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Summary. In this paper the implications of certain results obtained in earlier papers [1, 2, 3, 4] on confidence bounds on parametric functions connected with multivariate normal populations are fully worked out. This leads to a number of confidence bounds, expected to be useful, but hitherto unnoticed, (i) on the characteristic roots of one population dispersion matrix and on roots connected with (ii) two population dispersion matrices, (iii) the regression matrix of a p-set on a q-set and (iv) multivariate linear hypothesis on means, including, in particular, the problem of discriminant analysis.

1. Confidence bounds on roots connected with Σ of $N(\underline{\xi},\Sigma)$. Let us start from the statement (3.1.2) of [2] and note that the statement is exactly equivalent to

(1.1) $$n\theta_{1\alpha}^{-1}(p,n)\,\underline{a}'S\underline{a} \geq \underline{a}'\Sigma\underline{a} \geq n\theta_{2\alpha}^{-1}(p,n)\,\underline{a}'S\underline{a},$$

for all nonnull $\underline{a}(p \times 1)$'s, that is, to

(1.2) $$\lambda_1 \frac{\underline{a}'S\underline{a}}{\underline{a}'\underline{a}} \geq \frac{\underline{a}'\Sigma\underline{a}}{\underline{a}'\underline{a}} \geq \lambda_2 \frac{\underline{a}'S\underline{a}}{\underline{a}'\underline{a}},$$

where λ_1 and λ_2 stand respectively for $n\theta_{1\alpha}^{-1}(p, n)$ and $n\theta_{2\alpha}^{-1}(p,n)$.

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Choosing $\underline{a}$ so as to minimize $\underline{a}'\Sigma\underline{a}/\underline{a}'\underline{a}$, we observe that the second part of the inequality (1.2) implies that $\lambda_2 c_{min}(S) \leq c_{min}(\Sigma)$; and choosing $\underline{a}$ so as to minimize $\underline{a}'S\underline{a}/\underline{a}'\underline{a}$, we notice that the first part of the inequality implies that $c_{min}(\Sigma) \leq \lambda_1 c_{min}(S)$. Likewise, choosing $\underline{a}$ so as to maximize $\underline{a}'S\underline{a}/\underline{a}'\underline{a}$, we note that the second part of the inequality implies that $\lambda_2 c_{max}(S) \leq c_{max}(\Sigma)$; and choosing $\underline{a}$ so as to maximize $\underline{a}'\Sigma\underline{a}/\underline{a}'\underline{a}$, we have that (1.2) implies that $c_{max}(\Sigma) \leq \lambda_1\ c_{max}(S)$. Thus (1.2) $\Longrightarrow$

$$\lambda_1 c_{min}(S) \geq c_{min}(\Sigma) \geq \lambda_2 c_{min}(S)$$

(1.3) and

$$\lambda_1 c_{max}(S) \geq c_{max}(\Sigma) \geq \lambda_2\ c_{max}(S).$$

We use (2.4) of [2] and note that (1.3) has a confidence coefficient $\geq 1-\alpha$, and also incidentally that (1.3) implies (3.1.3) of [2] .

Going back to (1.2) let us take $\underline{a}(p \times 1)$ such that the ith component is zero. Then arguing in a similar manner, we observe that (1.2) $\Longrightarrow$

$$\lambda_1 c_{min}(S^{(i)}) \geq c_{min}(\Sigma^{(i)}) \geq \lambda_2 c_{min}(S^{(i)})$$

(1.4) and

$$\lambda_1 c_{max}(S^{(i)}) \geq c_{max}(\Sigma^{(i)}) \geq \lambda_2 c_{max}(S^{(i)}) ,$$

for $i = 1, 2, \ldots, p$, where $S^{(i)}$ and $\Sigma^{(i)}$ stand respectively for the truncated sample and population dispersion matrices obtained by cutting out the ith variate. Likewise, if we take an $\underline{a}(p \times 1)$ such that the ith and jth ($i \neq j$) components are zero and then argue in a similar manner, we observe that

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(1.2) also $\Longrightarrow$

$$\lambda_1 c_{min}(S^{(i,j)}) \geq c_{min}(\Sigma^{(i,j)}) \geq \lambda_2 c_{min}(S^{(i,j)})$$

(1.5) and

$$\lambda_1 c_{max}(S^{(i,j)}) \geq c_{max}(\Sigma^{(i,j)}) \geq \lambda_2 c_{max}(S^{(i,j)}),$$

for $i \neq j = 1, 2, \ldots, p$, where $S^{(i,j)}$ and $\Sigma^{(i,j)}$ stand respectively for the truncated sample and population dispersion matrices obtained by cutting out the ith and jth variates. We can continue this process on to the stage of cutting out any (p-1) variates, that is, retaining any one variate. It is seen that (1.2) $\Longrightarrow$ a pair of statements (1.3), and also p pairs of statements like (1.4), $\binom{p}{2}$ pairs of statements like (1.5), and so on down to $\binom{p}{p-1}$, i.e., p statements involving only one variate. All such statements will thus have a joint confidence coefficient $\geq 1-\alpha$, and will provide us, from a certain standpoint, with a complete analysis of what the psychologists call the problem of principal components.

2. Confidence bounds on roots connected with Σ_1 and Σ_2 of $N(\underline{\xi}_1,\Sigma_1)$ and $N(\underline{\xi}_2,\Sigma_2)$. Let us start from (3.2.1) of [2], put $\lambda_1 = (n_1/n_2)\, \theta_{1\alpha}^{-1}(p,n_1,n_2)$ and $\lambda_2 = (n_1/n_2)\, \theta_{2\alpha}^{-1}(p,n_1,n_2)$ and rewrite (3.2.1) as

$$\lambda_1 \geq \text{all } c(S_2\, (\mu')^{-1}\, D_{\gamma_i}\, \mu' S_1^{-1} \mu\, D_{\gamma_i}\, \mu^{-1}) \geq \lambda_2 \tag{2.1}$$

where γ_i's are $c(\Sigma_1\Sigma_2^{-1})$'s (with $i = 1, 2, \ldots, p$). We next recall (2.3.2) of [2] and also that $c(S_1S_2^{-1})$'s are invariant under a transformation: $S_1 = AS_1^*A'$ and $S_2 = AS_2^*A'$ where A is any nonsingular matrix, and put $A = \mu^{-1}$, and rewrite (2.1), without any loss of generality, for our purpose, in the canonical form

$$\lambda_1 \geq \text{all } c(S_2 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geq \lambda_2$$

or

$$\lambda_1 \geq \text{all } c(S_2 S_1^{-1} S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geq \lambda_2$$

or

$$(2.3) \qquad \lambda_1 \frac{\underline{a}' S_1 S_2^{-1} \underline{a}}{\underline{a}'\underline{a}} \geq \frac{\underline{a}'(S_1 D_{\sqrt{\gamma_1}} S_1^{-1} D_{\sqrt{\gamma_1}})\underline{a}}{\underline{a}'\underline{a}} \geq \lambda_2 \frac{\underline{a}' S_1 S_2^{-1} \underline{a}}{\underline{a}'\underline{a}} ,$$

for all nonnull $\underline{a}(p \times 1)$'s. Now choosing $\underline{a}$ so as to maximize the middle term of (2.3), we note that the left part of the inequality (2.3) $\Longrightarrow$ $\lambda_1 c_{max}(S_1S_2^{-1}) \geq c_{max}(S_1 D_{\sqrt{\gamma_1}} S_1^{-1} D_{\sqrt{\gamma_1}})$; and choosing $\underline{a}$ so as to minimize the middle term of (2.3), we note that the right part of (2.3) $\Longrightarrow$ $c_{min}(S_1 D_{\sqrt{\gamma_1}} S_1^{-1} D_{\sqrt{\gamma_1}}) \geq \lambda_2 c_{min}(S_1S_2^{-1})$. Thus (2.3) $\Longrightarrow$

$$(2.4) \quad \lambda_1 c_{max}(S_1 S_2^{-1}) \geq c_{max}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geq c_{min}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}})$$

$$\geq \lambda_2 c_{min}(S_1 S_2^{-1}).$$

Use is made of (2.3.2) and (2.3.5) of [2] to show in [4] that

$$(2.5) \quad c_{max}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geq \text{all } c(D_{\gamma_i}) \geq c_{min}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}).$$

Thus it is seen that (2.3) $\Longrightarrow$

$$(2.6) \quad \lambda_1 c_{max}(S_1 S_2^{-1}) \geq \text{all } c(\Sigma_1 \Sigma_2^{-1}) \geq \lambda_2 c_{min}(S_1 S_2^{-1}) ,$$

which, therefore, is a confidence statement with a confidence coefficient $\geq 1 - \alpha$, since (2.3) has the confidence coefficient $1 - \alpha$. (2.6) is proved in a slightly different way in [4].

We now go back to (2.3) and, as in the previous section, take $\underline{a}(p \times 1)$ such that the ith component is zero, argue the same way as from (2.3) to (2.6) and end up by observing that (2.3) also $\Longrightarrow$

$$(2.7) \quad \lambda_1 c_{max}(S_1^{(i)} S_2^{(i)-1}) \geq \text{all } c(\Sigma_1^{(i)} \Sigma_2^{(i)^{-1}}) \geq \lambda_2 c_{min}(S_1^{(i)} S_2^{(i)^{-1}}),$$

where $S_1^{(i)}$, $S_2^{(i)}$, $\Sigma_1^{(i)}$ and $\Sigma_2^{(i)}$ have the same meaning as in the previous section.

Likewise, as in the previous section, we note that (2.3) also $\Longrightarrow$

$$(2.8)\qquad \lambda_1 c_{max}(S_1^{(i,j)}S_2^{(i,j)^{-1}}) \geq \text{all } c(\Sigma_1^{(i,j)}\Sigma_2^{(i,j)^{-1}}) \geq \lambda_2 c_{min}(S_1^{(i,j)}S_2^{(i,j)^{-1}}),$$

and so on till we reach the stage where any (p-1) variates have been cut out, i.e., any one variate has been retained, which gives us just the confidence bounds on variance ratios in the univariate case. We have thus, with a joint confidence coefficient $\geq 1 - \alpha$, confidence statement (2.6), p confidence statements like (2.7), $\binom{p}{2}$ confidence statements like (2.8), and so on. This again, from a certain standpoint, provides part of the analysis of a problem which occurs in the multivariate generalization of the customary variance components analysis in univariate analysis of variance and covariance.

3. Confidence bounds connected with the regression matrix $\beta(p \times q)$ of a p-set on a q-set in a (p+q)-variate normal distribution. Let us start from (4.5) of [2], set $\lambda^2 = \theta_\alpha/(1-\theta_\alpha)$, $S_{1.2} = S_{11} - S_{12}S_{22}^{-1}S_{12}'$, use Lemmas C and E of [2] and obtain, with a confidence coefficient $\geq 1 - \alpha$ and for all unit vectors $\underline{d}_1(p \times 1)$ and $\underline{d}_2(q \times 1)$ the confidence statement

$$(3.1)\qquad \underline{d}_1'B\underline{d}_2 - \lambda\, c_{max}^{1/2}(S_{1.2})\, c_{max}^{1/2}(S_{22}^{-1}) \leq \underline{d}_1'\beta\underline{d}_2$$

$$\leq \underline{d}_1'B\underline{d}_2 + \lambda c_{max}^{1/2}(S_{1.2}) c_{max}^{1/2}(S_{22}^{-1}),$$

where $B(p \times q) = S_{12}S_{22}^{-1}$ (the sample regression matrix of the p-set on the q-set) and $\beta(p \times q) = \Sigma_{12}\Sigma_{22}^{-1}$ (the population regression matrix of the p-set on the q-set). Going back to Lemmas C and E of [2] again we notice that, with respect to variation over $\underline{d}_1$ and $\underline{d}_2$, the maximum values of $\underline{d}_1' B \underline{d}_2$ and $\underline{d}_1' \beta \underline{d}_2$ are respectively $c_{max}^{1/2}(BB')$ and $c_{max}^{1/2}(\beta\beta')$. Now, first choosing $\underline{d}_1$ and $\underline{d}_2$ so as to maximize $\underline{d}_1' B \underline{d}_2$ and then choosing $\underline{d}_1$ and $\underline{d}_2$ so as to maximize $\underline{d}_1' \beta \underline{d}_2$ and arguing in the same way as in the two previous sections we note that (3.1) $\Longrightarrow$

$$(3.2) \qquad c_{max}^{1/2}(BB') - \lambda c_{max}^{1/2}(S_{1.2}) c_{max}^{1/2}(S_{22}^{-1}) \leq c_{max}^{1/2}(\beta\beta')$$

$$\leq c_{max}^{1/2}(BB') + \lambda c_{max}^{1/2}(S_{1.2})\, c_{max}^{1/2}(S_{22}^{-1}),$$

which, therefore, is a confidence statement with a confidence coefficient $\geq 1-\alpha$. Now going back to (4.4) of [2] which is a confidence statement with a confidence coefficient $1-\alpha$, we rewrite it in the equivalent form

$$(3.3) \qquad \frac{\underline{a}'[(B-\beta)S_{22}(B'-\beta')]\underline{a}}{\underline{a}' S_{1.2} \underline{a}} \leq \lambda^2 ,$$

for all nonnull $\underline{a}(p \times 1)$'s. This means that (3.3), with a probability $1-\alpha$, implies (3.2), with a probability $\geq 1-\alpha$. As in the previous sections, take $\underline{a}(p \times 1)$ such that the ith component is zero, define $S_{1.2}^{(i)}$. $B^{(i)}$ and $\beta^{(i)}$ as the truncated matrices obtained by cutting out the ith variate of the p-set, and observe that (3.3) also $\Longrightarrow$

$$(3.4)\qquad c_{max}^{1/2}(B^{(i)}B^{(i)'}) - \lambda c_{max}^{1/2}(S_{1.2}^{(i)})c_{max}^{1/2}(S_{22}^{-1}) \leq c_{max}^{1/2}(\beta^{(i)}\beta^{(i)'})$$

$$\leq c_{max}^{1/2}(B^{(i)}B^{(i)'}) + \lambda c_{max}^{1/2}(S_{1.2}^{(i)})c_{max}^{1/2}(S_{22}^{-1}).$$

Likewise, as in the previous sections, we observe that (3.3) also $\Longrightarrow$

$$(3.5)\qquad c_{max}^{1/2}(B^{(i,j)}B^{(i,j)'}) - \lambda c_{max}^{1/2}(S_{1.2}^{(i,j)})c_{max}^{1/2}(S_{22}^{-1}) \leq c_{max}^{1/2}(\beta^{(i,j)}\beta^{(i,j)'})$$

$$\leq c_{max}^{1/2}(B^{(i,j)}B^{(i,j)'}) + \lambda c_{max}^{1/2}(S_{1.2}^{(i,j)})c_{max}^{1/2}(S_{22}^{-1}),$$

and so on. We have thus, with a joint confidence coefficient $\geq 1-\alpha$, the statement (3.2), p statements like (3.4), $\binom{p}{2}$ statements like (3.5) and so on. This kind of result could be generalized by truncating the variates of the q-set as well, but this will not be discussed here.

4. Confidence bounds on roots connected with multivariate linear hypothesis on means.

4a. On $\underline{\xi}$ of $N(\underline{\xi}, \Sigma)$. Let us start from (4.1.4) of [1], set $\lambda^2 = T_\alpha^2/(n+1)$, and rewrite (4.1.4) as

$$(4.1)\qquad \frac{\underline{a}'\bar{\underline{x}}}{(\underline{a}'\underline{a})^{1/2}} - \lambda\frac{(\underline{a}'S\underline{a})^{1/2}}{(\underline{a}'\underline{a})^{1/2}} \leq \frac{\underline{a}'\underline{\xi}}{(\underline{a}'\underline{a})^{1/2}} \leq \frac{\underline{a}'\bar{\underline{x}}}{(\underline{a}'\underline{a})^{1/2}} + \lambda\frac{(\underline{a}'S\underline{a})^{1/2}}{(\underline{a}'\underline{a})^{1/2}},$$

for all nonnull $\underline{a}(p \times 1)$'s. We recall that (4.1) is a confidence statement with a confidence coefficient 1-α. We recall that the maximum values of $\underline{a}'\bar{\underline{x}}/(\underline{a}'\underline{a})^{1/2}$, $\underline{a}'\underline{\xi}/(\underline{a}'\underline{a})^{1/2}$, $(\underline{a}'S\underline{a})^{1/2}/(\underline{a}'\underline{a})^{1/2}$, with respect to variation of $\underline{a}$'s, are respectively $(\bar{\underline{x}}'\bar{\underline{x}})^{1/2}$, $(\underline{\xi}'\underline{\xi})^{1/2}$ and $c_{max}^{1/2}(S)$, reason in the same way as in the previous sections and deduce that (4.1) $\Longrightarrow$

$$(4.2) \quad [\bar{\underline{x}}'\bar{\underline{x}}]^{1/2} - \lambda c_{max}^{1/2}(S) \leq [\underline{\xi}'\underline{\xi}]^{1/2} \leq [\bar{\underline{x}}'\bar{\underline{x}}]^{1/2} + \lambda c_{max}^{1/2}(S),$$

which is thus a confidence statement with a confidence coefficient $\geq$ 1-α. Arguing as in the previous sections and using the same notation as before for truncated $\bar{\underline{x}}$, $\underline{\xi}$ and S obtained by cutting out the ith variate, the ith and jth variates ($i \neq j$), and so on, we have with a joint confidence coefficient $\geq$ 1-α, in addition to (4.2), p statements like

$$(4.3) \quad [\bar{\underline{x}}^{(i)'}\bar{\underline{x}}^{(i)}]^{1/2} - \lambda c_{max}^{1/2}(S^{(i)}) \leq [\underline{\xi}^{(i)'}\underline{\xi}^{(i)}]^{1/2} \leq [\bar{\underline{x}}^{(i)'}\bar{\underline{x}}^{(i)}]^{1/2} + \lambda c_{max}^{1/2}(S^{(i)}),$$

$\binom{p}{2}$ statements like

$$(4.4) \quad \left[\bar{\underline{x}}^{(i,j)'}\bar{\underline{x}}^{(i,j)}\right]^{1/2} - \lambda c_{max}^{1/2}(S^{(i,j)}) \leq \left[\underline{\xi}^{(i,j)'}\underline{\xi}^{(i,j)}\right]^{1/2} \leq \left[\bar{\underline{x}}^{(i,j)'}\bar{\underline{x}}^{(i,j)}\right]^{1/2} + \lambda c_{max}^{1/2}(S^{(i,j)}),$$

and so on down to the stage of cutting out any (p-1) variates, i.e., retaining any one variate.

4b. Some observations on multivariate linear hypothesis on means. Confidence bounds connected with univariate and with multivariate linear hypothesis on means are discussed respectively in chapters 15 and 16 (to which chapter 14 forms the background) of [3]. Here we shall first set up a physically more general hypothesis and then discuss the associated confidence bounds.

Let $X(n \times p)$ (with $p < n$) consist of n row vectors $\underline{x}_i'(1 \times p)$ (with $i = 1, 2, \ldots, n$) which are independently distributed, each being $N[E(\underline{x}_i'), \Sigma]$ and let $E(X)(n \times p) = A(n \times m)\,\xi(m \times p)$, where $m < n$ and rank $(A) = r \leq m$. Let A_1 $(n \times r)$ be a basis of A and let us write (as we can, without any loss of generality) $A(n \times m) = n[\underset{r}{A_1} \quad \underset{m-r}{A_2}]$ and let us rewrite the expectation condition as

$$(4.5) \qquad \underset{p}{E(X)n} = n[\underset{r}{A_1} \quad \underset{m-r}{A_2}] \underset{p}{\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}} \begin{matrix} r \\ m-r \end{matrix} \quad .$$

Here the X is a set of (observable) stochastic variates, ξ is a set of unknown population parameters, A is a known matrix of constants given by the design of the experiment and is called the design matrix. It might consist of numbers like, say 0, 1, etc. and/or a set of observed (nonstochastic) quantities, as in the case of regression problems with concomitant variates. The population dispersion matrix Σ is also unknown. This is the model under which we propose to test the hypothesis

$$(4.6) \qquad H_0: \begin{matrix} s \\ q-s \end{matrix} \underset{\quad r \qquad m-r}{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}} \begin{bmatrix} \xi_1(r \times p) \\ \xi_2(\overline{m-r} \times p) \end{bmatrix} M(p \times u) = 0,$$

where $C(q \times m)$, partitioned as above, and $M(p \times u)$ are matrices given by the hypothesis to be tested and are called the hypothesis matrices. It is assumed that rank $(M) = u \leq p$ and rank $(C) = s \leq r$ $(\leq m < n$ of course), and furthermore that, rowwise, $[C_{11} \quad C_{12}]$ is a basis of C and, columnwise, $\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$

is also a basis of C. We go back to (14.2.13), (14.2.15) and (14.4.6) of [3] (that are repeatedly used in chapters 15 and 16 of [3]) and recall that for H_0 to be testable in the sense of chapter 14 of [3] we should have

$$(4.7) \qquad C_{i2} = C_{i1}\,(A_1'A_1)^{-1}\,A_1'A_2 \text{ (with } i = 1, 2).$$

However, in most realistic problems, the C matrix of the hypothesis is given in a form such that the last rows are absent and we can, therefore, without any essential loss of generality, replace (4.6) by

$$(4.8) \qquad s\,[\underset{r}{C_1} \quad \underset{m-r}{C_2}] \begin{bmatrix} \xi_1(r \times p) \\ \xi_2(\overline{m-r} \times p) \end{bmatrix} M(p \times u) = 0,$$

and (4.7) by

$$(4.9) \qquad C_2 = C_1(A_1'A_1)^{-1}\,A_1'A_2.$$

We now go back to X and observe that $X(n \times p)\,M(p \times u)$ [$= X^*(n \times u)$, say] consists of n rows of independently distributed vectors $\underline{x}_i'(1 \times p)\,M(p \times u)$ [$= \underline{x}_i^{*\prime}(1 \times u)$, say] such that $\underline{x}_i^*$ is $N\,[E(\underline{x}_i^*),\ M'\Sigma M]$, i.e., $N\,[E(\underline{x}_i^*),\ \Sigma^*]$, say, and that

$$(4.10) \qquad E(X^*) = [A_1 \quad A_2] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} M = [A_1 \quad A_2] \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix}, \text{ say,}$$

where

$$(4.10a) \quad \begin{matrix} r \\ u-r \end{matrix} \underset{p}{\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}} M(p \times u) = \begin{matrix} r \\ m-r \end{matrix} \underset{u}{\begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix}} .$$

The H_0 of (4.8) can now be rewritten so

$$(4.11) \qquad H_0: \; s \, \left[\underset{r}{C_1} \quad \underset{m-r}{C_2} \right] \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix} = 0$$

and the alternative H to H_0 can be expressed as

$$(4.12) \qquad H: \; s \, \left[\underset{r}{C_1} \quad \underset{m-r}{C_2} \right] \underset{u}{\begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix}} \begin{matrix} r \\ m-r \end{matrix} = \eta^*(s \times u) \quad .$$

If we now go back to (16.6.3) of [3] and substitute $M'(u \times p)\, X'(p \times n)$, i.e., $X^{*'}(u \times n)$ for $X'(p \times n)$, C_1 for C_{11}, and $\underline{a}^*(u \times 1)$ for $\underline{a}(p \times 1)$, u for p, η^* for η, and use (15.2.15) of [3], we observe that now the confidence statement (16.6.3) of [3] is replaced by

$$(4.13) \quad \underline{a}^{*'}(1 \times u)\, X^{*'}(u \times n)\, A_1(n \times r)(A_1'A_1)^{-1}(r \times r)\, C_1'(r \times s)\, \tilde{U}(s \times s)\underline{b}(s \times 1)$$

$$-(\underline{a}^{*'}S^*\underline{a})^{1/2} \left[sc_\alpha(u,s,n-r) \right]^{1/2} \leq \underline{a}^{*'}(1 \times u)\, \eta^{*'}(u \times s)\, \tilde{U}(s \times s)\underline{b}(s \times 1)$$

$$\leq \underline{a}^{*'} X^{*'} A_1(A_1'A_1)^{-1} C_1'\tilde{U}\underline{b} + (\underline{a}^{*'}S^*\underline{a})^{1/2} \left[sc_\alpha(u,s,n-r) \right]^{1/2},$$

for all nonnull $\underline{a}^*(u \times 1)$ and all unit vectors $\underline{b}(s \times 1)$, where η^* is given by (4.12), (ξ_1^*, ξ_2^*) by (4.10a), X^* stands for $X\,M$, and where

$$(4.14)\qquad \tilde{U}\tilde{U}' = [C_1(A_1'A_1)^{-1}C_1']^{-1}$$

and

$$(4.15)\quad (n-r)\,S^*(u \times u) = M'(u \times p)\,X'(p \times n)\,[I(n)-A_1(n \times r)(A_1'A_1)^{-1}(r \times r) \times A_1'(r \times n)]X(n \times p)\,M(p \times u)$$

We note that (4.13) is a set of confidence statements, with a joint confidence coefficient $1-\alpha$, on bilinear compounds of η^*, where $\eta^* = [C_1 \quad C_2]\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} M$, may be regarded as measuring the deviation from the null hypothesis H_0.

4c. Further consequences of (4.13). Starting from (4.13) and arguing in the same manner as in section 3 and setting $c_\alpha(u,s,n-r) = c_\alpha$ (say) we note that (4.13) $\Longrightarrow$

$$(4.16)\quad c_{max}^{1/2}[X^{*\prime}A_1(A_1'A_1)^{-1}C_1'\tilde{U}\tilde{U}'C_1(A_1'A_1)^{-1}A_1'X^*] - [sc_\alpha]^{1/2}c_{max}^{1/2}(S^*)$$

$$\leq c_{max}^{1/2}[\eta^{*\prime}(\tilde{U}\tilde{U}')\eta^*] \leq c_{max}^{1/2}[X^{*\prime}A_1(A_1'A_1)^{-1}C_1'\tilde{U}\tilde{U}'C_1(A_1'A_1)^{-1}A_1'X^*]$$

$$+ [sc_\alpha]^{1/2}c_{max}^{1/2}(S^*).$$

or, substituting for $\tilde{U}\tilde{U}'$ from (4.14),

$$(4.17)\quad c_{max}^{1/2}[s\ S^{**}] - [sc_\alpha]^{1/2} c_{max}^{1/2}(S^*) \leq c_{max}^{1/2}[\eta^{*'}[C_1(A_1'A_1)^{-1}C_1']^{-1}\eta^*]$$

$$\leq c_{max}^{1/2}[s\ S^{**}] + [sc_\alpha]^{1/2}\ c_{max}^{1/2}\ (S^*),$$

where the matrix due to the hypothesis, i.e., sS^{**} is given by

$$(4.18)\quad sS^{**} = M'X'A_1(A_1'A_1)^{-1}\ C_1'\ [C_1(A_1'A_1)^{-1}C_1']^{-1} C_1(A_1'A_1)^{-1}A_1'\ X\ M,$$

and the matrix due to the error, i.e., $(n-r)\ S^*$ is given by (4.15). Notice that (4.17) is a confidence statement with a confidence coefficient $\geq 1-\alpha$ and that the middle term of (4.17) is zero if and only if the hypothesis H_0 is true. For p=1, M(p x u) will drop out (except for a trivial scalar factor, since $u \leq p$) and we shall have the univariate problem when $c_{max}(sS^{**})$ will be replaced by just the sum of squares due to the hypothesis, $c_{max}(S^*)$ by just the variance due to the error and $c_{max}[\eta^{*'}\ [C_1(A_1'A_1)^{-1}C_1']^{-1}\ \eta^*]$ by just the scalar $\eta^{*'}\ [C_1(A_1'A_1)^{-1}\ C_1']^{-1}\ \eta^*$.

Starting from (4.13) and reasoning in exactly the same way as in sections 3 and 4a, we see that (4.13) also implies, in addition to (4.17), p statements like (4.17) on truncated $S^{(i)^*}$, $S^{(i)^{**}}$ and $\eta^{(i)^*}$ obtained by cutting out any ith variate, $\binom{p}{2}$ statements like (4.17) on truncated $S^{(i,j)^*}$, $S^{(i,j)^{**}}$ and and $\eta^{(i,j)^*}$ obtained by cutting out any pair of ith and jth variates (with $i \neq j$), and so on. These latter confidence statements will thus have a joint confidence coefficient $\geq 1-\alpha$.

It may be noticed that the problem discussed in section 4a is really a special case of the one discussed in 4c; nevertheless, for expository purposes, it is worthwhile to discuss first a simple problem like the one in 4a and then take up the most general one in 4c.

5. Concluding remarks. Similar confidence bounds that arise in connection with (i) factor analysis, (ii) classification problems (iii) univariate variance components analysis and (iv) multivariate variance components analysis, obtained by generalizing univariate variance components analysis, will be discussed in a later paper.

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