as well as

\[
p_m^k(t) = P(\text{exactly } m \text{ kth order visual responses in } (0,t]) \text{ as } t \to \infty,
\]

where by the "first \( n \) arrivals" in (2.2.1) we mean the "time interval determined by the first \( n \) arrivals," viz. \( (0,y_n] \). As one might suspect, the limiting properties of (2.2.1) and (2.2.2) are closely related. This is quite convenient, since the results of Chapter 1 lend themselves more readily to use with \( p_m^k(n) \) as defined in (2.2.2). In this section we shall pursue this relationship, so that results obtained about (2.2.1) can be 'translated' into corresponding results about (2.2.2).

It will be observed that a visual response can begin only at an arrival point, that is, only at one of the (random) time points \( \{y_i; i=1,2,\ldots\} \). (It need not, however, terminate at one of these points). Thus, since there is a one-to-one correspondence between the
number of visual responses in an interval and the number of starting points for these responses, we need only consider the latter for the Limit Problem. For our purposes, this observation will subsequently enable us to replace investigation of the continuous time process $X(t)$ by an investigation of a related discrete time vector process.

Now if $n_t$ is the (random) number of particles arriving in the time interval $(0,t]$, we may restate the Limit Problem (for non-random $t$) as a determination of the limiting behavior of (2.2.3) $q^k_m(n_t) = P$(exactly $m$ $k$th order visual responses in the first $n_t$ arrivals) as $t \to \infty$. Thus, the Limit Problem for non-random $t$ corresponds to the Limit Problem for a random $n$. Similarly, the Limit Problem for non-random $n$ corresponds to the Limit Problem for a random $t$. Now if, for example, the arrival times resulted from an iid positive renewal process, one might suspect that for large $t$, $p^k_m(t) = q^k_m(n_t)$ and $q^k_m(p_n)$ would be 'close' (in some sense). This, indeed, can be formalized and proved rigorously. For purposes of continuity, however, we shall just state some results at this point, and relegate a complete treatment to Appendix A.

Suppose that particles arrive according to a positive renewal process $\{Y_i; i=1,2,\ldots\}$ for which the 'interarrival times'

$[x_i = Y_i - Y_{i-1}; i=1,2,\ldots] (Y_0 = 0)$ form an iid sequence of positive random variables with mean $1/\mu$ ($\mu > 0$). Furthermore, assume that

particle lifetimes $[\tau_i; i=1,2,\ldots]$ form a sequence of iid random variables (independent of arrivals). The results of Appendix A can be used to establish the following special results that will have immediate use in the sequel.

**Theorem 2.2.1** If 'interarrival times' $[x_i = Y_i - Y_{i-1}; i=1,2,\ldots] (Y_0=0)$
form an iid sequence of exponential random variables with mean $1/\mu$, while lifetimes \{\tau_i; i=1,2,\ldots\} form a sequence of iid random variables (independent of arrivals) with finite mean $\alpha$, then for $\lambda > 0$,

$$\lim_{n \to \infty} q^k_m(n) \equiv \zeta^k \Rightarrow \lim_{t \to \infty} p^k_m(t) \equiv \zeta^k \equiv \zeta^k.$$

(The symbol $\equiv$ is to read 'exists and equals'.)

**Theorem 2.2.2** If 'interarrival times' \{\tau_i = \tau_i - \tau_{i-1}; i=1,2,\ldots\} \(\nu_o=0\) form an iid sequence of positive random variables with mean $1/\mu$ ($\mu > 0$) and finite variance, while lifetimes \{\tau_i; i=1,2,\ldots\} form a sequence of iid exponential random variables, independent of arrivals, and with mean $\alpha$, then

$$\lim_{n \to \infty} q^k_m(n) \equiv \zeta^k \Rightarrow \lim_{t \to \infty} p^k_m(t) \equiv \zeta^k,$$

$$\mu t \prod_{j=1}^{k-1} \hat{G}(\frac{j}{\mu}) \to \delta \quad \mu t \prod_{j=1}^{k-1} \hat{G}(\frac{j}{\mu}) \to \delta$$

where $\hat{G}(\cdot)$ is the Laplace-Stieltjes transform of the distribution function of the random variable $\mu \tau_i$.

It should be noted that Theorems 2.2.1 and 2.2.2 essentially coincide when both 'interarrival times' and lifetimes are exponential. In this case $\hat{G}(s) = \frac{1}{s}$.

As noted above, the proofs of Theorem 2.2.1 and 2.2.2 are relegated to Appendix A. For the moment, the important feature of these results is that they formalize the early remarks of this section for two very particular situations. These situations will present themselves in the following sections; in view of Theorems 2.2.1 and 2.2.2 we feel free to treat the Limit Problem in these situations for a non-random $n$ rather
than for a non-random $t$, since they provide the appropriate translation of results from the former to the latter.

2.3 Solution of the Limit Problem for known cases

In this section we establish the 'Poisson Limit Result' for all previously known cases. The method of approach is based on the results of the preceding chapter (in particular Corollary 1.1.3.1). It will later become apparent that the time spent in treating known cases of the Limit Problem is well worth while; not only does our method expose much of the rich limiting structure of the particular model at hand, but also the arguments used can (and will) be transferred, almost verbatim, to treat the Limit Problem for some new situations.

It is convenient to establish all results for the specific case $k=3$ (i.e., to deal only with the Limit Problem for third-order visual responses). It will be seen that this leads to considerable notational convenience. The proofs will remain valid for a general $k$ with the appropriate (and obvious) notational changes. Throughout this section we shall use the notation $(n \rightarrow \infty)$ as an abbreviation of the statement "$n \rightarrow \infty, \mu \downarrow 0$ so that $n\mu^2 \rightarrow \lambda (\lambda > 0)$.

The Simplest form of the Limit Problem

The simplest (non-trivial) case of the Limit Problem is the situation where arrival times $\{y_i; i=1,2,\ldots\}$ form a Poisson process of intensity $\mu$ (i.e., the 'interarrival times' $\{x_i = y_i-y_{i-1}, i=1,2,\ldots\}$ ($y_0 = 0$) are independent exponential random variables with mean $1/\mu$), and particle lifetimes are constant and equal (to $\tau > 0$). A typical realization of the process $X(t)$ (see Section 2.1) might, in part, appear as follows:
We now proceed to analyze the limiting behavior of the number of 3rd order visual responses in this simple model. If \( \{x_i; i=1,2,\ldots\} \) are the interarrival times of particles defined above, let \( \{z_i; i=1,2,\ldots\} \) be defined by \( z_i = \mu x_i; i=1,2,\ldots. \) The \( z_i \)'s are iid exponential with unit mean. Define a 2-dimensional discrete time stochastic process \( \{Y_i; i=1,2,\ldots\} \) and a collection of 2-dimensional sets \( S_{\mu} (\mu > 0) \) as follows:

\[
(2.3.1) \quad Y_i = (z_i + z_{i-1}, z_i); \quad i=2,3,\ldots; \quad S = \{(x,y) \mid x < \mu t, y < \mu t\}; \mu > 0.
\]

Note that \( \{Y_i; i=2,3,\ldots\} \) is a 1-dependent strictly stationary vector stochastic process, and that as \( (n \to \infty) \) (with \( \xi = \lambda \tau^2/2! \)), we have

\[
(2.3.2) \quad P(Y_i \in S_{\mu}) \sim \frac{\xi}{n}.
\]

This is true, since as \( (n \to \infty) \) with \( \xi = \lambda \tau^2/2! \),

\[
(2.3.3) \quad nP(Y_i \in S_{\mu}) = P(z_{i} + z_{i-1} < \mu t, z_i < \mu t) = nP(z_{i} + z_{i-1} < \mu t)
\]

\[
= n \int_0^{\mu t} se^{-s} ds = n[1 - e^{-\mu \tau (1+\mu \tau)}] \to \xi.
\]
(We remark that it is, in fact, not necessary to introduce the vector process \( \{Y_i, i=1,2,\ldots\} \). However, in the next section it will be necessary to do such a thing; we go through the procedure here so as to stress an analogy that will later become apparent.) Next we note that as \((n \to \infty)\), we have

\[
(2.3.4) \quad P(Y_i \in S_{\mu}, Y_{i+1} \in S_{\mu})/P(Y_i \in S_{\mu}) \to 0.
\]

To establish this result, we observe that via (2.3.2)

\[
(2.3.5) \quad 0 \leq nP(Y_i \in S_{\mu}, Y_{i+1} \in S_{\mu}) = nP(z_i + z_{i-1} < \mu \tau, z_{i+1} + z_i < \mu \tau) \\
\leq nP(z_i + z_{i-1} < \mu \tau) \cdot P(z_{i+1} < \mu \tau) \to 0
\]

as \((n \to \infty)\). Thus the process \(\{Y_i; i=2,3,\ldots\}\) and the sets \(S_{\mu} (\mu > 0)\) satisfy the conditions of Corollary 1.1.3.1 as \((n \to \infty)\), and we may immediately conclude

\[
(2.3.6) \quad \lim_{(n \to \infty)} P(Y_i \in S_{\mu} \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n) \\
= e^{-\xi} \frac{m^m}{m!}.
\]

We conclude by showing that, in the limit, the events "\(Y_i \in S_{\mu} \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n\)" and "\(m \text{ (3rd order) visual responses in the first } n \text{ arrivals}" have equal probability. Note that since

\(X_i \equiv \tau > 0; i=1,2,\ldots\), the event \(Y_i \notin S\) occurs if and only if \(X(Y_{i+1}) < 3\), and hence if and only if no third order visual response begins at time \(Y_{i+1}\). However, the event \(Y_i \in S_{\mu}\) guarantees only that \(X(Y_{i+1}) \geq 3\), so by itself does not insure that a third order visual response begins at time \(Y_{i+1}\). The following figures illustrate these points. In the first figure, (Figure 2.3.2; A,B).the correspondence
between the events \( Y_1 \in S_\mu \) and \( X(y_{i+1}) < 3 \) is illustrated; in the second figure (Figure 2.3.3), both panels depict the event \( Y_1 \in S_\mu \), yet only in the second panel (B) is the event 'a third order visual response beginning at time \( y_{i+1} \)' depicted. Figure 2.3.3 illustrates how 'bunching up' of events \( Y_1 \epsilon S_\mu \) creates the difficulties.

\[ \begin{align*}
\text{FIGURE 2.3.2} \\
\end{align*} \]

However, we note that using (2.3.2)
\[(2.3.7) \quad P(\bigcup_{i=1}^{n-1} \{X(y_{i+1}) \geq 3, X(y_i) \geq 3 \} \leq nP(X(y_n) \geq 3, X(y_{n-1}) \geq 3)
= nP(z_{n-2} + z_{n-1} < \mu \tau, z_{n-1} + z_n < \mu \tau)
\leq nP(z_{n-2} + z_{n-1} < \mu \tau) \cdot P(z_n < \mu \tau) \rightarrow 0\]

as \(n \rightarrow \infty\). Thus, in the limit, no 'bunching up' of events \(Y_i \in S_\mu \) occurs; the event \(Y_i \in S_\mu \) corresponds to the event "\(X(y_{i+1}) \leq 3\)," that is, to a visual response (beginning) at time \(y_{i+1}\). Since (as noted before) visual responses can begin only at arrival times, \((2.3.6)\) is equivalent to the Poisson Limit Result, viz.,

\[(2.3.8) \quad \lim_{n \rightarrow \infty} P(m \text{ 3rd order visual responses in first n arrivals}) = e^{-\frac{\mu}{2}\frac{m^2}{m!}}.
\]

With this, we are done. We shall, however, return in Section 2.4 for another look at the distribution of particle lifetimes measured in a different way.

**A more general form of the Limit Problem**

We now turn to a more general model that allows for randomness of the lifetimes \(\{\tau_i; i=1,2,\ldots\}\). The model we consider has been considered by Ikeda [10] and is as follows: as before the arrival times \(\{y_i; i=1,2,\ldots\}\) form a Poisson process with intensity \(\mu\), while now lifetimes \(\{\tau_i; i=1,2,\ldots\}\) form a sequence of iid positive random variables (independent of arrivals) with finite first moment \(\alpha\). We now proceed to establish the Poisson Limit Result for this case, which of course includes the previous case.

As in the previous case, let \(\{z_i, i=1,2,\ldots\}\) be defined by \(z_i = \mu \tau_i; i=1,2,\ldots\), where \(\{\tau_i; i=1,2,\ldots\}\) are the 'interarrival times'...
for particles. The \( z_i \)'s are iid exponential random variables with unit mean. Define the 2-dimensional vector valued discrete time stochastic process \( \{ Y_i; i=2,3,... \} \) and the collection of 2-dimensional sets \( S_{\mu} \) \((\mu > 0)\), as follows:

\[
(2.3.8) \quad Y_i = \left( \frac{z_i + z_{i-1}}{i-1}, \frac{z_i}{i} \right); \quad i=2,3,...; \quad S_{\mu} = \{(x,y) | x < \mu, y < \mu \} \quad (\mu > 0).
\]

Note that \( \{ Y_i; i=2,3,... \} \) is 1-dependent and strictly stationary.

Furthermore, letting \( F(t) = P(\tau_1 \leq t) \) we have

\[
(2.3.9) \quad nP( Y_i \in S_{\mu} ) = n\mu^2 \int_0^\infty \int_0^\infty \{1-F(x+y)\} \{1-F(x)\} e^{-\mu(x+y)} dy \, dx.
\]

Using the relation (see Ikeda [10], p. 298)

\[
(2.3.10) \quad \frac{\alpha^2}{2!} = \int_0^\infty \int_0^\infty \{1-F(x+y)\} \{1-F(x)\} \, dy \, dx.
\]

an application of the dominated convergence theorem shows that as \( (n \to \infty) \)

\[
(2.3.11) \quad nP( Y_i \in S_{\mu} ) \to \lambda \alpha^2 / 2! = \xi.
\]

Hence, as \( (n \to \infty) \),

\[
(2.3.12) \quad P( Y_i \in S_{\mu} ) \sim \xi / n.
\]

Furthermore, as \( (n \to \infty) \)

\[
(2.3.13) \quad P( Y_i \in S_{\mu}, Y_{i+1} \in S_{\mu}) / P( Y_i \in S_{\mu}) \to 0.
\]

Since,

\[
(2.3.14) \quad 0 \leq nP( Y_i \in S_{\mu}, Y_{i+1} \in S_{\mu}) \leq nP( Y_i \in S_{\mu}) \cdot P(z_{i+1} < \mu \tau_{i+1}) \to 0
\]
as \( n \to \infty \) via (2.3.12). Thus, the process \( \{Y_i: i=2,3,\ldots\} \) and the sets \( S_{\mu}(\mu > 0) \) satisfy the conditions of Corollary 1.1.3.1, so

\[
\lim_{n \to \infty} P(Y_i \in S_{\mu} \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n) = e^{-\frac{\mu^m}{m!}}.
\]

What remains is to establish a correspondence (in the limit) between the events "\( Y_i \in S_{\mu} \)" and "a third order visual response at time \( Y_{i+1} \);" \( i=2,\ldots \).

Establishing the above correspondence in the (present) case of random lifetimes is considerably more complicated than for the case of constant and equal lifetimes. The problem is that particle lifetimes independently may vary widely, from very small to very large. Nevertheless, the correspondence can be established, and the remainder of this section is devoted to this task.

Our method of attack is to establish the following relations (in the limit):

\[
\begin{align*}
(2.3.16) \quad &\text{the two events } "Y_i \in S_{\mu} \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n" \text{ and } "X(Y_{i+1}) \geq 3 \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n" \text{ have equal probability;}
\end{align*}
\]

\[
(2.3.17) \quad \text{if } X(Y_{i+1}) \geq 3 \text{ for some } i=2,3,\ldots,n \text{ then } X(Y_i) < 3;
\]

\[
(2.3.18) \quad \text{the event } "X(Y_{i+1}) \leq 3 \text{ for all } i=2,3,\ldots,n" \text{ occurs with probability } 1.
\]

From the above relations one may then conclude that (in the limit),

\[
(2.3.19) \quad \text{the desired correspondence is valid, and hence the Poisson Limit Result is true; i.e., the number of 3rd order}
\]
visual responses is finite and is distributed according to the Poisson law with parameter $\xi$;

(2.3.20) visual responses of order greater than three never occur;

(2.3.21) visual responses of order less than three occur infinitely often (this follows from the $m$-dependent analogue of the Borel-Cantelli lemma found in Lemma 1.2.1).

Thus, in the limit, we picture the process $X(t)$ (defined by 2.1.1) as jumping infinitely often amongst the states $X=r$, $r < 3$, and infrequently to the state $X=3$, though never higher. For large $n$ and $n\mu^2 \sim \lambda$ a typical realization of the process might appear, in part, as illustrated in Figure 2.2.4.

![Figure 2.2.4](image_url)

The number of jumps by the process to the state $X=3$ in along series of trials of such a process would have approximately a Poisson distribution with parameter $\xi$. These jumps to $X=3$ are the proverbial 'rare events' to which Poisson theory so frequently applies.

To prove (2.3.16) we use the following definitions and Lemma.

**Definition 2.3.1** The event $A_i$ is said to occur if $X(y_{i+1}) \geq 3$; $i=2,3,\ldots$, that is if two or more particles arriving prior to time $y_{i+1}$
survive until time $y_{i+1}$.

**Definition 2.3.2** The event $B_i$ is said to occur if $v_i \in S_\mu$; $i=2,3,...$, that is, if the two particles arriving immediately prior to time $v_{i+1}$ survive until time $y_{i+1}$.

**Definition 2.3.3** The event $C_i$ is defined by $C_i = A_i \overline{B}_i$; $i=2,3,...$.

If we let,

$A_n^{(j)}$ be the event "exactly $j$ amongst $A_i$ ($i=2,3,\ldots,n$) occur,"

$B_n^{(j)}$ be the event "exactly $j$ amongst $B_i$ ($i=2,3,\ldots,n$) occur,"

$C_n^{(j)}$ be the event "exactly $j$ amongst $C_i$ ($i=2,3,\ldots,n$) occur,"

we have the following relationship:

**Lemma 2.3.1**

$$A_n^{(m)} = \bigcup_{j=0}^{m} B_n^{(m-j)} C_n^{(j)} ; m=0,1,\ldots,n-1 .$$

**Proof.** The proof is simple. Since $B_i \subseteq A_i$ and $C_i = A_i \overline{B}_i$, we have $A_i = B_i \cup C_i$, a disjoint union. Thus, $m A_i$'s occur if and only if $m-j B_i$'s and $j C_i$'s occur for some integer $j$, $2 \leq j \leq n$.

By virtue of the definitions of the above events, (2.3.16) will be established if we can show that $P(A_n^{(m)}) - P(B_n^{(m)}) \rightarrow 0$ as $(n \rightarrow \infty)$. In fact, it will suffice to show that as $(n \rightarrow \infty)$,

$$P(C_n^{(0)}) \rightarrow 0 .$$

From Lemma 2.3.1 we have,

$$0 \leq |P(A_n^{(m)}) - P(B_n^{(m)})| = |P(\bigcup_{j=0}^{m} B_n^{(m-j)} C_n^{(j)}) - P(B_n^{(m)})|$$

$$= |P(\bigcup_{j=1}^{m} B_n^{(m-j)} C_n^{(j)}) - P(B_n^{(m)} C_n^{(0)})| .$$
Each of these last two events is contained in $\tilde{C}_n^{(0)}$. Hence the above expression is at most $2P(\tilde{C}^{(0)}_n)$, and so it tends to zero when (2.3.22) is satisfied.

To show that (2.3.22) holds, we first define a basic event, $F_i^m$; the event $F_i^m$ $(0 < i < m, m \geq 2)$ is said to occur if the $(m-i)$th particle survives until the arrival of the $m$th particle; i.e., $F_i^m = (Y_m - Y_{m-i-1} < \tau_{m-i-1})$. We now note that we may write

\begin{equation}
C_n^{(0)} = \bigcup_{m=4}^{n+1} C_{m-1}, \text{ (since } C_2 = \emptyset), \text{ and}
\end{equation}

\begin{equation}
C_{m-1} = \bigcup_{j=3}^{m-1} R_j^m, \text{ where}
\end{equation}

\[ R_3^m = F_1^{m_3} F_3^{m_2} \cup F_1^{m_3} F_3^{m_2} \]

\[ R_4^m = F_1^{m_4} F_2^{m_3} F_4^{m_2} \cup F_1^{m_4} F_2^{m_3} F_4^{m_2} \cup F_1^{m_4} F_2^{m_3} F_4^{m_2} \]

\[ \vdots \]

and so on. Replacing every event of the form $C_{i-1}^m$ by the sure event, we obtain from (2.3.25),

\begin{equation}
P(R_j^m) \leq \sum_{\ell=1}^{j-1} P(F_\ell^m F_j^m).
\end{equation}

By noticing that

\begin{equation}
P(F_\ell^m F_j^m) = P(Y_\ell < \tau_\ell, Y_j < \tau_j),
\end{equation}

we obtain from (2.3.25)

\begin{equation}
P(C_{m-1}) \leq \sum_{j=3}^{\infty} \sum_{\ell=1}^{j-1} P(Y_\ell < \tau_\ell, Y_j < \tau_j).
\end{equation}
\[ \sum_{j=3}^{\infty} \sum_{\ell=1}^{j-1} \int \int \left[ \frac{\mu^\ell e^{-\mu t}}{(\ell-1)!} \right] \left[ \frac{j-\ell-1}{(j-\ell-1)!} \right] \frac{1-F(s)}{1-F(s+t)} ds dt \]

where \( F(x) = P(T_1 \leq x) \). Thus, using the identity

\[ \sum_{\ell=1}^{j-1} \frac{t^{\ell-1}s^{j-\ell-1}}{(\ell-1)! (j-\ell-1)!} = \frac{1}{(j-2)!} \sum_{\ell=0}^{j-2} (t+s)^{j-2-\ell} = \frac{(t+s)^{j-2}}{(j-2)!} \]

we obtain from (2.3.28) and the principle of monotone convergence,

\[ P\left( C_{m-1} \right) \leq \mu^2 \sum_{j=3}^{\infty} \int \int \mu^{j-2} \frac{(s+t)^{j-2}}{(j-2)!} e^{-\mu(s+t)} [1-F(s)] \left[ 1-F(s+t) \right] ds dt \]

\[ = \mu^2 \int \int (1-e^{-\mu(s+t)}) [1-F(s)] [1-F(s+t)] ds dt . \]

Hence, by (2.3.24)

\[ P\left( C_{n}^{(0)} \right) \leq \nu^2 \int \int (1-e^{-\mu(s+t)}) [1-F(s)][1-F(s+t)] ds dt . \]

Now as \( n \to \infty \) the integrand in (2.3.24) tends, pointwise, to zero.

Since it is bounded by the integrable function \( 2[1-F(s)][1-F(s+t)] \) (see (2.3.10)) the integral tends to zero as \( n \to \infty \) (and \( \mu \downarrow 0 \)). Finally, \( \nu^2 \to \lambda < \infty \) as \( n \to \infty \), thereby establishing (2.3.22).

Now to prove (2.3.17) it will suffice to show that as \( n \to \infty \)

\[ P\left[ \bigcup_{j=2}^{n} \{ X_{j+1} \geq 3, X_j \geq 3 \} \right] \to 0 , \]

or, using the notation established above, that

\[ P\left[ \bigcup_{j=2}^{n} \left( A_{j+1} \cap A_j \right) \right] \to 0 . \]
Now \( A_i = B_i \cup C_i \), hence \( A_i \cap A_{i+1} = (B_i \cup C_i) \cap (B_{i+1} \cup C_{i+1}) \) \( \subseteq (B_i \cap B_{i+1}) \cup C_i \cup C_{i+1} \). Thus,

\[
P[\bigcup_{i=2}^n (A_i \cap A_{i+1})] \leq \sum_{i=2}^n P(B_i \cap B_{i+1}) + \sum_{i=2}^n P(C_i)
\]

\[
\leq nP(B_i \cap B_{i+1}) + 2P(C_{n+1})
\]

The first term on the (extreme) right-hand side of (2.3.34) tends to zero by (2.3.14) and the second term tends to zero by (2.3.22). Thus, (2.3.33) and hence (2.3.17) is established.

Finally, we prove (2.3.18). We begin by defining the following events: \( H_n = \{X_{(i+1)} > 3 \text{ for some } i=1,2,\ldots,n\} \), \( H^m_n = \{X_{(i+1)} > 3\} \), \( S_m = \{\text{the three particles arriving immediately prior to time } y_{m+1} \text{ survive until time } y_{m+1}\} \), \( T_m = \{\text{some three particles arriving prior to time } y_{m+1} \text{ survive until time } y_{m+1}\} \), and \( U_m = T_m \setminus S_m \).

First we note that since \( S_m \subseteq T_m \)

\[
H_n = \bigcup_{m=4}^n T_m = \bigcup_{m=4}^n (S_m \cup U_m)
\]

and hence,

\[
P(H_n) \leq \sum_{m=4}^n P(S_m) + \sum_{m=4}^n P(U_m).
\]

We now proceed to show that \( P(H_n) \rightarrow 0 \) as \( n \rightarrow \infty \), thereby establishing (2.3.18). To show that the first summation on the right-hand side of (2.3.36) tends to zero, observe that for \( m \geq 4 \),

\[
P(S_m) = \mu \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} [1-F(s+t+u)][1-F(s+t)][1-F(s)]e^{-u(s+t+u)}
\]

dsdtdudw, hence
\[
\begin{align*}
(2.3.38) \quad \sum_{m=4}^{m} P(S_m) & \leq \mu \left[ \sum_{0}^{\infty} \int \int \int [1-F(s+t+u)][1-F(s+t)][1-F(s)] e^{-\mu(s+t+u)} dsdtdu \right] \to 0
\end{align*}
\]

as \((n \to \infty)\) (and \(\mu \downarrow 0\)). We now show that the second summation on the right hand side of (2.3.36) tends to zero as \((n \to \infty)\). In a manner similar to that employed previously, we write

\[
(2.3.39) \quad \bigcup_{m=4}^{m-1} T_j^m, \text{ where } \bigcup_{j=4}^{m-1} T_j^m
\]

\[
(2.3.40) \quad T_j^m = F_{1}^{m,m,m,m} \cup F_{2}^{m,m,m,m} \cup F_{3}^{m,m,m,m} \cup F_{4}^{m,m,m,m} \cup \ldots,
\]

and so on. Replacing every event of the form \(T_j^m\) by the sure event, we obtain

\[
(2.3.41) \quad P(T_j^m) \leq \sum_{1 \leq k \leq j} \sum_{1 \leq k \leq j} P(F_{j}^{m,m,m,m} \cup F_{k}^{m,m,m,m} \cup F_{l}^{m,m,m,m} \cup F_{m}^{m,m,m,m} \cup \ldots).
\]

Using a 'trinomial' analogue of the arguments in (2.3.27) through (2.3.30), one can easily show that

\[
P(U_{m=1}) \leq \mu^2 \int \int \int [1-e^{-\mu(s+t+u)}][1-F(s+t+u)][1-F(s+t)][1-F(s)] dsdtdu
\]

and hence

\[
\begin{align*}
\sum_{m=4}^{m} P(U_m) & \leq \mu^2 \int \int \int [1-e^{-\mu(s+t+u)}][1-F(s+t+u)][1-F(s+t)][1-F(s)] dsdtdu \to 0
\end{align*}
\]

as \((n \to \infty)\), thereby establishing (2.3.18).
With this, the correspondence between events \( Y_i \in S \) and "a third order visual response at time \( Y_{i+1} \)" has been established. In addition to the Poisson Limit Result, we can immediately apply the general results of Chapter 1 to obtain the following more elaborate (and informative) result.

**Theorem 2.3.1** Suppose that particles arrive according to a Poisson process with intensity \( \mu \), and particle lifetimes are independent (of each other and arrivals) and identically distributed random variables with mean \( \alpha < \infty \). Let \( U(s,t) \) denote the number of visual responses in the time interval defined by \( Y_{[sn]}, Y_{[tn]} \) i.e., in the time interval defined by the \([sn]^{th}\) through \([tn]^{th}\) (\( s < t \)) arrivals. Write \( U(t) \equiv U(0,t) \). Then, \( U(t) \) has, asymptotically, properties of a Poisson process with intensity \( \xi \) as \( n \to \infty \). In particular, for \( t=1 \), as \( n \to \infty \) we have the Poisson Limit Result, viz.

\[
P\{m \text{ (third order) visual responses in first } n \text{ arrivals}\} 
\to e^{-\xi \frac{m}{m/!}}.
\]

**Proof.** The proof follows easily from the general results of Chapter 1. We have established the equivalence (in the limit) of events of the type "\( Y_i \in S \)" and "a third order visual response at time \( Y_{i+1} \)." Therefore, we need only show that \( V(t) = \) the number of occurrences of the event \( Y_i \in S \) with \( i=1,2,\ldots,[nt] \), is asymptotically a Poisson process. This is, of course, an immediate consequence of the stationary version of Theorem 1.1.2.

It should be observed that the preceding theorem provides a quite complete asymptotic picture of the process \( U(t) \). Naturally, as mentioned in the introduction, it would be of great value to have some idea
of the 'closeness' of the asymptotic result to the finite situation. However, the bounds, approximations and limits involved (implicitly and explicitly) in obtaining Theorem 2.3.1 are, unfortunately, too numerous and/or complex to permit such a determination. Empirical evaluation by Monte Carlo methods could be quite feasible, however.

We might also mention that one simple consequence of Theorem 3.2.1 is that for $0 \leq r < s < t < \infty$ $U(r, s)$ and $U(s, t)$ are asymptotically independent random variables as $(n \to \infty)$; i.e., $P(U(r, s) = u_1, U(s, t) = u_2) = P(U(r, s) = u_1) P(U(s, t) = u_2) \to 0$ as $(n \to \infty)$. In other words, the numbers of visual responses in non-overlapping intervals of time are asymptotically independent. The requirement of non-overlapping intervals is not crucial; the important point is that the intervals be 'asymptotically non-overlapping.' For example, an application of Theorem 1.3.1 establishes the following interesting result.

**Corollary 2.3.1.1** Let $0 \leq r_n < s_n$ and $0 \leq t_n < u_n$ and $t_n - s_n \to 0$ as $n \to \infty$. Then $\mathcal{U}(r_n, s_n)$ and $\mathcal{U}(t_n, u_n)$ are asymptotically independent (and Poisson in distribution).

Thus, for example, with $r_n = 0$, $s_n = 1 + \frac{1}{n}$, $t_n = 1 - \frac{1}{n}$, $u_n = 2$, we have that $\mathcal{U}(0, 1 + \frac{1}{n})$ and $\mathcal{U}(1 - \frac{1}{n}, 2)$ are asymptotically independent, even though these random variables represent the number of light sensations in highly overlapping time intervals.

For completeness we mention that the waiting times (in units of $n$) between successive light sensations have, asymptotically, independent exponential distributions with mean $1/\xi$. This, of course, is a consequence of the asymptotic Poisson character of $\mathcal{U}(t)$. In practice these inter-sensation times might provide a convenient characteristic of the
process $X(t)$ to observe for purposes of comparing the finite model with its (theoretical) asymptotic behavior.

Finally, we remark that the results of Theorem 2.3.1 are, in some sense, complimentary to those obtained by Cramér and Leadbetter [4] for certain continuous time normal stationary processes. In this latter work, the authors show that the stream of upcrossings of a high level by the sample functions of such a fixed process, when suitably normalized, asymptotically the Poisson distribution. In Theorem 2.3.1 we deal, not with a fixed process and increasing level, rather we deal with a fixed level (the threshold number of quanta necessary for a visual response) and a changing process (i.e., a process with parameter $\mu$ tending to a certain limit). The results obtained are strikingly similar, as one would suspect would be the case. Some reflection will convince one that an upcrossing of a Level for a discrete time stochastic process can be represented as a linear relation; limit results for numbers of upcrossings may therefore be obtained on the one hand by changing the level curve (the Cramér-Leadbetter [4] approach) or by changing the parameters of the underlying process while keeping the level fixed (the approach of Theorem 3.2.1). We refer forward to Chapter 3 for a further discussion of these and related points.

2.4 The distribution of 'scores'

In the model of the preceding section (viz. where arrivals formed a Poisson process of intensity $\mu$ and lifetimes were (random) iid), individual particle lifetimes could very likely be quite large if, for example, $\mathbb{E} t_1^{1+\delta} = \infty$ for $\delta > 0$, say. Nevertheless, it was established that in the limit light of no higher order than three occurs. This
indeed seems to be a curious result, and leads one to look more closely at the limiting behavior of the distribution of lifetimes of individual particles. To facilitate this investigation, we introduce the following notion of a 'score.'

**Definition 2.4.1** The 'score' of a particle is the number of arrivals the particle survives (counting its own arrival as the first). \( S(i) \) will denote the score of the \( i \)th particle.

To illustrate the notion of a 'score,' we provide the following figure; the first panel (A) represents a segment of a possible realization of the process \( X(t) \); the second panel (B) represents the corresponding record of 'scores.'

![Figure 2.4.1](image-url)

**FIGURE 2.4.1**

Clearly \( S(i) \) can be any positive integer and, for example, \( S(i) = 3 \) if and only if (using the notation of the previous sections)
\[ y_{i+2} \leq i + y_i < y_{i+3} \]. We remark that one can reconstruct the pattern (number, order) of visual responses of a particular realization of the process \( X(t) \) knowing only its sequence of scores. Thus, a visual response occurs at time \( y_m \) if and only if \( S(m-i) \geq i+1 \) for at least two amongst \( i=1,2,\ldots,m-1 \), while \( S(m-i-1) \geq i \) for at most one amongst \( i=1,2,\ldots,m-2 \). Accordingly, considering merely the pattern of scores is really sufficient for purposes of the Limit Problem.

We now establish a simple result that will enable us to bring to bear the results of Chapter 1 in a study of the asymptotic distribution of scores.

**Theorem 2.4.1** Suppose particles arrive according to a Poisson process of intensity \( \mu \), and lifetimes are iid (and independent of arrivals).

For fixed \( k > 1 \) the events \( \{ S(i) \geq k, \ i=1,2,\ldots \} \) are \((k-2)\)-dependent.

Furthermore, it is possible to define a \((k-2)\)-dependent strictly stationary (vector) stochastic process \( \{ Y_{i}^{\mu k}; \ i=1,2,\ldots \} \) and sets \( S_{\mu} (\mu > 0) \) so that the event \( \{ Y_{i}^{\mu k} \in S_{\mu} \} \) occurs if and only if \( S(i) \geq k \).

**Proof.** Clearly \( S(i) \geq k \) if and only if \( y_{i+k+1} - y_i \leq i + 1 \), which is equivalent to \( \frac{z_{i+1} + \cdots + z_{i+k-1}}{i+1} \leq \mu \), where \( z_i = \frac{\mu x_i}{y_i} \) (\( x_i \) being the interarrival time). Because the \( z_i \)'s are iid (and independent of lifetimes \( \{ T_i; \ i=1,2,\ldots \} \)), the events \( S(i) \geq k \) and \( S(j) \geq k \) (\( i \leq j \)) are dependent if and only if \( i+1 \leq j+1 \leq i+j-1 \), or generally \( |i-j| \leq k-2 \).

If we now define a sequence of \((1\text{-dimensional})\) vector random variables \( \{ Y_{i}^{\mu k}; \ i=1,2,\ldots \} \) and sets \( S_{\mu} (\mu > 0) \) as follows:

\begin{equation}
Y_{i}^{\mu k} = \left( \frac{z_{i+1} + \cdots + z_{i+k-1}}{i+1} \right); \ i=1,2,\ldots; \ S_{\mu} = \{ x; x \leq \mu \},
\end{equation}

the second assertion of the theorem follows, and we are done.
A similar theorem can be stated for events of the type \( S(i) \leq k; \ i = 1, 2, \ldots \) or \( S(i) = k; \ i = 1, 2, \ldots \). However, for the moment it is most convenient to work with the events described in Theorem 2.4.1. It is to be observed that this theorem 'sets the stage,' so to speak, for application of Corollary 1.1.3.1 to the asymptotic distribution of scores.

As a first example we consider the model considered in the beginning of Section 2.3 dealing with the case where lifetimes are constant and equal (to \( \tau > 0 \)) and arrivals are Poisson. Observe that in this case, as \( (n \to \infty) \), and \( \mu \downarrow 0 \),

\[
P[S(i) \geq k] = \frac{1}{\Gamma(k-1)} \int_{0}^{\mu \tau} s^{k-2} e^{-s} ds \cdot \frac{(\mu \tau)^{k-1}}{(k-1)!},
\]

while it can also be shown that

\[
\max_{|i-j| \leq k-2} P[S(i) \geq k, S(j) \geq k]/P[S(i) \geq k] \to 0.
\]

Therefore, an application of Corollary 1.1.3.1 in conjunction with Theorem 2.4.1 yields the result that as \( (n \to \infty) \).

\[
P(m \text{ scores of } k \text{ or more amongst first } n \text{ particles})
\quad \to \left\{ \begin{array}{ll}
0 & \text{if } k \geq 3 \\
e^{-\hat{\xi} \frac{m}{m!}} & \text{if } k = 3
\end{array} \right.
\]

where \( \hat{\xi} = \lambda \tau^2 /2 \). For \( k = 1, 2, \ldots \), the analogue of the Borel-Cantelli Lemma for dependent events embodied in Lemma 1.2.1 suffices to establish that events \( S(i) \geq k \) occur infinitely often. Thus, in this simplest (non-trivial) case there is really little difference between the limiting behavior of scores and the limiting behavior of visual responses. The next example treats a situation potentially quite
different in this respect.

As a second example we consider the more general model for the
Limit Problem, viz. the case where lifetimes \( \{ \tau_i ; i=1,2,\ldots \} \) are iid
random variables with finite first moment \( \alpha \) and arrivals from a Poisson
process independent of lifetimes. If \( F(t) = P(\tau_i \leq t) \) and \( k \geq 2, \)
\[
(2.4.5) \quad P[S(i) \geq k] = \frac{1}{\Gamma(k-1)} \int_0^\infty \left[ 1-F(s) \right] s^{k-2} \mu^{k-1} e^{-\mu s} ds.
\]

We consider two cases:

Case A: \( \varepsilon \tau_i^2 < \infty \). This situation is similar to the preceding case.

We have
\[
(2.4.6) \quad P[S(i) > 3 \text{ for some } i=1,2,\ldots,n] \leq \sum_{i=1}^n P[S(i) > 3] = nP[S(n) > 3]
\]
\[
= n\mu^2 \int_0^\infty \left[ 1-F(s) \right] s^2 e^{-\mu s} ds.
\]
The integrand above tends pointwise to zero as \( n \to \infty \), while it re-
mains bounded by the integrable function \( s[1-F(s)] \). Thus, via bounded
convergence,
\[
(2.4.7) \quad P[S(i) > 3 \text{ for some } i=1,2,\ldots,n] \to 0 \text{ as } (n \to \infty).
\]

Hence, in the limit, no score higher than three occurs. A more precise
result; e.g., an actual limit distribution for scores of three, say, is
not possible without further knowledge about \( F \).

Case B: \( \varepsilon \tau_i^2 = \infty \). This situation may be quite different from the
previous cases. To display this difference we must consider in general
the asymptotic behavior of (2.4.5) as \( n \to \infty \) and so \( \mu \downarrow 0 \). This
presents some problems, since the asymptotic behavior of (2.4.5) as
\( n \to \infty \) is intimately related to the asymptotic behavior of \( 1-F(t) \) as
$t \to \infty$. Now for particular distribution functions $F$ one may actually attempt to evaluate the Limit of (2.4.5) as $(n \to \infty)$, and attempt to verify the other conditions necessary for application of the results of Corollary 1.1.3.1. However, we are concerned with general results.

In the following discussion we present some (reasonable and general) conditions on the asymptotic behavior of $1-F$ that will enable us to obtain some general results. First note that, apart from constants, (2.4.5) is the Laplace transform of the function $[1-F(s)]s^{k-2}$. It is not possible to make general statements about the asymptotic behavior of this transform as $\mu \downarrow 0$ ($n \to \infty$) without making some assumptions about the behavior of $[1-F(s)]s^{k-2}$ as $s \to \infty$. (See Appendix B.) An appropriate assumption is that

$$(2.4.8) \quad [1-F(s)]s^{k-2} \sim L(s) s^{-\alpha} \quad \text{as } s \to \infty$$

where $L(s)$ is a so-called function of slow growth. That is, $L(s)$ is a non-negative real function defined for $s \geq 0$ and bounded in every finite interval such that for every fixed $\lambda > 0$, $L(\lambda s)/L(s) \to 1$ as $s \to \infty$.

The constant $\alpha$ is chosen so as to insure suitable moment conditions on $F$ are satisfied (here, we want $\mathbb{E} I_1^2 = \infty, \mathbb{E} I_1 < \infty$). In keeping with our previous technique, we may occasionally proceed into the more detailed analyses that follows for the specific case $k=3$. The results are valid (with appropriate and obvious changes) for a general $k$.

Let us, then, assume that as $s \to \infty$,

$$(2.4.9) \quad [1-F(s)] s \sim L(s)s^{-\alpha}, \quad 0 \leq \alpha \leq 1.$$

where $L(s)$ is a function of slow growth. If $\alpha > 0$ in (2.4.9), then $\mathbb{E} I_1 < \infty$; yet by appropriate choice of $\alpha$ and/or $L(s)$, one can have, for
example, \( E_{\epsilon_1}^5 = \infty \) for \( \delta \geq 2 \) and \( E_{\epsilon_1}^5 < \infty \) for \( \delta < 2 \). Such is the case when

\[
(2.4.10) \quad [1 - F(s)] \sim 1/s^2 \quad \text{as} \quad s \to \infty .
\]

Under the assumption of (2.4.9) we have the following result.

**Theorem 2.4.2** Suppose particles arrive according to a Poisson process with intensity \( \mu \), while lifetimes \( \{\tau_i; i=1,2,\ldots\} \) are iid (and independent of arrivals) with finite mean and common distribution function \( F \) satisfying (2.4.9). Then for any \( k > \alpha+2 \)

\[
(2.4.11) \quad P[S(i) \geq k] \sim \mu^{\alpha+1} L_1(1/\mu \mu) \frac{\Gamma(k-\alpha-1)}{\Gamma(k-1)} \quad \text{as} \quad \mu \downarrow 0 .
\]

**Proof.** The theorem is given in Smith [17]. The result follows from a general Abelian theorem for Laplace transforms by which we have, as \( \mu \downarrow 0 \),

\[
(2.4.12) \quad P[S(i) \geq k] = \frac{1}{\Gamma(k-1)} \int_0^\infty [1-F(s)]s^{k-2} \frac{\Gamma(k-\alpha-1)}{\Gamma(k-1)} \mu^{-\mu}s ds
\]

\[
\sim \frac{\mu^{\alpha+1} L_1(1/\mu \mu)}{\Gamma(k-1)} \int_0^\infty s^{k-\alpha-3} \frac{\Gamma(k-\alpha-2)}{\Gamma(k)} \mu^{-\mu}s ds
\]

We emphasize the fact that for all integers \( k > \alpha+2 \) the quantities \( P[S(i) \geq k] \) are of the same asymptotic order (as \( \mu \downarrow 0 \)). In particular, for \( k > \alpha+2 \),

\[
(2.4.13) \quad P[S(i) = k] \sim \mu^{1+\alpha} L_1(1/\mu \mu) \frac{\Gamma(k-\alpha-2)}{\Gamma(k-1)} - \frac{\Gamma(k-\alpha-1)}{\Gamma(k)} \quad \text{as} \quad \mu \uparrow 0 .
\]

Therefore, for all integers \( k > \alpha+2 \), scores of order \( k \) have the same asymptotic order of probability. This point will be pursued further.

To illustrate the curious behavior of scores, we consider the following very specific example. Suppose that the distribution function
F of lifetimes \( \{\tau_i; i=1,2,\ldots\} \) satisfies (2.4.10). This assumption, as noted, guarantees that \( \mathcal{E}_{\tau_i^0} < \infty \) for \( \delta < 2 \), while \( \mathcal{E}_{\tau_i^0} = \infty \) for \( \delta \geq 2 \). From Theorem 2.4.2 it follows that for \( k > 3 \),

\[
(2.4.14) \quad P[S(i) \geq k] \sim \mu^2 \frac{\Gamma(k-3)}{\Gamma(k-1)} \quad \text{as} \quad \mu \downarrow 0.
\]

Furthermore,

\[
(2.4.15) \quad P[S(i) \geq 3] \sim \mu^2 \log \left( \frac{1}{\mu} \right) \quad \text{as} \quad \mu \downarrow 0.
\]

Thus, as \( (n \to \infty) \)

\[
(2.4.16) \quad np[S(i) \geq k] \to \lambda \frac{\Gamma(k-3)}{\Gamma(k-1)}, \quad k \geq 4,
\]

while for \( k = 3 \), \( np[S(i) \geq 3] \) is unbounded as \( (n \to \infty) \).

It can be verified for \( k \geq 4 \), as \( (n \to \infty) \)

\[
(2.4.17) \quad \max_{0 < |i-j| \leq k-2} P[S(i) \geq k, S(j) \geq k]/P[S(i) \geq k] \to 0.
\]

Hence Corollary 1.1.3.1 is applicable to the determination of the asymptotic frequency of occurrence of events of the type \( S(i) \geq k; k \geq 4 \). In fact, from the aforementioned Corollary one may conclude that for \( k \geq 4 \)

\[
(2.4.18) \quad \lim_{(n \to \infty)} P[S(i) \geq k \text{ for exactly } m \text{ amongst } i=1,2,\ldots,n] = \frac{m!}{k^m} e^{-\delta_k} \delta_k^m / m!,
\]

where \( \delta_k = \lambda \Gamma(k-3)/\Gamma(k-1) \). Furthermore, the Borel-Cantelli-type result contained in Theorem 1.2.1 insures that as \( (n \to \infty) \) events of the type \( S(j) \geq k; j=1,2,3 \) occur infinitely often. By using (2.4.13) we can state the above results in a slightly different and more useful form.

For \( k \geq 4 \) and any non-negative integer \( m \),
\[(2.4.19) \quad \lim_{n \to \infty} P[S(i) = k \text{ for exactly } m \text{ among } i=1,2,\ldots,m] = e^{-\phi_k} \left(\frac{m}{k}\right)! \]

and for \(k \leq 3\)

\[(2.4.20) \quad \lim_{n \to \infty} P[S(i) = k \text{ for exactly } m \text{ among } i=1,2,\ldots,n] = 0, \]

where \(\phi_k = \xi_k - \xi_{k+1}\) (\(\xi\)'s are positive constants).

Thus, unlike the preceding cases, we have (a) scores of every order occurring in the limit (in particular scores of order \(\geq 4\) occur marginally according to a Poisson distribution), and (b) scores of order three (and less) occur infinitely often. Accordingly, the asymptotic distribution of scores displays a structure significantly different from that of visual responses (where the highest asymptotic order was three). Intuitively, the difference lies in the fact that we may obtain very long lived particles (judged in terms of scores), yet the particles do not 'accumulate' closely enough to produce patterns yielding visual responses of order higher than three.

For purposes of illustration, the following diagram represents a segment of a possible 'score record' when lifetimes have a distribution function satisfying (2.4.10). The horizontal scale is greatly compressed as compared to Figure 2.4.1 (B). In effect Figure 2.4.2 is obtained from the process \(X(t)\) by 'standing lifetimes on end,' measuring vertical distance in terms of score rather than time.
In the limit, jumps to $S=k$ ($k \leq 3$) occur infinitely often, however the orders of magnitude may be different. In the case at hand we note that $n\mathbb{P}[S(i) \geq 3] \sim \lambda \log \frac{1}{\mu}$, $n\mathbb{P}[S(i) \geq 2] \sim \frac{K\lambda}{\mu}$ ($K$ a positive, finite constant) and $n\mathbb{P}[S(i) \geq 1] \sim \frac{\lambda}{\mu^2}$ as $\mu \downarrow 0$, ($n \to \infty$). Thus, amongst the first $n$ particles one might expect the number of occurrences of, say, the event $S(i) \geq 2$ to be $O(\sqrt{n})$, whereas one knows that amongst the first $n$ particles the number of occurrences of the event $S(i) \geq 1$ is $n$, exactly. Of course, in the limit, jumps to $S=k$ ($k \geq 4$) are the 'rare events.' Such jumps occur a finite number of times and (marginally) according to the Poisson law.

The above marginal results can be refined. We have previously established that the marginal distribution of the number of jumps to $S=k$ ($k \geq 4$) is Poisson with parameter $\phi_k$ (in the limit). The following theorem extends this result to include a limiting joint distribution.

**Theorem 2.4.3** Suppose particles arrive according to a Poisson process with intensity $\mu$, and particle lifetimes $\{\tau_i; i=1,2,\ldots\}$ are iid (and independent of arrivals) with a distribution function satisfying (2.4.10). Let $m_1,\ldots,m_r$ be non-negative integers, $t_1,t_2,\ldots,t_r$ be $r$
positive numbers, and \( k_1, k_2, \ldots, k_r \) be \( r \) positive integers such that \( 3 < k_1 < k_2 < \ldots < k_r \). Then,

\[
\lim_{n \to \infty} \prod_{\ell=1}^{r} \left( \sum_{i=1,2,\ldots,[t_\ell n]} \frac{e^{-\phi_\ell t_\ell} \left( \phi_\ell t_\ell \right)^{m_\ell}}{m_\ell !} \right),
\]

where \( \phi_\ell \) is defined as in (2.4.20)

**Proof.** This is a rather direct application of Theorem 1.3.1.

2.5 **The Limit Problem for some new situations**

As was mentioned in the introduction to this chapter, our approach to the Limit Problem (via the general results of Chapter 1) enables us to establish the Poisson Limit Result in some new situations. It will be noticed that many of the results of the preceding sections are immediately 'transferable.'

We begin by establishing the following result, which essentially gives the Poisson Limit Result in the case where the interarrival and lifetime distributions considered in Theorem 2.3.1 are interchanged (and slightly strengthened).

**Theorem 2.5.1** Suppose that light quanta arrive at (random) time points \( 0 \leq \tau_1 < \tau_2 < \ldots \) where \( \tau_1 = \frac{1}{\mu} (x_1+x_2+\ldots+x_1) \) and the \( x_i \)'s are iid positive random variables with fixed distribution having unit mean and finite variance, and \( \mu > 0 \). Let particle lifetimes \( \{\tau_i; i=1,2,\ldots\} \) be iid exponential random variables (independent of arrivals) with mean \( \alpha \). Then
\[ (2.5.1) \quad \lim_{[n \to \infty]} P(\text{exactly } m \text{ third order visual responses amongst first arrivals}) = \e^{-\delta \frac{m}{m!}} \]

where \([n \to \infty]\) denotes "\(n \to \infty, \mu \downarrow 0\) in such a manner that
\[ n \hat{G}(\frac{1}{n}) \hat{G}(\frac{2}{n}) \to \delta,\]
where \(\hat{G}(s) = \int_0^\infty e^{-sx}dG(x)\) is the Laplace-Stieltjes transform of the distribution function \(G(x) = P(X_1 \leq x)\), and \(\delta > 0\) is a positive constant.

**Remarks.** Before proving Theorem 2.5.1 we make some observations. A limiting condition \([n \to \infty]\) is possible if and only if \(\hat{G}\) is the Laplace-Stieltjes transform of a positive random variable, since by definition
\[ (2.5.2) \quad \hat{G}(\frac{1}{\mu}) = P(X_1 = 0) + \int_0^\infty e^{\frac{-x}{\mu}} dG(x). \]

The right hand side of (2.5.2) tends to zero as \(\mu \downarrow 0\) if and only if \(P(X_1 = 0) = 0\). Furthermore, note that the limiting condition \([n \to \infty]\) reduces essentially to the condition \((n \to \infty)\) previously considered (e.g., in Section 2.3) when arrivals are Poisson. In that case,
\[ \hat{G}(\frac{1}{\mu}) \sim \mu \text{ as } \mu \downarrow 0 \text{ so that } n \hat{G}(\frac{1}{n}) \hat{G}(\frac{2}{n}) \to \delta \text{ as } n \to \infty \text{ and } \mu \downarrow 0 \text{ if and only if } \mu^2 \to \alpha^2 \delta /2! = \lambda \text{ as } n \to \infty, \mu \downarrow 0. \]

Finally, we are following a pattern established earlier by considering the particular case \(k=3\) in Theorem 2.5.1, that is, third order visual responses. It should be clear that Theorem 2.5.1 could have been proved for general \(k\); accordingly the appropriate interpretation of \([n \to \infty]\) would be "\(n \to \infty, \mu \downarrow 0\) in such a manner that
\[ \prod_{j=1}^{k-1} \hat{G}(\frac{1}{nj}) \to \delta.\]"

**Proof of Theorem 2.5.1** Via Theorem 2.2.2, a proof of this theorem will serve to establish that...
\[(2.5.3) \quad \lim_{[t \to \infty]} P(\text{exactly } m \text{ third order visual responses in } (0,t]) = e^{-\delta m/m!},\]

where the symbol \([t \to \infty]\) is interpreted to mean \(t \to \infty, \mu \uparrow 0\) in such a manner that \(\mu t \Pi_{j=1}^{k-1} \hat{G}_{\mu}^{(j)} \Rightarrow \delta.\) As a consequence, we deal exclusively with limits as \([n \to \infty]\). As before, define \(Z_i = \mu X_i\); where \(X_i = y_i - y_{i-1}; i=1,2,... (y_0 = 0)\) are the 'interarrival times.' The random variables \([Z_i; i=1,2,...]\) are iid with unit mean and finite variance.

We now define a two-dimensional vector valued stochastic process \([Y_i; i=2,3,...]\) and a collection of 2-dimensional sets \(S_\mu (\mu > 0)\) as follows:

\[(2.5.4) \quad Y_i = (\frac{Z_i + Z_{i-1}}{I_{i-1}}, \frac{Z_i}{I_i}); i=2,3,...; S_\mu = \{(x,y) | x<\mu, y<\mu\} (\mu > 0).\]

We now assert that the 1-dependent vector stochastic process \([Y_i; i=2,3,...]\) and the sets \(S_\mu (\mu > 0)\) satisfy the conditions of Corollary 1.1.3.1 under the limiting condition \([n \to \infty]\). To verify this assertion, we first note that

\[(2.5.5) \quad nP(Y_i \in S_\mu) = nP(Z_i + Z_{i-1} < \mu I_{i-1}, Z_i < \mu I_i)\]

\[= n \int \int e^{\mu} e^{\mu} dG(s)dG(t)\]

\[= \frac{2}{\mu^2} \int_0^{\infty} s \int_0^{\infty} t \frac{1}{\mu^2} dG(s)dG(t)\]

\[= n \hat{G}^{(2)} \hat{G}^{(1)} \rightarrow 0 \text{ as } [n \to \infty].\]

Furthermore,
(2.5.6) \[ P(Y_1 \in S_\mu, Y_{i+1} \in S_\mu) = P(z_1+z_1^* < \mu_{i-1}, z_{i+1}+z_{i+1}^* < \mu_{i+1}, z_{i+1} < \mu_{i+1}) = \lim_{s \to \infty} \lim_{t \to \infty} \lim_{v \to \infty} \int_0^s \int_0^t \int_0^v e^{\mu(s+t)} e^{\mu v} dG(s) dG(t) dG(v) \]

So, via (2.5.5),

(2.5.7) \[ P(Y_1 \in S_\mu, Y_{i+1} \in S_\mu)/P(Y_1 \in S_\mu) = \lim_{m \to \infty} \left( \begin{array}{c} \mu \mu \mu \\ \mu \mu \mu \end{array} \right) \to 0 \text{ as } [n \to \infty]. \]

Thus, via Corollary 1.1.3.1 we conclude that

(2.5.8) \[ \lim_{n \to \infty} P(Y_1 \in S_\mu \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n) = e^{-S_\mu m!/m!}. \]

All that remains to show is that the analogies of (2.3.16), (2.3.17) and (2.3.18) are satisfied for the case at hand. Accordingly we will then have established a correspondence (in the limit) between the events "\(Y_1 \in S_\mu \text{ for exactly } m \text{ amongst } i=2,3,\ldots,n\)" and "exactly \(m\) third order visual responses in the first \(n\) arrivals."

To verify that condition (2.3.16) is satisfied for the case of hand, all we need show is that, in the notation of Section 2.3,

\[ P(C_n^{(0)}) \to 0 \text{ as } [n \to \infty]. \]

Observe here,

(2.5.8) \[ P(C_n^{(0)}) \leq \sum_{j=3}^{\infty} \sum_{\ell=1}^{j-1} \sum_{s=t}^{x_j+z_1^* < \mu_{i-1}, x_1^*+\ldots+x_j^* < \mu_{i-1}} \int_0^s \int_0^t \int_0^v e^{\mu(s+t)} e^{\mu v} dG(s) dG(t) dG(v) \]

\[ = \sum_{j=3}^{\infty} \sum_{\ell=1}^{j-1} \left( \begin{array}{c} \mu \mu \mu \\ \mu \mu \mu \end{array} \right) \to 0 \text{ as } [n \to \infty]. \]
\[ n \hat{\theta}(\frac{1}{\mu}) \hat{\gamma}(\frac{2}{\mu}) \left\{ \frac{1}{[1 - \hat{\theta}(\frac{1}{\mu})][1 - \hat{\gamma}(\frac{2}{\mu})]} - 1 \right\} \to 0 \text{ as } [n \to \infty], \]

hence the analogue of (2.3.16) is satisfied. Arguments analogous to those used in Section 2.3 can be transferred, almost word for word, to establish the validity of the analogous (2.3.17) and (2.3.18). We shall not present the details. With this, we have established the Poisson Limit Result, viz.

(2.5.9) \[
\lim_{[n \to \infty]} P(\text{exactly } m \text{ third order visual responses in first } n \text{ arrivals}) = e^{-\xi_n} \frac{m^m}{m!}.
\]

In addition, the analogues of (2.3.19), (2.3.20) and (2.3.21) are, of course, valid. Hence, the asymptotic behavior of the process \( X(t) \) is indeed the same as for the case considered in Section 2.3 and pictured (in part) in Figure 2.2.4. We remark, in conclusion, that so far, the proof has made no explicit use of the fact that the \( x_i \)'s have a finite second moment. However, we recall that Theorem 2.2.2 required this condition in order that the 'non-random n' Poisson Limit Result imply the corresponding 'non-random t' Poisson Limit Result. (We refer to Appendix A for an expanded discussion of related points.) With this, the theorem is established.

As an illustration of the preceding theorem, let us suppose that the \( x_i \) (i=1,2,...) have a Gamma distribution with parameter \( \lambda \), so that \( \hat{\gamma}(s) = \frac{1}{(1+s)^{\lambda}}. \) Noting that \( \hat{\gamma}(\frac{1}{\mu}) \sim \lambda \) as \( \mu \downarrow 0 \), we conclude that if particle arrive at time points \( x_i = \frac{1}{\mu} (x_{i+1} + \ldots + x_i) \) (i=1,2,...) and have exponential (\( \xi \)) lifetimes, then the Poisson Limit Result (for the
number of third-order visual responses) is valid when \([n \to \infty]\) is taken to mean \(n \to \infty, \mu \downarrow 0\) in such a manner that \(n \left(\frac{a}{\sqrt{2}}\right)^{2\lambda} \to 8\)."
CHAPTER III

ASYMPTOTIC DISTRIBUTION OF THE NUMBER OF UPCROSSINGS OF A HIGH LEVEL BY CERTAIN DISCRETE-TIME STOCHASTIC PROCESSES

3.0 Summary

In Chapter 1 we developed a series of general theorems concerning the asymptotic behavior of the probabilities of 'rare' events in sequences of dependent (m-dependent, f(n)-dependent, strongly mixing) binomial (multinomial) trials. In this chapter we develop a particular application of these results dealing with the asymptotic distribution of the number of upcrossings of a high level by the sample function corresponding to certain dependent discrete time stochastic processes. The results are analogous to those obtained recently by Cramér and Leadbetter [4] and Cramér [3] for a broad class of continuous time, strictly stationary normal stochastic processes.

It should be noted that Chapter 2 dealt with a study of the (asymptotic) number of upcrossings of fixed integer levels by a particular continuous-time stochastic process. The general (discrete) theory of Chapter 1 could be applied because the level changes occurred only at points of time determined by a discrete-time stochastic process.

3.1 Introduction and preliminary comments concerning continuous time normal stochastic processes

We first develop some (known) definitions and results, pertaining to continuous time normal processes, that will enable us to give the
recent result due to Cramér and Leadbetter [4]. The purpose is primarily to motivate analogous definitions and results for the case we shall consider, and to display the similarity of the two results. We shall use terminology found in [4].

Let \( \xi(t), -\infty < t < \infty \) be a real, normal and stationary process with zero mean and unit variance. Suppose that its covariance function \( r(t) \) satisfies the conditions

\[
(3.1.1) \quad r(t) = 1 - \frac{\lambda_2^2}{2} t^2 + \frac{\lambda_4}{4!} t^4 + o(t^4)
\]

with finite \( \lambda_2 \) and \( \lambda_4 \) as \( t \to 0 \), and

\[
(3.1.2) \quad r(t) = o(t^{\alpha})
\]

for some \( \alpha > 0 \), as \( t \to \infty \). (It is known [4] that (3.1.1) implies that one can assume that \( \xi(t) \) has a continuous sample function derivative, and that (3.1.2) implies that \( \xi(t) \) is ergodic and that values of \( \xi(t) \) lying far apart on the time axis depend only weakly upon each other.) We shall let \( U(s,t) \) denote the number of upcrossings of a level \( u \) in the interval \( (s,t] \) by the process \( \xi(t) \). When \( s = 0 \), we write \( U(t) \) instead of \( U(0,t) \). Also write

\[
(3.1.3) \quad \mu = U(1) = \frac{\sqrt{\lambda_2}}{2\pi} e^{-u^2/2} \quad \text{(see 4).}
\]

As remarked in [4], for a process \( \xi(t) \) satisfying the above conditions, the 'stream' of upcrossings of a fixed level \( u \) is stationary and regular. Were the numbers of upcrossings occurring in disjoint time intervals also independent random variables, the three conditions characterizing the Poisson process would be satisfied. However, this will in general not be the case, but as the level \( u \to \infty \), the third
condition is satisfied asymptotically. Therefore, one might reasonably (and quite justifiably) suspect that the stream of upcrossings is asymptotically a Poisson process (in a specified way) as the level \( u \to \infty \). It appears natural to use \( 1/\mu \) as a scaling 'unit' of time. We remark that \( 1/\mu \to \infty \) as \( u \to \infty \) by relation (3.1.3). Accordingly, Cramér and Leadbetter [4] establish the following result.

**Theorem 3.1.1** Suppose that the normal and stationary process \( \xi(t) \) satisfies conditions (3.1.1) and (3.1.2). Let \( (a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r) \) be \( r \) disjoint time intervals depending on \( u \) in such a way that \( b_i - a_i = \tau_i / \mu \) (\( \tau_i > 0 \)), \( r \) and \( \tau_1, \ldots, \tau_r \) being independent of \( u \). Let \( m_1, m_2, \ldots, m_r \) be \( r \) non-negative integers independent of \( u \). Then

\[
P(\bigcap_{i=1}^{r} [a_i, b_i) = m_i) \to \prod_{i=1}^{r} \left( e^{-\tau_i m_i / \mu} \right) \quad \text{as} \quad u \to \infty
\]

A proof of the above result is quite involved, yet straightforward. It relies heavily upon the normality of the process \( \xi(t) \).

There are several other asymptotic results related to Theorem 3.1.1. If in the univariate special case of Theorem 3.1.1 one considers \( m_1 = 0 \) and replaces the parameter \( \tau \) by \( e^{-z} \), and taking as before

\[
(3.1.4) \quad \mu = \frac{\sqrt{\lambda_2}}{2\pi} \quad e^{-u^2/2} \quad \text{and} \quad T = e^{-z/\mu},
\]

it follows that, for any fixed real \( z \), the probability of no upcrossing of the level \( u \) in the interval \((0, T)\) satisfies

\[
(3.1.5) \quad P(U(T) = 0) \to e^{-e^{-z}} \quad \text{as} \quad u \to \infty.
\]

A small lemma (see [4] p. 272) then yields the following results:
Corollary 3.1.1.1

(3.1.6) \[ P(\max_{0 < t \leq T} \xi(t) \leq u) \to e^{-e^{-z}} \text{ as } u \to \infty, \text{ and hence} \]

(3.1.7) \[ P(\max_{0 < t \leq T} \xi(t)) \leq (2 \log T)^{\frac{1}{2}} + \frac{\frac{1}{2} \log \lambda_2 - \log 2\pi + z}{(2 \log T)^{\frac{1}{2}}} \]

\[ \to e^{-e^{-z}}. \]

Not quite so immediately we have another result. Let us choose as a scaling time unit,

(3.1.8) \[ \theta = \frac{1}{u} \left(\frac{2\pi}{\lambda_2}\right)^{\frac{1}{2}} \text{ (see [4]).} \]

Let \( F(t) \) be the distribution function of the duration of an upwards excursion above the (high) level \( u \). Then, we have the following result:

Theorem 3.1.2

(3.1.8) \[ F(\theta t) \to 1 - e^{-\left(\frac{2\pi}{\lambda_2}\right)^2 t^2} \text{ as } u \to \infty. \]

The proof of this theorem is not simple, and again depends heavily upon the normality of \( \xi(t) \).

It is remarked in [4] that the above results for continuous time normal stationary processes have their counterparts for discrete time normal stationary processes, but with simpler proofs. Let \( \xi_i \) \( (i=0, \pm 1, \pm 2, \ldots) \) be a stationary sequence of normal random variables with zero means, unit variance, such that

(3.1) \[ r_n = \mathcal{E}(\xi_0 \xi_n) = O(|n|^{-\alpha}), \text{ as } |n| \to \infty \]

for some \( \alpha > 0 \). For an analogue to the continuous time case, we consider the connected path of straight line segments between the points \( (i, \xi_i) \), and the up and down crossings of this line with a given level \( u \).
Note that this path has a single upcrossing in the interval \((j, j+1)\) if and only if \(s_{j-1} < u\) and \(s_j > u\). The asymptotic distribution of the number of upcrossings as well as the maximum of the discrete process differ somewhat from the corresponding asymptotic distributions for the continuous process. As an example, we quote the following result, to be compared with (3.1.7):

**Theorem 3.1.3** For a discrete stationary normal process satisfying (3.1.9) we have

\[
(3.1.10) \quad P\left( \max_{0 < i < T} s_i \leq (2 \log T)^{\frac{1}{2}} + z - \frac{\log \log T - \frac{1}{2} \log 4\pi}{(2 \log T)^{\frac{1}{2}}} \right) \rightarrow e^{-e^{-z}}, \text{ as } u \rightarrow \infty.
\]

3.2 **Related results for discrete time processes that are not necessarily normal**

In this section we shall apply some of the more general results of Chapter 1 to obtain limit results for some not-necessarily-normal discrete time processes, results analogous to those of the preceding section for normal processes. It is felt that, even though the class of discrete time processes for which the conclusions are valid is somewhat restricted, they are useful at least for displaying the robustness of the results of the preceding section under deviation from normality.

We are motivated by the previous section in formulating the following definitions.

Let \(\{x_i; i=0, \pm 1, \pm 2, \ldots\}\) be a discrete time stochastic process. We shall associate with each realization of this process a (continuous) sample function \(\{x(t), -\infty < t < \infty\}\) obtained from \(\{x_i; i=0, \pm 1, \pm 2, \ldots\}\) by joining, in sequence, the points \((i, x_i)\) by straight line segments.
Thus, a particular sample function associated with the process might (in part) appear as in the following figure:

![Sample function diagram](image)

**FIGURE 3.2.1**

Such a process might arise naturally from periodic phenomena, or as an approximation to a continuous phenomenon. For example, \( X_i \) might represent the amount of water in a reservoir at time \( i \) in the history (or projected future) of the reservoir. Frequently it is desirable to have some idea about the probability that such a process will exceed a certain (high) level, or drop below a certain (low) level. These events might correspond to catastrophes for the system being characterized by the process. In the example above, a high level might represent the capacity of the reservoir behind a dam; a low level might correspond to a minimal value necessary to produce, say, electric power. We are therefore faced with the problem of finding the probability that the sample function \( X(t) \) (associated with \( \{X_i\ (i=0, \pm 1, \pm 2, \ldots)\} \)) sustains an upcrossing of a (high) level \( u \) or sustains a downcrossing of a (low) level \( v \). Since a solution of the problem for upcrossings of \( X(t) \) implies the solution of the problem for downcrossings of \( -X(t) \), we restrict attention to the former of the two problems. In addition, a
related problem is to determine the distribution of (sojourn) time above a (high) level given that an upcrossing has occurred. This might well correspond to a 'duration of crisis' distribution when interpreted physically.

It seems that in many situations, the specific assumption of normality for \( \{X_i \ (i=0, \pm 1, \pm 2, \ldots)\} \) might be difficult to justify on the basis of evidence (or lack thereof) available. However, it may be possible to make some assumptions on certain conditional probabilities such as \( P(X_{i+j} > u \mid X_i > u) \) for large \( u \). As it will be seen, such assumptions are required in our approach.

Now, there is an upcrossing of a level \( u \) by the process \( X(t) \) in the interval \((i-1, i)\) if and only if \( X_{i-1} < u \) and \( X_i > u \). Thus, events of the latter type are of specific interest when studying upcrossings. For studying such events, we introduce the following definition.

**Definition 3.2.1** Corresponding to the stochastic process \( \{X_i \ (i=0, \pm 1, \pm 2, \ldots)\} \) we define a new (vector) stochastic process \( \{Y_i \ (i=0, \pm 1, \pm 2, \ldots)\} \) and sets \( S_u \ (-\infty < u < \infty) \) as follows: \( Y_i = (X_{i-1}, X_i) \) \( (i=0, \pm 1, \pm 2, \ldots) \) and \( S_u = \{(x, y) \mid x < u, y > u\} \).

It is to be observed that the event \( Y_i \in S_u \) occurs if and only if there is an upcrossing of the level \( u \) by the process \( X(t) \) in the interval \((i-1, i)\). Thus our study of the number of upcrossings of \( u \) by \( X(t) \) in some interval of time, say \((a, b)\), reduces to a study of the number of events of type \( Y_i \in S_u \) occurring with \( i = [a]+1, \ldots, [b]-1 \). This, in conjunction with the form of the theorems of Chapter 1, motivates our next definition.
Definition 3.2.2  The event $A^u_i$ is defined as $A^u_i = \{Y_i \in S_u\}$
\(i=0, \pm 1, \pm 2, \ldots\).

We now propose to show that, if $u \to \infty$ in a certain manner, and
the process \(\{Y_i(i=0, \pm 1, \pm 2, \ldots)\}\) (or equivalently \(\{X_i(i=0, \pm 1, \pm 2, \ldots)\}\))
satisfies certain conditions, a limit distribution for the number of
upcrossings of $u$ by $X(t)$ can be obtained.

We begin with one of the simplest cases. Suppose
\(\{X_i(i=0, \pm 1, \pm 2, \ldots)\}\) is a strictly stationary, $m$-dependent stochastic
process. If $F(\cdot)$ and $F(\cdot, \cdot)$ denote the distribution functions of
$X_1$ and $(X_1, X_2)$ respectively, let

\[
P(u) = P(A^u_1) = \int_{-\infty}^{\infty} \int_{u}^{\infty} dF(x_1, x_2),
\]

and suppose that $\sup\{x: F(x) < 1\} = +\infty$.

Now,

\[
P(u) \to 0 \text{ as } u \to \infty.
\]

Let us choose a level $u_n \to \infty$ in such a manner that for some $\xi > 0$

\[
nP(u_n) \to \xi \text{ as } n \to \infty.
\]

From this point on we assume, without loss of generality, that equality
holds in (3.2.3). We then have the following result.

Theorem 3.2.1  Let \(\{X_i(i=0, \pm 1, \pm 2, \ldots)\}\) be an $m$-dependent strictly
stationary stochastic process. Suppose that a level $u_n$ ($n=1, 2, \ldots$)
tending to infinity, is chosen so that (3.2.3) is satisfied. Suppose
that, in the notation of Definition 3.2.2,

\[
b_n = \max_{|i-j| \leq m+1} P(A^u_i | A^u_j) = o(1), \text{ as } n \to \infty.
\]
Then for every real $t$, $0 < t < \infty$, as $n \to \infty$,

\[(3.2.5) \quad P(\text{exactly } k \text{ upcrossings by } x(t) \text{ of } u_n \text{ in the interval } (0, nt)) \to e^{-t \xi}(t \xi)^k/k! .\]

**Proof.** This is a relatively simple application of the stationary version of Theorem 1.1.4. For each $n$ let \( A^n_i \ (i=1, 2, \ldots) \) be the sequence of events where \( A^n_i = u_n \ (i=0, 1, 2, \ldots) \). Since the process \( x_i \ (i=0, \pm 1, \pm 2, \ldots) \) is $m$-dependent and stationary, the sequence of events \( A^n_i \ (i=0, 1, \ldots) \) is $(m+1)$-dependent and stationary. Furthermore, $P(A^n_i) = \xi/n \ (i=1, 2, \ldots)$, and $\max_{i-j \leq m+1} P(A^n_i | A^n_j) = o(1)$ as $n \to \infty$ via (3.2.4). An appeal to the result of Theorem 1.1.4 with $s=0$ shows that $P(\text{exactly } k \text{ amongst } A^n_i \ (i=0, 1, \ldots [tn])) \to e^{-t \xi}(t \xi)^k/k!$. The remark preceding Definition 3.2.2 then serves to establish the theorem.

Now, the result obtained in Corollary 1.3.1.2 can be used to obtain a result more refined than the preceding theorem. We obtain the following result which is to be compared with the Cramér and Leadbetter result found in Theorem 3.1.1.

**Theorem 3.2.2** Let \( \{x_i \ (i=0, \pm 1, \pm 2, \ldots)\} \) be an $(m)$-dependent strictly stationary stochastic process, and suppose that a level $u_n \ (n=1, 2, \ldots)$ tending to infinity is chosen so that (3.2.3) is satisfied with $\xi=1$.

Let $(a_i, b_i) \ (i=1, 2, \ldots, r)$ be $r$ disjoint, bounded intervals of $(0, \infty)$ and let $m_i \ (i=1, 2, \ldots, r)$ be $r$ non-negative integers, (both intervals and integers independent of $n$). Suppose that condition (3.2.4) of Theorem 3.2.1 is satisfied. Then, as $n \to \infty$

\[
P(\cap_{i=1}^r \{\text{exactly } m_i \text{ upcrossings by } x(t) \text{ of } u_n \text{ in interval } (a_i n, b_i n)\}) \to \prod_{i=1}^r (e^{-\tau_i^m_i/m_i!}) ,
\]
where \( \tau_i = b_i - a_i \) (i=1,2,...,r).

**Proof.** The proof of Theorem 3.2.2 is similar to the proof of Theorem 3.2.1 and is a direct consequence of Corollary 1.3.1.2.

Naturally a question immediately arises concerning the nature of, in particular, assumption (3.2.4) in the preceding theorem. The assumption that \( b_n = o(1) \) as \( n \to \infty \) is equivalent to the condition that, for \( u_n \) defined by (3.2.3),

\[
(3.2.6) \quad \max_{0 < |i-j| \leq m+1} P(x_{i-1} < u_n, x_i > u_n \mid x_{j-1} < u_n, x_j > u_n) \to 0 \quad \text{as } n \to \infty .
\]

In words, the condition requires that, given an upcrossing of a high level \( u_n \) has occurred, the probability of another upcrossing occurring within \((m+1)\) time units of the given upcrossing tends to zero as the level tends to infinity. Thus (3.2.4) can be interpreted as a condition requiring that the occurrence of a rare event (an upcrossing of a high level) at some time implies that no other rare event occurs at a neighboring time point (i.e., within \((m+1)\) time units). (See Newell [14] for some comments relevant to this. He gives an example of a process not satisfying (3.2.6), and a general result analogous to Watson's theorem valid for such processes.)

It is interesting to look more closely at the conditions of Theorem 3.2.1 in the particular case where \( \{x_i (i=0, \pm 1, \pm 2, \ldots)\} \) is an m-dependent normal stationary process with mean zero, variance one. In that case it can be verified easily that

\[
(3.2.7) \quad P(u) \sim \frac{1}{\sqrt{2\pi} u} e^{-u^2/2} \quad \text{as } u \to \infty ,
\]
and so according to (3.2.3), \(n\) and \(u_n\) are (asymptotically) related by the equation
\[
(3.2.8) \quad \frac{1}{u_n} e^{-u_n^2/2} = \xi.
\]

Furthermore it can be shown that as \(u_n \to \infty\), \(b_n \to 0\). Thus the theorem is valid for stationary \(m\)-dependent normal processes. By observing that (3.2.8) implies that
\[
(3.2.9) \quad u_n \sim (2 \log n)^{1/2} + [\log \xi - \frac{1}{2} \log 4\pi - \frac{1}{2} \log \log n]/(2 \log n)^{1/2}
= \xi_n \quad \text{as } n \to \infty
\]
(see, for example, Cramer and Leadbetter [4]), it follows that for \(m\)-dependent normal stationary processes,
\[
(3.2.10) \quad P(\text{upcrossings by } x(t) \text{ of } \xi_n \text{ in } (0, t)) \to e^{-t^k(\xi)^k/k!}
= e^{-z} \quad \text{as } n \to \infty.
\]

If, in particular, we consider the above result for the special case \(k=0\), \(t=1\) and \(\xi = e^{-z}\) \((-\infty < z < \infty\), we have
\[
(3.2.11) \quad P(\text{no upcrossings by } x(t) \text{ of } \xi_n \text{ in } (0, n)) \to e^{-e^{-z}} \quad \text{as } n \to \infty.
\]

Now if \(E_n\) denotes the event "no upcrossings by \(x(t)\) of \(\xi_n\) in \((0, n)\)" we may write
\[
(3.2.12) \quad P(E_n) = P(E_n \text{ and } x_0 < \xi_n) + P(E_n \text{ and } x_0 > \xi_n)
= P(\max_{0 \leq t \leq n} x(t) \leq \xi_n) + P(\max_{0 \leq t \leq n} x(t) > \xi_n).
\]

The last term tends to zero as \(n \to \infty\) \((\xi_n \to \infty)\) and so, via (3.2.11),
\[
(3.2.13) \quad P(\max_{0 \leq t \leq n} x(t) \leq (2 \log n)^{1/2} + \log z \log 4\pi - \frac{1}{2} \log \log n]/(2 \log n)^{1/2}) \to e^{-e^{-z}}
\]
as \( n \to \infty \). This is, of course, a well-known result, and can be derived from results found in Watson [19], and is explicitly quoted in Berman [1].

Of course, it would be nice to have a characterization (in terms of, say, distribution functions) of stationary \( m \)-dependent processes that satisfy the conditions of Theorem 3.2.1. This appears to be a difficult problem. Newell [14] remarks that for many processes arising in application the conditions are satisfied; some of his results may prove useful in the problem of characterization.

We shall now state some other results valid for \( m \)-dependent stationary processes satisfying the conditions of Theorem 3.2.1. The first result concerns waiting times between consecutive upcrossings. Let \( u_n \) be determined by (3.2.3) with \( \xi = 1 \), and let \( F_k^n(t) \) denote the probability that the waiting time between an upcrossing of \( u_n \) and the \( k \)th successive upcrossing of \( u_n \) is \( \leq nt \). We may then write

\[
(3.2.14) \quad 1 - F_k^n(t) = P(U(o,nt) \leq k-1|A_{-1}^n),
\]

where \( U(o,nt) \) denotes the number of upcrossings by \( x(t) \) of the level \( u_n \) in \( (o,nt) \) and \( A_{-1}^n \) is as in Definition 3.2.2. Provided the process \( \{x_i \}_{i=0,1,2,...} \) satisfies the conditions of Theorem 3.2.1, the events "\( U(o,nt) \leq k-1 \)" and "\( U(m+1,nt) \leq k-1 \)" are asymptotically equivalent, since their difference is contained in the event \( \bigcup_{i=0}^m A_1^m \), and

\[
P(\sum_{i=0}^m A_1^m) \leq \sum_{i=0}^m P(A_i^m) = (m+1) P(A_0^m) \sim (m+1)/n \to 0 \text{ as } n \to \infty \text{ via (3.2.3). Thus } P(U(o,nt) \leq k-1|A_{-1}^n) - P(U(m+1,nt) \leq k-1|A_{-1}^n) \to 0 \text{ as } n \to \infty.\]
But from m-dependence, the events \( \bigcup(m+1,nt) \leq k-1 \) and \( A_{-1}^m \) are independent. Thus via (3.2.14) we have, as \( n \to \infty \)

\[
(3.2.15) \quad 1 - F_{k}^n(t) - P(\bigcup(m+1,nt) \leq k-1) = P(\bigcup(o,nt) \leq k-1 | A_{-1}^n) - P(\bigcup(m+1,nt) \leq k-1) \to 0.
\]

Via stationarity, \( P(\bigcup(m+1,nt) \leq k-1) = P(\bigcup(o,nt-(m+1)) \leq k-1) \). Now, an application of the stationary, univariate of Corollary 1.3.1.1 with \( \lambda_{1} = t \) serves to prove that as \( n \to \infty \)

\[
(3.2.16) \quad P(\bigcup(o,nt-(m+1)) \leq k-1) - P(\bigcup(o,nt) \leq k-1) \to 0
\]

while directly from Theorem 3.2.1 we have

\[
(3.2.17) \quad P(\bigcup(o,nt) \leq k-1) \to \sum_{i=0}^{k-1} e^{-t} \frac{t^i}{i!}.
\]

Thus, combining the results of the above paragraph we have the following theorem:

**Theorem 3.2.3** Let \( \{x_i \ (i=0,1,2,\ldots)\} \) be an m-dependent stationary stochastic process satisfying the conditions of Theorem 3.2.1 [with \( \xi = 1 \) in (3.2.3)]. Let \( F_{k}^n(t) \) denote the probability that the waiting time between an upcrossing of \( u_n \) and the kth successive upcrossing of \( u_n \) is \( \leq nt \). Then as \( n \to \infty \)

\[
(3.2.18) \quad F_{k}^n(t) \to F_{k}(t) = 1 - e^{-t}[1 + \frac{t}{1!} + \ldots + \frac{t^{k-1}}{(k-1)!}]
\]

that is, the (normed) waiting time has the asymptotic distribution of the sum of k independent exponential (unit mean) random variables.

The result is not surprising. It is another example lending justification and clarification to the assertion that the stream of upcrossings by \( x(t) \) of the (appropriately chosen) level \( u_n \) is asymptotically
(or converges weakly to) a Poisson process (as \( n \to \infty \)). (See Definition 1.3.1.)

As a final example of the asymptotic behavior of the stream of upcrossings and downcrossings by \( x(t) \) of a level \( u_n \) (tending to \( \infty \)) in the \( m \)-dependent case, we consider upward excursions that were mentioned in section 3.1 in connection with the continuous-time normal results of Cramér and Leadbetter [4]. We first notice that for \( p \geq m+1 \), the probability that an excursion above a level \( u_n \) lasts longer than \( p \) time units tends to zero as \( u_n \to \infty \). To prove this, observe that the probability of the above event, say \( E_p \), is given by

\[
P(E_p^n) = P(x_{p+1} > u_n, \ldots, x_1 > u_n, x_{i-2} < u_n, x_{i-1} > u_n).
\]

Now from \( m \)-dependence, \( x_{p+1} \) is independent of \( x_{i-2} \) and \( x_{i-1} \) for \( p \geq m+1 \). Therefore,

\[
P(E_p^n) = P(x_{p+1} > u_n) P(x_{p+1} > u_n, \ldots, x_1 > u_n, x_{i-2} < u_n, x_{i-1} > u_n)
\leq P(x_{p+1} > u_n) \to 0 \text{ as } u_n \to \infty.
\]

Therefore, in the limit, excursions above \( u_n \) tend to be of duration \( \leq m+1 \). Thus, in contrast with the result stated in Section 3.1, compared with the 'size' of the time interval involved in the statement of the limit distribution, viz \((0,tn)\), the excursion lengths are negligible. This, of course, is not unexpected. However, a more informative result is possible. Let \( E_q^n \); \( q=1,2,\ldots,m+1 \) denote the event "an excursion above Level \( u_n \) has duration \( \leq q \) time units." Then

\[
P(E_1^n) = P(x_1 < u_n | x_{i-2} < u_n, x_{i-1} > u_n), \text{ and}
P(E_q^n) = P(x_{i+q-1} < u_n, \ldots, x_1 > u_n | x_{i-2} < u_n, x_{i-1} > u_n), \quad q=2,\ldots,m+1.
\]
Then, assuming \( P(E_q^n) \to p_q, \ q=1,2,\ldots,m+1 \) as \( n \to \infty \), the asymptotic probability, \( \pi_j \), that an excursion above \( u_n \) persists for a time interval of length \( d, \ j < d \leq j+1; \ j=0,1,\ldots,m \) is given by

\[
\pi_1 = p_1, \quad \pi_{j+1} = p_{j+1} - p_j, \quad j=1,2,\ldots,m .
\]

We have considered only \( m \)-dependent stationary processes so far. However, the assumption of stationarity is not crucial. The general results in section 1.1 showed that certain limit results valid for \( m \)-dependent stationary sequences of events remained valid for non-stationary sequences of events, provided the non-stationarity was, so to speak, uniformly small. Also, the results of section 1.2 (Theorem 1.2.2, for example) yielded analogous limit results in a setting that did not require a uniform non-stationarity.

As one might suspect, these results may be applied in the problem of upcrossings to yield results analogous to, say, Theorem 3.2.1, when the stochastic process \( \{x_i; \ i=0,\pm 1,\pm 2,\ldots\} \) is \( m \)-dependent though not necessarily stationary. As an example we state the following result.

**Theorem 3.2.4** Let \( \{x_i; \ i=0,\pm 1,\pm 2,\ldots\} \) be an \( m \)-dependent stochastic process, unbounded above. Suppose that for some sequence of levels \( u_n \to \infty \), the events \( \{A_i^n; \ i=1,2,\ldots\} \) defined by \( A_i^n = \{x_{i-1} < u_n, x_i > u_n\} \) with \( p_i^n = P(A_i^n) ; \ i=1,2,\ldots \) satisfy the following conditions as \( n \to \infty \):

\[
a_n = \sum_{i=1}^{\infty} p_i^n = O(1) ,
\]

\[
\rho_n = \max_{1 \leq i \leq n} p_i^n = o(1) , \text{ and}
\]
(3.2.25) \[ b_n = \max_{0 < |i-j| \leq m+1} P(A^n_i | A^n_j) = o(1). \]

Then as \( n \to \infty \),

(3.2.26) \[ P(\text{exactly } k \text{ upcrossings by } \bar{x}(t) \text{ of level } u_n \text{ in } (0,n)) \sim e^{-\frac{a_n}{n} \frac{k}{k!}}. \]

**Proof.** This is almost an immediate consequence of Theorem 1.2.2. From that theorem we can immediately conclude that (3.2.26) is true when the expression on the left is replaced by \( P(\text{exactly } k \text{ amongst } A^n_i (i=1,2,...,n) \text{ occur}) \). However the two expressions differ by an amount which is at most equal to \( P(x_\alpha > u_n) \to 0 \) as \( n \to \infty \).

As an application of this theorem, consider the following situation. Let us assume that under some ideal conditions a certain \( m \)-dependent stochastic process \( \{x_i (i=0,\pm 1,\pm 2,...)\} \) is stationary, and does in fact satisfy the conditions of Theorem 3.2.1. However, let us suppose that we have reason to believe that we are dealing with a process \( \{y_i (i=0,\pm 1,\pm 2,...)\} \) that is related to \( \{x_i; i=0,\pm 1,\pm 2,...\} \) in such a manner that \( y_i = x_i + g(i) \), where \( g(i) \) is some bounded function of \( i \), that is, \( |g(i)| \leq M < \infty; i=0,\pm 1,\pm 2,... \). For example, the process, otherwise stationary, might have a mean function \( \mu(i) = \mathbb{E}x_i; i=0,\pm 1,\pm 2,... \) that varies in such a manner so as to remain bounded. It is then not difficult to see that the conditions of Theorem 3.2.4 would be satisfied, and we would expect the asymptotic Poisson character of the frequency of upcrossings to remain valid. It is also to be expected that a similar result would hold in the continuous case considered by Cramér and Leadbetter [4] if the normal process \( \{\xi(t) - m(t), -\infty < t < \infty\} \)
where stationary, where \( m(t) \) is some bounded function of \( t \). We are restricted to the discrete case, however. Needless to say, there are many forms of non-stationarity under which the conditions of Theorem 3.2.4 might be satisfied, but we shall not consider this point any further.

We shall now turn attention to applications of the results of section 1.4 to the problem of upcrossings by a discrete process of a high level. As was pointed out in section 3.1, the results of Cramér and Leadbetter [4] indicate that a sufficient condition for the asymptotic Poisson character of the stream of upcrossings of a high level for a discrete time stationary normal process is that \( r_n = \mathcal{E}(x_0, x_n) \) \( O(\lvert n \rvert^{-\alpha}) \) as \( \lvert n \rvert \to \infty \) for some \( \alpha > 0 \). As can be proved from the results of Berman [1], other sufficient conditions are that either \( r_n \log n \to 0 \) as \( n \to \infty \), or \( \sum_{n=1}^{\infty} r_n^2 < \infty \). These sufficient conditions have various interpretations (in terms of, for example, the spectral distribution function of the process; see Berman [1], p. 511), and give some intuitive notion of the 'rate' at which dependence in the process 'dies off.' In this respect the sufficient conditions required for validity of the results of this section may be less appealing. This is to be expected though, since in the non-normal case dependence generally cannot be summarized by such a simple quantity as a correlation. We begin with the following result.

**Theorem 3.2.5** Let \( \{x_i \ (i=0, \pm 1, \pm 2, \ldots \}) \) be a stochastic process, unbounded above, and \( S_u, \{Y_i \ (i=0, \pm 1, \pm 2, \ldots \}) \) the sets and associated stochastic process defined in Definition 3.2.1. Suppose that for some sequence of levels \( u_n \to \infty \) (as \( n \to \infty \)) the conditions of Theorem 1.1.4
are satisfied with events \( \{ A^n_i (i=1,2,...) \} \) defined by \( A^n_i = \{ Y_i \in S_{u_n} \} \).

Then, as \( n \to \infty \),

\[
(3.2.27) \quad P(\text{exactly k upcrossings by } x(t) \text{ of } u_n \text{ in } (0,n)) \to e^{-\xi k}/k!
\]

**Proof.** Analogous to Theorem 3.2.4, the proof is almost an immediate consequence of Theorem 1.4.1. From that theorem, we can immediately conclude that as \( n \to \infty \),

\[
(3.2.28) \quad P(\text{exactly k amongst } A^n_i (i=1,2,...,n) \text{ occur}) \to e^{-\xi k}/k! \text{.}
\]

Now the probabilities on the left-hand sides of (3.2.27) and (3.2.28) differ by an amount that is at most \( P(x_0 > u_n) \) which tends to zero as \( n \to \infty \).

Of course, Theorem 3.2.5 is somewhat cumbersome and obtuse. It is not at all clear when the conditions are satisfied. So, in order to give a more appealing version, we specialize to the case of stationary sequences. At least in this case some intuitive feeling for the conditions can be given.

**Theorem 3.2.6** (A stationary version of Theorem 3.2.7) Let

\( \{ x_i (i=0,1,2,...) \} \) be a strongly mixing strictly stationary stochastic process (with mixing function \( g \)), and unbounded above. Let

\( u_n (n=1,2,...) \) be a sequence of levels tending to infinity in such a manner that (3.2.3) is satisfied. Further, suppose that there exist sequences of integers \( \{ p_m \} \) and \( \{ q_m \} \) such that as \( m \to \infty \),

\[
(3.2.29) \quad m^r g(q_m - 1) \to 0 \text{ for } n > 0,
\]

\[
(3.2.20) \quad q_m/p_m \to 0 \text{ and } p_{m+1}/p_m \to 1 \text{,}
\]


and writing \( p = p_m, \ q = q_m, \ t = m(p+q) \)

\[
(3.2.21) \quad \prod_p = \frac{1}{p} \sum_{i=1}^{p-1} (p-i) \ P\{ x_i > u_t, x_{i-1} < u_t, x_i < u_t, x_o > u_t \} = o(1).
\]

Let \((a_i, b_i); \ i = 1, 2, \ldots, r \) be \( r \) bounded disjoint intervals in \((0, \infty)\) and \( m_i; \ i = 1, 2, \ldots, r \) be \( r \) non-negative integers, all independent of \( n \). Then as \( n \to \infty \)

\[
(3.2.32) \quad \begin{align*}
P\left( \bigcap_{i=1}^{r} \text{exactly } m_i \text{ upcrossings of } u_n \text{ by } x(t) \text{ in } (a_i, b_i, n) \right) & \to \prod_{i=1}^{r} \left( e^{-\tau_i^{m_i}/m_i!} \right) \\
\end{align*}
\]

where \( \tau_i = \xi(b_i - a_i); \ i = 1, \ldots, r \).

**Proof.** This theorem is an application of Theorem 1.4.5. Let

\([Y_i \ (i=0, +1, +2, \ldots)]\) be the stochastic process associated with

\([x_i \ (i=0, +1, +2, \ldots)]\) as in Definition 3.2.1, and observe that \([Y_i \ (i=0, +1, +2, \ldots)]\) is strongly mixing and stationary with mixing function \( g' \) defined by \( g'(k) = g(k-1) \). Define events \( A_i^n \ (i=1, 2, \ldots) \)

\[A_i^n = \{ Y_i \in S_{u_n} \} = \{ x_i > u_n, x_{i-1} < u_n \}. \]

By virtue of the properties of the process \([Y_i \ (i=0, +1, +2, \ldots)]\) these events are stationary and strongly mixing with mixing function \( g' \). For the sequences \( \{p_n\} \) and \( \{q_n\} \) of integers that were postulated in the hypothesis, we have as \( m \to \infty, \)

\[
(3.2.23) \quad m^r g'(q_m) = m^r g(q_m - 1) \to 0 \text{ for } r > 0.
\]

Since (3.2.3) is satisfied, and (3.2.31) implies that the sequences of sets \( \{A_i^n \ (i=0, 1, \ldots)\} \) satisfy the conditions of Theorem 1.4.5, the proof is complete.
We now discuss the conditions of the above theorem. Condition (3.2.29) requires that, judged in terms of the mixing function g(k), the dependence in \( \{x_i; i=0, \pm 1, \pm 2, \ldots \} \) decrease faster than any power of the 'distance' k between two events. Condition (3.2.31) is a bit more difficult to interpret. The quantity in (3.2.31) is a weighted average of the conditional probabilities of an upcrossing of \( u_t \) within p time units (to the right), given an upcrossing has occurred. Letting \( a_t^i = P(x_i > u_t, x_{i-1} < u_t \mid x_{i-1} < u_t, x_0 > u_t) \), we write (3.2.31) as

\[
I = \frac{1}{p} \sum_{i=1}^{p} \frac{(p-1)a_t^i}{2} \left| a_t^i \right| \leq \frac{p-1}{2} \max_{1 \leq i \leq p} a_t^i.
\]

Thus, if as \( m \to \infty \),

\[
(3.33) \quad m_p = \max_{1 \leq i \leq p} a_t^i = \max_{1 \leq i \leq p} P(x_i > u_t, x_{i-1} < u_t \mid x_{i-1} < u_t, x_0 > u_t) = o(\frac{1}{p})
\]

it would immediately follow that condition (3.2.31) would be satisfied. Also, without having to make reference to any sequence, it would be sufficient that

\[
(3.2.34) \quad m_n = \max_{1 \leq i \leq n} a_n^i = o\left(\frac{1}{n}\right) \text{ as } n \to \infty.
\]

Accordingly, we may state the following (weaker) form of Theorem 3.2.6.

**Corollary 3.2.6.1** Suppose that \( \{x_i; i=0, \pm 1, \pm 2, \ldots \} \) is a stationary, strongly mixing stochastic process, unbounded above, and with mixing function g(k) satisfying

\[
g(k) = o(k^{-r}), \ r=1,2, \ldots \ \text{as } k \to \infty.
\]

If \( u_n; n=1,2, \ldots \) is a sequence of levels tending to infinity and chosen so that as \( n \to \infty \)
(3.2.36) \[ nP(x_0 < u_n, x_1 > u_n) \rightarrow \xi, \text{ and} \]

(3.2.37) \[ n \max_{1 \leq i \leq n} P(x_i > u_n, x_{i-1} < u_n | x_0 < u_n, x_1 > u_n) \rightarrow 0 , \]

then the conclusion of Theorem 3.2.6 is valid.

As one would expect, a 'waiting time' result is also available for the situation at hand. Via the univariate version of Theorem 1.4.6, we have the following result.

**Theorem 3.2.7** Suppose the conditions of Theorem 3.2.6 (or Corollary 3.2.6.1) are satisfied. If \( F_n^k(t) \) is the probability of the event "the waiting time from an upcrossing of level \( u_n \) by \( x(t) \) until the \( k \)th successive upcrossing of \( u_n \) is \( \leq nt \)," then as \( n \rightarrow \infty \),

(3.2.38) \[ F_n^k(t) \rightarrow F_k(t) = 1 - e^{-\xi t}(1 + \frac{\xi t}{1!} + \ldots + \frac{(\xi t)^{k-1}}{(k-1)!}) . \]
APPENDIX A

REMARKS CONCERNING NON-RANDOM t AND NON-RANDOM n

The main purpose of this appendix is to establish the validity of Theorems 2.2.1 and 2.2.2. As was brought out in Section 2.2, these theorems represent the 'link' between our approach to the Limit Problem (non-random n), and the approach taken, for example, by Ikeda [10] (non-random t). There are two limiting conditions of interest, \( n \to \infty \) and \( [n \to \infty] \). We shall deal with these, while, as before, sometimes restricting attention to the special case of \( k=3 \) (i.e. third-order light sensations).

First, we shall show that for the model of Section 2.3 (i.e., Poisson arrivals of intensity \( \mu \) and iid lifetimes with means \( \alpha < \infty \), the Poisson Limit Result for non-random n implies the (known) Poisson Limit Result for non-random t. Second, we shall deal with the model of Section 2.5 (i.e., Poisson lifetimes with mean \( \alpha \) and iid interarrival times with mean \( 1/\mu \)). We propose to show that the non-random n result contained therein implies an appropriate non-random t Poisson Limit Result.

We begin with a restatement of the first theorem we wish to prove, while making free use of the notation established in Section 2.2

**Theorem 2.2.1** If interarrival times \( \{x_i = x_i - x_{i-1} \, (i=1,2,\ldots) \} \) \((x_0 = 0)\) form an iid sequence of exponential random variables with mean \( 1/\mu \), while lifetimes \( \{t_i \, (i=1,2,\ldots) \} \) form a sequence of iid random variables
(independent of arrivals) with finite mean, then for \( \lambda > 0 \)

\[
(A.1) \quad \lim_{n \to \infty} q_m^k(n) \xrightarrow{\exists} \zeta_m^k \quad \Rightarrow \quad \lim_{t \to \infty} p_m(t) \xrightarrow{\exists} \zeta_m^k,
\]

where the symbol \( \exists \) is to be read 'exists and equals.' Here, \( (n \to \infty) \)
means "\( n \to \infty, \mu \downarrow 0 \) in such a manner that \( n \mu^{k-1} \to \lambda, \)" and \( (t \to \infty) \)
means "\( t \to \infty, \mu \uparrow 0 \) in such a manner that \( t \mu^{k} \to \lambda. \)"

Our proof of Theorem 2.2.1 requires two lemmas. The lemmas (as
should be noted) represent particular cases of general results involving
the relationship between renewal theory and the law of large numbers
Although this relationship would be interesting to pursue in generality,
we shall be content with the specific results in the following para-
graphs.

Let us define two quantities \( n_u^t = \mu t + (\mu t)^{1+\varepsilon}, \quad n_y^t = \mu t - (\mu t)^{1+\varepsilon}; \)
\( 0 < t < \infty, \quad \frac{1}{2} > \varepsilon > 0. \) We shall, without loss of generality, assume that
both \( n_u^t \) and \( n_y^t \) are integers. Furthermore, let \( n_t \) = number of arrivals
in \( (0, t] \); \( 0 < t < \infty. \)

**Lemma A.1** If interarrival times \( \{x_i = x_{i+1} - x_{i-1}; \quad i=1,2,\ldots\} \) \( (x_0=0) \) form
an iid sequence of exponential random variables with mean \( 1/\mu, \) then

\[
(A.2) \quad P(n_y^t \leq n_t \leq n_u^t) \to 1 \text{ as } \mu t \to \infty.
\]

**Proof.** The proof is a simple application of Chebychev's inequality.

Via that probability inequality we have

\[
(A.3) \quad P(n_y^t \leq n_t \leq n_u^t) = P\left(\frac{|n_t - \mu t|}{(\mu t)^{1/2}} \leq (\mu t)^{\varepsilon}\right) \geq 1 - \frac{1}{(\mu t)^{2\varepsilon}} \to 1
\]
as \( \mu t \to \infty. \)

We now let \( n = \mu t \) and observe that, as \( \mu t \to \infty, \) \( n_y^t = n + o(n) \) and
\( n_u^t = n + o(n) \). Assume, without loss of generality, that \( n \) is a non-negative integer. Note that \((n \to \infty)\) if and only if \((t \to \infty)\) according to the above definition. If \( P^k_m(t) = \sum_{j=m}^{\infty} p^k_j(t) \) and \( Q^k_m(n) = \sum_{j=m}^{\infty} q^k_j(n) \), we have the following result.

**Lemma A.2** Under the assumptions of Theorem 2.2.1,

\[
(A.4) \quad \lim_{(n \to \infty)} Q^k_m(n) = L^k_m \Rightarrow \lim_{(t \to \infty)} P^k_m(t) = L^k_m.
\]

**Proof.** Since \( y_n \) is the waiting time until the arrival of the \( n \)th particle, letting \( n^t = n^t_u \),

\[
(A.5) \quad P^k_m(t) \leq P(m \text{ or more } k \text{th order visual responses in } (o,t], \text{ and }
\]

\[
\frac{y_{n^t}}{t} < t) + P(\frac{y_{n^t}}{t} > t)
\]

\[
\leq P(m \text{ or more } k \text{th order visual responses in first } n^t \text{ arrivals, and } \frac{y_{n^t}}{t} < t) + P(\frac{y_{n^t}}{t} > t).
\]

But \( P(\frac{y_{n^t}}{t} < t) = P(n^t > n^t_u) \to 1 \) as \( \mu t \to \infty \), and \( P(\frac{y_{n^t}}{t} > t) \to 0 \) as \( \mu t \to \infty \), so that from (A.5)

\[
(A.6) \quad \limsup_{(t \to \infty)} P^k_m(t) \leq \limsup_{(n \to \infty)} Q^k_m(n).
\]

We now recall that \( n^t = n + o(n) \) as \((n \to \infty)\). Furthermore, in Chapter 2 it was shown that visual responses (light sensations) correspond to events which satisfy the conditions of Theorem 1.1.2. We may therefore appeal to the results of Corollary 1.1.3.1 and conclude that

\[
(A.7) \quad \limsup_{(n \to \infty)} Q^k_m(n) = \limsup_{(n \to \infty)} Q^k_m(n) = \lim_{(n \to \infty)} Q^k_m(n) = L^k_m.
\]

Hence, via (A.6) and (A.7),
\[(A.8) \quad \limsup_{(t \to \infty)} P^k_m(t) \leq \lim_{(n \to \infty)} Q^k_m(n) = L^k_m.\]

On the other hand, letting \(n_u = n^t_u\),

\[(A.9) \quad P^k_m(t) \geq P(m \text{ or more } k\text{th order visual responses in } (v_0, t], \text{ and } \nu_{n_u} \geq t \geq \nu_{n_{\ell}}) \geq P(m \text{ or more } k\text{th order visual responses in first } n_{\ell} \text{ arrivals and } \nu_{n_u} \geq t \geq \nu_{n_{\ell}}).\]

But, \(P(\nu_{n_u} \geq t \geq \nu_{n_{\ell}}) = P(n_{\ell} \leq n_t \leq n_u) \to 1\) and \(n_{\ell} = n + o(n)\) as \(\mu t \to \infty\) so that another appeal to Corollary 1.1.3.1 yields

\[(A.10) \quad \liminf_{(t \to \infty)} P^k_m(t) \geq \lim_{(n \to \infty)} Q^k_m(n) = L^k_m.\]

Combining the results of (A.10) and (A.8) we have proved (A.4).

**Proof of Theorem 2.2.1** Theorem 2.2.1 is now almost immediate. All we need note is that \(P^k_m(t) = P^k_m(t) - P^k_{m+1}(t)\) and \(Q^k_m(n) = Q^k_m(n) - Q^k_{m+1}(n)\).

With \(L^k_m = L^k_m - L^k_{m+1}\), Theorem 2.2.1 then follows from Lemma A.2.

We remark that the limiting condition \((n \to \infty)\) or \((t \to \infty)\) is actually used in two crucial steps in the above proof. We do indeed need more than \(\mu t \to \infty\). (The steps referred to are (A.7) and (A.10)).

Of course, Theorem 2.2.2 is the analogue of Theorem 2.2.1 for the case where interarrival times \(\{x_{i} = y_{i} - y_{i-1}; i = 1, 2, \ldots\}\) \((y_{0} = 0)\) for an iid sequence of random variables with mean \(1/\mu\), and lifetimes \(\{\tau_{i}; i = 1, 2, \ldots\}\) are iid exponential (independent of arrivals) with mean \(\alpha\). We shall state and prove a theorem, Theorem 2.2.2', slightly more general than Theorem 2.2.2.
Theorem 2.2.2' Suppose 'interarrival times' \( X_i = Y_i - Y_{i-1}; i=1,2,\ldots \) (\( Y_0 = 0 \)) form an iid sequence of positive random variables with mean \( 1/\mu \) (\( \mu > 0 \)), and lifetimes \( \tau_i; i=1,\ldots \) are iid exponential (independent of arrivals) with mean \( \alpha \). Let \( H_{\mu}(t) = \sum_{n_t} \), where \( n_t \) is the number of 'arrivals' in \((0,t]\). If there is a function \( g_{\mu}(t) \) such that as \( t \to \infty \) and \( \mu \downarrow 0 \) with \( H_{\mu}(t) \prod_{j=1}^{k-1} \mathcal{G}(\frac{1}{\mu j}) \to \delta > 0 \) we have

\[ g_{\mu}(t) \to \infty \]

(A.12)

\[ g_{\mu}(t) = o(H_{\mu}(t)) \]

(A.13)

\[ P(n \leq n < n) \to 1, \] where \( n = H_{\mu}(t) - g_{\mu}(t) \) and \( n_u = H_{\mu}(t) + g_{\mu}(t) \), then,

\[ \lim_{n \to \infty} q_m^k(n) \equiv \mathcal{L}_m^k \implies \lim_{t \to \infty} p_m^k(t) \equiv \mathcal{L}_m^k \]

where the symbol \( [n \to \infty] \) is integrated to mean "\( n \to \infty, \mu \downarrow 0 \) so that \( n \prod_{j=1}^{k-1} \mathcal{G}(\frac{1}{\mu j}) \to \delta > 0 \)" and \( [t \to \infty] \) is interpreted to mean "\( t \to \infty, \mu \downarrow 0 \) so that \( H_{\mu}(t) \prod_{j=1}^{k-1} \mathcal{G}(\frac{1}{\mu j}) \to \delta.\)" Of course, as above, \( \mathcal{G}(s) \) denotes the Laplace-Stieltjes transform of the distribution function of \( \mu X_i \).

**Proof.** Proceeding as in the proof of Theorem 2.2.1 we conclude that by defining \( n = H_{\mu}(t) \) and assuming without loss of generality that \( n, n, n_u \) are integers,

\[ \lim_{n \to \infty} q_m^k(n) \equiv \mathcal{L}_m^k \implies \lim_{t \to \infty} p_m^k(t) \equiv \mathcal{L}_m^k \]

from which the conclusion of the theorem is immediate.

We remark that since \( H_{\mu}(t) \sim \mu t \) as \( t \to \infty \), one may replace the former by the latter in the definition of \( [t \to \infty] \) in the theorem. Of
course, a question arises concerning the conditions under which the hypotheses of Theorem 2.2.2' are fulfilled. We remark that it is sufficient that $\xi_m^2 < \infty$. For in this case, we shall let $g_\mu(t) = \{V_\mu(t)\}^{1/2} + \varepsilon (\varepsilon > 0)$, where $V_\mu(t) = \text{var } n_t$; it is known (see Smith [15]) that $V_\mu(t) \sim K \mu t (K > 0)$ as $t \to \infty$. Furthermore $H_\mu(t) \sim \mu t \to \infty$, while obviously $g_\mu(t) = o(H_\mu(t))$. Finally $P(n \leq n_t \leq n_u) = P\left(\frac{n_t - n_r}{\sigma n_t}\right) \leq (K\mu t)^6 \to 1$ as $[t \to \infty]$. Therefore, the condition of Theorem 2.2.2' are valid, and so its conclusion holds. This is, of course, precisely Theorem 2.2.2.
APPENDIX B

SOME RESULTS FOR POSITIVE RANDOM VARIABLES

The results of this section apply to remarks found in Section 2.4. We state and prove the following (known) result for positive random variables.

**Theorem B.1** Let \( F \) be the distribution function of a positive random variable \( X \). For real numbers \( r, s \geq 1 \) we have,

(i) \( \mathbb{E} X^r < \infty \) implies \( x^r [1 - F(x)] \to 0 \) as \( x \to \infty \)

(ii) \( x^r [1-F(x)] \to 0 \) as \( x \to \infty \) implies \( \mathbb{E} X^s < \infty \) for \( s < r \).

**Proof.** For \( \lambda > 0, \mathbb{E} X^r \geq \int_0^\lambda x^r dF(x) + \lambda^r [1-F(\lambda)] \geq 0 \). Taking limits (as \( \lambda \to \infty \)) of both sides of the preceding inequality, we obtain (i).

Formally, for \( 1 \leq s < r \) we may write for any \( 0 \leq x < \infty, \frac{1}{s} \int_0^x X^s dF(X) = -\frac{x^s}{s} [1-F(x)] + \int_0^x x^{s-1} [1-F(X)] dX \). Now \( x^r [1-F(x)] \to 0 \) as \( x \to \infty \) implies that for \( x \geq X_o x^{s-1} [1-F(x)] \leq x^{-(1+r-s)} \), and hence the second term on the right of the preceding equality represents a convergent integral as \( x \to \infty \). By hypothesis the first term on the right of the preceding equality \( \to 0 \) as \( x \to \infty \). Taking limits of both sides as \( x \to \infty \) it is seen that \( \mathbb{E} X^s < \infty \), thus proving (ii).

**Corollary B.1**

\( x^r [1-F(x)] \to 0 \) as \( x \to \infty \).
and \[ \int_0^\infty x^{r-1} [1-F(x)] \, dx < \infty \] imply \( E x^s < \infty \) for \( s \leq r \).

Now if, for a positive random variable \( X \) with distribution function \( F, r_o < \sup_r [E X^r < \infty] \), the above result insures that \( x^\delta [1-F(x)] \to 0 \) as \( x \to \infty \) for any \( \delta < r_o \). One might be tempted to conclude that \( x^\delta [1-F(x)] \to \infty \) as \( x \to \infty \) for \( \delta < r_o \). This is not the case in general unless additional assumptions are made on the behavior of \( [1-F(x)] \) as \( x \to \infty \). The following theorem, a modification of a result due to Smith [16], demonstrates that without additional assumptions on \( [1-F(x)] \), little can be said about the asymptotic behavior of \( x^\eta [1-F(x)] \) \( (\eta > r_o) \) as \( x \to \infty \). The moment assumptions made in the theorem are seen to be relevant to the situation encountered in the discussion of asymptotic behavior of scores in Chapter II.

**Theorem B.2** Let \( w \) be any increasing function and \( \mathcal{F} = \{ F : E X < \infty, E X^2 = \infty \} \) (that is, the class of all distribution function \( F \) of random variables \( X \) with finite mean, infinite variance). Then there exists an \( F \) in \( \mathcal{F} \) such that

\[
\lim_{x \to \infty} \inf \quad w(x) [1-F_w(x)] = 0 .
\]

(Note that if \( w(x) = x^\delta \) \( (\delta > 2) \), the theorem insures that for at least one distribution function \( F, x^\delta [1-F(x)] \) does not tend to infinity.)

**Proof.** The proof follows Smith [16]. Suppose \( w(x) > 1 \) for \( x > 1 \). Let \( \{ e_n \} \) be a monotone decreasing sequence of positive real numbers such that \( e_1 < 1 \) and \( e_n \downarrow 0 \) as \( n \to \infty \). Define \( \xi_1, \xi_2, \ldots \), and \( S_n = \xi_1 + \xi_2 + \ldots + \xi_n \) as follows: \( w(\xi_1) = 1 \) and \( \xi_{n+1} = \frac{w(S_n)}{e_n} \), \( n=2,3, \ldots \). Thus \( \xi_1 = e_1 > 1 > \xi_1 \), and for \( n > 1 \), \( \xi_{n+1} = \frac{w(S_n)}{e_n} > \frac{w(S_{n-1})}{e_{n-1}} = \xi_n \).
Now define
\[
1 - F_w(x) = \begin{cases} \\
\frac{\varepsilon_n}{n^2 w(S_n)} & \text{if } x \in [S_n, S_{n+1}), n=1,2,\ldots \\
1 & \text{if } x < 1 .
\end{cases}
\]

Then \(1 - F_w(x) \downarrow 0\) as \(x \to \infty\) since
\[
\frac{\varepsilon_n}{n^2 w(S_n)} > \frac{\varepsilon_n}{(n+1)^2 w(S_{n+1})} > \frac{\varepsilon_{n+1}}{(n+1)^2 w(S_{n+1})} .
\]

Note that
\[
\mathcal{E}_F \left[\frac{1}{x} \right] = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - F_w(x)\right] dx = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^2 w(S_n)} \cdot \xi_{n+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .
\]

Furthermore,
\[
\sum_{n=1}^{\infty} \frac{S_{n+1}}{S_n} x[1 - F_w(x)] dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^2 w(S_n)} \left( S_{n+1}^2 - S_n^2 \right)
\]
\[
\geq \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^2 w(S_n)} \xi_{n+1} S_n = \sum_{n=1}^{\infty} \frac{S_n}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty .
\]

and so \(\mathcal{E}_F \left[\frac{1}{x} \right] = \infty\). Thus \(F_w \in \mathcal{F}\). In addition
\[
w(S_n) \left[1 - F_w(S_n)\right] = w(S_n) \frac{\varepsilon_n}{n^2 w(S_n)} = \frac{\varepsilon_n}{n^2} \to 0 \text{ as } n \to \infty
\]
so, \(\lim \inf w(x) \left[1 - F_w(x)\right] = 0\), as was to be shown.

The results of Theorem B.2 show that if \(x\) is a positive random variable with distribution function \(F\) and \(\mathcal{E}_x < \infty, \mathcal{E}_x^2 = \infty\), the behavior of \(x^{2+\varepsilon} \left[1 - F_w(x)\right]\) as \(x \to \infty\) is not determined in general. Thus
to deal with the asymptotic behavior of expressions such as 2.12, it seems reasonable (by virtue of necessity) to make some additional assumptions concerning the asymptotic behavior of $1-F(x)$ as $x \to \infty$.

The somewhat natural occurrence of a relation between the asymptotic behavior of $1-F(x)$, (2.12) and functions of slow growth, both from a probabilistic and mathematical standpoint, make the slow growth assumptions on $1-F(x)$ seem reasonable (see Smith [17], Feller [5]).
APPENDIX C

MULTIVARIATE ANALOGUE OF THE BONFERRONI INEQUALITIES

Suppose we are given two finite classes of events, \( \{A_1, \ldots, A_M\}, \)
\( \{B_1, \ldots, B_N\} \). We are interested in upper and lower bounds on the quantities,

\[
P_{[m,n]} = \text{Pr (exactly } m \text{ } A\text{'s and exactly } n \text{ } B\text{'s occur)}
\]

and

\[
P_{(m,n)} = \text{Pr (at least } m \text{ } A\text{'s and at least } n \text{ } B\text{'s occur)};
\]

\(0 \leq m \leq M, 0 \leq n \leq N\). Define

\[
S_{m,n} = \sum_{1 \leq i_1 < \ldots < i_m \leq M} \sum_{1 \leq j_1 < \ldots < j_n \leq N} \prod_{1 \leq l \leq m} P(A_{i_l}) \prod_{1 \leq l \leq n} P(B_{j_l}).
\]

It is known (see Fréchet [6]) that

\[
P_{[m,n]} = \sum_{i=m}^{M} \sum_{j=n}^{N} (-1)^{i+j-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j},
\]

\[
= \sum_{t=m+n}^{M+N} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j},
\]

from which it follows (by solving the linear system (C.1)) that

\[
S_{m,n} = \sum_{i=m}^{M} \sum_{j=n}^{N} \binom{i}{m} \binom{j}{n} P_{[i,j]}
\]
\[
= \sum_{t=m+n}^{M+N} \sum_{i+j=t}^{i+m} (i,j) P[i,j].
\]

In the following theorem we establish a sequence of upper and lower bounds on \(P[m,n]\) based on the expansion in (C.1). These bounds are analogous to the Bonferroni inequalities (see Feller [5], p. 100) and in fact reduce to them in case \(M = 0\) or \(N = 0\). In Theorems C.1 and C.2, the indices \(i\) and \(j\) are restricted to the range \(m \leq i \leq M, m \leq j \leq N\).

**Theorem C.1** For \(\ell = 0, 1, 2, \ldots\) we have

\[
P[m,n] \leq \sum_{t=m+n}^{m+n+2\ell} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j},
\]

and

\[
P[m,n] \geq \sum_{t=m+n}^{m+n+2\ell+1} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j}.
\]

**Proof.** We shall make use of the following combinatorial identities:

(i) \(\sum_{\nu=t}^{k} (-1)^{\nu-t} \binom{k-m}{\nu-m} = \binom{k-m-1}{t-m-1}\), (Feller [5], p. 61),

and

(ii) \(\binom{k}{m} \binom{\nu}{\nu-m} = \binom{k}{m} \binom{k-m}{\nu-m}\).

It will suffice to show that for \(\ell \geq m+n\),

\[
= \sum_{t=m+n}^{M+N} \sum_{i+j=t} (-1)^{t-\ell} \binom{i}{m} \binom{j}{n} S_{i,j} \geq 0.
\]

Using (C.2) we have the left-hand side of (C.3) equal to

\[
= \sum_{t=m+n}^{M+N} \sum_{i+j=t} (-1)^{t-\ell} \binom{i}{m} \binom{j}{n} \{ \sum_{j=i}^{M} \sum_{z=j}^{N} \binom{y}{z} \} P[y,z].
\]
\[= \sum_{i=m}^{M} \sum_{j=\min(\ell-i,n)}^{N} (-1)^{i+j-\ell} \binom{i}{m} \binom{j}{n} \sum_{z=j}^{\min((\ell-1)\ell, n)} \binom{z}{j} \binom{\ell}{z} P_{y, z}\]

\[= \sum_{i=m}^{M} \sum_{j=\ell-i}^{N} (-1)^{i+j-\ell} \binom{i}{m} \binom{j}{n} \binom{\ell}{j} \binom{\ell}{z} P_{y, z}.\]

The range of summations for \(j\) has been extended to \(\ell-i\) to \(N\) since the
summand is zero if \(j < n\). Rearranging summation orders for \(z\) and \(j\) and
using (ii) yields the following expression for (C.4):

\[= \sum_{i=m}^{M} \sum_{y=i}^{\ell-i} (-1)^{z-(\ell-i)} \binom{z-n}{j-n} \sum_{z=j}^{\ell-i} \binom{i}{y} \binom{n}{z} P_{y, z}\]

\[= \sum_{i=m}^{M} \sum_{y=i}^{\ell-i-n} (-1)^{z-(\ell-i-n-1)} \binom{i}{m} \binom{j}{n} \binom{\ell-i-n}{j-n} P_{y, z}\]

\[\geq 0.\]  This completes the proof.

We have analogous results for \(P_{n,m}\). First,

(C.5) \[P_{m,n} = \sum_{i=m}^{M} \sum_{j=n}^{N} (-1)^{i+j-(m+n)} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,k}.\]

This is established by noting

\[P_{n,m} = \sum_{y=m}^{M} \sum_{z=n}^{N} P_{y, z},\] and using (C.1)

\[= \sum_{y=m}^{M} \sum_{z=n}^{N} \sum_{i=y}^{\ell-i} (-1)^{i+j-(y+z)} \binom{i}{y} \binom{j}{z} S_{i,j}.\]

Interchanging the order of summation of both \(y\) and \(i\) and \(z\) and \(j\), and
using identity (i), we obtain (C.5).

By solving the system of linear equations (C.5) we obtain,
\[(C.6) \quad S_{m,n} = \sum_{i=m}^{M} \sum_{j=n}^{N} \binom{i-1}{m-1} \binom{j-1}{n-1} P(i,j).\]

Again, a sequence of upper and lower bounds for \(P_{m,n}\) based on the expansion (C.5) are available. They reduce to the well-known inequalities (see Feller [5], p. 100) in case either \(M = 0\) or \(N = 0\).

**Theorem C.2** For \(\ell = 0, 1, 2, \ldots\) we have

\[
P_{m,n} \leq \sum_{\ell=m+n}^{m+n+2\ell} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,j}
\]

and

\[
P_{m,n} \geq \sum_{\ell=m+n}^{m+n+2\ell+1} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,j}.
\]

**Proof.** As in Theorem C.1, it suffices to show that for \(\ell \geq m+n\),

\[(C.7) \quad \sum_{t=\ell}^{M+N} \sum_{i+j=t} (-1)^{t-\ell} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,j} \geq 0.
\]

Proceeding as before, and using (C.6), the left hand side of (C.7) is equal to,

\[(C.8) \quad \sum_{t=\ell}^{M+N} \sum_{i+j=t} (-1)^{t-\ell} \binom{i-1}{m-1} \binom{j-1}{n-1} \{ \sum_{i=z}^{M} \sum_{j=z}^{N} \binom{y-1}{i-1} \binom{z-1}{j-1} P(y,z) \}
\]

\[
= \sum_{i=m}^{M} \sum_{j=n}^{N} (-1)^{i+j-\ell} \binom{i-1}{m-1} \binom{j-1}{n-1} \{ \sum_{y=i}^{M} \sum_{z=j}^{N} \binom{y-1}{i-1} \binom{z-1}{j-1} P(y,z) \}
\]

\[
= \sum_{i=m}^{M} \sum_{j=n}^{N} \sum_{y=i}^{M} \sum_{z=j}^{N} (-1)^{i+j-\ell} \binom{i-1}{m-1} \binom{j-1}{n-1} \binom{y-1}{i-1} \binom{z-1}{j-1} \binom{y-1}{i-1} \binom{z-1}{j-1} P(y,z).
\]
Interchanging the order of summation for \( z \) and \( j \) and using identities (i) and (ii) yields the following expression for (C.8)

\[
\sum_{i=m}^{M} \sum_{y=i}^{N} \sum_{z=\ell-i}^{z-n-1} (\ell-1)_{m-1} i_{-1} (y-1)_{i-1} (z-1)_{j-1} \geq 0.
\]

This completes the proof.

It is clear that Theorems C.1 and C.2 can be generalized to any number of dimensions, that is, if \( \{A_{11}, \ldots, A_{1N_1}\}, \{A_{21}, \ldots, A_{2N_2}\}, \ldots, \{A_{k1}, \ldots, A_{kN_k}\} \) are \( k \) finite classes of events and \( P(m_1, \ldots, m_k) \), \( P[m_1, \ldots, m_k] \), and \( S_{m_1, \ldots, m_k} \) are defined in the obvious way, then for any \( \ell \geq 0 \)

\[
P[m_1, \ldots, m_k] \leq \sum_{t=\Sigma m_j}^{\Sigma m_j + 2\ell} \sum_{m_1 \leq i_1 \leq N_1}^{t-(\Sigma m_j)} \sum_{m_2 \leq i_2 \leq N_2}^{(t-1)m_1} \prod_{j=1}^{k} S_{i_1, \ldots, i_k},
\]

\[
P[m_1, \ldots, m_k] \geq \sum_{t=2m_j}^{\Sigma m_j + 2\ell + 1} \sum_{m_1 \leq i_1 \leq N_1}^{t-(\Sigma m_j)} \sum_{m_2 \leq i_2 \leq N_2}^{(t-1)m_1} \prod_{j=1}^{k} S_{i_1, \ldots, i_k},
\]

\[
P(m_1, \ldots, m_k) \leq \sum_{t=\Sigma m_j}^{\Sigma m_j + 2\ell} \sum_{m_1 \leq i_1 \leq N_1}^{t-(\Sigma m_j)} \sum_{m_2 \leq i_2 \leq N_2}^{(t-1)m_1} \prod_{j=1}^{k} S_{i_1, \ldots, i_k},
\]

and
As an application of Theorem C.1 we consider the following problem. Let \( \{A^n_1, \ldots, A^n_n\}, \{B^n_1, \ldots, B^n_n\} \) be two classes of \( n \) events. Suppose that for these events \( S^n_{\alpha, \beta} \rightarrow \xi^n_1 \xi^n_2/\alpha! \beta! \) as \( n \rightarrow \infty \). What can be said about \( P^n_{[\alpha, \beta]} \) as \( n \rightarrow \infty \)? Via Theorem C.1 we have, for any \( \ell > 0 \),

\[
(C.9) \quad P^n_{\alpha, \beta} \leq \sum_{t=\alpha+\beta}^{\alpha+\beta+2\ell} (-1)^{t-(\alpha+\beta)} \binom{\alpha}{\ell} \binom{\beta}{\ell} S^n_{i,j}, \text{ and}
\]

\[
(C.10) \quad P^n_{\alpha, \beta} \leq \sum_{t=\alpha+\beta}^{\alpha+\beta+2\ell+1} (-1)^{t-(\alpha+\beta)} \binom{\alpha}{\ell} \binom{\beta}{\ell} S^n_{i,j}.
\]

From the hypothesis, as \( n \rightarrow \infty \) the right hand sides of (C.9) and (C.10) tend, respectively, to
(C.11) \[ \frac{\xi_1^{\alpha+\beta+2l}}{\alpha!\beta!} \sum_{t=\alpha+\beta} \sum_{0 \leq i \leq n} (-1)^{i+j+t} \frac{\xi_1^{(1-\alpha)} \xi_2^{(j-\beta)}}{(1-\alpha)! (j-\beta)!} \]

\[ = \frac{\xi_1^{\alpha+\beta} \xi_2^{2l}}{\alpha!\beta!} \sum_{r=0}^{2l} \sum_{0 \leq i \leq n} \frac{(-\xi_1)^i (-\xi_2)^j}{i! j!} \]

and

(C.12) \[ \frac{\xi_1^{\alpha+\beta+2l+1}}{\alpha!\beta!} \sum_{t=\alpha+\beta} \sum_{0 \leq i \leq n} (-1)^{i+j+t} \frac{\xi_1^{(1-\alpha)} \xi_2^{(j-\beta)}}{(1-\alpha)! (j-\beta)!} \]

\[ = \frac{\xi_1^{\alpha+\beta} \xi_2^{2l+1}}{\alpha!\beta!} \sum_{r=0}^{2l+1} \sum_{0 \leq i \leq n} \frac{(-\xi_1)^i (-\xi_2)^j}{i! j!} \]

Since \( l \) is arbitrary and as \( l \to \infty \) both (C.11) and (C.12) tend to

\( (\xi_1 e^{-\xi_1/\alpha!}) (\xi_2 e^{-\xi_2/\beta!}) \), the conclusion is that \( f_{[\alpha, \beta]}^n \) also tends to

this quantity as \( n \to \infty \).
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