A NOTE ON CONFIDENCE BOUNDS CONNECTED WITH ANOVA AND MANOVA
FOR BALANCED AND PARTIALLY BALANCED INCOMPLETE
BLOCK DESIGNS

by

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1. Introduction and summary

It is well known [2,3] how, in the case of any general "testable" linear hypothesis for either ANOVA or MANOVA one can put simultaneous confidence bounds on a set of parametric functions, which might be regarded as measures of deviation from the total hypothesis and its various components. The parametric functions are such that, in each problem, one of these can be appropriately called the "total" and the rest "partials" of various orders. For each problem the "total" one (i) is related to but not quite the same as the noncentrality parameter of the usual F-test of the total hypothesis in ANOVA and (ii) is the largest characteristic root of a certain parametric matrix which is related to but not quite the same as another parametric matrix whose nonzero characteristic roots occur as a set of noncentrality parameters in the power function for the test (no matter which one of the standard tests we use) of the total hypothesis in MANOVA. The same remark applies to "partials" of various orders considered in the proper sense.

In this note, for both ANOVA and MANOVA, the hypothesis considered is that of equality of the treatment effects—vector equality in the

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case of MANOVA. Starting from such a hypothesis explicit algebraic expressions are obtained for the total and partial parametric functions that go with the simultaneous confidence statements in the case of both ANOVA and MANOVA and for balanced and partially balanced designs. It is also indicated how to obtain, in a convenient form, the algebraic expression for the confidence bounds on each such parametric function, without a derivation of these expressions in an explicit form.

2. **Notation and preliminaries**

1) Let $x$ denote a column-vector of $n$ independent normal variables with a common variance $\sigma^2$ and the means given by

$$E x = \frac{A}{n \times m} \theta_{m \times 1},$$

where $A$ is a matrix of known constants and $\theta$ is a vector of unknown parameters. Suppose

$$\mathcal{H}_0: B\theta = 0 \quad [\text{Rank } B = s]$$

is the "testable" hypothesis to be tested. If we write

$$B \theta = F_{s \times 1},$$

then the "total" parametric function $\Delta$ associated with $\mathcal{H}_0$ is:

$$\Delta = F D^{-1} F,$$

where $D \sigma^2$ is the variance-covariance matrix of the best unbiased linear estimates of $\Phi$. It may be observed that $\Delta/\sigma^2$ is the noncentrality-parameter of the $F$-test for $\mathcal{H}_0$. The confidence bounds on $\Delta$, with a confidence coefficient greater than or equal to $(1 - \alpha)$, are then given by $[2,3]$:

$$S_{H_0}^{\frac{1}{2}} \left[ \frac{s}{n - r} F_{\alpha}^{\frac{1}{2}} S_{E}^{\frac{1}{2}} \right] \leq \Delta^{\frac{1}{2}} \leq S_{H_0}^{\frac{1}{2}} + \left[ \frac{s}{n - r} F_{\alpha} \right]^{\frac{1}{2}} S_{E}^{\frac{1}{2}}.$$
where \( r = \text{Rank} \ A \), \( F_\alpha \) is the 100\(\alpha\)% significance point of \( F \) with d.f. \( s \) and \( n-r \) respectively, \( S_{H_0} \) is the sum of squares due to \( \mathcal{H}_0 \) and \( S_E \) is the sum of squares due to error. We also have the simultaneous confidence statements

\[
S_{H_0}^{1/2} \left[ \begin{array}{l} -\frac{s}{n-r} \ F \end{array} \right] \frac{1}{2} S_E^{1/2} \leq \Delta(a) \leq S_{H_0}^{1/2} \left[ \begin{array}{l} \frac{s}{n-r} \ F \end{array} \right] \frac{1}{2} S_E^{1/2},
\]

where \( \Delta(a) = \Phi(a) \ D^{-1}(a) \ D(a) \), \( \Phi(a) \) is any subvector of \( \Phi \), \( D(a) \) is the corresponding submatrix of \( D \), and \( S_{H_0} \) is the corresponding sum of squares due to the partial hypothesis \( \mathcal{H}_0 : \Phi(a) = 0 \).

In the case of experimental designs, we have

\[
\mathcal{E} \chi_{\alpha} = t_i + b_j \quad i = 1, 2, \ldots, v, \quad j = 1, 2, \ldots, b
\]

if the \( \alpha \)-th observation belongs to the \( i \)-th treatment and \( j \)-th block. The hypothesis of equality of treatment effects may be given by:

\[
\mathcal{H}_0 : (I_{v-1}, -J_{v-1, 1})^T = 0,
\]

where \( t' = (t_1, t_2, \ldots, t_v) \) and \( J_{r,s} = \{1\}_{r \times s} \). We shall write \( J_{r \times r} \) as \( J_r \). We shall assume that the design is connected. Then it is well known [1] that the equations for \( t \) are

\[
C \ t = 0,
\]

in the usual notation. Also

\[
\text{Cov}(Q) = \sigma^2 C^{-1}.
\]

Then from (3) and (8), \( \Phi_i = t_i - t_v \), \( i = 1, 2, \ldots, v-1 \). We may express \( \Delta \) in a symmetrical form by taking \( \Phi = (I_{v-1}, -J_{v-1, 1}) \xi \), where \( \xi_i = t_i - \frac{1}{v} (t_1 + t_2 + \ldots + t_v), \ i = 1, 2, \ldots, v \). From (4) then we get
\begin{equation}
\Delta = \begin{pmatrix} I_{p-1} & \frac{D}{J_{p-1,1}} \end{pmatrix} \begin{pmatrix} I_{p-1} & -J_{p-1,1} \end{pmatrix} \xi.
\end{equation}

ii) In all experimental situations, what we capture under any design and sampling scheme is certain experimental units or individuals. According as just one character or many characters (or variates or responses) are observed on each such unit or individual, the problem is said to be a univariate or a multivariate one.

Let $X$ denote a matrix of $n$ independent $p$-dimensional normal variables with a common variance-covariance matrix $\Sigma$, $p$ being the number of characters observed on each individual, and let the means be given by

\begin{equation}
E_\chi n x p = A_{n x m} \Theta m x p,
\end{equation}

where $\Theta$ is a matrix of unknown parameters. Suppose

\begin{equation}
H_0 : B \otimes U_{p x n} = 0 \quad [\text{Rank } U = u \leq p]
\end{equation}

is the "testable" hypothesis to be tested. If we write

\begin{equation}
B \otimes U = \Phi s x u,
\end{equation}

then the "total" parametric function $\Delta$ associated with $H_0$ is given by [2,3]

\begin{equation}
\Delta = C_{m x} \begin{pmatrix} \Phi & D^{-1} \Phi \end{pmatrix}.
\end{equation}

It may be observed that the characteristic roots of $\Phi D^{-1} \Phi (U^t \Sigma U)^{-1}$ are the noncentrality parameters in the power function of the test (no matter which one of the standard tests we use) of the total hypothesis given by (13).

The confidence statement is given by

\begin{equation}
C_{m x} \begin{pmatrix} s_{H_0} \end{pmatrix} - \left[ \frac{s}{n-x} C_\alpha \right]^{\frac{1}{2}} C_{m x} \begin{pmatrix} s_E \end{pmatrix} \leq \Delta \leq \left[ \frac{s}{n-x} C_\alpha \right]^{\frac{1}{2}} C_{m x} \begin{pmatrix} s_E \end{pmatrix},
\end{equation}

\begin{equation}
C_{m x} \begin{pmatrix} s_{H_0} \end{pmatrix} + \left[ \frac{s}{n-x} C_\alpha \right]^{\frac{1}{2}} C_{m x} \begin{pmatrix} s_E \end{pmatrix}.
\end{equation}
where $S_{H_0}$ and $S_E$ are the sum of products matrices due to the hypothesis and error respectively, and $C_\alpha$ is the 100$\alpha$% significance point of the distribution of the largest characteristic root, with d.f. $u$, $s$, and $n-r$. In this case, we have simultaneous confidence statements, similar to (6), given by

$$\left(17\right) \quad C_{\max}^{|}\left[S_{(a)H_0}\right] - \left[\frac{s}{n-r} C_\alpha\right]^{|}\frac{1}{2} C_{\max}^{|}\left[S_E\right] \leq \Delta_{(a)}^{|} \leq$$

$$\quad C_{\max}^{|}\left[S_{(a)H_0}\right] + \left[\frac{s}{n-r} C_\alpha\right]^{|}\frac{1}{2} C_{\max}^{|}\left[S_E\right],$$

where $\Delta_{(a)} = C_{\max}^{|}\left[\Phi_{(a)}^T \Phi_{(a)}\right]$; $\Phi_{(a)}$ being a submatrix of $\Phi$ obtained by choosing some rows of $\Phi$. In addition, we have, by dropping some columns of $\Phi$, simultaneous confidence statements given by

$$\left(18\right) \quad C_{\max}^{|}\left[S_{(b)H_0}\right] - \left[\frac{s}{n-r} C_\alpha\right]^{|}\frac{1}{2} C_{\max}^{|}\left[S_E\right] \leq \Delta_{(b)}^{|} \leq$$

$$\quad C_{\max}^{|}\left[S_{(b)H_0}\right] + \left[\frac{s}{n-r} C_\alpha\right]^{|}\frac{1}{2} C_{\max}^{|}\left[S_E\right],$$

where $\Delta_{(b)} = C_{\max}^{|}\left[\Phi_{(b)}^T \Phi_{(b)}\right]$; $\Phi_{(b)}$ being a submatrix of $\Phi$ obtained by choosing some columns of $\Phi$; $S_{(b)H_0}$ and $S_{(b)E}$ are the corresponding submatrices of $S_{H_0}$ and $S_E$. These two operations may be combined.

In the case of experimental designs, we have

$$\left(19\right) \quad \xi_{(k)} = t_i^{(k)} + b_j^{(k)}, \quad i = 1, 2, \ldots, v$$

$$\quad j = 1, 2, \ldots, b$$

$$\quad k = 1, 2, \ldots, p$$

where $\chi_{(k)}$ denotes the $k$-th character measured on the $\alpha$-th experimental unit or individual that turns up for the $i$-th treatment and the $j$-th block; and $t_i^{(k)}$, $b_j^{(k)}$ stand respectively for the contributions to the expectation of the $k$-th variate made by the $i$-th treatment and the $j$-th block.

It may be observed from (1) and (12) that we have the same "structure
matrix" \textit{A} in the multivariate situation. This "structure matrix" depends on the design as well as on what the experimental statisticians have been calling the model—e.g. (7) and (19).

In this set-up, so far as the hypothesis is concerned, we shall take \( \mathbf{u} = \mathbf{I} \) for simplicity.

3. \textbf{Balanced incomplete block designs}

i) Here

\[
\mathbf{C} = r \mathbf{I}_v - \frac{1}{k} [(r - \lambda) \mathbf{I}_v + \lambda \mathbf{J}_v] = \frac{\lambda v}{k} \mathbf{I}_v - \frac{\lambda}{k} \mathbf{J}_v .
\]

Imposing the usual condition \( \mathbf{J}_v \mathbf{t} = 0 \) to get unique solutions, we have

\[
\mathbf{t} = \frac{k}{\lambda v} \mathbf{Q} .
\]

Therefore,

\[
\Phi = \frac{k}{\lambda v} (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1}, 1)^T \mathbf{Q} ,
\]

and hence

\[
\mathbf{D} = \frac{k^2}{\lambda^2 v^2} (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1}, 1)^T \mathbf{C} (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1}, 1) = \frac{k}{\lambda v} (\mathbf{I}_{v-1} + \mathbf{J}_{v-1}) ,
\]

whence

\[
\mathbf{D}^{-1} = \frac{\lambda v}{k} (\mathbf{I}_{v-1} - \frac{1}{v} \mathbf{J}_{v-1}) .
\]

Thus

\[
\Delta = \frac{\lambda v}{k} \Phi^T (\mathbf{I}_{v-1} - \frac{1}{v} \mathbf{J}_{v-1}) \Phi
\]

\[
= \frac{\lambda v}{k} \mathbf{E}^T (\mathbf{I}_v - \frac{1}{v} \mathbf{J}_v) \mathbf{E}
\]

(\text{from} (11))

\[
= \frac{\lambda v}{k} \mathbf{E}^T \mathbf{E}
\]

(\text{since} \( \mathbf{J}_v \mathbf{E} = 0 \))

\[
= \frac{\lambda v}{k} \sum_{i=1}^{v} \xi_i^2 .
\]

Then we can have a confidence statement (5) with \( n = bk, r = b + v - 1, s = v - 1 \) and \( \Delta \) given by (22).
For the "partial" statements (6), if we have
\[ \Phi'_l(a) = (\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_t}) \quad (t < v - 1) \]
then
\[ D(a) = \frac{k}{\lambda v} (I_t + J_t) \quad \text{(from (20).)} \]
Hence
\[ D^{-1}(a) = \frac{\lambda v}{k} (I_t - \frac{1}{t+1} J_t) . \]
Then
\[ \Delta(a) = \frac{\lambda v}{k} \Phi'_l(a) (I_t - \frac{1}{t+1} J_t) \Phi(a) . \]
For a symmetrical form of expression, we take
\[ \Phi'_l(a) = (I_t, -J_t)^{\frac{1}{2}} \xi(a) , \]
where \( \xi_{i,j}(a) = t_{i,j} - \frac{1}{t+1} [t_{i_1} + t_{i_2} + \cdots + t_{i_t} + t_v] \)
so that
\[ \Delta(a) = \frac{\lambda v}{k} \xi_{i}^{\frac{1}{2}}(a) (I_{t+1} - \frac{1}{t+1} J_{t+1}) \xi(a) \]
\[ = \frac{\lambda v}{k} \xi_{i}^{\frac{1}{2}}(a) \xi(a) \quad \text{(since } J_{t+1} \xi(a) = 0) \]
\[ = \frac{\lambda v}{k} \left[ \sum_{j=1}^{t} \xi_{i,j}^2(a) + \xi_v^2(a) \right] . \]

ii) In the multivariate situation we shall have the confidence bounds (16) with \( n = bk, r = b + v - 1, s = v - 1 \) and
\[ \Delta = C_{\max} \left[ \Phi' \frac{\lambda v}{k} (I_{v-1} - \frac{1}{v} J_{v-1}) \Phi \right] . \]
Here again we may write \( \Phi = (I_{v-1}, -J_{v-1,1}) \xi \), where \( \xi = (\xi^{(1)}, \ldots, \xi^{(p)}) \)
and \( \xi_{i}^{(j)} = t_i^{(j)} - \frac{1}{v} \sum_{l=1}^{v} t_{i,l}^{(j)} . \) Then \( \Delta = \frac{\lambda v}{k} C_{\max}(\xi' \xi) . \)
We shall have one set of "partial" statements given by (17) with
\[ \Delta_a = \frac{\lambda \nu}{k} C_{\max} \left[ \Phi'_{(a)} \left( I_t - \frac{1}{t+1} J_{t+1} \Phi_{(a)} \right) \right] \text{ (from (24))} \]
or (equivalently) \[ \Delta_a = \frac{\lambda \nu}{k} C_{\max} \left[ \xi'_{(a)} \xi_{(a)} \right] \text{ (from (25)),} \]
where \[ \xi_{(a)} = \left( \xi^{(1)}_{(a)}, \xi^{(2)}_{(a)}, \ldots, \xi^{(p)}_{(a)} \right). \]

Similarly, we shall have another set of "partial" statements given by (18) with
\[ \Delta_b = C_{\max} \left[ \Phi'_{(b)} \left( I_{v-1} - \frac{1}{v} J_{v-1} \Phi_{(b)} \right) \right] \text{ (from (21))} \]
\[ = \frac{\lambda \nu}{k} C_{\max} (\xi'_{(b)} \xi_{(b)}). \]

As mentioned earlier, we may have "combined partial" statements.

4. Partially balanced incomplete block designs

i) Let us consider a PBIBD with m associate classes and association matrices \( B_i \) (\( i = 0,1,\ldots,m \)). Then it is well known [1] that
\[ C = \sum_{k=0}^{m} \alpha_k B_k, \text{ where } B_0 = I_v, \]
\[ \alpha_0 = \frac{r(k-1)}{k}, \quad \alpha_i = -\frac{\lambda_1}{k} \quad i = 1,2,\ldots,m; \]
and on imposing the condition \( J_{1v} t = 0 \) on (9) we have
\[ t = \left( \sum_{k=0}^{m} \alpha_k B_k \right) Q = EQ, \text{ say.} \]

It is well known that, when the design is connected, \( \text{Rank } C = v - 1 \), so that the condition \( J_{1v} t = 0 \) is sufficient to give unique solutions. We have further
(26) \[ J_{1v} \mathbb{C} = 0 \quad \text{and} \quad J_{1v} \mathbb{Q} = 0 \ . \]

Let \[ \mathbb{C} = \begin{bmatrix} C_1 \\ C' \end{bmatrix} \quad \text{and} \quad (\mathbb{C}^t, \mathbb{Q}) = \begin{bmatrix} C_1^t & d \\ C' & C_0 \end{bmatrix} \ . \]

and \[ \mathbb{E} = \begin{bmatrix} E_1 \\ e' \end{bmatrix} \quad \text{and} \quad (\mathbb{E}^t, \mathbb{e}) = \begin{bmatrix} E_1^t & f \\ e' & e_0 \end{bmatrix} \ . \]

Then \[ \begin{bmatrix} C_1 \\ J_{1v} \end{bmatrix} t = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \quad \text{(where} \quad Q = \begin{bmatrix} Q_1 \\ Q_v \end{bmatrix} \text{)} \quad \Rightarrow \quad t = \mathbb{E}Q = (\mathbb{E}^t, \mathbb{e}) \begin{bmatrix} Q_1 \\ Q_v \end{bmatrix} = \mathbb{E}^t Q_1 + e Q_v \ . \]

But \[ Q_v = -J_{1,v-1} Q_1 \quad \text{in view of} \quad (26) \ . \]

Therefore, \[ t = \mathbb{E}^t Q_1 - e J_{1,v-1} Q_1 = (\mathbb{E}^t - e J_{1,v-1}) Q_1 \]
\[ = (\mathbb{E}^t - e J_{1,v-1} : x) \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \ . \]

Hence \[ \begin{bmatrix} C_1 \\ J_{1v} \end{bmatrix}^{-1} = (\mathbb{E}^t - e J_{1,v-1} : x) \ . \]

whence \[ (\mathbb{E}^t - e J_{1,v-1} : x) \begin{bmatrix} C_1 \\ J_{1v} \end{bmatrix} = \mathbb{I}_v \ . \]

Thus \[ \mathbb{E}^t C_1 - e J_{1,v-1} C_1 + x J_{1v} = \mathbb{I}_v \ . \]

But \[ (26) \quad \Rightarrow \quad J_{1,v-1} C_1 = -C' \ . \]

Hence \[ \mathbb{E}^t C_1 + e C' = \mathbb{I}_v - x J_{1v} \ , \]

that is \[ (28) \quad \mathbb{E} C = \mathbb{I}_v - x J_{1v} \ . \]
Also \( (27) \implies \begin{bmatrix} C_1 \\ J_{1v} \end{bmatrix} \begin{bmatrix} E_1' - e J_{1,v-1} \\ x \end{bmatrix} = I_v. \)

Hence \( C_1 x = 0. \) But \( C_1 J_{1v} = 0 \) and \( \text{Rank } C_1 = v - 1. \) Therefore, 
\( x = x J_{1v} \). Furthermore, \( J_{1v} x = 1 \), whence \( x J_{1v} J_{1v} = 1 \), that is, 
\( x = \frac{1}{v}. \) (28) thus reduces to 
\( (29) \quad E C = I_v - \frac{1}{v} J_v. \)

Now \( \hat{Q} = (I_{v-1}, -J_{v-1,1}) E = (I_{v-1}, -J_{v-1,1}) E \quad Q \implies 
D = (I_{v-1}, -J_{v-1,1}) E C \quad E \begin{bmatrix} I_{v-1} \\ -J_{1,v-1} \end{bmatrix}. \)

Therefore,
\[
D = (I_{v-1}, -J_{v-1,1})(I_v - \frac{1}{v} J_v) E \begin{bmatrix} I_{v-1} \\ -J_{1,v-1} \end{bmatrix} \quad \text{(from (29))}
\]
\[
= (I_{v-1}, -J_{v-1,1})(I_v - \frac{1}{v} J_v)(E_1' - e J_{1,v-1})
\]
\[
= (I_{v-1}, -J_{v-1,1})(E_1' - e J_{1,v-1}) \quad \text{(from (27))}
\]
\[
(30) \quad E_{11} - f J_{1,v-1} - J_{v-1,1} f' + e_0 J_{v-1}.
\]

Furthermore, premultiplying by \( (I_{v-1}, -J_{v-1,1}) \) both sides of the equation 
\[
\begin{bmatrix} E_{11} - f J_{1,v-1} \\ f' - e_0 J_{1,v-1} \end{bmatrix} \begin{bmatrix} C_{11} \\ d \end{bmatrix} = I_v,
\]
we have 
\[
D C_{11} = I_{v-1} \quad \text{and, therefore,}
\]
\[
(31) \quad D^{-1} = C_{11}.
\]

Hence
(32) \[ \Delta = \Phi' C_{11} \Phi = \xi' C \xi \] (from (11)).

Here, we may note that \( C_{ii} = \alpha_0 = \frac{r(k-1)}{k} \) and \( C_{ij} = \alpha_p = -\frac{\lambda_p}{k} \) if i-th and j-th treatments are \( l \)-th associates. Then we can have a confidence statement (5) with \( n = bk, r = b + v - 1, s = v - 1 \) and \( \Delta \) given by (32).

The "partial" statements (6), however, cannot be made in a compact form, unless we know the association scheme. For example, if we have \( \Phi'(a) = (\Phi_1, \Phi_2, \ldots, \Phi_t) \), then

\[
\Omega_{(a)}^{-1} = X - YW^{-1}Z ,
\]

where \( C_{11} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \) and

\[
C_{ij} = \begin{cases} X & \text{if } i = j \text{ and } 1 \leq i, j \leq t \\ Z & \text{if } i = j \text{ and } t + 1 \leq i, j \leq t + v - 1 \\ W & \text{otherwise} \end{cases}
\]

and thus

\[
\Delta_{(a)} = \Phi'(a)[X - YW^{-1}Z]\Phi'(a) .
\]

ii) In the multivariate situation we shall have the confidence bounds (16) with \( n = bk, r = b + v - 1, s = v - 1 \) and

\[
\Delta = \max[C_{11} \Phi' \Phi] = \max[\xi' C \xi] .
\]

We shall have, as before, one set of "partial" statements given by (17) with

\[
\Delta_{(a)} = \max[\Phi'(a)[X - YW^{-1}Z]\Phi'(a)] .
\]

The other set of "partial" statements given by (18) will be with

\[
\Delta_{(b)} = \max[\Phi'(b) C_{11} \Phi'(b)] = \max[\xi'(b) C \xi'(b)] .
\]

As before, we may have the "combined partial" statements.
5. General "connected" incomplete block designs

It is well known [1] that, in general, we have $C_t = Q$, which, on imposing the condition $J_{1v} t = 0$, yields $t = EQ$. Then, arguing as before, from (26) to (32), we have

$$\Delta = \Phi' C_{11} \Phi = \xi' C \xi.$$

Then we can have a confidence statement (5) with $n = \sum r_i = \sum k_j$, $v = b + v - 1$, $s = v - 1$ and $\Delta$ given by (33). We can have "partial" statements and confidence bounds in the multivariate situation analogous to those for PBIBD.

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References

