ON THE MONOTONIC CHARACTER OF THE POWER
FUNCTIONS OF TWO MULTIVARIATE TESTS

by

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The power function of the largest root test of normal multivariate linear hypothesis or of independence between two sets of variates involves, in each case, aside from the degrees of freedom, certain nonnegative, noncentrality parameters. This report supplies a relatively simple and elegant proof that the power function monotonically increases as each parameter, separately, increases - a result that was conjectured and proved (but not published) by one of the authors several years ago by a very lengthy and laborious method.

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ON THE MONOTONIC CHARACTER OF THE POWER
FUNCTIONS OF TWO MULTIVARIATE TESTS

By S. N. Roy and W. F. Mikhail

1. Summary. The largest characteristic root test for multivariate analysis of variance or independence between two sets of variates has in each case a number of well known properties $1, 2, 3$ including, in particular, the following. (i) The power function involves as arguments, aside from the degrees of freedom and the level of significance, a set of non-negative non-centrality parameters that are statistically meaningful, (ii) the power functions has a lower bound which is a monotonically increasing function of each of these parameters, separately, and (iii) it is possible, by using the distribution of the test statistic under the null hypothesis alone, to obtain, with a confidence coefficient greater than or equal to a preassigned level, simultaneous confidence bounds on a set of parametric functions that might be interpreted as measures of departure, respectively from the total hypothesis or from partial hypothesis defined, for multivariate analysis of variance, by cutting out one or more variates and one or more factor levels, and for independence between a $p$-set and a $q$-set, by cutting out one or more of the $p$-set and one or more of the $q$-set.

It is the purpose of the present paper to prove that, for each test, the power function is a monotonically increasing function of each non-centrality (or deviation) parameter separately, a fact which was stated

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without proof in \( \text{[1,]} \). There the remark was made that the very long and tedious proof available to the author of \( \text{[1,]} \) was not being offered in the hope that a far simpler and more elegant proof would be forthcoming in the near future. The kind of proof looked for at that stage is being offered here.

2. Preliminaries for the multivariate analysis of variance situation \( \text{[1,]} \).

Let \( u, s \) and \( n-r \) denote respectively the "effective" number of variates, the degrees of freedom of the hypothesis and the degrees of freedom for the error and let \( t = \min (u, s) \). Then with

\[
(2.1) \quad X = \begin{bmatrix} x_{11} & \cdots & x_{1s} \\ \cdot & \cdots & \cdot \\ x_{u1} & \cdots & x_{us} \end{bmatrix}, \quad Y = \begin{bmatrix} y_{11} & \cdots & y_{1n-r} \\ \cdot & \cdots & \cdot \\ y_{u1} & \cdots & y_{us} \end{bmatrix}
\]

the canonical form for the elementary distribution is

\[
(2.2) \quad \frac{1}{u} \frac{u(u+n-r)}{2} \exp \left( -\frac{1}{2} \sum_{i=1}^{u} \sum_{j=1}^{n-r} y_{ij}^2 + \frac{t}{2} \sum_{i=1}^{u} (x_{ii} - \bar{x}_i)^2 \right) 
\]

\[
+ \frac{u}{u} x_{i1}^2 + \frac{u}{u} \sum_{i=1}^{u} x_{i1}^2 \int_{i=1}^{n-r} \int_{j=1}^{n-r} dx_{ij} dy_{ij} 
\]

and the acceptance region (at \( \alpha \)) for the linear hypothesis \( H_0 \) of analysis of variance (under which \( \gamma ' s = 0 \)) can be expressed as

\[
(2.3) \quad \text{ch} \frac{(XX')(YY')^{-1}}{\text{max}} \leq a,
\]

where \( a \) is given by

\[
(2.4) \quad P \left\{ \text{ch}_{\text{max}} \left\{ (XX')(YY')^{-1} \right\} \leq a \mid \gamma ' s = 0 \right\} = 1 - \alpha.
\]
If we denote the domain (2.3) by $\mathcal{D}$, then with an obvious notation for the differential elements, the probability of the second kind of error can be written as

$$
\int_{\mathcal{D}} \text{const. exp} \left[ -\frac{1}{2} \sum_{i,j} \gamma_{ij}^2 + \sum_{i=1}^{t+1} \sum_{i=j}^{u} \gamma_{ii}^2 \right] \ dX dY.
$$

The problem now is to prove that the integral (2.5) is a monotonically decreasing function of each $\gamma_i$, separately. If, now, we regard the domain $\mathcal{D}$ as one in an Euclidean space of dimensionality $u$ (for $X$) and $u(n-r)$ (for $Y$), then it is clear that we can rewrite (2.5) as

$$
\int_{\mathcal{D}^*} \text{const. exp} \left[ -\frac{1}{2} \sum_{i=1}^{u} \sum_{j=1}^{n-r} \gamma_{ij}^2 + \sum_{i=1}^{u} \sum_{j=1}^{s} \chi_{ij}^2 \right] \ dX dY,
$$

where $\mathcal{D}^*$ is merely the domain $\mathcal{D}$ translated by $\sqrt{\gamma_1}$ along $x_{i_1}$, that is, along the 1-th axis (with $i=1,2,\ldots,t$). Notice that if, in the integral (2.6), we replace the domain $\mathcal{D}^*$ by $\mathcal{D}$, integral over the new domain becomes equal to $1-\alpha$, where $\alpha$ is the probability of the first kind of error. It is useful now to put

$$
Y Y' = (V' V)^{-1},
$$

where $V$ is a uu triangular matrix with zeroes above the diagonal, observe that

$$
\text{ch} \left[ (XX') (YY')^{-1} \right] = \text{ch} \left[ XX' (V' V)^{-1} \right] = \text{ch} \left[ VX (V' V)^{-1} \right].
$$
where

\[ (2.8) \quad \widetilde{v}^T X = \begin{bmatrix} v_{11} & 0 & \cdots & 0 \\ v_{21} & v_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{ul} & v_{u2} & \cdots & v_{uu} \end{bmatrix} \begin{bmatrix} x_{11} \\ \vdots \\ \vdots \\ x_{ul} \end{bmatrix} \]

and rewrite (2.3), that is, the domain $\mathcal{D}$ as

\[ (2.9) \quad \mathcal{D} : \quad \text{ch}_{\max} \left( \widetilde{v}^T X (\widetilde{v}^T X)' \right) \leq \mu. \]

Notice that the domain $\mathcal{D}$ is a shift of $\mathcal{D}$ by $\sqrt{\gamma}_1$ along $x_{11}$ with $i=1,2,\ldots,t$. The problem now can be rephrased in the following way. How does the integral

\[ (2.10) \quad \int_{\mathcal{D}} \exp \left\{ \frac{u}{2} \sum_{i=1}^{n-r} \sum_{j=1}^{s} y_{ij}^2 + \sum_{i=1}^{u} \sum_{j=1}^{s} x_{ij}^2 \right\} \, dX dY \]

over the domain $\mathcal{D}$ given by (2.9) change under successive translations of $\sqrt{\gamma}_1$ along $x_{11}$, of $\sqrt{\gamma}_2$ along $x_{22}$, \ldots, $\sqrt{\gamma}_t$ along $x_{tt}$? It is clear that the successive changes are cumulative and that at any stage, say the $i$-th, the change depends on $\sqrt{\gamma}_i$. It will be also seen from the mechanics of the demonstration that if we can prove that the integral decreases for the first shift of $\mathcal{D}$, namely, by $\sqrt{\gamma}_1$ along $x_{11}$, then the general theorem itself will be proved.

3. Proof of the monotonicity property for the multivariate analysis of variance situation. The proof is developed in three main steps discussed in the following subsections.

3.1 The proof for the univariate case. In this case, $u=1$ and we can drop the first subscript in $X, Y$. The domain $\mathcal{D}$ of (2.9) now takes on the form
(3.1.1) \[ D : \sum_{j=1}^{n-r} x_j^2 \leq \mu \sum_{j=1}^{n-r} y_j^2, \]
and the integral (2.10) the form
\[ \int_{D} \exp \left( \frac{1}{2} \sum_{j=1}^{n-r} \frac{x_j^2}{y_j^2} \right) \prod_{j=1}^{n-r} dy_j \prod_{j=1}^{n-r} dx_j. \]
Notice that now \( D \) is just a shift of \( D \) along \( x_1 \) by \( \gamma \). It is evident from the form of (3.1.2) that the integral (3.1.2) decreases under this shift if, for any given set of \( y_j \)'s (\( j=1,2,\ldots, n-r \)) and \( x_j \)'s (\( j=2,3,\ldots,s \)),
\[ \int_{-a+\gamma}^{a} \exp \left( -\frac{1}{2} x_1^2 \right) dx_1 < \int_{-a}^{a} \exp \left( -\frac{1}{2} x_1^2 \right) dx_1, \]
where \( a \) is the positive square root of \( \mu \sum_{j=2}^{n-r} x_j^2 / \sum_{j=1}^{n-r} y_j^2 \); then it is clear that it doesn't matter whether we take \( \gamma \) to be positive or negative. It is easy to verify an even more general result than this, namely that
\[ \int_{-a+\lambda}^{a} g(x) dx < \int_{-a}^{a} g(x) dx, \]
where \( a \) is positive and \( \lambda \) might be taken to be either positive or negative and \( g(x) \) is a continuous function of \( x \), symmetrical about \( 0 \) and monotonically decreasing with \( \mid x \mid \). It is also clear from the nature of the proof that the left side of (3.1.3) steadily decreases with \( \frac{1}{2} \gamma \).

3.2. The nature of the multivariate domain (2.9). We go back now to the multivariate case and to the domain of (2.9). Let us investigate the nature of the domain in \( x_{11}, x_{21}, \ldots, x_{ul} \) for a given set of values of \( \mu, Y \) (that is, \( V \)) and of the elements of the matrix \( X \) except those in the first column. Toward this end, put \( X^* = VX \) and observe that, if \( v \) is any
characteristic root of \(X^{**}X^{**}\), then denoting the matrix by \(\Lambda_{s_1s_2...s_u}\), we have that

\[
\begin{vmatrix}
  s_{i1}^* - \nu & s_{i2}^* & \cdots & s_{iu}^* \\
  s_{i2}^* & s_{22}^* - \nu & \cdots & s_{2u}^* \\
  \cdots & \cdots & \cdots & \cdots \\
  s_{iu}^* & s_{2u}^* & \cdots & s_{uu}^* - \nu
\end{vmatrix} = 0
\]

or \(|s_{ij}^*| - \nu \) (sum of the \(p-1\) rowed principal minors of \(\Lambda_{s_1s_2...s_u}\) ) + \(\nu^2 x\) (sum of the \(p-2\) rowed principal minors of \(\Lambda_{s_1s_2...s_u}\) ) - \(\ldots + (-1)^u \nu^u = 0\)

But \(|s_{ij}^*| = \begin{vmatrix}
  x_{i1}^* & \cdots & x_{is}^* \\
  \cdots & \cdots & \cdots \\
  x_{u1}^* & \cdots & x_{us}^*
\end{vmatrix} \begin{vmatrix}
  x_{11}^* & \cdots & x_{1s}^* \\
  \cdots & \cdots & \cdots \\
  x_{ls}^* & \cdots & x_{qs}^*
\end{vmatrix} \]

which, given \(x_{ij}^*\)'s \((i=1,2,\ldots,u; j=2,\ldots,s)\), is easily seen to be a homogeneous quadric function of \((x_{i1}^*, \ldots, x_{ul}^*) + \text{a constant which is really a function of the other } x_{ij}^*\)'s just mentioned. The coefficients of the quadric function are each polynomial functions of \(x_{ij}^*\)'s \((i=1,2,\ldots,u; j=2,3,\ldots,s)\). Likewise, if we take any \(q\)-rowed principal minor of \(\Lambda_{s_1s_2...s_u}\), say the one with rows and columns numbered \((1,2,\ldots,q)\), then that minor

\[
\begin{vmatrix}
  x_{i1}^* & \cdots & x_{is}^* \\
  \cdots & \cdots & \cdots \\
  x_{q1}^* & \cdots & x_{qs}^*
\end{vmatrix} \begin{vmatrix}
  x_{11}^* & \cdots & x_{ql}^* \\
  \cdots & \cdots & \cdots \\
  x_{ls}^* & \cdots & x_{qs}^*
\end{vmatrix}
\]

which, given \(x_{ij}^*\)'s \((i=1,2,\ldots,q; j=2,\ldots,s)\), is a homogeneous quadratic function of \((x_{i1}^*, \ldots, x_{ql}^*)\) (in which the coefficients are polynomials in \(x_{ij}^*\)'s, with \(i=1,2,\ldots,q\) and \(j=2,\ldots,s) + \text{a constant which is really a}
function of the other $x_{ij}^*$'s just mentioned. Thus, given $v$ and $x_{ij}^*$'s 
$(i=1,2,\ldots,u; \, j=2,\ldots,s)$, the equation (3.2.1) in $v$, yields a homogeneous
quadric surface in $x_{11}^*, \ldots, x_{ul}^*$. Now recall from (2.8) that, given
$y_{ij}$'s, that is $v$, the $(x_{11}^*, \ldots, x_{ul}^*)$ are linear functions of $(x_{11}, \ldots, x_{ul})$
and likewise $(x_{1j}^*, \ldots, x_{uj}^*)$ are linear functions of $(x_{1j}, \ldots, x_{uj})$ (with
$j=2,\ldots,s$). Thus, given $v, \, x_{ij}$'s $(i=1,\ldots,u; \, j=2,\ldots,s)$, the equation
(3.2.1) yields a homogeneous quadric surface in $(x_{11}, \ldots, x_{ul})$ in which the
coefficients and the constant term are all functions of $v, \, Y$ and the other
$x_{ij}$'s already referred to. This is for any characteristic root $v$.

Keeping this result in mind, let us examine more closely the nature
of the domain (2.9) which is the same as the one given by (2.3). Let us
rewrite (2.3) in the equivalent form

$$
(3.2.2) \quad \sup_a \frac{(x_{11}+a_2x_{21}+\ldots+a_ux_{ul})^2 + \ldots + (x_{1s}+a_2x_{2s}+\ldots+a_ux_{us})^2}{(y_{11}+a_2y_{21}+\ldots+a_uy_{ul})^2 + \ldots + (y_{1r}+a_2y_{2r}+\ldots+a_uy_{ur})^2} \leq \mu,
$$

where $a'$ stands for the $(u-1)$-dimensional vector $(a_2, \ldots, a_u)$. Now, given
$\mu, \, Y$ and $x_{ij}$'s $(i=1,\ldots,u; \, j=2,\ldots,s)$, (3.2.2) represents the domain for
$(x_{11}, \ldots, x_{ul})$ in an u-dimensional Euclidean space, the boundary being
given by the surface defined by the equality sign. An equivalent form of the
same surface is the homogeneous quadric associated with (3.2.1) after $v$ is
replaced by $\mu$. Next, (3.2.2) tells us the following:

(i) if a particular point $(x_{11}, \ldots, x_{ul})$ belongs to the domain just mentioned,
then $c(x_{11}, \ldots, x_{ul})$, where $0 < c \leq 1$ also belongs to the domain;

(ii) $(0, \ldots, 0)$ belongs to the domain;

(iii) radiating from $(0, \ldots, 0)$ along any given direction there is only a
finite length belonging to the domain.
Thus, given $\mu, Y$ and the other $x_{ij}$'s (already described), (2.3) or (2.9) represents a domain for $(x_{11}, ..., x_{ul})$ which is the interior of an $u$-dimensional ellipsoid whose boundary is given by (3.2.1) after $\mu$ is substituted for $v$. It is well known that there is an orthogonal transformation by which the ellipsoid can be referred to principal axes, or in other words, the transformed equation to the surface becomes free from the product terms in the transformed variables and involves only the square terms with positive coefficients. Let $\frac{x_i}{1x_1} = \bigcap x_{11}^2, ..., x_{ul}^2$ and let

\[(3.2.3)\]
\[
\frac{x_i}{ux_1} = L \frac{x_i}{ux_i} \frac{x_i}{ux_1},
\]

where $L$ is the orthogonal matrix that transforms the ellipsoid into principal axes. This $L$ can be determined and the rows of $L$, say $\{v_{i1}, ..., v_{ij}\}$ ($i=1, 2, ..., u$) are the cosines of the different principal axes. Note that $z_i z_j = x_{ij} x_{ij}$. It would be useful to rewrite (2.6), after substitution of $\Omega$ for $\Omega$ and omission of the constant, in the form

\[(3.2.4)\]
\[
\int \exp \left\{ \frac{-\frac{1}{2}}{i=1} \left\{ \sum_{s=1}^{u} \frac{u}{s} \sum_{s=1}^{u} \sum_{j=2} \sum_{x_{ij}}^2 + \sum_{s=1}^{u} \sum_{s=1}^{u} \sum_{x_{ij}}^2 \right\} \right\}
\]
\[
\int x \mathrm{d}X \int_{i=1}^{u} \int_{j=1}^{u} \int_{k=1}^{u} \mathrm{d}x_{ij} \mathrm{d}x_{ik} \mathrm{d}z_1,
\]

where, given $\mu, Y$ and the $x_{ij}$'s, the domain $\Omega$, as a domain in $(z_1, ..., z_u)$, forms the interior of an ellipsoid referred to principal axes (that is, in a form which is free from the product terms of $z$'s and involves only the square terms with positive coefficients. In other words, $\Omega$ is symmetrical about the origin in each $z_i$ separately. A displacement $\sqrt[\gamma_1]{\gamma_1}$ along the direction of $x_{11}$ might be regarded as the resultant of a displacement.
\( \chi_{11} / \gamma_1 \) along \( z_1 \), that is, along the direction with cosines \( (\chi_{11}', \chi_{12}', \ldots, \chi_{1u}') \) a displacement \( \chi_{21} / \gamma_1 \) along \( z_2 \), that is, along the direction with cosines \( (\chi_{21}', \chi_{22}', \ldots, \chi_{2u}') \), and so on, and finally a displacement \( \chi_{u1} / \gamma_1 \) along \( z_u \), that is, along the direction with cosines \( (\chi_{u1}', \chi_{u2}', \ldots, \chi_{uu}') \). It should be remembered that these \( \chi_{ij}'s \) are functions of \( \mu, y \) and the \( x_{ij}'s \) of (3.2.4).

### 3.3 The final step in the proof of the monotonicity property

Looking at (3.2.4) and using (3.1.4) we observe that a displacement of \( D \) by \( \chi_{11} / \gamma_1 \) along \( z_1 \) will decrease the integral under (3.2.4), because, for any given set \( \mu, y, x_{ij}'s \) and \( z_2, z_3, \ldots, z_u \),

\[
(3.3.1) \quad \exp \left( -\frac{\alpha}{2} z_1^2 / \gamma_1 \right) < \exp \left( -\frac{\alpha}{2} z_1^2 / \gamma_1 \right)
\]

where \( a \) and \( \chi_{11} / \gamma_1 \), without any loss of generality, can be assumed to be positive, \( a \) being a function of \( \mu, y, x_{ij}'s \) and \( z_2, \ldots, z_u \). Using the same argument for successive displacements by \( \chi_{21} \) along \( z_2 \), by \( \chi_{31} / \gamma_1 \) along \( z_3 \), and so on, and finally by \( \chi_{u1} / \gamma_1 \) along \( z_u \) we have a successive decrease of the integral. In other words, the resultant displacement, which is along \( x_{11} \) and by \( \gamma_1 \) decreases the integral. At this point we go back to (2.10), forget about the \( x_i \)'s, use the result just stated about a displacement by \( \gamma_1 \) along \( x_{11} \), apply successive displacements by \( \gamma_2 \) along \( x_{12} \), \( \gamma_3 \) along \( x_{13} \) and so on, and finally \( \gamma_t \) along \( x_{tt} \) and eventually obtain an integral over the displaced domain \( D^* \) which is less than the one over the original domain \( D \). It is also clear from the mechanics of the proof that the integral over \( D^* \) decreases as each \( |\chi_{ij}| \) (i=1, 2, ..., t) increases separately. This proves the monotonicity property.
4. The case of the test for independence between two sets of variates.

With a \((p+q)\) set \((p<q)\) of variables let us assume, for a sample of size \(n+1\) \((>p+q)\), the canonical distribution law \(\sum_{1}^{\infty}\)

\[
\sum_{1}^{\infty} \left(\sum_{i=1}^{p} \frac{1}{\sqrt{2\pi \rho_i^2}} \exp \left(-\frac{1}{2\rho_i^2} \sum_{j=1}^{q} \rho_i \left(x_{ij}^2 + y_{ij}^2 - 2\rho_i x_{ij} y_{ij}\right) + \sum_{i=p+1}^{q} \frac{1}{\sqrt{2\pi \rho_i^2}} \sum_{j=1}^{n} \rho_i \sum_{i=1}^{p} \sum_{j=1}^{n} dx_{ij} dy_{ij}\right) \right)
\]

where \(\rho_i\)'s are the population canonical correlation coefficients, the hypothesis of independence \(H_0\) is equivalent to the hypothesis that \(\rho_i\)'s = 0, the acceptance region for \(H_0\) is

\[
\sum_{1}^{\infty} \leq \mu,
\]

\(\mu\) is given by

\[
P \left( \sum_{1}^{\infty} \leq \mu \right) | H_0 = 1 - \alpha,
\]

and \(X\) and \(Y\) are given by

\[
X \sim \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pn} \end{bmatrix}
\text{ and } Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{q1} & \cdots & y_{qn} \end{bmatrix}.
\]

It would be profitable to reduce still further both the canonical distribution law \((4.1)\) and the domain \(\mathcal{D}\) of \((4.2)\). Toward this end put

\[
Y_{pxq} \sim L_{q\times q} q_{pxn},
\]
where $T$ is a triangular matrix of the type already described after (2.7) and $L$ is an orthonormal matrix or in other words, $LL' = I(q)$. Next complete $L (qxn)$ into an orthogonal matrix

\[
\begin{bmatrix}
L \\
M
\end{bmatrix}_{n-q}, \text{ so that } \begin{bmatrix}
L \\
M
\end{bmatrix}_{n-q} = \begin{bmatrix}
L' \\
M
\end{bmatrix}_{n-q} = I(n).
\]

Now put

\[
(4.5) \quad x^*_{p\times n} = \begin{bmatrix}
x^*_1 & \cdots & x^*_q \\
\vdots & \ddots & \vdots \\
x^*_p & \cdots & x^*_n
\end{bmatrix} \quad \text{(say)}
\]

\[
= \begin{bmatrix}
U^* & \cdots & V^*
\end{bmatrix}_{p \times q} \quad \text{(say)}.
\]

Thus

\[
X = \begin{bmatrix}
U^* & \cdots & V^*
\end{bmatrix} \begin{bmatrix}
L \\
M
\end{bmatrix} = U^*L + V^*M.
\]

The Jacobian of the transformation given by (4.3) and the first line of (4.5) is

\[
\begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix}
\]

\[
(4.6) \quad 2 \sum_{i=1}^{q} t_{ii}^n / \left| \frac{\partial (LL')}{\partial \mathbf{D}} \right|_{L^I}
\]

where $t_{ij}$'s are the elements of $T$ and $L_D$ and $L_I$ are as in $L_I$. Thus, under this transformation, we have
\[(4.7)\] \[dx dy \longrightarrow \text{const.} \sum_{i=1}^{q} t_{ii}^{n-i} dT \frac{\partial}{\partial L_{i}} dU* dV*,\]

Next observe that

\[(4.8)\] \[\sum_{i,j} x_{ij}^{2} = \text{tr} XX' = \text{tr} (U^* U^* + V^* V^*)\]

\[= \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^{* 2} + \sum_{i=1}^{p} \sum_{j=q+1}^{q} v_{ij}^{2};\]

\[\sum_{i=1}^{p} \sum_{j=1}^{n} y_{ij}^{2} = \sum_{i=p+1}^{n} \sum_{j=1}^{q} t_{ij}^{2}, \quad \sum_{i=p+1}^{n} \sum_{j=1}^{p} t_{ij}^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij}^{2};\]

\[\sum_{j=1}^{n} y_{ij}^{2} x_{ij} = \text{the} \ ii\text{-th element of} \ X Y' = \text{the} \ ii\text{-th element of} \ (U^* L + V^* M)^T;\]

\[= \sum_{j=1}^{n} u_{ij}^{*} t_{ij}.\]

We have now, for $T$, $L_I$, $U^*$ and $V^*$, the distribution

\[\frac{\text{const.} \exp \left( - \frac{1}{2} \right) \sum_{i=1}^{p} \frac{1}{1-\rho_i} \left( \sum_{j=1}^{q} u_{ij}^{* 2} + \sum_{j=q+1}^{q} v_{ij}^{2} - 2\rho_i \sum_{j=q+1}^{q} t_{ij}^{2} \right) + \sum_{i=1}^{p} \frac{1}{1-\rho_i} \sum_{j=q+1}^{q} t_{ij}^{2} \right) \right) dU^* dV^* \]

\[\times \sum_{i=1}^{q} t_{ii}^{n-i} dT dL_{i} / \left( \frac{\partial (L_{i})}{\partial (L_{i})} \right) L_{i}.\]

To express the domain $D$ of (4.2) in terms of the transformed variables we observe that
\[ ch_{\max} \left( (XX')^{-1}(XY')(YY')^{-1}(XY') \right) \]

\[ = ch_{\max} \left( (X*X*)^{-1} X^* \left( \frac{L^T}{M^T} \right) T' \left( \frac{T'T}{L'L} \right)^{-1} \frac{T'}{L'L} \right) X* \]

\[ = ch_{\max} \left( (X*X*)^{-1} X^* \begin{pmatrix} I(q) \\ 0 \end{pmatrix} \begin{pmatrix} I(q) \\ 0 \end{pmatrix} X* \right) \]

\[ = ch_{\max} \left( (U^*U^*)^{-1} \right) \]

Hence we can rewrite (4.2) as

\[ (4.10) \quad \mathcal{D} \quad ch_{\max} \left( (U^*U^*)^{-1} \right) \leq \frac{\mu}{1-\mu} \]

Starting from (4.9) we can also integrate out over \( L_1 \) and have, for \( U^* \) and \( V^* \) the distribution

\[ (4.11) \quad \text{const} \quad \exp \left( \sum_{i=1}^{p} \frac{1}{1-\rho_i} \sum_{j=1}^{q} \left( u^*_i t_{ij} - \rho^*_i t_{ij} \right)^2 + \frac{\rho^*_i}{1-\rho_i} \sum_{i=1}^{p} \frac{t_{ij}^2}{1-\rho_i} \right) \int_{\mathcal{U}^* \mathcal{V}^*} \prod_{i=1}^{n} \left( 1-t_{ii} \right) dt \]

where \( \rho^*_i = \rho_i \), for \( j=1,2,\ldots,i \) and \( i=1,2,\ldots,p \); and = 0, otherwise.

At this stage set

\[ (4.12) \quad U = D_{pxq} / \sqrt{1-\rho_1^2} \quad V^* = D_{px(n-q)} / \sqrt{1-\rho_1^2} \]

\[ \rho^*_i / \sqrt{1-\rho_1^2} = \gamma_{ij} \]

where \( \gamma_{ij} = \rho_1 / \sqrt{1-\rho_1^2} = \gamma_{i} \) (say), for \( j=1,2,\ldots,i \) and \( i=1,2,\ldots,p \);

and = 0 otherwise.

Next observe that under this transformation \( \mathcal{U} \) the characteristic roots of the matrix in (4.10) stay invariant, so that we can rewrite (4.10) as
(4.13) \[ \frac{1}{\lambda} \leq \frac{\mu}{1-\mu}, \text{ ie, } \leq \mu^* \text{ (say)}. \]

We have now for \( U, V \) and \( T \), the distribution

\[
(4.14) \quad \text{const.} \exp \left( \sum_{i=1}^{p} \sum_{j=1}^{q} (u_{ij} - \gamma t_{ij})^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} t_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2 \right) \, x \, dU \, dV \, \frac{q}{1} t_{11}^{n-1} \, dT.
\]

The probability of the second kind of error is given by integrating (4.14) over the domain (4.13). It is easy to see that, aside from the positive constant factor, this is equivalent to

\[
(4.15) \quad \int_{D^*} \exp \left( \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^2 \right. + \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2 \left. \right) \, x \, \frac{q}{1} t_{11}^{n-1} \, dT,
\]

where, for any given set of \( U, V \), \( D^* \) is just \( D \) displaced by \( \gamma t_{11} \) along \( u_{11} \), by \( \gamma t_{21} \) along \( u_{21} \) and \( \gamma t_{22} \) along \( u_{22} \), and so on, and finally by \( \gamma t_{pl} \) along \( u_{pl} \), \( \gamma t_{p2} \) along \( u_{p2} \), ..., \( \gamma t_{pp} \) along \( u_{pp} \). Notice that when \( H_0 \) is true, that is, when all \( \gamma_i \)'s = 0, we should have \( D^* \) replaced by \( D \) in the integral (4.15). Using the same kind of argument as in section 3 it follows that, for any given \( T \), the integral

\[
(4.16) \quad \int_{D} \exp \left( \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2 \right) \, x \, dU \, dV
\]

decreases as \( D \) is displaced by \( \gamma t_{11} \) along \( u_{11} \) and we can continue using the same reasoning for the other displacements. Finally, introducing the density function of \( T \) and going back to (4.15), it is easy to see from considerations of symmetry that the integral (4.15) monotonically decreases.
as each $|\gamma_1|$, that is each $|\rho_1|$ separately increases. The proves the monotonicity property of the power function of the test for independence between two sets of variates.

Concluding remarks. The power functions of the $\lambda$-criteria for the multivariate linear hypothesis and for the test of independence between two sets of variates have also somewhat similar monotonicity properties that will be discussed in a subsequent paper.

References

