The Use of Orthogonal Polynomial Contrasts in the Confounding of Factorial Experiments

by

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Errata

The Use of Orthogonal Polynomial Contrasts in the Confounding of Factorial Experiments

Page 7, line 4: for $s^{n/k}$ read $s^{n/k}$.

Page 10, line 13: for electrotonic read electronic.

Page 16, line 13: for at read of.

Page 19, line 3 of second paragraph: for very read vary.

Page 23, third line from bottom: for $(J'_-, J'_+)$ read $(J'_-, J'_+)$.  

Page 25, line 1: for $(J'_-, J'_+)$ read $(J'_-, J'_+)$.  

Page 29, equation (3.2): for $(A_{BC}, r s t)$ read $(A_{BC}, r s t)$.  

Page 30, equation (3.3): for $(A_{BC}, r s t)$ read $(A_{BC}, r s t)$.  

Page 32, equation (3.3), fourth line: for $(r b s -)$ read $(r b s -)$.  

Page 37, third line from bottom: for $a_{r+}$ read $a_{r+}$.  

Page 39, line 15: for were read where.

Page 43, first line of Theorem 4.1: delete $T$ in $(A_{BC}, r s t, A_{BC}, h)$.  

Page 47, line 3: for $P_{ri}$ read $P_{ri}$.  

Page 48, $N$ even, $P_2$: for $3(N^2 - 4)$ read $|3(N^2 - 4)|$.  

Page 50, third line from bottom: for $(P_2, P_4, ...)$ read $P_2, P_4, ...$.  

Page 54, third line from bottom: for Hence read Hence.

Page 57, title of Table 8a: for Blocking read Blocking.

Page 58, Table 8b, last column: for $sE_{ao}$ read $-sE_{ao}$.  

Page 65, line 11: for orthogonal read orthogonal.

Page 70, line 11: for $J'a b c J$ read $J'a b c J$.  

Page 89, line 6: for $a_{fi g j}$ read $(a_{fi g j})$.  

Page 90, Table 21, fourth line: for $n_{++} + n_{+-}$ read $n_{++} + n_{--}$.  

Page 91, line 8: for $P_{h+} = P_{h-}$ read $P_{h+} = P_{h-}$.  

Page 92, equation (8.8): for $(r_{g+} - r_{g-})$ read $(n_{g+} - n_{g-})$.  

Page 102, sixth line from bottom: for $A_2 B_2 C_1$ read $A_2 B_2 C_1$.  


CHAPTER I

Introduction and Review of Literature

1.1 Introduction

Factorial experiments are used in experimental situations that require the examination of the effects of varying two or more factors. The various representatives of each factor are designated as levels, regardless of whether they are continuous or discrete. In the complete exploration of such a situation all combinations of the different factor levels must be examined in order to elucidate the effect of each factor and the possible ways in which each factor may be modified by the variation of others.

When a factor is investigated at two levels, its main effect is uniquely defined as the difference between the mean of the results at the higher level and the mean of the results at the lower level. Its main effect can thus be expressed as a unique contrast between the higher level mean and the lower level mean. Interaction effects between two or more factors, each at two levels, are also uniquely defined by similar contrasts. Thus if, in a $2^3$ factorial experiment, the factors are denoted by $A$, $B$ and $C$ and their levels by $a_0$, $a_1$; $b_0$, $b_1$; $c_0$, $c_1$ respectively, where the subscript 0 refers to the lower level, and if $y_{ijk}$ denotes the response to the treatment combination $a_i b_j c_k$, the contrasts can be presented as follows:
It will be observed that any two of the contrasts are orthogonal; i.e., the sum of products of their corresponding coefficients is zero. This property of the contrasts makes for easy analysis, and ensures that all the main effects and interactions can be independently estimated. When the number of factors increases, the complete factorial design may become too large to be accommodated under uniform experimental conditions. The totality of treatment combinations is then subdivided into smaller groups, each group being assigned to a given block of experimental units, the grouping being made so that block effects will not affect the main effects of the factors and those interactions regarded as being of importance. This process is termed "confounding."

Confounding of a contrast in the $2^3$ series is accomplished by placing these treatment combinations corresponding to the positive coefficients in one block and those corresponding to the negative coefficients in another block. These parts of the contrast are now non-estimable from the data,
and we say that the contrast is completely confounded with block effects, or in the usual parlance, with blocks.

Even though the device of confounding may take care of the heterogeneous material or conditions, with a large number of factors the total number of observations may become prohibitive. Under certain conditions it is then possible to examine the main effects of the factors and their more important interactions in only a fraction of the number of treatment combinations required for the complete factorial experiment. This type of design is known as a fractional factorial and is always equivalent to one block in a system of confounding. In the \(2^3\) experiment, for instance, the block containing the treatment combinations corresponding to either the positive signs of ABC or to the negative signs of ABC would be a one-half replicate.

With factors at three or more levels, their main effects are no longer uniquely expressible as contrasts between the different levels. It is well-known that the infinite number of contrasts that can be set up between \(N\) observations can be divided into an infinite number of subsets consisting of \((N-1)\) orthogonal contrasts, with the property that the sum of squares of each of the contrasts in the subset add up to the total sum of squares of the observations, corrected for the mean. In this thesis exclusive attention will be paid to the subset of orthogonal contrasts based on the orthogonal polynomial values tabulated by Fisher and Yates (1938) and other authors. Apart from satisfying the orthogonality criteria, these contrasts have useful interpretational value for an experiment involving factors with continuously varying levels; these being equally spaced in the experiment. The tables of the orthogonal polynomial values have been set up in such a way that any value is either a positive whole number, a negative whole number, or a zero.
If a particular contrast is to be confounded with blocks and if it contains no zero elements, a blocking procedure identical to the one for the $2^n$ experiment is applicable. If the contrast contains zero elements, a third block is obtained by placing those treatment combinations corresponding to the zero elements in yet another block.

In general, when the number of levels exceed two, complete confounding does not occur with orthogonal polynomial contrasts. Even though the treatment combinations corresponding to the negative coefficients of, say the linear contrast, are put in one block and those corresponding to the positive coefficients are put in a second block, it is possible to estimate a linear contrast in each block based, of course, on only one half the number of observations. These two estimates can then be pooled to give a single estimate of the linear effect, adjusted for blocks. If there are no block effects, the variance of the adjusted linear contrast will be much larger than one not adjusted for blocks, because of the restricted range of the independent variable in each block. The actual efficiency of the adjusted estimate will depend on the reduction of error variance due to blocking.

A complete discussion of partial confounding in this sense should include a method of estimating all pertinent contrasts and their standard errors. This thesis, however, will be concerned only with the blocking procedures.

Although use has been made of orthogonal contrasts to obtain confounded designs, in other than the $2^n$ series, no systematic study appears to have been made either on the theory of general orthogonal single contrasts, or on the special system of orthogonal polynomial contrasts.
Whatever number of levels the factors may have, in many experimental situations practical interest is mainly centered on the estimation of the main effects and the lower order interaction contrasts, especially those of second degree (linear x linear). One reason for this interest is the difficulty of interpreting higher degree interaction contrasts. Most designs appearing in the literature were developed from the point of view of obtaining balanced sets of confounded blocks. It therefore appears to be of interest to investigate the possibility of obtaining designs with the more limited objective in mind. Considering the m x n x p experiment from this point of view, one would be satisfied if a confounded design could be found in which the main effects and most, if not all, of the linear by linear interaction contrasts were unconfounded or only slightly confounded. Simple analyses and relatively precise estimates of these contrasts can be obtained from a balanced set of replicates, but in many practical situations the total number of observations thus required would be prohibitive.

With this background the objectives of this thesis may be summarized as follows: (i) To investigate the theory of confounding general orthogonal contrasts; (ii) to establish mathematical procedures and rules to obtain the confounding patterns of higher order interaction contrasts, based on the orthogonal polynomial system.

1.2 Review of Literature

Prior to 1926, when Fisher (1926) first suggested confounding, the factorial experiment was known as the "complex" experiment; indeed, after many years it was still referred to as the complex experiment. This type of experiment had been used by the Rothamsted Experimental Station on wheat experiments at Broadbalk since 1843 and on barley experiments at Hoosfield since 1852; the randomization element, however, was lacking in
these early years. Fisher and Wishart (1930) gave a detailed explanation of a confounded experiment, and Yates (1933) discussed the principles of confounding, giving different types of confounding and setting out methods of analysis. Later, Yates (1935) gave some more illustrations of confounded designs and discussed their relative efficiency compared with randomized complete blocks.

Ever since the first appearance of the book "The Design of Experiment" by Fisher (1935) and the monograph "The Design and Analysis of Factorial Experiments" by Yates (1937), experimenters have used factorial designs in a great variety of situations in many fields of research. This has led to a voluminous literature on all aspects of the subject. Mathematical statisticians became interested and found new explanations for existing designs in terms of the results of combinatorial mathematics. The utilization of these results, in turn, led to the development of new designs. Special mention will be made of the work of Bose and his co-workers who gave the complete solution of the symmetrical factorial design.

Recently Binet et al (1955) re-examined most of the factorial plans published to determine what effect the confounding had on the orthogonal polynomial contrasts; in recent years these contrasts were found to be important in certain fields of chemical and industrial research. The authors also presented new single-replicate plans, some of which were obtained by confounding of the higher order high degree orthogonal polynomial contrasts. Anderson (1957) reviews the whole field of complete factorials, confound and fractional factorials and explains the interrelationship of these subjects. The present thesis is an attempt to formalize some of the techniques of Binet and his co-workers, and can hence be regarded as providing the theoretical background for some of their practical findings.
1.2.1. **Symmetrical factorial designs**

A factorial design is symmetric when each of the \( n \) factors is at \( s \) levels, where \( s = p^m \) and \( p \) is a prime number. Confounding occurs if each complete replication is allocated to \( b = s^{n/k} \) blocks of \( k \) plots each, where \( k \) divides \( s^n \). The problem that arises is how to allocate the treatment combinations to the various blocks in such a way that only certain desired treatment contrasts become non-estimable. Bose and Kishen (1940) and Bose (1947, 1950) showed that by identifying the \( s \) levels with the elements of a Galois field \( GF(p^m) \), any treatment combination can be represented by a point in finite Euclidean space \( E_0(n, p^m) \). If any linear homogeneous form \( U = a_1x_1 + a_2x_2 + \cdots + a_nx_n \) is considered, where the \( a_i \)'s belong to \( GF(p^m) \) then \( U = c \), where \( c \) is a constant, will represent a flat space of \( (n-1) \) dimensions. If the constant varies over all the elements, \( U = c \) represents a pencil \( P(a_1, a_2, \cdots, a_n) \) of parallel \( (n-1) \)-flats. To each flat of the pencil corresponds a set of \( (n-1) \) treatment combinations, and the contrasts between these \( s \) sets carry \( (s-1) \) degrees of freedom.

Thus the \( (s^n-1) \) degrees of freedom can be split into \( (s^n-1)/(s-1) \) sets of \( (s-1) \) degrees of freedom each, such that each set is carried by one parallel pencil, and the degrees of freedom belonging to different sets are orthogonal. If particular coordinates of the pencil \( P(a_1, a_2, \cdots, a_n) \) are nonzero, then the \( (s-1) \) degrees of freedom carried by that pencil belong to that particular \( (t-1) \)th order interaction. Thus if only one coordinate of the pencil is nonzero, the \( (s-1) \) degrees of freedom will belong to that particular main effect.

In practice these pencils can be obtained by means of the orthogonal Latin squares, since Bose (1938) showed that a Latin square of prime order
can be regarded as the addition table of the roots of the cyclotomic polynomial, these roots being the elements of a Galois field.

It follows that when \( s = 2 \), and the experiment is confounded into two equal blocks, a single contrast only will become nonestimable. When, however, two contrasts of the \( 2^n \) experiment are confounded, Barnard (1936) showed that this implies the automatic confounding of another contrast, the generalized interaction. He also gave various systems of confounding, using factors up to 6 in number.

As far as symmetrical experiments are concerned, Yates (1937), in his comprehensive monograph, gives designs, analyses and examples for various confoundings of the \( 2^n \), \( 3^n \) and \( 4^n \) experiments. When \( s = 3 \), he makes use of the combinatorial properties of the orthogonal latin squares to define his orthogonal components \( I \) and \( J \), each with two degrees of freedom. In terms of Bose's development these components correspond to the pencils of parallel 2-flats. For \( s = 4 \), Yates derived confounded experiments by defining single degree of freedom contrasts, based on two pseudo-factors of two levels each, and blocking according to signs—similarly for \( s = 8 \). He also suggested a routine of calculation for experiments with several factors at two levels each, this depending only upon a succession of additions and subtractions of pairs of numbers.

Nair (1938, 1940) discussed balanced confounded arrangements for the \( 4^n \) and \( 5^n \) experiments. Fisher (1942) established the connection between the theory of finite Abelian groups and the relations recognizable in the choice of interactions for confounding contrasts of the \( 2^n \) experiment. He showed that, using blocks of \( 2^r \) plots, it was possible to test all combinations of as many as \((2^r-1)\) factors in such a way that all interactions confounded shall involve not less than three factors each. He also
supplied a catalog of systems of confounding available up to 15 factors, each at two levels. Fisher (1945) generalized this proposition to factors having \( s = p^m \) levels, where \( p \) is a prime number.

Kempthorne (1947) introduced a new notation for the different sets of degrees of freedom in the \( s^n \) experiment to simplify the enumeration and choice of systems of confounding. Comparing his notation with that of Yates and Bose, we have, for instance, for the \( 3^2 \) experiment, the following components, each with two degrees of freedom:

<table>
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<th>Yates</th>
<th>Kempthorne</th>
<th>Bose</th>
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<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>( P(1,0) )</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>( P(0,1) )</td>
</tr>
<tr>
<td>AB</td>
<td>AB(I)</td>
<td>AB</td>
<td>( P(1,1) )</td>
</tr>
<tr>
<td>AB</td>
<td>AB(J)</td>
<td>AB^2</td>
<td>( P(1,2) )</td>
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Cochran and Cox (1950) survey the field of factorial experiments mainly from an applied point of view, discuss detailed examples, give complete plans of those experiments most likely to be generally used in practice together with details on the confounding and the relative information. In his book Kempthorne (1952) devotes considerable space to a discussion of the methods of confounding, the statistical models, methods of estimation and the assumptions involved in the various confounded designs he presents. Binet et al (1955) used the Bainbridge extension of Yates' (1937) technique to examine the effects of the confounding systems proposed by the previous authors on the orthogonal polynomial contrasts.

1.2.2. Asymmetrical factorial designs

A factorial design is asymmetric when the \( n \) factors are respectively at \( s_1, s_2, \ldots, s_n \) levels, where \( s_1, s_2, \ldots, s_n \) are not all equal to one another. Since Galois fields do not exist for products of prime numbers, no single
mathematical system is available to deal with the confounding of the asymmetrical design. Of course, if the levels of a certain number of factors are the same prime number or power of a prime number, the general theory can be applied to that part of the experiment. Thus in a $5 \times 3^2$ experiment the $3^2$ part can be confounded as usual. In general, however, the single replicates of such confounded designs are not very satisfactory from a practical point of view, since low order interactions will be confounded. This difficulty can be overcome by obtaining balanced sets of replications, but such a procedure usually requires a triplication or more of the total number of observations.

With a general mathematical theory lacking, much of the success in obtaining satisfactory confounded designs will depend on the ingenuity of the designer. With the availability of electrotonic calculators, it is conceivable that good practical designs may be obtained mechanically.

As far as asymmetric designs consisting of combinations of smaller number of factors of 2, 3 and 4 levels are concerned, it would appear that Yates (1937) and Li (1944) have supplied most of those likely to be required in practice. The designs presented by these authors are balanced with respect to the partial confounding of interactions, and more than one replicate will usually be required.

When the number of levels of each factor is either 2, 4 or 8, Yates (1937) employed the signs of contrasts with single degrees of freedom as a basis for confounding. Consider the case of a factor A with four levels. Since $4 = 2^2$, the factor A can be regarded as consisting of two pseudo-factors X and Y, each at two levels, and hence the pseudo-effects are:
\[ X = (x-1)(y+1) = xy + x - y - 1 \]
\[ Y = (x+1)(y-1) = xy - x + y - 1 \]
\[ XY = (x-1)(y-1) = xy - x - y + 1 \]

Let \( a_0 = 1 \), \( a_1 = x \), \( a_2 = y \), \( a_3 = xy \). Then

\[ A' = a_3 + a_2 - a_1 - a_0 = Y \]
\[ A'' = a_3 - a_2 - a_1 + a_0 = XY \]
\[ A''' = a_3 - a_2 + a_1 - a_0 = X \]

and it will be seen that

\[ A' A'' = A''' , \quad A' A''' = A'' , \quad A'' A''' = A' \]

If the levels are equally spaced and if \( A_1, A_2 \) and \( A_3 \) denote the linear, quadratic and cubic orthogonal polynomial contrasts, then

\[ A_1 = 2A' + A''' \]
\[ A_2 = A'' \]
\[ A_3 = -A' + 2A''' \]

For confounding purposes it should be noted that there is a one-to-one correspondence between the signs of \( A' \) and \( A_1, A'' \) and \( A_2, A''' \) and \( A_3 \). Hence, since the confounding of any contrast proceeds on the basis of the signs only, identical blocking arrangements will result with both systems of contrasts.

Employing Yates' techniques, Li (1944) constructed confounded designs for 10 types of the asymmetrical factorial experiment with the purpose of filling some of the gaps existing among the plans previously available. Nair and Rao (1948) developed a set of sufficient combinatorial conditions for the balanced confounded designs, and discussed the
requirements for optimum designs. Thompson and Dick (1950) showed how to obtain factorial experiments in blocks of equal size, equal to or less than the number of levels in the leading factor from the orthogonal Latin squares. Thompson (1952) presented the layouts and details of statistical analysis for two and three-factor designs, the number of plots per block in no case exceeding five.

Binet et al (1955) re-examined the asymmetric designs proposed by Yates and Li for the effects of the confounding on the orthogonal polynomial contrasts. Detailed tables are presented, showing which orthogonal polynomial contrasts are confounded, and the size of the coefficients of the constants of the block contrasts appearing in the expected values of the treatment contrasts. Using either single orthogonal polynomial contrasts of high order interaction, or the methods of Yates, as a basis for confounding, they present new single-replicate designs with confounding patterns. Their investigation is entirely of a practical nature. The present thesis supplies the theory of confounding a single orthogonal polynomial contrast and the simultaneous confounding of two or more polynomial contrasts.

1.2.3 Fractional Factorial Experiments

Referring to the mathematical summary at the beginning of Section 1.2.1, fractionalization of a factorial experiment will occur when instead of taking all the \( b \) blocks, we take only \( B \) blocks where \( B \) is less than \( b \). We then have a \( B/b \) fraction of a complete replicate. Bose (1950) summarized the mathematical procedures to obtain fractions of factorial experiments, when the number of levels is either a prime number or a power of a prime number. He showed that contrasts are no longer separately estimable, but that the sum of the contrasts belonging to any
complete alias set is estimable. He pointed out, however, that if all contrasts of any alias set with the exception of one contrast are of sufficiently high order to be negligible, then this one contrast is estimable. Bose and his co-workers have recently expanded the original theory and developed a whole series of new designs; these are to be published by the National Bureau of Standards.

Finney (1945) originally developed the principles of fractional replication with particular reference to the $2^N$ and $3^N$ factorial experiments in terms of finite Abelian groups. Finney (1946) presented a popular exposition of these ideas, together with some plans and examples; and Chinloy, Innes and Finney (1953) gave an interesting example of the use of a one-third replicate of a $3^5$ experiment on sugar cane manuring. An interesting generalization of Yates' (1937) routine of calculation was included in this paper.

Plackett and Burman (1946) discussed what they call optimum multi-factorial experiments which Kempthorne (1947) showed to be fractionally replicated designs. In his book, Kempthorne (1952) treats the general case of fractional replication, and describes a one-ninth replicate of a $3^6$ design applied to measurements on the consistency of types of canned food. Brownlee, Kelly and Loraine (1948) enumerated subgroups suitable for high-order fractional replication for $2^N$ designs of practical interest. Davies and Hay (1950) gave special attention to the use of fractional factorial designs in sequence with special reference to industrial experiments. Kitagawa and Mitome (1953) and Davies et al (1954) compiled plans for fractional designs of the $2^N$ and $3^N$ series. Daniel (1956) discussed fractional replication in industrial research.
Box and his co-workers [see Box (1952, 1954), Box and Wilson (1951),
Box and Hunter (1954), Box and Youle (1955)] have recently opened new
fields in which complete and fractional factorial experiments are fruit-
fully employed. Their work is devoted to the problem of determining opti-
mal factor combinations in chemical and industrial research, and to describe
the response surface in the neighborhood of this optimum. These procedures
are sequential in nature and have been described and reviewed by Anderson
(1953) and Davies et al (1954).

No work, as such, has been done on the fractionalization of the
asymmetrical factorial experiment.

1.2.4 **Orthogonal polynomials**

The orthogonal polynomials used in statistics are discrete cases of
the well-known Legendre polynomials in mathematics. The latter are usually
defined between the limits of -1 and 1 by the formula of Rodrigues

\[ P_0(x) = 1, \quad P_r(x) = \frac{1}{2^r r!} \frac{d^r}{dx^r}(x^2-1)^r \]

Using results from the finite calculus, one can show that the discrete
orthogonal polynomials for n observations between the limits a and b
can be written as

\[ P_0(x) = 1, \quad P_r(x) = c_r D^r_r(x-a)(x-b) \]

where \( D^r_r \) denotes the r\textsuperscript{th} advancing difference, and \( c_r \) is an arbitrary
constant.

The orthogonal polynomial systems proposed by various authors for
the equi-distant case differ only in the choice of the arbitrary constant
\( c_r \), this choice being dependent on the purpose the author had in mind.
Tchebycheff (1859) first gave the theoretical basis of the orthogonal polynomials of least squares. He treats the nonequidistant as well as the equidistant case, and for the latter derived a reduction formula. Poincaré (1896) developed similar polynomials for the nonequidistant case. Gram (1915) developed orthogonal polynomials for the purpose of smoothing empirical curves. Jordan (1929, 1932) simplified Tchebycheff's methods for practical application, presented tables and explained the method of successive summation of the observations. He chose the arbitrary constant in such a way that $SS_P(x) = n$, where SS denotes the sum of squares.

In the system proposed by Esscher (1920) the value of $c_r$ was determined by the convention that the coefficient of $x^r$ shall be unity. In his later system Esscher (1930) chose the same convention as Jordan. A similar choice of arbitrary constant was made by Lorenz (1931), who developed his system of orthogonal polynomials by means of determinants.

Fisher (1921) derived the polynomials in which $x$ is measured from the mean and adopted the convention that the coefficient of $x^r$ shall be unity. Allan (1930) derived the general expression of this polynomial. In his studies on the regression integral, Fisher (1924) presented orthogonal polynomials such that $SS_P(x)$ equaled unity. Fisher and Yates (1938) present standard tables of the polynomial values for $n$ through 52, and $r$ through 5. Anderson and Houseman (1942) extended these tables to $n$ through 104. DeLury (1950) presents values and integrals of these polynomials up to $n = 26$ for all degrees.

Aitken (1933), by deriving new forms for the orthogonal polynomials showed in a remarkably lucid way the relationship between the theory of interpolation and the theory of orthogonal polynomial curve fitting. The choice of the arbitrary constant equal to unity results in obtaining
integers throughout the standard tables presented by Van der Reyden (1943). for \( n \) through 52 and \( r \) through 9. Pearson and Hartley (1954) reproduced part of these tables in slightly modified form for \( n \) through 52 and \( r \) through 6.

Due to the arbitrary nature of the constant \( c_r \), it follows that if the greatest common factor is removed from the numerical values of a polynomial in any system described above, the resulting figures would be identical for all systems.

1.3 Notation To Be Used

A factor and contrast
a element of coefficient matrix of factor A
B factor and contrast
b element of coefficient matrix at factor B
C factor and contrast
c element of coefficient matrix of factor C
D diagonal matrix
E or e even

\( F_r \) vector of \( \text{tr} \ a_r \); \( F_R \) vector of \( \text{tr} \ a_R \\
\( f \) subscript of A contrast
\( G_{s+} \) matrix of \( \text{tr} \ b_s \); \( G_S \) matrix of \( \text{tr} \ b_S \\
g subscript of B contrast
\( H_t \) vector of \( \text{tr} \ c_t \); \( H_T \) vector of \( \text{tr} \ c_T \\
h subscript of C contrast; polynomial coefficient, Section 5.1

I identity matrix

i level of factor A
J vector

j level of factor B
K  confounded
k  level of factor C; polynomial coefficient, Section 5.1.
L  linear combination
M  |P|
m  number of levels of factor A
N  total number of yields
n  number of levels of factor B
O  zero
P  polynomial
p  number of levels of factor C
Q  N/2 or (N-1)/2
R  subscript of A contrast
r  subscript of A contrast; degree of polynomial
S  subscript of B contrast
s  subscript of B contrast
T  subscript of C contrast
t  subscript of C contrast
U or u  odd
v  subscript of contrast A (generalized interaction)
w  subscript of contrast B (generalized interaction)
X  level of general treatment
x  times; X-\bar{X}
y  yield
z  subscript of contrast C (generalized interaction)
\alpha  sign a_r; sign P_{ri} coefficient, Section 7.2
\beta  sign b_s; sign P_{ri} coefficient, Section 7.2
\gamma  sign c_t
\delta  sign a_f
\[ \varepsilon \quad \text{sign } b \]

\[ \gamma \quad \text{sign } c \]

\[ \gamma = \alpha \delta \]

\[ x = \beta \varepsilon \]

\[ \gamma' = \gamma \eta \]

\[ \lambda \quad \text{polynomial multiplier} \]
PART I. THE GENERAL CASE OF ORTHOGONAL CONTRASTS

CHAPTER 2

Matrix Notation

Let \( y_{ijk} \) be the yield resulting from the application of the treatment combinations consisting of the \( i^{th} \) level of the factor A with \( m \) levels, the \( j^{th} \) level of the factor B with \( n \) levels, and the \( k^{th} \) level of the factor C with \( p \) levels.

Arrange these yields into a \((mnp \times 1)\) column vector, in which the \( y_{ijk} \) are ordered in the following special manner: first put both the subscripts \( i \) and \( j \) equal to zero and vary the subscript \( k \) from 0 to \((p-1)\) to obtain \( p \) combinations; then, keeping the subscript \( i \) fixed at 0, vary the subscript \( j \) from 1 to \((n-1)\) for each of the \( p \) variations of \( k \); finally, vary the subscript \( i \) from 1 to \((m-1)\) for all variations of the subscripts \( j \) and \( k \). In transposed form the subscripts of the vector \( y \) will thus read as

\[
\begin{align*}
000,001,\ldots,00(p-1); & \quad 010,011,\ldots,01(p-1); \quad 020,021,\ldots,02(p-1); \quad \ldots,0(n-1)0, \\
0(n-1)1,\ldots,0(n-1)(p-1); & \quad 100,101,\ldots,10(p-1); \quad 110,111,\ldots,11(p-1); \quad 120,121, \\
\ldots,12(p-1); & \quad \ldots,1(n-1)0,\ldots,1(n-1)(p-1); \quad \ldots,(m-1)(n-1)0,(m-1)(n-1)1,\ldots, \\
(m-1)(n-1)(p-1) & 
\end{align*}
\]

If \( I \) denotes the identity matrix, "tr" the trace (the sum of the diagonal elements), and \( SS_{tr} = np(a_{ro}^2 + a_{rl}^2 + \ldots + a_{r,m-1}^2) \), let \( a_{tr} = a_{tr}(mnp \times mnp) \) be the diagonal matrix
\[
\begin{bmatrix}
a_{ro}I(n_p \times np) & 0 \\
\vdots & \vdots \\
a_{rl}I(n_p \times np) & 0 \\
0 & a_{r,m-1}I(n_p \times np)
\end{bmatrix}, \quad r=1,2,\ldots,m-1
\]

such that \( \text{tr } a_r = 0 \), and \( \text{tr } a_r a_r = 0 \) when \( r \neq f \) and \( \text{tr } a_r a_r = SSa_r \) when \( r = f \). If \( D \) denotes a diagonal matrix, let

\[
a_r = D(a_{ro}, a_{rl}, \ldots, a_{r,m-1}) \quad \text{and} \quad SSa_r = (a_{ro}^2 + a_{rl}^2 + \cdots + a_{r,m-1}^2)
\]

then \( \text{tr } a_r = np(\text{tr } a_r) \) and \( \text{tr } a_r a_r \) will equal zero when \( r \) is not equal to \( f \), and will equal \( SSa_r \) when \( r \) equals \( f \), where \( SSa_r = np(SSa_r) \).

Let \( b_s = b_s(n_p \times n_p) \) denote the diagonal matrix

\[
\begin{bmatrix}
0 & b_{s1}I(p \times p) \\
b_{s2}I(p \times p) & 0 \\
0 & b_{s1}I(p \times p) \\
b_{s,n-1}I(p \times p) & 0 \\
0 & b_{s,n-1}I(p \times p)
\end{bmatrix}
\]

such that \( \text{tr } b_s = 0 \) and \( \text{tr } b_s b_s \) will equal zero when \( s \neq g \), but will equal \( SSb_s \) when \( s = g \), where \( SSb_s = np(b_{s0}^2 + b_{s1}^2 + \cdots + b_{s,n-1}^2) \). Let
\[ b_s = D(b_{s0}, b_{s1}, \ldots, b_{sn-l}) \quad \text{and} \quad SSb_s = (b_{s0}^2 + b_{s1}^2 + \cdots + b_{sn-l}^2) \]

Then \( \text{tr } b_s = mp(\text{tr } b_s) \) and \( \text{tr } b_s b_g = 0 \) when \( s \neq g \), but equal to \( SSb_s \) when \( s = g \) where \( SSb_s = mp(SSb_g) \).

If \( c_t \) is the diagonal vector \( D(c_{t0}, c_{t1}, \ldots, c_{tp-l}) \), let \( c_t \) denote the diagonal matrix \( c_t \text{ (mnp x mnp)} \)

\[
c_t = \begin{bmatrix}
  c_t & & \\
   & \ddots & \\
  & & c_t
\end{bmatrix}, \quad t = 1, 2, \ldots, (p-l)
\]

such that \( \text{tr } c_t = 0 \) and \( \text{tr } c_t c_h \) equals zero when \( t \neq h \), but equals \( SSc_t \) when \( t = h \), where \( SSc_t = mn(c_{t0}^2 + c_{t1}^2 + \cdots + c_{tp-l}^2) = mn(SSc_t) \).

Let \( J' \) be the \((1 \times mnp)\) row vector, \( J' = (1, 1, \ldots, 1) \). Then

\[
J'a_r b_s c_t J = \text{tr } a_r b_s c_t = (\text{tr } a_r)(\text{tr } b_s)(\text{tr } c_t)
\]

and hence

\[
J'a_r b_s c_t a_r b_g c_h J = \text{tr}(a_r a_r b_s b_g c_t c_h) = (\text{tr } a_r a_r)(\text{tr } b_s b_g)(\text{tr } c_t c_h)
\]

This expression will equal zero when one or more of the following is true: \( r \neq f, s \neq g, t \neq h \); or when either \( r = 0 \), or \( s = 0 \), or \( t = 0 \). It will equal \((SSa_r)(SSb_s)(SSc_t)\) when \( r = f, s = g, \) and \( t = h \). As a special case, \( a_0 = b_0 = c_0 = I(\text{mnp x mnp}) \).

A contrast between the \( y_{ijk} \) will be \((A_r B_s C_t) = J'a_r b_s c_t y\), when not all three \( r, s, t \) are simultaneously zero. This will be a contrast, since if \( y_{ijk} = 1 \) for all \( i, j, k \) then
\[(A_rB_sC_t)(y = 1) = J'a_r b_s c_t y = (\text{tr } a_r)(\text{tr } b_s)(\text{tr } c_t) = 0.\]

In the above expression the notation \((y = 1)\) is used rather than \((y_{ijk} = 1)\) to simplify typography.

Define respectively

\[A_r = (A_r B C_o) = J'a_r b_c o y,\]
\[B_s = (A_o B S_c) = J'a_o b_s c y\]

and

\[C_t = (A_r B C_t) = J'a_o b_c t y,\]

where \(r = 1, 2, \ldots, (m-1); s = 1, 2, \ldots, (n-1);\) and \(t = 1, 2, \ldots, (p-1)\) as contrasts of the main effects \(A, B, C;\) alternatively these contrasts could be designated as contrasts of the zero order interaction.

Define respectively

\[(A_r B_s) = (A_r B S_c) = J'a_r b_s c o y,\]
\[(A_r C_t) = (A_r B C_t) = J'a_r b_c t y\]

and

\[(B_s C_t) = (A_o B C_t) = J'a_o b_s c t y;\]

\(r, s, t\) having the same ranges as before, as contrasts of the two-factor interactions \(AB, AC,\) and \(BC;\) alternatively these could be designated as contrasts of the first order interaction.

Define

\[(A_r B C_t) = J'a_r b_s c_t y,\]

\(r, s, t\) having the same ranges as before, as the three-factor interaction contrasts; alternatively these could be designated contrasts of the second order interaction.
At times these contrasts will be indicated only by their subscripts, as for instance in Table 1. All these contrasts are orthogonal to each other by virtue of the definitions of the coefficient matrices $a, b, c$. The orthogonal contrasts of an $m \times n \times p$ factorial experiment can be displayed as in Table 1.

The major purpose of this thesis is to investigate the consequences of using incomplete blocks in such factorial experiments. When all treatment combinations are not included in each block of experimental units, certain of the treatment contrasts become confounded (mixed up) with block contrasts. If a given contrast is orthogonal to block contrasts, it is said to be unconfounded. We will consider confounding systems based only on the signs of the elements of a contrast (plus zero, if zero elements occur). When any contrast in Table 1 is confounded, the problem arises to determine the effect this confounding has on the other contrasts.

From a matrix point of view, confounding amounts to a process of partitioning of matrices. Consider, for example, the main effect contrast $A_{r} = J'a_{r}b_{c}c'y$, and assume that no zero elements occur in the diagonal of $a_{r}$. Arrange the coefficients of the matrix $a_{r}$ and the corresponding elements of $y$ so that the first $npm_{-}$ elements in the diagonal of $a_{r}$ will have negative signs and the next $npm_{+}$ elements in the diagonal will have positive signs. Let $a_{r}^{-}$ denote the $(npm_{-} \times npm_{-})$ matrix of negative elements, and $a_{r}^{+}$ the $(npm_{+} \times npm_{+})$ matrix of positive elements. It is well-known that when a matrix in a product of matrices is partitioned, the other matrices have to be partitioned conformably in order that the product may retain its meaning. Accordingly, let $(J'_{-}, J'_{+})'$ and $(y'_{-}, y'_{+})'$ denote the conformable partitionings of the vectors $J'$ and $y$, so that we can write the contrast as
Table 1. The orthogonal contrasts of the \( m \times n \times p \) factorial experiment.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Contrast</th>
<th>Degrees of freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>A ( A_1 )</td>
<td>( = 100 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_{m-1} )</td>
<td>( = (m-1)00 )</td>
<td>1</td>
</tr>
<tr>
<td>B ( B_1 )</td>
<td>( = 010 )</td>
<td>1</td>
</tr>
<tr>
<td>( B_{n-1} )</td>
<td>( = 0(n-1)0 )</td>
<td>1</td>
</tr>
<tr>
<td>C ( C_1 )</td>
<td>( = 001 )</td>
<td>1</td>
</tr>
<tr>
<td>( C_{p-1} )</td>
<td>( = 00(p-1) )</td>
<td>1</td>
</tr>
<tr>
<td>AB ( A_1B_1 )</td>
<td>( = 110 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_1B_{n-1} )</td>
<td>( = 1(n-1)0 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_2B_1 )</td>
<td>( = 210 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_2B_{n-1} )</td>
<td>( = 2(n-1)0 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_{m-1}B_1 )</td>
<td>( = (m-1)10 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_{m-1}B_{n-1} )</td>
<td>( = (m-1)(n-1)0 )</td>
<td>1</td>
</tr>
<tr>
<td>AC ( A_1C_1 )</td>
<td>( = 101 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_{m-1}C_{p-1} )</td>
<td>( = (m-1)0(p-1) )</td>
<td>1</td>
</tr>
<tr>
<td>BC ( B_1C_1 )</td>
<td>( = 011 )</td>
<td>1</td>
</tr>
<tr>
<td>( B_{n-1}C_{p-1} )</td>
<td>( = 0(n-1)(p-1) )</td>
<td>1</td>
</tr>
<tr>
<td>ABC ( A_1B_1C_1 )</td>
<td>( = 111 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_{m-1}B_{n-1}C_{p-1} )</td>
<td>( = (m-1)(n-1)(p-1) )</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>( mnp-1 )</td>
</tr>
</tbody>
</table>
\[
A_r = \begin{pmatrix}
  a_{r-} & 0 \\
  0 & a_{r+}
\end{pmatrix}
\begin{pmatrix}
  y_- \\
  y_+
\end{pmatrix}
\]

\[= A_{r-} + A_{r+}\]

where \(A_{r-} = J' a_{r-} y_-\) is the value of the contrast \(A_r\) in the block containing those \(y\) values corresponding to the negative values of \(a_r\) and \(A_{r+} = J' a_{r+} y_+\) is the value of the contrast \(A_r\) in the block containing the \(y\) corresponding to the positive elements of \(a_r\). Placing \(y = 1\), we have

\[A_{r-}(y = 1) = J' a_{r-} J = \text{tr} a_{r-} \neq 0\]

\[A_{r+}(y = 1) = J' a_{r+} J = \text{tr} a_{r+} \neq 0.\]

Since

\[A_r(y = 1) = \text{tr} A_r(y = 1) = \text{tr} a_r = np(\text{tr} a_r) = 0,\]

and since

\[\text{tr} A_r = (\text{tr} A_{r-} + \text{tr} A_{r+}),\]

it follows that

\[\text{tr} a_{r-} = -(\text{tr} a_{r+}).\]

Since any contrast is of the form \(J' a_r b_s c_t y\) where either none, or one, or two of the subscripts \(r, s, t\) are zero, it follows that partitioning of any one, or any two, or of all three of the matrices \(a_r, b_s, c_t\) will require conformable partitioning of the other and \(y\). The confounding of any particular contrast will thus partition the coefficient matrix of every remaining contrast of the experiment, but this partitioning may or may not confound the remaining contrasts with block effects. Problems of this type will now be investigated. In succeeding chapters, \(r, s, t\) will be used to denote specific contrasts and \(R, S, T\) general contrasts.
CHAPTER 3

The Effects of Confounding a Single Contrast

3.1 Theorems

Theorem 3.1. The confounding of any contrast belonging to any lower order interaction does not confound any contrast belonging to any higher order interaction.

Consider any contrast, \( (A_{R B} C_T) \)

\[ (A_{R B} C_T)(y = 1) = J^t a_R b_S c_T J = (\text{tr } a_R)(\text{tr } b_S)(\text{tr } c_T). \]

Suppose we select any element of vector \( c_T \), say \( c_{T_k} \); i.e., set \( p = 1 \).

Then

\[ (A_{R B} C_{T_k})(y = 1) = c_{T_k} \text{tr } a_R \text{tr } b_S = 0 \]

Since this is true for all coefficients \( c_{T_k} \), \( A_{R B} C_T \) will remain a contrast regardless of how the levels of \( C \) are assigned to each block.

Hence it is possible to confound on any \( C \)-contrast, say \( C_t \), because there is a one-to-one correspondence between the elements of \( c_c \) and \( c_T \).

Symbolically this may be expressed as

\[ (A_{R B} C_T)(y = 1) | C_t = 0 \]

The above result can be extended to confounding on any \( BC \)-contrast, say \( B_S C_t \), assuming all \( A \)-levels are assigned to each block. Symbolically

\[ (A_{R B} C_{T_k})(y = 1) = b_{S_j} c_{T_k} \text{tr } a_R = 0 \]

Hence

\[ (A_{R B} C_T)(y = 1) | B_S C_t = 0, \]
since there is a one-to-one correspondence between $b_{sj}$ and $b_{sj}$ and between $c_{tk}$ and $c_{tk}$. By permutation of the letters of the contrast $(A_R B_S C_T)$, the above results follow immediately for all permutations; hence the theorem is proven.

**Theorem 3.2.** When $m$, $n$, $p$ are even numbers and none of vectors $a_r$, $b_s$, or $c_t$ contains zero elements, and when the vector $c_t$ has the same number of positive as negative elements, then, when $(A_R B_S C_T)$ is confounded, $(A_R B_S)$, $A_r$ and $B_s$ will be unconfounded.

Suppose that the first $m$ coefficients of $a_r$ are negative and the next $m_+$ are positive; that the first $n$ of $b_s$ are negative and the next $n_+$ are positive; and finally, that the first $p$ of $c_t$ are negative and the next $p_+$ are positive. Then $a_r$, $b_s$, and $c_t$ can be written as

$$a_r = D(a_{r-}, a_{r+});$$
$$b_s = D(b_{s-}, b_{s+}, b_{s-}, b_{s+});$$
$$c_t = D(c_{t-}, c_{t+}, \ldots, c_{t-}, c_{t+})$$

where $D$ denotes a diagonal matrix and the elements of the above matrices have orders that will now be explained.

Consider some negative element $a_{ri}$ in the $(np_+ x np_-)$ matrix $a_{r-}$. This element will occur successively $np$ times on the diagonal. Since

$$np = (n_+ + n_+)(p_- + p_+) = n_p_- + n_p_+ + n_+p_- + n_+p_+,$$

we can say that the element $a_{ri}$ will first occur in $n_p-$ successive positions, then in $n_p+$ successive positions, then in $n_+p_-$ successive positions and finally in $n_+p_+$ successive positions on the diagonal of $a_{r-}$. The positive elements of $a_{r+}$ will be similarly arranged on the diagonal. That is, $a_{r-}$ can be regarded as a $(4 x 4)$ diagonal matrix having submatrices
as diagonal elements, respectively with orders \( (m_{-p-})^2, (m_{-p+})^2, (m_{+p-})^2, (m_{+p+})^2 \). Likewise \( a_{r+} \) can be regarded as a \((4 \times 4)\) diagonal matrix with diagonal elements having as orders the same expressions as for \( a_{r-} \), but with \( m_+ \) substituted for \( m_- \). Consequently the matrix \( a_r \) can be regarded as the \((8 \times 8)\) diagonal matrix given in Table 2.

Table 2. Conformable submatrices of \((A B C_t)\).

\[
y' = (y'_{--}, y'_{--}, y'_{+-}, y'_{++}, y'_{+-}, y'_{++}, y'_{+-}, y'_{++})
\]

\[
a_r = D(a_{r-}, a_{r-} a_{r-} a_{r-} a_{r-} a_{r+} a_{r+} a_{r+} a_{r+})
\]

\[
b_s = D(b_{s-}, b_{s-} b_{s-} b_{s+} b_{s+} b_{s-} b_{s+} b_{s+} b_{s+})
\]

\[
c_t = D(c_{t-}, c_{t+} c_{t-} c_{t+} c_{t-} c_{t-} c_{t+} c_{t+} c_{t+})
\]

The order of the 8 respective submatrices are \( (m_{-p-})^2, (m_{-p+})^2, (m_{+p-})^2, (m_{+p+})^2, (m_{+p-})^2, (m_{+p+})^2 \).

Examining \( c_t \) in a similar way, it can be expressed as the \((8 \times 8)\) matrix shown in Table 2. Arrange the elements of \( y \) in such a manner that \( y \) can be regarded as an \((8 \times 1)\) vector, shown in Table 2, where for example the subvector \( y_{--} \) denotes the \( m_{-p-} \) elements of \( y \) that correspond with the \( m_{-p-} \) diagonal elements of the product \( a_{r-} b_{s-} c_{t-} \).

Placing those yields that correspond with positive values of \((a_{r-} b_{s-} c_{t-})\) into one block, and those that correspond with negative values into another block, the constituents of the respective blocks will be given by

\[
\text{Block 1:} \quad (y'_{++}, y'_{+-}, y'_{--}, y'_{-+})'
\]

\[
\text{Block 2:} \quad (y'_{+-}, y'_{++}, y'_{-+}, y'_{--})'
\]

To examine the confounding pattern for contrast \((A_{r-} B S_t)\) consider
\[ \begin{align*}
J^i a^c b^d c^e & = \left[ J^i a^c b^d c^e + J^i a^c b^d c^e + J^i a^c b^d c^e + J^i a^c b^d c^e \right]
\end{align*} \\
& \quad + \left[ J^i a^c b^d c^e + J^i a^c b^d c^e \right]
\end{align*} \\
& \quad + \left[ J^i a^c b^d c^e + J^i a^c b^d c^e \right]
\end{align*} \\
& = \left[ (\text{tr} a^c b^d c^e) + (\text{tr} a^c b^d c^e) + (\text{tr} a^c b^d c^e) \right]
\end{align*} \\
& \quad + \left[ (\text{tr} a^c b^d c^e) + (\text{tr} a^c b^d c^e) \right]
\end{align*} \\
& = (A_{t}B_{s}C_{t})^+ + (A_{t}B_{s}C_{t})^-
\]

where \( J^i \) denotes \( J(m,n,p \times 1) \) and where \( (A_{t}B_{s}C_{t})^+ \) indicates the value of the contrast in the block containing the positive elements of \( (a^c b^d c^e) \).

Consider \( (A_{t}B_{s})(y = 1)|(A_{t}B_{s}C_{t})^+ \), which is the value of \( (A_{t}B_{s}) \) when \( y_{ijk} = 1 \) in the positive block of the confounded contrast \( (A_{t}B_{s}C_{t}) \).

Let \( t = 0 \) in the first square bracket of \( (3.1) \). Then

\[ \begin{align*}
(A_{t}B_{s})(y = 1)|(A_{t}B_{s}C_{t}) & = \left( \text{tr} a^c b^d c^e \right)p_+ + \left( \text{tr} a^c b^d c^e \right)p_-
\end{align*} \\
& \quad + \left( \text{tr} a^c b^d c^e \right)p_-
\end{align*} \\
& \quad + \left( \text{tr} a^c b^d c^e \right)p_+
\]

When \( p_- = p_+ \), \( (3.2) \) becomes

\[ \begin{align*}
(A_{t}B_{s})(y = 1)|(A_{t}B_{s}C_{t})^+ & = \left( \text{tr} a^c b^d c^e + \text{tr} a^c b^d c^e \right)p_+
\end{align*} \\
& = \left( \text{tr} a^c b^d c^e \right)p_+ = 0
\]

since \( (A_{t}B_{s}) \) is a contrast. A similar result can be derived for \( (A_{t}B_{s})(y = 1) \) in the negative block of \( (A_{t}B_{s}C_{t}) \).
Consider the value of the contrast \( A_r(y = 1) \) in the positive block of \( A_r B_s C_t \). Let \( s = 0 \) in (3.2), then

\[
(A_r(y = 1) \mid (A_r B_s C_t) = (\text{tr} \ a^-_T)(p^- n^- + p^n n^-) + (\text{tr} \ a^+_T)(p^- n^- + p^n n^-) \tag{3.3}
\]

When \( p^- = p^+ \), (3.3) becomes

\[
(A_r(y = 1) \mid (A_r B_s C_t) = (\text{tr} \ a^-_T + \text{tr} \ a^+_T)n^+ = (\text{tr} \ a^-)n^+ = 0
\]

Similarly

\[
(B_s(y = 1) \mid (A_r B_s C_t) = (\text{tr} \ b^-_S)(p^+ m^- + p^- m^-) + (\text{tr} \ b^+_S)(p^- m^- + p^+ m^+) \tag{3.4}
\]

\[
= 0 \text{ when } p^- = p^+
\]

Hence the theorem is proved.

**Corollary 3.2.1.** The confounding of \( (A_r B_s C_t) \) will not confound (1) those contrasts obtained from \( (A_r B_s C_t) \) by putting any one of the subscripts \( r, s, t \) equal to zero if the vector whose subscript is equated to zero has the same number of positive as negative elements; (2) those contrasts obtained from \( (A_r B_s C_t) \) by putting either \( r \) and \( s \), or \( r \) and \( t \), or \( s \) and \( t \) equal to zero, if either one of the two vectors whose subscripts are equated to zero has the same number of positive as negative elements.

This corollary leads to the following practical situation: when contrast \( (A_r B_s C_t) \) is confounded, the following contrasts will be unconfounded if the condition(s) given on the right is(are) satisfied.
\[(A_r B_s) \text{ when } p_- = p_+ \]
\[(A_r C_t) \text{ when } n_- = n_+ \]
\[(B_s C_t) \text{ when } m_- = m_+ \]
\[A_r \text{ when either } n_- = n_+ \text{ or } p_- = p_+ \]
\[B_s \text{ when either } m_- = m_+ \text{ or } p_- = p_+ \]
\[C_t \text{ when either } m_- = m_+ \text{ or } n_- = n_+ \]

These conditions lead to

**Corollary 3.2.2.** When each of the vectors \(a_r, b_s, c_t\) has the same number of negative as positive elements, the confounding of \((A_r B_s C_t)\) will not confound any lower order contrast obtained from \((A_r B_s C_t)\) by placing either one or two of the subscripts \(r, s, t\) equal to zero.

When the contrast \((A_r B_s C_t)\) is confounded, and we would like to know under what conditions the lower order contrasts with subscript(s) identical to one (or two) of \(r, s, t\) is(are) confounded, the results of Theorem 3.2 apply. This theorem, however, states nothing about the confounding or not of lower order contrasts with subscripts different from \(r, s, t\). Also Theorem 3.2 dealt only with even \(m, n, p\).

To investigate these points, consider the confounding of contrast \((A_r B_s)\) of an \(m \times n\) experiment. We consider a two-factor experiment to simplify typography: with a three-factor experiment we would obtain 27 submatrices instead of the 9 in the present case. Partition the vector \(a_r\) into a negative part \(a_-\) with \(m_-\) elements in the diagonal, a positive part \(a_+\) with \(m_+\) elements, and a zero part \(a_0\) with \(m_0\) elements. Partition the vector \(b_s\) into a negative part \(b_-\) with \(n_-\) elements, a positive part \(b_+\) with \(n_+\) elements, and a zero part \(b_0\) with \(n_0\) elements.
As before, if the matrix $a_r$ is expressed in terms of the diagonal matrices $a_{r-}$, $a_{r+}$, $a_{r0}$, the matrix $b_s$ can be expressed in terms of the diagonal matrices $b_{s-}$, $b_{s+}$, $b_{s0}$, conformable to the parts of $a_r$. The elements of $y$ can then be arranged into submatrices conformable to the products of the parts of $a_r$ and $b_s$. Placing those yields that correspond to positive values of $(a_r b_s)$ into one block, those that correspond to negative values into another block, and those that correspond to the zero values into yet another block, the elements of the respective blocks will be as follows:

Block 1: $(y_{++}^t, y_{--}^t)^t$

Block 2: $(y_{+-}^t, y_{-+}^t)^t$

Block 3: $(y_{+0}^t, y_{-0}^t, y_{0+}^t, y_{0-}^t, y_{00}^t)^t$

The values of $(A_r B_s)(y = 1)$ in the respective blocks will be given by the expressions

$$(A_r B_s)(y = 1) = (\text{tr } a_{r+})(\text{tr } b_{s+}) + (\text{tr } a_{r-})(\text{tr } b_{s-})$$

$$(A_r B_s)(y = 1) = (\text{tr } a_{r+})(\text{tr } b_{s-}) + (\text{tr } a_{r-})(\text{tr } b_{s+})$$

$$(A_r B_s)(y = 1) = (\text{tr } a_{r+})(\text{tr } b_{s0}) + (\text{tr } a_{r-})(\text{tr } b_{s0}) + (\text{tr } a_{r0})(\text{tr } b_{s+}) + (\text{tr } a_{r0})(\text{tr } b_{s-}) + (\text{tr } a_{r0})(\text{tr } b_{s0})$$

$$= 0$$

Consider contrast $(A_r B_s)(A_r B_s)$. Let $a_{r1}$, $a_{r2}$, $a_{r3}$ denote the parts of $a_r$ conformable to $a_{r-}$, $a_{r+}$, $a_{r0}$, and similarly for $b_s$. The values of $(A_r B_s)(y = 1)$ in the three blocks of $(A_r B_s)$ can be obtained by substituting the parts of $a_r|a_r$ and $b_s|b_s$ in (3.3.1).
\[(A_R B_S)(y = 1) (A_R B_S)_+ = (\text{tr } a_{R2})(\text{tr } b_{S2}) + (\text{tr } a_{R1})(\text{tr } b_{S1})\]
\[(A_R B_S)(y = 1) (A_R B_S)_- = (\text{tr } a_{R2})(\text{tr } b_{S1}) + (\text{tr } a_{R1})(\text{tr } b_{S2}) \quad (3.3.2)\]
\[(A_R B_S)(y = 1) (A_R B_S)_0 = -(\text{tr } a_{R3})(\text{tr } b_{S3}) \quad \text{(for } R, S, > 1)\]

Next consider contrast \((A_R) (A_R B_S)\). The values of \(A_R(y = 1)\) in the first two blocks of \((A_R B_S)\) can be obtained by letting \(S = 0\) in (3.3.2); for the zero block one must rework the relationship from (3.3.1).

\[(A_R)(y = 1) (A_R B_S)_+ = (\text{tr } a_{R2})n_+ + (\text{tr } a_{R1})n_-\]
\[(A_R)(y = 1) (A_R B_S)_- = (\text{tr } a_{R2})n_- + (\text{tr } a_{R1})n_+ \quad (3.3.3)\]
\[(A_R)(y = 1) (A_R B_S)_0 = (n-n_0)(\text{tr } a_{R3})\]

Similar expressions for contrast \((B_S) (A_R B_S)\) are

\[(B_S)(y = 1) (A_R B_S)_+ = (m_+)(\text{tr } b_{S2}) + (m_-)(\text{tr } b_{S1})\]
\[(B_S)(y = 1) (A_R B_S)_- = (m_+)(\text{tr } b_{S1}) + (m_-)(\text{tr } b_{S2}) \quad (3.3.4)\]
\[(B_S)(y = 1) (A_R B_S)_0 = (m-m_0)(\text{tr } b_{S3})\]

Having the above relations, we can now prove the following theorems.

Theorem 3.3. In an \(m \times n\) experiment if contrast \((A_R B_S)\) is confounded, contrast \((A_R B_S)\) will also be confounded unless one or both of these situations prevail:

(i) \(A_R \mid A_R\) or \(B_S \mid B_S\) or both are unconfounded

(ii) \(\text{tr } a_{R3} = 0 \) and \(\text{tr } b_{S1} = \text{tr } b_{S2}\); or
\[\text{tr } b_{S3} = 0 \text{ and } \text{tr } a_{R1} = \text{tr } a_{R2}.\]

It is easy to show that, if (i) or (ii) prevails, \((A_R B_S)\) \((A_R B_S)\) is unconfounded. What happens if one of these conditions is not fulfilled? It is obvious that either \(\text{tr } a_{R3}\) or \(\text{tr } b_{S3}\) must be zero in order to have \((A_R B_S)\) unconfounded.
If $\text{tr} \ a_{R3} = 0$, $\text{tr} \ a_{R1} = -\text{tr} \ a_{R2}$, and

$$(A^R_{s}B_{s})(y = 1)(A^R_{s}B_{s}) = \text{tr} \ a_{R2}(\text{tr} \ b_{S2} - \text{tr} \ b_{S1})$$

Unless (i) is fulfilled, so that $\text{tr} \ a_{R2} = 0$, or $\text{tr} \ b_{S1} = \text{tr} \ b_{S2} = 0$, or (ii) is fulfilled so that $\text{tr} \ b_{S2} = \text{tr} \ b_{S1}$, $(A^R_{s}B_{s})(A^R_{s}B_{s})$ will be confounded with blocks.

**Corollary 3.3.1.** If $(A^R_{s}B_{s})$ is confounded, contrast $A_{R}$ will also be confounded unless one or both of these situations prevail:

(i) $A_{R}|A_{T}$ is unconfounded

(ii) $\text{tr} \ a_{R3} = 0$ and $n_+ = n_-$.

A similar statement can be made for $B_{S}$.

This corollary follows immediately from (3.3.3) and (3.3.4). One can rephrase this corollary as follows:

**Corollary 3.3.2.** If $A_{R}|A_{T}$ is confounded, $A_{R}|(A^R_{s}B_{s})$ will also be confounded unless $\text{tr} \ a_{R3} = 0$ and $n_+ = n_-$. Similarly if $B_{S}|B_{s}$ is confounded, $(B_{S})(A^R_{s}B_{s})$ will also be confounded unless $\text{tr} \ b_{S3} = 0$ and $m_+ = m_-$.  

**3.2 Bundles of Contrasts**

In terms of Theorem 3.3 we may define a bundle of contrasts of **homogeneous order** to be that set of contrasts with properties such that when any single contrast belonging to the set is confounded, all other contrasts belonging to the set will also be confounded. In other words, in terms of confounding, any contrast will always possess a homogeneous bundle. A homogeneous bundle of a contrast $(A^R_{s}B_{s}C_{s})$ or (rst) will be indicated by [rst].
In terms of corollary 3.3.1 we may define a bundle of heterogeneous order to be that set of bundles of homogeneous orders with properties such that when any contrast belonging to the homogeneous bundle of highest order is confounded, all other contrasts in the bundle of highest order and all contrasts belonging to all the other homogeneous bundles in the set will be confounded. In terms of confounding, any contrast of higher order will always possess a heterogeneous bundle, but a main effect contrast cannot possess a heterogeneous bundle.
CHAPTER 4

The Effects of Confounding Two Contrasts Simultaneously

We now investigate the case when two contrasts are confounded simultaneously. For the sake of simplicity of representation, we are assuming that none of the contrastshas zero elements, otherwise $9^3 = 729$ submatrices will result from the conformable partitionings. In the present case the more manageable number of $64$ will occur. This assumption is one of convenience only, and similar procedures and results will hold for contrasts with zero elements.

Consider the two contrasts $(A_r B C_t)$ and $(A_r B C_h)$ where at least one of the following inequalities holds: $r \neq f$, $s \neq g$, $t \neq h$; and a contrast $(A_v B C_z)$ derived from the two previous ones in the manner shown below.

Let $a_r$, $a_f$, $a_v$ be the matrices

$$a_r = D(a_{ro}, a_{rl}, \ldots, a_{rm-l})$$
$$a_f = D(a_{fo}, a_{fl}, \ldots, a_{fm-l})$$
$$a_v = D(a_{vo}, a_{vl}, \ldots, a_{vm-l}) = a_r a_f$$

where $I = I(mp \times np)$ and $a_{vi} = a_{ri} a_{fi}$.

Originally the row vector $a_r^t = (a_{ro}, a_{rl}, \ldots, a_{rm-l})$ has $m_-$ negative elements followed by $m_+$ positive elements ($m_- + m_+ = m$). Corresponding to the $m_-$ negative elements of $a_r^t$, there are a certain number of negative and a certain number of positive elements of the row vector $a_f^t = (a_{fo}, a_{fl}, \ldots, a_{fm-l})$. Hence $a_v^t = (a_{vo}, a_{vl}, \ldots, a_{vm-l})$ would consist of a sequence of alternating positive and negative elements. Let us rearrange $a_v^t$ so that it has a single sequence of negative followed by a single sequence of positive elements; then rearrange $a_r^t$ and $a_f^t$. 

Accordingly $a_i^v$ will have $(m_{--} + m_{+-})$ negative elements where there are $m_{+-}$ elements of $a_i^v$ which are the products of negative elements from $a_r^t$ and positive elements from $a_i^t$, and $m_{+-}$ elements which are the products of positive elements from $a_i^t$ and negative elements from $a_r^t$. Similarly there are $(m_{--} + m_{++})$ positive elements of $a_i^v$. Hence the $(a_i^t)(a_i^v)$ row vector has a sequence of $m_{--}$ negative elements, $m_{+-}$ positive elements, $m_{+-}$ negative elements, and finally $m_{++}$ positive elements. [If there were zero elements in $a_r^t$ and $a_r^t$, there would be $(m_{--} + m_{+-} + m_{+-} + m_{++} + m_{oo})$ zero elements in $a_r^t$.] Similarly $(a_i^t)(a_i^v)$ has a sequence of $m_{--}$ positive elements, $m_{+-}$ negative elements, $m_{+-}$ negative elements, and finally $m_{++}$ positive elements. It is important to remember that the first subscript refers to the $A_r$ contrast and the second to the $A_t$ contrast. Extensions to more than two contrasts can be made without too much difficulty.

As an example, consider the two row vectors $a_i^t = (-5, -3, -1, +1, +3, +5)$ and $a_i^v = (+5, -1, -4, -4, -1, +5)$. Before rearrangement, the vector $a_i^v$ will be $a_i^v = (-25, +3, +1, -4, -3, +25)$. After rearrangement we have the following

$$a_i^v = (-25, -4, -3, +3, +4, +25)$$
$$a_i^t = ( +5, -4, -1, -1, -4, +5)$$
$$a_i^v = ( -5, +1, +3, -3, -1, +5)$$

and $m_{--} = 1$, $m_{+-} = 2$, $m_{+-} = 2$, and $m_{++} = 1$.

In the diagonal matrices each element of the vector is repeated $np$ times. The matrices, rearranged according to $a_i^v$, are indicated as

$$a_r^t | a_i^v = D(a_r^{--}, a_r^{-+}, a_r^{+-}, a_r^{++})$$
$$a_r^v | a_i^v = D(a_r^{--}, a_r^{-+}, a_r^{+-}, a_r^{++})$$
$$a_r^v | a_i^v = D(a_r^{--}, a_r^{-+}, a_r^{+-}, a_r^{++})$$
$$a_r^v | a_i^v = D(a_r^{--}, a_r^{-+}, a_r^{+-}, a_r^{++})$$
where the first sign in the subscripts refers to the element in \( a_r \), the
second sign to the element in \( a_v \), and the sign in \( a_v \) is the product of the
two basic signs. The respective orders of the diagonal submatrices are
\((npm_{++} \times npm_{--}), (npm_{+-} \times npm_{-+}), (npm_{--} \times npm_{++})\),
\((npm_{+-} \times npm_{-+}), (npm_{--} \times npm_{++})\).

A similar procedure can be followed for the \( B \)-contrasts and the
\( C \)-contrasts. As seen in the original definition of \( b_g \), we need consider
only one of the submatrices here and then repeat the rearranged submatrix
\( m \) times to form the rearranged \( b \) matrix. Let us consider one of these
submatrices \( b(np \times np) \). The two basic submatrices will be designated as
\( b^*_s \) and \( b^*_g \), and their product as \( b^*_w \). Hence

\[
\begin{align*}
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\end{align*}
\end{align*}
\]

where the first sign in the subscripts refers to the element in \( b^*_s \) and the
second sign to the element in \( b^*_g \), and the sign in \( b^*_w \) is the product of the
two basic signs. The respective orders of the diagonal matrices are
\((pn_{++} \times pn_{--}), (pn_{+-} \times pn_{-+}), (pn_{--} \times pn_{++})\) and \((pn_{+-} \times pn_{-+})\). The cor-
responding \( b \)-matrices are simply \( D(b^*_s, b^*_s, \ldots, b^*_s) \) where \( b^*_s \) is repeated
\( m \) times.

In the \( c \)-matrices we need study only the diagonalized \( c \)-vector
\( c^*(p \times p) = D(c_{t0}, c_{1}, \ldots, c_{tp-1}) \). We will consider the two basic matrices
\( c^*_t \) and \( c^*_h \) with product \( c^*_z \).

\[
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\begin{align*}
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\end{align*}
\end{align*}
\]
where the signs in the subscripts have the same meaning as before, and the respective orders of the diagonal matrices are \((p_+ \times p_+), (p_- \times p_-), (p_- \times p_-),\) and \((p_+ \times p_+).\) The corresponding \(c\)-matrices are simply \(D(c^*, c^*, \ldots, c^*)\) where \(c^*\) is repeated \(nm\) times.

Since we have the relations

\[
\begin{align*}
m &= m_+ + m_- + m_+ - m_+ + m_+ \\
n &= n_+ + n_- + n_+ + n_+ \\
p &= p_+ + p_- + p_- + p_+ \\
mnp &= \sum \delta^{m_n n_p} \beta \epsilon \gamma
\end{align*}
\]

where each subscript \((\alpha, \beta, \cdots, \gamma)\) is either \(-\) or \(+.\) Hence there will be \(64\) terms in the sum. The elements of the \(y\) matrix can now be arranged into \(64\) subvectors conformable to the products of the parts of \(a_r | a_v, b_s | b_w, c_t | c_z\) with the parts of \(a_r | a_v, b_s | b_w,\) and \(c_h | c_z\). Thus

\[
y = (y_{-+--}^{+-}, y_{-++-}^{++}, y_{-+--}^{+-}, y_{--++}^{++}, y_{+++}^{++}, y_{++}^{++}, y_{++}^{++}, y_{++}^{++})
\]

were the subvector \(y_{+\cdots}^{++} \beta \epsilon \gamma\) has \(m \delta^{n_p} \beta \epsilon \gamma\) columns.

Observe again that the first subscript of \(m\) or \(n\) or \(p\) (i.e., \(\alpha\) or \(\beta\) or \(\gamma\)) indicates the sign of the \(a_r\) or the \(b_s\) or the \(c_t\) in the relevant part, and hence the product \((\alpha \beta \gamma)\) will indicate the signs of the elements of the contrast \((A_{rs} B_{st} C_{tu})\) in that part. The second subscript of \(m\) or \(n\) or \(p\) (i.e., \(\delta\) or \(\epsilon\) or \(\gamma\)) indicates the sign of the \(a_r\) or the \(b_s\) or the \(c_t\) in the same part as before, and hence the products \((\delta \epsilon \gamma)\) will indicate the signs of the elements of the contrast \((A_{rs} B_{st} C_{tu})\) in that part. The products \(\Psi = \delta \epsilon \gamma = \beta \epsilon \gamma\) or \(\Psi = \delta \gamma\) indicate the sign of \(a_v\) or \(b_w\) or \(c_z\) in the same part as before, and hence the products \(\Psi \times \Psi = \delta \beta \epsilon \gamma\) will indicate the signs of the elements of the contrast \((A_{rs} B_{st} C_{tu})\) in that part.
The sequence $\alpha, \delta, \gamma, \beta, \varepsilon, \kappa, \varphi, \eta, \psi$ is thus the key to the study of the confounding situation. When all those elements of $y$ which have positive signs in $(A_{r s t} B C_h)$ and $(A_{r s t} B C_t)$ are placed in one block, those which have positive signs in $(A_{r s t} B C_t)$ but negative signs in $(A_{r s t} B C_h)$ in another block, those which have negative signs in $(A_{r s t} B C_t)$ but positive signs in $(A_{r g h} B C_t)$ in yet another block, and those which have negative signs in both $(A_{r s t} B C_t)$ and $(A_{r g h} B C_t)$ in the final block, then we will obtain the blocking arrangement due to the simultaneous confounding of the two contrasts.

Table 3 supplies the $64$ possible combinations of the sequence; it can be seen at once that if $(A_{r s t} B C_t)$ and $(A_{r g h} B C_t)$ are confounded simultaneously, $(A_{r s t} B C_t)$ will be confounded automatically. $(A_{r s t} B C_t)$ can be termed the generalized interaction of $(A_{r s t} B C_t)$ and $(A_{r g h} B C_t)$ and written symbolically as

$$(A_{r s t} B C_t) = (A_{r s t} B C_t) \times (A_{r g h} B C_t)$$

When, e.g., $f = 0$, $s = 0$, $h = 0$, $(A_{r s t} B C_t) = (A_{r t} B C_t) x B_g = (A_{r g h} B C_t)$, the well-known "interaction" of $(A_{r t} B C_t)$ with $B_g$. It should be emphasized that $(A_{r s t} B C_t)$ is a contrast, but in its generalized form will not be orthogonal to all of the basic contrasts in the experiment.

Let $(A_{r s t} B C_t, A_{r g h} B C_t)$ denote the simultaneous confounding of the contrasts within the brackets. Then

$$(A_{r s t} B C_t)(y = 1)(A_{r s t} B C_t, A_{r g h} B C_t) = J^{'a_{r s t} b_{r g h} c_{r g h}}$$

where the sum contains $64$ terms. Similar expressions hold for $(A_{r g h} B C_t)$ and $(A_{r s t} B C_t)$ when $(A_{r s t} B C_t)$ and $(A_{r g h} B C_t)$ are simultaneously confounded.

Let $(A_{r s t} B C_t, A_{r g h} B C_t)$ denote the block obtained by selecting those treatment combinations that have a positive sign in $A_{r s t} B C_t$ as well as in $A_{r g h} B C_t$. 
Table 3. Signs of the conformable parts of \((A_{B,C})_{r,s,t}, (A_{B,C})_{f,g,h}, (A_{B,C})_{v,w,z}\).

<table>
<thead>
<tr>
<th>(a_{r,s,t})</th>
<th>(a_{f,g,h})</th>
<th>(a_{v,w,z})</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ + + + + + + +</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>+ + + + + + + +</td>
<td>(+)</td>
<td>(+)</td>
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<tr>
<td>+ + + + + + + +</td>
<td>(+)</td>
<td>(+)</td>
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<tr>
<td>+ + + + + + + +</td>
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<td>(+)</td>
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<tr>
<td>+ + + + + + + +</td>
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<td>(+)</td>
</tr>
<tr>
<td>+ + + + + + + +</td>
<td>(+)</td>
<td>(+)</td>
</tr>
</tbody>
</table>

( ) indicates parts of (++) block.

If \(F_r^1\) denote the \((1 \times 4)\) row vector \((tr\ a_{r++}, tr\ a_{r+-}, tr\ a_{r-+}, tr\ a_{r--})\)
and \(G\) the \((4 \times 4)\) matrix
\[ G_{s^{++}} = \begin{bmatrix}
  \text{tr } b_{s^{++}} & \text{tr } b_{s^{+-}} & \text{tr } b_{s^{-+}} & \text{tr } b_{s^{--}} \\
  \text{tr } b_{s^{+-}} & \text{tr } b_{s^{++}} & \text{tr } b_{s^{-+}} & \text{tr } b_{s^{--}} \\
  \text{tr } b_{s^{-+}} & \text{tr } b_{s^{--}} & \text{tr } b_{s^{++}} & \text{tr } b_{s^{+-}} \\
  \text{tr } b_{s^{--}} & \text{tr } b_{s^{+-}} & \text{tr } b_{s^{--}} & \text{tr } b_{s^{++}}
\end{bmatrix} \]

and \( H_t^f \) the \((1 \times 4)\) row vector \((\text{tr } c_t^{++}, \text{tr } c_t^{+-}, \text{tr } c_t^{-+}, \text{tr } c_t^{--})\), then, using Table 3 to select those treatment combinations that have a positive sign in \((A_{r_{s}C_{t}})\) as well as in \((A_{r_{g}C_{h}})\), it will be found that

\[ (A_{r_{s}C_{t}}(y = 1) | (A_{r_{g}C_{h}})^{++} = F_{R}^{G H}_{S} T \quad (4.2) \]

Similar expressions hold for \((A_{r_{g}C_{h}}(y = 1)\) and \((A_{r_{g}C_{h}}(y = 1)\) in the \((++)\) block. To derive the value of contrast \((A_{r_{S}C_{T}}(y = 1)\) in that block, let \( a_{R22} \) denote the part of \( a_{R} \) conformable to \( a_{r^{++}} \) and \( a_{r^{++}} \); i.e., those elements of \( a_{R} \) for which the corresponding elements in both \( a_{r} \) and \( a_{f} \) are positive. Similarly \( a_{R12} \) denotes a vector for which the corresponding elements of \( a_{r} \) are negative and of \( a_{f} \) are positive, etc. Similarly for the parts of \( b_{S} \) and \( c_{T} \). Since

\[ a_{R} = a_{R11} + a_{R12} + a_{R21} + a_{R22} \]
\[ = a_{R1} + a_{R2} \]
\[ = a_{R,1} + a_{R,2} \]

with similar results for \( b_{S} \) and \( c_{T} \), we have a symbolic one-to-one correspondence between \((A_{r_{S}C_{T}})\) and \((A_{r_{g}C_{h}})\) if \( 1 \) in the subscript of a vector of the former is substituted for \( - \) in the latter and \( 2 \) for \( + \). Hence

\[ (A_{r_{S}C_{T}}(y = 1) | (A_{r_{g}C_{h}})^{++} = F_{R}^{G H}_{S} T \quad (4.3) \]

where
\[ F'_R = (\text{tr } a_{22}^{R}, \text{tr } a_{21}^{R}, \text{tr } a_{12}^{R}, \text{tr } a_{11}^{R}) \]  

\[ G_{S++} = \begin{bmatrix} \text{tr } b_{S22} & \text{tr } b_{S21} & \text{tr } b_{S12} & \text{tr } b_{S11} \\ \text{tr } b_{S21} & \text{tr } b_{S22} & \text{tr } b_{S11} & \text{tr } b_{S12} \\ \text{tr } b_{S12} & \text{tr } b_{S11} & \text{tr } b_{S22} & \text{tr } b_{S21} \\ \text{tr } b_{S11} & \text{tr } b_{S12} & \text{tr } b_{S21} & \text{tr } b_{S22} \end{bmatrix} \]  

\[ H'_T = (\text{tr } c_{T22}, \text{tr } c_{T21}, \text{tr } c_{T12}, \text{tr } c_{T11}) \]  

\textbf{Theorem 4.1.} If \((A^* R B^* C^*_T)\) | \((A^* R B^* C^*_S)\) is confounded, \((A^* R B^* C^*_T, A^* B^* C^*_h)\) must also be confounded; where \(R, S, \) and \(T\) can be any values except that all three cannot be zero.

This is an obvious result, which can be simply proven as follows.

Since \((A^* R B^* C^*_T)\) | \((A^* R B^* C^*_S)\) is confounded, \((A^* R B^* C^*_T)(y = 1)\) | \((A^* R B^* C^*_S)\) = \(W_1 \neq 0\).

But subsequent confounding with respect to \((A^* B^* C^*_h)\) subdivides the above block into two blocks (\(-\) and \(+\)). It is impossible to subdivide a non-zero quantity into two parts, each of which is zero, the latter being necessary if \((A^* R B^* C^*_T)\) is to be unconfounded in these two blocks. A similar statement holds for the \((A^* B^* C^*_T)\) block. It is equally obvious that this theorem holds when contrasts containing zero elements are considered.

\textbf{Corollary 4.1.1.} All contrasts belonging to the heterogeneous bundle of \((A^* R B^* C^*_S)\) will be confounded when \((A^* R B^* C^*_S, A^* B^* C^*_h)\) is confounded. Similarly for the heterogeneous bundle of \((A^* B^* C^*_h)\).

\textbf{Theorem 4.2.} The simultaneous confounding of \((A^* B^* C^*_S, A^* B^* C^*_h)\) in an \(m \times n \times p\) experiment, where \(m, n\) and \(p\) are even, will not confound
(i) \( A_{RS} \) if \( p_{++} = p_{+-} = p_{-+} = p_{--} \)

(ii) \( A_R \) if either \( p_{++} = p_{+-} = p_{-+} = p_{--} \), or

\[ n_{++} = n_{+-} = n_{-+} = n_{--} \]

or both systems of relations hold.

Consider (4.3) and let \( T = 0 \). Then \( H_0' = (p_{++}, p_{+-}, p_{-+}, p_{--}) \) and hence equal to \( J' p_{++} \) if the condition in (i) is satisfied. Hence

\[
(A_{RS})(y = 1) \left| (A_{RS} C_t, A_{R} B_{C_t} h)_{++} = F_{RS}^{G} p_{++} = 0. \right. (4.7)
\]

Let \( S = 0 \); then (4.7) becomes

\[
A_R(y = 1) \left| (A_{RS} C_t, A_{R} B_{C_t} h)_{++} = F_{J} J p_{++} = 0 \right. \]

Similarly for the second part of the theorem if \( n_{++} = n_{+-} = n_{-+} = n_{--} \)

Theorem 4.3. When two contrasts \( (A_{RS} C_t) \) and \( (A_{R} B_{C_t} h) \) are confounded simultaneously so that \( (A_{RS} C_t) \times (A_{R} B_{C_t} h) = (A_{R} B_{C_t} h) \), and if \( (A_{RS} C_t) \) is a contrast not orthogonal to \( (A_{R} B_{C_t} h) \) then \( (A_{RS} C_t) \) will also be confounded.

When \( (A_{RS} C_t) \) is not orthogonal to \( (A_{R} B_{C_t} h) \) then

\[
(\text{tr } a_{R} a_{R})(\text{tr } b_{S} b_{S})(\text{tr } c_{T} c_{T}) \neq 0.
\]

Hence \( \text{tr } a_{R} a_{R} \neq 0 \), \( \text{tr } b_{S} b_{S} \neq 0 \), and \( \text{tr } c_{T} c_{T} \neq 0 \). If \( a_{R} \) and \( a_{R} \) are partitioned as before, then

\[
\text{tr } a_{R} a_{R} = \text{tr } D(a_{R12} a_{R12}, a_{R21} a_{R21}, a_{R11} a_{R11}, a_{R22} a_{R22}) \quad (4.8)
\]

Since the left-hand side of (4.8) does not equal zero, it follows that at least one of the traces of the diagonalized subvectors will not equal zero.

A similar result can be established for \( \text{tr } b_{S} b_{S} \) and \( \text{tr } c_{T} c_{T} \), and hence the theorem is proved. If \( (A_{RS} C_t) \) is orthogonal to \( (A_{R} B_{C_t} h) \), it may or may not be confounded.
Corollary 4.3.1. If \((A_{vwz})\) can be expressed as some linear combination of contrasts, and if one of the contrasts contained in the linear combination, \((A_{rst})\) is not orthogonal to \((A_{vwz})\), then \((A_{rst})\) will be confounded when \((A_{rs})\) and \((A_{gh})\) are confounded simultaneously.
PART II. THE ORTHOGONAL POLYNOMIAL SYSTEM OF CONTRASTS

CHAPTER 5

The Orthogonal Polynomials

5.1 Introduction

If we have N yields $y_0, y_1, \ldots, y_{N-1}$ corresponding to N equi-spaced levels $X_0, X_1, \ldots, X_{N-1}$ of a treatment, and we wish to fit a regression curve of the polynomial type

$$y = k_0 + k_1X + k_2X^2 + \cdots + k_{N-1}X^{N-1}$$

it is advantageous to employ orthogonal polynomials instead of the power polynomials, and write

$$y = h_0 + h_1P_1 + h_2P_2 + \cdots + h_{N-1}P_{N-1}$$

where, in the Fisher (1921) system, or Fisher and Yates (1938)

$$P_{r+1} = P_1P_r - \frac{r^2(N^2-r^2)}{4(4r^2-1)}P_{r-1}, \quad P_0 = 1$$

$$\sum_{r=1}^N P_r^2 = \frac{(r!)^4}{(2r)!(2r+1)!}N(N^2-1)(N^2-4)\cdots(N^2-r^2)$$

Letting $x = X - \bar{X}$, where $\bar{X} = (N-1)/2$, we have for the first few values of r

$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = x^2 - (1/12)(N^2-1)$$
$$P_3 = x^3 - (1/20)(3N^2-7)x$$
$$P_4 = x^4 - (1/14)(3N^2-13)x^2 + (3/560)(N^2-1)(N^2-9)$$
$$P_5 = x^5 - (5/18)(N^2-7)x^3 + (1/1008)(15N^4-230N^2+407)x$$
Let \( P_{ri} \) denote the numerical value of the orthogonal polynomial of degree \( r \) at the level \( i \). With the object of illustrating confounding procedures, write \( P_{ri} = \langle r_i \rangle M_{ri} \), where \( M_{ri} = P_{ri} \) and \( \langle r_i \rangle \) will be either - or + or 0. Some of these values are presented in Table 4 for the first four orthogonal polynomials, when \( N \) is even and when \( N \) is odd. In the table a common factor has been taken out to simplify typography. Note that when \( N \) is odd, the central element of \( M_r \) when \( r \) is odd is unspecified. This does not matter in the confounding, since the blocking will proceed only on the signs. Those elements of \( \langle r_i \rangle \) given explicitly in the table, retain their signs independent of the size of \( N \); the others depend on the size of \( N \).

Table 5 supplies the values of \( P_{ri} = \lambda P_{ri} \) for \( N \) through 8. From the table it can be ascertained that for \( N \) even, the elements of \( P_r \) are symmetric about the origin; for \( N \) odd, the central element is 0 and the other elements are symmetric about this central element.

5.2 Partial and Complete Confounding

Let \( P_r = \langle M_r \rangle \) denote the diagonalized vector

\[
D\langle r_0 \rangle M_{r0} \langle r_1 \rangle M_{r1} \ldots \langle r_{m-1} \rangle M_{rm-1} \langle r_m \rangle M_{rm} = D\langle r_0 \rangle \langle r_1 \rangle \ldots \langle r_{m-1} \rangle D(M_{r0} \langle M_{r1} \rangle \ldots \langle M_{rm-1} \rangle) ;
\]

let

\[
P_s = \langle M_s \rangle = D\langle s_0 \rangle M_{s0} \langle s_1 \rangle M_{s1} \ldots \langle s_{n-1} \rangle M_{sn-1} = D\langle s_0 \rangle \langle s_1 \rangle \ldots \langle s_{n-1} \rangle D(M_{s0} \langle M_{s1} \rangle \ldots \langle M_{sn-1} \rangle) ;
\]

let

\[
P_t = \langle M_t \rangle = D\langle t_0 \rangle M_{t0} \langle t_1 \rangle M_{t1} \ldots \langle t_{p-1} \rangle M_{tp-1} = D\langle t_0 \rangle \langle t_1 \rangle \ldots \langle t_{p-1} \rangle D(M_{t0} \langle M_{t1} \rangle \ldots \langle M_{tp-1} \rangle).
\]
Table 4. Signs and absolute values of the orthogonal polynomials.

### N even

<table>
<thead>
<tr>
<th>X</th>
<th>λ</th>
<th>( \beta_1 M_1 )</th>
<th>( \beta_2 M_2 )</th>
<th>( \beta_3 M_3 )</th>
<th>( \beta_4 M_4 )</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>- (N-1)/2  ( - (N-1) )</td>
<td>+ 2(N-1)(N-2)</td>
<td>- 2(N-1)(N-2)(N-3)</td>
<td>+ ( 8(N-1)(N-2) ) (N-3)(N-4)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>- (N-3)/2  ( - (N-3) )</td>
<td>(</td>
<td>2(N-2)(N-7)</td>
<td>)</td>
</tr>
<tr>
<td>-1.5</td>
<td>- 3</td>
<td>(</td>
<td>(28-N^2)</td>
<td>)</td>
<td>(</td>
</tr>
<tr>
<td>-0.5</td>
<td>- 1</td>
<td>(</td>
<td>(N^2-4)</td>
<td>)</td>
<td>+ ( 3(N^2-4) )</td>
</tr>
<tr>
<td>0.5</td>
<td>+ 1</td>
<td>(</td>
<td>(N^2-4)</td>
<td>)</td>
<td>- (</td>
</tr>
<tr>
<td>1.5</td>
<td>+ 3</td>
<td>(</td>
<td>(28-N^2)</td>
<td>)</td>
<td>(</td>
</tr>
<tr>
<td>N-1</td>
<td>2</td>
<td>+ (N-1) ( + (N-1) )</td>
<td>+ 2(N-1)(N-2)</td>
<td>+ 2(N-1)(N-2)(N-3)</td>
<td>+ ( 8(N-1)(N-2) ) (N-3)(N-4)</td>
</tr>
</tbody>
</table>

### N odd

<table>
<thead>
<tr>
<th>X</th>
<th>λ</th>
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<th>( \beta_3 M_3 )</th>
<th>( \beta_4 M_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>- (N-1)/2  ( - (N-1) )</td>
<td>+ 2(N-1)(N-2)</td>
<td>- 2(N-1)(N-2)(N-3)</td>
<td>+ ( 8(N-1)(N-2) ) (N-3)(N-4)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>- (N-3)/2  ( - (N-3) )</td>
<td>(</td>
<td>13-N^2</td>
<td>)</td>
</tr>
<tr>
<td>-2/2</td>
<td>- 2</td>
<td>(</td>
<td>13-N^2</td>
<td>)</td>
<td>+ ( 6(N^2-9) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( 0 \text{ unspecified} )</td>
<td>(</td>
<td>N^2-1</td>
<td>)</td>
</tr>
<tr>
<td>2/2</td>
<td>+ 2</td>
<td>(</td>
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<td>- ( 6(N^2-9) )</td>
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<tr>
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<td>+ 2(N-1)(N-2)</td>
<td>+ 2(N-1)(N-2)(N-3)</td>
<td>+ ( 8(N-1)(N-2) ) (N-3)(N-4)</td>
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</table>

\[ \lambda \] denotes absolute value.
Table 5. Numerical values of $P_r = \lambda P_r$

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<td>+1 0 -2 0 +6</td>
<td>+1 -1 -3 +1 +1 -5 +15</td>
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<tr>
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<td>+1 0 -4 0 +6 0 -20</td>
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<tr>
<td>1 1 3</td>
<td>+1 +2 +2 +1 +1</td>
<td>+1 +1 -3 -1 +1 +5 +15</td>
</tr>
<tr>
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<td>+1 +2 +2 +1 +1</td>
<td>+1 +2 0 -1 -7 -4 -6</td>
</tr>
<tr>
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<td>+1 +2 +2 +1 +1</td>
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<tr>
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<table>
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<td>$P_1$</td>
<td>$P_2$</td>
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<tr>
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<td>+1 -1 -1 -3</td>
<td>+1 -1 -4 +1 +2 -10</td>
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<tr>
<td>1 2</td>
<td>+1 +3 +1 +1</td>
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</tr>
<tr>
<td>1 2</td>
<td>+1 +3 +1 +1</td>
<td>+1 +5 +5 +5 +1 +1</td>
</tr>
<tr>
<td>1 2</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{5}{3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
</tr>
<tr>
<td>$P_0$</td>
</tr>
<tr>
<td>+1 -7 +7 -7 +7 -7 +7 -7 +1 -1</td>
</tr>
<tr>
<td>+1 -5 +1 +5 -13 +23 -5 +7</td>
</tr>
<tr>
<td>+1 -3 -3 +7 -3 -17 +9 -21</td>
</tr>
<tr>
<td>+1 -1 -5 +3 +9 -15 -5 +35</td>
</tr>
<tr>
<td>+1 +1 -5 -3 +9 +15 -5 -35</td>
</tr>
<tr>
<td>+1 +3 -3 -7 -3 +17 +9 +21</td>
</tr>
<tr>
<td>+1 +7 +7 +7 +7 +7 +1 +1</td>
</tr>
<tr>
<td>1 2 1 2 7 7 11 11</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>
Then, as before in the $m \times n \times p$ factorial experiment, any contrast between the $y_{ijk}$ can be written

$$ (A_{t}B_{s}C_{r}) = J' P_s P_t Y $$

$$ = J' \omega_r \omega_s \omega_t M_{rs} M_{st} Y $$

when not all three $r, s, t$ are simultaneously zero.

If we wish to confound the contrast $A_{t}B_{s}C_{r}$ the confounding will be based on whether $\omega_r, \omega_s, \omega_t$ is $-$, or $+$, or $0$, from the following factor combinations.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>ijk</td>
<td>0 1 2 $\cdots$ $(m-1)$</td>
<td>0 1 2 $\cdots$ $(n-1)$</td>
<td>0 1 $\cdots$ $(p-1)$</td>
</tr>
<tr>
<td>rst</td>
<td>$\omega_r$ $\omega_r$ $\omega_r$ $\omega_r$, $m-1$</td>
<td>$\omega_s$ $\omega_s$ $\omega_s$ $\omega_s$, $n-1$</td>
<td>$\omega_t$ $\omega_t$ $\omega_t$ $\omega_t$, $p-1$</td>
</tr>
</tbody>
</table>

Any treatment combination $ijk$ ($i = 0, 1, \cdots, m-1; j = 0, 1, \cdots, n-1; k = 0, 1, \cdots, p-1$) has block indicator $\omega_r \omega_s \omega_t = -$ or $+$, or $0$, depending on the sizes of $m, n, p$ and the degrees $r, s, t$. If $M_r = M_s = M_t = I$, contrast $(A B C)$ will be completely confounded with blocks based on the above confounding scheme; otherwise, it will be partially confounded. Note that $P_2$ for 4 levels and $P_3$ for 3 levels are the only cases within the range of Table 5 that will allow complete confounding.

5.3 Partitioning of the Polynomial Values

Inspection of Tables 4 and 5 reveals that for even $N$, the elements of any even degree polynomials ($P_2, P_4, \cdots$) will consist of $N_1$ duplicated negative elements (a total of $N_1 = 2N_1$ negative elements) and $N_2$ duplicated positive elements ($N_2 = 2N_2$); hence $2N_1 + 2N_2 = N$. These
polynomials will be denoted by \( P_e \) when the elements are rearranged so that the first \( 2N_1 \) elements are negative and the last \( 2N_2 \) are positive. The vector of these elements is

\[
P'_e = (e_{11}, e_{12}, \ldots, e_{1N_1}, e_{11}, e_{12}, \ldots, e_{1N_1}, e_{2N_2}, e_{2N_2})
\]

\[
= (P'_e^-; P'_e^+)
\]

where \( e_{11} \) denotes a negative element and \( e_{21} \) a positive element.

The odd-degree polynomials \( (P_1, P_3, \ldots) \) will consist of \( Q = N/2 \) negative elements and \( Q \) positive elements, the absolute value of each negative element being matched by a duplicate positive element. These polynomials will be denoted by \( P_u \) if the elements are rearranged so that the first \( Q \) elements are negative and the last \( Q \) are positive. The vector of these elements is

\[
P'_u = (-u_1, -u_2, \ldots, -u_Q; u_1, u_2, \ldots, u_Q)
\]

\[
= (P'_u^-; P'_u^+)
\]

Note that for \( P_u \), it is unnecessary to use a subscript notation to denote which elements are negative and which positive. If the observations are ordered according to either a \( P_e \) or a \( P_u \), the other even degree polynomials when ordered conformably will be indicated as \( P_E|P_e \) or \( P_E|P_u \); similarly the other odd degree polynomials when ordered conformably will be indicated as \( P_U|P_e \) or \( P_U|P_u \). E includes \( e \) and U includes \( u \), unless otherwise specified. These vectors are indicated in Table 6a.

In Table 6a the elements of \( P_E|P_e \) and \( P_U|P_e \) have the same subscript notation as for \( P_e \); similarly \( P_E|P_u \) and \( P_U|P_u \) have the same subscript notation as \( P_u \). Each pair of equal elements in any \( P_e \) will correspond to a pair of equal (in absolute value) elements of any \( P_u \); similarly each pair
Table 6a. Partitioning of the orthogonal polynomials.  

<table>
<thead>
<tr>
<th>$p_e$</th>
<th>$p_E p_e$</th>
<th>$p_U p_e$</th>
<th>$p_U$</th>
<th>$p_E p_u$</th>
<th>$p_U p_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{11}$</td>
<td>$E_{11}$</td>
<td>$U_{11}$</td>
<td>$-u_1$</td>
<td>$E_1$</td>
<td>$-U_1$</td>
</tr>
<tr>
<td>$e_{11}$</td>
<td>$E_{11}$</td>
<td>$-U_{11}$</td>
<td>$-u_2$</td>
<td>$E_2$</td>
<td>$-U_2$</td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>$E_{12}$</td>
<td>$U_{12}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{1N_1}$</td>
<td>$E_{1N_1}$</td>
<td>$U_{1N_1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{1N_1}$</td>
<td>$E_{1N_1}$</td>
<td>$-U_{1N_1}$</td>
<td>$-u_Q$</td>
<td>$E_Q$</td>
<td>$-U_Q$</td>
</tr>
<tr>
<td>$\delta e_0$</td>
<td>$\delta E_0$</td>
<td>0</td>
<td>$u_1$</td>
<td>$E_1$</td>
<td>$U_1$</td>
</tr>
<tr>
<td>$e_{21}$</td>
<td>$E_{21}$</td>
<td>$U_{21}$</td>
<td>$u_2$</td>
<td>$E_2$</td>
<td>$U_2$</td>
</tr>
<tr>
<td>$e_{21}$</td>
<td>$E_{21}$</td>
<td>$-U_{21}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{22}$</td>
<td>$E_{22}$</td>
<td>$U_{22}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{22}$</td>
<td>$E_{22}$</td>
<td>$-U_{22}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{2N_2}$</td>
<td>$E_{2N_2}$</td>
<td>$U_{2N_2}$</td>
<td>$u_Q$</td>
<td>$E_Q$</td>
<td>$U_Q$</td>
</tr>
<tr>
<td>$e_{2N_2}$</td>
<td>$E_{2N_2}$</td>
<td>$-U_{2N_2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1-\delta)e_0$</td>
<td>$(1-\delta)E_0$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If $N = 2Q + 1$  

$\delta e_0$ is negative and $e_{21}$ is positive; $u_1$ is positive. $E_1$ and $U_1$ are negative or positive elements in $P_E$ and $P_U$, corresponding to elements in $P_e$ and $P_u$. If $N = 2Q$, omit the last line and the $e_0$ lines. If $N = 2Q + 1$, a negative $e_0$ belongs to the $e_1$ group and a positive $e_0$ to the $e_2$ group; this is indicated by $\delta e_0$ where $\delta = 1$ if $e_0$ is negative and $\delta = 0$ if $e_0$ is positive.
of equal elements of $P_u$ will correspond to a pair of equal (in absolute value) elements of any $P_u$. Hence if one partitions $P_e'$ into the two parts $P_{e-}'$ and $P_{e+}'$, $P_e|P_u$ will be unconfounded; similarly for $P_e|P_u$ if $P_u$ is partitioned into $P_{u-}$ and $P_{u+}$. $P_e|P_e$ and $P_u|P_u$ will be confounded under these conditions. These results are summarized in Table 6b.

Table 6b. Sums of elements of conformable parts.

<table>
<thead>
<tr>
<th>Block</th>
<th>$N = 2Q$</th>
<th>$N = 2Q + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_e</td>
<td>P_e$</td>
</tr>
<tr>
<td>1 (-)</td>
<td>$2\sum E_{11}$</td>
<td>0</td>
</tr>
<tr>
<td>2 (+)</td>
<td>$-2\sum E_{11}$</td>
<td>0</td>
</tr>
<tr>
<td>3 (0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\delta = 1$, if $e_o$ is negative; 0 if $e_o$ is positive.

For odd $N (N = 2Q + 1)$, the odd degree polynomials have a zero element and the even degree polynomials an extra negative or positive element; the latter is indicated by $e_o$ or $E_o$. Hence

$$P_e' = (P_{e-}', P_{e+}'); \ P_u' = (P_{u-}', P_{u+}', 0)$$

where either $P_{e-}'$ or $P_{e+}'$ has $Q + 1$ elements, depending on whether $e_o$ is - or +. Hence confounding based on these subvectors will result in unequal sized blocks. In this case $P_e|P_u$ will be confounded, but $P_u|P_e$ will still remain unconfounded. These results are also summarized in Tables 6a and 6b.

5.4 Bundles of Homogeneous Order

A bundle of homogeneous order was previously defined as that set of contrasts with properties such that when any single contrast belonging to
the set is confounded, all other contrasts belonging to the set will also be confounded.

The theorems in Chapter 3 could be extended to study the confounding of \((A_T B_{C_1} T)\) \((A_T B_{C_2} T)\); however, the results in Table 6b can now be employed to examine the confounding of \((A_T B_{C_1} T)\) \((A_T B_{C_2} T)\) for different values of \(m, n, p\) and the subscripts. For example, if \(A_{1_1} B_{1_1} C_1\) is confounded for \(m = n = p = 4\), then \((A_{1_1} B_{1_2} C_3)\), \((A_{1_1} B_{1_3} C_1)\), \((A_{1_1} B_{1_3} C_2)\), \((A_{1_1} B_{1_3} C_3)\), \((A_{3_3} B_{1_3} C_1)\), \((A_{3_3} B_{1_3} C_2)\) will be confounded automatically. Hence the homogeneous bundle of contrasts corresponding to the contrast \((111)\) will be \((111, 113, 131, 311, 133, 313, 331, 333)\), indicated as \([111]\).

Considering \(m \times n \times p\) factorial experiments, Table 7 shows the bundles of contrasts of homogeneous orders obtained in the \(4 \times 4 \times 4\) and \(4 \times 3 \times 3\) experiments. The second-last column in the table gives the bundles obtained by the application of the theory of Galois fields, where these exist.
A Galois field can exist only for a prime number or a power of a prime number. Hence the six contrasts of the interaction \(AB\) in the \(4 \times 3 \times 3\) factorial cannot be split into bundles according to Galois theory, since 6 is neither a prime number nor a power of a prime number.
Table 7. Allocation of contrasts to homogeneous bundles.

<table>
<thead>
<tr>
<th>Effect</th>
<th>Constituent contrasts of bundles</th>
<th>Number of contrasts per bundle</th>
<th>Number of contrasts per Galois bundle</th>
<th>df</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>300</td>
<td>1,2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>200 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>030</td>
<td>1,2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>020 010</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>003</td>
<td>1,2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>002 001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>330</td>
<td>1,2,3,4</td>
<td>3,3,3</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>310</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>230 320 130</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>220 210 120 110</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>303</td>
<td>1,2,3,4</td>
<td>3,3,3</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>301</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>203 302 103</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>202 201 102 101</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BC</td>
<td>033</td>
<td>1,2,3,4</td>
<td>3,3,3</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>031</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>023 032 013</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>022 021 012 011</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ABC</td>
<td>333</td>
<td>1,2,3,4</td>
<td>3,3,3,3</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>331</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>313</td>
<td></td>
<td>4,4,4,8</td>
<td>3,3,3,3</td>
</tr>
<tr>
<td></td>
<td>313</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>233 323 332 311</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>231 321 312 311</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>223 232 322 312 323 123 132 113</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>222 221 212 122 211 121 112 111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>300</td>
<td>1,2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>200 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B 1/</td>
<td>020</td>
<td>1,2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>020 010</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C 1/</td>
<td>002</td>
<td>1,2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>002 001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>320 310</td>
<td>1,1,2,2</td>
<td>nonexistent</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>220 210 120 110</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>302 301</td>
<td>1,1,2,2</td>
<td>nonexistent</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>202 201 102 101</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BC 1/</td>
<td>022</td>
<td>1,1,1,1</td>
<td>2,2</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>022 021 012 011</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ABC</td>
<td>322</td>
<td>1,1,1,2</td>
<td>nonexistent</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>321 312 311</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>222 221 212 122 211 121 112 111</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1/If 020 is confounded, 010 will be unconfounded; if 010 is confounded, 020 will also be confounded. Similarly for C and 011 of BC.
CHAPTER 6

The Confounding Pattern of a Contrast

6.1 Bundles of Heterogeneous Order

Inspection of Table 5 reveals that when \( n = 2, 4, \) or 8 the same number of negative as positive elements occur in both the even and the odd degrees; when \( n = 6, \) only in the odd degrees. Hence, in these cases by Theorem 3.2 and its corollaries, the confounding of any higher order interaction contrast will not confound any lower order interaction contrast. In the other cases the heterogeneous bundle must be determined for each particular contrast confounded.

6.1.1 Confounding on \((A_u B_u)\).

Consider contrast \((A_u B_u) \ (u = \text{odd})\) of an \( m \times n \) factorial experiment when \( m \) and \( n \) are both odd \((m = 2r + 1, n = 2s + 1)\). Let \( \gamma_{ij} = 1 \). Then the elements of the expression \((A_u B_u) \ (y = 1)\) are given by the elements in the body of Table 8a. Any element there is obtained by multiplication of the appropriate marginal elements, where the marginal elements are those respectively of the orthogonal polynomials \( A_u \) and \( B_u \).

Confounding of \((A_u B_u)\) means blocking the elements in the pattern shown in Table 8a: the positive block consists of the elements in the upper left-hand corner and the elements in the lower right-hand corner; the negative block consists of the elements in the upper right-hand corner and lower left-hand corner; the zero block (omitted if both \( m \) and \( n \) are even) consists of the elements in the central "cross."

Consider \( A_u \ (y = 1) \mid (A_u B_u) \). From Table 8a we see that the elements in the upper left-hand corner have the same absolute values as the
Table 8a. Blocking pattern due to confounding of \((A_uB_u)\) when \(m\) and \(n\) are odd.

<table>
<thead>
<tr>
<th>(A_u)</th>
<th>(-u_b1)</th>
<th>(-u_b2)</th>
<th>(\ldots)</th>
<th>(-u_bj)</th>
<th>(\ldots)</th>
<th>(-u_{bs})</th>
<th>(0)</th>
<th>(u_{b1})</th>
<th>(u_{b2})</th>
<th>(\ldots)</th>
<th>(u_{bj})</th>
<th>(\ldots)</th>
<th>(u_{bs})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-u_{a1})</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
</tr>
<tr>
<td>(-u_{a2})</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(-u_{ar})</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-,)</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>((-0,)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

\(\text{If } n \text{ is even, the zero column is removed; if } m \text{ is even, the zero row is removed.}\)

rs elements in the lower right-hand corner, but will have opposite signs, and will accordingly cancel in the positive block; similarly for the negative and zero blocks; hence \(A_u(A_uB_u)\) will be unconfounded. In a similar manner \(B_u(A_uB_u)\) can be shown to be unconfounded. These results are valid for all values of \(m\) and \(n\).

Next consider \(A_E(A_uB_u)\). For \(m\) and \(n\) both odd, the sum of the elements in the various sectors of Table 8a are presented in Table 8b, using the notation of Tables 6a and 6b. Since \(A_E (y = 1) (A_uB_u) \neq 0\), this contrast is confounded with blocks. Similarly \(B_E(A_uB_u)\) is confounded with blocks when \(m\) and \(n\) are both odd. If \(m\) is even and \(n\) is odd, the row of the zero block is removed in Table 8b and \(E_{ao} = 0\); hence \(A_E(A_uB_u)\) is
unconfounded with blocks; however, \( B_E \mid (A_u B_u) \) will still be confounded.

Similarly, if \( m \) is odd and \( n \) is even, \( B_E \mid (A_u B_u) \) is unconfounded with blocks, but \( A_E \mid (A_u B_u) \) will still be confounded (\( E_{ao} \neq 0 \), although the column of the zero block is removed). If both \( m \) and \( n \) are even, both \( A_E \mid (A_u B_u) \) and \( B_E \mid (A_u B_u) \) are unconfounded with blocks.

Table 6b. Sums of elements of \( A_E \mid (A_u B_u) \) \((y = 1)\) corresponding to blocking pattern in Table 6a.

<table>
<thead>
<tr>
<th>( A_E )</th>
<th>( A_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{a1} )</td>
<td>( -(sE_{ao})/2 )</td>
</tr>
<tr>
<td>( E_{a2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( E_{ai} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( E_{ar} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( E_{ao} )</td>
<td>( -sE_{ao} )</td>
</tr>
<tr>
<td>( E_{al} )</td>
<td>( E_{a2} )</td>
</tr>
<tr>
<td>( E_{ai} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( E_{ar} )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

Finally let us consider the two-factor interactions. From the work on homogeneous bundles we know that any odd order contrast \( (A_u B_u) \mid (A_u B_u) \) will be confounded for any \( m \) and \( n \). That is, the contrasts \( (UU) \) belong to \([uu]\), the homogeneous bundle of the contrast \((uu)\). The amount of confounding is easily determined from Table 6b.
\[(A_u B_u) (y = 1) \big| (A_u B_u)_+ = 2(\sum u_{a1})(\sum u_{b1})\]
\[(A_u B_u) (y = 1) \big| (A_u B_u)_- = -2(\sum u_{a1})(\sum u_{b1})\]

The same applies to \((A_e B_e) \big| (A_u B_u)\) for both \(m\) and \(n\) odd; however, if either \(m\) or \(n\) is even, \((A_e B_e) \big| (A_u B_u)\) is unconfounded with blocks. Finally we find that \((A_e B_u) \big| (A_u B_u)\) and \((A_u B_e) \big| (A_u B_u)\) are not confounded with blocks for any \(m\) or \(n\). For example

\[(A_e B_u) (y = 1) \big| (A_u B_u)_+ = (-E_{ac}/2)(-\sum u_{b1}) + (-E_{ac}/2)(\sum u_{b1}) = 0\]

### 6.1.2 Confounding on \((A_u B_e)\)

As indicated in Table 6b, the blocking pattern for \(B_e\) will have only two parts, regardless of whether \(n\) is even or odd. If \(n\) is even, there are \(n_- = 2n_1\) negative and \(n_+ = 2n_2\) positive elements; if \(n\) is odd and the middle element is negative, \(n_- = 2n_1 + 1\); if \(n\) is odd and the middle element is positive, \(n_+ = 2n_2 + 1\). The blocking pattern based on confounding of \((A_u B_e)\) is presented in Table 9.

Using Table 9, we see that

\[A_u (y = 1) \big| (A_u B_e)_+ = (2n_2 - 2n_1 + \delta_2 - \delta_1)\sum u_{a1}\]
\[A_u (y = 1) \big| (A_u B_e)_- = (2n_1 - 2n_2 + \delta_1 - \delta_2)\sum u_{a1}\]
\[A_u (y = 1) \big| (A_u B_e)_0 = 0\]

Hence \(A_u \big| (A_u B_e)\) will be unconfounded with blocks only when \(n\) is even \((\delta_1 = \delta_2 = 0)\) and \(n_1 = n_2\); that is, \(n_- = n_+\); this is true for all \(m\). From Table 6b we see that \(B_u (y = 1) \big| B_e = 0\) for all sectors; hence

\(B_u \big| (A_u B_e)\) is unconfounded with blocks for all \(m\) and \(n\). The sum of the elements of \(B_e (y = 1) \big| (A_u B_e)\) has the same absolute value but opposite signs for the two parts of each block. Hence \(B_e \big| (A_u B_e)\) is unconfounded with blocks for all \(m\) and \(n\). \(A_u \big| (A_u B_e)\) will be unconfounded with blocks
Table 9. Blocking pattern due to the confounding of \((A_u B_e)\) and contrast totals.

<table>
<thead>
<tr>
<th>(A_u)</th>
<th>(e_{bl1} e_{b12} \cdots e_{blm} e_{b1m} e_{bo})</th>
<th>(e_{b21} e_{b22} \cdots e_{b2n} e_{b2n} e_{bo})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-u_{al})</td>
<td>(A_u \ (y = 1) = -(2n_1 + \delta_1)\sum U_{ai})</td>
<td>(A_u \ (y = 1) = -(2n_2 + \delta_2)\sum U_{ai})</td>
</tr>
<tr>
<td>(-u_{si})</td>
<td>(A_E \ (y = 1) = -(2n_1 + \delta_1)E_{ac}/2)</td>
<td>(A_E \ (y = 1) = -(2n_2 + \delta_2)E_{ac}/2)</td>
</tr>
<tr>
<td>(-u_{ar})</td>
<td>(B_E \ (y = 1) = r(2\sum E_{blj} + \delta_1 E_{bo}))</td>
<td>(B_E \ (y = 1) = -r(2\sum E_{blj} + \delta_1 E_{bo}))</td>
</tr>
<tr>
<td>0</td>
<td>(A_u = 0; \ A_E = (2n_1 + \delta_1)E_{ac};) (B_E: \text{set } r = 1 \text{ above})</td>
<td>(A_u = 0; \ A_E = (2n_2 + \delta_2)E_{ac};) (B_E: \text{set } r = 1 \text{ above})</td>
</tr>
<tr>
<td>(u_{al})</td>
<td>(A_u \ (y = 1) = (2n_1 + \delta_1)\sum U_{ai})</td>
<td>(A_u \ (y = 1) = (2n_2 + \delta_2)\sum U_{ai})</td>
</tr>
<tr>
<td>(u_{si})</td>
<td>(A_E \ (y = 1) ) as in ((-\cdot)) sector</td>
<td>(A_E \ (y = 1) ) as in ((-\cdot)) sector</td>
</tr>
<tr>
<td>(u_{ar})</td>
<td>(B_E \ (y = 1) )</td>
<td>(B_E \ (y = 1) )</td>
</tr>
</tbody>
</table>

\(\text{\textsuperscript{1}}\)Cmit the zero row if \(m = 2r\). If \(n = 2s\), \(e_{bo}\) does not exist; hence, set \(\delta_1 = \delta_2 = 0\). If \(n = 2s + 1\) and \(e_{bo}\) is \(-\), \(\delta_1 = 1\) and \(\delta_2 = 0\); if \(e_{bo}\) is \(+\), \(\delta_1 = 0\) and \(\delta_2 = 1\).

only if \(m\) is even (so that \(E_{ac}\) does not exist) for all values of \(n\); if \(m\) is odd, \(A_E| (A_u B_e)\) will be confounded with blocks for all values of \(n\).

Since \(B_E \ (y = 1) \big| B_e = 0\) for all sectors in Table 6b, \((A_E B_U) \big| (A_u B_e)\) and \((A_U B_U) \big| (A_u B_e)\) are unconfounded for all \(m\) and \(n\). As for \((A_E B_E)\), we note that

\[(A_E B_E) \ (y = 1) \big| (A_u B_e) \bigg(\bigg(\sum E_{bl} + \delta_1 E_{bo}\bigg)(-(E_{ac}/2) + (E_{ac}/2)) = 0\)

and similarly for the other blocks. Hence \((A_E B_E) \big| (A_u B_e)\) is also unconfounded with blocks for all \(m\) and \(n\). Finally we note that
\[(A_{ue} B_{e}) (y = 1) \mid (A_{ue} e) = -2(2 \sum_{blj} + \delta_{lc} b_{oc}) \sum_{al1}\]
\[(A_{ue} B_{e}) (y = 1) \mid (A_{ue} B_{e}) = 2(2 \sum_{blj} + \delta_{lc} b_{oc}) \sum_{al1}\]
\[(A_{ue} B_{e}) (y = 1) \mid (A_{ue} e) = 0.\]

Hence \((A_{ue} B_{e})\) is confounded with blocks for all \(m\) and \(n\); i.e., \((UE)\) belongs to \([ue]\).

6.1.3 Confounding on \((A_{ue} B_{e})\).

The blocking pattern here can be set up as in Table 9 by using \(e_{b1}\) and \(e_{b2}\) for \(B_{e}\) and \(e_{a1}\) and \(e_{a2}\) for \(A_{e}\); this is done in Table 10.

Table 10. Blocking pattern due to the confounding of \((A_{ue} B_{e})\) and contrast totals.

<table>
<thead>
<tr>
<th>(B_{e})</th>
<th>(A_{e})</th>
<th>(e_{b1})</th>
<th>(e_{b1l})</th>
<th>(e_{b1n})</th>
<th>(e_{b2})</th>
<th>(e_{b2l})</th>
<th>(e_{b2n})</th>
<th>(e_{c1})</th>
<th>(e_{c2})</th>
<th>(e_{c2n})</th>
<th>(e_{c2m})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_{al})</td>
<td>(A_{e}(y=1) = (2n_1 + \delta_{a1} b_{oc})(2 \sum_{ali} + \delta_{al} E_{ao}))</td>
<td>(A_{e}(y=1) = (2n_2 + \delta_{a2} b_{oc})(2 \sum_{ali} + \delta_{al} E_{ao}))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e_{aln})</td>
<td>(B_{e}(y=1) = (2m_1 + \delta_{al} b_{oc})(2 \sum_{blj} + \delta_{bl} E_{bo}))</td>
<td>(B_{e}(y=1) = -(2m_2 + \delta_{al} b_{oc})(2 \sum_{blj} + \delta_{bl} E_{bo}))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e_{aln})</td>
<td>(A_{e}(y=1) = -(2n_1 + \delta_{a1} b_{oc})(2 \sum_{ali} + \delta_{al} E_{ao}))</td>
<td>(A_{e}(y=1) = -(2n_2 + \delta_{a2} b_{oc})(2 \sum_{ali} + \delta_{al} E_{ao}))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e_{aln})</td>
<td>(B_{e}(y=1) = (2m_2 + \delta_{al} b_{oc})(2 \sum_{blj} + \delta_{bl} E_{bo}))</td>
<td>(B_{e}(y=1) = -(2m_2 + \delta_{al} b_{oc})(2 \sum_{blj} + \delta_{bl} E_{bo}))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1/ If \(m = 2r\), \(e_{ao}\) does not exist (\(\delta_{al} = \delta_{a2} = 0\)); similarly for \(\delta_{bl}\) and \(\delta_{b2}\) if \(n = 2s\). If \(m = (2r+1)\) and \(e_{ao}\) is -, \(\delta_{al} = 1\) and \(\delta_{a2} = 0\); if \(e_{ao}\) is +, \(\delta_{al} = 0\) and \(\delta_{a2} = 1\); similarly for \(\delta_{bl}\) and \(\delta_{b2}\) if \(n = (2s+1)\).
From Table 10 we see that

\[ A_E \mid (A_e B_e) = (2n_1 - 2n_2 + \delta_{b1} - \delta_{b2})(2 \sum_{ai} \delta_{al} + \delta_{al} E_{ao}) \]

This expression equals zero only when \( n \) is even (\( \delta_{b1} = \delta_{b2} = 0 \)) and \( n_1 = n_2 \), so that \( n_- = n_+ \), but \( m \) can be any value. Similarly

\[ B_E \mid (A_e B_e) = (2m_1 - 2m_2 + \delta_{a1} - \delta_{a2})(2 \sum_{blj} \delta_{bl} + \delta_{bl} E_{bo}) \]

hence \( B_E \mid (A_e B_e) \) is unconfounded with blocks only when \( m \) is even and \( m_1 = m_2 \) (so that \( m_- = m_+ \)), but \( n \) can be any value. As in system 6.1.2, \( A_U \mid (A_e B_e) \) and \( B_U \mid (A_e B_e) \) are unconfounded with blocks for all \( m \) and \( n \).

As for interactions, those involving \( A_U \) or \( B_U \) will be unconfounded; however, \( (A_e B_e) \mid (A_e B_e) \) will be confounded with blocks for all \( m \) and \( n \):

\[ A_{EE} \mid (A_e B_e) = 2(2 \sum_{ai} \delta_{al} + \delta_{al} E_{ao})(2 \sum_{blj} \delta_{bl} + \delta_{bl} E_{bo}) \]

Hence \( (EE) \) belongs to \( [ee] \).

6.1.4 Summary

Table 11 summarizes all the findings on polynomial confounding of main effects and two-factor interactions when specific two-factor interactions are confounded. The results in Table 11 can be verified by applying the conditions of Theorem 3.3, Corollary 3.3.1, and Table 6b. Consider, for instance, the value of \( A_U \mid (A_e B_e) \), when \( m \) is odd and \( n \) is even; since \( \text{tr} A_{U3} = 0 \), \( A_U \mid (A_e B_e) \) will be unconfounded when \( n_+ = n_- \).
Table 11. Confounding of general contrasts when specific $A B$ contrasts are confounded.  

Specific confounded effect

<table>
<thead>
<tr>
<th>Value of $m$ and $n$</th>
<th>$(A_{u}B_{u})$</th>
<th>$(A_{u}B_{e})$</th>
<th>$(A_{e}B_{u})$</th>
<th>$(A_{e}B_{e})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{u}E_{u}(A_{u}E_{u})(A_{u}E_{u})$</td>
<td>$A_{u}A_{u}(A_{u}E_{u})$</td>
<td>$B_{u}B_{u}(A_{u}B_{u})$</td>
<td>$A_{e}E_{u}(A_{u}E_{u})$</td>
</tr>
<tr>
<td>$n$ even $n_{=}n_{+}$</td>
<td># # # # # # # # # # # #</td>
<td>K K K K K K K K K K K K</td>
<td># # # # # # # # # # # #</td>
<td>K K K K K K K K K K K K</td>
</tr>
<tr>
<td>$m$ even $m_{=}m_{+}$</td>
<td>0 K K 0 K K 0 K K K K K K</td>
<td>0 K K 0 K K K K K K K K</td>
<td>0 K K 0 K K K K K K K K</td>
<td>0 K K 0 K K K K K K K K</td>
</tr>
<tr>
<td>$n$ odd $m_{=}m_{+}$</td>
<td># # # # # # # # # # # #</td>
<td>K K K K K K K K K K K K</td>
<td># # # # # # # # # # # #</td>
<td>K K K K K K K K K K K K</td>
</tr>
</tbody>
</table>

1/ K denotes confounding. # indicates an impossible set of $m$ and $n$; for example $m_{-}$ must equal $m_{+}$ for all odd contrasts, $A_{u}$. All main effects and two-factor interactions omitted from this table are not confounded for any $m$ or $n$. 

6.2 Rules for Obtaining Confounding Patterns

6.2.1 Introduction

When more than two factors are involved in an experiment, the previous method to obtain the heterogeneous bundle of a particular contrast becomes unwieldy, and mistakes can very easily be made. As a substitute, a symbolic multiplication process will now be derived which allows easy generalization.

Suppose we have an m x n experiment in which \( A \times B \), denoted by (ee), is confounded with blocks so that \( m_+ \neq m_- \) and \( n_+ \neq n_- \). We introduce the symbol (CO) to denote the "dummy" contrast \( A_0 B_0 \). Consider the following scheme:

\[
\begin{array}{ccc|c}
(a) & (EO) & (CO) & (e0) \\
(b) & (OE) & (CO) & (0e) \\
(c) & (EB) & (EO) & (OE) & (CO) & (ee) \\
\end{array}
\]

The scheme is to read as follows:

(i) By the condition for confounding \( (EO) | (e0) \), row (a), and the conditions for confounding \( (OE) | (0e) \), row (b), we imply the confounding of \( (EO) | (ee) \) and \( (OE) | (ee) \), row (c), by Corollary 3.3.2 because \( m_+ \neq m_- \) and \( n_+ \neq n_- \).

(ii) The confounding of \( (ee) \) also results in the confounding of \( (EE) \), row (c).

Hence the contrasts on the left side of row (c) are then confounded when \( (ee) \) is confounded. The contrasts left of the vertical line of row (c) are found by multiplying those of row (a) to the left of the vertical line by the corresponding contrasts in row (b); and similarly for the right side. In actual practice, one starts with \( (ee) \), then constructs
(Oe) and (eo), then the left side of (a) and (b) and finally the left side of (c).

The introduction of the dummy contrast is an all-important step, since this fictitious contrast allows us to obtain the heterogeneous bundle of a higher order interaction contrast from the homogeneous bundles of those lower order interaction contrasts, whose symbolic product is equal to the higher order interaction contrast.

The above scheme gives the heterogeneous bundle of the contrast (ee) only when \( m_+ \neq m_- \) and \( n_+ \neq n_- \). Generalization to other values of \( m \) and \( n \) and other polynomial contrasts is required. It will be shown that by using the properties of orthogonal polynomial contrasts and by adopting appropriate conventions with respect to the dummy contrast, the heterogeneous bundle of any higher order interaction orthogonal polynomial contrast can be obtained from the symbolic multiplication scheme. This is accomplished by the provisions of Corollary 3.3.2 when constituting the heterogeneous bundle.

6.2.2 Polynomial confounding patterns for an \( m \times n \) experiment

One would prefer to substitute the bundle notation for the contrast notation in Section 6.2.1 and thus generalize these results to both odd and even contrasts. If all confounding contrasts were of even degree, \([e0] = (EO)\) for all \( m \); hence the use of the bundle notation would be satisfactory. However, \([u0] = (U0)\) for even \( m \) and \([u0] = (U0) + (EO)\) for odd \( m \); a general bundle notation becomes unsatisfactory. In Table 12 we extend the scheme of Section 6.2.1 to each type of polynomial confounding pattern, for any \( m \) and \( n \). In order to fulfill the provisions of Corollary 3.3.2, (00) is used or omitted in different situations. Also one will have
Table 12. General procedures for determining all confounded contrasts, using polynomial confounding in a two-factor experiment.

<table>
<thead>
<tr>
<th>(rs) = (uu)</th>
<th>(EO)</th>
<th>(00)</th>
<th>(UO)</th>
<th>(u0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when m is even</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>(OE)</td>
<td>(00)</td>
<td>(OU)</td>
<td>(Ou)</td>
</tr>
<tr>
<td>Omit when n is even</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>(a) x (b)</td>
<td></td>
<td>(UU)</td>
<td>(uu)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(rs) = (ue)</th>
<th>(EO)</th>
<th>(00)</th>
<th>(UO)</th>
<th>(u0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when m is even</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>(00)</td>
<td></td>
<td>(OE)</td>
<td>(Oe)</td>
</tr>
<tr>
<td>Omit when n_+ = n_-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>(a) x (b)</td>
<td></td>
<td>(a) x (b)</td>
<td>(ue)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(rs) = (eu)</th>
<th>(EO)</th>
<th>(00)</th>
<th>(O0)</th>
<th>(e0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when m_+ = m_-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>(OE)</td>
<td>(00)</td>
<td>(OU)</td>
<td>(Ou)</td>
</tr>
<tr>
<td>Omit when n is even</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>(a) x (b)</td>
<td></td>
<td>(a) x (b)</td>
<td>(eu)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(rs) = (ee)</th>
<th>(EO)</th>
<th>(00)</th>
<th>(O0)</th>
<th>(e0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when m_+ = m_-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>(00)</td>
<td></td>
<td>(OE)</td>
<td>(Oe)</td>
</tr>
<tr>
<td>Omit when n_+ = n_-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>(00)</td>
<td>(a) x (b)</td>
<td></td>
<td>(ee)</td>
</tr>
</tbody>
</table>
separate sections for (E) and (U) contrasts; the use of these will be
ddictated by the absence or presence of each in the homogeneous bundles of
the confounding contrasts. The multiplicative schemes are simply conveni-
ent arithmetical devices to produce the results of Table II. A separate
scheme is constructed for each of the four general two-factor confounding
contrasts \((rs) = (uu), (ue), (eu)\) and \((ee)\).

6.2.3 A general rule for obtaining the confounding
patterns of any polynomial contrast in an
m x n factorial experiment

In previous chapters \((RS)\) has referred to any two-factor contrast
with \((rs)\) being a given confounding contrast. If we now use the restricted
\(R^*\) to refer to only odd degree contrasts if \(r\) is odd and to even degree
contrasts if \(r\) is even, and similarly for \(S^*\) and \(s\), we can generalize the
four parts of Table II into a single part, as in Table III.

Table III. Generalization of the parts of Table II.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(EO)</th>
<th>(00)</th>
<th>(R*O)</th>
<th>(rO)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Omit when (r = e)</td>
<td></td>
<td>Omit when (r = u)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when (m) is even</td>
<td></td>
<td>Omit when (m_+ = m_-)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th>(OE)</th>
<th>(00)</th>
<th>(OS*)</th>
<th>(Os)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Omit when (s = e)</td>
<td></td>
<td>Omit when (s = u)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omit when (n) is even</td>
<td></td>
<td>Omit when (n_+ = n_-)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ (R^*O) = \begin{cases} 
(UO) & \text{if } r = u \\
(EO) & \text{if } r = e 
\end{cases} \]

\[ (OS^*) = \begin{cases} 
(OU) & \text{if } s = u \\
(OE) & \text{if } s = e 
\end{cases} \]

6.2.4 Confounding pattern of any polynomial contrast
in an m x n x p factorial experiment

Since all the theorems that affect the confounding pattern of any
particular contrast of an m x n experiment are embodied in this rule, ex-
tension of the rule to an m x n x p or larger experiment will be valid.
Such extension is straightforward and given below to obtain the confounding pattern of any contrast (rst) of an \( m \times n \times p \) experiment.

Table 14. Confounding pattern for any polynomial contrast in an \( m \times n \times p \) factorial experiment.\(^1\)

\[
\begin{array}{|c|c|c|c|c|}
\hline
(a) & [e00] & (000) & (000) & (R^*00) & (r00) \\
\hline
\text{Omit when } r \text{ is even} & \text{Omit when } r \text{ is odd} \\
\text{Omit when } m \text{ is even} & \text{Omit when } m^+ = m^- \\
\hline
(b) & [0e0] & (000) & (000) & (OS^*0) & (0S0) \\
\hline
\text{Omit when } s \text{ is even} & \text{Omit when } s \text{ is odd} \\
\text{Omit when } n \text{ is even} & \text{Omit when } n^+ = n^- \\
\hline
(c) & [00e] & (000) & (000) & (0OT^*) & (0Ot) \\
\hline
\text{Omit when } t \text{ is even} & \text{Omit when } t \text{ is odd} \\
\text{Omit when } p \text{ is even} & \text{Omit when } p^+ = p^- \\
\hline
(d) & (a) \times (b) & (a) \times (b) & & (rs0) \\
\hline
(e) & (d) \times (c) & (d) \times (c) & & (rst) \\
\hline
\end{array}
\]

\(^1\)See footnote to Table 13. \( T^* \) is treated as \( R^* \) and \( S^* \) in that table.

In practice is is simpler to first obtain row (d) and then row (e).

6.2.4 Polynomial confounding patterns for an \( m \times n \times p \) experiment

Table 15 supplies the confounding patterns of the four basic contrasts of highest order of an \( m \times n \times p \) experiment: (uuu), (uue), (uee), (eee). Any other contrast can be found by permuting factor letters; i.e., (ueu) is obtained by interchanging \( B \) and \( C \) in (uue). As before, \( K \) denotes the contrast belonging to the heterogeneous bundle of the confounding contrast.
Table 15. Confounding patterns of the four basic contrasts of highest order of an m x n x p experiment.

<table>
<thead>
<tr>
<th>(uue)</th>
<th>m odd</th>
<th>m even</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n+ n- n+ n-</td>
<td>n+ n- n+ n-</td>
</tr>
<tr>
<td></td>
<td>p+ p- p+ p-</td>
<td>p+ p- p+ p-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(OOO)</th>
<th>K</th>
<th>K</th>
<th>K</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EOO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(OEO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EOE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EEE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(uee)</th>
<th>m odd</th>
<th>m even</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n+ n- n+ n-</td>
<td>n+ n- n+ n-</td>
</tr>
<tr>
<td></td>
<td>p+ p- p+ p-</td>
<td>p+ p- p+ p-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(OOO)</th>
<th>K</th>
<th>K</th>
<th>K</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EOO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(OEO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EOE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EEE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(uuu)</th>
<th>(eee)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All odd p even n, p even All even</td>
<td>m+ m- m+ m- m+ m- n+ n- n+ n- n+ n- p+ p- p+ p- p+ p- p+ p-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(OOO)</th>
<th>K</th>
<th>K</th>
<th>K</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EOO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(OEO)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EOE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(EEE)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>(UUU)</td>
<td>K</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
</tbody>
</table>
6.3 Determination of the Coefficients of the Block Constants

Whenever a contrast is confounded with blocks, its expectation will contain some linear combination of the block constants. The coefficients of this linear combination may be obtained by application of the Bainbridge technique (Binet et al., 1955), or by any easy extension of the rule used to derive the heterogeneous bundle of the contrast confounded.

Consider first contrast \((A_{R_{S}}C_{t})\) of the \(m \times n \times p\) experiment. When \(a_r, b_s, c_t\) are partitioned into negative and positive parts, the expression \(J' a_{r} b_{s} c_{t} J \) can be written as

\[
J' a_{r} b_{s} c_{t} J = (tr a_{r-})(tr b_{s-})(tr c_{t+}) + (tr a_{r+})(tr b_{s+})(tr c_{t-})
+ (tr a_{r-})(tr b_{s-})(tr c_{t-}) + (tr a_{r+})(tr b_{s+})(tr c_{t+})
+ (tr a_{r-})(tr b_{s-})(tr c_{t-}) + (tr a_{r+})(tr b_{s+})(tr c_{t+})
+ (tr a_{r+})(tr b_{s+})(tr c_{t-}) + (tr a_{r+})(tr b_{s+})(tr c_{t-})
= (A_{R_{S}}C_{t})_{+} + (A_{R_{S}}C_{t})_{-}
\]

The values \((A_{R_{S}}C_{t})_{+}\) and \((A_{R_{S}}C_{t})_{-}\) will respectively be the coefficients of the block constants in the expectation of the confounded contrast \((A_{R_{S}}C_{t})\).

In any particular case, when the traces of the parts of the vectors are known, the block coefficients may be obtained by inspection of the expression

\[(tr a_{r-}, tr a_{r+})(tr b_{s-}, tr b_{s+})(tr c_{t-}, tr c_{t+})\]

Consider, for example, the confounding of the contrast \((A_4B_2C_1)\) of the \(5 \times 3 \times 2\) experiment.
Since
\[ a_4 = D(+1,-4,+6,-4,+1), \]
\[ b_2 = D(+1,-2,+1), \]
\[ c_1 = D(-1,+1) \]
\[ \text{tr } a_{4-} = -8 \quad \text{tr } b_{2-} = -2 \quad \text{tr } c_{1-} = -1 \]
\[ \text{tr } a_{4+} = +8 \quad \text{tr } b_{2+} = +2 \quad \text{tr } c_{1+} = +1. \]

Consider the expression \((+8,-8)(+2,-2)(+1,-1)\); the coefficient of the positive block becomes

\[
(8 \times 2 \times 1) + (8 \times -2 \times -1) + (-8 \times 2 \times -1) + (-8 \times -2 \times 1) = 6h;
\]

that of the negative block, \(-6h\). From Table 15 the heterogeneous bundle of contrast \((421)\) is \((421, 221, 021, 401, 201, 001)\). The contrast \((221)\) in this bundle is the contrast \((221)\left| (421)\right)\) which, since the b and c coefficients are the same in \((221)\) and \((421)\), may be written \((2\left| 4\right)(2\left| 1\right)\). Since \[ a_2 = D(+2,-1,-2,-1,+2), \]

it follows that \(\text{tr } (a_{21} a_{4}) = -2 \) and \(\text{tr } (a_{22} a_{4}) = +2\). To obtain the coefficients of the block constants in the expectation of the contrast \((221)\), we have to examine the expression \((+2,-2)(+2,-2)(+1,-1)\). Comparing this expression with that obtained for contrast \((421)\), it follows that the coefficients will become \(6h/4 = 16\), and \(-6h/4 = -16\). For the contrast \((021)\) we examine the expression \((+2,-2)(+1,-1)\), and find the coefficients of the block constants to be \(4\) and \(-4\). Similar procedures are applied for the contrasts \((401),(201), (001)\).

This work can be systematized as follows:
In a 5 x 3 x 2 experiment,

\[ a_4 = D(\pm 1, -1, \pm 6, -1, \pm 1) \quad b_2 = D(\pm 1, -1, \pm 2) \quad c_1 = D(-1, 1) \]

\[ a_2 = D(\pm 2, -1, -1, \pm 1, \pm 2) \]

Confounded contrast: \( 421 \)

<table>
<thead>
<tr>
<th>Heterog. bundle</th>
<th>Partitioned vectors</th>
<th>Block coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>(421)</td>
<td>(+8, -8)(+2, -2)(+1, -1)</td>
<td>(64, -64)</td>
</tr>
<tr>
<td>(221)</td>
<td>(+8, -8)(+2, -2)(+1, -1)</td>
<td>(16, -16)</td>
</tr>
<tr>
<td>(021)</td>
<td>(+2, -2)(+1, -1)</td>
<td>(4, -4)</td>
</tr>
<tr>
<td>(401)</td>
<td>(+2, -2)(+1, -1)</td>
<td>(16, -16)</td>
</tr>
<tr>
<td>(201)</td>
<td>(+2, -2)(+1, -1)</td>
<td>(4, -4)</td>
</tr>
<tr>
<td>(001)</td>
<td>(+1, -1)</td>
<td>(+1, -1)</td>
</tr>
</tbody>
</table>
CHAPTER 7

The Confounding Pattern of Two Contrasts

7.1 Simultaneous Partitioning of Two Orthogonal Polynomials

We now investigate further partitioning of the polynomial elements as given in Table 6a. First we will consider the results for even \( N \). As before, \( P_e \) denotes an even polynomial with \( 2N_1 \) negative elements followed by \( 2N_2 \) positive elements (\( 2N_1 + 2N_2 = N \)). Let \( P_e^1 | P_e \) denote another even polynomial with its elements arranged so that the first \( 2N_{11} \) elements are negative, the next \( 2N_{12} \) are positive, the next \( 2N_{21} \) are negative and the final \( 2N_{22} \) are positive (\( N_{11} + N_{12} = N_1 \) and \( N_{21} + N_{22} = N_2 \)). Finally, let \( P_U | (P_e, P_e^1) = P_U | P_e^1 | P_e \) be any odd polynomial with elements corresponding to those of \( P_e^1 | P_e \). Each pair of equal elements in \( P_e^1 | P_e \) will correspond to a pair of equal (in absolute value) elements of any \( P_U \); i.e., there will be a \( +U \) and \( -U \) for each pair of elements in \( P_e^1 | P_e \). Hence \( P_U | (P_e, P_e^1) \) will not be confounded with block effects.

If \( N \) is odd there will be a central element of \( P_e^1 | P_e \); this central value will be either negative or positive. However, the corresponding central value of \( P_U | (P_e, P_e^1) = P_U | P_e^1 | P_e \) will be zero. Hence, whatever the value of \( N \), \( P_U | (P_e, P_e^1) \) will be unconfounded with blocks.

It can readily be verified that this is the only partitioning, in terms of the simultaneous partitioning of two orthogonal polynomials, that will yield parts whose elements sum to zero.

Let us now consider the general case of an \( m \times n \) experiment with contrasts \( (A \cdot B) \) and \( (A \cdot B) \) confounded simultaneously (in 4, 6, or 9 blocks). Using the notation of previous chapters,
\[
(\mathbf{A}_{R}^{B})_{(y = 1)} (\mathbf{A}_{r}^{B}, \mathbf{A}_{r}^{B})^{++} = \text{tr} \frac{a_{R22}}{a_{R21}} \text{tr} b_{S22} + \text{tr} a_{R21} \text{tr} b_{S21} + \text{tr} a_{R12} \text{tr} b_{S12} + \text{tr} a_{R11} \text{tr} b_{S22}
\]

(7.1)

\[
(\mathbf{A}_{R}^{B})_{(y = 1)} (\mathbf{A}_{r}^{B}, \mathbf{A}_{r}^{B})^{+-} = \text{tr} \frac{a_{R22}}{a_{R21}} \text{tr} b_{S21} + \text{tr} a_{R21} \text{tr} b_{S22} + \text{tr} a_{R12} \text{tr} b_{S11} + \text{tr} a_{R11} \text{tr} b_{S12}
\]

(7.2)

and similarly for the other blocks. For main effects

\[
\mathbf{A}_{R} (y = 1) (\mathbf{A}_{r}^{B}, \mathbf{A}_{r}^{B})^{++} = n^{++} \text{tr} \frac{a_{R22}}{a_{R21}} + n^{+-} \text{tr} a_{R21} + n^{-+} \text{tr} a_{R12} + n^{--} \text{tr} a_{R11}
\]

(7.3)

and similarly for \(B_{S}\) by replacing \(n\) by \(m\) and \(a_{R}\) by \(b_{S}\). If \(R\) is odd and both \(r\) and \(f\) are even in (7.3),

\[
\text{tr} \frac{a_{R22}}{a_{R21}} = \text{tr} a_{R21} = \text{tr} a_{R12} = \text{tr} a_{R11} = 0;
\]

hence \(A_{R}\) is unconfounded with blocks. If \(r = f\), the right-hand side of (7.3) becomes

\[
n^{++} \text{tr} \frac{a_{R22}}{a_{R21}} + n^{--} \text{tr} a_{R21}.
\]

In addition to the above solution, \(\text{tr} \frac{a_{R22}}{a_{R21}} = \text{tr} a_{R11} = 0\) if \(m\) is even, \(R\) is even and \(r = f\) is odd. Hence if \(m\) is even, \(R\) is even and \(r = f\) is odd, \(A_{R}\) is also unconfounded with blocks. These results are summarized as follows:

**Theorem 7.1.** In an \(m \times n\) experiment, \(A_{R} (\mathbf{A}_{r}^{B}, \mathbf{A}_{r}^{B})\) will be unconfounded when either (i) \(R\) is odd and both \(r\) and \(f\) (including \(r = f\)) are even, for all \(m\) and \(n\), or (ii) \(R\) is even and \(r = f\) is odd, for \(m\) even and all \(n\).
A further examination of (7.1) and (7.3) indicates the following

**Theorem 7.2.** In an \(m \times n\) experiment, when contrasts \(A_{R}B_{S}\) and \(A_{I}B_{G}\) are confounded simultaneously, contrast \(A_{R}B_{S}\) will be unconfounded when either \(A_{R}\mid (A_{R}B_{S}, A_{I}B_{G})\) or \(B_{S}\mid (A_{R}B_{S}, A_{I}B_{G})\) is unconfounded; \(A_{R}\mid (A_{R}B_{S}, A_{I}B_{G})\) will be unconfounded if \(A_{R}\mid (A_{R}, A_{I})\) is unconfounded and similarly for \(B_{S}\).

If \(A_{R}\) is unconfounded, \(tr a_{R22} = tr a_{R21} = tr a_{R12} = tr a_{R11} = 0\); similarly if \(B_{S}\) is unconfounded. We have been unable to prove that the conditions in these theorems guarantee that all other situations produce confounded effects; experiments fulfilling the conditions in these theorems result in unconfounded effects, but other situations may also prevail in which effects could be unconfounded.

### 7.2 The Generalized Interaction

When two contrasts \(A_{R}B_{S}\) and \(A_{I}B_{G}\) of an \(m \times n\) experiment are confounded simultaneously, then, as shown in Chapter 4, their generalized interaction \(A_{V}B_{W}\) will be confounded automatically. The usual notation for the generalized interaction of two contrasts is

\[
(A_{V}B_{W}) = (A_{R}B_{S}) \times (A_{I}B_{G}).
\]

It is important to observe that this is a symbolic relation only, not an algebraic one; algebraically, however, \(a_{V} = a_{R}a_{I}\) and \(b_{W} = b_{S}b_{G}\). The notation is meant to imply that when a new contrast is formed out of the coefficient matrices of two given contrasts, this new contrast will be confounded, whenever the two given contrasts are confounded simultaneously.

Assume the matrices \(a_{V}\) and \(b_{W}\) are expressible as linear combinations of other matrices, say
\[ a_v = a_{r+f} = a_{r+f} + \alpha a_{r+f-2} + \cdots = L(a_{r+f}), \text{ say} \]
\[ b_w = b_{s+g} = b_{s+g} + \beta b_{a+g-2} + \cdots = L(b_{s+g}), \text{ say} \]

where \( \alpha \) and \( \beta \) are constants. In this case,

\[
a_v b_w = L(a_{r+f})L(b_{s+g})
\]
\[
= a_{r+f}b_{s+g} + \beta a_{r+f}b_{s+g-2} + \alpha a_{r+f-2}b_{s+g} + \cdots
\]
\[
= L(a_{r+f}b_{s+g})
\]

and

\[
(A_v B_w) = (A_{r+f} B_{s+g}) + \beta (A_{r+f} B_{s+g-2}) + \alpha (A_{r+f-2} B_{s+g}) + \cdots
\]
\[
= L(A_{r+f} B_{s+g})
\]

Hence, when the generalized interaction of the two contrasts \((A_r B_s)\) and \((A_f B_g)\) is expressible in the form of a linear combination of contrasts, this linear combination will be confounded when the two contrasts are confounded simultaneously. It does not follow, however, that each contrast in the linear combination will be confounded (see Corollary 4.3.1).

7.3 Reduction Formulae for Products of Orthogonal Polynomials

The recursion formula of the orthogonal polynomials in the Fisher system can be written as

\[
P_1 P_r = P_{r+1} + \frac{r^2(N-r^2)}{n(r^2-1)^2} P_{r-1}
\]

It might appear that we would require generalization of this formula to the case \(P_s P_r\) for the purpose of expressing the generalized interaction as a linear combination of contrasts, but this is not necessarily so. Consider the product \(P_s P_r\) when there are \(N\) observations, and
suppose that \( s = r = (N-1) \). Then \( P_s \times P_r \) will equal some linear combination of polynomials such as \( P_2(N-1) + P_2(N-2) + \cdots \). Since there are only \( N \) observations, all polynomials with degrees higher than \((N-1)\) will vanish, and the number of terms in the linear combination is greatly reduced.

The number of levels of any one factor being usually small in practice, the direct calculation of the reduction formulae, for those cases likely to be useful, seems indicated. Table 16 accordingly supplies the reduction formulae of the products for \( N \) through 5.

Table 16. Products of polynomials.

<table>
<thead>
<tr>
<th>( N = 3 )</th>
<th>( N = 4 )</th>
<th>( N = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 \times P_1 = P_2 + \frac{2}{3} P_0 )</td>
<td>( P_1 \times P_1 = P_2 + \frac{5}{4} P_0 )</td>
<td>( P_1 \times P_1 = P_2 + 2P_0 )</td>
</tr>
<tr>
<td>( P_1 \times P_2 = \frac{1}{3} P_1 )</td>
<td>( P_1 \times P_2 = P_3 + \frac{1}{3} P_1 )</td>
<td>( P_1 \times P_2 = P_3 + \frac{7}{5} P_1 )</td>
</tr>
<tr>
<td>( P_1 \times P_3 = \frac{9}{20} P_2 )</td>
<td>( P_1 \times P_3 = \frac{36}{35} P_2 )</td>
<td>( P_1 \times P_4 = \frac{1}{7} P_3 )</td>
</tr>
<tr>
<td>( P_2 \times P_2 = -\frac{1}{3} P_2 + \frac{2}{9} P_0 )</td>
<td>( P_2 \times P_2 = P_0 )</td>
<td>( P_2 \times P_2 = P_4 + \frac{3}{7} P_2 + \frac{11}{5} P_0 )</td>
</tr>
<tr>
<td>( P_2 \times P_3 = -\frac{1}{2} P_3 + \frac{9}{25} P_1 )</td>
<td>( P_2 \times P_3 = \frac{2}{5} P_3 + \frac{36}{25} P_1 )</td>
<td>( P_2 \times P_4 = \frac{10}{7} P_4 + \frac{114}{245} P_2 )</td>
</tr>
<tr>
<td>( P_3 \times P_3 = -\frac{9}{25} P_2 + \frac{9}{20} P_0 )</td>
<td>( P_3 \times P_3 = \frac{2}{5} P_4 - \frac{72}{175} P_2 + \frac{50}{175} P_0 )</td>
<td>( P_3 \times P_4 = -\frac{26}{35} P_3 + \frac{114}{175} P_1 )</td>
</tr>
<tr>
<td>( P_4 \times P_4 = \frac{106}{245} P_4 - \frac{288}{343} P_2 + \frac{288}{175} P_0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since in practice we use \( P' = \lambda P \), the relations in Table 16 should be adjusted accordingly before application to a specific problem. In this chapter we use Table 16 in a symbolical sense only.
7.4 The Status of the Contracts in the Linear Combination

Let u and u₁ denote odd and e and e₁ even confounding contrasts; u₂ and e₂ odd and even contrasts in the generalized interaction. There are four types of confounding contrasts, depending on the odd–even character of (rs) and (fg). If r and f are both odd or both even, we call them similar contrasts; if one is odd and the other even, they are dissimilar contrasts; likewise for s and g. The four types are

<table>
<thead>
<tr>
<th>Type</th>
<th>Character of contrasts</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Similar, Similar</td>
<td>(ue) x (u₁e₁)</td>
</tr>
<tr>
<td>II</td>
<td>Similar, Dissimilar</td>
<td>(ue) x (u₁u₁)</td>
</tr>
<tr>
<td>III</td>
<td>Dissimilar, Similar</td>
<td>(ue) x (e₁e₁)</td>
</tr>
<tr>
<td>IV</td>
<td>Dissimilar, Dissimilar</td>
<td>(ue) x (e₁u₁)</td>
</tr>
</tbody>
</table>

Based on the results in Table 16, the form of the generalized interaction \( L[(r+f)(s+g)] \) is given in Table 17, for each of these types.

Table 17. Form of generalized interaction contrast, \( L[(r+f)(s+g)] \)

<table>
<thead>
<tr>
<th>Type</th>
<th>( r \neq f )</th>
<th>( r = f )</th>
<th>( r \neq f )</th>
<th>( r = f )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s \neq g</td>
<td>s \neq g</td>
<td>s = g</td>
<td>s = g</td>
</tr>
<tr>
<td>I</td>
<td>L(e₂e₂)</td>
<td>L(e₂e₂, 0e₂)</td>
<td>L(e₂e₂, e₂0)</td>
<td>L(e₂e₂, 0e₂, e₂0, 00)</td>
</tr>
<tr>
<td>II</td>
<td>L(e₂u₂)</td>
<td>L(e₂u₂, 0u₂)</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>III</td>
<td>L(u₂e₂)</td>
<td>X</td>
<td>L(u₂e₂, u₂0)</td>
<td>X</td>
</tr>
<tr>
<td>IV</td>
<td>L(u₂u₂)</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

\( X \) indicates this combination cannot exist.

For example, in a 5 x 4 experiment: (12) x (31) = L(43), where (12) = (ue), (31) = (u₁u₁); hence, from Table 16
L(43) = (A_4B_3) + \frac{1}{5}(A_4B_1) + \frac{36}{35}(A_2B_3) + \frac{114}{175}(A_2B_1)

This is a Type II contrast with \( r \neq f \) and \( s \neq g \); hence \( L(43) \) is a sum of even-odd contrasts. Each individual contrast in \( L(43) \) [(43), (41), (23) and (21)] is confounded. However, it should be emphasized that when the contrasts (RS), (RO), (OS) belong to the linear combination \( L[(r+f)(s+g)] \), the automatic confounding of this linear combination will not necessarily confound the contrasts (RS), (RO), (OS). For example, consider the confounding of contrasts (32) and (33) of a 4 x 4 experiment. Here

\[
(32) \times (33) = L(23) = \frac{36}{125}(A_2B_3) - \frac{81}{625}(A_2B_1) - \frac{9}{25}B_3 + \frac{81}{500}B_1
\]

Note that the complete compounding of the subscripts does not hold in this case, since the sum of the corresponding degrees for each factor cannot exceed the number 3 = 4 - 1, where 4 is the number of levels.

By Theorem 7.2, (23) | (32, 33) and (21) | (32, 33) will be unconfounded; however, (03) | (32, 33) and (01) | (32, 33) will be confounded.

7.5 Confounding Pattern of Two Contrasts

Confounded Simultaneously

Consider the simultaneous confounding of contrasts \((A_fB_r)\) and \((A_fB_g)\) of an \( m \times n \) experiment. By Corollary 4.1.1, all contrasts belonging to the heterogeneous bundle of \((A_fB_r)\) [if \((A_fB_r)\) has a heterogeneous bundle], and all contrasts belonging to the heterogeneous bundle of \((A_fB_g)\) [if \((A_fB_g)\) has a heterogeneous bundle] will be confounded when \((A_fB_r)\) and \((A_fB_g)\) are confounded simultaneously. Hence the first step to obtain the confounding pattern of \((A_fB_r, A_fB_g)\) will consist in applying the rule developed in Chapter 6 to each of the two contrasts separately (alternatively, Table 15 can be used).
Let us now denote the generalized interaction \((rs) \times (fg)\) by \(L(r's')\), where \(L(r's')\) is a linear combination of the contrasts

\[(r's'), (r'(s'-2)), ((r'-2)s'), ((r'-2)(s'-2)) \cdots\]

The expression \((R'S')\) will refer to any one of the contrasts in \(L(r's')\).

According to Table 17, when both \(r \neq f\), \(s \neq g\), contrasts of the type \((R'0)\) and \((0S')\) will not appear in \(L(r's')\); when \(r = f\) and \(s \neq g\), contrasts of the type \((R'0)\) will not appear; when \(r \neq f\) and \(s = g\), contrasts of the type \((0S')\) will not appear. Let \([r's']\) denote the homogeneous bundle of the particular contrast \((r's')\), which can be found in Table 13.

If \(r' = e'\) and \(s' = u'\), for example, \([r's']\) will be the homogeneous bundle \([e'u']\) = \((EU)\). It follows by the results of the partitioning of the orthogonal polynomials that if contrast \((e'u')\) is confounded, all contrasts belonging to the homogeneous bundle \([e'u']\) will be confounded. Similarly let \([r'0]\) denote the homogeneous bundle of \((r'0)\) and \([0s']\) the homogeneous bundle of \((0s')\).

When both \(r \neq f\) and \(s \neq g\), \(L(r's')\) will be a linear combination of the contrasts belonging to the homogeneous bundle \([r's']\); when \(r = f\) and \(s \neq g\), \(L(r's')\) will be a linear combination of the contrasts belonging to the homogeneous bundles \([r's']\) and \([0s']\); when \(r \neq f\) and \(s = g\), \(L(r's')\) will be a linear combination of the contrasts belonging to the homogeneous bundles \([r's']\) and \([r'0]\). For example, in a 5 x 4 experiment:

\((12) \times (31) = L(43),\) where \(L(43)\) is a linear combination of the contrasts belonging to the homogeneous bundle \([43], \text{viz.}, (43, 41, 23, 21)\); in a 4 x 4 experiment: \((32) \times (33) = L(23),\) where \(L(23)\) is a linear combination of the contrasts belonging to the homogeneous bundle \([23], \text{viz.}, (23, 21)\) and to the homogeneous bundle \([03], \text{viz.}, (03, 01)\).
By Theorem 7.2, contrast \( (r's') \) will be unconfounded by \( (rs, fg) \) when either \( r' \mid (r, f) \) or \( s' \mid (s, g) \) is unconfounded; \( (r'0) \) will be unconfounded if \( r' \mid (r, f) \) is unconfounded and \( (0s') \) will be unconfounded if \( s' \mid (s, g) \) is unconfounded. Hence, under the same conditions, \( [r's'] \), \( [r'0] \) and \( [0s'] \) will also be unconfounded by \( (rs, fg) \).

In terms of the theoretical development above, the following stages can be distinguished in the procedure for obtaining the confounding pattern of \( (rs, fg) \) in an \( m \times n \) experiment:

(i) Obtain the confounding pattern of the contrast \( (rs) \).

(ii) Obtain the confounding pattern of the contrast \( (fg) \).

(iii) Determine the linear combination \( L(r's') = (rs) x (fg) \) from Table 16.

(iv) Obtain the homogeneous bundle \( [r's'] \) of the contrast \( (r's') \) when \( r \neq f, s \neq g \); the homogeneous bundle \( [0's] \) of the contrast \( (0s') \) when \( r = f, s \neq g \); the homogeneous bundle \( [r'0] \) of the contrast \( (r'0) \) when \( r \neq f, s = g \).

(v) Determine the confounding status of the homogeneous bundle \( [r's'] \) when \( r \neq f, s \neq g \); of the homogeneous bundle \( [0s'] \) when \( r = f, s \neq g \); of the homogeneous bundle \( [r'0] \) when \( r \neq f, s = g \).

In actual practice, stages (iii), (iv) and (v) can be combined into the symbolic multiplication scheme given below, which embodies the consequences of the theorems stated above.

The part to the right of the double vertical line is first written down; then, \( [r'0] \) in row (a) and \( [0s'] \) in row (b) are obtained from the symbolic products \( f x r \) and \( g x s \) respectively in Table 16, and the confounding according to Theorems 7.1 and 7.2, remembering that \( r' \) is
even if \( r \) and \( f \) are both even or both odd. Depending on the conditions
given in the scheme, multiply the elements in row (a) by the elements in
row (b) to obtain the elements in row (c), subject to the restrictions that
multiplication across vertical lines is not permitted.

Table 18. Rule for obtaining homogeneous bundles in the generalized
interaction of the contrasts \((rs, fg)\) of an \( m \times n \) experiment.

<table>
<thead>
<tr>
<th></th>
<th>((00))</th>
<th>([r'0])</th>
<th>((f0))</th>
<th>((r0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Omit when ( r \neq f )</td>
<td>Omit when ( r = f ) is odd for ( m ) even</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>((00))</td>
<td>([0s'])</td>
<td>((0g))</td>
<td>((0s))</td>
</tr>
<tr>
<td>(c)</td>
<td>((a) \times (b))</td>
<td>((fg))</td>
<td>((rs))</td>
<td></td>
</tr>
</tbody>
</table>

When the confounding patterns of contrasts \((rs)\) and \((fg)\) are un-
known, the last two columns of Table 18 can be enlarged to accommodate the
scheme of Table 13.

As an example, consider the confounding of the contrasts \((12)\) and
\((31)\) of a \(5 \times 4\) experiment. Since \( r \neq f, s \neq g, (12) \times (31) = L(43)\). The
scheme for the generalized interaction becomes

\[
\begin{array}{c|c|c}
[40] & (30) & (10) \\
[03] & (01) & (02) \\
[43] & (31) & (12) \\
\end{array}
\]

The homogeneous bundle of contrast \((43)\) is \((43, 41, 23, 21)\). From Table 13
the heterogeneous bundles of contrasts \((31)\) and \((12)\) are respectively
\((31, 33, 11, 13, 20, 40)\) and \((12, 32, 20, 40)\). Hence the confounding pattern of
contrasts \((31)\) and \((12)\) is

\((33, 31, 13, 11, 40, 20, 43, 41, 32, 12, 23, 21)\).
Generalization of the rule to experiments with more than two factors is immediate, as was done in Table 11.

The results in this chapter presume that the only contrasts which are confounded when \((A_{s}B_{t}C_{e})\) and \((A_{f}B_{g}C_{h})\) are simultaneously confounded are: (i) those contrasts confounded by either \((A_{s}B_{t}C_{e})\) or \((A_{f}B_{g}C_{h})\), (ii) possibly some or all of the contrasts in the linear combination of the generalized interaction \((A_{v}B_{w}C_{z})\). We have been unable to prove that no other contrasts orthogonal to the confounding contrasts can be confounded; however, we believe that this is the case, and recommend the above procedures be used in determining which contrasts are confounded.
8.1 Confounding a Single Highest Order Interaction Contrast

Consider the confounding of contrast \((A_{rs}B_{st})\) of an \(m \times n \times p\) experiment. As before, let \(m_-, m_+, m_o\) denote the frequencies of the negative, positive and zero elements respectively of the vector \(a_r\); \(n_-, n_+, n_o\) those of the vector \(b_s\); \(p_-, p_+, p_o\) those of the vector \(c_t\). Schematically we have

<table>
<thead>
<tr>
<th>Block indicator</th>
<th>Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_r)</td>
<td>(b_s)</td>
</tr>
<tr>
<td>(-)</td>
<td>(m_-, n_-, p_-)</td>
</tr>
<tr>
<td>(+)</td>
<td>(m_+, n_+, p_+)</td>
</tr>
<tr>
<td>(0)</td>
<td>(m_o, n_o, p_o)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>m, n, p</strong></td>
</tr>
</tbody>
</table>

Confounding \((A_{rs}B_{st})\) consists of placing the treatment combinations corresponding to a - sign in one block, those corresponding to a + sign in another block. Confounding a single contrast, therefore, implies the arrangement of the treatment combinations into 3 or 2 blocks, depending on whether the 0 block indicator is present or absent.

Let \(N_-, N_+, N_o\) denote the number of treatment combinations in the negative block, the positive block, and the zero block respectively. Then, from the above scheme
\[ N_+ = m_+ n_+ p_+ + m_+ n_- p_+ + m_- n_+ p_- + m_- n_- p_- \]
\[ N_- = m_- n_+ p_- + m_- n_- p_- + m_+ n_+ p_+ + m_+ n_- p_+ \]
\[ N_o = mnp - (m-m_0)(n-n_0)(p-p_o) \]
\[ = mnp - (m_+ + m_-)(n_+ + n_-)(p_+ + p_-) \]

If \( N_o = 0 \), we need only have \( N_+ = N_- \) to have equal-sized blocks (there will be only two blocks in this case). The condition for \( N_+ = N_- \) is

\[ (m_+ - m_-)(n_+ - n_-)(p_+ - p_-) = 0 \quad (8.1) \]

(8.1) is satisfied if and only if one or more of these conditions hold:

\[ m_+ = m_-; \quad n_+ = n_-; \quad p_+ = p_- \]

\[ \quad (8.2) \]

If \( N_o > 0 \), (8.1) must still be met; however, a second condition is also imposed:

\[ N_+ + N_- - 2N_o = 0; \text{ or,} \]

\[ 3(m-m_0)(n-n_0)(p-p_o) = 2mnp \quad (8.3) \]

It is not possible to derive a simple condition, such as (8.2), which will satisfy (8.3). One must determine suitable combinations of \((m,n,p)\) for given \((m_o,n_o,p_o)\) or vice versa. For example, if \( m_o = n_o = 0 \) and \( p_o > 0 \), \( p = 3p_o \), and \( m \) and \( n \) can be any value. In general, if only two of the three main effect contrasts have no zero elements, the number of levels of the other factor must be three times the number of zero elements; in addition, at least one of the three contrasts must have the same number of positive as negative elements to satisfy (8.2).

Similarly, if \( m_o = 0 \) and \( n_o = 1 \),

\[ n = \frac{3(p-p_o)}{p-3p_o} \quad \text{and} \quad p = \frac{3(n-1)p_o}{n-3}; \]
hence, \( n > 3 \) and \( p > 3p_0 \). If \( m_0 = 0 \) and \( n_0 = 2 \),

\[
n = \frac{6(p-p_0)}{p - 3p_0} \quad \text{and} \quad p = \frac{3(n-2)p_0}{n - 6};
\]

hence \( n > 6 \) and \( p > 3p_0 \). These results can be extended to any \( n_0 \), and can be interchanged to give formulas for the situation where any main effect contrast contains no zero elements and each of the others has at least one zero element. Of course, condition (8.2) must also be met.

If \( m_0 = n_0 = p_0 = 1 \),

\[
p = \frac{3(m-1)(n-1)}{mn - 3(m+n-1)};
\]

hence \( m > 3 \) and \( n > 3(m-1)/(m-3) \). If \( m_0 = n_0 = 1 \) and \( p_0 = 2 \),

\[
p = \frac{6(m-1)(n-1)}{mn - 3(m+n-1)};
\]

hence \( m > 3 \) and \( n > 3(m-1)/(m-3) \), as before. These results also can be extended to other values of \( m_0, n_0 \) and \( p_0 \), and can be easily interchanged to give results for any situation with every main effect contrast having at least one zero element.

Table 19. Examples of contrasts which satisfy condition (8.3), with at least two contrasts having zero elements.

<table>
<thead>
<tr>
<th>( m_0 )</th>
<th>( n_0 )</th>
<th>( p_0 )</th>
<th>( m )</th>
<th>( n )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>Any</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>Any</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>Any</td>
<td>7</td>
<td>9</td>
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<tr>
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<td>1</td>
<td>2</td>
<td>Any</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

\(^1\)These are main effect contrasts, the 3-factor contrast elements consisting of all possible products of the main effect elements; \( m_0, n_0 \), and \( p_0 \) are the number of zero elements in the contrasts and \( m, n \) and \( p \) the total number of elements.
Table 19 presents specific examples of the results mentioned above for number of levels less than 10. These results can be generalized by interchanging any of the contrasts. For example, one design presented in Table 19 is $m_o = 0$, $n_o = p_o = 1$, $n = 5$, $p = 6$ and $m$ any value. Similarly one can have $m_o = 0$, $n_o = p_o = 1$, $n = 6$, $p = 5$ and $m$ any value; $m_o = p_o = 1$, $n_o = 0$, $m = 5$, $p = 6$ and $n$ any value, etc. It should be emphasized again that these designs must also have at least one contrast with equal numbers of positive and negative elements.

When one compares Table 19 with Table 5, he notes that no orthogonal polynomial contrasts meet the requirements of Table 19. Only odd-level contrasts have zero elements; hence, none of the results in Table 19 with the number of zero elements being 0 or 1 can be used (in all cases at least one of the values $m$, $n$ or $p$ is even). For $p_o = 2$, we note that $p = 9$, whereas the only case in Table 5 is $p = 7$. If one had an experiment with one factor having as many as 15 levels, he could use $P^1$ for $p = 7$ in Table 5, because if $m_o = 0$, $n_o = 1$ and $p_o = 2$, (8.3) is satisfied if $n = 15$ and $p = 7$. Based on these conclusions, it appears that, in most situations, at least two of the contrasts must have no zero elements if (8.3) is to be satisfied. If the third contrast has one zero element, the contrast must be the linear contrast for a three-level factor; no orthogonal polynomial contrast meets the requirement for more than one zero element ($p = 3p_o$).

These conditions lead to the formulation of the following rules for orthogonal polynomial contrasts:

**RULE 1.** When a highest order interaction contrast is confounded, two equal blocks will result if (i) no zero elements appear in any coefficient vector of any factor in the confounding contrast, and (ii) the coefficient
vector of at least one factor in the confounding contrast has the same number of positive as negative elements.

**RULE 2.** When a highest order interaction contrast is confounded, three equal blocks will result if (i) one of the factors has three levels and the coefficient vector of this factor in the confounding contrast is of the first degree, and (ii) no zero elements appear in any coefficient vector of any remaining factors in the confounding contrast.

Table 20 supplies the frequencies of the block indicators in the various degrees of the orthogonal polynomials for \( n \) through 6.

**Table 20.** Frequencies of block indicators in the orthogonal polynomials.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Degree</th>
<th>( n_- )</th>
<th>( n_+ )</th>
<th>( n_0 )</th>
<th>( n )</th>
<th>Degree</th>
<th>( n_- )</th>
<th>( n_+ )</th>
<th>( n_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
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<td>1</td>
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<td></td>
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<td>1</td>
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<td>0</td>
<td></td>
</tr>
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<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
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<td>4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
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<td>1</td>
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<td></td>
</tr>
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<td>3</td>
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<td>0</td>
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<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
8.2 Simultaneous Confounding of Two Highest Order Interaction Contrasts

Consider first the simultaneous confounding of the two contrasts \((A_i B_s)\) and \((A_f B_s)\) of an m x n experiment. The block indicator of the treatment combination \(y_{ij}\) will be the combination of the sign of \((a_{ri} b_{sj})\) with the sign of \((a_{fi} b_{sj})\). The following nine combinations are possible: 
\((-,-), (-+), (+-)\), \((+-),(+0),(0-),(00)\), where the first "sign" refers to \((a_{ri} b_{sj})\) and the second "sign" to \((a_{fi} b_{sj})\). Since 
\[ (a_{ri}-b_{sj})(a_{ri}-b_{gh}) = (a_{ri}-b_{gh})(a_{ri}-b_{sj}) \]
the blocking arrangement resulting from the simultaneous confounding of \(A_i B_s\) and \(A_f B_s\) will be identical to the blocking arrangement resulting from the simultaneous confounding of \(A_i B_g\) and \(A_f B_s\).

The blocking arrangement resulting from the simultaneous confounding of \((A_i B_s)\) and \((A_f B_g)\) can be obtained by first confounding \((A_i B_s)\) and then \((A_f B_g)\); that is, the blocks obtained by confounding \((A_i B_s)\) are then divided into as many sub-blocks as would be required by the subsequent confounding of \((A_i B_g)\), or vice versa. Alternatively, by our result above, \((A_i B_g)\) could first be confounded and then \((A_f B_s)\). This procedure is useful in determining whether nine equal blocks can result from the simultaneous confounding of two highest order interaction contrasts in an m x n experiment.

By Rule 2, the confounding of \((A_i B_s)\) will yield three equal blocks if \((A_i B_s) = (A_1 B_s)\) when \(m = 3\) and \(s\) is even, or if \((A_i B_s) = (A_1 B_1)\) when \(n = 3\) and \(r\) is even, but not if \((A_i B_s) = (A_1 B_1)\) when \(m = n = 3\). Since confounding of \((A_1 B_1) x (A_i B_s)\) or \((A_1 B_s) x (A_f B_1)\) implies a 3 x 3 experiment, there is no point in examining the confounding pattern;
obviously no one will plan a $3 \times 3$ experiment in nine blocks. Hence the only possible useful set of nine equal blocks would be based on confounding $(A_1 B_g) x (A_1 B_e)$ or $(A_R B_1) x (A_R B_1)$. Consider the former. As before, $a_1 = (-1, 0, +1)$; $b_g$ and $b_e$ are restricted only to have no zero elements. Let $n_{\beta\xi} (\beta = + or -; \xi = + or -)$ denote the number of elements with a $\beta$-sign on $b_g$ and an $\xi$-sign in $b_e$. The size of each of the nine blocks is given in Table 21, in the second column from the right.

Table 21. Size of blocks when $(A_1 B_g) x (A_1 B_e)$ is confounded under certain conditions.

<table>
<thead>
<tr>
<th>Block $(A_1 B_g)(A_1 B_e)$</th>
<th>Block size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 3$, $r = f = 1$</td>
<td>$m = 3$, $r = 1$, $f = 2$</td>
</tr>
<tr>
<td>$n_{so} = n_{go} = 0$</td>
<td>$n_{so} = n_{go} = 0$; $n_{g+} = n_{g-}$</td>
</tr>
<tr>
<td>$-$  $-$</td>
<td>$n_{++} + n_{--}$</td>
</tr>
<tr>
<td>$-$  $+$</td>
<td>$n_{-+} + n_{-+}$</td>
</tr>
<tr>
<td>$+$  $-$</td>
<td>$n_{+-} + n_{++}$</td>
</tr>
<tr>
<td>$+$  $+$</td>
<td>$n_{++} + n_{-+}$</td>
</tr>
<tr>
<td>0   $+$</td>
<td>Impossible</td>
</tr>
<tr>
<td>0   $-$</td>
<td>Impossible</td>
</tr>
<tr>
<td>$-$  0</td>
<td>Impossible</td>
</tr>
<tr>
<td>$+$  0</td>
<td>Impossible</td>
</tr>
<tr>
<td>0   0</td>
<td>2$n$</td>
</tr>
<tr>
<td></td>
<td>Impossible</td>
</tr>
</tbody>
</table>

We note that only five blocks are possible and that the $(00)$ block is by far the largest. A similar statement holds when $A$ and $B$ are interchanged. Hence one cannot construct a design of this kind with nine equal-sized blocks.

Next let us investigate if we can construct a design with six equal-sized blocks. In this case we consider $(A_1 B_g) x (A_2 B_g)$, where neither $b_g$ nor $b_e$ have zero elements and $n_{g+} = n_{g-}$ (no other restrictions
on \( b_2 \). The block sizes are given in the right-hand column of Table 21. Since \( n_{g}^{+} = n_{g}^{-} \), the six blocks are of equal size. A and B can be interchanged in the results.

In an \( m \times n \times p \) experiment, we require the following to have six equal-sized blocks:

(i) \( m = 3, r = 1 \) and \( f = 2 \)

(ii) no other vectors have any zero elements

(iii) \( n_{g}^{+} = n_{g}^{-} \) or/and \( P_{h}^{+} = P_{h}^{-} \).

Obviously A, B and C can be interchanged in these results.

**RULE 3:** When two highest order interaction contrasts are confounded simultaneously, six equal blocks will result if (i) one and only one of the factors has three levels, and the coefficient vector of this factor is of the first degree in one of the two confounding contrasts and of the second degree in the other contrast; (ii) with the exception of the first degree coefficient vector of the factor with three levels, no zero elements appear in any coefficient vector of any factor in the two confounding contrasts; (iii) for at least one of the other factors in the contrast involving the second degree main effect in (i), the number of positive elements equals the number of negative elements in this coefficient vector.

Consider now the simultaneous confounding of the contrasts \((A_x B_g, A_x B_g)\) of an \( m \times n \) experiment, when none of the coefficient vectors contain zero elements. Schematically we have

<table>
<thead>
<tr>
<th>Block indicator</th>
<th>Frequencies</th>
<th>((a_x, a_x))</th>
<th>((b_x, b_x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>++</td>
<td>(m^++)</td>
<td>(n^++)</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td>(m^+)</td>
<td>(n^+)</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>(m^-)</td>
<td>(n^-)</td>
<td></td>
</tr>
<tr>
<td>--</td>
<td>(m)</td>
<td>(n)</td>
<td></td>
</tr>
</tbody>
</table>

Total \( m \) \( n \)
The sizes of the blocks will be given by the relations:

\[ N_{--} = m_{--}n_{++} + m_{--}n_{+-} + m_{--}n_{-+} + m_{--}n_{--} \]
\[ N_{--} = m_{--}n_{+-} + m_{--}n_{-+} + m_{--}n_{--} + m_{--}n_{--} \]
\[ N_{--} = m_{--}n_{--} + m_{--}n_{--} + m_{--}n_{--} + m_{--}n_{--} \]
\[ N_{--} = m_{--}n_{--} + m_{--}n_{--} + m_{--}n_{--} + m_{--}n_{--} \]

Block sizes will be equal if the following equations are satisfied simultaneously:

\[ (N_{++} + N_{+-}) - (N_{+-} + N_{--}) = 0 \]
\[ (N_{++} + N_{-+}) - (N_{-+} + N_{--}) = 0 \]
\[ (N_{++} + N_{--}) - (N_{-+} + N_{--}) = 0 \]

or, after substitution, if

\[ (m_{r-} - m_{r-})(n_{s-} - n_{s-}) = 0 \] (8.4)
\[ (m_{f-} - m_{f-})(n_{g-} - n_{g-}) = 0 \] (8.5)
\[ (m_{+-} - m_{--} - m_{+-} + m_{--})(n_{++} - n_{+-} - n_{-+} + n_{--}) = 0 \] (8.6)

When three-factor contrasts are considered, these equations become

\[ (m_{r-} - m_{r-})(n_{s-} - n_{s-})(p_{t+} - p_{t-}) = 0 \] (8.7)
\[ (m_{f-} - m_{f-})(n_{g-} - n_{g-})(p_{h+} - p_{h-}) = 0 \] (8.8)
\[ (m_{++} - m_{+-} - m_{--} + m_{--})(n_{++} - n_{+-} - n_{-+} + n_{--})(p_{++} - p_{+-} - p_{-+} + p_{--}) = 0 \] (8.9)

If \[ m_{++} = m_{+-} = m_{--} = m_{--} \] then \[ m_{r+} = m_{r-} = m_{f+} = m_{f-} \], and hence (8.4) to (8.9) will be satisfied. Similarly for the partitioned \( n \) and \( p \) values.

These conditions lead to Rule 4.

**RULE 4:** When two highest order interaction contrasts are simultaneously confounded, four equal blocks will result (i) if no zero elements appear
in any coefficient vector of any factor in the two contrasts; and (ii) when
the first coefficient vector of the first contrast is partitioned according
to the first coefficient vector of the second contrast (or vice versa),
and the same process applied to the two corresponding coefficient vectors
of the other factors in the two confounding contrasts, the parts of at
least one of these partitioned coefficient vectors contain the same number
of elements.

Table 22 supplies the signs of the elements of the orthogonal poly-
nomial values for \( n = 2, 4, 6 \) and 8.

Table 22. Signs of the elements of the orthogonal polynomial vectors.

<table>
<thead>
<tr>
<th>( n = 2 )</th>
<th>( n = 4 )</th>
<th>( n = 6 )</th>
<th>( n = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>( - )</td>
<td>( - )</td>
<td>( - )</td>
<td>( - )</td>
</tr>
<tr>
<td>( + )</td>
<td>( - )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
<tr>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
</tr>
</tbody>
</table>

If the elements of each degree are partitioned in terms of the elements
of each of the other degrees, the relation \( n_{++} = n_{+-} = n_{-+} = n_{--} \) holds
when \( n \) is equal to 4. When \( n \) is equal to 8, the relation does not hold
for the partitionings 1|3, 1|5, 3|7, 5|7, but is true for all the other
partitionings. When \( n \) is equal to 2 or to 6, the relation does not hold.

It is also possible to obtain four equal-sized blocks in an
\( m \times n \) experiment by the following procedure:
(a) \( m_{+} = m_{-} \) and \( n_{+} = n_{-} \)  \( (8.10) \)

(b) \( m_{++} = m_{--} \) and \( m_{+-} = m_{-+} \) or \( n_{++} = n_{--} \) and \( n_{+-} = n_{-+} \)  \( (8.11) \)

(c) In general one can select any factor from (8.7) and a different one from (8.8) for (8.10) and then apply (8.11) accordingly. Hence one can set up rule 4a.

**RULE 4a.** When two highest order interaction contrasts are simultaneously confounded, four equal-sized blocks can also result if (i) no zero elements appear in any coefficient vector; (ii) equality of positive and negative elements in the original coefficient vectors follows (8.10); (iii) equality of the \( m_{+8} \) or \( n_{-8} \) follows (8.11); (iv) one can interchange factors as suggested in (c) above.

Rule 4a enables one to construct 6 level designs in four equal blocks.

8.3 Experiments in Equal Blocks With Main Effects Unconfounded

8.3.1 Two-factor experiments

Examination of Table 11 reveals that those two-factor experiments in which the confounding of any highest order interaction contrast does not confound main effect contrasts, also satisfy the rule for obtaining two equal blocks. In all cases the number of levels of each factor must be even; if any factor has an odd number of levels, at least one main effect contrast will be confounded. In certain cases the number of positive elements must also equal the number of negative elements in the main effect vector, as indicated below.
Confounding contrast | Requirements so that no main effect contrasts will be confounded
---|---
(uu) | m and n even
(ue) | m even; \( n_+ = n_- \)
(eu) | n even; \( m_+ = m_- \)
(ee) | \( m_+ = m_- \); \( n_+ = n_- \)

Consider next the simultaneous confounding of any two highest order interaction contrasts with corresponding subscripts not alike. If \( m_0 = n_0 = 0 \) and \( m_+ = m_- \) and \( n_+ = n_- \) (\( m \) and \( n \) both even), their generalized interaction will either be a single contrast of the same order or a linear combination of contrasts of the same order (see Table 17) and hence no main effect contrasts will be confounded because of this generalized interaction. By Rule 3, four equal blocks will result. The homogeneous bundles for various confounding patterns are indicated below.

<table>
<thead>
<tr>
<th>Homogeneous bundle</th>
<th>Sets of contrasts confounding this bundle</th>
</tr>
</thead>
<tbody>
<tr>
<td>[EE]</td>
<td>(uu, u₁u₁); (ue, u₁e₁); (eu, e₁u₁); (ee, e₁e₁)</td>
</tr>
<tr>
<td>[EU]</td>
<td>(uu, u₁e₁); (ue, u₁u₁); (eu, e₁e₁); (ee, e₁u₁)</td>
</tr>
<tr>
<td>[UE]</td>
<td>(uu, e₁u₁); (ue, e₁e₁); (eu, u₁u₁); (ee, u₁e₁)</td>
</tr>
<tr>
<td>[UU]</td>
<td>(uu, e₁e₁); (ue, e₁u₁); (eu, u₁e₁); (ee, u₁u₁)</td>
</tr>
</tbody>
</table>

8.3.2 Three-factor experiments

Examination of Table 15 reveals that those \( m \times n \times p \) experiments, in which the confounding of any higher order interaction contrast does not confound main effect contrasts, also satisfy the rule for two equal blocks. The combination of factor levels in the confounding interaction contrasts which will not result in confounding main effect contrasts are presented below.
Confounding contrasts

Requirements so that no main effect contrasts will be confounded

\begin{align*}
\text{uuu} & \quad m, n \text{ and } p \text{ even} \\
\text{uue} & \quad m \text{ and } n \text{ even} \\
\text{uee} & \quad m \text{ even}; \ n_+ = n_- \text{ or } p_+ = p_- \\
\text{eee} & \quad \text{two or three of these: } m_+ = m_-,
\end{align*}

n_+ = n_-; \ p_+ = p_-.

The interaction contrasts which are confounded are presented in Table 15.

Consider finally the simultaneous confounding of any two highest order interaction contrasts. From the left-hand column of the extension of Table 18 to three factors, we see that no more than one pair of the corresponding subscripts (out of the three) can be alike and still not confound a main effect because of the generalized interaction.

Table 23 supplies the confounding contrasts of highest order of an \(m \times n \times p\) experiment such that the simultaneous confounding of any two—with not more than one pair of corresponding subscripts alike—will yield four equal blocks without confounding main effect contrasts. To distinguish between an odd and an even number of levels: affix an asterisk to each of the letters corresponding with factors with an even number of levels, but the number of positive and negative elements differ for at least one contrast; affix an additional asterisk to the same letter if the coefficient vector of the factor corresponding with that letter contains the same number of positive as negative elements for all contrasts. For instance, \(p\) will denote an odd number of levels; \(p^*\) an even number of levels, but some \(p_+ \neq p_-; p^{**}\) an even number of levels such that all \(p_+ = p_-\).
Table 23. Confounding contrasts of highest order not confounding main effects and yielding four equal blocks, when taken in pairs.

<table>
<thead>
<tr>
<th>m x n* x p*</th>
<th>m x n** x p**</th>
<th>m* x n* x p*</th>
<th>m* x n* x p**</th>
</tr>
</thead>
<tbody>
<tr>
<td>euu</td>
<td>euu</td>
<td>euu</td>
<td>euu</td>
</tr>
<tr>
<td>eue</td>
<td>eue</td>
<td>ueu</td>
<td>eue</td>
</tr>
<tr>
<td>eeu</td>
<td>ueu</td>
<td>uue</td>
<td>uue</td>
</tr>
<tr>
<td>eee</td>
<td>uuu</td>
<td>uuu</td>
<td>uuu</td>
</tr>
</tbody>
</table>

1/ Not more than one pair of corresponding subscripts should be alike; e.g., for the first column, A_e \neq A_{e1} or B_u \neq B_{u1}.

2/ No asterisk denotes odd number of levels.
* denotes even number, but n_+ \neq n_- for at least one contrast.
** denotes even number and p_+ = p_- for all contrasts.

Details for the m* x n** x p** and m** x n** x p** are not given in Table 23, since in these cases all eight possible confounding contrasts (uuu, uue, ... , eee) can be used.

Four equal blocks in which main effect contrasts are not confounded will also be obtained when an appropriate three-factor and an appropriate two-factor interaction contrast, with corresponding subscripts not alike, are confounded simultaneously. Four equal blocks in which no main effect contrasts are confounded will finally result when two appropriate two-factor interaction contrasts, with all corresponding subscripts not alike when both are not zero, are confounded simultaneously.

When two three-factor interaction contrasts are confounded simultaneously, the requirement of at least two pairs of corresponding subscripts to be unlike cannot always be met in practice, with the result that main effect contrasts will be confounded. In such cases it becomes necessary to confound either one three-factor and one two-factor contrast
or two two-factor contrasts. Consider, for instance, the confounding of \((e_u, e_1, e_2)\) in a \(m \times n^* \times p**\) experiment. If \(m = 3\), \(n^* = 2\), \(p** = 4\), the confounding contrasts will be \((2l_u, 2l_e)\) and the generalized interaction \(I(2Cu', 00u')\); by Table 18 [00U] will be confounded. If, however, \((2l_u, 0le)\) or \((20u, 0le)\) are chosen as confounding contrasts, no main effect contrast will be confounded.

By Rule 2, three equal blocks can only result from the confounding of a highest order interaction contrast when one of the factors has three levels, and the coefficient vector of this factor in the confounding contrast is of the first degree. If such is the case, however, Tables 11 and 15 show that main effect contrasts will be confounded. Hence it is impossible to confound a polynomial contrast and obtain experiments in three equal blocks for which main effect contrasts are unconfounded. Similarly for the case when six equal blocks are desired; here again, by Rule 3, one of the factors must have three levels.

From a practical point of view, one would be interested in confounded experiments in which the equal block sizes are not too large, and in which main effect contrasts as well as first-order first degree interaction contrasts and possibly second-order first degree interaction contrasts [e.g., \((110)\) or \((111)\)] are unconfounded.

As can be expected from the extensive investigations undertaken by Yates (1937), Li (1944), Kempthorne (1952), Binet et al (1955) and other authors, most of the two-factor and three-factor experiments which have some or all of these desirable properties are already available in the literature. The following designs, however, could not be found in the literature and appear worthy of tabulation. Two of these experiments are compared with ones suggested by Li (1944) and analyzed from the degree point of view by Binet et al (1955).
Table 2.4. New three-factor experiments in four equal blocks with main effects unconfounded.

<table>
<thead>
<tr>
<th>Type of experiment</th>
<th>Confounding contrasts</th>
<th>Contrast confounded?</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 x 2 x 2</td>
<td>(201, 210)</td>
<td>No No Yes No</td>
</tr>
<tr>
<td>6 x 2 x 2</td>
<td>(e11, u10)</td>
<td>Yes Yes Yes No</td>
</tr>
<tr>
<td>8 x 2 x 2</td>
<td>(e11, e10)</td>
<td>No No No No</td>
</tr>
<tr>
<td>4 x 4 x 2</td>
<td>(2u1, 220)</td>
<td>No No No No</td>
</tr>
<tr>
<td></td>
<td>(201, u120)</td>
<td>No No No No</td>
</tr>
<tr>
<td>6 x 4 x 2</td>
<td>(e21, u101)</td>
<td>No Yes No No</td>
</tr>
<tr>
<td>8 x 4 x 2</td>
<td>(e11, e1u11)</td>
<td>No No No No</td>
</tr>
</tbody>
</table>

8.4 Confounding of Linear Combinations of Contrasts

In the previous section it was shown to be impossible to obtain three equal blocks with main effects unconfounded when confounding a single orthogonal polynomial contrast. With a 3 x 3 experiment, for instance, the confounding of \((A_1 B_1)\) will yield three equal blocks but will confound \(B_1\) and \(B_2\); the confounding of \((A_1 B_2)\) will yield three equal blocks, but will confound \(A_1\) and \(A_2\). The question arises whether it would be possible to obtain a contrast equal to some linear combination of orthogonal polynomial contrasts such that its confounding will leave main effects free.
From the scheme below it can be seen that if such a contrast exists, the signs of its elements must follow some such pattern as given in the last two columns.

<table>
<thead>
<tr>
<th>Treatment combination</th>
<th>Elements of contrasts ((y=1))</th>
<th>Alternative signs of (A_xB_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>(-1) (-1) (-2) 0</td>
<td>(-) 0</td>
</tr>
<tr>
<td>01</td>
<td>(+2) 0 (+2) (-2)</td>
<td>(+) (-)</td>
</tr>
<tr>
<td>02</td>
<td>(-1) (+1) 0 (+2)</td>
<td>0 (+)</td>
</tr>
<tr>
<td>10</td>
<td>0 (+2) (+2) (+2)</td>
<td>(+) (+)</td>
</tr>
<tr>
<td>11</td>
<td>0 0 (+2) (+2) (-2) (+2)</td>
<td>0 (+) (+)</td>
</tr>
<tr>
<td>12</td>
<td>0 (-2) (-2) (+2) (-2)</td>
<td>(+) (-) (+) (-)</td>
</tr>
<tr>
<td>20</td>
<td>(+1) (-1) 0 (-2) (+2)</td>
<td>0 (+)</td>
</tr>
<tr>
<td>21</td>
<td>(-2) 0 (-2) (+2) (-2)</td>
<td>(-) (+)</td>
</tr>
<tr>
<td>22</td>
<td>(+1) (+1) (+2) 0</td>
<td>(+) 0</td>
</tr>
</tbody>
</table>

It can be seen at once that our requirements will be met when

\[
(A_xB_y) = (A_2B_1) + (A_1B_2) \tag{8.12}
\]

or

\[
(A_xB_y) = (A_2B_1) - (A_1B_2) \tag{8.13}
\]

Confounding of (8.12) will yield the I-component design, while confounding of (8.13) will yield the J-component design of Yates (1937). These designs do not confound main effect contrasts.

A similar investigation of the \(3^3\) experiment shows that the respective confounding of the following linear combinations of contrasts will yield the replicates of Yates' (1937) balanced design in books of nine. Again main effect contrasts are unconfounded. The replicate numbers correspond to the tabulation by Cochran and Cox (1950).
Replicate I: \((A_1B_2C_2) + (A_2B_1C_2) - (A_2B_2C_1) + 3(A_1B_1C_1)\)

Replicate II: \((A_1B_1C_2) + (A_2B_1C_2) + (A_2B_2C_1) - 3(A_1B_1C_1)\)

Replicate III: \((A_1B_2C_2) - (A_2B_1C_2) + (A_2B_2C_1) + 3(A_1B_1C_1)\)

Replicate IV: \(-(A_1B_2C_2) + (A_2B_1C_2) + (A_2B_2C_1) + 3(A_1B_1C_1)\)

Similarly experiments of the type \(3^3 \times q^*\) and \(3^2 \times p^* \times q^*\) can be arranged into six equal blocks without confounding main effects by the simultaneous confounding of the linear combination (8.12) or (8.13) and an appropriate two-factor interaction contrast. Thus, for instance, we have the following sets of confounding contrasts for certain experiments.

\[
\begin{align*}
3 \times 3 \times 3 \times q^* & : (xX00,002u) \\
3 \times 3 \times 3xq^{**} & : (xX00,002u) \text{ or } (xX00,002e) \\
3 \times 3 \times p^* \times q^* & : (xX00,00uu) \\
3 \times 3 \times p^{**} \times q^* & : (xX00,00uu) \text{ or } (xX00,00eu)
\end{align*}
\]

Binet et al (1955) devote considerable space to the \(3^2 \times 2^2\) experiment in blocks of 12; 12 new plans were developed and the confounding pattern for a design of Li (1944) is presented. A design in six blocks of six units each, obtained by simultaneously confounding the contrasts \((xX00)\) and \((0011)\), is identical to the one developed by Kempthorne (1952) and confounds, inter alia, \(A_1B_1, C_1D_1, A_2B_1C_1D_1\). From a degree point of view, this design appears to be useful; when the experimental material is rather variable; it may be better than the design in blocks of 12.

8.5 Balanced confounding

Consider the \(3 \times 2 \times 2\) experiment. By Rule 1 and Table 14, confounding of the contrast \((211)\) will yield two equal blocks; contrast \((011)\) will be confounded, but no others. This blocking arrangement is identical to
Replicate II of the plan given by Cochran and Cox (1950). The confounding scheme for this blocking arrangement is given in the first line in the body of the scheme below:

<table>
<thead>
<tr>
<th>Confounding contrast</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1 2</td>
<td>0 1</td>
<td>0 1</td>
</tr>
<tr>
<td>211</td>
<td>+ - +</td>
<td>- +</td>
<td>- +</td>
</tr>
<tr>
<td>x11</td>
<td>- + +</td>
<td>- +</td>
<td>- +</td>
</tr>
<tr>
<td>X11</td>
<td>+ + -</td>
<td>- +</td>
<td>- +</td>
</tr>
</tbody>
</table>

Let (x11) and (X11) be contrasts, the signs of which are as indicated above. The confounding of (x11) yields Replicate I, and the confounding of (X11) yields Replicate III of the balanced design supplied by Cochran and Cox (1950). Let \( x' \) be the row vector \((-2,1,1)\) and \( X' \) the row vector \((1,1,-2)\). If \( P_r \) denotes the orthogonal polynomial of degree \( r \) then, when \( x \) and \( X \) are expressed in terms of \( P_1 \) and \( P_2 \) for \( m = 3 \), we find

\[
x = \frac{(3P_1 - P_2)}{2}
\]

\[
X = \frac{-(3P_1 + P_2)}{2}
\]

Hence

\[
(x_{11}) = \frac{(3A_1B_1C_1 - A_2B_1C_1)}{2}
\]

\[
(X_{11}) = \frac{-(3A_1B_1C_1 + A_2B_1C_1)}{2}
\]

This result shows that a balanced group of sets for the \( 3 \times 2^2 \) experiment can be obtained by confounding the required highest order interaction contrast in one replicate, and those linear combinations of highest order interaction contrasts derived from the signs of the required contrast in two other replicates.
In practice one would obtain this results as follows: interchange levels 0 and 1 of the first factor in Replicate II to obtain Replicate I, and interchange levels 1 and 2 of the first factor in Replicate II to obtain Replicate III. Similar results can be derived for a $3 \times 2^n$ experiment in equal blocks of size $3 \times 2^{n-1}$. Generalization is possible. Hence we have shown that linear combinations of orthogonal contrasts play an important part in obtaining balanced sets of treatment combinations; however, this subject is not pursued further in this thesis.
CHAPTER 9

Summary, Conclusions and Suggestions for Future Research

9.1 Summary and Conclusions

The survey of literature reveals that for all factors of a factorial experiment with the same prime number (or power of a prime number) of levels, results from Finite Group Theory can be employed to confound the experiment into equal blocks, or into fractions, leaving main effects and some lower order interactions unconfounded. For factors with different numbers of levels, no such general theory exists.

In many experimental situations it would be desirable to confound higher degree rather than higher order contrasts. A general mathematical technique, based on conformable partitioning of the coefficient matrices of orthogonal contrasts, was developed to obtain confounded designs for both the symmetrical and asymmetrical cases. This approach not only leads to designs required from the degree point of view, but reproduces most of the designs based on the confounding of order components. By defining the positive and negative signs of the matrix elements and the zero elements of a contrast as block indicators, it was shown that confounding of a contrast with blocks corresponds mathematically to a partitioning of the coefficient matrix of the contrast according to the block indicators. The study of the confounding pattern of a contrast was thus reduced to a mathematical examination of the values of the traces of conformably partitioned coefficient matrices.

Using this technique, certain general theorems were derived in Part I. Considering the confounding of a single contrast at a time, it was shown that the confounding of a contrast of lower order does not confound any contrast of higher order. Conditions were established for the
case in which the confounding of a contrast of higher order does not confound any contrast of lower order, and for the case in which the confounding of a contrast of given order does not confound contrasts of the same order. These theorems led to the definition of the heterogeneous bundle of a confounding contrast, such a bundle consisting of all the contrasts confounded by that contrast. When two contrasts are confounded simultaneously, it is shown that their symbolic product yields another heterogeneous bundle, that of the generalized interaction.

In Part II these general results are specialized to the orthogonal polynomial contrasts. Tables are presented of the heterogeneous bundles of confounding contrasts of highest order. An easy rule for determining the heterogeneous bundle of any given contrast was found, this rule being a convenient arithmetization of the consequences of the various theorems. A rapid procedure to determine the coefficients of the block constants in the expectation of the confounded contrasts is demonstrated. The generalized interaction of two polynomial contrasts was reduced to a linear combination of polynomial contrasts, and theorems were developed to determine the confounding status of these contrasts. The rule to determine heterogeneous bundles was generalized. Conditions necessary to obtain equal blocks, when one or more contrasts of highest order are confounded, but main effects are not confounded, were established. These conditions lead to new single-replicate three-factor experiments in four equal blocks in which main effects are unconfounded.

10.2 Further Research Needed

Since this thesis dealt with a field of research rather than with a single topic, many single topics require further elaboration.
1. A complete discussion of partial confounding (in the sense of this thesis); including a method of estimating all pertinent contrasts and their standard errors.

2. A systematic search for new four- and five-factor confounded designs in equal blocks with main effects unconfounded, using the short-cut methods developed in this thesis.

3. A fuller treatment of the theory of confounding linear combinations of orthogonal contrasts, along the lines proposed for single orthogonal contrasts.


5. The development of a comprehensive theory of fractional replication, based on signs of contrasts, with special emphasis on the asymmetrical case.

6. The extension of the present theory of confounding based on the signs only of contrasts, to one based on both the signs and types of elements of the contrasts; hence, the extension of the system of block indicators $(-,+,0)$ to something like $(-e,+e,-u,+u,0)$, where $e$ denotes an even element and $u$ an odd element.
LITERATURE CITED


Nair, K. R. 1940. Balanced confounded arrangements for the 5th type of experiments. Sankhya **2**: 57-70.


