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THE ASYMPTOTIC POWER OF THE KOLMOGOROV TESTS
OF GOODNESS OF FIT

by

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The asymptotic power of the one-sided and two-sided
Kolmogorov tests of goodness of fit of a hypothesis
distribution \( H(x) \) against sequences of alternatives
\( G_n(x) \) for which \( \sup_x \sqrt{n} |H(x) - G_n(x)| \) tends to a limit
is investigated by application of Doob's "heuristic
procedure"; bounds on the power are found, and some
numerical examples provided.

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SUMMARY

Let there be a random sample of size $n$ from some unknown distribution function $F(x)$, let $F_n(x)$ be the empirical distribution function of the sample, and let $H(x)$ be the null-hypothesis distribution. Then the one-sided Kolmogorov test rejects $H(x)$ for large values of

$$D_n^+ = \sup_x \sqrt{n} \left| F_n(x) - H(x) \right|,$$

and the two-sided Kolmogorov test rejects $H(x)$ for large values of

$$D_n = \sup_x \sqrt{n} \left| F_n(x) - H(x) \right|.$$  

In this paper we investigate the asymptotic power of these tests against sequences of alternatives $G_n(x)$ for which

$$\lim_{n \to \infty} \sup_{x} \sqrt{n} \left| H(x) - G_n(x) \right|$$

exists. In particular, we extend Donsker's justification of Doob's "heuristic procedure" as applied to this problem; we find upper and lower bounds on the asymptotic power; and we provide some illustrative numerical examples for the case where $G_n(x)$ consists in a translation of $H(x)$.

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CHAPTER I

THE KOLMOGOROV TESTS

Let \( \mathcal{F} \) be the class of all univariate distribution functions \( F(x) \) on the real line, and \( \mathcal{F}^* \) the subclass of all continuous \( F(x) \). Then, given any \( F \) and \( G \) in \( \mathcal{F} \), define

\[
D(F,G) = \sup_{-\infty < x < \infty} |F(x) - G(x)|
\]

The functional \( D(F,G) \) obeys the relations

\[
\begin{align*}
\text{a)} & \quad D(F,F) = 0 \\
\text{b)} & \quad D(F,G) = D(G,F) \\
\text{c)} & \quad D(F,G) \leq D(F,H) + D(H,G)
\end{align*}
\]

for all \( F, G, H \in \mathcal{F} \) and is therefore a true distance between distribution functions. Furthermore, \( 0 \leq D(F,G) \leq 1 \) and

\[
D(F,G) = 0 \quad \text{if and only if} \quad F(x) \equiv G(x).
\]

We may define also a "directed" or "one-sided" distance

\[
D^+(F,G) = \sup_{-\infty < x < \infty} \sqrt{F(x) - G(x)}
\]

then

\[
D(F,G) = \max \left\{ D^+(F,G), D^+(G,F) \right\} \geq D^+(F,G).
\]

And let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from some distribution \( F \in \mathcal{F} \). If

\[
c(x) = \begin{cases} 
1 & \quad x \geq 0 \\
0 & \quad x < 0
\end{cases}
\]
then the random distribution function

\[ F_n(x) = \frac{1}{n} \sum_{i=1}^{n} c(x-x_i) \]

is called the **empirical distribution function** corresponding to the sample; it is the proportion of the sample which is less than or equal to \( x \). As the sample size increases, the empirical distribution function approaches the true distribution function; in particular, the Glivenko-Cantelli lemma \( \text{[L2]} \) states that with probability 1

\[ \lim_{{n \to \infty}} D(F_n, F) = 0. \]

Now suppose that we desire to test a hypothesis

\[ H_0: \quad F(x) = H(x), \]

where \( H \) is completely specified. Then it seems reasonable to adopt the following procedure:

\[ \text{Reject } H_0 \text{ if and only if } D_n > Q_n, \]

where

\[ D_n = \sqrt{n} D(F_n, H) = \sup_{-\infty < x < \infty} \sqrt{n} \left| F_n(x) - H(x) \right| \]

and \( Q_n = Q_n(\alpha) \) is determined so that the Type I error is

\[ \alpha = 1 - J_n(Q_n) = \text{pr} \left\{ D_n > Q_n \right\} \quad (F = H). \]

This test was first proposed by Kolmogorov in \( \text{[L4]} \). Alternatively, we may

\[ \text{Reject } H_0 \text{ if and only if } D_n^+ > R_n, \]

where

\[ D_n^+ = \sup_{-\infty < x < \infty} \sqrt{n} \left| F_n(x) - H(x) \right| \]

\[ R_n = R_n(\alpha) \]

\[ \text{pr} \left\{ D_n^+ > R_n \right\} \quad (F = H) \]

---

1. The numbers in square brackets refer to the bibliography.
where
\[
(1.13) \quad D_n^+ = \sqrt{n} D^+(F_n, H) = \sup_{-\infty < x < \infty} \sqrt{n} \left( \frac{F_n(x)}{H(x)} - 1 \right)
\]
and \( R_n = R_n(\alpha) \) is also determined so that the Type I error is
\[
(1.14) \quad \alpha = 1 - J_n^+(R_n) = \Pr \left\{ D_n^+ > R_n \right\} \quad (F = H).
\]

Since the test (1.12) is based on the one-sided distance we call it the one-sided Kolmogorov test, thus distinguishing it from the two-sided Kolmogorov test (1.9). For a fuller expository treatment of these tests than can be given in this introductory chapter we refer to the paper by Darling \( \text{[7]} \) which also includes an extensive bibliography.

The two Kolmogorov tests are distribution-free in \( \mathcal{F}^* \); that is, the distribution functions \( J_n \) and \( J_n^+ \) do not depend on \( H \) so long as \( H \in \mathcal{F}^* \).

Even if \( H \notin \mathcal{F}^* \) the tests are conservative, for
\[
J_n(y) \geq \Pr \left\{ D_n \leq y \right\} \quad (F = H \in \mathcal{F})
\]
and
\[
J_n^+(y) \geq \Pr \left\{ D_n^+ \leq y \right\} \quad (F = H \in \mathcal{F}).
\]

In his original paper Kolmogorov gave recurrence relations suitable for calculating \( J_n(y) \) and proved that
\[
(1.15) \quad J(y) = \lim_{n \to \infty} J_n(y) = \begin{cases} 
1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2k^2y^2} & y > 0 \\
0 & y \leq 0
\end{cases}
\]

In \( \text{[10]} \) Smirnov showed that
\[ J_n^+(y) = \begin{cases} 
\frac{1 - \frac{y}{\sqrt{n}}}{\sqrt{n}} \sum_{k=0}^{n^*} \binom{n}{k} \left( 1 - \frac{y}{\sqrt{n}} - \frac{k}{n} \right)^{n-k} \left( \frac{y}{\sqrt{n}} + \frac{k}{n} \right)^{k-1}, \\
\text{where } n^* \text{ is the greatest integer not greater than } n(1 - \frac{y}{\sqrt{n}}), \text{ if } 0 < y < \sqrt{n} \\
0 \text{ if } y \leq 0 \\
1 \text{ if } y \geq \sqrt{n} 
\end{cases} \]

but earlier he had found the limiting distribution

\[ J^+(y) = \lim_{n \to \infty} J_n^+(y) = \begin{cases} 
1 - e^{-2y^2} \quad y > 0 \\
0 \quad y \leq 0 
\end{cases} \]

Tables of these distributions are available; for \( J_n \) see [I7], for \( J \) see [I9], and for \( J_n^+ \) see [5]. Thus, given any \( \alpha \), \( 0 < \alpha < 1 \), we can determine \( Q_n \), \( R_n \), and also \( Q \) and \( R \) by the equations

\[ J_n(Q_n) = J_n^+(R_n) = J(Q) = J^+(R) = 1 - \alpha, \]

where \( Q = \lim_{n \to \infty} Q_n \) and \( R = \lim_{n \to \infty} R_n \) also.

Let an alternative hypothesis to \( H_0 \) be

\[ H_1: F(x) \equiv G(x), \]

where \( G \in \mathcal{F} \) and \( D(G,H) > 0 \). Then the power of the two-sided Kolmogorov test of \( H \) against \( G \) (in more usual terminology, "of \( H_0 \) against \( H_1 \)) is

\[ P_n(H,G,\alpha) = \Pr \left\{ D_n > Q_n \right\} (F = G). \]

Similarly, the power of the one-sided test of \( H \) against \( G \) is
\[ P^+_n(H, G, \alpha) = \text{pr} \left\{ D_n > R_n \right\} \quad (F = G) . \]

Furthermore, if \( \{ G_n \} \) is any sequence of alternative distributions, we define the asymptotic power of the two-sided and one-sided tests of \( H \) against \( \{ G_n \} \) to be

\[ P(H, \{ G_n \}, \alpha) = \lim_{n \to \infty} P_n(H, G_n, \alpha) \]

and

\[ P^+_n(H, \{ G_n \}, \alpha) = \lim_{n \to \infty} P^+_n(H, G_n, \alpha) \]

respectively, if these limits exist; otherwise the asymptotic power is not defined. Suppose that \( G_n = G, \ n = 1, 2, \ldots \). Then it is not difficult to see, from (1.7) and (1.2c), that \( D(F_n, H) \to D(G, H) > 0 \) with probability 1. But \( Q_n \to Q \) and hence \( \text{pr} \left\{ D_n > Q_n \right\} \to 1 \), or \( P(H, \{ G_n \}, \alpha) = 1 \). In other words, the two-sided Kolmogorov test is consistent against every fixed alternative distribution. Similarly we can see that the one-sided test is consistent against every alternative such that \( D^+(G, H) > 0 \).

In this paper it will be our purpose to investigate the asymptotic power against sequences of alternatives which converge to the hypothesis distribution; specifically, we shall restrict our attention to the case where \( (H, \{ G_n \}) \) belongs to the class

\[ J = \left\{ (H, \{ G_n \}) : H \in \mathcal{F}^* ; G_n \in \mathcal{F}^* , \ n = 1, 2, \ldots ; \quad \text{and} \lim_{n \to \infty} \sqrt{n} \ D(H, G_n) = \Delta \exists \right\} . \]

However, in order to do this it will be convenient first to re-express
these definitions in terms of a new notation which is explained in the next paragraph.

For any $F \in \mathcal{F}^*$ define

$$F^{-1}(t) = \begin{cases} 
\sup_x \{ x : F(x) = t \} & 0 \leq t < 1 \\
\lim_{l > t \to 1} F^{-1}(t) & t = 1
\end{cases}$$

(1.25)

(note that)

$$F\sqrt{F^{-1}(t)} = t, \quad 0 \leq t \leq 1.$$ (1.26)

And let the stochastic process $\{Z_n(t), \ t \in T\}$ be defined by

$$Z_n(t) = \sqrt{n} (F_n F^{-1}(t) - t), \ t \in T,$$ (1.27)

where $T$ is the closed interval $[0,1]$. Then all probability statements about $Z_n(t)$ are entirely independent of $F$, so long as $F \in \mathcal{F}^*$. In particular, it is easy to verify that

$$\begin{cases} 
a) \ Z_n(0) = Z_n(1) = 0 \text{ with probability 1} \\
b) \ \mathbb{E}(Z_n(t)) = 0 \\
c) \ \text{cov}(Z_n(t), Z_n(t')) = \min(t, t') - tt' \ .
\end{cases}$$ (1.28)

Next, if $a(t)$ and $b(t)$ are any two functions defined for $t \in T$, and $\Omega$ is any subset of $T$, we define the event

$$\mathbb{P}(a(t), b(t), \Omega) = \{a(t) \leq Z_n(t) \leq b(t), \ t \in \Omega\}.$$ (1.29)

Then it is easy to see that

$$\mathbb{P}(a(t), b(t), \Omega) = \{A(x) \leq F_n(x) \leq B(x), \ F(x) \in \Omega\},$$
where \( A(x) = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} a(F(x)) F(x) + F(x) \) and \( B(x) = \frac{1}{\sqrt{n}} b(F(x)) F(x) + F(x) \).

Now if

\[
S_i = \{ x: F(x) \in \Omega \text{ but } A(x) \leq \frac{1}{n} \leq B(x) \text{ is false} \}
\]

and

\[
I_i = \{ x: y \in S_i \implies x > y \text{ and } y \in S_{i-1} \implies x \leq y \}
\]

then

\[
E_{\sqrt{n}, b(t), \Omega} = \{ x_i \in I_i, 1 \leq i \leq n \}
\]

where \( x_i \) is the \( i \)-th smallest observation; but since \( I_i \) is an interval, the event \( E_{\sqrt{n}, b(t), \Omega} \) has a well-defined probability which we may write as

\[
(1.30) \quad \beta_{\sqrt{n}, b(t), \Omega} = \text{pr} \left\{ a(t) \leq Z_n(t) \leq b(t), \quad t \in \Omega \right\}.
\]

Finally, for any \( F, G \in \mathcal{F}^+ \) define

\[
(1.31) \quad S_n(t; F, G) = \sqrt{n} (F^{-1}(G^{-1}(t)) - t), \quad t \in T ;
\]

from (1.26) it follows that

\[
(1.32) \quad S_n(t; F, F) = 0, \quad t \in T .
\]

Note also that

\[
(1.33) \quad D(F, G) = \frac{1}{\sqrt{n}} \sup_{t \in T} | S_n(t; F, G) |
\]

and

\[
(1.34) \quad D^+(F, G) = \frac{1}{\sqrt{n}} \sup_{t \in T} \left| \sqrt{S_n(t; F, G)} \right| .
\]

Then, starting from the definition (1.10), we have
\[
D_n = \sup_{-\infty < x < \infty} \sqrt{n} \left| F_n(x) - H(x) \right|
= \sup_{t \in T} \sqrt{n} \left| F_n^{-1}(t) - H^{-1}(t) \right|
= \sup_{t \in T} \left| \sqrt{n} (F_n^{-1}(t) - t) - \sqrt{n} \left( H^{-1}(t) - t \right) \right|
\]

or

\[(1.35) \quad D_n = \sup_{t \in T} \left| Z_n(t) - S_n(t; H, F) \right| ; \]

and

\[
\Pr\left\{ D_n > Q_n \right\} = \Pr\left\{ \sup_{t \in T} \left| Z_n(t) - S_n(t; H, F) \right| > Q_n \right\}
= 1 - \Pr\left\{ \left| Z_n(t) - S_n(t; H, F) \right| \leq Q_n, t \in T \right\}
= 1 - \Pr\left\{ S_n(t; H, F) - Q_n \leq Z_n(t) \leq S_n(t; H, F) + Q_n, t \in T \right\}
= 1 - \beta_n \sqrt{S_n(t; H, F) - Q_n, S_n(t; H, F) + Q_n} ;
\]

Hence, using (1.32), we can rewrite (1.11) as

\[(1.36) \quad \alpha = 1 - J_n(Q_n) = 1 - \beta_n \sqrt{Q_n, Q_n} ; \]

the power of the two-sided test becomes

\[(1.37) \quad P_n(H, G, \alpha) = 1 - \beta_n \sqrt{S_n(t; H, G) - Q_n, S_n(t; H, G) + Q_n} ;
\]

and the asymptotic power, if it exists, is

\[(1.38) \quad P(H, \{G_n\}, \alpha) = 1 - \lim_{n \to \infty} \beta_n \sqrt{S_n(t; H, G_n) - Q_n, S_n(t; H, G_n) + Q_n} .
\]

Note that, since we consider only \((H, \{G_n\}) \in \mathcal{J}\), we have from (1.24) and (1.33) that

\[(1.39) \quad \lim_{n \to \infty} \sup_{t \in T} \left| S_n(t; H, G_n) \right| = \Delta \text{ exists} .
\]
Proceeding in the same manner we obtain also that
\begin{equation}
D_n^+ = \sup_{t \in T} \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i(t) - S_n(t;H,F) \right) \quad ;
\end{equation}

we can rewrite (1.14) as
\begin{equation}
\alpha = 1 - J_n^+(R_n) = 1 - \beta_n \sqrt{-\infty}, R_n, T \quad ;
\end{equation}

the power of the one-sided test becomes
\begin{equation}
P_n^+(H,G,\alpha) = 1 - \beta_n \sqrt{-\infty}, S_n(t;H,G) + R_n, T \quad ;
\end{equation}

and the asymptotic power, if it exists, is
\begin{equation}
P^+(H, \{G_n\}, \alpha) = 1 - \lim_{n \to \infty} \beta_n \sqrt{-\infty}, S_n(t;H,G_n) + R_n, T \quad .
\end{equation}

A test of $H_0$: $F(x) = H(x)$ is called partially ordered if, for any two alternative distributions $A$ and $B$ such that $D^+(A,B) = 0$, its power against $A$ is less than or equal to its power against $B$; the terminology is due to Chapman, who showed in $\mathcal{J}$ that the one-sided Kolmogorov test (among others) has this property. In our new notation this is particularly easy to prove, as follows:

**Theorem 1.1.** If $H,A,B \in \mathcal{J}$ and $D^+(A,B) = 0$ then $P_n^+(H,A,\alpha) \leq P_n^+(H,B,\alpha)$.

**Proof.** If $D^+(A,B) = 0$ then $A(x) \leq B(x)$ for all $x$ and hence $A^{-1}(t) \geq B^{-1}(t), t \in T$. But then
\[ S_n(t;H,A) = \sqrt{n} \left( \frac{H(A^{-1}(t))}{n} - t \right) \geq \sqrt{n} \left( \frac{H(B^{-1}(t))}{n} - t \right) = S_n(t;H,B) \]

and the theorem follows immediately from (1.42). (The theorem remains true for $H, A, B \in \mathcal{J}$).
CHAPTER II

THE PROCESS Z(t)

Consider the behavior of the stochastic process \( Z_n(t) \) as \( n \) tends to infinity. If, for any \( k \), we have \( 0 < t_1 < t_2 < \ldots < t_k < 1 \), and if \( (z_1, z_2, \ldots, z_k) \) is an arbitrary vector, then by the multidimensional central limit theorem

\[
\lim_{n \to \infty} \Pr \left\{ Z_n(t_i) \leq z_i, \ 1 \leq i \leq k \right\} = \Pr \left\{ Z(t_i) \leq z_i, \ 1 \leq i \leq k \right\}
\]

where \( Z(t) \) is a Gaussian stochastic process with the same mean and covariance functions (1.28) as \( Z_n(t) \); that is

\[
\begin{align*}
& a) \quad Z(0) = Z(1) = 0 \text{ with probability } 1 \\
& b) \quad \mathbb{E}[Z(t)] = 0 \\
& c) \quad \text{cov}[Z(t), Z(t')] = \min(t, t') \cdot t' 
\end{align*}
\]

Thus if \( Z_i = Z(t_i) \), \( 1 \leq i \leq k \), then \( (Z_1, Z_2, \ldots, Z_k) \) have a joint multivariate normal distribution with all means zero and with covariance matrix \( \Sigma_k = (\sigma_{ij}) \) where \( \sigma_{ij} = t_i(1-t_j) \) for \( 1 \leq i \leq j \leq k \). From this it is easily verified that the joint probability density function of \( (Z_1, Z_2, \ldots, Z_k) \) is

\[
f_k(z_1, z_2, \ldots, z_k) = \frac{1}{(2\pi)^{k/2} \prod_{i=1}^{k} (t_i-t_{i-1})} \prod_{i=1}^{k} \frac{1}{2} \left( \frac{(z_i-z_{i-1})^2}{t_{i-1}-t_{i-2}} \right)
\]

where \( z_0 = z_{k+1} = 0 \). In the special case \( k=1 \), (2.1) says that for \( 0 < \tau < 1 \) and all real \( z \)
(2.4) \[ \lim_{n \to \infty} \Pr \left\{ Z_n(\tau) \leq z \right\} = \Pr \left\{ Z(\tau) \leq z \right\} = \phi\left(\frac{z}{\sqrt{\tau(1-\tau)}}\right), \]

where

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} \, du; \]

or, in other words, that \( Z(\tau) \) is normally distributed with mean 0 and variance \( \tau(1-\tau) \), as is \( Z_n(\tau) \) asymptotically, which statements we write

(2.6) \[ Z(\tau) : \mathcal{N}(0, \tau(1-\tau)) \]

and

(2.7) \[ Z_n(\tau) \sim \mathcal{N}(0, \tau(1-\tau)) \]

The process \( Z(t) \) specified by (2.2) is related to the classical Wiener process \( \left\{ W(\tau), 0 \leq \tau < \infty \right\} \), a Gaussian stochastic process such that

\[
\begin{cases}
  a) & W(0) = 0 \text{ with probability } 1 \\
  b) & \mathcal{L}[W(\tau)] = 0 \\
  c) & \text{cov} \left[ W(\tau), W(\tau') \right] = \min (\tau, \tau')
\end{cases}
\]

by the following transformation of Doob \( \mathcal{L} \) :

(2.8) \[ Z(t) = \frac{1}{\tau+1} W(\tau) \quad \text{where} \quad t = \frac{1}{\tau+1}. \]

It may also be verified that a conditional Wiener process, so constrained that \( W(1) = 0 \), is identical with \( Z(t) \). Now we may and shall assume that \( Z(t) \) and \( W(\tau) \) are separable; then the sample functions of \( Z(t) \) are continuous with probability 1, since this is a well-known property of a separable process \( W(\tau) \). Finally, \( W(\tau) \) is a Markov process, and thus so is \( Z(t) \).
In analogy with (1.29), if \( a(t) \) and \( b(t) \) are any two functions defined for \( t \in T \), and \( \Omega \) is any subset of \( T \), we may define the event
\[
E[a(t),b(t),\Omega] = \{ a(t) \leq Z(t) \leq b(t), \ t \in \Omega \}.
\]

Now only trivial changes are necessary in order to derive from Donsker's argument in \( \mathbb{F} \) the following

**Theorem 2.1.** Let \( \mathcal{A} \) be the space of all real single-valued functions \( f(t) \) which are continuous for \( t \in T \) except for at most a finite number of finite jumps, and let \( F \) be a functional defined on \( \mathcal{A} \) which is continuous in the uniform topology at almost all points of \( \Xi \), the space of continuous sample functions of the \( Z(t) \) process. Then the set 
\[
\{ F[Z(t),\mathcal{F}] \leq Q \}
\]
is measurable for every \( Q \); that is, \( F[Z(t),\mathcal{F}] \) is a (generalized) random variable.

By a "generalized" random variable we mean one which may take on infinite values with positive probability; that is
\[
\lim_{Q \to -\infty} \Pr \{ F[Z(t),\mathcal{F}] \leq Q \geq 0 \}
\]
and
\[
\lim_{Q \to +\infty} \Pr \{ F[Z(t),\mathcal{F}] \leq Q \} \leq 1.
\]

But this theorem allows us to define, in analogy with (1.30),
\[
E[a(t),b(t),\Omega] = \Pr \{ a(t) \leq Z(t) \leq b(t), \ t \in \Omega \}
\]
in all cases; in fact we have

**Corollary 2.2.** If \( a(t) \) and \( b(t) \) are any two functions (possibly infinite) defined for \( t \in T \), and \( \Omega \) is any subset of \( T \), then
(2.12) \( \beta^*; a(t), b(t), \mathcal{J} = \beta[a(t)] - q, b(t) + q, \mathcal{J} \)

is the distribution function of the (generalized) random variable

(2.13) \( X = \max \left\{ \sup_{t \in \Omega} \left[ \sqrt{a(t)} - Z(t), \sup_{t \in \Omega} \left[ \sqrt{Z(t)} - b(t) \right] \right] \right\} \).

**Proof.** For any function \( f(t) \) defined for \( t \in \mathcal{T} \), let

\[
F_{1}(t) = \sup_{t \in \Omega} \left[ \sqrt{a(t)} - f(t) \right], \\
F_{2}(t) = \sup_{t \in \Omega} \left[ \sqrt{f(t)} - b(t) \right], \\
F(t) = \max \left\{ F_{1}, F_{2} \right\}.
\]

Then certainly the functional \( F \) is defined on \( \Lambda \). Now, to say that \( F \) is continuous in the uniform topology at almost points of \( \Xi \) is to say that whenever we have a sequence of functions \( \{ f_n(t) \} \) in \( \Xi \) which converges uniformly to a function \( F(t) \in \Xi \), that is

\[
\lim_{n \to \infty} \sup_{t \in \mathcal{T}} \left| f_n(t) - f(t) \right| = 0,
\]

then with probability 1

\[
\lim_{n \to \infty} F(t) = F(t).
\]

So let such a sequence be chosen. Then

\[
s(t) - f(t) = a(t) - f_n(t) + f_n(t) - f(t)
\]

\[
\sup_{t \in \mathcal{T}} \left[ a(t) - f(t) \right] \leq \sup_{t \in \mathcal{T}} \left[ a(t) - f_n(t) \right] + \sup_{t \in \mathcal{T}} \left[ f_n(t) - f(t) \right]
\]

and letting \( n \) tend to infinity yields

\[
\sup_{t \in \mathcal{T}} \left[ a(t) - f(t) \right] \leq \lim_{n \to \infty} \sup_{t \in \mathcal{T}} \left[ a(t) - f_n(t) \right] + \lim_{n \to \infty} \sup_{t \in \mathcal{T}} \left[ f_n(t) - f(t) \right]
\]
or

\[ F_1(F_n) \leq \lim_{n \to \infty} F_1(F_n) + 0. \]

On the other hand, since

\[ a(t) - f_n(t) = a(t) - f(t) + f(t) - f_n(t), \]

\[ \sup_{t \in \Omega} \left| a(t) - f_n(t) \right| \leq \sup_{t \in \Omega} \left| a(t) - f(t) \right| + \sup_{t \in \Omega} \left| f(t) - f_n(t) \right| \]

so that

\[ \lim_{n \to \infty} \sup_{t \in \Omega} \left| a(t) - f_n(t) \right| \leq \sup_{t \in \Omega} \left| a(t) - f(t) \right| + \lim_{n \to \infty} \sup_{t \in \Omega} \left| f(t) - f_n(t) \right| \]

or

\[ \lim_{n \to \infty} F_1(F_n) \leq F_1(F) + 0. \]

Thus

\[ \lim_{n \to \infty} F_1(F_n) = F_1(F) ; \]

and since it can be shown in a similar manner that

\[ \lim_{n \to \infty} F_2(F_n) = F_2(F) ; \]

we have finally

\[ \lim_{n \to \infty} F_1(F_n) = \lim_{n \to \infty} \max \left\{ F_1(F_n), F_2(F_n) \right\} = \max \left\{ F_1(F), F_2(F) \right\} = F(F) ; \]

that is, \( F \) is continuous in the uniform topology. But then the corollary
follows immediately from the theorem because the random variable 
\( F(Z(t)) \) is exactly the \( X \) defined by (2.13) and the \( \beta^*(Q) \) defined by
(2.12) is just the probability of the event \( \{ X \leq Q \} \).

**Theorem 2.3.** If \( a(t) \) and \( b(t) \) are any two functions defined for \( t \in T \) and \( 0 \leq t' < t'' \leq 1 \), define

\[
\frac{\sqrt{a(t)}b(t); t', t'', z', z''}{(2.14)} = \text{pr} \left\{ a(t) \leq Z(t) \leq b(t), \ t' < t'' \bigg| Z(t') = z', \ Z(t'') = z'' \right\}
\]

where it is understood that if \( t' = 0 \) then \( z' = 0 \) and if \( t'' = 1 \) then \( z'' = 0 \). Then

\[
\frac{\sqrt{a(t)}b(t); t', t'', z', z''}{(2.15)} = \text{pr} \left\{ M_{\tau}; a \leq W(\tau) \leq M_{\tau}; b, \ 0 < \tau < \infty \right\}
\]

where

\[
M_{\tau}; f = \left( \frac{t}{t'} + \frac{t''}{t''-t'} \right) f(t''(t''-t') + t't''^2) \left( \frac{t}{t''} + \frac{t''}{t''-t'} \right).
\]

**Proof.** It is well-known, and easily verified, that if

\[
w(t) = \frac{1}{\tau} W(\tau) \quad \text{where} \quad t = \frac{1}{\tau},
\]

then \( w(t) \) is a Wiener process; and if it is given that \( W(x) = y \) then the process

\[
w(t) = W(\tau) - y \quad \text{where} \quad t = \tau - x,
\]

is a Wiener process also. So consider the event

\[
\{ a(t) \leq Z(t) \leq b(t), \ t' < t < t'' \big| Z(t') = z', \ Z(t'') = z'' \}
\]
Applying (2.9), we see that this is equivalent to

\[
\begin{aligned}
\left\{ a\left(\frac{1}{\tau+1}\right) & \leq \frac{1}{\tau+1} W(\tau) \leq b\left(\frac{1}{\tau+1}\right) \, , \, \frac{1}{\tau+1} - 1 < \tau < \frac{1}{\tau+1} - 1 \\
& \quad \left| W\left(\frac{1}{\tau+1} - 1\right) = z''_{\tau} \right. \right.
\right. \nonumber \\
& \left. \left. \quad \left| W\left(\frac{1}{\tau+1} - 1\right) = z'_{\tau} \right) \right. \right. 
\end{aligned}
\]

Now apply (2.13) with \( x = \frac{1}{\tau+1} - 1 \), \( y = \frac{z''_{\tau}}{t^\tau} \). This yields

\[
\left\{ a\left(\frac{1}{t+1}\right) \leq \frac{1}{t+1} \int_0^{\tau_t(t)} \frac{z''_{\tau}}{t^\tau} \leq b\left(\frac{1}{t+1}\right) \, , \, \frac{1}{t+1} - 1 < t + \frac{1}{t+1} - 1 < \frac{1}{t+1} - 1 \\
& \quad \left| \omega(0) = 0 \right. \right.
\right. \nonumber \\
& \left. \left. \quad \left| \omega\left(\frac{1}{t+1} - 1\right) + \frac{z''_{\tau}}{t^\tau} = z'_{\tau} \right) \right. \right. 
\end{aligned}
\]

or

\[
\left\{ a\left(\frac{t''}{1+\tau t^\tau} \right) \leq \frac{t''W(\tau)+z''}{1+\tau t^\tau} \leq b\left(\frac{t''}{1+\tau t^\tau} \right), \, 0 < \tau < \frac{1}{t^\tau} - \frac{1}{t^\tau} \left| W\left(\frac{1}{t^\tau} - 1\right) = z'_{t^\tau} - z''_{t^\tau} \right) \right. \right. 
\right. \nonumber \\
& \left. \left. \quad \left| W\left(\frac{1}{t^\tau} - 1\right) = z'_{t^\tau} - z''_{t^\tau} \right) \right. \right. 
\end{aligned}
\]

Applying (2.17) to this gives

\[
\left\{ a\left(\frac{t''}{1+t} \right) \leq \frac{t''W(t)+z''}{1+t} \leq b\left(\frac{t''}{1+t} \right), \, 0 < \frac{1}{t} < \frac{1}{t^\tau} - \frac{1}{t^\tau} \\
& \quad \left| \frac{1}{t} - \frac{1}{t^\tau} \right. \right.
\right. \nonumber \\
& \left. \left. \quad \left| W\left(\frac{1}{t^\tau} - 1\right) = z'_{t^\tau} - z''_{t^\tau} \right) \right. \right. 
\end{aligned}
\]

or

\[
\left\{ a\left(\frac{t''}{1+t^\tau} \right) \leq \frac{t''W(t)+z''}{t^\tau + t} \leq b\left(\frac{t''}{t^\tau + t} \right), \, \frac{t''t}{t^\tau - t} < \tau < \infty \left| W\left(\frac{t''}{t^\tau - t} \right) = \frac{t''z'_{t^\tau} - t''z''}{t^\tau - t} \right) \right. \right. 
\right. \nonumber \\
& \left. \left. \quad \left| W\left(\frac{t''}{t^\tau - t} \right) = \frac{t''z'_{t^\tau} - t''z''}{t^\tau - t} \right) \right. \right. 
\end{aligned}
\]

Next, applying (2.10) again, this time with \( x = \frac{t''t}{t^\tau - t} \), and \( y = \frac{t''z'}{t^\tau - t} \), we have
\[
\begin{align*}
\left\{ \frac{t''(t''-t')}{t''-t} + t'' - t \right\} \\
\leq b(\frac{t''(t''-t')}{t''-t} + t'') \quad 0 < \tau < \infty \quad a(0) = 0
\end{align*}
\]

or
\[
\begin{align*}
\left\{ \frac{a(\tau''(t''-t') + t''^2)}{\tau(t''-t') + t''^2} \right\} \\
= \frac{\tau''(t''-t')}{\tau(t''-t') + t''^2} \leq \frac{b(\tau''(t''-t') + t''^2)}{\tau(t''-t') + t''^2}
\end{align*}
\]

which is equivalent to
\[
\left\{ \sqrt{\tau}; a_{\mathcal{F}} \leq W(\tau) \leq \sqrt{\tau}; b_{\mathcal{F}}, \quad 0 < \tau < \infty \right\}
\]

and this yields the theorem.

Theorem 2.3 and the procedure used in proving it were suggested by the proof of Malmquist's Theorem 1 in [15] which is essentially a special case of it. We also remark that if \(a(0) \leq 0 \leq b(0)\) and \(a(1) \leq 0 \leq b(1)\) then

\[
(2.19) \quad \xi = \alpha(t), b(t); 0, 1, 0, 0, f_{\mathcal{F}} = \beta = \alpha(t), b(t), \mathcal{T}_{\mathcal{F}}.
\]

Consider now \(M_{\tau}; f_{\mathcal{F}}\) when \(f(t)\) is linear: say

\[
(2.20) \quad f(t) = \lambda(1-t) + \mu t
\]

then by substitution into (2.16) we obtain

\[
(2.21) \quad M_{\tau}; f_{\mathcal{F}} = \frac{T}{t''}(\lambda(1-t'') + \mu t'') - z''_{\mathcal{F}} + \frac{T''}{t''-t'}(\lambda(1-t') + \mu t'') - z'_{\mathcal{F}}.
\]
which is also linear. But the following formulas were derived by Doe in [10]: if we define

(2.22) \[ G(c,d,\gamma,\delta) = \text{pr} \left\{ -\gamma + \delta \leq W(\tau) \leq c\tau + d, \; 0 < \tau < \infty \right\} \]

then

\[
G(c,d,\gamma,\delta) = \begin{cases} 
1 + \sum_{k=1}^{\infty} \sum_{i=1}^{4} (-1)^i e^{-2G_{kl}(c,d,\gamma,\delta)} 
\end{cases} 
\]

(2.23) \[ G(c,d,\gamma,\delta) = \begin{cases} 
G_{k1}(c,d,\gamma,\delta) & \text{if } c,d,\gamma,\delta > 0, \text{ and} \\
0 & \text{otherwise}, \text{ where} \\
G_{k2}(c,d,\gamma,\delta) & k^2(\gamma+c\delta) + k(\gamma d - c\delta) \\
G_{k3}(c,d,\gamma,\delta) & k^2(c\gamma - d\delta) - k(\gamma d - c\delta) \\
G_{k4}(c,d,\gamma,\delta) & k^2(\gamma+c\delta) + k(\gamma d - c\delta) \\
\end{cases} \]

and if

(2.24) \[ G^+(c,d) = \text{pr} \left\{ W(\tau) \leq c\tau + d, \; 0 < \tau < \infty \right\} \]

then

(2.25) \[ G^+(c,d) = \begin{cases} 
1 - e^{-2cd} & \text{if } c,d > 0 \\
0 & \text{otherwise} \\
\end{cases} \]

It now requires only simple algebra to establish, using Theorem 2.3, that

(2.26) \[ t\sqrt{\delta(1-t)+\phi t, \lambda(1-t)+\mu t^{1/2}, t^{1/2}, t^{1/2}, z^{1/2}, z^{1/2}} \]

\[ = G(\lambda(1-t)-z, t^{1/2}, \lambda(1-t)-z, t^{1/2}) - \delta(1-t)+\phi t^{1/2}, z^{1/2}, t^{1/2}) \]
and that
\[
\begin{align*}
(2.27) \quad \Phi \left( \lambda \left( l-1 \right) + \mu t ; t', t'', z', z'' \right) &= \frac{g^+ \left( \lambda (l-t'') + \mu t'' - z'' , \frac{t''/\lambda (l-t') + \mu t' - z'}{t''-t'} \right)}{1-e^{-\frac{2/\lambda (l-t') + \mu t' - z'}{t''-t'} / \lambda (l-t'') + \mu t'' - z''}} \\
&= \begin{cases} 
1-e^{-\frac{2/\lambda (l-t') + \mu t' - z'}{t''-t'} / \lambda (l-t'') + \mu t'' - z''} & \text{if } z' < \lambda (l-t') + \mu t' , z'' < \lambda (l-t'') + \mu t'' , \text{ and} \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

**Lemma 2.4.** If $0 < \delta < \frac{1}{2}$ and $\lambda > 0$ then
\[
\Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda \right\} < \frac{2\delta}{\lambda^2}.
\]

**Proof.** Certainly
\[
\begin{align*}
\Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda \right\} &= \Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda , Z(\delta) > \lambda \right\} \\
&\quad + \Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda , Z(\delta) \leq \lambda \right\}.
\end{align*}
\]
Now using (2.4) we have
\[
\begin{align*}
\Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda , Z(\delta) > \lambda \right\} &= \Pr \left\{ Z(\delta) > \lambda \right\} = \Phi \left( \frac{-\lambda}{\sqrt{\delta(1-\delta)}} \right).
\end{align*}
\]
But
\[
\begin{align*}
\Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda , Z(\delta) \leq \lambda \right\} &= \lambda \int_{-\infty}^{\lambda} \Pr \left\{ \sup_{0 \leq t \leq \delta} Z(t) > \lambda \left| Z(\delta) = z \right. \right\} f_1(z) dz
\end{align*}
\]
where $f_1(z)$ is the probability density function of $Z(\delta)$ defined by (2.3) with $k=1$ and
\[
pr \left\{ 0 \leq t \leq \delta \mid Z(t) > \lambda, \; Z(0) = z \right\}
= 1 - \Phi(-\infty, \lambda, 0, \delta, 0, z)
= e^{-\frac{2}{\delta} \lambda (\lambda - z)}
\]
by (2.27); hence
\[
pr \left\{ 0 \leq t \leq \delta \mid Z(t) > \lambda, \; Z(0) \leq \lambda \right\} = \int_{-\infty}^{\lambda} e^{-\frac{2}{\delta} \lambda (\lambda - z)} \frac{1}{\sqrt{2\pi \delta (1-\delta)}} e^{-\frac{z^2}{2\delta (1-\delta)}} dz
= e^{-2\lambda^2} \phi\left( \frac{-\lambda (1-2\delta)}{\sqrt{\delta (1-\delta)}} \right)
\]
Adding these results, we have
\[
(2.28) \quad pr \left\{ 0 \leq t \leq \delta \mid Z(t) > \lambda \right\} = \phi\left( \frac{-\lambda}{\sqrt{\delta (1-\delta)}} \right) + e^{-2\lambda^2} \phi\left( \frac{-\lambda (1-2\delta)}{\sqrt{\delta (1-\delta)}} \right)
\]
where (2.28) holds if \(0 < \delta < 1\). Now it is easily established that for \(x > 0\)
\[
\phi(-x) < e^{-\frac{x^2}{2}} < \frac{1}{x}
\]
and the statement of the lemma then follows easily from (2.28) using these inequalities.

**Lemma 2.5.** Let \(f(t)\) be an arbitrary function defined for \(t \in \mathbb{T}\), and let \(\Omega\) be any countable subset of the closed interval \([0, 1-\delta]\) where \(0 < \delta < \frac{1}{2}\). Then for every real \(\psi\) and \(\phi\)
\[
\left| \beta_{-\infty}^{0}, f(t) + \psi, \Omega \right| - \beta_{-\infty}^{0}, f(t) + \phi, \Omega \right| \leq \frac{\psi - \phi}{\sqrt{\delta (1-\delta)}}
\]
**Proof.** The lemma is trivially true if \(\psi = \phi\); so without loss of
generality we may assume $\psi > \phi$, and then certainly

$$\beta_{-\infty}^{\infty}, f(t) + \psi, \Omega \cap \beta_{-\infty}^{\infty}, f(t) + \phi, \Omega \cap \geq 0.$$ 

Let $\Omega = \{t_1, t_2, \ldots\}$ and, for each $k$, let

$$\Omega_k = \{t_1, \ldots, t_k\} = \{u_1^{(k)}, u_2^{(k)}, \ldots, u_k^{(k)}\},$$

where $0 < \delta < u_1 < u_2 < \ldots < u_k < 1 - \delta < 1$. Let $f_i = f(u_i)$, $1 \leq i \leq k$, and assume the $f_i$'s are all finite. Then, using (2.3), we have that for any $s$

$$\beta_{-\infty}^{\infty}, f(t) + \psi, \Omega_k \cap \beta_{-\infty}^{\infty}, f(t) + \phi, \Omega_k \cap = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} c_k e^{-\frac{1}{2} \sum_{i=1}^{k} u_i - u_i - 1} dz_k \cdots dz_1,$$

where $u_0 = 0$, $u_{k+1} = 1$, $z_0 = z_{k+1} = 0$, and $c_k = (2\pi)^{\frac{k}{2}} \sqrt{(u_2 - u_1) \cdots (1 - u_k)}$. 

Transform the variables of integration so that $x_i = f_i + \epsilon - z_i$, $1 \leq i \leq k$; then

$$\beta_{-\infty}^{\infty}, f(t) + \psi, \Omega_k \cap \beta_{-\infty}^{\infty}, f(t) + \phi, \Omega_k \cap = \int_{0}^{\infty} \cdots \int_{0}^{\infty} c_k e^{-\frac{1}{2} \left(\frac{(x_1 - f_1 - \epsilon)^2}{u_1} + \frac{(x_k - f_k - \epsilon)^2}{1 - u_k}\right)}$$

$$+ \sum_{i=2}^{k} \left\{ \frac{(x_i - x_{i-1}) - (f_i - f_{i-1})}{u_i - u_{i-1}} \right\} dx_k \cdots dx_1.$$

Differentiating with respect to $\epsilon$, we obtain
\[
\frac{d}{dx} \beta_{-\infty, f(t)+\varepsilon, \Omega_k}\nabla
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_k \left[ \frac{x_1 - f_1 - \varepsilon}{u_1} + \frac{x_k - f_k - \varepsilon}{1-u_k} \right] e^{-\frac{1}{2} \left\{ \frac{(x_1 - f_1 - \varepsilon)^2}{u_1} + \frac{(x_k - f_k - \varepsilon)^2}{1-u_k} + \frac{k}{u_1 - u_{i-1}} \left\{ \frac{(x_1 - x_{i-1}) - (f_1 - f_{i-1})}{u_1 - u_{i-1}} \right\}^2 \right\} dx_k \ldots dx_1
\]

or, transforming back from the x's to the z's,

\[
\frac{d}{dx} \beta_{-\infty, f(t)+\varepsilon, \Omega_k}\nabla
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_k \left[ \frac{z_1}{u_1} + \frac{z_k}{1-u_k} \right] e^{-\frac{1}{2} \left\{ \frac{1}{u_1 - u_{i-1}} \left\{ \frac{(z_1 - z_{i-1})}{u_1 - u_{i-1}} \right\}^2 \right\} dz_k \ldots dz_1
\]

\[
= \mathbb{C} \left[ \frac{Z(u_1)}{u_1} + \frac{Z(u_k)}{1-u_k} \right] \nabla
\]

But, from (2.6), \(Z(u_1) : N(\Omega, u_1(1-u_1))\nabla\) and \(Z(u_k) : N(\Omega, u_k(1-u_k))\nabla\);
and \(\text{cov}(Z(u_1), Z(u_k)) = u_1(1-u_k)\) by (2.2c); so

\[
\mathbb{C} \left[ \frac{Z(u_1)}{u_1} + \frac{Z(u_k)}{1-u_k} \right] : N(\Omega, \frac{1}{u_1} + \frac{1}{1-u_k})\nabla
\]
and thus for all $\varepsilon$.

\[
\frac{d}{d\varepsilon} \beta_{[-\infty, f(t) + \varepsilon]} \Omega \leq \varepsilon \left| \frac{z(u_1)}{u_1} + \frac{z(u_k)}{1-u_k} \right| = \sqrt{\frac{2}{\pi \varepsilon u_1 + \frac{1}{1-u_k}}} \leq \sqrt{\frac{2}{\pi \delta(1-\delta)}}.
\]

From this it follows that

\[
\beta_{[-\infty, f(t) + \psi]} \Omega - \beta_{[-\infty, f(t) + \phi]} \Omega \leq \sqrt{\frac{2}{\pi \delta(1-\delta)}} |\psi - \phi|.
\]

We have assumed that the $f_i$'s are all finite; but if any $f_1 = -\infty$ then $\beta_{[-\infty, f(t) + \varepsilon]} \Omega = 0$ and the result above is trivial; whereas if any $f_1 = +\infty$ then the corresponding $u_1$ can be ignored and the argument will proceed as before. Now since $\Omega \subset \Omega$,

\[
\beta_{[-\infty, f(t) + \psi]} \Omega - \beta_{[-\infty, f(t) + \phi]} \Omega \leq 0.
\]

And since $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ there must be a $k = k(\psi, \phi)$ so large that

\[
\beta_{[-\infty, f(t) + \phi]} \Omega_k - \beta_{[-\infty, f(t) + \phi]} \Omega \leq (1 - \frac{\sqrt{2}}{\pi}) \frac{1}{\sqrt{\delta(1-\delta)}} |\psi - \phi|.
\]

Then using this $k$ we obtain

\[
\beta_{[-\infty, f(t) + \psi]} \Omega - \beta_{[-\infty, f(t) + \phi]} \Omega \leq \frac{|\psi - \phi|}{\sqrt{\delta(1-\delta)}}
\]

by adding the last three inequalities; and this completes the proof of the lemma.

Given any function $f(t)$ defined for $t \in T$, let $I(f)$ be the set of all points of discontinuity of $f(t)$ in $T$, always including the two
endpoints 0 and 1 of T. Here f(t) need not be finite: but if f(t) = +∞ at t = t₀, then t₀ ∉ Γ(f) if and only if f(t) = +∞ everywhere in some neighborhood of t₀; and similarly if f(t₀) = -∞. Then we shall say that f(t) is piecewise-continuous (in T) if and only if Γ(f) is countable.

Theorem 2.6. Given any two piecewise-continuous functions a(t) and b(t), let Γ be any dense countable subset of T which includes Γ(a) and Γ(b), and define

$$Q^*_{a(t), b(t)}$$

(2.29) $$\lim_{δ \to 0} \max \left\{ \sup_{0 \leq t \leq δ} a(t), \sup_{1-δ \leq t \leq 1} a(t), \sup_{0 \leq t \leq δ} b(t), \sup_{1-δ \leq t \leq 1} b(t) \right\} ;$$

then

(i) $$β^*(Q) = β^*_{\sup\bar{Q}; a(t), b(t), T} = β^*_{\sup\bar{Q}; a(t), b(t), Γ} ;$$

(ii) $$β^*(Q) = 0 \text{ for } Q < Q^*; \text{ and}$$

(iii) $$β^*(Q) \text{ is continuous for } Q \neq Q^* .$$

Proof. Let the stochastic process \{X(t), t ∈ T\}, be defined by

$$X(t) = \max \left\{ a(t) - Z(t), b(t) - Z(t) \right\} ;$$

then from Corollary 2.2 we have that $$β^*_{\sup\bar{Q}; a(t), b(t), T}$$ is the distribution function of the random variable $$X = \sup_{t ∈ T} X(t)$$ and $$β^*_{\sup\bar{Q}; a(t), b(t), Γ}$$ is the distribution function of the random variable $$X_T = \sup_{t ∈ T} X(t)$$. Let E be the event that $$Z(0) = Z(1) = 0$$ and $$Z(t)$$ is continuous for all $$t ∈ T$$; then E occurs with probability 1. Finally,
let $\tau$ be any point in $(T, \Gamma)$. Now since $\Gamma$ is dense in $T$ it includes a sequence $\{t_n\}$ such that $\lim_{n \to \infty} t_n = \tau$. So, if $E$ occurs,

\[
\lim_{n \to \infty} X(t_n) = \lim_{n \to \infty} \max \left\{ a(t_n) - Z(t_n), Z(t_n) - b(t_n) \right\}
\]

\[
= \max \left\{ \lim_{n \to \infty} a(t_n) - \lim_{n \to \infty} Z(t_n), \lim_{n \to \infty} Z(t_n) - \lim_{n \to \infty} b(t_n) \right\}
\]

\[
= \max \left\{ a(\tau) - Z(\tau), Z(\tau) - b(\tau) \right\}
\]

\[
= X(\tau)
\]

because then $Z(t)$, $a(t)$, and $b(t)$ are all continuous at $t = \tau$; since $Z(t)$ is necessarily bounded when $E$ occurs the possibility of $a(\tau)$ and $b(\tau)$ being infinite need not trouble us. Now certainly

\[
\lim_{n \to \infty} X(t_n) \leq \sup_{t \in \Gamma} X(t)
\]

or, that is,

\[
X(\tau) \leq \sup_{t \in \Gamma} X(t)
\]

and since this is true for every $\tau \in (T, \Gamma)$ we have that in fact

\[
\sup_{t \in (T, \Gamma)} X(t) \leq \sup_{t \in \Gamma} X(t)
\]

But then

\[
X = \sup_{t \in T} X(t) = \max \left\{ \sup_{t \in \Gamma} X(t), \sup_{t \in (T, \Gamma)} X(t) \right\} = \sup_{t \in \Gamma} X(t) = X_\Gamma.
\]

Thus we have shown that $X = X_\Gamma$ with probability 1, and hence we have (i).
There is no loss in generality in supposing that
\[ Q^* = \lim_{\delta \to 0} \sup_{0 \leq t \leq \delta} a(t) ; \]
and suppose that \( Q < Q^* \): say \( Q = Q^* - c \), where \( c > 0 \). Then there is a \( \delta_0(c) \) such that for \( \delta < \delta_0 \)
\[ \sup_{0 \leq t < \delta} a(t) > Q^* - \frac{c}{2} = Q + \frac{c}{2} . \]

But for any \( \delta < \delta_0 \)
\[ \beta^*(Q) = \Pr \{ a(t) - Q \leq Z(t) \leq b(t) + Q, t \in T \} \]
\[ \leq \Pr \{ Z(t) \geq a(t) - Q, 0 \leq t \leq \delta \} \]
\[ \leq \Pr \{ 0 \leq t \leq \delta \} \sup_{0 \leq t \leq \delta} Z(t) \geq \sup_{0 \leq t \leq \delta} a(t) - Q \}
\[ \leq \Pr \{ 0 \leq t \leq \delta \} \frac{c}{2} \}
\[ \leq \frac{\delta c}{2} \]

by Lemma 2.4. Thus we have immediately statement (ii), and a fortiori \( \beta^*(Q) \) is continuous for \( Q < Q^* \).

Now if \( X_1 = \sup_{t \in T} \inf \{ a(t) - Z(t) \} \) and \( X_2 = \sup_{t \in T} \inf \{ Z(t) - b(t) \} \), then
\[ \beta_1(Q) = \beta_{\inf} a(t) - Q, + \infty, T \] is the distribution function of \( X_1 \) and
\[ \beta_2(Q) = \beta_{\inf} b(t) + Q, T \] is the distribution function of \( X_2 \); and if both \( \beta_1(Q) \) and \( \beta_2(Q) \) are continuous for \( Q > Q^* \) then \( \beta^*(Q) \), which is the distribution function of \( X = \max (X_1, X_2) \), is certainly continuous for \( Q > Q^* \) also. A distribution function is by definition continuous to the right. We shall show that \( \beta_2(Q) \) is continuous to the left for
\( Q > Q^* \); the proof for \( \beta_1(Q) \) is similar, and statement (iii) then follows.

So suppose that \( Q > Q^* \); say \( Q = Q^* + c \), where \( c > 0 \). For any \( \delta, 0 < \delta < \frac{1}{2} \), let \( \mathcal{B}_\delta = \{ t: t \in \Gamma, \delta \leq t \leq 1-\delta \} \). Then for any \( \varepsilon \), \( 0 < \varepsilon < \frac{c}{3} \),

\[
\beta_{\delta}^{\leftarrow \infty, \mathcal{B}_\delta} - \beta_{\delta}^{\leftarrow \infty, \mathcal{B}_\delta} = \mathbb{E} \left\{ \mathcal{B}_\delta \cap \mathbb{E} \left\{ \mathcal{B}_\delta \right\} \right\} = \mathbb{P} \left\{ Z(t) \leq b(t)+Q-\varepsilon, t \in \mathcal{B}_\delta \right\}
\]

But for some \( t \in \mathcal{B}_\delta \),

\[
Z(t) > b(t)+Q-\varepsilon
\]

\[
\leq \mathbb{P} \left\{ Z(t) > b(t)+Q-\varepsilon \text{ for some } t, 0 \leq t \leq \delta \right\}
\]

\[
+ \mathbb{P} \left\{ Z(t) > b(t)+Q-\varepsilon \text{ for some } t, 1-\delta \leq t \leq 1 \right\}
\]

But, from (2.29), there is a \( \delta_1(c) \) such that for \( \delta < \delta_1 \) we have

\[
b(t) \geq -Q^* - \frac{c}{3} \text{ for } 0 \leq t \leq \delta \text{ and } 1-\delta \leq t \leq 1; \text{ and then}
\]

\[
b(t) + Q - \varepsilon > (-Q^* - \frac{c}{3}) + Q - \frac{c}{3} = \frac{c}{3},
\]

so that

\[
\mathbb{P} \left\{ Z(t) > b(t)+Q-\varepsilon \text{ for some } t, 0 \leq t \leq \delta \right\} \leq \mathbb{P} \left\{ 0 \leq t \leq \delta \right\} \left( \sup_{0 \leq t \leq \delta} Z(t) > \frac{c}{3} \right)
\]

\[
< \frac{188}{c^2}
\]

and

\[
\mathbb{P} \left\{ Z(t) > b(t)+Q-\varepsilon \text{ for some } t, 1-\delta \leq t \leq 1 \right\} \leq \mathbb{P} \left\{ 1-\delta \leq t \leq 1 \right\} \left( \sup_{1-\delta \leq t \leq 1} Z(t) > \frac{c}{3} \right)
\]

\[
< \frac{188}{c^2}
\]

by Lemma 2.4, and hence
\[ \beta_{\infty, b(t)+Q, \Gamma_8} - \beta_{\infty, b(t)+Q-\varepsilon, \Gamma_8} < \frac{36\varepsilon}{c^2}. \]

Now since \( \Gamma_8 \subset \Gamma \) for all \( \varepsilon \),

\[ \beta_{\infty, b(t)+Q, \Gamma} - \beta_{\infty, b(t)+Q, \Gamma_8} \leq 0; \]

and by Lemma 2.5

\[ \beta_{\infty, b(t)+Q, \Gamma_8} - \beta_{\infty, b(t)+Q-\varepsilon, \Gamma_8} \leq \frac{\varepsilon}{\sqrt{\delta(1-\varepsilon)}}. \]

Adding these last three inequalities, and applying statement (i), we obtain

\[ \beta_{\infty, b(t)+Q, \Gamma} - \beta_{\infty, b(t)+Q-\varepsilon, \Gamma} < \frac{36\varepsilon}{c^2} + \frac{\varepsilon}{\sqrt{\delta(1-\varepsilon)}}. \]

Thus for any \( c, \eta > 0 \) we can choose \( \delta_0(c, \eta) \) and then \( \varepsilon_0(\delta_0, c, \eta) \) so that for \( 0 < \varepsilon < \varepsilon_0 \)

\[ \beta_2(Q) - \beta_2(Q-\varepsilon) < \eta; \]

that is, \( \beta_2(Q) \) is continuous to the left for \( Q > Q^* \).

The arguments for statements (ii) and (iii) have tacitly assumed \( Q^* \) finite, but the changes that are necessary if this is not so are obvious, and thus the proof of the theorem is complete.

We note that from Theorem 2.6 it follows that \( \beta_*(Q) \) is continuous in \( Q \) for all \( Q \) if and only if \( \beta_*(Q^*) = 0 \). Consider two trivial examples: if \( a(t) = -\infty \) and \( b(t) = 0 \) for \( t \in \mathbb{T} \) then by (2.19) and (2.27)

\[ \beta_*(Q) = \delta_{\infty, Q; 0, 1, 0, 0} = \begin{cases} 1 - e^{-2Q^2} & Q > 0 \\ 0 & Q \leq 0 \end{cases} . \]
which function is continuous for all \( Q \), and \( \beta^*(Q^*) = \beta^*(0) = 0 \); but if \( a(t) = -\infty \) and \( b(t) = +\infty \) for \( 0 < t < 1 \), yet \( a(0) = b(0) = a(1) = b(1) = 0 \), then

\[
\beta^*(Q) = \begin{cases} 
0 & Q < 0 \\
1 & Q \geq 0
\end{cases},
\]

which has a discontinuity at \( Q = Q^* = 0 \).

**Theorem 2.7.** Let \( a(t) \) and \( b(t) \) be any two functions defined for \( t \in T \), and let \( k \geq 1 \) and \( 0 < t_1 < t_2 < \cdots < t_k < 1 \) be chosen arbitrarily; then \( \beta_{[a(t), b(t), T]} \) can be expressed as

\[
\beta_{[a(t), b(t), T]} = \int \ldots \int \left\{ \prod_{i=1}^{k+1} \xi_{\frac{a(t), b(t)}{t_{i-1}, t_i, z_{i-1}, z_i}} \right\} f_k(z_1, \ldots, z_k) dz_k \ldots dz_1
\]

where \( t_0 = 0, t_{k+1} = 1 \), and \( z_0 = z_{k+1} = 0 \).

**Proof.** Certainly we have

\[
\beta_{[a(t), b(t), T]} = \int \ldots \int \{ a(t) \leq Z(t) \leq b(t), t \in T \}
\]

\[
\int Z(t_i) = z_i, 1 \leq i \leq k f_k(z_1, \ldots, z_k) dz_k \ldots dz_1
\]

where \( f_k(z_1, \ldots, z_k) \) is the joint probability density function of \( Z(t_1), Z(t_2), \ldots, Z(t_k) \) given by (2.3). But, because \( Z(t) \) is a Markov process,
\[ \text{Pr} \{ a(t) \leq Z(t) \leq b(t), t \in T \mid Z(t_i) = z_i, 1 \leq i \leq k \} \]

\[ = \prod_{i=1}^{k+1} \text{Pr} \{ a(t) \leq Z(t) \leq b(t), t_{i-1} < t < t_i \mid Z(t_{i-1}) = z_{i-1}, Z(t_i) = z_i \} \]

\[ = \prod_{i=1}^{k+1} \bar{Q}(a(t), b(t); t_{i-1}, t_i, z_{i-1}, z_i) \]

by (2.14), and this clearly gives the theorem.

We shall say that a piecewise-continuous function \( f(t) \) is **piecewise-linear with \( (k+1) \) segments** if \( \Gamma(f) = \{ t_0, t_1, \ldots, t_{k+1} \} \)

where \( 0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1 \) and \( f(t) \) is linear in each open interval \( (t_i, t_{i+1}) \), \( 0 \leq i \leq k \). \( f(t_i) \), \( 0 \leq i \leq k+1 \), may be arbitrarily defined. Then (2.24), (2.25), and Theorem 2.7 enable us to calculate \( \beta^* \bar{Q}(a(t), b(t), T) \) whenever \( a(t) \) and \( b(t) \) are piecewise-linear. However, it will be more useful to specialize further to the case where \( a(t) \) and \( b(t) \) form a coincident **piecewise-linear pair** of functions; that is, where \( \Gamma(a) = \Gamma(b) \) and

\[ a(t) = b(t) = \lambda_i \Delta(1-t) + \mu_i \Delta t \quad \text{for} \quad t_i < t < t_{i+1}, 0 \leq i \leq k, \]

but \( a(t_i) \) and \( b(t_i) \), \( 0 \leq i \leq k+1 \), may be arbitrarily defined. The apparently superfluous parameter \( \Delta \) has been inserted for convenience in a later chapter. Note that here

\[ \bar{Q}(a(t), b(t)) = \max \left\{ a(0), a(1), -b(0), -b(1), |\lambda_1 \Delta|, |\mu_{k+1} \Delta| \right\} \]

and

\[ \bar{Q}(\infty, b(t)) = -\min \left\{ b(0), b(1), \lambda_1 \Delta, \mu_{k+1} \Delta \right\}. \]
Now if we define

\[ C(Q, \lambda, \mu, t', t'', z', z'') \]

\[ = \xi \sqrt{\lambda(l-t) + \mu t - Q, \lambda(l-t) + \mu t + Q; t', t'', z', z''} \] 

(2.31) \[ = \Pr \left\{ \lambda(l-t) + \mu t - Q \leq Z(t) \leq \lambda(l-t) + \mu t + Q, t' < t < t'' \right\} \]

\[ Z(t') = z', Z(t'') = z'' \]

we have, using (2.26) and (2.23) to write out the formula explicitly,

(2.32) \[ C(Q, \lambda, \mu, t', t'', z', z'') = \begin{cases} 
\frac{-2C_{ik}}{1 + \sum_{k=1}^{\infty} \sum_{i=1}^{k} (-1)^{i} e^{-\frac{t}{t'''}} t'''} \\
\text{if } Q > \max \left\{ \left| \lambda(l-t') + \mu t' - z' \right|, \right. \\
\left. \left| \lambda(l-t'') + \mu t'' - z'' \right| \right\}, \text{ and} \\
0 \text{ otherwise, where} \\
\end{cases} \]

\[ C_{11} = \left\{ (2i-1)Q + \sqrt{\lambda(l-t') + \mu t' - z'} \right\} \left\{ (2i-1)Q + \sqrt{\lambda(l-t'') + \mu t'' - z''} \right\} \]

\[ C_{12} = 2iQ \left\{ 2iQ + (\mu - \lambda)(t'' - t') - (z'' - z') \right\} \]

\[ C_{13} = \left\{ (2i-1)Q - \sqrt{\lambda(l-t') + \mu t' - z'} \right\} \left\{ (2i-1)Q - \sqrt{\lambda(l-t'') + \mu t'' - z''} \right\} \]

\[ C_{14} = 2iQ \left\{ 2iQ - (\mu - \lambda)(t'' - t') + (z'' - z') \right\} ; \]

and if we define also

(2.33) \[ C^+(R, \lambda, \mu, t', t'', z', z'') \]

\[ = \xi \sqrt{-\infty, \lambda(l-t) + \mu t + R; t', t'', z', z''} \] 

\[ = \Pr \left\{ Z(t) \leq \lambda(l-t) + \mu t + R, t' < t < t'' \middle| Z(t') = z', Z(t'') = z'' \right\}, \]

then from (2.27) we have
(2.34) \[ C^+(R, \lambda, \mu, t', t'', z', z'') \]

\[
= \begin{cases} 
\frac{2\sqrt{R+\lambda(1-t')}\sqrt{\mu t'-z'}/\sqrt{R+\lambda(1-t'')+\mu t''-z''}}{t''-t'} & \text{if } R > \max \{ z'-\lambda(1-t')-\mu t', z''-\lambda(1-t'')-\mu t'' \} \\
0 & \text{otherwise.}
\end{cases}
\]

Thus we can now establish

**Corollary 2.8.** If \( a(t) \) and \( b(t) \) form a coincident piecewise-linear pair of functions satisfying (2.30), let

\[
a_i^\Delta = \max \{ \Delta \lambda_{i-1} (1-t_i - \mu t_i - z_i), a(t_i), \Delta \lambda_i (1-t_i + \mu t_i - z_i) \}
\]

(2.35) \[ b_i^\Delta = \min \{ \Delta \lambda_{i-1} (1-t_i - \mu t_i - z_i), b(t_i), \Delta \lambda_i (1-t_i + \mu t_i - z_i) \} \]

for \( 1 \leq i \leq k \), and define

(2.36) \[ Q_{\sqrt{a}(t), b(t)}^{-\Delta} = \max \left\{ Q_{\sqrt{a}(t), b(t)}^{-\Delta}; \frac{1}{2} (a_i - b_i), 1 \leq i \leq k \right\}. \]

then

(2.37) \[ \beta_{\sqrt{a}(t)} - Q, b(t) + Q, T^{-\Delta} \]

\[
= \begin{cases} 
\Delta a_1 + Q, \Delta b_1 + Q, \mathcal{C}(Q, \Delta a_1, \Delta b_1, t_1-1, t_1, z_1-1, z_1) & \text{if } Q \geq Q_0, \text{ and} \\
\Delta a_k - Q, \Delta b_k - Q, \int \cdots \int f_k(z_1, \ldots, z_k) \, dz_k \ldots dz_1 & \text{otherwise.}
\end{cases}
\]

Also,
\[(2.38) \quad \beta_{\left[-\infty, \mathcal{b}(t) + R_T \right]}\]
\[
\left\{ \begin{array}{l}
\Delta b_1 + R \\
\Delta b_k + R \\
\quad \ldots \\
\quad \ldots \\
\quad \ldots \\
\quad \ldots \\
-\infty \\
-\infty \\
\{ \begin{array}{l}
k+1 \\
\Pi \\i=1 \\
C(R, \Delta \lambda_1, \Delta \mu_1, t_{i-1}, t_i, z_{i-1}, z_i)
\end{array} \}
\end{array} \right\}
\times f_k(z_1, \ldots, z_k) dz_k \ldots dz_1
\]
\[
= \left\{ \begin{array}{l}
\text{if } R \geq Q \left[-\infty, \mathcal{b}(t) \right], \text{ and} \\
0 \quad \text{otherwise.}
\end{array} \right.
\]

Proof. From the definition (2.31) and from Theorem 2.7 it follows that \( \beta_{\left[a(t)-Q, \mathcal{b}(t) + Q, T \right]} \) can be expressed as

\[
\left\{ \begin{array}{l}
b(t_1) + Q \\
b(t_k) + Q \\
\quad \ldots \\
\quad \ldots \\
\quad \ldots \\
\quad \ldots \\
a(t_1) - Q \\
a(t_k) - Q \\
\end{array} \right\}
\times f_k(z_1, \ldots, z_k) dz_k \ldots dz_1
\]

for all \( Q \), and the refinements included in (2.37) are trivial: if \( Q < Q_0 \), or if, for some \( i, 1 \leq i \leq k \), \( \Delta \lambda_i - Q \leq z_i \leq \Delta \mu_i + Q \) fails to hold, then from (2.32) we see that

\[
\prod_{i=1}^{k+1} C(Q, \Delta \lambda_i, \Delta \mu_i, t_{i-1}, t_i, z_{i-1}, z_i) = 0.
\]

And the proof of (2.38) is similar and even simpler.

It will be useful to have in explicit form some of the formulas for \( \beta_{\left[a(t)-Q, \mathcal{b}(t) + Q, T \right]} \) and \( \beta_{\left[-\infty, \mathcal{b}(t) + R_T \right]} \) when \( a(t) \) and \( b(t) \) form a coincident piecewise-linear pair with the number of segments very small. In each case we write the formula only for

\( Q \geq Q_0(a(t), b(t)) \) or \( R \geq Q \left[-\infty, \mathcal{b}(t) \right] \).
If \( k = 0 \) (i.e., there is \( k+1 = 1 \) segment), write \( \lambda \) for \( \lambda_1 \) and \( \mu \) for \( \mu_1 \). Then substitution into (2.32) gives

\[
\beta_{[s(t)-Q,b(t)+Q,T]} = 1 - \sum_{i=1}^{\infty} \left\{ e^{-2((2i-1)Q+\lambda \Delta)/(2i-1)Q+\mu \Delta}} - e^{-4iQ/(2iQ) - \Delta(\lambda-\mu)} \right\};
\]

(2.39)

(one segment)

\[
+ e^{-2((2i-1)Q,\lambda \Delta)/(2i-1)Q+\mu \Delta}} - e^{-4iQ/(2iQ) + \Delta(\lambda-\mu)} \right\};
\]

and (2.34) gives

\[
(2.40) \quad \beta_{[a(t)-Q,b(t)+R,T]} = 1 - e^{-2(R+\lambda \Delta)/(R+\mu \Delta)}.
\]

(one segment)

If \( k = 1 \) (i.e., there are \( k+1 = 2 \) segments), write \( \tau \) for \( t_1 \), \( a \) for \( a_1 \), and \( b \) for \( b_1 \). Then by Corollary 2.3 we can express

\[
\beta_{[a(t)-Q,b(t)+Q,T]} \text{ as}
\]

\[
\begin{align*}
\Delta b + Q & \int_{C} C(Q,\Delta_{1,1,0,\tau,0,\tau,0,\tau,0,z)} Q(Q,\Delta_{2,2,\tau,1,1,1,2,1,0}) \frac{e^{-2\tau(1-\tau)}}{\sqrt{2\pi(1-\tau)}} dz.
\end{align*}
\]

Let

\[
\begin{cases}
\begin{align*}
\lambda_{1} & = \lambda \Delta + (2i-1)Q \\
(2.41) \\
\mu_{2} & = (2j-1)Q \\
p_{11} & = \lambda \Delta + (2i-1)Q \\
p_{12} & = 2iQ \\
p_{13} & = \lambda \Delta - (2i-1)Q \\
p_{14} & = -2iQ \\
q_{j1} & = \mu \Delta + (2j-1)Q \\
q_{j2} & = 2jQ \\
q_{j3} & = \mu \Delta - (2j-1)Q \\
q_{j4} & = -2jQ.
\end{align*}
\end{cases}
\]

then
\[ C(Q, \Delta_1, \Delta_1, 0, \tau, 0, z) = 1 + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k e^{-\frac{2}{\tau} p_{1k}^2 \Delta_1^2 - \Delta_1^2 (\Lambda_1 - \mu_1)^2 - \tau} \]

and

\[ C(Q, \Delta_2, \Delta_2, \tau, 1, z, 0) \]

\[ = 1 + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^j e^{-\frac{2}{\tau} q_{jk}^2 \Delta_2^2 - \Delta_2^2 (\Lambda_2 - \mu_2)^2 - \tau} \]

and with some computation it can be shown that

\[ (2.42) \quad \beta \left[ a(t) - Q, b(t) + Q, T \right] = \frac{\phi \left( \frac{\Delta b + Q}{\sqrt{\tau(1-\tau)}} \right) - \phi \left( \frac{\Delta a - Q}{\sqrt{\tau(1-\tau)}} \right)}{\sqrt{\tau(1-\tau)}} \]

(two segments)

\[ \begin{array}{c}
\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k e^{-2p_{1k}^2 \Delta_1^2 - \Delta_1^2 (\Lambda_1 - \mu_1)^2 - \tau} \\
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^j e^{-2q_{jk}^2 \Delta_2^2 - \Delta_2^2 (\Lambda_2 - \mu_2)^2 - \tau}
\end{array} \]

\[ \times \left\{ \phi \left( \frac{\Delta b + Q - 2p_{1k}(1-\tau)}{\sqrt{\tau(1-\tau)}} \right) - \phi \left( \frac{\Delta a - Q - 2p_{1k}(1-\tau)}{\sqrt{\tau(1-\tau)}} \right) \right\} \]

\[ \times \left\{ \phi \left( \frac{\Delta b + Q - 2q_{jk}(1-\tau)}{\sqrt{\tau(1-\tau)}} \right) - \phi \left( \frac{\Delta a - Q - 2q_{jk}(1-\tau)}{\sqrt{\tau(1-\tau)}} \right) \right\} \]

Furthermore,
\[ b \left[ -\infty, b(t) + R, \tau \right] = \phi \left( \frac{R + \Delta b}{\sqrt{\tau(1-\tau)}} \right) \]

(two segments)

\[ -2(R+\lambda_1 \Delta)(R+\mu_1 \Delta) \phi \left( \frac{R(2\tau-1)+\Delta b-2(1-\tau)\lambda_1 \Delta}{\sqrt{\tau(1-\tau)}} \right) \]

\[ -2(R+\lambda_2 \Delta)(R+\mu_2 \Delta) \phi \left( \frac{R(1-2\tau)+\Delta b-2\tau \mu_2 \Delta}{\sqrt{\tau(1-\tau)}} \right) \]

\[ -2\Delta \left( R+\lambda_1 \Delta \right) \left( \mu_1 - \mu_2 \right) - (R+\mu_2 \Delta \right) \]

\[ + e \phi \left( \frac{-R+\Delta b-2(1-\tau)\lambda_1 \Delta - 2\tau \mu_2 \Delta}{\sqrt{\tau(1-\tau)}} \right). \]

Finally, define

\[ F(x, y; \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}}}{dvdu}; \]

then some rather tedious calculation will establish that

\[ b \left[ -\infty, b(t) + R, \tau \right] = F \left[ \frac{R + \Delta b_1}{\sqrt{t_1(1-t_1)}}, \frac{R + \Delta b_2}{\sqrt{t_2(1-t_2)}} ; \rho \right] \]

(three segments)

\[ -2(R+\lambda_1 \Delta)(R+\mu_1 \Delta) \]

\[ \times F \left[ \frac{R(2t_1-1)+\Delta b_1-2\lambda_1 \Delta(1-t_1)}{\sqrt{t_1(1-t_1)}}, \frac{R(2t_1-1)+\Delta b_2-2\lambda_1 \Delta(1-t_2)}{\sqrt{t_2(1-t_2)}} ; \rho \right] \]

\[ -2(R+\lambda_2 \Delta)(R+\mu_2 \Delta) \]

\[ \times F \left[ \frac{R(1-2t_1)+\Delta b_1-2\mu_2 \Delta t_1}{\sqrt{t_1(1-t_1)}}, \frac{R(2t_1-1)+\Delta b_2-2\lambda_2 \Delta(1-t_2)}{\sqrt{t_2(1-t_2)}} ; -\rho \right] \]

(continued on next page)
\[-2(R+\lambda_3 \Delta)(R+\mu_3 \Delta)
- e \left[ \frac{R(1-2t_1)+\Delta b_1-2\mu_3 \Delta t_1}{\sqrt{t_1(1-t_1)}}, \frac{R(1-2t_2)+\Delta b_2-2\mu_3 \Delta t_2}{\sqrt{t_2(1-t_2)}} ; \rho \right] \]

+ e \left[ \frac{-R+\Delta b_1-2\lambda_1 \Delta(1-t_1)-2\mu_3 \Delta t_1}{\sqrt{t_1(1-t_1)}}, \frac{R+\Delta b_2+2\Delta(\lambda_1-\lambda_2)(1-t_2)}{\sqrt{t_2(1-t_2)}} ; -\rho \right] \]

\[-2\Delta(R+\lambda_1 \Delta)(\mu_1-\mu_2)-(R+\mu_2 \Delta)(\lambda_1-\lambda_2) \]

- e \left[ \frac{-R+\Delta b_1-2\lambda_1 \Delta(1-t_1)-2\mu_3 \Delta t_1}{\sqrt{t_1(1-t_1)}}, \frac{-R+\Delta b_2-2\lambda_1 \Delta(1-t_2)-2\mu_3 \Delta t_2}{\sqrt{t_2(1-t_2)}} ; \rho \right] \]

+ e \left[ \frac{R+\Delta b_1-2(\mu_2-\mu_3) \Delta t_1}{\sqrt{t_1(1-t_1)}}, \frac{-R+\Delta b_2-2\lambda_2 \Delta(1-t_2)-2\mu_3 \Delta t_2}{\sqrt{t_2(1-t_2)}} ; -\rho \right] \]

\[-2\Delta(R+\lambda_2 \Delta)(\mu_2-\mu_3)-(R+\mu_3 \Delta)(\lambda_2-\lambda_3) \]

- e \left[ \frac{R(1+2t_1)+\Delta b_1-2\lambda_1 \Delta(1-t_1)-2\mu_2 \Delta t_1+2\mu_3 \Delta t_1}{\sqrt{t_1(1-t_1)}} \right] \]

\[\frac{R(1-2t_2)+\Delta b_2+2\lambda_1 \Delta(1-t_2)-2\lambda_2 \Delta(1-t_2)-2\mu_3 \Delta t_2}{\sqrt{t_2(1-t_2)}} ; -\rho \right] ,

where

\[\rho = \sqrt{\frac{t_1(1-t_2)}{t_2(1-t_1)}} .\]

These formulas rapidly become intolerably complex, and we shall
not attempt to write down any more of them. Were it necessary to perform calculations with larger $k$ we might well prefer to consider approximations, as indeed we should be forced to do if $a(t)$ and $b(t)$ were not piecewise-linear.
CHAPTER III

ASYMPTOTIC $Z_n(t)$ PROCESS PROBABILITIES

We return now to consideration of the asymptotic power functions of the Kolmogorov tests, which were defined as

\[(1.38) \quad P(H, \{G_n\}, \alpha) = 1 - \lim_{n \rightarrow \infty} \beta_n E_n(t; H, G_n) - Q_n, S_n(t; H, G_n) + Q_n, T] \]

and

\[(1.43) \quad P^+(H, \{G_n\}, \alpha) = 1 - \lim_{n \rightarrow \infty} \beta_n E_n(t; H, G_n) + R_n, T] \]

In \[\sim 10\] Doob suggested that "in calculating asymptotic $x_n(t)$ process distributions when $n \rightarrow \infty$ we may simply replace the $x_n(t)$ processes by the $x(t)$ process." (Doob uses $x_n$ and $x$ for the $Z_n$ and $Z$ of this paper.) In the case at hand, we might interpret his suggestion as follows: assume that the limit

\[S(t) = \lim_{n \rightarrow \infty} S_n(t; H, G_n)\]

exists; then it is not unreasonable to suppose that

\[P(H, \{G_n\}, \alpha) = 1 - \beta E[S(t) - Q, S(t) + Q, T] \]

and

\[P^+(H, \{G_n\}, \alpha) = 1 - \beta E[S(t) + R, T] \]

The major aim of this chapter is to justify the "heuristic" procedure which we have just exemplified. However, we are already able to present two simple cases in which it yields correct answers: according to (1.36),
\[ J_n(Q_n) = \beta_{\sqrt{-Q_n}, Q_n, T} \]

hence, using our procedure, we might expect that

(3.1) \[ \lim_{n \to \infty} J_n(Q_n) = \beta_{\sqrt{-Q}, Q, T} \]

Now \( \lim_{n \to \infty} J_n(Q_n) = J(q) \) is given by (1.15), whereas \( \beta_{\sqrt{-Q}, Q, T} \) can be evaluated by substituting \( \Delta = 0 \) into (2.39); and the reader may verify that the two formulas are indeed equivalent. Similarly, according to (1.41),

\[ J^+_n(R_n) = \beta_{\sqrt{-\infty}, R_n, T} \]

so that we expect

(3.2) \[ \lim_{n \to \infty} J^+_n(R_n) = \beta_{\sqrt{-\infty}, R, T} \]

and (3.2) may also be verified, using (1.17) and (2.40).

Doob's suggestion was justified under fairly general conditions by Donsker, whose theorem in \( \sqrt{9} \) may be stated as follows:

**Theorem 3.1.** (Donsker) Let \( \Lambda \) be the space of all real, single-valued functions \( f(t) \) which are continuous for \( t \in \mathbb{T} \) except for at most a finite number of finite jumps. Let \( F \) be a functional defined on \( \Lambda \) which is continuous in the uniform topology at almost all points of \( \Xi \), the space of continuous sample functions of the \( Z(t) \) process. Then

\[ \lim_{n \to \infty} \mathsf{pr}\left\{ F(Z_n(t)) \leq Q \right\} = \mathsf{pr}\left\{ F(Z(t)) \leq Q \right\} \]

at all the points of continuity of the distribution function on the right.

Application of this theorem to the case at hand, following the
reasoning which established Corollary 2.2, yields

Corollary 3.2. If \( a(t) \) and \( b(t) \) are any two functions defined for \( t \in T \), then

\[
\lim_{n \to \infty} \beta_{n}^{\sqrt{a(t) - Q}, b(t) + Q, T} = \beta_{l}^{\sqrt{a(t) - Q}, b(t) + Q, T} = \beta^{*}(Q)
\]

at all the points of continuity of \( \beta^{*}(Q) \).

We see that the preceding corollary is not sufficiently general for our purposes; however, it is easy to extend it to

Theorem 3.3. If the two sequences of functions \( \{a_{n}(t)\} \) and \( \{b_{n}(t)\} \) converge uniformly to functions \( a(t) \) and \( b(t) \) respectively then

\[
\lim_{n \to \infty} \beta_{n}^{\sqrt{a_{n}(t) - Q}, b_{n}(t) + Q, T} = \beta_{l}^{\sqrt{a(t) - Q}, b(t) + Q, T} = \beta^{*}(Q)
\]

at all the points of continuity of \( \beta^{*}(Q) \).

Proof. By uniform convergence, there is an \( n_{1}(\varepsilon) \) such that for \( n \geq n_{1} \)

\[
\sup_{t \in T} |a_{n}(t) - a(t)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{t \in T} |b_{n}(t) - b(t)| \leq \frac{\varepsilon}{2}
\]

and let \( n_{2}(\varepsilon) \) be so chosen that for \( n \geq n_{2} \)

\[
|Q_{n} - Q| \leq \frac{\varepsilon}{2}
\]

where \( \varepsilon > 0 \) is arbitrary. Then for \( n \geq n_{0} = \max(n_{1}, n_{2}) \) we have, for all \( t \in T \),

\[
a(t) - Q - \varepsilon \leq a_{n}(t) - Q_{n} \leq a(t) - Q + \varepsilon
\]

\[
b(t) + Q - \varepsilon \leq b_{n}(t) + Q_{n} \leq b(t) + Q + \varepsilon
\]

and hence

\[
\beta_{n}^{\sqrt{a(t) - Q + \varepsilon}, b(t) + Q - \varepsilon, T} = \beta_{n}^{\sqrt{a_{n}(t) - Q_{n}}, b_{n}(t) + Q_{n}, T} \leq \beta_{n}^{\sqrt{a(t) - Q - \varepsilon}, b(t) + Q + \varepsilon, T}.
\]
Now if \( \beta^*(q) = \beta[\sqrt{a-Q,b+Q},T] \) is continuous at \( Q+\epsilon \) and \( Q-\epsilon \) then by Corollary 2.2

\[
\lim_{n \to \infty} \beta_n[\sqrt{a-Q+\epsilon,b+Q-\epsilon},T] = \beta[\sqrt{a-Q+\epsilon,b+Q-\epsilon},T]
\]

\[
\lim_{n \to \infty} \beta_n[\sqrt{a-Q-\epsilon,b+Q+\epsilon},T] = \beta[\sqrt{a-Q-\epsilon,b+Q+\epsilon},T]
\]

so

\[
\beta[\sqrt{a-Q+\epsilon,b+Q-\epsilon},T] \leq \lim_{n \to \infty} \beta_n[\sqrt{a_n-Q_n,b_n+Q_n},T]
\]

\[
\leq \lim_{n \to \infty} \beta_n[\sqrt{a_n-Q_n,b_n+Q_n},T]
\]

\[
\leq \beta[\sqrt{a-Q-\epsilon,b+Q+\epsilon},T].
\]

But since \( \beta^*(q) \) is a distribution function, it can have at most a countable number of points of discontinuity, and we can let \( \epsilon \) tend to zero through values such that \( \beta^*(q) \) is continuous at \( Q+\epsilon \) and \( Q-\epsilon \); this yields the theorem.

**Corollary 3.4.** If \( (H,\{G_n\}) \in \mathcal{F} \) and the sequence of functions \( \{S_n(t;H,G_n)\} \) converges uniformly to some function \( S(t) \), then

\[
P(H,\{G_n\},\alpha) = 1 - \beta[\sqrt{S(t)-Q},S(t)+Q,T]
\]

for all \( \alpha \) such that \( \beta[\sqrt{S(t)-Q},S(t)+Q,T] \) is continuous at \( Q = Q(\alpha) \); and

\[
P^+(H,\{G_n\},\alpha) = 1 - \beta[\sqrt{-\infty},S(t)+R,T]
\]

for all \( \alpha \) such that \( \beta[\sqrt{-\infty},S(t)+R,T] \) is continuous at \( R = R(\alpha) \).

Even Theorem 3.3 is not sufficiently general for some of the applications we have in mind; and we now undertake to extend it under slightly less restrictive conditions. For this we shall require two lemmas, of which the first is an extension of Lemma 2.4.

**Lemma 3.5.** There is a finite positive constant \( \phi \) such that if
0 \leq t' < t'' \leq 1 \text{ and } \varepsilon > 0 \text{ then for every positive integer } n

\Pr \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} \inf_{t' \leq t \leq t''} Z_n(t) > \varepsilon \right\} < \frac{\phi(t''-t')}{\varepsilon^2};

and also

\Pr \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z(t) - \inf_{t' \leq t \leq t''} Z(t) > \varepsilon \right\} < \frac{\phi(t''-t')}{\varepsilon^2}.

Proof. Let

\xi = \Pr \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z_n(t) - \inf_{t' \leq t \leq t''} Z_n(t) > \varepsilon \right\}

= \Pr \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z_n(t) - \inf_{t' \leq t \leq t''} Z_n(t) > \varepsilon, \right. \\
|Z_n(t') - Z_n(t'')| \geq \frac{\varepsilon}{4} \bigg\}

+ \Pr \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z_n(t) - \inf_{t' \leq t \leq t''} Z_n(t) > \varepsilon, \left. \\
|Z_n(t') - Z_n(t'')| < \frac{\varepsilon}{4} \right\}

\leq \Pr \left\{ |Z_n(t') - Z_n(t'')| \geq \frac{\varepsilon}{4} \right\}

+ \Pr \left\{ t' \leq t \leq t'' Z_n(t) > Z_n(t') + \frac{\varepsilon}{2}, |Z_n(t') - Z_n(t'')| < \frac{\varepsilon}{4} \right\}

+ \Pr \left\{ t' \leq t \leq t'' Z_n(t) < Z_n(t') - \frac{\varepsilon}{2}, |Z_n(t') - Z_n(t'')| < \frac{\varepsilon}{4} \right\}

= \xi_1 + \xi_2 + \xi_3.

say. Now by Chebyshev's inequality

(3.3) \xi_1 = \Pr \left\{ |Z_n(t') - Z_n(t'')| \geq \frac{\varepsilon}{4} \right\} \leq \frac{16(t''-t')(1-t''+t')}{\varepsilon^2} < \frac{16(t''-t')}{\varepsilon^2}.

Then, since probabilities about \(Z_n(t) = \sqrt{F_n} F_n^{-1}(t) \cdot \sqrt{-t}\) do not depend on \(F\), so long as \(F \in \mathcal{F}\), we may as well simplify matters by taking
\[ F(x) = \begin{cases} 
 0 & x \leq 0 \\
 x & 0 \leq x \leq 1 \\
 1 & x \geq 1 
\end{cases} \]

so that \( F^{-1}(t) = t, \ t \in T \), and \( Z_n(t) = \sqrt{n} \left( \frac{F_n(t)}{F_n(t)} - t \right) \). Then

\[ \xi_2 = \text{pr} \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z_n(t) > Z_n(t') + \frac{c}{2}, \ |Z_n(t'' - Z_n(t'))| < \frac{c}{4} \right\} \]

\[ = \sum_{0 \leq i \leq j \leq n} a_{ij} b_{ij} \]

\[ |(j-i)-n(t''-t')| < \frac{c}{4} \sqrt{n} \]

where

\[ a_{ij} = \text{pr} \left\{ Z_n(t') = \sqrt{n} \left( \frac{i}{n} - t' \right), \ Z_n(t'') = \sqrt{n} \left( \frac{j}{n} - t'' \right) \right\} \]

\[ = \text{pr} \left\{ F_n(t') = \frac{i}{n}, \ F_n(t'') = \frac{j}{n} \right\} \]

\[ = \frac{n!}{i!(j-i)!(n-j)!} \ (t')^i \ (t''-t')^{j-i} \ (1-t'')^{n-j} \]

and

\[ b_{ij} = \text{pr} \left\{ t' \leq t \leq t'' \sup_{t' \leq t \leq t''} Z_n(t) > Z_n(t') + \frac{c}{2} \right\} \]

\[ \left| Z_n(t') = \sqrt{n} \left( \frac{i}{n} - t' \right), \ Z_n(t'') = \sqrt{n} \left( \frac{j}{n} - t'' \right) \right\} \]

\[ = \text{pr} \left\{ t' \leq t \leq t'' \sqrt{F_n(t)-t'} > \frac{i}{n} - t' + \frac{c}{2 \sqrt{n}} \right\} \]

\[ \left| F_n(t') = \frac{i}{n}, \ F_n(t'') = \frac{j}{n} \right\} \].

If \( j = i \) then certainly \( b_{ij} = 0 \); so assume \( j > i \). But, for \( F_n(t') \)

and \( F_n(t'') \) given, that portion of the empirical distribution

function \( F_n(t) \) where \( t' \leq t \leq t'' \) is itself, except for a linear

transformation, an empirical distribution function; namely, if we let
\[ G_{j-1}(u) = \frac{n}{j-1} F_n(t) - \frac{1}{j-1}, \text{ } u \in T, \]

where
\[ u = \frac{t-t'}{t'-s'}, \]

then \( G_{j-1}(u) \) is the empirical distribution function of a random sample of size \( (j-1) \) from \( F(x) \). And since
\[ F_n(t') = \frac{1}{n} \text{ if and only if } G_{j-1}(0) = 0, \]

and
\[ F_n(t'') = \frac{1}{n} \text{ if and only if } G_{j-1}(1) = 1, \]

we have
\[
b_{ij} = \Pr \left\{ \sup_{u \in T} \sqrt{\frac{j-1}{n} G_{j-1}(u)} + \frac{1}{n} - t' - u(t'' - t') \geq \frac{1}{n} - t' + \frac{\varepsilon}{2\sqrt{n}} \right\}
\]
\[
= 1 - \Pr \left\{ G_{j-1}(u) \leq \frac{nu(t'' - t')}{j-1} + \frac{\varepsilon \sqrt{n}}{2(j-1)}, u \in T \right\}
\]
\[
\leq 1 - \Pr \left\{ G_{j-1}(u) \leq u + \frac{\varepsilon \sqrt{n}}{4(j-1)}, u \in T \right\}
\]
\[
= 1 - \Pr \left\{ \sup_{u \in T} \sqrt{\frac{G_{j-1}(u)}{n}} - u \leq \frac{\varepsilon \sqrt{n}}{4(j-1)} \right\}
\]
\[
< c \varepsilon^2 \frac{n}{4(j-1)},
\]

where \( c \) is the finite positive constant of Lemma 2 in \( \text{[1]} \). But then certainly \( b_{ij} < \frac{8c(j-1)}{\varepsilon^2 n} \) and hence
\[
\frac{\varepsilon^2}{2} < 0 \leq i < j \leq n \frac{8c(j-1)}{\varepsilon^2 n} \frac{n!}{i!(j-i)!(n-j)!} \frac{(t')^i(t''-t')^j-(1-t'')^{n-j}}{(i+j)!}
\]
\[
|\{(j-1)-n(t''-t')| < \frac{\varepsilon \sqrt{n}}{4}
\]
\[
< \frac{8c}{\varepsilon^2 n} \sum_{i=0}^{n} \sum_{j=i+1}^{n} \frac{n!}{i!(j-i-1)!(n-j)!} \frac{(t')^i(t''-t')^j-(1-t'')^{n-j}}{(i+j)!}
\]
\[
= \frac{8c(t''-t')}{\varepsilon^2}.\]
Since a similar argument will give the same upper bound for \( \xi_3 \) we have

\[
\xi = \xi_1 + \xi_2 + \xi_3 < \frac{16(c+1)(t''-t')}{\varepsilon^2}.
\]

The preceding applies equally well to the process \( Z(t) \) also as far as (3.3); then, using Theorem 2.7,

\[
\xi_2 = \int_{-\infty}^{\infty} \int_{z''}^{\xi} e^{-\frac{2}{t''-t'} (\frac{\xi}{2} - z'')} f_2(z',z'') dz'' dz'
\]

\[
\leq e^{-\frac{2}{t''-t'} (\frac{\xi}{2})} \int_{-\infty}^{\infty} \int_{z''}^{\xi} f_2(z',z'') dz'' dz'
\]

\[
\leq e^{-\frac{\varepsilon^2}{4(t''-t')}}
\]

\[
< \frac{4(t''-t')}{\varepsilon^2};
\]

and since \( \xi_3 \) again may be shown to have the same upper bound, we have for \( Z(t) \) that

\[
\xi = \xi_1 + \xi_2 + \xi_3 < \frac{24(t''-t')}{\varepsilon^2}.
\]

Then from (3.5) and (3.6) we see that the lemma is verified if we take

\[
\phi = \max \left\{ 16(c+1), 24 \right\}.
\]

If \( \Lambda \) is any finite subset of \( T \) (which we always take to include
the points 0 and 1; let $M = M(A)$ be determined by

\[(3.7) \quad \frac{1}{M} = \frac{1}{2} \min_{\tau, \tau' \in \Lambda} |\tau - \tau'|.
\]

Then if $f(t)$ is any function defined for $t \in T$, and $m > M(A)$, we define

\[(3.8) \quad A_m(t;f,A) = \begin{cases} f(t) & \min_{\tau \in \Lambda} |t - \tau| \geq \frac{1}{m} \\
\sup f(t) & |t - \tau'| < \frac{1}{m} \end{cases}
\]

and

\[(3.9) \quad B_m(t;f,A) = \begin{cases} f(t) & \min_{\tau \in \Lambda} |t - \tau| \geq \frac{1}{m} \\
\inf f(t) & |t - \tau'| < \frac{1}{m} \end{cases}
\]

With these definitions we have

**Lemma 3.6.** If $a(t)$ and $b(t)$ are defined for $t \in T$, and $m > M(A)$, and $Q > Q^* A(t), b(t)$ defined by (2.29), then

\[
|\beta_n A(t) - Q, b(t) + Q, T - \beta_n A_m(t; a, A) - Q, B_m(t; b, A) + Q, T| < \lambda m^{-\frac{1}{3} + n^{-\frac{1}{2}}}
\]

and

\[
|\beta A(t) - Q, b(t) + Q, T - \beta A_m(t; a, A) - Q, B_m(t; b, A) + Q, T| < \lambda m^{-\frac{1}{3}}
\]

where $\lambda = \lambda(A, Q - Q^*)$ but does not depend on the forms of $a(t)$ and $b(t)$.

**Proof.** First, since $A_m(t; a, A) > a(t)$ and $B_m(t; b, A) < b(t)$ for $t \in T$, certainly
\[ \beta_n \left( a(t) - Q, b(t) + Q, T \right) - \beta_n \left( A(t; a, \lambda) - Q, B(t; b, \lambda) + Q, T \right) \geq 0. \]

Now let \( \Lambda = \left\{ t_0, t_1, \ldots, t_{k+1} \right\} \) where \( 0 = t_0 < t_1 < \ldots < t_{k+1} = 1. \)

Then \( A_m(t; a, \Lambda) = a(t) \) and \( B_m(t; b, \Lambda) = b(t) \) unless \( t \in I_{im} \) for some \( i, \)

\[ 0 \leq i \leq k+1, \quad \text{where} \quad I_{im} = \left\{ t : t \in T, |t - t_i| < \frac{1}{m} \right\}. \]

Let also

\[ A_m(t_i; a, \Lambda) = \sup_{t \in I_{im}} a(t) = a_{im} \quad \text{and} \quad B_m(t_i; b, \Lambda) = \inf_{t \in I_{im}} b(t) = b_{im}, \]

\[ 0 \leq i \leq k+1; \quad \text{and let} \quad Q - Q^* \left( a(t), b(t) \right) = c > 0. \]

Then

\[ \beta_n \left( a(t) - Q, b(t) + Q, T \right) - \beta_n \left( A_m(t; a, \Lambda) - Q, B_m(t; b, \Lambda) + Q, T \right) = \Pr \left\{ \bigcup_{i=0}^{k+1} E_{im} \right\} \]

where the event

\[ E_{im} = \left\{ a(t) - Q \leq Z_n(t) \leq b(t) + Q, \quad t \in I_{im} \right\}. \]

but \( \inf_{t \in I_{im}} Z_n(t) < a_{im} - Q \) or \( \sup_{t \in I_{im}} Z_n(t) > b_{im} + Q \),

and hence

\[ \beta_n \left( a(t) - Q, b(t) + Q, T \right) - \beta_n \left( A_m(t; a, \Lambda) - Q, B_m(t; b, \Lambda) + Q, T \right) \]

\[ \leq \sum_{i=0}^{k+1} A_{im} + \sum_{i=0}^{k+1} B_{im} \]

where

\[ A_{im} = \Pr \left\{ Z_n(t) \geq a(t) - Q, t \in I_{im}; \quad \text{but} \quad \inf_{t \in I_{im}} Z_n(t) < a_{im} - Q \right\} \]

and

\[ B_{im} = \Pr \left\{ Z_n(t) \leq b(t) + Q, t \in I_{im}; \quad \text{but} \quad \sup_{t \in I_{im}} Z_n(t) > b_{im} + Q \right\} \]

\[ \leq \Pr \left\{ \inf_{t \in I_{im}} Z_n(t) \leq \inf_{t \in I_{im}} b(t) + Q, \quad \sup_{t \in I_{im}} Z_n(t) > b_{im} + Q \right\} \]
\[
= \text{pr} \left\{ \inf_{t \in I_{im}} Z_n(t) \leq b_{im} + Q, \sup_{t \in I_{im}} Z_n(t) > b_{im} + Q, \right. \\
\left. \sup_{t \in I_{im}} Z_n(t) - \inf_{t \in I_{im}} Z_n(t) > c m - \frac{1}{3} \right\} \\
+ \text{pr} \left\{ \inf_{t \in I_{im}} Z_n(t) \leq b_{im} + Q, \sup_{t \in I_{im}} Z_n(t) > b_{im} + Q, \right. \\
\left. \sup_{t \in I_{im}} Z_n(t) - \inf_{t \in I_{im}} Z_n(t) \leq c m - \frac{1}{3} \right\} \\
\leq \text{pr} \left\{ \sup_{t \in I_{im}} Z_n(t) - \inf_{t \in I_{im}} Z_n(t) > c m - \frac{1}{3} \right\} \\
+ \text{pr} \left\{ \sup_{t \in I_{im}} Z_n(t) \leq b_{im} + Q + c m - \frac{1}{3}, \right. \\
\left. \inf_{t \in I_{im}} Z_n(t) > b_{im} + Q - c m - \frac{1}{3} \right\} \\
\leq \text{pr} \left\{ \sup_{t \in I_{im}} Z_n(t) - \inf_{t \in I_{im}} Z_n(t) > c m - \frac{1}{3} \right\} \\
+ \text{pr} \left\{ b_{im} + Q - c m - \frac{1}{3} < Z_n(t_i) \leq b_{im} + Q + c m - \frac{1}{3} \right\}. \\
\]

By Lemma 3.5,
\[
\text{pr} \left\{ \sup_{t \in I_{im}} Z_n(t) - \inf_{t \in I_{im}} Z_n(t) > c m - \frac{1}{3} \right\} < \frac{2 \phi m}{c^2}. \\
\]

Consider the case where \( i = 0 \); then \( t_i = t_0 = 0 \). Now
\[
Q = Q^* \left[ a(t) \right]_m b(t) + c > -B_{m}(O;b;A) + c = -b_{0m} + c, \\
or \ b_{0m} + Q > c, \text{ and so}
\]
\[
pr \left\{ b_{0m} + Q - cm - \frac{1}{3} < Z_n(0) \leq b_{0m} + Q - cm + \frac{1}{3} \right\} \leq pr \left\{ Z_n(0) > b_{0m} + Q - cm + \frac{1}{3} \right\} \\
\leq \frac{0}{0}.
\]

and the same obviously holds for \( i = k+1 \). But for \( 1 \leq i \leq k \) we may write

\[
pr \left\{ b_{im} + Q - cm - \frac{1}{3} < Z_n(t_i) \leq b_{im} + Q + cm - \frac{1}{3} \right\} \\
= pr \left\{ \frac{b_{im} + Q - cm}{\sqrt{t_i(1-t_i)}} - \frac{1}{3} < \frac{\sqrt{n}F^{-1}(t_i) - nt_i}{\sqrt{nt_i(1-t_i)}} \leq \frac{b_{im} + Q + cm}{\sqrt{t_i(1-t_i)}} - \frac{1}{3} \right\}
\]

where \( \sqrt{n}F^{-1}(t_i) \sim Binomial(n,t_i) \); and then by the Berry-Esseen Uniform Central Limit Theorem \( \mathcal{L}1 \) we have this probability not greater than

\[
(3.10) \left[ \phi\left( \frac{b_{im} + Q - cm}{\sqrt{t_i(1-t_i)}} - \frac{1}{3} \right) - \phi\left( \frac{b_{im} + Q - cm}{\sqrt{t_i(1-t_i)}} + \frac{1}{3} \right) \right] + \frac{c_0}{\sqrt{n}} \left[ \frac{t_i^2 + (1-t_i)^2}{\sqrt{t_i(1-t_i)}} \right]
\]

where \( c_0 \) is an absolute constant. Let \( \gamma(A) = \max_{1 \leq i \leq k} \frac{1}{\sqrt{t_i(1-t_i)}} \), then, since \( \frac{d}{dx} \phi(x) \leq \frac{1}{\sqrt{2\pi}} \), the probability above is not greater than

\[
(\gamma c) \sqrt{\frac{2}{\pi}} \frac{1}{3} + (\gamma c_0) \frac{1}{2}
\]

and hence

\[
\sum_{i=0}^{k+1} B_{im} \leq \left[ \frac{2\phi(k+2)}{c^2} + \gamma c \sqrt{\frac{2}{\pi}} \right] m - \frac{1}{3} + (\gamma c_0) n - \frac{1}{2}.
\]
The same upper bound clearly holds for \( \sum_{i=0}^{k+1} A_{im} \), and hence the lemma is verified for \( Z_n(t) \) if

\[
\lambda(A,c) = 2 \max \left\{ \frac{2\Phi(k+2)}{c^2} + ky^{c/2}, ky_{0}^{c} \right\} .
\]

The proof for the process \( Z(t) \) obviously proceeds in exactly the same manner, except that the second term in (3.10) does not appear.

**Theorem 3.7.** If the two sequences of functions \( \{a_n(t)\} \) and \( \{b_n(t)\} \) are such that for some finite subset \( \Lambda \) of \( T \) and for every sufficiently large \( m \) the corresponding sequences of functions \( \{A_m(t;a_n,\Lambda)\} \) and \( \{B_m(t;b_n,\Lambda)\} \) converge uniformly to \( A_m(t;a,\Lambda) \) and \( B_m(t;b,\Lambda) \) respectively, where \( a(t) \) and \( b(t) \) are any two piecewise-continuous functions, then

\[
\lim_{n \to \infty} \beta\sqrt{\text{a}_n(t) - Q_n, b_n(t) + Q_n, T} = \beta\sqrt{\text{a}(t) - Q, b(t) + Q, T} = \beta^*(Q)
\]

for \( Q \neq Q^*\sqrt{\text{a}(t), b(t)} \).

**Proof.** Consider first the case where \( Q < Q^*\sqrt{\text{a}(t), b(t)} \) defined by (2.29): say \( Q = Q^* - c \), where \( c > 0 \). There will be no loss in generality in supposing that \( Q^* = \lim_{\delta \to 0} \sup_{0 \leq t \leq \delta} a(t) \). Given any \( \eta > 0 \), choose \( m = m(\eta) \) to be the least integer greater than

\[
\max \left\{ M(\Lambda), \frac{2\Phi}{\eta^c} \right\},
\]

where \( \phi \) is the constant of Lemma 3.5. Then because the point \( 0 \) is in \( \Lambda \) (as we always assume)

\[
A_m(0;a,\Lambda) = \sup_{0 \leq t \leq \frac{1}{m}} a(t) \geq \lim_{\delta \to 0} \sup_{0 \leq t \leq \delta} a(t) = Q^*
\]

and by the uniform convergence of \( \{A_m(t;a_n,\Lambda)\} \) to \( A_m(t;a,\Lambda) \) there is an \( n_0 = n_0(\sqrt{m(\eta)}) \) such that for \( n \geq n_0 \) we have both
\[ A_{m}(O; a_{n}, \Lambda) \geq A_{m}(O; a, \Lambda) - \frac{c}{3} \]

and

\[ Q_{n} \leq Q + \frac{c}{3} \]

Then

\[ \sup_{0 \leq t < \frac{1}{m}} a_{n}(t) - Q_{n} = A_{m}(O; a_{n}, \Lambda) - Q_{n} \geq Q^{*} - \frac{c}{3} - Q - \frac{c}{3} = \frac{c}{3} \]

so

\[ \beta_{n}(a_{n}(t) - Q_{n}, b_{n}(t) + Q_{n}, T) = \Pr \left\{ a_{n}(t) - Q_{n} \leq Z_{n}(t) \leq b_{n}(t) + Q_{n}, teT \right\} \]

\[ \leq \Pr \left\{ Z_{n}(t) \geq a_{n}(t) - Q_{n}, 0 \leq t < \frac{1}{m} \right\} \]

\[ \leq \Pr \left\{ \sup_{0 \leq t < \frac{1}{m}} Z_{n}(t) \geq \sup_{0 \leq t < \frac{1}{m}} a_{n}(t) - Q_{n} \right\} \]

\[ \leq \Pr \left\{ \sup_{0 \leq t < \frac{1}{m}} Z_{n}(t) > \frac{c}{3} \right\} ; \]

but since \( Z_{n}(O) = 0 \) with probability 1, and hence \( \inf_{0 \leq t < \frac{1}{m}} Z_{n}(t) \leq 0 \) with probability 1,

\[ \Pr \left\{ \sup_{0 \leq t < \frac{1}{m}} Z_{n}(t) > \frac{c}{3} \right\} \leq \Pr \left\{ \sup_{0 \leq t < \frac{1}{m}} Z_{n}(t) - \inf_{0 \leq t < \frac{1}{m}} Z_{n}(t) > \frac{c}{3} \right\} \]

\[ < \frac{\phi}{mc^{2}} \]

using Lemma 3.5; and thus by the definition of \( m(\eta) \)

\[ \beta_{n}(a_{n}(t) - Q_{n}, b_{n}(t) + Q_{n}, T) < \eta . \]

Then it follows that for \( Q < Q^{*} \)

\[ \lim_{n \to \infty} \beta_{n}(a_{n}(t) - Q_{n}, b_{n}(t) + Q_{n}, T) = 0 ; \]

but, by statement (ii) of Theorem 2.6,

\[ \beta(a(t) - Q, b(t) + Q, T) = 0 ; \]

and thus the conclusion of this theorem is verified for \( Q < Q^{*} \).
Now consider the case where $Q > Q^*$: say $Q = Q^* + c$, where $c > 0$. Then, for any $m > M(\Lambda)$,

$$
\left| \beta_n[\bar{A}(t) - Q_n, b_n(t) + Q_n, T] - \beta[\bar{a}(t) - Q, b(t) + Q, T] \right|
\leq \left| \beta_n[\bar{A}(t) - Q_n, b_n(t) + Q_n, T] - \beta_n[\tilde{A}(t; a_n, \Lambda) - Q_n, B_m(t; b_n, \Lambda) + Q_n, T] \right|
+ \left| \beta_n[\tilde{A}(t; a_n, \Lambda) - Q_n, B_m(t; b_n, \Lambda) + Q_n, T] - \beta[\tilde{A}(t; a, \Lambda) - Q, B_m(t; b, \Lambda) + Q, T] \right|
+ \left| \beta[\tilde{A}(t; a, \Lambda) - Q, B_m(t; b, \Lambda) + Q, T] - \beta[a(t) - Q, b(t) + Q, T] \right|.
$$

It is not difficult to verify that for any pair of functions $f_1(t)$ and $f_2(t)$ defined for $t \in T$

$$
Q^*[\tilde{A}(t; f_1, \Lambda), B_m(t; f_2, \Lambda)] \geq Q^*[\tilde{A}(t; f_1', \Lambda), B_m(t; f_2', \Lambda)].
$$

Furthermore,

$$
\lim_{m \to \infty} Q^*[\tilde{A}(t; f_1, \Lambda), B_m(t; f_2, \Lambda)] = Q^*[\tilde{A}(t), f_2(t)],
$$

and hence, for any $\eta > 0$, we can choose an $m(\eta)$ so large that

(i) $m > M(\Lambda)$

(ii) $\max\left\{ \frac{1}{\lambda(\Lambda, c)}, \frac{1}{\lambda(\Lambda, \eta)} \right\} < \frac{\eta}{4}$

(iii) $Q^*[\bar{A}(t, b(t)] \geq Q^*[\bar{A}(t; a, \Lambda), B_m(t; b, \Lambda)] - \frac{c}{4}.$

Next, since $\{A_m(t; a_n, \Lambda)\}$ converges uniformly to $A_m(t; a, \Lambda)$ and $\{B_m(t; b_n, \Lambda)\}$ converges uniformly to $B_m(t; b, \Lambda)$ it follows that

$$
\lim_{n \to \infty} Q^*[\tilde{A}(t; a_n, \Lambda), B_m(t; b_n, \Lambda)] = Q^*[\tilde{A}(t; a, \Lambda), B_m(t; b, \Lambda)].
$$
and hence we can choose an $n_0(m)$ so large that for $n \geq n_0$

\begin{align*}
(i) \quad n^{\frac{1}{2}} \lambda(\Lambda, \frac{C}{4}) &< \frac{n}{4} \\
(ii) \quad |Q - Q_n| &< \frac{C}{4} \\
(iii) \quad Q^* \min_{m} \{t; a_n, \Lambda, B_m(t; b_n, \Lambda, \Lambda) \} &> Q^* \min_{m} \{t; a_n, \Lambda, B_m(t; b_n, \Lambda, \Lambda) \} - \frac{C}{4} \\
(iv) \quad |\beta_n(t; a_n, \Lambda, Q_n, B_m(t; b_n, \Lambda) + Q_n, T) - \beta_n(t; a_n, \Lambda, Q_n, B_m(t; b_n, \Lambda) + Q_n, T)| &< \frac{n}{4}
\end{align*}

where (iv) follows from Theorem 3.3 and (ii) and (iii) imply that

\begin{align*}
Q_n &> Q - \frac{C}{4} = Q^* \min_{m} \{t; a(t), b(t), \Lambda, \Lambda \} + \frac{3C}{4} \\
&> Q^* \min_{m} \{t; a(t), B_m(t; b, \Lambda, \Lambda) \} + \frac{C}{2} \\
&> Q^* \min_{m} \{t; a_n, \Lambda, B_m(t; b_n, \Lambda, \Lambda) \} + \frac{C}{4} \\
&> Q^* \min_{m} \{t; a_n(t), b_n(t), \Lambda, \Lambda \} + \frac{C}{4} .
\end{align*}

But then from (3.11) and (3.12) and Lemma 3.6 we have

\begin{align*}
|\beta_n(t; a_n(t), b_n(t) + Q_n, T) - \beta_n(t; a(t), b(t) + Q, T)| &< \eta ,
\end{align*}

where $\eta > 0$ is arbitrary, and thus the proof of the theorem is complete.

**Corollary 3.8.** If $(H_\alpha, \{F_n\}) \in \mathcal{F}$ and the sequence of functions \(\{S_n(t; H, G_n)\}\) is such that for some finite subset $\Lambda$ of $T$ and every sufficiently large $m$ the sequence of functions \(\{B_m(t; S_n, \Lambda)\}\) converges uniformly to $B_m(t; b, A)$, where $b(t)$ is piecewise-continuous in $T$, then

\begin{align*}
P^+(H_\alpha, \{F_n\}, \alpha) = 1 - \beta_{-\infty, b(t) + R, T, T}.
\end{align*}
for $R \not\in \mathbb{Q}^* \cup (-\infty, b(t)]$; and if also the sequence $\{A_m(t; S_n, A)\}$ converges uniformly to $A_m(t; a, A)$, where $a(t)$ is piecewise-continuous in $T$, then

$$P(H_i, \{G_n\}, \alpha) = 1 - \beta \sqrt{a(t)} - Q(t) + Q(T)$$

for $Q \not\in \mathbb{Q}^* \cup (-\infty, a(t), b(t)]$.

Since $G_n \in \mathcal{F}^*$, $G_n^{-1}(t)$ is monotonically increasing in $t$ and hence cannot have more than countably many points of discontinuity; thus since $F \in \mathcal{F}^*$ also,

$$S_n(t) = \sqrt{n} \left( F \Big| G_n^{-1}(t) \right) - t$$

is necessarily piecewise-continuous in $T$. But if the sequence $\{S_n(t)\}$ converges uniformly to some function $S(t)$, then certainly

$$\Gamma(S) \subset \bigcup_{n=1}^{\infty} \Gamma(S_n),$$

which latter is a countable set, and thus $S(t)$ is piecewise-continuous in $T$ also. This shows that Corollary 3.8 actually includes Corollary 3.4 as a special case.
CHAPTER IV

BOUNDS ON THE POWER

Let $J^*$ be any subclass of the class $J$ defined by (1.24). Then we define the asymptotic least upper bound on the power in $J^*$ of the one-sided Kolmogorov test to be

$$U^\leftarrow J^*, \alpha J = \lim_{n \to \infty} \sup (H, \{G_n\}) \in J^* \quad P_n^+(H, G_n, \alpha).$$

Other asymptotic bounds are defined similarly:

$$U^\leftarrow J^*, \alpha J = \lim_{n \to \infty} \sup (H, \{G_n\}) \in J^* \quad P_n(H, G_n, \alpha),$$

$$L^+ J^*, \alpha J = \lim_{n \to \infty} \inf (H, \{G_n\}) \in J^* \quad P_n^+(H, G_n, \alpha),$$

$$L^\leftarrow J^*, \alpha J = \lim_{n \to \infty} \inf (H, \{G_n\}) \in J^* \quad P_n(H, G_n, \alpha).$$

We shall deal in particular with the following subclasses of $J$:

a) $J(\Delta) = \left\{ (H, \{G_n\}) \in J \right\}$ such that

$$\sqrt{n} \ D(H, G_n) = \Delta, \ n=1,2,...$$

b) $J^+(\Delta, \tau) = \left\{ (H, \{G_n\}) \in J(\Delta) \right\}$ such that for some $x$

$$H(x) = \tau = G_n(x) - \frac{\Delta}{\sqrt{n}}, \ n=1,2,...$$

(4.5)

c) $J^+(\Delta) = \bigcup \frac{1}{\sqrt{n}} \bigcup \bigcup J^+(\Delta, \tau)$

d) $C^+(\Delta, \tau) = \left\{ (H, \{G_n\}) \in J^+(\Delta, \tau) \right\}$ such that

$$\sqrt{n} \ D^+(H, G_n) = 0, \ n=1,2,...$$

e) $C^+(\Delta) = \bigcup \frac{1}{\sqrt{n}} \bigcup \bigcup C^+(\Delta, \tau)$. 
Pairs \((H, G_n)\) abstracted from members of \(C^+(\Delta)\) have been called "stochastically comparable" by Birnbaum and Scheuer \(\int_4^\gamma\); it is for distinguishing such pairs that a one-sided test of fit seems most reasonable, although the one-sided Kolmogorov test is suitable for the larger class \(J^+(\Delta)\). Finally, define also, for any subclass \(J^+\) of \(J\),

\[
\mathcal{J}_n^* = \{S(t) : S(t) = S_n(t; H, G_n) \text{ for some } (H, \{G_n\}) \in J^+\}.
\]

The procedure by which we shall find asymptotic bounds on the power of the one-sided test is justified by the following simple

**Lemma 4.1.** Let \(J^+\) be any subclass of \(J\). Suppose that for every sufficiently large \(n\)

\[
\inf_{S \in \mathcal{J}_n^*} S(t) = S_{1n}(t) \in J^+ \implies \mathcal{J}_n^* \]

then

\[
U^+ = 1 - \lim_{n \to \infty} \beta_n \Delta, S_{1n}(t) + R_n, T_n.
\]

Similarly, if for every sufficiently large \(n\)

\[
\sup_{S \in \mathcal{J}_n^*} S(t) = S_{2n}(t) \in J^+ \implies \mathcal{J}_n^* \]

then

\[
L^+ = 1 - \lim_{n \to \infty} \beta_n \Delta, S_{2n}(t) + R_n, T_n.
\]

**Proof.** Obvious.

Then the asymptotic least upper bound on the power of the one-sided test in any of the classes defined by (4.5) is given by

**Theorem 4.2.** Let \(J^+\) be any of the classes defined by (4.5). Then

\[
U^+ = \begin{cases} 
\frac{e^{-2(R-\Delta)^2}}{\Delta < R} \\
1 \quad \Delta \geq R
\end{cases}
\]
Proof. If \((H, \{G_n\}) \in \mathcal{J}^*\) then \(\sqrt{n} D(H, G_n) = \Delta, n = 1, 2, \ldots\), since all the classes defined by (4.5) are subclasses of \(\mathcal{J}(\Delta)\). Hence \(S_n(t; H, G_n) \geq -\Delta\), and if we let

\[
S_{ln}(t) = \begin{cases} 
-t \sqrt{n} & 0 \leq t \leq \frac{\Delta}{\sqrt{n}} \\
\frac{\Delta}{\sqrt{n}} & \frac{\Delta}{\sqrt{n}} \leq t \leq 1 
\end{cases}
\]

then certainly \(S_{ln}(t) \leq \inf_{S \in \mathcal{J}} S(t)\). But for every \(a < b\) define

\[
(4.7) \quad U(x; a, b) = \begin{cases} 
x-a & x \leq a \\
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & x \geq b 
\end{cases}
\]

then let \(H(x) = U(x; 0, 1)\) and \(G_n(x) = U(x; -\frac{\Delta}{\sqrt{n}}, 1 - \frac{\Delta}{\sqrt{n}}), n = 1, 2, \ldots\). Then it is easily verified that \((H, \{G_n\}) \in \mathcal{J}^*\) and that \(S_n(t; H, G_n) = S_{ln}(t) \in \mathcal{J}^*\). (The pair \((H, \{G_n\})\) is given by Birnbaum in \(\mathcal{J}^*\), where he exhibits an exact closed expression for the power of the one-sided test of \(H\) against \(G_n\) based on a sample of size \(n\).)

Hence from Lemma 4.1 we have

\[
U^\mathcal{J}^* \alpha T = 1 - \lim_{n \to \infty} \beta_n <_{\mathcal{J} \alpha} S_{ln}(t) + R_n, T J .
\]

But the sequence of functions \(\{S_{ln}(t)\}\) satisfies the conditions of Corollary 3.8 with \(\Lambda = \{0, 1\}, b(t) = -\Delta_t \in T\); hence

\[
\lim_{n \to \infty} \beta_n <_{\mathcal{J} \alpha} S_{ln}(t) + R_n, T J = \beta <_{\mathcal{J}, -\Delta+R}, T J ,
\]

and the numerical evaluation follows from (2.40).

We may compare this asymptotic least upper bound with Chapman's statement in \(\mathcal{J}^*\) that for large \(n\) and \(\Delta < R_n\) (in our notation)
\[ P_n^+(H_n \{ g_n \}, \alpha) = e^{-2(R_n - \Delta)^2}. \]

The following asymptotic greatest lower bound holds little interest in itself and is given mainly for the sake of completeness:

**Corollary 4.3.**

\[ L^+ \int \alpha = e^{-2(R+\Delta)^2}. \]

**Proof.** Replace \( S_{2n}(t) \) in the proof of Theorem 4.2 by

\[
S_{2n}(t) = \begin{cases} 
\Delta & 0 \leq t \leq 1 - \frac{\Delta}{\sqrt{n}} \\
\sqrt{n} (1-t) & 1 - \frac{\Delta}{\sqrt{n}} \leq t \leq 1,
\end{cases}
\]

which corresponds to the pair \( (H_n \{ g_n \}) \) where \( H(x) = U(x; 0, 1) \) and \( G_n(x) = U(x; \frac{\Delta}{\sqrt{n}}, 1 + \frac{\Delta}{\sqrt{n}}) \); the proof then proceeds in a similar manner.

**Theorem 4.4.** For \( 0 < \tau < 1 \),

\[ L^+ \int \alpha = 1 - \text{pr} \left\{ \sup_{t \in \mathbb{T}} Z(t) \leq R+\Delta, Z(\tau) \leq R-\Delta \right\} \]

\[ = 1 - \Phi \left( \frac{R - \Delta}{\sqrt{\tau(1-\tau)}} \right) - \Phi \left( \frac{-R - 3\Delta}{\sqrt{\tau(1-\tau)}} \right) \]

\[ + e^{-2(R+\Delta)^2} \left[ \Phi \left( \frac{-R(1-2\tau) - 2\Delta(1-\tau)}{\sqrt{\tau(1-\tau)}} \right) \right. \]

\[ + \left. \Phi \left( \frac{-R(2\tau-1) - 2\Delta \tau}{\sqrt{\tau(1-\tau)}} \right) \right], \]

\[ L^+ \int \alpha = 1 - \text{pr} \left\{ \sup_{t \in \mathbb{T}} Z(t) \leq R, Z(\tau) \leq R - \Delta \right\} \]
\[ 1 - \frac{R - \Delta}{\sqrt{\tau(1-\tau)}} - \frac{-R-\Delta}{\sqrt{\tau(1-\tau)}} \]

\[ + e^{-2R^2} \frac{-R(l-2\tau)-\Delta}{\sqrt{\tau(1-\tau)}} \]

\[ + e^{-2R^2} \frac{-R(2\tau-1)-\Delta}{\sqrt{\tau(1-\tau)}} \]

and

\[(4.10) \quad 1 - \frac{R - \Delta}{\sqrt{\tau(1-\tau)}} \leq L^+ \int_{\Delta,\tau}^\infty \alpha \]

\[ \leq L^+ \int_{\Delta,\tau}^\infty \alpha \]

\[ \leq 1 - \frac{R - \Delta}{\sqrt{\tau(1-\tau)}} + \alpha. \]

**Proof.** Let

\[ S_{3n}(t;\tau) = \begin{cases} \min\left[ \Delta, \sqrt{n}(\tau-t) \right] & 0 \leq t < \tau + \frac{\Delta}{\sqrt{n}}, \\ \min\left[ \Delta, \sqrt{n}(1-t) \right] & \tau + \frac{\Delta}{\sqrt{n}} \leq t \leq 1; \end{cases} \]

then it is easily verified that \( S_{3n}(t;\tau) \geq \sup_S \int_{\Delta,\tau}^\infty S(t). \)

But if we take

\[(4.11) \quad H(x) = \begin{cases} 0 & x \leq 0, \\ x & 0 \leq x \leq \tau, \\ \tau & \tau \leq x \leq 1 + \tau, \\ x-1 & 1 + \tau \leq x \leq 2, \\ 1 & x \geq 2. \end{cases} \]

and
\[ G_n(x) = \begin{cases} 
0 & x \leq \frac{\Delta}{\sqrt{n}} \\
\frac{x - \Delta}{\sqrt{n}} & \frac{\Delta}{\sqrt{n}} \leq x \leq \tau + \frac{2\Delta}{\sqrt{n}} \\
\tau + \frac{\Delta}{\sqrt{n}} & \tau + \frac{2\Delta}{\sqrt{n}} \leq x \leq 1 + \tau + \frac{2\Delta}{\sqrt{n}} \\
x - 1 - \frac{\Delta}{\sqrt{n}} & 1 + \tau + \frac{2\Delta}{\sqrt{n}} \leq x \leq 2 + \frac{\Delta}{\sqrt{n}} \\
1 & x \geq 2 + \frac{\Delta}{\sqrt{n}} 
\end{cases} \]

so that \( H(1) = \tau = G_n(1) - \frac{\Delta}{\sqrt{n}}, n=1,2,\ldots, \) then \((H, \{G_n\}) \in J^+(\Delta, \tau)\).

and \( S_n(t; H, G_n) = S_{3n}(t; \tau) \in J_n^{\infty} J^+(\Delta, \tau) \). Hence by Lemma 4.1

\[ L^\infty \bigcap J^+(\Delta, \tau), \alpha \bigcap = 1 - \lim_{n \to \infty} \beta_n \bigcap \beta_{3n}(t; \tau) + R_n, T \bigcap. \]

But the sequence of functions \( \{S_{3n}(t; \tau)\} \) satisfies the conditions of Corollary 3.8 with \( \Lambda = \{0, \tau, 1\} \) and

\[ b(t) = \begin{cases} 
- \Delta & t = \tau \\
+ \Delta & t \neq \tau, t \neq T. 
\end{cases} \]

Hence \( L^\infty \bigcap J^+(\Delta, \tau), \alpha \bigcap = 1 - \beta \bigcap \beta_{3n}(t; \tau) + R_n, T \bigcap, \) which is the first part of (4.8). The explicit formula is obtained by substituting the values \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = -b = 1 \) into (2.43).

The argument for (4.9) is the same, except that we take

\[ S_{4n}(t; \tau) = \begin{cases} 
\min \bigcap 0, \sqrt{n} (\tau - t) \bigcap & 0 \leq t \leq \tau + \frac{\Delta}{\sqrt{n}} \\
\min \bigcap 0, \sqrt{n} (1 - t) \bigcap & \tau + \frac{\Delta}{\sqrt{n}} \leq t \leq 1. 
\end{cases} \]
H(x) as before,

\[
G_n(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{x}{\sqrt{n}} & 0 \leq x \leq t + \frac{\Delta}{\sqrt{n}} \\
t + \frac{\Delta}{\sqrt{n}} & 0 \leq x \leq 1 + t + \frac{\Delta}{\sqrt{n}} \\
x - 1 & 1 + t + \frac{\Delta}{\sqrt{n}} \leq x \leq 2 \\
1 & x \geq 2,
\end{cases}
\]

and

\[
b(t) = \begin{cases} 
-\Delta & t = t \\
0 & t \in T, t \neq t 
\end{cases}
\]

then we must substitute \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0, b = -1 \), into (2.43).

Finally, (4.10) follows easily from (4.8) and (4.9).

Birnbaum and Scheuer \( \mathcal{L}_n \) give an exact closed expression for the lower bound in \( C^*(\Delta, t) \) for any sample size \( n \); this was approximated for large \( n \) by Chapman \( \mathcal{L}_6 \).

**Theorem 4.5.**

1. \( L^+ [ J^+ (\Delta), \alpha ] = e^{-2(R+\Delta)^2} \) for \( \Delta < R \)
2. \( L^+ [ J^+ (\Delta), \alpha ] \geq 1 - \phi (2R-2\Delta) \) for \( \Delta > R \)
3. \( L^+ [ J^+ (\Delta), \alpha ] \leq 1 - \phi (2R-2\Delta) - \phi (-2R-6\Delta) + 2e^{-2(R+\Delta)^2} \phi (-4\Delta) \)
4. \( L^+ [ J^+ (\Delta), \alpha ] = e^{-2R^2} = \alpha \) for \( \Delta < R \)
(v) \[ L^+ \bigcap C^+(\Delta), \alpha \bigcap \geq 1 - \phi(2R - 2\Delta) \text{ for } \Delta > R \]
(vi) \[ L^+ \bigcap C^+(\Delta), \alpha \bigcap \leq 1 - \phi(2R - 2\Delta) - \phi(-2R - 2\Delta) + 2e^{-2R^2} \phi(-2\Delta) \leq 1 - \phi(2R - 2\Delta) + \alpha \text{ for } \Delta \geq R. \]

Proof. Since \( J^+(\Delta, \tau) \subset J^+(\Delta) \subset J(\Delta) \) for every \( \tau, 0 < \tau < 1 \), we have
\[
L^+ \bigcap J^+(\Delta, \tau), \alpha \bigcap \geq L^+ \bigcap J^+(\Delta), \alpha \bigcap \\
\geq L^+ \bigcap J(\Delta), \alpha \bigcap \\
= e^{-2(R+\Delta)^2};
\]
but, from (4,8),
\[
\lim_{\tau \to 0} L^+ \bigcap J^+(\Delta, \tau), \alpha \bigcap = e^{-2(R+\Delta)^2}
\]
for \( \Delta < R \); hence (i). Next, if \( (H, \{ G_n \}) \in C^+(\Delta) \) then \( G_n(x) \geq H(x), \)
\(-\infty < x < \infty \), so that \( P_n^+(H, G_n, \alpha) \geq P_n^+(H, H, \alpha) = \alpha \) for \( n = 1, 2, \ldots \)
by Theorem 1.1, and thus \( L^+ \bigcap C^+(\Delta), \alpha \bigcap \geq \alpha \) from (4,3); but since
\[
C^+(\Delta, \tau) \subset C^+(\Delta) \text{ for } 0 < \tau < 1 \text{ certainly } L^+ \bigcap C^+(\Delta, \tau), \alpha \bigcap \\
\geq L^+ \bigcap C^+(\Delta), \alpha \bigcap; \text{ and, from (4,9)},
\]
\[
\lim_{\tau \to 0} L^+ \bigcap C^+(\Delta, \tau), \alpha \bigcap = e^{-2R^2} = \alpha
\]
for \( \Delta < R \); hence (iv). We can obtain (iii) and (vi) by putting
\( \tau = \frac{1}{2} \) in Theorem 4.4. Also, since \( C^+(\Delta) \subset J^+(\Delta) \), (v) will
follow from (ii). Finally, we prove (ii) as follows: we have
\[
L^+ \bigcap J^+(\Delta), \alpha \bigcap = \lim_{n \to \infty} \left( \inf_{H, \{ G_n \}} \bigcap J^+(\Delta) \right) P_n^+(H, G_n, \alpha)
\]
and \( J^+(\Delta) = \bigcup_{0 < \tau < 1} J^+(\Delta, \tau) \). If \( (H, \{ G_n \}) \in J^+(\Delta, \tau) \) then for some \( x_0 \)
\[ H(x_0) = \tau = G_n(x_0) - \frac{\Delta}{\sqrt{n}}, \quad n=1,2,\ldots, \]

and from (1.21)

\[ P_n^+(H,G_n,\alpha) = \Pr \left\{ D_n^+ > R_n \right\} \quad (F = G_n) \]

\[ = \Pr \left\{ \sup_{-\infty < x < \infty} \sqrt{n} \left( H_n(x) - H(x) \right) > R_n \right\} \]

\[ \geq \Pr \left\{ \sqrt{n} \left( H_n(x_0) - H(x_0) \right) > R_n \right\} \]

\[ = \Pr \left\{ x > \tau_n + R_n \sqrt{n} \right\} \]

where \( x = nF_n(x_0) \) is Binomial \( (n, \tau + \frac{\Delta}{\sqrt{n}}) \). Write \( p = \tau + \frac{\Delta}{\sqrt{n}} \);

\[ \frac{\Delta}{\sqrt{n}} < p < 1. \]

Then

\[ P_n^+(H,G_n,\alpha) \geq 1 - \Pr \left\{ x \leq np - (\Delta - R_n) \sqrt{n} \right\} \]

\[ = 1 - \Pr \left\{ \frac{x - np}{\sqrt{np(1-p)}} \leq \frac{- (\Delta - R_n)}{\sqrt{p(1-p)}} \right\}. \]

Since \( \Delta > R \), we may choose \( n_0 \) so large that, for \( n \geq n_0, \Delta > R_n \).

Then by Chebyshev's inequality

\[ P_n^+(H,G_n,\alpha) \geq 1 - \frac{p(1-p)}{(\Delta - R_n)^2}; \]

and by the Berry-Esseen Uniform Central Limit Theorem

\[ P_n^+(H,G_n,\alpha) \geq 1 - \Phi \left( -\frac{\Delta - R_n}{\sqrt{p(1-p)}} \right) - c_o \left( \frac{p^2 + (1-p)^2}{\sqrt{p(1-p)}} \right) \]

where \( c_o \) is an absolute constant; hence
\[ L^+ \left( J^*(\Delta), \alpha J \right) \geq \lim_{n \to \infty} \inf_{\Delta < p \leq 1} \max \left\{ 1 - \frac{p(1-p)}{(\Delta - R_n)^2}, \right. \]
\[ \left. 1 - \phi \left( -\frac{\Delta - R_n}{\sqrt{p(1-p)}} \right) - \frac{c_0}{\sqrt{n}} \left( \frac{p^2 + (1-p)^2}{\sqrt{p(1-p)}} \right) \right\} \]

and it is not difficult to see that this yields (ii).

The procedure we use for finding asymptotic least upper bounds on the power of the two-sided Kolmogorov test is justified by Lemma 4.6. Let \( J^* \) be any subclass of \( J \). Suppose that there exist two continuous functions \( S_1(t) \) and \( S_2(t) \) defined for \( t \in T \) such that for every sufficiently large \( n \)

\[ S_1(t) \leq \inf_{S_n \in J_n^*} S(t) \leq \sup_{S_n \in J_n^*} S(t) \leq S_2(t); \]

and that there exist two disjoint dense countable subsets \( A = \{ a_1, a_2, \ldots \} \) and \( B = \{ b_1, b_2, \ldots \} \) of the open interval \((0, 1)\) such that for every positive integer \( k \) there is an \( n_0(k) \) so large that for \( n \geq n_0(k) \) there exists a function \( V_k \in J_n^* \) with

\[ V_k(0) = V_k(1) = 0, \quad V_k(a_i) = S_1(a_i), \quad 1 \leq i \leq k, \quad \text{and} \]
\[ V_k(b_i) = S_2(b_i), \quad 1 \leq i \leq k. \]

Then

\[ U_n^+ \left( J^*, \alpha J \right) = 1 - \beta \sqrt{S_2(t) - Q}, \quad S_1(t) + Q, \quad T \]

**Proof.** Let \( (H_i, \{ G_n \}) \) be a member of \( J^* \). Now by (1.37)

\[ P_n(H_i, G_n, \alpha) = 1 - \beta \sqrt{S_n(t; H_i, G_n) - Q_n}, \quad S_n(t; H_i, G_n) + Q_n, \quad T \]

but since for all \( (H_i, \{ G_n \}) \in J^* \) we have \( S_1(t) \leq S_n(t; H_i, G_n) \leq S_2(t) \)

for \( n \) sufficiently large, certainly
\[ \sup_{(H, \{G_n\}) \in J^*} F_n(H, G_n; \alpha) \leq 1 - \beta_n \sqrt{S_2(t)} - Q_n, S_1(t) + Q_n, T \]

and then, by Theorem 3.3, from the definition (4.2) we have

\[(4.13) \quad U^\ast J^*, \alpha J \leq 1 - \beta \sqrt{S_2(t)} - Q, S_1(t) + Q, T \]

But consider

\[ \beta_{\sqrt{V_k - Q}, V_k + Q, T} = \text{pr}\left\{ \frac{E[\sqrt{V_k - Q}, V_k + Q, T] \cap E[S_2 - Q, S_1 + Q, T]}{E[S_2 - Q, S_1 + Q, T]} \right\} \]

\[ + \text{pr}\left\{ \frac{E[\sqrt{V_k - Q}, V_k + Q, T] \cap E[S_2 - Q, S_1 + Q, T]}{E[S_2 - Q, S_1 + Q, T]} \right\} \]

\[ \leq \beta_{\sqrt{S_2 - Q}, S_1 + Q, T} \]

\[ + \text{pr}\left\{ \frac{E[\sqrt{V_k - Q}, \infty, T] \cap E[S_2 - Q, \infty, T]}{E[S_2 - Q, \infty, T]} \right\} \]

\[ + \text{pr}\left\{ \frac{E[\sqrt{V_k - Q}, \infty, T] \cap E[S_2 - Q, S_1 + Q, T]}{E[S_2 - Q, S_1 + Q, T]} \right\} \]

\[ = \beta_{\sqrt{S_2 - Q}, S_1 + Q, T} + \alpha_k + \beta_k, \]

say. Here

\[ \alpha_k = \text{pr}\left\{ Z(t) \geq V_k(t) - Q, \text{for some } t \in T \right\} \]

\[ \leq \text{pr}\left\{ Z(t) \geq V_k(b_1) - Q, \text{for some } t \in T \right\} \]

\[ = \text{pr}\left\{ Z(t) \geq S_2(b_1) - Q, \text{for some } t \in T \right\} \]

because \( V_k(b_1) = S_2(b_1), \) \( 1 \leq i \leq k. \) But then, by Theorem 2.6(i), since \( S_2(t) \) is continuous and \( B \) is dense in \( T, \) for any \( \eta > 0 \) there is a \( k_1(\eta) \) so large that for \( k \geq k_1(\eta) \) we have \( \alpha_k < \frac{\eta}{2}. \) Similarly, for \( k \geq k_2(\eta), \) say, \( \beta_k < \frac{\eta}{2}; \) and then, choosing \( k(\eta) \geq \max(k_1, k_2), \) we have

\[ \beta_{\sqrt{V_k - Q}, V_k + Q, T} \leq \beta_{\sqrt{S_2 - Q}, S_1 + Q, T} + \eta. \]

Next, if \( n \geq n_0(\eta) \) then \( V_k \in J_n, J^* J \); say

\[ V_k(t) = S_n(t; H, G_{kn}), \text{ where } (H, \{G_{kn}\}) \in J^*. \]
and thus
\[
\lim_{n \to \infty} P_n(H, G_n, \alpha) \geq P_n(H, G_{kn}, \alpha)
\]
and thus
\[
\lim_{n \to \infty} \left( \sup_{(H, \{G_n\})} P_n(H, G_n, \alpha) \right) \geq \lim_{n \to \infty} P_n(H, G_{kn}, \alpha)
\]
\[
\geq \lim_{n \to \infty} \left\{ 1 - \beta_n \sqrt{\frac{V_k}{2}} - Q, V_k + Q, T \right\}
\]
\[
= 1 - \beta V_k - Q, V_k + Q, T
\]
\[
> 1 - \beta S_2 - Q, S_1 + Q, T
\]
but since \( \eta \) is arbitrary we can conclude that
\[
(4.14) \quad \lim_{n \to \infty} \left( \sup_{(H, \{G_n\})} P_n(H, G_n, \alpha) \right) \geq 1 - \beta S_2(t) - Q, S_1(t) + Q, T
\]
Then (4.13) and (4.14) yield the lemma.

**Theorem 4.7.** Let \( J \) \* be any of the classes defined by (4.5abc); then
\[
(4.15) \quad \lim_{n \to \infty} \left( \sup_{(H, \{G_n\})} P_n(H, G_n, \alpha) \right) = 1 - \sqrt{\Delta} - Q, -\Delta + Q, T
\]
\[
= \begin{cases} 
1 & \text{if } \Delta \geq Q; \text{ and otherwise} \\
1 - \sum_{k=1}^{\infty} \frac{(-1)^k 2^{k-1} e^{-2k(2-\Delta)^2}}{k!} 
\end{cases}
\]
And let \( C \) \* be any of the classes defined by (4.5de); then
\[
(4.16) \quad \lim_{n \to \infty} \left( \sup_{(H, \{G_n\})} P_n(H, G_n, \alpha) \right) = 1 - \sqrt{\Delta} - Q, -\Delta + Q, T
\]
\[
= \begin{cases} 
1 & \text{if } \Delta \geq Q; \text{ and otherwise} \\
\sum_{k=1}^{\infty} \left\{ e^{-2k(2\Delta - Q)^2/(k-1)^2} + e^{-2k(2-\Delta)^2/(k-1)^2} \right. \\
\left. - 2^{2k(2\Delta - Q)^2/(k-1)^2} \right\}
\end{cases}
\]
Proof. It is easily verified that the conditions of Lemma 4.6 are satisfied if, for example, \( S_1(t) = -\Delta; S_2(t) = \Delta; \) \( A \) and \( B \) are arbitrary disjoint countable dense subsets of \((0,1)\); \( V_k(t) \) is the function whose graph is the broken line passing through the required points; and \( n_0(k) = \frac{\lambda \Delta^2}{\delta_k^2} \) where \( \delta_k = \min_{-1 \leq i, j \leq k} |a_i - b_j| \) and \( a_0 = b_0 = 0, a_{-1} = b_{-1} = 1. \) Hence we have the first equation of (4.15); and the explicit evaluation follows from (2.39) with \( \lambda = \mu = 0 \) and \( Q \) replaced by \( Q - \Delta. \) Similarly we obtain the first equation of (4.16) with all expressions identified in the same way except that here \( S_2(t) = 0; \) and the explicit evaluation follows from (2.39) upon substituting \(-l\) for \( \lambda \) and \( \mu, \frac{\Delta}{2} \) for \( \Delta, \) and \( (Q - \frac{\Delta}{2}) \) for \( Q. \)

We have not succeeded in finding exactly the asymptotic greatest lower bounds on the power of the two-sided Kolmogorov test in any of the classes defined by (4.5); but from the following three lemmas approximations to the bounds can be obtained which will in no case be in error by more than \( \alpha. \)

The first method of approximation simply makes use of the lower bounds already found for the power of the one-sided test. In particular, we have

**Lemma 4.8.** Let \( \{J^* \} \) be any subclass of \( \{J \}. \) Then

\[
L_\cap J^* \cup \Delta J \geq L_\cap J^* \cup \Delta J \cup R^{-1} Q(\alpha) J .
\]

**Proof.** Let \( \{H, \{G_n\}\} \) be any member of \( \{J^* \}. \) Then
\[ P_n(H, G_n, \alpha) = 1 - \beta_n \sqrt{3} n(t; H, G_n) - Q_n, S_n(t; H, G_n) + Q_n, T \]
\[ \geq 1 - \beta_n \sqrt{-\infty}, S_n(t; H, G_n) + Q_n, T \]
\[ = P_n^+(H, G_n, R_n^{-1} Q_n(\alpha), J) \].

Hence
\[ \inf_{(H, \{G_n\})} J^* P_n(H, G_n, \alpha) \geq \inf_{(H, \{G_n\})} J^* P_n^+(H, G_n, R_n^{-1} Q_n(\alpha), J) \]
and
\[ L^+ J^*, \alpha J = \lim_{n \to \infty} \inf_{(H, \{G_n\})} J^* P_n(H, G_n, \alpha) \]
\[ \geq \lim_{n \to \infty} \inf_{(H, \{G_n\})} J^* P_n^+(H, G_n, R_n^{-1} Q_n(\alpha), J) \]
\[ = L^+ J^*, R^{-1} Q(\alpha), J \],

where the last step follows without difficulty since
\[ \lim_{n \to \infty} R_n^{-1} Q_n(\alpha), J = R^{-1} Q(\alpha), J \].

The second method of approximation, which we may call the "one-point" method, is due to Massey, who used it in \[ \sqrt{16}, J \] to derive (4.17) below. The proofs of (4.17), and also (ii) in Theorem 4.5, involve an extension of it.

**Lemma 4.9.** For \( 0 < \tau < 1 \)

(4.17) \[ L^+ C^+(\Delta, \tau), \alpha J \geq L^+ J^+(\Delta, \tau), \alpha J \]
\[ \geq 1 - \phi \left( \frac{Q - \Delta}{\sqrt{\tau(1-\tau)}} \right) + \phi \left( \frac{-Q - \Delta}{\sqrt{\tau(1-\tau)}} \right) \].

And if \( \Delta > Q \) then
\[(4.18) \quad L^\int \mathcal{C}^+(\Delta) \mathcal{A} = L^\int \mathcal{J}^+(-\Delta) \mathcal{A} = L^\int \mathcal{J}^+(-\Delta) \mathcal{A} \geq 1 - \phi(2Q-2\Delta) + \phi(-2Q-2\Delta).\]

**Proof.** Let \((H, \{G_n\})\) be any member of \(J^+(\Delta, \tau)\). Then

\[P_n(H, G_n, \alpha) = \Pr \left\{ D_n > Q_n, (F = G_n) \right\} = \Pr \left\{ -\infty < x < \infty \sqrt{n} \left| F_n(x) - H(x) \right| > Q_n \right\} \]

\[= 1 - \Pr \left\{ -\infty < \sqrt{n} (F_n(x) - H(x)) \leq Q_n, -\infty < x < \infty \right\} \geq 1 - \Pr \left\{ -Q_n \leq \sqrt{n} (F_n(x_0) - H(x_0)) \leq Q_n \right\} \]

\[= 1 - \Pr \left\{ \tau_n - Q_n \sqrt{n} \leq x \leq \tau_n + Q_n \sqrt{n} \right\} \]

where \(x\) is Binomial \(n, \tau + \Delta \sqrt{n}\). Write \(p = \tau + \frac{\Delta}{\sqrt{n}}\). Then

\[P_n(H, G_n, \alpha) \geq 1 - \Pr \left\{ np - Q_n \sqrt{n} - \Delta \sqrt{n} \leq x \leq np + Q_n \sqrt{n} - \Delta \sqrt{n} \right\} \]

\[= 1 - \Pr \left\{ \frac{-Q_n - \Delta}{\sqrt{p(1-p)}} \leq \frac{x - np}{\sqrt{np(1-p)}} \leq \frac{Q_n - \Delta}{\sqrt{p(1-p)}} \right\} \]

so

\[\inf_{(H, \{G_n\}) \in J^+(\Delta, \tau)} P_n(H, G_n, \alpha) \geq 1 - \Pr \left\{ \frac{-Q_n - \Delta}{\sqrt{p(1-p)}} \leq \frac{x - np}{\sqrt{np(1-p)}} \leq \frac{Q_n - \Delta}{\sqrt{p(1-p)}} \right\} \]

and by the central limit theorem

\[L^\int J^+(\Delta, \tau) \alpha = \lim_{n \to \infty} \inf_{(H, \{G_n\}) \in J^+(\Delta, \tau)} P_n(H, G_n, \alpha) \geq 1 - \phi \left( \frac{-Q - \Delta}{\sqrt{p(1-p)}} \right) + \phi \left( \frac{-Q - \Delta}{\sqrt{(1-p)(1-\tau)}} \right) \]

which is the right-hand inequality in \((4.17)\). The right-hand
inequality in (4.18) follows from the Chebyshev inequality and the Uniform Central Limit Theorem just as in the proof of (ii) in Theorem 4.5. That $L^{-(\Delta, \alpha)} = L^{+\Delta, \alpha}$ is clear from considerations of symmetry; and the remaining inequalities follow from the inclusion relations among the classes defined by (4.5).

It can be shown by numerical calculation that neither Lemma 4.8 nor Lemma 4.9 yields approximations which are uniformly better (i.e., larger) than those yielded by the other.

Finally, we approximate the asymptotic greatest lower bounds from above by considering an "almost-worst" sequence of alternative distributions.

**Lemma 4.10.** For $0 < \tau < 1$,

\[(4.19) \quad L^{-(\Delta, \tau), \alpha} \leq 1 - \Pr\left\{ |Z(t)| \leq Q, t \in T; Z(\tau) \leq Q - \Delta \right\} \]

\[\leq 1 - \Phi\left( \frac{Q - \Delta}{\sqrt{1 - \tau}} \right) + \alpha.\]

Also, for $\Delta < Q$,

\[(4.20) \quad L^{-(\Delta), \alpha} \leq \alpha;\]

and, for $\Delta \geq Q$,

\[(4.21) \quad L^{-(\Delta), \alpha} \leq 1 - \Phi(2Q - 2\Delta) + \alpha.\]

**Proof.** Let $H(x)$ be defined by (4.11) and $G_n(x)$, $n = 1, 2, \ldots$, by (4.12). Then $(H, \{G_n\}) \in C^{(\Delta, \tau)}$. Also, the sequence of functions \(\left\{ S_n(t; H, G_n) \right\}\) satisfies the conditions of Corollary 3.8 with \(a(t) \equiv 0\) for $t \in T$, and $b(t) = 0$ for $t \in T$, $t \neq \tau$, where $b(\tau) = -\Delta$.
this yields the first inequality of (4.19). An explicit evaluation of the probability could be obtained by substituting \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = a = 0, b = -1 \), into (2.42). The second inequality follows since
\[
1 - \text{pr}\left\{ |Z(t)| \leq Q, t \in T; Z(\tau) \leq Q-\Delta \right\} \leq \left[ \text{I-pr} \left\{ Z(\tau) \leq Q-\Delta \right\} \right] + \left[ \text{I-pr} \left\{ |Z(t)| \leq Q, t \in T \right\} \right]
\]
and
\[
1 - \text{pr}\left\{ |Z(t)| \leq Q, t \in T \right\} = 1 - J(Q) = \alpha
\]
by (1.18) and (3.1). We can obtain (4.20) from (4.19) by letting \( \tau \) tend to 0, and (4.21) by setting \( \tau = \frac{1}{2} \).

The reader can now verify that, as was stated in the paragraph preceding Lemma 4.3, the approximations now obtained are in no case in error by more than \( \alpha \). In particular, the well-known lower bound (4.17) due to Massey is within \( \alpha \) of the true value.
CHAPTER V

ASYMPTOTIC EFFICIENCY; EXAMPLES

Let $H^*$ be a hypothesis distribution, and let

$$T_i = \{ T_n^{(i)}(\alpha), 0 < \alpha < 1, n = 1, 2, \ldots \}, \quad i = 1, 2,$$

be any two tests (actually systems of tests) of it, where $T_n^{(1)}(\alpha)$ is based on $n$ observations and has Type I error $\alpha$. Also, let $\{G(x; \delta), \delta \in \Omega\}$ be a class of alternative distributions; then write $P_n^{(1)}(\delta, \alpha)$ for the power of $T_n^{(1)}(\alpha)$ against $G(x; \delta)$, and let $n_1(\delta, \alpha, P)$ be the least integer $n$ such that

$$(5.1) \quad P_n^{(1)}(\delta, \alpha) \geq P;$$

if no finite $n$ satisfying $(5.1)$ exists we write

$$n_1(\delta, \alpha, P) = +\infty.$$ 

The relative efficiency of $T_1$ with respect to $T_2$ is then defined to be

$$e_{1,2}(\delta, \alpha, P) = \frac{n_2(\delta, \alpha, P)}{n_1(\delta, \alpha, P)}.$$ 

For example, if test $T_1$ requires twice as many observations as test $T_2$ in order to achieve power $P$ against $G(x; \delta)$ when the Type I error is $\alpha$, then its efficiency with respect to $T_2$ at $(\delta, \alpha, P)$ is $\frac{1}{2}$. In what follows $\Omega$ will be an interval of the real line which includes 0, and

$$\lim_{\delta \to 0} G(x; \delta) = H(x).$$
Then, in agreement with the usual definition, we may say that the asymptotic relative efficiency of $T_1$ with respect to $T_2$ is

$$E_{1,2}(\alpha,p) = \lim_{\delta \to 0} e_{1,2}(\delta,\alpha,p)$$

if this limit exists.

Now suppose that expressions of the form

$$P^{(i)}(\Delta,\alpha) = \lim_{n \to \infty} P_n^{(i)}\left(\frac{\Delta}{\sqrt{n}},\alpha\right)$$

have been obtained. Then we conclude that for small $\delta$

$$P_n^{(i)}(\delta,\alpha) \sim P^{(i)}(\delta \sqrt{n},\alpha)$$

and hence, if $\Delta^{(i)}(\alpha,p)$ is determined from the relation

$$P^{(i)}(\Delta^{(i)},\alpha) = p$$

then

$$n_i(\delta,\alpha,p) \sim \left[\frac{\Delta^{(i)}(\alpha,p)}{\delta}\right]^2$$

thus finally the asymptotic relative efficiency of $T_1$ with respect to $T_2$ is

$$E_{1,2}(\alpha,p) = \left[\frac{\Delta^{(ii)}(\alpha,p)}{\Delta^{(i)}(\alpha,p)}\right]^2$$

In the remainder of this chapter we shall restrict our attention to the asymptotic power and asymptotic relative efficiency of $T^+$, the one-sided Kolmogorov test. Let $T_j$ be some competitive test whose asymptotic power function has the form

$$P^{(j)}(\Delta,\alpha) = \phi^{-1}(\alpha) + c^{(j)}\Delta^+$$

Then

$$\Delta^{(j)}(\alpha,p) = \frac{1}{c^{(j)}} \left[\phi^{-1}(p) - \phi^{-1}(\alpha)^+\right]$$
and the asymptotic relative efficiency of $T^+$ with respect to $T^j$ (simplifying the notation) is

$$E_j(\alpha,P) = \left[ \frac{\Phi^{-1}(P) - \Phi^{-1}(\alpha)}{c(j)\Delta^+(\alpha,P)} \right]^2$$

where $\Delta^+(\alpha,P)$ is determined by the equation

$$P^+(\Delta^+,\alpha) = P.$$  

If $P^+(\Delta^+,\alpha) = 1$ then we write $c(j) = +\infty$ and $E_j(\alpha,P) = 0$; and if $P^+(\Delta^+,\alpha) \leq \alpha$ we write $c(j) = 0$ and $E_j(\alpha,P) = +\infty$. Note that if $P^+(\Delta^+,\alpha)$ were also of the form

$$P^+(\Delta^+,\alpha) = \Phi^{-1}(\alpha) + c^+\Delta^+$$

then the asymptotic relative efficiency of $T^+$ with respect to $T^j$ would be

$$E_j(\alpha,P) = \left[ \frac{c^+}{c(j)} \right]^2$$

which is independent of $\alpha$ and $P$; this situation is the most familiar one in statistical literature.

The **sign test** based on the p-fractile consists in rejecting the hypothesis if the sign of $\sum H^{-1}(p) - 1/I_j$ is positive "too often"; that is, if $F_n\sum H^{-1}(p) - 1/I_j$ is "too large."

**Theorem 5.1.** Assume $G\sum H^{-1}(p) - 1/I_j = t + \delta c(t) + o(\delta), 0 < t < 1$. Then if $T(p)$ is the sign test based on the p-fractile, and $c(p) > 0$, the asymptotic relative efficiency of $T^+$ with respect to $T(p)$ is

$$E(p)(\alpha,P) \geq 0 < t < 1 \quad E(p)(\alpha,P) \geq \sup \frac{P(1-p)}{t(1-t)} \left[ \frac{c(t)}{c(p)} \Phi^{-1}(p) - \Phi^{-1}(\alpha) \right]$$

$$\Phi^{-1}(p) + \frac{E(\alpha)}{\sqrt{t(1-t)}} \left] \right|^2.$$
Proof. The random variable $nF_n^{\frac{1}{n}}H^{-1}(p)$ is Binomial $(n,t)$ where $t = F^\frac{1}{n}H^{-1}(p)$ and $F$ is the true distribution. If $F(x) = H(x)$, then $t = p$; and if

$$S_n = \frac{\sqrt{n} (F_n^{\frac{1}{n}}H^{-1}(p) - p)}{\sqrt{p(1-p)}}$$

then $S_n \sim N(0,1)$; hence the test $T(p)$ is asymptotically equivalent to rejecting if $S_n > \phi^{-1}(1-\alpha)$. But if $F(x) = G(x; \Delta/\sqrt{n})$ then by hypothesis $t = p + \frac{\Delta c(p)}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})$ and thus $S_n - \frac{\Delta c(p)}{\sqrt{p(1-p)}} \sim N(0,1)$ so that the asymptotic power of $T(p)$ is

$$P(p)(\Delta,\alpha) = \phi^{-1}(\alpha) + \frac{\Delta c(p)}{\sqrt{p(1-p)}}$$

and hence from (5.3) and (5.4) we have

$$(5.5) \quad E(p)(\alpha,p) = \left[ \frac{\sqrt{p(1-p)}}{c(p)} \cdot \frac{\phi^{-1}(p) - \phi^{-1}(\alpha)}{\Delta^*(\alpha,p)} \right]^2$$

where

$$F^*(\Delta,\alpha) = P.$$

But choose any $t$, $0 < t < 1$. Then

$$F_n^*(\Delta/\sqrt{n},\alpha) = \text{pr} \left\{ \sup_{0 < \tau < 1} \sqrt{n} (F_n^{\frac{1}{n}}H^{-1}(\tau) - \tau) > R_n(\alpha) \right\}$$

$$\leq \text{pr} \left\{ \sqrt{n} (F_n^{\frac{1}{n}}H^{-1}(t) - t) > R_n(\alpha) \right\}$$

$$= \text{pr} \left\{ \frac{\sqrt{n} (F_n^{\frac{1}{n}}H^{-1}(t) - \alpha(t)) - \Delta c(t)}{\sqrt{t(1-t)}} > \frac{R_n(\alpha) - \Delta c(t)}{\sqrt{t(1-t)}} \right\}$$

$$= \text{pr} \left\{ Z_n > \frac{R_n(\alpha) - \Delta c(t)}{\sqrt{t(1-t)}} \right\}$$
where $Z_n \sim N(0,1)$; so
\[ P^+(\Delta, \alpha) \geq \Phi \left( \frac{-R(\alpha) + \Delta c(t)}{\sqrt{t(1-t)}} \right) \]
and
\[ \Delta^+(\alpha, p) \leq \frac{\sqrt{t(1-t)}}{c(t)} \left[ \Phi^{-1}(p) + \frac{R(\alpha)}{\sqrt{t(1-t)}} \right] \]
which yields the lemma by substitution into (5.5), since $t$ was arbitrary.

Note that it is easily shown by differentiation that the lower bound on $E^P(\alpha, p)$ given by Lemma 5.1 increases with increasing $P$.

**Theorem 5.2.** Suppose that we can write
\[ H(x) = \int_{-\infty}^{x} h(u) du, \]
where $h(u)$ is continuous for $H^{-1}(0) \leq u \leq H^{-1}(1)$ and identically zero elsewhere; and that
\[ G_n(x) = H(x + \frac{\theta n}{\sqrt{n}}), \quad \text{where} \quad \lim_{n \to \infty} \theta_n = \theta > 0. \]

Then, defining $S(t) = -\theta \sqrt{H^{-1}(t)}$ for $t \in T$, we have
\[ P^+(H, [G_n], \alpha) = 1 - \beta \sqrt{\infty, S(t) + R, T} \]

**Proof.** Consider first the case where $\theta_n = \theta$ for all $n$. Then from
\[ G_n^{-1}(t) = H^{-1}(t) - \frac{\theta}{\sqrt{n}} \]
so that from (5.6) and (1.31)
$$S_n(t;H,G_n) = -\sqrt{n} \int_{H^{-1}(t) - \frac{\theta}{\sqrt{n}}}^{H^{-1}(t)} \mathbf{H}(x)dx.$$ 

Now if \( h(x) \) is continuous in the closed interval \([H^{-1}(0), H^{-1}(1)]\) it must be uniformly continuous there; that is, for any \( \varepsilon > 0 \) there is a \( \delta_0(\varepsilon) > 0 \) such that if \( 0 < \delta < \delta_0(\varepsilon) \) and if the closed interval \([x, x+\delta]\) does not include \( H^{-1}(0) \) or \( H^{-1}(1) \) then \( |h(x) - h(x+\delta)| < \varepsilon \). Let \( A = \{0, 1\} \) and, choosing any positive integer \( m > M(A) = 2 \), define \( I_1 = [0, \frac{1}{m}] \), \( I_2 = [\frac{1}{m}, 1 - \frac{1}{m}] \), and \( I_3 = (1 - \frac{1}{m}, 1] \). Assume \( n \) sufficiently large that

$$\frac{\theta}{\sqrt{n}} < \min \left\{ \delta_0(\varepsilon), H^{-1}(\frac{1}{m}) - H(0), H^{-1}(1) - H^{-1}(1 - \frac{1}{m}) \right\}.$$ 

Then for \( t \in I_1 \)

$$E_m(t;S_n,A) - E_m(t;S,A)$$

$$= \inf_{t \in I_1} S_n(t) - \inf_{t \in I_1} S(t)$$

$$= \inf_{t \in I_1} \left\{ -\sqrt{n} \int_{H^{-1}(t) - \frac{\theta}{\sqrt{n}}}^{H^{-1}(t)} h(x)dx \right\} - \inf_{t \in I_1} \left\{ -\theta \sqrt{H^{-1}(t)} \right\}$$

$$= \sup_{t \in I_1} \theta \sqrt{H^{-1}(t)} - \sup_{t \in I_1} \sqrt{n} \int_{H^{-1}(t) - \frac{\theta}{\sqrt{n}}}^{H^{-1}(t)} h(x)dx,$$

which is obviously non-negative. Since \( h \sqrt{H^{-1}(t)} \) is continuous for \( t \in I_1 \) we have that for some \( t_1 \in I_1 \)
\[
\sup_{t \in I_1} \theta h^{-1}(t) \quad \geq \quad \sup_{t \in I_1} \frac{\theta}{\sqrt{n}} \int_{H^{-1}(t)}^{H^{-1}(t_1)} h(x) dx
\]

If \( t_1 > 0 \), choose \( n \) so large that \( \frac{\theta}{\sqrt{n}} H^{-1}(t) - H^{-1}(0) \); then for some \( \phi, 0 \leq \phi \leq 1 \),

\[
\sup_{t \in I_1} \frac{\theta}{\sqrt{n}} \int_{H^{-1}(t)}^{H^{-1}(t_1)} h(x) dx \geq \frac{\theta}{\sqrt{n}} \int_{H^{-1}(t)}^{H^{-1}(t_1)} h(x) dx
\]

by the mean value theorem for integrals, and hence, using the uniform continuity of \( h(x) \), we have

\[
(5.8) \quad |B_m(t; S_n, \Lambda) - B_m(t; S, \Lambda)| \leq \phi \varepsilon.
\]

On the other hand, if \( t_1 = 0 \) then

\[
\sup_{t \in I_1} \frac{\theta}{\sqrt{n}} \int_{H^{-1}(t)}^{H^{-1}(0) + \frac{\theta}{\sqrt{n}}} h(x) dx \geq \int_{H^{-1}(t)}^{H^{-1}(0)} h(x) dx
\]

by the mean value theorem again and (5.8) still holds. A similar argument obtains if \( t \in I_3 \). But for \( t \in I_2 \),

\[
B_m(t; S_n, \Lambda) - B_m(t; S, \Lambda) = S_n(t) - S(t)
\]
\[
H^{-1}(t) = -\sqrt{n} \int h(x)dx + \theta h\mathbb{E}H^{-1}(t) \]
\[
H^{-1}(t) - \frac{\theta}{\sqrt{n}}
\]
\[
= \theta \left\{ h\mathbb{E}H^{-1}(t) - h\mathbb{E}H^{-1}(t) - \frac{\theta}{\sqrt{n}} \right\}
\]

using the mean value theorem once more, and thus we have established that for \( n \) sufficiently large
\[
\sup_{t \in T} |B_m(t; S_n, \Lambda) - B_m(t; S, \Lambda)| \leq \theta \epsilon.
\]

Then the lemma is verified by applying Corollary 3.8.

But in the general case
\[
S_n(t; H, G_n) = -\sqrt{n} \int h(x)dx - \sqrt{n} \int h(x)dx.
\]
\[
H^{-1}(t) - \frac{\theta}{\sqrt{n}} \quad H^{-1}(t) - \frac{\theta n}{\sqrt{n}}
\]

Now if \( h(x) \) is uniformly continuous in \([H^{-1}(0), H^{-1}(1)]\), and zero elsewhere, it must be bounded by some positive constant \( K \). Then
\[
\left| H^{-1}(t) - \frac{\theta}{\sqrt{n}} \right| \int h(x)dx \leq K|\theta_n - \theta|
\]

and since this tends to zero independently of \( t \) the previous argument can obviously be modified to give the same result.

We remark that Theorem 5.2 can be extended easily enough to the two-sided test, or to cases where \( \theta < 0 \), or where the possible points of discontinuity of \( h(x) \) form an arbitrary finite set \( \Omega \); the main
difficulty consists in properly defining $S(t)$ when $H^{-1}(t) \in \Omega$.

We now consider five numerical examples of pairs $(H, \{G_n\})$ which satisfy the hypotheses of Theorem 5.2. These are presented in Table 5.1, and the corresponding situations are illustrated in the accompanying diagram, Figure 5.2. In words, the asymptotic power $P_1^+(\Delta, \alpha)$ of the one-sided Kolmogorov test of $H_1(x)$ against the sequence of alternative distributions $\{G_{\text{in}}(x)\}$, where $G_{\text{in}}(x) = H_1(x + \frac{\theta_{\text{in}}}{\sqrt{n}})$ and $\lim_{n \to \infty} \theta_{\text{in}} = \theta_1 > 0$, is the probability that a sample function of the stochastic process $Z(t)$ will cross the boundary $S_1(t; \Delta) + R(\alpha)$; from Theorem 5.2 it follows that the relationship between $\Delta$ and $\theta$ is given by

$$\Delta = \theta_1 \sup_{x} h_1(x).$$

One may note the following partial ordering of the $P_1^+$'s from such a diagram immediately: for all $\Delta, \alpha$

$$(5.10) \quad P_1^+(\Delta, \alpha) \geq \begin{cases} P_2^+(\Delta, \alpha) \\ P_4^+(\Delta, \alpha) \geq \begin{cases} P_3^+(\Delta, \alpha) \\ P_5^+(\Delta, \alpha) \end{cases} \end{cases}.$$
<table>
<thead>
<tr>
<th>i</th>
<th>$H_i(x)$</th>
<th>$\Delta_i$</th>
<th>$S_i(t;\Delta)$</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ \begin{cases} 0 &amp; x \leq 0 \ x &amp; 0 \leq x \leq 1 \ 1 &amp; x \geq 1 \end{cases} ]</td>
<td></td>
<td>- $\Delta$</td>
<td>Rectangular</td>
</tr>
<tr>
<td>2</td>
<td>[ \begin{cases} 0 &amp; x \leq 0 \ 1 - e^{-x} &amp; x \geq 0 \end{cases} ]</td>
<td></td>
<td>- $\Delta(1-t)$</td>
<td>Exponential</td>
</tr>
<tr>
<td>3</td>
<td>[ \begin{cases} \frac{1}{2} e^x &amp; x \leq 0 \ 1 - \frac{1}{2} e^{-x} &amp; x \geq 0 \end{cases} ]</td>
<td>$\frac{1}{2}</td>
<td>\theta</td>
<td>$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$</td>
<td>$\frac{</td>
<td>\theta</td>
<td>}{\sqrt{2\pi}}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$</td>
<td>$\frac{</td>
<td>\theta</td>
<td>}{\pi}$</td>
</tr>
</tbody>
</table>
Since $S_1(t;\Delta)$, $S_2(t;\Delta)$, and $S_3(t;\Delta)$ satisfy (2.30) with $k = 0$, 0, and 1 respectively, the formulas for the corresponding asymptotic power functions can be obtained immediately by substitution into (2.40) and (2.43); we then have

\begin{equation}
(5.11) \quad P_{1}^{+}(\Delta, \alpha) = \begin{cases} 
    e^{-2(R-\Delta)^2} & \Delta < R \\
    1 & \Delta \geq R 
\end{cases},
\end{equation}

\begin{equation}
(5.12) \quad P_{2}^{+}(\Delta, \alpha) = \begin{cases} 
    e^{-2R(R-\Delta)} & \Delta < R \\
    1 & \Delta \geq R 
\end{cases},
\end{equation}

and

\begin{equation}
(5.13) \quad P_{3}^{+}(\Delta, \alpha) = 1 - \Phi(2R-2\Delta) + 2e^{-2R(R-2\Delta)}\Phi(-2\Delta) - e^{2R\Delta}\Phi(-2R-2\Delta).
\end{equation}
Some computations based on these formulas are exhibited in Tables 5.3 and 5.4.

**Table 5.3**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>P</th>
<th>R</th>
<th>$\Delta_1$ (Rectangular)</th>
<th>$\Delta_2$ (Exponential)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.10</td>
<td>.90</td>
<td>1.0730</td>
<td>.8435</td>
<td>1.0239</td>
</tr>
<tr>
<td>.95</td>
<td>.99</td>
<td>1.0021</td>
<td>.9128</td>
<td>1.0491</td>
</tr>
<tr>
<td>.99</td>
<td>1.1808</td>
<td>.9944</td>
<td>1.0683</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.05</td>
<td>1.2239</td>
<td>.9944</td>
<td>1.0637</td>
</tr>
<tr>
<td></td>
<td>.95</td>
<td>1.1530</td>
<td>1.1808</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.99</td>
<td>1.1530</td>
<td>1.2029</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.01</td>
<td>1.5174</td>
<td>1.2198</td>
<td></td>
</tr>
<tr>
<td>.90</td>
<td>1.4827</td>
<td>1.2879</td>
<td>1.5005</td>
<td></td>
</tr>
<tr>
<td>.95</td>
<td>1.4465</td>
<td>1.3573</td>
<td>1.5141</td>
<td></td>
</tr>
<tr>
<td>.99</td>
<td>1.4465</td>
<td>1.3573</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It will be necessary to approximate the power functions which correspond to $S_4(t;\Delta)$ and $S_5(t;\Delta)$ because these curves are not piecewise-linear. Now if $S(t) \leq A(t)$, $0 \leq t \leq 1$, then certainly

$$1 - \beta_{-\infty}^0 S(t) + R, T \geq 1 - \beta_{-\infty}^0 A(t) + R, T.$$

For any $\tau$, $0 \leq \tau \leq \frac{1}{2}$, let

$$A_i(t;\tau) = \begin{cases} S_4(t;\Delta) & t \leq 1 - \tau \\ + \infty & 0 \leq \tau < t, 1 - \tau < t \leq 1 \end{cases}$$

for $i = 4, 5$; then $S_i(t;\Delta) \leq A_i(t;\tau)$ and

$$P_i^+(\Delta, \alpha) \geq 1 - \beta_{-\infty}^0 A_i(t;\tau) + R, T.$$

The actual formula required, which can be calculated using

Corollary 2.8, is
\[ 1 - \beta_{\leq \alpha, A_1(t) \geq R, T} = 1 - F \left[ \frac{\lambda_1}{\sqrt{\tau(1-\tau)}}, \frac{\lambda_1}{\sqrt{\tau(1-\tau)}} ; \frac{\tau}{1-\tau} \right] \]

(5.14)

\[ + e^{-2\lambda_1^2} F \left[ \frac{(1-2\tau)\lambda_1}{\sqrt{\tau(1-\tau)}}, \frac{(1-2\tau)\lambda_1}{\sqrt{\tau(1-\tau)}} ; \frac{-\tau}{1-\tau} \right] \]

where \( \lambda_1 = R + S_1(\tau; \Delta) \) and \( F_{\alpha, \beta, \gamma, \rho} \) is defined by (3.33).

The lower bounds exhibited in Table 5.4 have been computed from (5.14) with \( \tau = .375 \); this value of \( \tau \) was chosen mainly for computational convenience, but some numerical work indicates that it may be near the optimal value for use in such a formula. We admit that these bounds are undoubtedly quite crude, particularly in the case of the Cauchy distribution; some refinement could be obtained by the use of formula (2.45), but to go further would apparently require considerable numerical work. Of course, from (5.10) we have also that \( P_4^+(\Delta, \alpha) \geq P_3^+(\Delta, \alpha) \), but we see that this bound is even poorer for all the entries in the table.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( R )</th>
<th>( \Delta )</th>
<th>( P_3^+(\Delta, \alpha) ) (Laplace)</th>
<th>Lower Bound on ( P_3^+(\Delta, \alpha) ) (Normal)</th>
<th>Lower Bound on ( P_5^+(\Delta, \alpha) ) (Cauchy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.13534</td>
<td>1.00</td>
<td>1.00</td>
<td>.74179</td>
<td>.76326</td>
<td>.69362</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td></td>
<td>.94210</td>
<td>.96028</td>
<td>.92445</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td></td>
<td>.99404</td>
<td>.99743</td>
<td>.99114</td>
</tr>
<tr>
<td>.03405</td>
<td>1.30</td>
<td>1.30</td>
<td>.69966</td>
<td>.75318</td>
<td>.65938</td>
</tr>
<tr>
<td></td>
<td>1.95</td>
<td></td>
<td>.96040</td>
<td>.97902</td>
<td>.94735</td>
</tr>
<tr>
<td></td>
<td>2.60</td>
<td></td>
<td>.99867</td>
<td>.99966</td>
<td>.99773</td>
</tr>
<tr>
<td>.00598</td>
<td>1.60</td>
<td>1.60</td>
<td>.66905</td>
<td>.74288</td>
<td>.62380</td>
</tr>
<tr>
<td></td>
<td>2.40</td>
<td></td>
<td>.97597</td>
<td>.99968</td>
<td>.96437</td>
</tr>
<tr>
<td></td>
<td>3.20</td>
<td></td>
<td>.99979</td>
<td>.99997</td>
<td>.99953</td>
</tr>
</tbody>
</table>
Finally, we shall consider the asymptotic relative efficiency of the one-sided Kolmogorov test with respect to several competitive tests for the same five examples.

The t-test consists in rejecting if

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

is "too small." In particular, if when \( x \) is distributed according to \( H \) we have \( \mathcal{C}(x) = \mu \) and \( \text{var}(x) = \sigma^2 \), and if

$$t_n = \frac{\sqrt{n}(\bar{x}_n - \mu)}{\sigma}$$

then \( t_n \sim N(0,1) \) and hence the t-test is asymptotically equivalent to rejecting if \( t_n < \Phi^{-1}(\alpha) \). If for each \( n \) the alternative \( G_n \) is true then \( \mathcal{C}(x) = \mu - \frac{\theta_n}{\sqrt{n}} \), \( \text{var}(x) = \sigma^2 \), and the random variable

\( (t_n + \frac{\theta}{\sigma}) \sim N(0,1) \); hence the asymptotic power of the t-test is

$$P(t)(\Delta, \alpha) = \Phi(\Phi^{-1}(\alpha) + \frac{\theta}{\sigma})$$

where \( \theta \) is determined from (5.9). Thus for the t-test the constant required in equation (5.4) is

$$c_1(t) = \frac{1}{\sigma \sup_x h_1(x)}, \quad 1 \leq i \leq 4.$$  

(5.15)

The preceding argument does not apply to the Cauchy distribution, for which \( \mathcal{C}(x) \) does not exist, but it is not difficult to show that the asymptotic power of the t-test is here equal to \( \alpha \); hence we may take
(5.16) \[ c_5(t) = 0. \]

The U-test, which is actually a one-sample analogue of Wilcoxon's familiar two-sample test, consists in rejecting if

\[ U_n = \sqrt{n} \int_{-\infty}^{\infty} \left[ \frac{F_n(x)}{H(x)} - 1 \right] \, dH(x) \]

is "too large." It is easily shown that

\[ U_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \]

where \( z_i = \frac{1}{2} - H(x_i) \), and that when \( x \) is distributed according to \( H \) we have

\[ C(z) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} - H(x) \right] \, dH(x) = 0 \]

and

\[ \text{var}(z) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} - H(x) \right]^2 \, dH(x) = \frac{1}{12} \]

so that \( U_n \sim N(0, \frac{1}{12}) \) and hence the U-test is asymptotically equivalent to rejecting if

\[ U_n > \frac{1}{\sqrt{12}} \phi^{-1}(1-\alpha). \]

Now if \( H(x) \) has a derivative \( h(x) \) then under suitable restrictions (which are satisfied by the five examples under consideration) we have that if for each \( n \) the alternative \( G_n \) is true then
\[ c_i^{(U)} = \frac{\phi^{-1}(\alpha)}{-\sup_x h(x)} , \quad 1 \leq i \leq 5 \]

The median test consists in rejecting the hypothesis if \( m_n \), the sample median, is "too small." In particular, suppose that \( H(x) \) has a derivative \( h(x) \) which is continuous in some neighborhood of \( \eta \), where \( H(\eta) = \frac{1}{2} \), and let

\[ M_n = 2(m_n - \eta) \sqrt{n} h(\eta) ; \]

then when the hypothesis is true \( M_n \sim N(0,1) \) and hence the median test is asymptotically equivalent to rejecting if \( M_n < \phi^{-1}(\alpha) \). But if for each \( n \) the alternative \( G_n \) is true then \( \sqrt{M_n} + 2\theta h(\eta) \sim N(0,1) \) and hence the asymptotic power of the median test is

\[ P^{(m)}(\Delta, \alpha) = \phi^{-1}(\alpha) + 2\theta h(\eta) \]

where \( \theta(\Delta) > 0 \) is determined from (5.9); that is, for the median test
the constant required in equation (5.4) is

\[(5.18) \quad c_i^{(m)} = \frac{2h_i(\eta)}{\sup_x b_i(x)} , \quad 1 \leq i \leq 5 .\]

If the median and mode of \( H \) are the same, then \( c^{(m)} = 2 \). The median test is asymptotically equivalent to a sign test based on the median (the \( p \)-fractile with \( p = \frac{1}{2} \)) and hence Theorem 5.1 would apply here also.

The most powerful tests for translation among rectangular and exponential populations are based on the extreme values; against the sequences of alternatives considered here their asymptotic power is 1 for all \( \alpha, \Delta > 0 \); thus, the constants needed for equation (5.4) are

\[(5.19) \quad c_1^* = c_2^* = +\infty .\]

In the Laplace case the most powerful test is equivalent to a sign test based on the median (cf. Hoeffding and Rosenblatt \([13]\)) and hence to the median test; in the Normal case the \( t \)-test is most powerful; these have already been discussed. One can easily derive, using the well-known Neyman-Pearson Lemma, that the test which rejects for small values of

\[V_n = \sum_{i=1}^{n} z_i ,\]

where \( z_i = \frac{x_i}{1 + x_i^2} \), is asymptotically most powerful for detecting infinitesimally small translations among Cauchy distributions. When \( H_0(x) \) is true, \( \mathcal{C}(z) = 0 \) and \( \text{var}(z) = \frac{1}{16} \) so that \( \frac{4V_n}{\sqrt{n}} \sim \mathcal{N} / 0, 1 / 4 \) and
hence this test is asymptotically equivalent to rejecting if

\[ V_n < \frac{\sqrt{n}}{4} \phi^{-1}(\alpha). \]

If for each \( n \) the alternative \( G_{\alpha n}(x) \) is true then

\[ \frac{4V_n}{\sqrt{n}} \sim \mathcal{N}(0,1) \]

and therefore the asymptotic power is

\[ P_5^*(\Delta, \alpha) = \phi^{-1}(\alpha) + \theta \]

where \( \theta(\Delta) > 0 \) is determined from (5.9), so that we obtain finally

(5.20) \[ c_5^* = \frac{1}{\sup_x h_5(x)} = \pi. \]

The values of the constants needed for equation (5.4) have been calculated from (5.15) through (5.20) and are given in Table 5.5. Table 5.6 then gives the asymptotic relative efficiency of the one-sided Kolmogorov test (or a lower bound on it), with respect to the various competitors considered here, for the same five examples, using the same values of \( P \) and \( \alpha \) as appear in Tables 5.3 and 5.4.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Test 1</th>
<th>Test 2</th>
<th>median</th>
<th>most powerful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( \sqrt{12} )</td>
<td>( \sqrt{12} )</td>
<td>2</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Exponential</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>1</td>
<td>( \infty )</td>
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In this chapter we have restricted our attention to the one-sided Kolmogorov test, but no theoretical difficulties remain which would prevent extension of the results to the two-sided test, and also to many other kinds of examples; the only differences would lie in the numerical work required.
BIBLIOGRAPHY


