THE USE OF A PRELIMINARY TEST FOR INTERACTIONS
IN THE ESTIMATION OF FACTORIAL MEANS

by
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1. INTRODUCTION

1.1. Statement of the Problem

Consider a factorial experiment with two factors, A and B, having p and q levels, respectively. Suppose that each of the pq treatment combinations has \( r(\geq 2) \) replications.

Denoting by \( y_{ijk} \) the response to the treatment combination \( (A_iB_j) \) in the \( k \)-th experimental unit, let

\[
y_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \tag{1.1.1}
\]

\[
i = 1, 2, \ldots, p \]
\[
j = 1, 2, \ldots, q \]
\[
k = 1, 2, \ldots, r,
\]

where \( \mu_{ij} \) is the expected response to the combination \( (A_iB_j) \) per experimental unit, and \( \varepsilon_{ijk} \) is the random component, assumed \( \text{NIID}(0, \sigma^2) \).

We shall further assume that for our experiment the 'fixed effects model' (Eisenhart's Model I) is appropriate, so that we may write

\[
\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}, \tag{1.1.2}
\]

where

\[
\mu = \text{general effect},
\]
\[
\alpha_i = \text{(main) effect of } A_i,
\]
\[
\beta_j = \text{(main) effect of } B_j,
\]
\[
\gamma_{ij} = \text{interaction effect of } A_i \text{ and } B_j,
\]

\( \text{NIID}(0, \sigma^2) \) means 'normally, independently and identically distributed with mean 0 and variance \( \sigma^2 \)'.

and
\[ \sum_{i=1}^{p} \alpha_i = 0 \quad , \quad \sum_{j=1}^{q} \beta_j = 0 \quad , \quad \sum_{j=1}^{q} \gamma_{ij} = 0 \quad \text{for} \quad i = 1, 2, \ldots, p \quad , \quad \sum_{i=1}^{p} \gamma_{ij} = 0 \quad \text{for} \quad j = 1, 2, \ldots, q . \] (1.1.3)

Now suppose our objective is to estimate a cell mean, say \( \mu_{ij} \).

Let \( \ddot{y}_{ij} = \frac{\sum_{k=1}^{r} y_{ijk}}{r} \quad (i = 1, 2, \ldots, p \quad ; \quad j = 1, 2, \ldots, q) \quad , \)\( \ddot{y}_i = \frac{\sum_{j=1}^{q} \ddot{y}_{ij}}{q} \quad , \quad \ddot{y}_j = \frac{\sum_{i=1}^{p} \ddot{y}_{ij}}{p} \) (1.1.4)

and \( \ddot{y}_{..} = \frac{\sum_{i=1}^{p} \sum_{j=1}^{q} \ddot{y}_{ij}}{pq} . \)

The least squares estimators of the parameters in model (1.1.2) are

\[
\hat{\mu} = \ddot{y}_{..} , \\
\hat{\alpha}_i = \ddot{y}_{i..} - \ddot{y}_{..} , \quad \hat{\beta}_j = \ddot{y}_{.j} - \ddot{y}_{..} , \quad \hat{\gamma}_{ij} = \ddot{y}_{ij} - \ddot{y}_{i..} - \ddot{y}_{.j} + \ddot{y}_{..} .
\] (1.1.5)

Thus the least squares estimator of \( \mu_{ij} \) under model (1.1.2), which is in consequence unbiased and minimum-variance under that model, is

\( \ddot{y}_{ij} \) for simplicity, we prefer these symbols to the more elaborate \( \ddot{y}_{ij} , \ddot{y}_{i..} , \ddot{y}_{.j} , \) and \( \ddot{y}_{..} . \)
\[ \hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij} \]

\[ = \bar{y}_{ij}. \]  

We have

\[ E(\hat{\mu}_{ij}) = \mu_{ij} \]  

(1.1.6a)

and

\[ \text{Var}(\hat{\mu}_{ij}) = \sigma^2/r. \]  

(1.1.6b)

If interaction effects are supposed to be absent, then model (1.1.2) reduces to

\[ \hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j, \]  

(1.1.2a)

with the same restrictions on \( \alpha \)'s and \( \beta \)'s as in (1.1.3). Under this no-interaction model, the least squares estimator of \( \mu_{ij} \) is

\[ \hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j \]

\[ = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}. \]

(1.1.7)

which is unbiased and minimum-variance under this model.

Now suppose (1.1.2), and not (1.1.2a), is the appropriate model for our experiment. Noting that \( \hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j \) and \( \hat{\gamma}_{ij} \) are unbiased estimators of the corresponding parameters, that

\[ \text{Var}(\hat{\mu}) = \sigma^2/pqr, \]

\[ \text{Var}(\hat{\alpha}_i) = \sigma^2(q-1)/pqr, \text{ Var}(\hat{\beta}_j) = \sigma^2(p-1)/pqr, \]

\[ \text{Var}(\hat{\gamma}_{ij}) = \sigma^2(p-1)(q-1)/pqr, \]  

(1.1.8)

and that these components are mutually independently distributed, we have then
\[ E(\tilde{\mu}_{ij}) = \mu_{ij} - \gamma_{ij} \]  \hspace{1cm} (1.1.9a)

and

\[ \text{Var}(\tilde{\mu}_{ij}) = \sigma^2[1 + (q-1) + (p-1)]/pqr \]

\[ = \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right] . \]  \hspace{1cm} (1.1.9b)

Suppose further that minimum mean square error (MSE), rather than unbiasedness plus minimum variance, is our criterion of a good estimator. This means in effect that we have in mind a loss function of the form

\[ c(T - \mu_{ij})^2 , \]

where \( c \) is a positive quantity which may depend on the parameters, and \( T \) is the proposed estimator of \( \mu_{ij} \), and our goal is to control the risk

\[ c \ E(T-\mu_{ij})^2 = c \text{MSE}(T) . \]

It is seen that in this situation, even though (1.1.2) is the proper model, \( \tilde{\mu}_{ij} \), rather than \( \hat{\mu}_{ij} \), will be the more desirable estimator at times. For

\[ \text{MSE}(\hat{\mu}_{ij}) = \text{Var}(\hat{\mu}_{ij}) \]

\[ = \frac{\sigma^2}{r} \]  \hspace{1cm} (1.1.10)

and

\[ \text{MSE}(\tilde{\mu}_{ij}) = \text{Var}(\tilde{\mu}_{ij}) + \gamma_{ij}^2 \]

\[ = \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right] + \gamma_{ij}^2 , \]  \hspace{1cm} (1.1.11)

so that

\[ \text{MSE}(\tilde{\mu}_{ij}) \leq \text{MSE}(\hat{\mu}_{ij}) \]
as long as

\[ \gamma_{ij}^2 \leq \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq} = \text{Var}(\gamma_{ij}) \]  \hspace{1cm} (1.1.12) 

As a consequence, if the relevant parameters would be known, the following estimation procedure would be advisable:

\begin{align*}
\text{if } \gamma_{ij}^2 \leq \text{Var}(\gamma_{ij}), & \text{ use } \hat{\mu}_{ij} ; \\
\text{if } \gamma_{ij}^2 > \text{Var}(\gamma_{ij}), & \text{ use } \tilde{\mu}_{ij} .
\end{align*}

However, since these parameters are unknown, the natural procedure that suggests itself is to substitute the outcome of a preliminary test for the parameter inequalities. This leads to the following estimation procedure:

1) First, test for the hypothesis that

\[ \gamma_{ij}^2 / \text{Var}(\gamma_{ij}) , \]

or possibly some average of all pq such quantities (see Chapter 4), is less than or equal to unity.

2) In case significance is found in step 1), take \( \hat{\mu}_{ij} \) as the estimate of \( \mu_{ij} \); otherwise take \( \tilde{\mu}_{ij} \).

Hence, if \( \Phi(x) \) is the probability of rejection of the hypothesis \( \gamma_{ij}^2 \leq \text{Var}(\gamma_{ij}) \) (or similar hypotheses in the \( p \times q \) case), the ultimate estimator \( T \) has the form \( ^{1/} \)

\[ [1 - \Phi(x)] \hat{\mu}_{ij} + \Phi(x) \tilde{\mu}_{ij} . \]

\( ^{1/} \)A next step, not taken in this work, might be to look into more general functions of \( \mu_{ij}, \hat{\mu}_{ij}, \) and some test statistic.
The consequences of such an estimation procedure following a test of significance (a TE procedure, for short) form the subject-matter of the present investigation. The heart of the difficulty is that the interpretation of the joint application of a test and an estimation procedure has to be different from the interpretation that would be valid for their individual applications. The fact that a test and an estimation procedure are carried out on the same data causes these two procedures to be interdependent in the probability sense, which entails rather cumbersome distribution problems.

We shall derive the distribution of the suggested estimator, its bias and mean square error. Next, we shall study how the bias and the mean square error are affected by our choice of \( r \), the number of replications, and \( \alpha \), the level of significance of the preliminary test.

A rather detailed study will be made for the simple case where \( p = q = 2 \) (Chapters 2 and 3). For the general case, directions in which our findings for the \( 2 \times 2 \) case will need to be modified will be indicated and these modifications worked out to some extent (Chapters 4 and 5).

In Chapter 6, we shall set up the general problem of estimation after a preliminary test as a problem in decision-making and shall investigate the nature of Bayes rules and minimax rules. We shall first make some general remarks and then devote our attention to some special cases, viz., the estimation of the mean of a normal population when the variance is known as well as when it is unknown, and the estimation of cell means in a factorial experiment.
1.2. Review of Literature

We have taken as the starting point of our discussion a paper by Anderson (1960) in which he considered the MSE's of parameters as a basis of comparison between alternative models and alternative modes of analysis for factorial experiments.

The problem of estimation after a preliminary test of significance has been studied in a variety of contexts (all different from ours) by a number of investigators. All have examined, as we shall do, how the biases and MSE's are affected by following a TE procedure.

Mosteller (1948), Kitagawa (1950) and Bennett (1952) dealt with a TE procedure for estimating the mean of a normal population $N(\mu_1, \sigma^2)$, when samples from this and another population $N(\mu_2, \sigma^2)$, with the same variance, are given. If we denote by $n_1$, $n_2$ the two sample sizes and by $\bar{x}_1$, $\bar{x}_2$ the two sample means, the procedure consists of making a test for the equality of $\mu_1$, $\mu_2$, and taking $\bar{x}_1$ as the estimate in case of significance and taking the pooled value

$$\frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

otherwise.

Bancroft (1944) considered a similar situation, where samples of sizes $n_1$, $n_2$ drawn from normal populations $N(\mu_1, \sigma^2_1)$ and $N(\mu_2, \sigma^2_2)$ are given and the problem is to estimate $\sigma^2_1$. The hypothesis of equality of the two variances is first tested by means of the sample variances $s^2_1$ and $s^2_2$. In the case of significance, one takes $s^2_1$ as the estimate of $\sigma^2_1$; otherwise, the pooled value
\[
\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}
\]
is taken as the estimate.

Kitagawa (1950) also considered the problem of 'sometimes pooling' in interpenetrating sampling, where actually one has a sample from a bivariate normal population with means \( \mu_x \) and \( \mu_y \), say, and one wants to estimate \( \mu_x \). Supposing \( \bar{x} \) and \( \bar{y} \) are the two sample means, the method followed is to take \( \bar{x} \) or \( (\bar{x} + \bar{y})/2 \) as the estimate, according as the hypothesis of equality of \( \mu_x \) and \( \mu_y \) is rejected or not in a preliminary test of significance.

Bancroft (1944) and Kitagawa (1963) investigated the application of TE procedures to linear regression analysis. Larson and Bancroft (1963) dealt with some aspects of the same problem. Kitagawa (1963) also considered the use of TE procedures in response surface analysis.

Asano (1960) considered the application of TE procedures to some problems in biometrical and pharmaceutical research. The procedures boil down to the estimation of means after testing for variances, and the estimation of a variance-like function of effects in analysis of variance after a preliminary test of these effects.

Bennett (1956) discussed TE procedures in which ultimately one wants interval estimators, rather than point estimators, of means and variances.

Huntsberger (1955) generalized the TE procedure by considering a weighted sum of simple estimators, the weights being determined by the observed value of the test statistic. In the case of normally
distributed estimators, he showed that the weighting procedure on the whole effects a greater degree of control on the MSE than does the TE procedure.

Kitagawa has brought together many of these findings in his recent publication (1963) where he has discussed some other aspects of the TE procedure as well. A mention may also be made of the review article by Bancroft (1964).

There is one aspect in which our TE procedure differs from those envisaged in the investigations cited above. In our case, the hypothesis involved in the preliminary test has the form

\[(\text{parameter})^2 \leq \lambda (\text{variance of its estimator})\]

\(\lambda\) being an appropriate positive constant, whereas in the investigations referred to, the hypothesis is of the usual null type:

\[
\text{parameter} = 0
\]

We are thus concerned with 'material significance' rather than with ordinary significance. The notion of material significance and some methods of testing for material significance in certain specific problems have been discussed by Hodges and Lehmann (1954).
2. THEORETICAL ASPECTS OF 2x2 CASE

2.1. Notation

In the 2x2 case we may consider the problem of estimating the cell mean \( \mu_{11} \). We shall see (in Section 3.1) that the results for this mean hold with little modification for the other cell means as well.

We shall denote by \( \alpha \) \((0 \leq \alpha \leq 1)\) the level of significance at which the preliminary test is made. \( T \) will denote the estimator of \( \mu_{11} \), which equals \( \hat{\mu}_{11} \) in the case of significance and \( \mu_{11} \) otherwise.

For simplicity, we shall write

\[
\begin{align*}
  w &= \bar{y}_{..} + (\bar{y}_1 - \bar{y}_{..}) + (\bar{y}_{1..} - \bar{y}_{..}) = \mu_{11} \\
  z &= \bar{y}_{11} - \bar{y}_1 - \bar{y}_{1..} + \bar{y}_{..} = \hat{\gamma}_{11} = \mu_{11} - \mu_{11}
\end{align*}
\]

(2.1.1)

and

Note that \( \hat{\gamma}_{11} \) is the least squares estimator of \( \gamma_{11} \) in the interaction model (1.1.2).

The sample error variance (under the interaction model) will be denoted by \( s^2 \) and the number of its degrees of freedom by \( v \). Thus here

\[
v = 4(r-1)
\]

(2.1.2)

and

\[
s^2 = \frac{2}{v} \sum_{i=1}^{r} \sum_{j=1}^{2} \sum_{k=1}^{2} (\bar{y}_{ijk} - \bar{y}_{ij})^2 / v
\]

(2.1.3)

2.2. Preliminary Test Procedure and Definition of the Estimator \( T \)

From Chapter 1, p. 5, it is seen that prior to estimation we are to test the hypothesis

\[
H_0: \gamma_{11}^2 \leq \sigma^2 / 4r
\]
or, equivalent,

\[ H_0: \psi \leq 1 \]  \hspace{1cm} (2.2.1a)

against the alternative

\[ H_1: \psi > 1 \]  \hspace{1cm} (2.2.1b)

where

\[ \psi = 4rz^2/\sigma^2 . \]  \hspace{1cm} (2.2.2)

Considerations of invariance under the customary transformation

groups dictate that this test should be based solely on the statistic

\[ 4rz^2/s^2 \]

(vide Lehmann (1959), Chapter 7, Section 5, and Hodges and Lehmann
(1954)). Let \( \alpha \) be the specified level of significance, i.e., suppose
we require that

\[ E_\theta[\varphi(4rz^2/s^2)] \leq \alpha \quad \text{for} \ \theta \in \Theta_0 , \]  \hspace{1cm} (2.2.3)

where \( \varphi(4rz^2/s^2) = \text{probability of rejecting } H_0 \text{ at the point } 4rz^2/s^2 \) and
\( \Theta_0 \) denotes the part of the parameter space \( \Theta \) corresponding to \( H_0 \), just
as \( \Theta_1 \) will denote the part of \( \Theta \) corresponding to \( H_1 \).

Note that generally in testing a hypothesis the goal is to maximize
the power uniformly over \( \Theta_1 \), i.e., in our case to maximize

\[ E_\theta[\varphi(4rz^2/s^2)] \quad \text{for each } \theta \in \Theta_1 , \]  \hspace{1cm} (2.2.4)

subject to (2.2,3), and it is known that the UMP invariant test is given
by:

\[ \varphi_0 = \begin{cases} 1 & \text{if } 4rz^2/s^2 > c \\ 0 & \text{if } 4rz^2/s^2 \leq c \end{cases} . \]  \hspace{1cm} (2.2.5)
However, in a TE procedure the purpose should be to minimize, uniformly over $\theta$, the mean square error of the ultimate estimator (see Section 1.1)

$$T = \{1 - \varphi(4xz^2/s^2)\}w + \varphi(4xz^2/s^2)(w+z)$$

(2.2.6a)

Thus we want to minimize (by making a suitable choice of $\varphi$)

$$\text{MSE}_\theta(T) = E_\theta(T - \mu_{11})^2$$

$$= E_\theta[w + \varphi(4xz^2/s^2)z - \mu_{11}]^2$$

(2.2.6b)

uniformly over $\theta$.

Note that for non-randomized tests $\varphi$

$$\text{MSE}_\theta(T) = E_\theta[(1 - \varphi(4xz^2/s^2))(w - \mu_{11})^2 + \varphi(4xz^2/s^2)(w + z - \mu_{11})^2]$$

(2.2.6c)

We should, therefore, look for an invariant test $\varphi(4xz^2/s^2)$ which has level $\alpha$ and which minimizes (2.2.6b). We shall call an invariant test minimizing (2.2.6b) for each $\theta \in \Theta$ a 'uniformly TE-best invariant test'. However, the following theorem shows that a uniformly TE-best invariant test does not exist.

**Theorem 2.2.1.** For $\alpha > 0$ there is no invariant test $\varphi(4xz^2/s^2)$, randomized or non-randomized, such that

$$E_\theta[\varphi(4xz^2/s^2)] \leq \alpha \quad \text{for} \quad \theta \in \Theta_0$$

(2.2.7a)

and such that

$$\text{MSE}_\theta(T)$$

(see (2.2.6b))

is a minimum for each $\theta \in \Theta$. 
Proof: Consider the test of level 0 (and hence of level $\alpha$)

$$\phi(4rz^2/s^2) = 0.$$  \hspace{1cm} (2.2.8)

For all points of $\Theta$ for which $\psi = 0$ (for definition of $\psi$ see (2.2.2)), this is the TE-best invariant test. For from (2.2.6b) it follows that if $\psi = 0$ (implying $\gamma_{11} = 0$); then

$$\text{MSE}_0(T) = E_0[(w - \mu_{11})^2 + 2 \phi(4rz^2/s^2)(w - \mu_{11})z + \phi^2(4rz^2/s^2)z^2]$$

$$= E_0[(w - \mu_{11})^2] + E_0[\phi^2(4rz^2/s^2)z^2]^{1/2}$$

(since $w$, $z$ and $s$ are independently distributed and $E(w) = \mu_{11}$ when $\gamma_{11} = 0$.)

$$\sum E_0(w - \mu_{11})^2,$$

the equality holding iff $\phi = 0$ a.e.

It will be enough to show that there exists another test of level $\alpha$ which for $\psi \to \infty$ is better than the above test. Put

$$\phi(4rz^2/s^2) = \begin{cases} 1 & \text{if } 4rz^2/s^2 > F_{\alpha} \\ 0 & \text{if } 4rz^2/s^2 \leq F_{\alpha} \end{cases}.$$  \hspace{1cm} (2.2.9)

Here and elsewhere in this thesis $F_{\alpha}$ stands for the upper $\alpha$-point of a certain non-central $F$ distribution--see (2.2.13a). We shall see in what follows that for this test

$$E_0[\phi(4rz^2/s^2)] \leq \alpha \text{ for } \theta \in \Theta_0,$$

so that it satisfies (2.2.7a) as well as $\phi = 0$ does. From (1.1.11) it

$\frac{1}{1/}$Since this theorem admits randomized tests, $\phi$ may assume values other than 0 or 1. Hence $\phi^2$ does not reduce to $\phi$ here, as it did in (2.2.6c).
follows that for the estimator corresponding to the preliminary test (2.2.8)

\[ \text{MSE}_\theta(T) \to \infty \quad \text{if } \psi \to \infty . \]

On the other hand, Theorem 3.5.1 shows that for the estimator corresponding to the preliminary test (2.2.9)

\[ \text{MSE}_\theta(T) \to \sigma^2 / r \quad \text{as } \psi \to \infty . \]

Consequently, it does not seem unreasonable to use the UMP invariant test of level \( \alpha \) as the preliminary test in our TE procedure and to study the properties of the resulting estimator. Thus we shall be using the preliminary test (2.2.5) where \( C \) is so chosen that

\[ \sup_{\theta \in \Theta_0} \Pr[4rz^2 / s^2 > C | \theta] = \alpha . \]

Since the statistic \( 4rz^2 / s^2 \) has a distribution dependent on \( \psi = 4r \gamma_{11}^2 / \sigma^2 \) alone and since the power function of this test is monotonically increasing in \( \psi \), the above is equivalent to the condition

\[ \Pr[4rz^2 / s^2 > C | \psi = 1] = \alpha . \quad (2.2.10) \]

Here \( 4rz^2 / \sigma^2 \) has the non-central \( F \) distribution with \( (1, v) \) d.f. (or, equivalently, the non-central \( t^2 \) distribution with \( v \) d.f.) and non-centrality parameter

\[ \psi = 4r \gamma_{11}^2 / \sigma^2 . \quad (2.2.11) \]

If \( F_{\alpha} \) denotes the upper \( \alpha \)-point of the non-central \( F \) distribution with \( (1, v) \) d.f. and \( \psi = 1 \), then (2.2.10) implies that
\[ C = F_{\alpha} \quad \] (2.2.12)

and then (2.2.5) coincides with (2.2.9).

The probability element of \( F = 4rz^2/s^2 \), given, e.g., in Graybill (1961), is

\[ \exp(-\psi/2) \sum_{m=0}^{\infty} \frac{\left(\psi/2\right)^m}{m!} \cdot \frac{1}{B(m+1/2,\nu/2)} \cdot \frac{(f/\nu)^{m-1/2}}{(1+f/\nu)^{\nu/2+m+1/2}} \, d(f/\nu), \]

\[ 0 \leq f < \infty. \quad (2.2.13) \]

Hence \( F_{\alpha} \) is given by the equation

\[ \alpha = \int_{F_{\alpha}}^{\infty} \exp\left(-1/2\right) \sum_{m=0}^{\infty} \frac{1}{2^m m! B(m+1/2, \nu/2)} \cdot \frac{(f/\nu)^{m-1/2}}{(1+f/\nu)^{\nu/2+m+1/2}} \, d(f/\nu) \quad . \]

(2.2.13a)

For notational simplicity, we shall deal with

\[ x = \frac{4rz^2/\nu s^2}{1 + 4rz^2/\nu s^2} = \frac{F/\nu}{1 + F/\nu} \quad (2.2.14) \]

and

\[ x_{\alpha} = \frac{F_{\alpha}/\nu}{1 + F_{\alpha}/\nu} \quad (2.2.14a) \]

rather than with \( F \) and \( F_{\alpha} \). This \( x_{\alpha} \) is then given by

\[ \exp(-1/2) \sum_{m=0}^{\infty} \frac{1}{2^m m! B(m+1/2, \nu/2)} I_{x_{\alpha}} (m+1/2, \nu/2) = 1 - \alpha, \quad (2.2.14b) \]

where \( I_{x_{\alpha}} (r,s) \) is the incomplete beta function

\[ \frac{1}{B(r,s)} \int_{0}^{x_{\alpha}} x^{r-1} (1-x)^{s-1} \, dx. \quad (2.2.15) \]
2.3. Distribution of the Estimator $T$

Note that the TE estimator (2.2.6a) of the cell mean $\mu_{11}$, in case the preliminary test is defined by (2.2.9), is

$$T = \begin{cases} 
    w & \text{if } F \leq F_\alpha \\
    w+z & \text{if } F > F_\alpha 
\end{cases} \quad (2.3.1)$$

where $F = \frac{4rz^2/s^2}$ and $F_\alpha$ is still defined as in equation (2.2.13a).

Hence the probability element of $T$ is

$$g_1(t|F \leq F_\alpha) \Pr(F \leq F_\alpha) dt + g_2(t|F > F_\alpha) \Pr(F > F_\alpha) dt \quad (2.3.2)$$

where $g_1$ and $g_2$ are the conditional density functions of $w$ and $w+z$ respectively.

Noting that $w$, $z$, and $s^2$ are mutually independently distributed (the first two normally, as $N(\mu_{11} - \gamma_{11}, \frac{3\sigma^2}{4r})$ and $N(\gamma_{11}, \frac{\sigma^2}{4r})$, respectively, and the last one as $\chi^2_{\nu} \frac{\sigma^2}{\nu}$, $\chi^2_{\nu}$ being the (central) $\chi^2$ statistic with $\nu$ d.f.), we may write the expression (2.3.2) as

$$h_1(t) \int_R^R h_2(z) h_3(s^2) dz \, ds^2 \, dt$$

$$+ \int_R^R \int_R^R h_4(t, z) h_3(s^2) dz \, ds^2 \, dt$$

$$= h_1(t) \int_R^R h_2(z) h_3(s^2) dz \, ds^2 \, dt$$

$$+ h_4(t, .) dt - \int_R h_4^*(t/z) h_2(z) h_3(s^2) dz \, ds^2 \, dt$$
\[ = h_4(t, \cdot)dt + \iint_R \left[ h_1(t) - h_4^*(t|z) \right] h_2(z) h_3(s^2)dz \, ds^2 \, dt \quad . \quad (2.3.3) \]

Here \( h_1, h_2 \) and \( h_3 \) are the density functions of \( w, z \) and \( s^2; h_4(t, z) \) the joint density of \( t = w + z \) and \( z; h_4^*(t, \cdot) \) the marginal density of \( t = w + z \) and \( h_4^* \) the conditional density of \( t \) given \( z \); and the region of integration \( R \) is defined by

\[ R = \{(z, s^2) \mid 4rz^2 \leq s^2 \phi \} \quad , \quad (2.3.4) \]

and \( R^c \) is the complement of \( R \).

Since the marginal distribution of \( t = w + z \) is \( N(\mu_{11}, \sigma^2/r) \) and the conditional distribution of \( t = w + z \) given \( z \) is \( N(\mu_{11} + z - \gamma_{11}, 3\sigma^2/4r) \), \( (2.3.3) \) may be written as

\[ (2\pi \sigma^2/r)^{-1/2} \exp\left[ -(t - \mu_{11})^2 / \frac{2\sigma^2}{r} \right] dt \]

\[ + \iint_R \left[ (2\pi \cdot \frac{3\sigma^2}{4r})^{-1/2} \exp\left[ -(t - \mu_{11} + \gamma_{11})^2 / (2 \cdot \frac{3\sigma^2}{4r}) \right] \right] \]

\[ - (2\pi \cdot \frac{3\sigma^2}{4r})^{-1/2} \exp\left[ -(t - \mu_{11} + z - \gamma_{11})^2 / (2 \cdot \frac{3\sigma^2}{4r}) \right] x \]

\[ h_2(z) h_3(s^2)dz \, ds^2 \, dt \quad . \quad (2.3.5) \]

2.4. Expected Value of the Estimator and Its Mean Square Error

The result

\[ (2\pi \eta_2^2)^{-1/2} \int_{-\infty}^{\infty} y \exp\left[ -(y - \eta_1)^2 / (2\eta_2^2) \right] dy = \eta_1 \]

yields from \( (2.3.5) \) the expected value of \( T \):
\[ E_\theta(T) = \int_{-\infty}^{\infty} t \Pr(dt) \]

\[ = \mu_{11} + \iint_{R} \left[ (\mu_{11} - \gamma_{11}) - (\mu_{11} + z - \gamma_{11}) \right] h_2(z) h_3(s^2) dz \, d(s^2) \]

(where, as before, the subscript \( \theta \) serves to remind the reader that these expected values depend on the \( \mu_{ij}, \gamma_{ij} \) and \( \sigma^2 \))

\[ = \mu_{11} - \iint_{R} z h_2(z) h_3(s^2) dz \, d(s^2) \quad . \quad (2.4.1) \]

Again, the result

\[ (2\pi\eta_2^2)^{-1/2} \int_{-\infty}^{\infty} (y-\eta_3)^2 \exp[-(y-\eta_1)^2/(2\eta_2^2)] dy \]

\[ = \eta_2^2 + (\eta_1 - \eta_3)^2 \]

yields from (2.3.5) the MSE of \( T \):

\[ E_\theta(T-\mu_{11})^2 = \int_{-\infty}^{\infty} (t-\mu_{11})^2 \Pr(dt) \]

\[ = \sigma^2/r + \iint_{R} \left[ \left( \frac{3\sigma^2}{4r} + \gamma_{11} \right) - \left( \frac{3\sigma^2}{4r} + (z-\gamma_{11})^2 \right) \right] \times \]

\[ h_2(z) h_3(s^2) dz \, d(s^2) \]

\[ = \sigma^2/r + 2\gamma_{11} \iint_{R} z h_2(z) h_3(s^2) dz \, d(s^2) \]

\[ - \iint_{R} z^2 h_2(z) h_3(s^2) dz \, d(s^2) \quad . \quad (2.4.2) \]
Putting

\[ J_k = \int \int_R z^k h_2(z) h_3(s^2) dz \, ds^2 \]  \hspace{1cm} (2.4.3)

(k is a positive integer),

we thus have

\[ E_\theta(T) = \mu_{11} - J_1 \]  \hspace{1cm} (2.4.4a)

and

\[ \text{MSE}_\theta(T) = \sigma^2 / r + 2\gamma_{11} J_1 - J_2 \]  \hspace{1cm} (2.4.4b)

Now

\[ J_k = \int \int_R (2\pi \frac{\sigma^2}{4r})^{-1/2} z^k \exp[-(z-\gamma_{11})^2/(2 \frac{\sigma^2}{4r})] h_3(s^2) dz \, ds^2 \]

\[ = \exp(-\psi/2)(2\pi \frac{\sigma^2}{4r})^{-1/2} \int \int_R z^k \exp[-z^2/(2 \frac{\sigma^2}{4r})] x \]

\[ \sum_{m=0}^{\infty} \frac{(\gamma_{11}^2 \frac{\sigma^2}{4r})^m}{m!} h_3(s^2) dz \, ds^2 . \]

On making the transformation

\[ u = 4rz^2/\sigma^2 , \]

we have

\[ J_k = \exp(-\psi/2)(2\pi \frac{\sigma^2}{4r})^{-1/2} \int_0^\infty \int_0^{2r} \frac{\sigma^2}{x \sigma^2} \exp(-u/2) \sum_{m=0}^{\infty} \frac{(4r \gamma_{11} / \sigma^2)^m}{m!} \left( \frac{u \sigma^2}{4r} \right)^{\frac{1}{2}(m+k)} \]

\[ + (-1)^{m+k} \left( \frac{u \sigma^2}{4r} \right)^{\frac{1}{2}(m+k)} \} \, h_3(s^2) \frac{\sigma}{\sqrt{4r}} \cdot \frac{du}{2\sqrt{u}} \, ds^2 . \]

Hence
\[ J_1 = \exp(-\psi/2)(2\pi \sigma^2)^{-1/2} \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \exp(-u/2) \sum_{m=0}^\infty \frac{2^{(4\sigma^2\gamma_1^2)/(\sigma^2) + m + 1} \Gamma(m + 1/2)}{(2m + 1) \sigma^{2m + 1}} 
\times \frac{1}{2^{1/2}} \sigma \cdot \frac{du}{2^{1/2}} d(s^2) \]

\[ = (\gamma_1^2/2) \exp(-\psi/2) \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \cdot \frac{\exp(-u/2)(u/2)^{m+1/2}}{\Gamma(m + 1/2)} \]
\[ \times \frac{1}{2^{1/2}} \sigma \cdot \frac{du}{2^{1/2}} d(s^2) \]

\[ = \gamma_1 \exp(-\psi/2) \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \cdot G(1/2, m + 3/2, u) h_3(s^2) du d(s^2) , \]
to use a notation from Rao (1952), viz.,

\[
G(\alpha, p, x) = \begin{cases} 
\frac{\alpha^p}{\Gamma(p)} \exp(-\alpha x) x^{p-1} & \text{for } x \geq 0 \\
0 & \text{for } x < 0.
\end{cases} \quad (2.4.5)
\]

Also

\[ J_2 = \exp(-\psi/2)(2\pi \sigma^2)^{-1/2} \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \exp(-u/2) \sum_{m=0}^\infty \frac{2^{(4\sigma^2\gamma_1^2)/(\sigma^2) + m} \Gamma(m + 1/2)}{(2m + 1) \sigma^{2m + 1}} 
\times \frac{1}{2^{1/2}} \sigma \cdot \frac{du}{2^{1/2}} d(s^2) \]

\[ = \frac{\sigma^2}{4r} \exp(-\psi/2) \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \cdot \frac{\exp(-u/2)(u/2)^{m+1/2}}{\Gamma(m + 1/2)} \]
\[ \times h_3(s^2) du d(s^2) \]

\[ = \frac{\sigma^2}{4r} \exp(-\psi/2) \int_0^\infty \int_0^\infty s^2 F^{(s^2)/(\sigma^2)} \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \cdot (2m + 1) G(1/2, m + 3/2, u) h_3(s^2) du d(s^2) \]
\[ = \frac{\sigma^2}{4\pi} \exp(-\psi/2) \int_0^\infty \int_0^\infty \left[ \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \frac{(\psi/2)^m}{m!} \right] G(1/2, m+3/2, u) \]

\[ + \psi \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \frac{(\psi/2)^m}{m!} \right] h_2(s^2) du d(s^2). \]

Now

\[ h_2(s^2) d(s^2) = G(1/2, \psi/2, \psi s^2/\sigma^2) d(\psi s^2/\sigma^2) \]

since \(\psi s^2/\sigma^2\) has the (central) \(X^2\) distribution with \(\psi\) d.f. Also we have the result, given, e.g., in Rao (1952),

\[ \int G(\alpha, p_1, x) G(\alpha, p_2, x) dx dy \]

\[ \frac{x}{x+y} \leq \frac{w}{w} \]

\[ = \frac{1}{B(p_1, p_2)} \int_0^1 \nu^{p_1-1}(1-\nu)^{p_2-1} d\nu \]

\[ = I_\nu(p_1, p_2), \] (2.4.6)

where again \(I_\nu(p_1, p_2)\) is an incomplete beta function. Consequently,

\[ J_1 = \gamma_1 \exp(-\psi/2) \int \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \frac{(\psi/2)^m}{m!} \right] G(1/2, m+3/2, u) G(1/2, \psi/2, \psi s^2/\sigma^2) \]

\[ \frac{u}{\psi s^2/\sigma^2} \leq \frac{\alpha}{\nu} \]

\[ x dx d(\psi s^2/\sigma^2) \]

\[ = \gamma_1 \exp(-\psi/2) \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \] \(I_\nu(m+3/2, \psi/2)\)

and

\[ J_2 = \frac{\sigma^2}{4\pi} \exp(-\psi/2) \sum_{m=0}^\infty \frac{(\psi/2)^m}{m!} \left[ I_\nu(m+3/2, \psi/2) + \psi I_\nu(m+5/2, \psi/2) \right] \].
The change of the order of integration and summation made here is valid by virtue of Theorem 27B in Halmos (1950).

Let us define the symbol
\[ Q_j(\psi/2, v/2, x_\alpha) \]
by the equation
\[ Q_j(\psi/2, v/2, x_\alpha) = \exp(-\psi/2) \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} I_x (m+j, v/2), \]
where \( j > 0 \). We may then write
\[ J_1 = \gamma_{11} Q_{3/2}(\psi/2, v/2, x_\alpha) \]
and
\[ J_2 = \frac{\sigma^2}{4r} Q_{3/2}(\psi/2, v/2, x_\alpha) + \gamma_{11} Q_{5/2}(\psi/2, v/2, x_\alpha). \]

In terms of this \( Q \) function, (2.2.14b), (2.4.1) and (2.4.2) are as follows:
\[ Q_{1/2}(1/2, v/2, x_\alpha) = 1 - \alpha \]
(2.4.8)

(which determines the cut-off point \( x_\alpha \)),
\[ E_\theta(T) = \mu_{11} - \gamma_{11} Q_{3/2}(\psi/2, v/2, x_\alpha) \]
(2.4.9)

and
\[ E_\theta(T-\mu_{11})^2 = \frac{\sigma^2}{r} + 2\gamma_{11} Q_{3/2}(\psi/2, v/2, x_\alpha) \]
\[ - \frac{\sigma^2}{4r} Q_{3/2}(\psi/2, v/2, x_\alpha) - \gamma_{11} Q_{5/2}(\psi/2, v/2, x_\alpha) \]
\[ = \frac{\sigma^2}{r} \left[ 1 - \frac{Q_{3/2}(\psi/2, v/2, x_\alpha)}{4} \right] + \frac{\psi}{4} \left[ 2Q_{3/2}(\psi/2, v/2, x_\alpha) - Q_{5/2}(\psi/2, v/2, x_\alpha) \right] \]
(2.4.10)

1/Sometimes we shall write simply \( Q_j \).
2.5. Some Properties of the $Q$ Function

(a) Since

$$0 \leq I_{x_{\alpha}}^{(m+j, \nu/2)} \leq 1,$$

we also have, on multiplying each term by $(\nu/2)^m/m!$ and adding over $m$ from 0 to $\infty$,

$$0 \leq Q_j^{(\nu/2, \nu/2, x_{\alpha})} \leq 1,$$  \hspace{1cm} (2.5.1)

the lower and upper bounds being attained only if $x_{\alpha} = 0$ and $x_{\alpha} = 1$ respectively.

For $0 < x_{\alpha} < 1$, somewhat better bounds are obtained as follows:

Making the transformation

$$x = z x_{\alpha},$$

we have

$$\int_0^x x^{m+j-1}(1-x) \nu/2-1 \, dx = x_{\alpha} \int_0^1 z^{m+j-1}(1-x_{\alpha} z) \nu/2-1 \, dz$$

$$> x_{\alpha}^{m+j} \int_0^1 z^{m+j-1}(1-z) \nu/2-1 \, dz.$$

Hence

$$I_{x_{\alpha}}^{(m+j, \nu/2)} > x_{\alpha}^{m+j}$$

and

$$Q_j^{(\nu/2, \nu/2, x_{\alpha})} > x_{\alpha}^j \exp[-(\nu(1-x_{\alpha})/2)]$$  \hspace{1cm} (2.5.2)

Again, making the transformation

$$1 - x = (1-z)(1-x_{\alpha}),$$

we have in a similar manner
\[ I_{x_{\alpha}}(m+j, \nu/2) < 1 - (1-x_{\alpha})^{\nu/2} \]

and

\[ Q_{j}(\psi/2, \nu/2, x_{\alpha}) < 1 - (1-x_{\alpha})^{\nu/2} \]  \hspace{1cm} (2.5.3)

(b) Other things remaining constant, \( I_{x_{\alpha}} \) increases monotonically as \( x_{\alpha} \) increases (i.e., as \( \alpha \) decreases). Hence \( Q_{j}(\psi/2, \nu/2, x_{\alpha}) \) also increases monotonically as \( x_{\alpha} \) increases from 0 to 1 (or, equivalently, as \( \alpha \) decreases from 1 to 0), \( j, \psi \) and \( \nu \) remaining fixed.

(c) As shown by Jordan (1962), p. 85, for \( 0 < x_{\alpha} < 1 \)

\[ I_{x_{\alpha}}(m+j, \nu/2) > I_{x_{\alpha}}(m+j+1, \nu/2) \]

and

\[ Q_{j}(\psi/2, \nu/2, x_{\alpha}) > Q_{j+1}(\psi/2, \nu/2, x_{\alpha}) \]

for any \( j, j' \) with \( 0 < j < j' \) (see Section 2.7).

(d)

\[ \frac{\partial}{\partial \psi} Q_{j}(\psi/2, \nu/2, x_{\alpha}) \]

\[ = \frac{1}{2} \sum_{m=0}^{\infty} \left\{ - \exp(-\psi/2)(\psi/2)^{m} + m \exp(-\psi/2)(\psi/2)^{m-1} I_{x_{\alpha}}(m+j, \nu/2)/m! \right\} \]

\[ = \frac{1}{2} \left[ Q_{j+1}(\psi/2, \nu/2, x_{\alpha}) - Q_{j}(\psi/2, \nu/2, x_{\alpha}) \right] \]

\[ < 0 \]  \hspace{1cm} (2.5.5)

because of (2.5.4), showing that other things remaining the same, \( Q \) is a monotonically decreasing function of \( \psi \), provided \( 0 < x_{\alpha} < 1 \). In Section 3.5 we shall show that actually \( Q \to 0 \) as \( \psi \to \infty \).
2.6. General Observations

On the basis of the results mentioned in the preceding section, some general comments can be made on the bias and MSE of the estimator T (with $0 < \alpha < 1$).

The bias of T is

$$E_\theta(T) - \mu_{11} = -\gamma_{11} Q_3/2(\psi/2, \psi/2, x_\alpha)$$

and in magnitude is

$$|E_\theta(T) - \mu_{11}| = |\gamma_{11}| Q_3/2.$$  \hspace{1cm} (2.6.2)

Because of (2.5.1) then

$$0 < |E_\theta(T) - \mu_{11}| < |\gamma_{11}|,$$  \hspace{1cm} (2.6.3)

i.e., the bias of T lies between that of $\hat{\mu}_{11}$ and that of $\tilde{\mu}_{11}$. Further, (2.5.5) implies that the bias becomes smaller and smaller as $\psi$ increases from 0 to $\infty$.

Again, consider the MSE of T. We have

$$\frac{\text{MSE}_\theta(T)}{\sigma^2/r} = 1 - \frac{Q_3/2}{4} + \frac{\psi}{4} (2Q_3/2 - Q_3/2).$$  \hspace{1cm} (2.6.4)

Hence

$$\frac{\text{MSE}_\theta(T)}{\sigma^2/r} - (\frac{\psi}{4} + \frac{\psi}{4}) = \frac{1}{4} [(1-\psi)(1-Q_3/2) + \psi(Q_3/2-Q_3/2)]$$

$$> 0 \text{ for } 0 \leq \psi \leq 1.$$  \hspace{1cm} (2.6.5)

This means that in the domain $0 \leq \psi \leq 1$ the MSE of T is necessarily greater than that of $\tilde{\mu}_{11}$.

Also
\[
\frac{\text{MSE}_\theta(T)}{\sigma^2/r} - 1 = \frac{1}{4} \left\{ (\psi-1)Q_{3/2} + \psi(Q_{3/2} - Q_{5/2}) \right\} > 0 \text{ for } 1 \leq \psi < \infty . \tag{2.6.6}
\]

It follows that in the domain \( \psi \geq 1 \) the MSE of \( T \) is necessarily greater than that of \( \hat{\mu}_{11} \).

For \( \psi = 0 \)

\[
\frac{\text{MSE}_\theta(T)}{\sigma^2/r} = 1 - I_{x,\alpha}^{(3/2,3/2)} , \tag{2.6.7}
\]

so that at this value of \( \psi \), the MSE is increased by taking a smaller value of \( x,\alpha \) (i.e., a larger value of \( \alpha \)).

Another result that follows from (2.5.1) is that for \( \psi \leq 1/2 \)

\[
\frac{\text{MSE}_\theta(T)}{\sigma^2/r} < 1 - \frac{Q_{5/2}}{\delta} < 1 , \tag{2.6.8}
\]

so that if \( \psi \leq 1/2 \), then \( T \) is necessarily better than \( \hat{\mu}_{11} \).

2.7. Appendix: Monotonicity of Incomplete Beta Function

The incomplete beta function

\[
I_x(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1} dy \bigg/ \int_0^1 y^{p-1}(1-y)^{q-1} dy ,
\]

where \( 0 \leq x \leq 1 \) and \( p, q > 0 \), is known to have the properties that

\[
I_x(p,q) > I_x(p+1,q)
\]

and

\[
I_x(p,q) < I_x(p,q+1) ,
\]

provided \( 0 < x < 1 \) (see, e.g., Jordan (1962), p. 85). We shall show that these results can be generalized, that in fact \( I_x(p,q) \) is, for
0 < x < 1, a monotonically decreasing (increasing) function of the real variable p (of the real variable q).

Let p' > p > 0. Then, denoting the complete beta function by $B(p, q)$, we have

$$B(p, q) B(p', q)[I_x(p, q) - I_x(p', q)]$$

$$= \int_0^1 \int_0^x y^{p-1}(1-y)^{q-1} z^{p'-1}(1-z)^{q-1} \, dy \, dz$$

$$- \int_0^1 \int_0^x y^{p'-1}(1-y)^{q-1} z^{p-1}(1-z)^{q-1} \, dy \, dz$$

$$= \int_x^1 \int_0^x y^{p-1}(1-y)^{q-1} z^{p'-1}(1-z)^{q-1} \, dy \, dz$$

$$- \int_x^1 \int_0^x y^{p'-1}(1-y)^{q-1} z^{p-1}(1-z)^{q-1} \, dy \, dz .$$

Now, in the domain

$$\{(y, z) \mid 0 < y < x, \ x < z < 1\} ,$$

we have

$$z/y > 1 ,$$

so that also

$$\frac{z}{y} > p' - p > 1 ,$$

i.e.,

$$y^{p-1}z^{p'-1} > y^{p'-1}z^{p} .$$

This implies that the expression (2.7.1) is positive and hence that
\[ I_x(p, q) > I_x(p', q) \quad (2.7.2) \]

if \( p' > p > 0 \).

In a similar way, one can show that

\[ I_x(p, q) < I_x(p, q') \quad (2.7.3) \]

if \( q' > q > 0 \).
3. FURTHER STUDY OF 2X2 CASE WITH SOME NUMERICAL RESULTS

3.1. Computational Procedure

Note that

$$\exp(-\psi/2)(\psi/2)^m/m!$$

is the probability that a Poisson r.v. with parameter $\psi/2$ takes the value $m$ ($m = 0,1,2,\ldots\ldots$). These terms have been tabulated, e.g., in the tables published by the Defense System Department of GE (1962). Also values of the incomplete beta function

$$I_x(p,q)$$

for various combinations of $p$ and $q$ and for $x$ ranging from 0 to 1 have been tabulated in Pearson (1934). Using these two sets of tables and applying, wherever necessary, the relation

$$I_x(p,q) = 1 - I_{1-x}(p,q)$$

we have found values of $Q_{1/2}(1/2, \psi/2, x)$ and from these, by inverse interpolation, those of the upper $\alpha$-point, $x_\alpha'$ of our test statistic

$$x = \frac{F/\psi}{1+F/\psi}$$

as defined in (2.2.14).

The same procedure of combining tables of the Poisson distribution with those of the incomplete beta function yields values of

$Q_{1/2}(\psi/2, \psi/2, x_\alpha')$ and $Q_{1/2}(\psi/2, \psi/2, x_\alpha)$, whence, using (2.4.9) and (2.4.10), we have obtained the bias of $T$ as a proportion of $\gamma_{11}$ and $\text{rMSE}(T)/\sigma^2$ for various combinations of $\psi = 4(r-1)$, $\alpha$ and $\psi$. But here we had to resort entirely to the electronic computer, since the tables are not as
extensive as needed for our purpose. The computer generated series of
terms of the Poisson law and of values of the incomplete beta function
and then combined them in the desired manner.

It is to be noted that the formulas for bias and MSE are essentially
the same for each of the cell means $\mu_{ij}$ (i = 1, 2 and j = 1, 2). This is
because $\gamma_{12} = \gamma_{21} = -\gamma_{11}$ and $\gamma_{22} = \gamma_{11}$; while the preliminary test in
each case is based on

$$F = 4r \gamma_{11}^2 / s^2.$$  

Hence the biases of the estimators of all four cell means have the same
magnitude but are either positive or negative, i.e., they are

$$\pm \gamma_{11}^2 / 2,$$

while the MSE's are all equal.

All values appearing in the tables (in this chapter and in Chapter
5) are correct to the last decimal shown.

3.2. Percentage Points of $x$

In order to study the variation of the bias and MSE with $r$, $\alpha$ and
$\psi = 4r \gamma_{11}^2 / s^2$, we have considered three values of $r$, viz., 2, 5 and 10,
which correspond to $\nu = 4, 16$ and 36, respectively. For each of these
values of $r$, we have considered procedures based on tests of levels
$\alpha = .01, .05, .25$ and .50. The two trivial values, $\alpha = 0$ (which
corresponds to making no test and taking $\mu_{11}$ as the estimate) and $\alpha = 1$
(which corresponds to making no test and taking $\mu_{11}$ as the estimate) may
also be taken into account, and for these the percentage points of $x$ are

$$x_0 = 1 \quad (3.2.1)$$
and

\[ x_1 = 0 \quad , \tag{3.2.2} \]

irrespective of the number of replications.

The percentage points \( x_\alpha \) for \( \alpha = .01, .05, .25 \) and .50 are shown in Table 3.2.1.

Table 3.2.1. Upper \( \alpha \)-points of \( x \) (2x2 case)

<table>
<thead>
<tr>
<th>Number of replications ( (r) )</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.908</td>
<td>.789</td>
<td>.492</td>
<td>.246</td>
</tr>
<tr>
<td>5</td>
<td>.479</td>
<td>.342</td>
<td>.160</td>
<td>.0670</td>
</tr>
<tr>
<td>10</td>
<td>.258</td>
<td>.173</td>
<td>.0754</td>
<td>.0303</td>
</tr>
</tbody>
</table>

3.3. Bias as a Proportion of \( \gamma_{11} \)

We have seen in Section 2.6 that

\[ E(T) = \mu_{11} - \gamma_{11} Q_{3/2} \quad . \]

Also \( 0 \leq Q_{3/2} \leq 1 \). Hence \( Q_{3/2} \) is the proportion of \( \gamma_{11} \) that constitutes the bias of the estimator \( T \). The following tables show how the relative bias

\[ \gamma_{11}^{-1} E(\mu_{11} - T) = Q_{3/2} \quad (3.3.1) \]

varies with \( \psi \) for our chosen values of \( r \) and \( \alpha \). Note that \( Q_{3/2} \) is also the proportion of \( \gamma_{i,} \) that constitutes the bias of the estimator \( T_{i,j} \) of \( \mu_{ij} \) for any \( i,j \) \( (1,j = 1,2) \).
The values corresponding to $\alpha = 0$ and $\alpha = 1$ are not shown here. For these two cases, the biases are $\gamma_{11}$ and 0, irrespective of $r$ and $\psi$, since the estimator $T$ equals $\hat{\mu}_{11}$ in one case and $\tilde{\mu}_{11}$ in the other.

Table 3.3.1. Values of $\gamma_{11} E(\mu_{11} - T)$
(r=2)

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.985</td>
<td>.923</td>
<td>.603</td>
<td>.260</td>
</tr>
<tr>
<td>0.2</td>
<td>.983</td>
<td>.915</td>
<td>.586</td>
<td>.243</td>
</tr>
<tr>
<td>0.4</td>
<td>.981</td>
<td>.907</td>
<td>.565</td>
<td>.228</td>
</tr>
<tr>
<td>0.6</td>
<td>.979</td>
<td>.899</td>
<td>.544</td>
<td>.213</td>
</tr>
<tr>
<td>1.0</td>
<td>.975</td>
<td>.883</td>
<td>.505</td>
<td>.186</td>
</tr>
<tr>
<td>1.4</td>
<td>.971</td>
<td>.866</td>
<td>.468</td>
<td>.163</td>
</tr>
<tr>
<td>2.0</td>
<td>.964</td>
<td>.842</td>
<td>.418</td>
<td>.133</td>
</tr>
<tr>
<td>3.0</td>
<td>.952</td>
<td>.800</td>
<td>.344</td>
<td>.095</td>
</tr>
<tr>
<td>4.0</td>
<td>.939</td>
<td>.758</td>
<td>.283</td>
<td>.068</td>
</tr>
<tr>
<td>6.0</td>
<td>.912</td>
<td>.676</td>
<td>.189</td>
<td>.034</td>
</tr>
<tr>
<td>10.0</td>
<td>.850</td>
<td>.524</td>
<td>.082</td>
<td>.009</td>
</tr>
</tbody>
</table>
Table 3.3.2. Values of $\gamma_{11}^{-1} E(\mu_{11}-T)$  
(r=5)

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.987</td>
<td>.925</td>
<td>.588</td>
<td>.233</td>
</tr>
<tr>
<td>0.2</td>
<td>.983</td>
<td>.912</td>
<td>.561</td>
<td>.216</td>
</tr>
<tr>
<td>0.4</td>
<td>.980</td>
<td>.899</td>
<td>.534</td>
<td>.200</td>
</tr>
<tr>
<td>0.6</td>
<td>.976</td>
<td>.885</td>
<td>.509</td>
<td>.186</td>
</tr>
<tr>
<td>1.0</td>
<td>.967</td>
<td>.858</td>
<td>.461</td>
<td>.159</td>
</tr>
<tr>
<td>1.4</td>
<td>.958</td>
<td>.830</td>
<td>.416</td>
<td>.137</td>
</tr>
<tr>
<td>2.0</td>
<td>.942</td>
<td>.786</td>
<td>.357</td>
<td>.109</td>
</tr>
<tr>
<td>3.0</td>
<td>.911</td>
<td>.711</td>
<td>.275</td>
<td>.074</td>
</tr>
<tr>
<td>4.0</td>
<td>.874</td>
<td>.637</td>
<td>.210</td>
<td>.050</td>
</tr>
<tr>
<td>6.0</td>
<td>.790</td>
<td>.498</td>
<td>.120</td>
<td>.023</td>
</tr>
<tr>
<td>10.0</td>
<td>.601</td>
<td>.278</td>
<td>.037</td>
<td>.005</td>
</tr>
</tbody>
</table>

Table 3.3.3. Values of $\gamma_{11}^{-1} E(\mu_{11}-T)$  
(r=10)

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.988</td>
<td>.926</td>
<td>.577</td>
<td>.228</td>
</tr>
<tr>
<td>0.2</td>
<td>.984</td>
<td>.912</td>
<td>.548</td>
<td>.211</td>
</tr>
<tr>
<td>0.4</td>
<td>.980</td>
<td>.897</td>
<td>.520</td>
<td>.195</td>
</tr>
<tr>
<td>0.6</td>
<td>.975</td>
<td>.882</td>
<td>.494</td>
<td>.181</td>
</tr>
<tr>
<td>1.0</td>
<td>.965</td>
<td>.851</td>
<td>.444</td>
<td>.155</td>
</tr>
<tr>
<td>1.4</td>
<td>.954</td>
<td>.820</td>
<td>.399</td>
<td>.132</td>
</tr>
<tr>
<td>2.0</td>
<td>.934</td>
<td>.770</td>
<td>.339</td>
<td>.104</td>
</tr>
<tr>
<td>3.0</td>
<td>.895</td>
<td>.687</td>
<td>.256</td>
<td>.070</td>
</tr>
<tr>
<td>4.0</td>
<td>.849</td>
<td>.605</td>
<td>.192</td>
<td>.047</td>
</tr>
<tr>
<td>6.0</td>
<td>.744</td>
<td>.453</td>
<td>.106</td>
<td>.021</td>
</tr>
<tr>
<td>10.0</td>
<td>.516</td>
<td>.229</td>
<td>.031</td>
<td>.004</td>
</tr>
</tbody>
</table>
From the above tables it is seen that, for fixed \( r \) and \( \alpha \), the bias decreases as \( \psi \) increases from zero. This is just a verification of what we proved theoretically in (2.5.3), viz., that \( Q_{3/2} \) is a monotonically decreasing function of \( \psi \).

For a given \( \alpha \), the bias may start (at \( \psi = 0 \)) at a slightly higher level for a higher value of \( r \). But the rate at which it decreases with increasing \( \psi \) is higher, the higher the value of \( r \); and, on the whole, there is a definite gain in taking a larger number of replications.

As regards the effect of the level of significance \( \alpha \), it is seen that as \( \alpha \) increases, \( r \) and \( \psi \) being kept fixed, the bias decreases. This, again, is a verification of the result in Section 2.5(b) that \( Q_{3/2} \) is a monotonically decreasing function of \( \alpha \). The relative bias may be considered moderate (even when \( \psi \) is small) for a value of \( \alpha \) like \( .50 \) or even \( .25 \).

However, the main subject of our investigation is not the bias but the MSE which will be discussed in the next section.

3.4. Relative Mean Square Error: \( \text{MSE}(T)/(\sigma^2/r) \)

Formula (2.6.4) shows that for fixed \( r \), \( \alpha \) and \( \psi \), the MSE of the proposed estimator \( T \) is given by

\[
\frac{\text{MSE}(T)}{\sigma^2/r} = 1 - \frac{Q_{3/2}}{4} + \frac{\psi}{4} (2Q_{3/2} - Q_{5/2})
\]

\[
= 1 + 0.25 \left[ (2\psi - 1)Q_{3/2} - \psi Q_{5/2} \right].
\]  \hspace{1cm} (3.4.1)

In this section we shall study the behaviour of this ratio for varying \( r \), \( \alpha \) and \( \psi \).
We first note that for $\alpha = 0$, $x_\alpha = 1$ (the case that corresponds to taking $T$ equal to $\mu_{11}$, the least squares estimator under the no-interaction model), $Q_{3/2} = Q_{5/2} = 1$, for all $r$ and $\psi$; hence

$$\frac{\text{MSE}(T)}{\sigma^2/r} \bigg|_{\alpha=0} = 3/4 + \psi/4 \quad . \quad (3.4.2)$$

On the other hand, for $\alpha = 1$, $x_\alpha = 0$ (the case that corresponds to taking $T$ equal to $\mu_{11}$, the least squares estimator under the interaction model), $Q_{3/2} = Q_{5/2} = 0$, for all $r$ and $\psi$; hence

$$\frac{\text{MSE}(T)}{\sigma^2/r} \bigg|_{\alpha=1} = 1 \quad , \quad (3.4.3)$$

identically in $\psi$. We have, therefore, excluded $\alpha = 1$ from the following tables, where we give, for $r = 2$, 5 and 10 and $\psi = 0$, .2, .4, .6, 1.0, 1.4, 2.0, 3.0, 4.0, 6.0 and 10.0, the values of the ratio $\frac{\text{MSE}(T)}{\sigma^2/r}$ for five levels of significance, viz., $\alpha = 0$, .01, .05, .25 and .50.

Table 3.4.1. Values of the ratio $\text{MSE}(T)/(\sigma^2/r)$

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>0</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.75</td>
<td>.754</td>
<td>.769</td>
<td>.848</td>
<td>.935</td>
</tr>
<tr>
<td>0.2</td>
<td>.80</td>
<td>.804</td>
<td>.821</td>
<td>.894</td>
<td>.959</td>
</tr>
<tr>
<td>0.4</td>
<td>.85</td>
<td>.855</td>
<td>.872</td>
<td>.936</td>
<td>.981</td>
</tr>
<tr>
<td>0.6</td>
<td>.90</td>
<td>.905</td>
<td>.922</td>
<td>.976</td>
<td>1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>1.005</td>
<td>1.020</td>
<td>1.047</td>
<td>1.031</td>
</tr>
<tr>
<td>1.4</td>
<td>1.10</td>
<td>1.105</td>
<td>1.115</td>
<td>1.109</td>
<td>1.055</td>
</tr>
<tr>
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<td>1.252</td>
<td>1.252</td>
<td>1.185</td>
<td>1.078</td>
</tr>
<tr>
<td>3.0</td>
<td>1.50</td>
<td>1.495</td>
<td>1.463</td>
<td>1.273</td>
<td>1.096</td>
</tr>
<tr>
<td>4.0</td>
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<td>1.731</td>
<td>1.652</td>
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<td>1.097</td>
</tr>
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<td>2.183</td>
<td>1.966</td>
<td>1.352</td>
<td>1.078</td>
</tr>
<tr>
<td>10.0</td>
<td>3.25</td>
<td>2.993</td>
<td>2.135</td>
<td>1.272</td>
<td>1.047</td>
</tr>
</tbody>
</table>
Table 3.4.2. Values of the ratio $\text{MSE}(T)/(\sigma^2/r)$
(r=5)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi$</th>
<th>0</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.75</td>
<td>0.753</td>
<td>0.769</td>
<td>0.853</td>
<td>0.942</td>
</tr>
<tr>
<td>0.2</td>
<td>0.80</td>
<td>0.805</td>
<td>0.824</td>
<td>0.901</td>
<td>0.965</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.85</td>
<td>0.857</td>
<td>0.878</td>
<td>0.946</td>
<td>0.985</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.90</td>
<td>0.908</td>
<td>0.932</td>
<td>0.987</td>
<td>1.003</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>1.011</td>
<td>1.035</td>
<td>1.058</td>
<td>1.030</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1.10</td>
<td>1.113</td>
<td>1.133</td>
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<td>1.050</td>
<td></td>
</tr>
<tr>
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<td>1.270</td>
<td>1.182</td>
<td>1.069</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
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<td>1.506</td>
<td>1.468</td>
<td>1.247</td>
<td>1.080</td>
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</tr>
<tr>
<td>4.0</td>
<td>1.75</td>
<td>1.733</td>
<td>1.624</td>
<td>1.272</td>
<td>1.077</td>
<td></td>
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<td>2.123</td>
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<td>1.056</td>
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<tr>
<td>10.0</td>
<td>3.25</td>
<td>2.590</td>
<td>1.845</td>
<td>1.140</td>
<td>1.020</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4.3. Values of the ratio $\text{MSE}(T)/(\sigma^2/r)$
(r=10)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi$</th>
<th>0</th>
<th>.01</th>
<th>.05</th>
<th>.25</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.75</td>
<td>0.753</td>
<td>0.769</td>
<td>0.856</td>
<td>0.943</td>
</tr>
<tr>
<td>0.2</td>
<td>0.80</td>
<td>0.805</td>
<td>0.825</td>
<td>0.905</td>
<td>0.966</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.85</td>
<td>0.857</td>
<td>0.880</td>
<td>0.949</td>
<td>0.986</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.90</td>
<td>0.909</td>
<td>0.934</td>
<td>0.989</td>
<td>1.003</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>1.013</td>
<td>1.039</td>
<td>1.059</td>
<td>1.030</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1.10</td>
<td>1.117</td>
<td>1.138</td>
<td>1.115</td>
<td>1.050</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>1.25</td>
<td>1.269</td>
<td>1.275</td>
<td>1.178</td>
<td>1.067</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>1.50</td>
<td>1.511</td>
<td>1.468</td>
<td>1.237</td>
<td>1.077</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>1.75</td>
<td>1.734</td>
<td>1.615</td>
<td>1.256</td>
<td>1.073</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>2.25</td>
<td>2.098</td>
<td>1.777</td>
<td>1.229</td>
<td>1.052</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>3.25</td>
<td>2.436</td>
<td>1.727</td>
<td>1.118</td>
<td>1.018</td>
<td></td>
</tr>
</tbody>
</table>
3.5. Comments on MSE

An examination of the values tabulated in Section 3.4 indicates how the MSE of T changes with r, \( \alpha \) and \( \psi \).

First, for fixed \( r \) and \( \alpha (0 < \alpha < 1) \), as \( \psi \) increases from zero the quantity \( \text{rMSE}(T)/\sigma^2 \) increases for a time, reaches a maximum and then tends to decrease. For small \( \alpha \), this tendency to decrease may not be apparent from Tables 3.4.1-3.4.3, since we have not considered sufficiently large \( \psi \)-values. However, the fact that this is so will follow from Theorem 3.5.1. We first state the following lemma which is proved in Section 3.7:

**Lemma 3.5.1.** For \( a > 0, \alpha \alpha 0, \lim_{\psi \to \infty} \psi \psi Q_\psi(\psi/2, q, x_\alpha) = 0. \)

Since

\[
\text{rMSE}(T)/\sigma^2 = 1 - \frac{Q_{2/2}}{4} + \frac{\psi}{4} (2Q_{3/2} - Q_{5/2})
\]

we immediately have from the lemma

**Theorem 3.5.1.** For \( \alpha > 0 \) \( \text{rMSE}(T|\alpha)/\sigma^2 \to 1 \) as \( \psi \to \infty \).

Secondly, for given \( \alpha \) and \( \psi \), the change in \( \text{rMSE}(T)/\sigma^2 \) is only slight for a change in \( r \). Indeed, in the range from \( \psi = 0 \) to \( \psi = 4.0 \), this ratio is almost the same for \( r = 2 \) as for \( r = 10 \). This is not to say that the MSE itself remains constant. On the contrary, the MSE varies almost as \( \sigma^2/r \) does: it varies almost inversely as \( r \). Also, for higher values of \( \psi \) the ratio itself decreases appreciably as \( r \) increases.

Lastly, for fixed \( r \) and \( \psi \), the MSE is highly sensitive to a change in \( \alpha \) (see Figure 3.5.1). We found in Section 2.6 that for \( 0 < \alpha < 1 \) the MSE is greater than \( [3/4 + \psi/4] \sigma^2/r \) if \( \psi \leq 1 \) and greater than \( \sigma^2/r \) if
ψ > 1. The higher the value of α, the higher generally is the difference between MSE and \([3/4 + ψ/4]σ^2/r\) for ψ ≤ 1 but the lower is the difference between MSE and \(σ^2/r\) on the larger part of the range \((1,∞)\). Further, while the maximum regret\(^1\) in \([0,1]\) increases with increasing α, the maximum regret in \((1,∞)\) decreases with increasing α. The value of ψ at which this second maximum occurs also draws nearer to ψ = 1 as α approaches unity. We can, however, show that this ψ must always be greater than 5/2. In fact, using a result from Section 2.5(d), we have

\[
\frac{3}{\psi} (\text{rMSE}/σ^2) = \frac{1}{8} \left( (3ψ-3)Q_{5/2} - (2ψ-5)Q_{3/2} - ψQ_{7/2} \right). \tag{3.5.3}
\]

But

\[
(3ψ-3)Q_{5/2} - (2ψ-5)Q_{3/2} - ψQ_{7/2} > (2ψ-3)Q_{5/2} - (2ψ-5)Q_{3/2}
\]

\[
= (5-2ψ)(Q_{3/2} - Q_{5/2}) + 2Q_{5/2}
\]

\[
≥ 2Q_{5/2} \text{ if } ψ ≤ 5/2 , \tag{3.5.4}
\]

showing that the point at which the derivative is zero (giving a maximum) must be to the right of 5/2.

It is thus seen that, on the whole, by choosing a low level of significance, we reduce the MSE for ψ∈[0,1], while we increase the MSE

\(^1\)The difference

\[
\text{rMSE}(T|α)/σ^2 - \min_α \text{rMSE}(T|α)/σ^2
\]

may be called our 'regret' resulting from our choice of α. It is a function of the parameter vector α (in the present case of ψ alone).
for \( \psi_{\epsilon}(1, \infty) \). On the other hand, a high level of significance, by and large, leads to a slight increase in the MSE for \( \psi_{\epsilon}[0,1] \) but to a comparatively big reduction for \( \psi_{\epsilon}(1, \infty) \). This would seem understandable from the nature of the power function of the test we are using. Note that this power function is monotonically increasing, as a function of either \( \psi \) or \( \alpha \). Because of what we said in Section 1.1, \( w \) may be called the 'proper estimator' for \( \psi \leq 1 \) and \( w + z \) the 'proper estimator' for \( \psi > 1 \). The probability of choosing the proper statistic then equals \( 1 - (\text{value of the power function}) \) for \( \psi \leq 1 \) and equals the power itself for \( \psi > 1 \). The monotonic character of the power function, therefore, implies that the probability of choosing the proper statistic is smaller (larger) in \([0,1]\) and larger (smaller) in \((1, \infty)\), the larger (smaller) the value of \( \alpha \).

3.6. Optimal Choice of \( \alpha \)

We have seen that when the test procedure suggested in (2.2.5) is used, the ratio \( \text{rMSE}(T)/\sigma^2 \) depends on \( \psi \), \( r \) and \( \alpha \). The non-centrality parameter, \( \psi \), is generally an unknown quantity and hence is beyond the experimenter's control. The number of replications, \( r \), also is in most cases determined by the resources available to the experimenter. However, the level of significance of the preliminary test, \( \alpha \), is completely within his control, and he may choose any \( \alpha \) that he considers appropriate. The aim, of course, is to keep the ratio \( \text{rMSE}(T)/\sigma^2 \) at a low level. The difficulty stems from the fact that an \( \alpha \) that is good
for some \( \psi \)-value may be bad for some other. As always in decision theory, the problem is how to cope with this non-uniformity. We may suggest four approaches which are discussed below:

(a) **Bayes Approach**: Suppose that the experimenter has some prior knowledge of the likely magnitude of \( \psi \) and that this knowledge can be mathematically represented by a distribution function, say \( g(\psi) \). In that case the value of \( \alpha \) (or, equivalently, of \( x_\alpha \)) that minimizes the average

\[
ER(\psi) = \int_0^\infty R(\psi) \, dg(\psi) ,
\]

where

\[
R(\psi) = r\text{MSE}(T)/\sigma^2 ,
\]

may be said to be optimal for the purpose of the preliminary test.

We illustrate this procedure of choosing \( \alpha \) with a gamma-type prior distribution:

\[
dg(\psi) = \beta^s e^{-\beta \psi/2}(\psi/2)^{s-1} d(\psi/2) ,
\]

\[
\beta > 0, \ s > 0, \ \psi \geq 0 .
\]

From (2.4.7) and (2.4.10)

\[
ER(\psi) = 1 - \frac{1}{4} \sum_{m=0}^\infty \frac{\beta^s}{(1+\beta)^{s+m}} \cdot \frac{\Gamma(s+m)}{m!} \cdot I_{x_\alpha}(m+3/2, \nu/2) \\
+ \sum_{m=0}^\infty \frac{\beta^s}{(1+\beta)^{s+m+1}} \cdot \frac{\Gamma(s+m+1)}{m!} \cdot I_{x_\alpha}(m+3/2, \nu/2) \\
- \frac{1}{2} \sum_{m=0}^\infty \frac{\beta^s}{(1+\beta)^{s+m+1}} \cdot \frac{\Gamma(s+m+1)}{m!} \cdot I_{x_\alpha}(m+5/2, \nu/2) .
\]
The derivative of $\text{ER}(\psi)$ w.r.t. $x_\alpha$ equals

\[
\frac{x_\alpha^{1/2}(1-x_\alpha)^{v/2-1}}{\Gamma(v/2)} \left[ -\frac{1}{4} \sum_{m=0}^{\infty} \frac{\Gamma(m+v/2+3/2)\Gamma(m+s+1)}{\Gamma(m+3/2)\Gamma(m+1)} \cdot \left( \frac{x_\alpha}{1+\beta} \right)^m \\
+ \sum_{m=0}^{\infty} \frac{\Gamma(m+v/2+3/2)\Gamma(m+s+1)}{\Gamma(m+5/2)\Gamma(m+1)} \cdot \frac{1}{1+\beta} \cdot \left( \frac{x_\alpha}{1+\beta} \right)^m \\
- \frac{1}{2} \sum_{m=0}^{\infty} \frac{\Gamma(m+v/2+5/2)\Gamma(m+s+1)}{\Gamma(m+5/2)\Gamma(m+1)} \cdot \left( \frac{x_\alpha}{1+\beta} \right)^{m+1} \right].
\]

Equating this to zero and making some simplifications, we find that the optimal $x_\alpha$ (hence the optimal $\alpha$) is given by the equation:

\[
2F_1\left(\frac{v}{2+3/2}, s; 3/2; \frac{x_\alpha}{1+\beta}\right) - \frac{4}{1+\beta} 2F_1\left(\frac{v}{2+3/2}, s+1; 3/2; \frac{x_\alpha}{1+\beta}\right)
\]

\[
+ \frac{2}{3} \left( v+3 \right) \cdot \frac{x_\alpha}{1+\beta} \cdot 2F_1\left(\frac{v}{2+5/2}, s+1; 5/2; \frac{x_\alpha}{1+\beta}\right)
\]

\[= 0, \quad (3.6.4)\]

where $2F_1$ is a hypergeometric function. For given $\beta$ and $s$, this equation may be solved by using some method of iteration. Since our choice of the prior distribution was rather arbitrary, we shall not pursue this matter.

However, some general observations may be made without going into the mathematics of a Bayes solution. Thus, if the experimenter's prior knowledge indicates that the probability is very high that $\psi \leq 1$, he should choose a small $\alpha$. On the other hand, if this probability is small, then an $\alpha$ near to unity ought to be selected.
(b) **Minimax Approach:** In the absence of any prior information regarding the likely magnitude of \( \psi \), one may like to guard against the worst and choose \( \alpha \) in such a way that

\[
\begin{align*}
\max_{\psi \geq 0} \frac{\text{rMSE}_\psi(T)/\sigma^2}{\sigma^2} = \min_{\alpha} \frac{\text{rMSE}_{\psi}(T|\alpha)}{\sigma^2}
\end{align*}
\]  

(3.6.5)

is a minimum. The result (2.6.6) then shows that this \( \text{minimax } \alpha \) is \( \alpha = 1 \). In other words, according to this approach, we should use the estimate \( \hat{\mu}_{11} \), irrespective of the sample observations. Since we had set out to investigate if we could improve upon \( \hat{\mu}_{11} \), this result is not very interesting to us. Of course, if the nature of a concrete problem calls for a choice of a loss function leading to (3.6.5) and for the application of the minimax criterion, then the estimator \( \hat{\mu}_{11} \) is the obvious choice.

(c) **Minimizing the \( L_1 \)-norm of the Regret Function:** Consider for each \( \psi \) the regret

\[
\frac{\text{rMSE}_\psi(T|\alpha)}{\sigma^2} - \min_{\alpha} \frac{\text{rMSE}_{\psi}(T|\alpha)}{\sigma^2} = s(\psi, \alpha),
\]  

(3.6.6)

say. Evidently, this regret function equals

\[
\begin{align*}
s(\psi, \alpha) &= \begin{cases} 
\text{rMSE}(T|\alpha)/\sigma^2 - (3/4 + \psi/4) & \text{for } \psi \in [0, 1] \\
\text{rMSE}(T|\alpha)/\sigma^2 - 1 & \text{for } \psi \in (1, \infty) 
\end{cases}
\end{align*}
\]  

(3.6.7)

An approach that might seem appealing would be to choose an \( \alpha \) that minimizes the total area

\[
\int_0^\infty s(\psi, \alpha) d\psi
\]
under the regret function curve\(^1\). However, the following theorem shows that this approach (of minimizing the L\(_1\)-norm of the regret function) again leads to the estimator \(\hat{\mu}_{11}\), regardless of the observations taken.

**Theorem 3.6.1:** The level of significance \(\alpha\) for which

\[
\int_0^\infty s(\psi, \alpha) d\psi
\]

is a minimum is \(\alpha = 1\).

**Proof:**

\[
\int_0^\infty s(\psi, \alpha) d\psi
\]

\[
= \int_0^1 \left\{ \frac{\text{rMSE}}{\sigma^2} - \left(\frac{3}{4} + \psi/4\right) \right\} d\psi + \int_1^\infty (\text{rMSE}/\sigma^2 - 1) d\psi
\]

\[
= \frac{1}{4} \int_0^1 \left\{ (1-\psi)(1-Q_3/2) + \psi(Q_3/2 - Q_5/2) \right\} d\psi
\]

\[
+ \frac{1}{4} \int_1^\infty \left\{ (\psi-1)Q_3/2 + \psi(Q_3/2 - Q_5/2) \right\} d\psi
\]

\[
= \frac{1}{4} \int_0^1 (1-\psi) d\psi + \frac{1}{4} \int_0^\infty \left\{ (\psi-1)Q_3/2 + \psi(Q_3/2 - Q_5/2) \right\} d\psi
\]

\[
= \frac{1}{8} + \frac{1}{4} \sum_{m=0}^\infty \int_0^\infty \frac{\exp(-\psi/2)(\psi/2)^m}{m!} \left[ (\psi-1)I_x(m+3/2, v/2) \right] d\psi
\]

\(^1\)The difference between our approach and that of Huntsberger (1955) in his Theorem 2 is to be noted. Our approach would correspond to Huntsberger's if we minimized

\[
\int_0^\infty \frac{\text{rMSE}}{\sigma^2} - 1 d\psi
\]

\[
+ \psi \left\{ I_{x^\alpha} \left( \frac{m+3}{2}, v/2 \right) - I_{x^\alpha} \left( \frac{m+5}{2}, v/2 \right) \right\} \right] \psi
\]

\[
= \frac{1}{8} + \frac{1}{2} \sum_{m=0}^{\infty} \left[ \left( 2m+1 \right) I_{x^\alpha} \left( \frac{m+3}{2}, v/2 \right) \right.
\]

\[
= \frac{1}{8} \sum_{m=0}^{\infty} \left[ \left( 2m+1 \right) \left( I_{x^\alpha} \left( \frac{m+3}{2}, v/2 \right) - I_{x^\alpha} \left( \frac{m+5}{2}, v/2 \right) \right) \right]
\]

the equality holding iff \( x^\alpha = 0 \), i.e., iff \( \alpha = 1 \). A remark similar to the one made at the end of Subsection (b) pertains to this result.

(d) **Minimax Regret Approach** (minimizing the Chebyshev norm of the regret function): Another approach which might be suggested, and which would seem quite appropriate in the absence of any prior information regarding the variability of \( \psi \), would be to choose \( \alpha \) in such a manner as to minimize

\[
\text{Max } s(\psi, \alpha) .
\]

An exact determination of this 'minimax regret \( \alpha \)' presents some problems because of the complicated expression for the regret function and has not been attempted. As for a promising suggestion for numerical work, see the end of Section 3.7.

Anyway, it is seen from Tables 3.4.1-3.4.3 that this minimax regret \( \alpha \) lies between .25 and .50. A value less than .25 makes the maximum in \((1, \infty)\) too large, while a value greater than .50 increases the maximum in \([0,1]\) to too high a level.

Apart from any theoretical criteria, it would appear from the tables that a value of \( \alpha \) between .25 and .50 (perhaps even closer to .50)
also effects a satisfactory degree of control over the MSE throughout the range of \( \psi \)-values. Moreover, for such a value of \( \alpha \) the relative bias of the estimator is also only of a moderate magnitude.

Note that in our discussion we have been concerned with a test for the hypothesis

\[
H_0: \quad \psi \leq 1
\]

against the alternative

\[
H_1: \quad \psi > 1
\]

However, the usual tables refer to tests for

\[
H_0: \quad \psi = 0
\]

against the alternative

\[
H_1: \quad \psi > 0
\]

Noting that in both cases the critical regions have the form

\[
4rz^2/s^2 > c
\]

and that the power function for such critical region is monotonically increasing in \( \psi \), it is possible to convert the values of \( \alpha \) related to our test into levels of significance related to the usual ANOVA test. This requires only the determination of the value of the power function at \( \psi = 0 \) when the value at \( \psi = 1 \) is \( \alpha \). Since the power function of our test is

\[
1 - Q_{1/2}(\psi/2, v/2, x_\alpha) \,,
\]

it means that we have to find the value of

\[
1 - Q_{1/2}(0, v/2, x_\alpha) = 1 - I_{x_\alpha}(1/2, v/2) \quad (3.6.9)
\]
when

\[ 1 - Q_{1/2}(1, w/2, x^2) = \alpha \]  \hspace{1cm} (3.6.10)

We show these values corresponding to \( \alpha = .25 \) and \( \alpha = .50 \) in the following table, for the set of \( r \)-values considered in the tables for bias and MSE.

Table 3.6.1. Level of significance for ANOVA test corresponding to \( \alpha = .25 \) and \( .50 \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>.12</td>
</tr>
<tr>
<td>5</td>
<td>.10</td>
</tr>
<tr>
<td>10</td>
<td>.10</td>
</tr>
</tbody>
</table>

Thus we may conclude that for \( r \) between 2 and 10 (and likely beyond) the preliminary test may be conducted as the usual ANOVA test, however, at a level of significance between .10 and .30, most likely closer to .30 than to .10. This would seem to corroborate the suggestion made by Anderson (1960), p. 52, that the preliminary ANOVA test be made at about the 25\% level. One thing is clear: the customary levels of significance, .01 and .05, are completely unsuitable for a TE procedure in the 2x2 case, unless there is prior information that \( \psi \) is very likely to lie in \([0,1]\). For such a value may reduce the MSE appreciably in \([0,1]\) but will make it inordinately high in \((1,\infty)\).
3.7. An Alternative Formula for MSE

The formula for the MSE of $T$ in terms of the $Q$ function will not be suitable for many purposes. One would like to have an expression for it in a closed form rather than in the form of an infinite series. In this section a general method is suggested and the actual formula obtained for the special case of $r = 2$.

The ratio $\text{rMSE}/\sigma^2$ can be written in the form:

$$
\frac{\text{rMSE}}{\sigma^2} = 1 + \frac{\sqrt{\psi}}{2\sqrt{2\pi}} \exp(-\psi/2) \int_0^\infty \exp(-u/2) \sinh(\sqrt{\psi}u)\left\{1 - \gamma(x, \psi/2)\right\} du

- \frac{1}{4\sqrt{2\pi}} \exp(-\psi/2) \int_0^\infty \exp(-u/2) \cosh(\sqrt{\psi}u)/\psi \left\{1 - \gamma(x, \psi/2)\right\} du,
$$

(3.7.1)

where $u = 4rz^2/\sigma^2$, $x = \psi u/2\Gamma$ and $\gamma(x, \psi/2)$ is the incomplete gamma function:

$$
\int_0^x \exp(-t) t^{\psi/2-1} dt / \Gamma(\psi/2)
$$

and $\sqrt{\psi} = \sqrt{4r \gamma^{11}/\sigma}$.

Now putting $s = \sqrt{\psi}/2$

$$
\frac{1}{2\Gamma(s)} \int_0^\infty \left[ \exp\left(-\frac{1}{2}(u - \sqrt{\psi})^2\right) - \exp\left(-\frac{1}{2}(u + \sqrt{\psi})^2\right) \right] \int_0^\infty \exp(-t) t^{s-1} dt
$$

$$
\frac{1}{\sqrt{\psi}/2} = 2(r-1), \text{ an even integer.}
$$
\[
\begin{align*}
&= \frac{1}{2\Gamma(s)} \int_0^\infty \left[ \exp\left\{ - \frac{1}{2}\left(\sqrt{u} - \sqrt{\psi}\right)^2 \right\} - \exp\left\{ - \frac{1}{2}\left(\sqrt{u} + \sqrt{\psi}\right)^2 \right\} \right] \\
&\quad \times \left[ \left(\frac{s}{F}\right)^{s-1} + (s-1)\left(\frac{s}{F}\right)^{s-2} + \ldots + (s-1)t \right] \exp\left( - \frac{s}{F}u \right) du \\
&= \frac{\psi}{\Gamma(s)} \int_0^\infty \left[ \exp\left\{ - \frac{\psi}{2} \left( (y-1)^2 + 2sy^2/F_\alpha \right) \right\} - \exp\left\{ - \frac{\psi}{2} \left( (y+1)^2 + 2sy^2/F_\alpha \right) \right\} \right] \\
&\quad \times \left[ \left(\frac{s\psi}{F}\right)^{s-1} y^{2s-1} + (s-1)\left(\frac{s\psi}{F}\right)^{s-2} y^{2s-3} + \ldots + (s-1)t \right] dy \\
&\quad \text{(3.7.2)}
\end{align*}
\]

by the substitution
\[
\sqrt{u} = y\sqrt{\psi}.
\]

Since only odd powers of \(y\) are involved in the polynomial, this reduces to
\[
\begin{align*}
&= \frac{\psi}{\Gamma(s)} \int_0^\infty \left[ \exp\left\{ - \frac{\psi}{2} \left( (y-1)^2 + 2sy^2/F_\alpha \right) \right\} \left(\frac{s\psi}{F}\right)^{s-1} y^{2s-1} + (s-1)\left(\frac{s\psi}{F}\right)^{s-2} y^{2s-3} \\
&\quad + \ldots + (s-1)t \right] dy \\
&\quad \text{(3.7.3)}
\end{align*}
\]

In a similar way, we have
\[
\begin{align*}
\exp(-\psi/2) \int_0^\infty \exp(-u/2) \cosh(\sqrt{\psi u}) \sqrt{u} \left\{ 1 - \gamma(x,\psi/2) \right\} du \\
&= \frac{\psi^{3/2}}{\Gamma(s)} \int_0^\infty \left[ \exp\left\{ - \frac{\psi}{2} \left( (y-1)^2 + 2sy^2/F_\alpha \right) \right\} + \exp\left\{ - \frac{\psi}{2} \left( (y+1)^2 + 2sy^2/F_\alpha \right) \right\} \right] \\
&\quad \times \left[ \left(\frac{s\psi}{F}\right)^{s-1} y^{2s} + (s-1)\left(\frac{s\psi}{F}\right)^{s-2} y^{2s-2} + \ldots + (s-1)t \right] dy \\
&\quad \text{(3.7.4)}
\end{align*}
\]
Now only even powers of \( y \) are involved in the polynomial, and so the right-hand side of equation (3.7.4) equals

\[
\frac{\psi^{3/2}}{\Gamma(s)} \int_{-\infty}^{\infty} \exp\left[ -\frac{\psi}{2} \left( (y-1)^2 + 2sy^2/F_\alpha \right) \right] \left[ \left( \frac{\psi}{F_\alpha} \right)^{s-1} y^{2s} + (s-1) \left( \frac{\psi}{F_\alpha} \right)^{s-2} y^{2s-2} + \ldots + (s-1)! \ y^2 \right] dy .
\]  

(3.7.5)

If we make the substitution

\[ y = t + F_\alpha/\left( \psi + F_\alpha \right) = v + u_\alpha \]

in (3.7.3) and (3.7.5), then (3.7.1) becomes

\[
\text{rMSE}/\sigma^2 = 1 + \frac{\psi^{3/2}}{2\sqrt{s\pi} \Gamma(s)} \exp\left[ -\frac{\psi}{2} (1-u_\alpha) \right] \int_{-\infty}^{\infty} \exp(-\frac{1}{2} vt^2/x_\alpha) \times \sum_{i=0}^{s-1} \left( \frac{\psi}{F_\alpha} \right)^i (t+u_\alpha)^{2i+1} dt
\]

\[ - \frac{\psi^{3/2}}{2\sqrt{s\pi} \Gamma(s)} \exp\left[ -\frac{\psi}{2} (1-u_\alpha) \right] \int_{-\infty}^{\infty} \exp(-\frac{1}{2} vt^2/x_\alpha) \times \sum_{i=0}^{s-1} \left( \frac{\psi}{F_\alpha} \right)^i (t+u_\alpha)^{2i+2} dt .
\]

(3.7.6)

This can now be simplified by noting that

\[
\int_{-\infty}^{\infty} \exp(-\frac{1}{2} vt^2/x_\alpha) t^n dt = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\left( \frac{\psi}{2x_\alpha} \right)^{-\frac{n+1}{2}} r^{(\frac{n+1}{2})} & \text{if } n \text{ is even}.
\end{cases}
\]

(3.7.7)

Thus, e.g., for the case \( r = 2, s = 2 \), formula (3.7.6) gives
\[ rMSE/\sigma^2 = 1 + \frac{\psi^{3/2}}{2^{3/2}\sqrt{2\pi}} \exp \left\{ -\frac{\psi}{2} (1-x_\alpha) \right\} \]

\[ \times \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \frac{\psi t^2}{x_\alpha} \right) \left\{ (t+x_\alpha) + \frac{2\psi}{F_\alpha} (t+x_\alpha)^4 \right\} dt \]

\[ = 1 + \frac{\psi^{3/2}}{2^{3/2}\sqrt{2\pi}} \exp \left\{ -\frac{\psi}{2} (1-x_\alpha) \right\} \left[ x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right) \Gamma(1/2) \right] \]

\[ + \frac{2\psi}{F_\alpha} \left\{ 3x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right)^{3/2} \Gamma(3/2) + x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right)^{1/2} \Gamma(1/2) \right\} \]

\[ - \frac{1}{2} \left\{ \frac{1}{2} \frac{\psi}{x_\alpha} \right\}^{3/2} \Gamma(3/2) + x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right) \Gamma(1/2) \]

\[ - \frac{\psi}{F_\alpha} \left\{ \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right)^{5/2} \Gamma(5/2) + 6x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right)^{3/2} \Gamma(3/2) \right\} \]

\[ + x_\alpha \left( \frac{1}{2} \frac{\psi}{x_\alpha} \right) \Gamma(1/2) \left[ x_\alpha (1+\psi x_\alpha) - \frac{2\psi}{F_\alpha} \left( 3+6\psi x_\alpha + \psi^2 \frac{x_\alpha}{2} \right) \right] \quad (3.7.8) \]

Note that, for small \( \psi \) and for some analytical purposes, a formula of this type can be more easily used than the infinite series in (2.6.4); e.g., it gives a direct proof of Theorem 3.5.1. (We have retained the earlier proof because it hardly needs any modification in the general context of a p x q experiment, and the generalization of the present formula is bound to be cumbersome.) For numerical purposes too, it
would be more convenient to use than formula (2.6.4), at least with the usual number of replications, since in that case it would require the evaluation of considerably fewer terms. Thus it is much easier to use in locating the point on the \( \psi \)-axis for which the MSE is a maximum or in the determination of the minimax regret \( \alpha \). An iterative procedure for determining the minimax regret \( \alpha \) is suggested here. First note that for each \( \alpha \) the regret function has two local maxima, one for \( \psi \in [0,1] \) and the other for \( \psi \in (1,\infty) \). Our problem is then to determine the \( \alpha \) for which the larger of the two is least. Supposing we draw a graph for each of the local maxima as a function of \( \alpha \), we are to find, proceeding from \( \alpha = 1 \) backwards, the abscissa of the first point of intersection of the two curves. The idea of proceeding from \( \alpha = 1 \) backwards is suggested by the results from Tables 3.4.1-3.4.3 which seem to indicate that the maximum in \( (1,\infty) \) is a monotonically decreasing function of \( \alpha \).

3.8. Appendix: Proof of Lemma 3.5.1

From (2.2), for \( \alpha > 0 \), i.e., for \( x_\alpha < 1 \), \( Q_j(\psi/2,q,x_\alpha) \) is a monotonically increasing function of \( q \). If \( q_0 = [q+1] \), the highest integer contained in \( q+1 \), we then have

\[
Q_j(\psi/2,q_0,x_\alpha) < Q_j(\psi/2,q_0,x_\alpha) \quad (3.8.1)
\]

Now

\[
Q_j(\psi/2,q_0,x_\alpha) = \exp(-\psi/2) \int_0^{x_\alpha} \left( \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} \cdot \frac{\sqrt{m+j-1}}{B(m+j,q_0)}(1-y)^{q_0-1} \right) dy,
\]
the change of the order of summation and integration being admissible by virtue of Theorem 27B in Halmos (1950). Also

\[
\sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} \frac{x^{m+j-1}}{B(m+j,q_0)}
\]

\[
= \frac{1}{\Gamma(q_0)} \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} \frac{x^{m+j-1}}{(m+j)(m+j+1) \ldots (m+j+q_0-1)x^{m+j-1}}
\]

\[
= \frac{1}{\Gamma(q_0)} \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} \left[ \frac{d}{dq_0} \frac{x^{m+j+q_0-1}}{x^{m+j+q_0-1}} \right]
\]

\[
= \frac{1}{\Gamma(q_0)} \cdot \frac{d}{dq_0} \left\{ y^{j+q_0-1} \exp(\psi y/2) \right\}
\]

\[
= \frac{1}{\Gamma(q_0)} \cdot \exp(\psi y/2) \sum_{i=0}^{q_0} a_i y^{j+q_0-i-1}
\]

where

\[
a_i = \binom{q_0}{i} (\psi/2)^{q_0-i} (j+q_0-i) \ldots (j+q_0-r)
\]

with \( i = 0, 1, \ldots, q_0 \).

Hence

\[
Q_j(\psi/2, q_0, x_\alpha) = \frac{1}{\Gamma(q_0)} \int_0^x \exp\left[ -\psi(1-y)/2 \right] \sum_{i=0}^{q_0} a_i y^{j+q_0-i-1}(1-y)^{q_0-1} dy
\]

\[
< \exp\left[ -\psi(1-x_\alpha)/2 \right] \sum_{i=1}^{q_0} b_i (\psi/2)^i \quad \text{ (3.8.2)}
\]

where

\[
b_i = \binom{q_0}{i} \cdot \frac{\Gamma(j+q_0)}{\Gamma(q_0)\Gamma(j+i)} \cdot B_{x_\alpha}(j+i, q_0).
Since $\Sigma_{i=0}^{q_0} b_i (\psi/2)^i$ is a polynomial in $\psi$, the right-hand side of (3.8.2) $\to 0$, implying that $Q_j(\psi/2, q, x_{\alpha})$ also $\to 0$, as $\psi \to \infty$, and so does $\psi^a Q_j(\psi/2, q, x_{\alpha})$ for $a > 0$. 
4. THEORETICAL ASPECTS OF pxq CASE

4.1 Notation

As in the 2x2 case, here also we shall denote by $\alpha (0 \leq \alpha \leq 1)$ the level of significance at which any preliminary test is made. $T_{ij}$ will denote the estimator of $\mu_{ij}$, the population mean for the cell $(i,j)$. This $T_{ij}$ equals $\mu_{ij}$ in the case of significance at the preliminary test and equals $\mu_{ij}$ otherwise.

In line with (2.1.1), we shall write

$$w_{ij} = \bar{y} - (\bar{y}_{.j} - \bar{y}) + (\bar{y}_{.i} - \bar{y})$$

and

$$z_{ij} = \bar{y}_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}$$

(4.1.1)

$w_{ij}$ is the same as $\mu_{ij}$ and $z_{ij}$ is the same as $\gamma_{ij}$, the least squares estimator of $\gamma_{ij}$ under the interaction model (1.1.2).

With (1.1.2) supposed to be the appropriate model for our experiment, we shall denote by $s_{1}^{2}$ the interaction mean square and by $\nu_{1}$ its d.f. The sample error variance will be denoted by $s^{2}$ and its d.f. by $v$. Thus

$$\nu_{1} = (p-1)(q-1), \quad v = pq(r-1),$$

(4.1.2)

and

$$s_{1}^{2} = r \sum_{i=1}^{p} \sum_{j=1}^{q} (\bar{y}_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y})^{2} / \nu_{1}$$

$$s^{2} = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} (\bar{y}_{ijk} - \bar{y}_{ij})^{2} / v.$$

$$= r \sum_{i=1}^{p} \sum_{j=1}^{q} z_{ij}^{2} / \nu_{1}$$

(4.1.3)
4.2. Estimation of a Single Cell Mean

When we proceed to the case of factors, A and B, at least one of which has more than two levels, the problem of the estimation of any particular cell mean and that of the estimation of all pq cell means simultaneously are to be distinguished from each other. The first problem is discussed in the present section mainly for the sake of completeness. It is not to be expected that the need for it will often arise in practice since usually one will conduct a different type of experiment if one is interested only in one treatment combination.

As we stated in Chapter 1, we have taken low MEE as the criterion of a good estimator, and we are making a choice between the two sets of estimators $\hat{\mu}_{ij}$ and $\tilde{\mu}_{ij}$, on the basis of this criterion.

Looking at (1.1.12), we see that, when $p=q=2$, the procedure for estimating a cell mean is the same for all four cells. In fact, in this case, since $\gamma_{ij} = \gamma_{1j}$, the hypotheses are the same for all four cells, and since $z_{ij} = z_{ij}$, the corresponding tests of significance are also the same for all four cells. For the same reasons, the simultaneous estimation of all four means follows the same procedure.

However, the above results do not hold when $p > 2$ and/or $q > 2$, and so the estimation of a single cell mean and that of all cell means simultaneously will then be entirely different problems.

Suppose our problem is to estimate a single cell mean $\mu_{ij}$. Referring back to (1.1.12), we would choose either $\tilde{\mu}_{ij}$ or $\hat{\mu}_{ij}$, according as

$$\gamma_{ij}^2 \leq \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq}$$
or
\[ \gamma_{ij}^2 > \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq} \]
if \( \gamma_{ij} \) and \( \sigma^2 \) were known quantities. As these are unknown, we make our choice on the basis of a test for

\[ H_0: \frac{pq r \gamma_{ij}^2}{(p-1)(q-1)\sigma^2} \leq 1 \]

against the alternative

\[ H_1: \frac{pq r \gamma_{ij}^2}{(p-1)(q-1)\sigma^2} > 1 \]

The UMP invariant test of level \( \alpha \) for \( H_0 \), under appropriate transformation groups (vide Lehmann (1959), Chapter 7, Section 5), has the rejection region

\[ \frac{pq r z_{ij}^2}{(p-1)(q-1)\sigma^2} > F_{\alpha}(1,\nu) \]

where \( F_{\alpha}(1,\nu) \) is the upper \( \alpha \)-point of the non-central F distribution with \( (1,\nu) \) d.f. and non-centrality parameter

\[ \frac{pq r \gamma_{ij}^2}{(p-1)(q-1)\sigma^2} = 1 \]

The problem and the expressions being similar to those in Chapters 2 and 3, we can say off-hand that the findings of those chapters in essence hold here also, if we bear in mind that here

\[ \frac{pq r \gamma_{ij}^2}{(p-1)(q-1)\sigma^2} \]
replaces \( \psi = 4r \gamma_{ij}^2/\sigma^2 \) and \( \nu = pq(r-1) \) replaces \( \nu = 4(r-1) \).
4.3. Simultaneous Estimation of Cell Means

When our problem is to estimate all \( pq \) cell means simultaneously, we need some method of combining the \( pq \) \( MSE \)'s to obtain a single criterion of estimation. This mode of combining the \( MSE \)'s is bound to be more or less arbitrary. We shall consider weighted averages of \( MSE \)'s. This means, in effect, that we have in mind loss functions of the form

\[
C \sum_{i=1}^{p} \sum_{j=1}^{q} \omega_{ij} (T_{ij} - \mu_{ij})^2
\]

(4.3.1)

where \( c > 0 \) may depend on the parameters \( \lambda \), \( \omega_{ij} > 0 \) and \( \sum_{i=1}^{p} \sum_{j=1}^{q} \omega_{ij} = 1 \). The weight \( \omega_{ij} \) will then indicate the importance of the cell \((i,j)\) in the whole set. In a real-world situation, it might be the proportion, among all users of the factors A and B, of those who use the combination \((A_i, B_j)\). The risk function corresponding to (4.3.1) is the weighted average of \( MSE \)'s (to be denoted by \( AMSE \)):

\[
AMSE = C \sum_{i=1}^{p} \sum_{j=1}^{q} \omega_{ij} E (T_{ij} - \mu_{ij})^2 ,
\]

(4.3.2)

except for the multiplier \( c \). Here \( T_{ij} \) is the estimator of \( \mu_{ij} \) under a given estimation procedure.

The minimization of the \( AMSE \) may then be taken as the basis of a choice between rival estimation procedures. For the least squares procedure under the full model,

\[
T_{ij} = \bar{y}_{ij} = \hat{\mu}_{ij} .
\]

\( \lambda \) In the sequel \( c \) will depend on \( \sigma \) (see Section 5.5) alone.
On the other hand, for the least squares procedure under the no-interaction model,

\[ T_{ij} = \bar{y} + (\bar{y}_1 - \bar{y}_-.J) + (\bar{y}_i - \bar{y}_-.) \]

\[ = \tilde{\mu}_{ij} \]

Now

\[ \text{AMSE}(\beta) = \sum_{i,j} \omega_{ij} E(\hat{\mu}_{ij} - \mu_{ij})^2 \]

\[ = \frac{1}{r} \sum_{i,j} \omega_{ij} \sigma^2 = \frac{\sigma^2}{r}, \quad (4.3.3) \]

while

\[ \text{AMSE}(\mu) = \sum_{i,j} \omega_{ij} \left\{ \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right] + \gamma_{ij}^2 \right\} \]

\[ = \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right] + \sum_{i,j} \omega_{ij} \gamma_{ij}^2 \]

\[ (4.3.4) \]

Formulae (4.3.3) and (4.3.4) suggest the following TE procedure:

1. First, test the hypothesis

\[ H_0: \sum \sum \omega_{ij} \gamma_{ij}^2 < \frac{\sigma^2}{r} \frac{(p-1)(q-1)}{pq} \]

against the alternative

\[ H_1: \sum \sum \omega_{ij} \gamma_{ij}^2 > \frac{\sigma^2}{r} \frac{(p-1)(q-1)}{pq} \]

2. In case significance is found in step (1), estimate each \( \mu_{ij} \) by the corresponding \( \hat{\mu}_{ij} \); otherwise estimate each \( \mu_{ij} \) by the corresponding \( \sim \mu_{ij} \).

\[ ^1/ \] In this work we have only considered the choice between neglecting the interactions for all cell means and including the interactions for all cell means. A question for future research would be how to allow for cases where interactions may be neglected for some cell means and included for others.
However, a satisfactory test for $H_0$ against $H_1$ (for unequal weights) is not available. Even the heuristic test, viz., the one based on

$$pqr \sum_{i,j} \lambda_{ij}^2 s_{ij}^2 / \sigma^2$$

presents such a distribution problem, especially because the distribution will not depend solely on

$$\sum_{i,j} \lambda_{ij}^2 / \sigma^2 \quad ,$$

as would make it meaningless to be too factitious on this point. In step (1) we shall, therefore, be concerned with a test for

$$H_0': r \sum_{i,j} \gamma_{ij}^2 / \sigma^2 \leq (p-1)(q-1) \quad (4.3.5)$$

against

$$H_1': r \sum_{i,j} \gamma_{ij}^2 / \sigma^2 > (p-1)(q-1) \quad (4.3.6)$$

In this way, we shall make the preliminary test independent, as it were, of the choice of the weighting system.

Note that $H_0'$ and $H_1'$ correspond to the traditional ANOVA test for interactions, except for the aspect of material significance. (In traditional ANOVA we test the hypothesis

$$r \sum_{i,j} \omega_{ij}^2 / \sigma^2 = 0$$

against the alternative

$$r \sum_{i,j} \omega_{ij}^2 / \sigma^2 > 0 . \)$$

Although we shall apply this simplification to the test procedure in step (1), we shall not apply it to the loss function to be used in assessing the AMSE of the estimators used in step (2); there we shall still allow the weights attached to the different cells to be unequal.
As we shall see, the properties (the bias and AMSE) of the estimators $T_{ij}$ will then depend on two parameters,

$$
\psi = r \Sigma \Sigma \gamma_{ij}^2/\sigma^2
$$

and

$$
\psi^* = pqr \Sigma \Sigma \omega_{ij} \gamma_{ij}^2/\sigma^2.
$$

4.4. Some Relations between $\psi$ and $\psi^*$

Obviously,

$$
\Sigma \Sigma \omega_{ij} \gamma_{ij}^2 \leq \max_{i,j} \gamma_{ij}^2.
$$

Since

$$
\psi \geq r \max_{i,j} \gamma_{ij}^2/\sigma^2,
$$

we then have

$$
\psi \geq \psi^*/pq.
$$

In case either $p$ or $q$ is 2, the inequality (4.4.1) can be improved upon; for then there will be two pairs $(i,j)$ for which $\gamma_{ij}^2$'s are a maximum, and we shall have

$$
\psi \geq \psi^*/pq.
$$

From (4.4.1) it follows that:

(i) if $\psi = 0$, then $\psi^* = 0$;

and

(ii) if $\psi^* \to \infty$, then $\psi \to \infty$.

Also

$$
\psi^* - \psi = \frac{pqr}{\sigma^2} \Sigma \Sigma (\omega_{ij} - \bar{\omega})(\gamma_{ij}^2 - \bar{\gamma}^2),
$$

where

$$
\bar{\omega} = \Sigma \Sigma \omega_{ij}/pq = 1/pq
$$

and

$$
\bar{\gamma}^2 = \Sigma \Sigma \gamma_{ij}^2/pq.
$$
We have, therefore, the following rough interpretation:

If the weights are all equal (= 1/pq), then \( \psi^* = \psi \). If generally larger weights go with larger \( \gamma^2_{ij} \)'s, then \( \psi^* > \psi \). On the other hand, if generally larger weights go with smaller \( \gamma^2_{ij} \)'s, then \( \psi^* < \psi \).

4.5. Preliminary Test Procedure

In choosing a satisfactory test for \( H'_0 \) against \( H'_1 \), our purpose is, of course, the minimization of the AMSE (see equation (4.3.2)) uniformly over the parameter space. However, for the same reasons as those mentioned in Section 2.2, we shall be using a test that is best from the point of view of power rather than of AMSE.

Referring to Lehmann (1959), we see that invariance considerations demand that a test for \( H'_0 \) should be based on the statistic

\[
\frac{r \sum \sum \gamma^2_{ij}/v_1}{s^2} = \frac{s^2_1}{s^2}.
\]

(4.5.1)

The UMP invariant test of level \( \alpha \) has the rejection region

\[
\frac{2}{s^2} > C
\]

where \( C \) is so chosen that

\[
\Pr\left[ \frac{s^2_1}{s^2} > C \mid \psi = v_1 \right] = \alpha
\]

(4.5.2)

(vide Lehmann (1959) and Hodges and Lehmann (1954)).

Now \( s^2_1/s^2 \) has the non-central F distribution with \( v_1 \) and \( v \) d.f. and with non-centrality parameter \( \psi \). Let us denote this statistic by \( F(v_1, v; \psi) \). Then \( C \) must be the upper \( \alpha \)-point of \( F(v_1, v; v_1) \):

\[
C = F_{\alpha}(v_1, v; v_1),
\]

(4.5.3)

or simply \( F_{\alpha} \).
As in Chapter 2, it will be convenient to deal with the statistic

$$\frac{v_1 s^2}{1 + v_1 s^2} = x,$$  

(4.5.4)

say, rather than with $s^2$, and with

$$\frac{v_1 \frac{F}{\alpha}}{1 + v_1 \frac{F}{\alpha}} = x_\alpha,$$  

(4.5.5)

say, rather than with $F_\alpha$.

Since the probability element of $F(v_1, v; \psi)$ is

$$\exp(-\psi/2) \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} \cdot \frac{1}{B(\frac{v_1}{2} + m, \frac{\psi}{2})} \cdot \frac{\psi_1^{1/2} (v_1 - 2 + m)}{(1 + \frac{v_1}{2})^{1/2} (v_1 + \psi + m)} \cdot d(\frac{1}{\psi_1}),$$  

(4.5.6)

(see, e.g., Graybill (1961)), that of $x$ is

$$\exp(\psi/2) \sum_{m=0}^{\infty} \frac{\psi/2)^m}{m!} \cdot \frac{1}{B(\frac{v_1}{2} + m, \frac{\psi}{2})} \cdot x^{1/2} (1-x)^{v_1/2 - 1} \cdot (1-x)^{(1-x)^{1/2} - 1},$$  

(4.5.7)

Hence $x_\alpha$ is given by

$$\exp(-\psi_1/2) \sum_{m=0}^{\infty} \frac{(\beta v_1)^m}{m!} \cdot \frac{1}{B(\beta v_1 + m, \beta v)} \cdot I_\alpha \left( \frac{\beta v_1 + m}{2}, \frac{\beta v}{2} \right) = 1 - \alpha,$$

i.e., by

$$Q_{\psi_1/2} \left( \frac{v_1}{2}, \frac{\psi}{2}; x_\alpha \right) = 1 - \alpha,$$  

(4.5.8)

using a notation introduced in Section 2.4.
4.6. Distribution of the Estimator of a Cell Mean

Considering any of the pq cell means, let us study the nature of the distribution of its estimator, the estimation being based on the results of the preliminary test mentioned above. Without any loss of generality, we may consider the case with \( i=j=1 \), for which the cell mean is \( \mu_{11} \) and the estimator is \( T_{11} \).

Note that \( T_{11} \) is defined by:

\[
T_{11} = \begin{cases} 
\bar{w}_{11} & \text{if } F \leq F_{\alpha} \\
\bar{w}_{11} + z_{11} & \text{if } F > F_{\alpha} 
\end{cases}
\]

where \( \bar{w}_{11} \) and \( z_{11} \) are as given by (4.1.1). The probability element of \( T_{11} \) is then

\[
ge_1(t|F \leq F_{\alpha}) \Pr (F \leq F_{\alpha}) dt + e_2(t|F > F_{\alpha}) \Pr (F > F_{\alpha}) dt,
\]

where \( e_1 \) and \( e_2 \) are the conditional density functions of \( \bar{w}_{11} \) and \( \bar{w}_{11} + z_{11} \) respectively.

Now \( \bar{w}_{11} \) and \( z_{11} \) are independently distributed, the former as
\[
N(\mu_{11} - \gamma_{11}, \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right])
\]

and the latter as
\[
N(\gamma_{11}, \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq}).
\]

As to the test statistic \( F = \frac{s_{1}^2}{s^2} \), we know that

\[
\frac{\nu s^2}{\sigma^2}
\]

has the (central) \( x^2 \) distribution with \( \nu \) d.f. and is distributed independently of \( \bar{w}_{11} \) and \( z_{11} \). Before taking up \( s_{1}^2 \), let us state the following generalization of the Cochran-Fisher theorem, first proved by Medow (1940). We state it in the form given by Graybill (1961):
Theorem 4.6.1: Let \( y_1, y_2, \ldots, y_n \) be independent random variables, \( y_r \) being distributed as \( N(\mu_r,1) \), and let
\[
\mathbf{Y}' = \sum_{i=1}^{k} \mathbf{Y}'_i \mathbf{A}_i \mathbf{Y}.
\]

Then a necessary and sufficient condition for
\[
\mathbf{Y}' \mathbf{A}_i \mathbf{Y}
\]
to be distributed as a non-central \( \chi^2 \) with \( d.f. \ n_i \) and non-centrality parameter \( \lambda_i \), where \( n_i \) is the rank of \( \mathbf{A}_i \) and
\[
\lambda_i = \mu' \mathbf{A}_i \mu
\]
and for \( \mathbf{Y}' \mathbf{A}_i \mathbf{Y} (i=1,2,\ldots,k) \) to be mutually independent is that
\[
\sum_{i=1}^{k} n_i = n.
\]

It will be convenient to consider
\[
v_{1} s_{1}^2 / \sigma^2
\]
to be composed of two parts:
\[
u = \frac{pqr z_{11}^2}{(p-1)(q-1) \sigma^2}
\]
and
\[
U = v_{1} s_{1}^2 / \sigma^2 - u.
\]

Here \( \mathbf{Y}' = (y_1,y_2,\ldots,y_n) \), \( \mu' = (\mu_1,\mu_2,\ldots,\mu_n) \) and \( \mathbf{Y}, \mu \) are the corresponding column vectors. \( \mathbf{A}_i (i=1,2,\ldots,k) \) is an nxn symmetric matrix.
It is known that $\nu_1 s_1^2 / \sigma^2$ has the non-central $\chi^2$ distribution with $\nu_1$ d.f. and non-centrality parameter

$$ r \sum_{i,j} \gamma_{ij}^2 / \sigma^2 = \psi. $$

On the other hand, $u$ has the non-central $\chi^2$ distribution with 1 d.f. and non-centrality parameter

$$ \frac{pq \gamma_{11}^2}{(p-1)(q-1)\sigma^2} = \psi_1 \quad \text{(say)}. \quad (4.6.6) $$

By making a suitable orthogonal transformation on the variables $y_{ijk} (i=1,2,\ldots,p; j=1,2,\ldots,q; k=1,2,\ldots,r)$, we can then show from Theorem 4.6.1 that $U$ has the non-central $\chi^2$ distribution with $\nu_1 - 1$ d.f. and non-centrality parameter

$$ \psi - \psi_1 = \psi_2 \quad \text{(say)}, \quad (4.6.7) $$

and that $U$ is independent of $z_{11}$.

Also both $z_{11}$ and $U$ are independent of $w_{11}$ and $s^2$.

Hence (4.6.2) may be written as

$$ h_1(t) \int \int \int_R h_2(z_{11}) h_3(U) h_4(s^2) dz_{11} dU d(s^2) dt $$

$$ + \int \int \int_R h_5(t, z_{11}) h_3(U) h_4(s^2) dz_{11} dU d(s^2) dt $$

$$ - h_1(t) \int \int \int_R h_2(z_{11}) h_3(U) h_4(s^2) dz_{11} dU d(s^2) dt $$

$$ + h_5(t, \ldots) dt - \int \int \int_R h_5(t, z_{11}) h_3(U) h_4(s^2) dz_{11} dU d(s^2) dt $$

$$ = h_5(t, \ldots) dt + \int \int \int_R [h_1(t) - h_5^*(t | z_{11})] h_2(z_{11}) h_3(U) h_4(s^2) dz_{11} dU d(s^2) dt \quad (4.6.8) $$
Here \( h_1, h_2, h_3 \) and \( h_4 \) are the density functions of \( w_{11}, z_{11}, U \) and \( s^2 \), \( h_2(t, z_{11}) \) is the joint density of \( t = w_{11} + z_{11} \) and \( z_{11} \), \( h_2(t, \cdot) \) the marginal density of \( t = w_{11} + z_{11} \), and \( h_2^* \) the conditional density of \( t \) given \( z_{11} \); the region of integration \( R \) is defined by

\[
R = \left\{ (z_{11}, U, s^2) | u + U \leq \frac{\nu_1 s^2}{\sigma^2} \right\} \tag{4.6.9}
\]

and \( R^c \) is the complement of \( R \). By reductions similar to those in Section 2.3 the above expression may be put in the form:

\[
(2\pi \sigma^2/r)^{-\frac{1}{2}} \exp \left\{ - \frac{(t - \mu_{11})^2}{2 \sigma^2/r} \right\} dt
\]

\[
+ \iint_{R} \left[ (2\pi \cdot \frac{p+q-1}{pq} \cdot \frac{\sigma^2}{r})^{-\frac{1}{2}} \exp \left\{ - \frac{(t - \mu_{11} + \gamma_{11})^2}{2 \cdot \frac{p+q-1}{pq} \cdot \frac{\sigma^2}{r}} \right\} \right]
\]

\[
- (2\pi \cdot \frac{p+q-1}{pq} \cdot \frac{\sigma^2}{r})^{-\frac{1}{2}} \exp \left\{ - \frac{(t - \mu_{11} + \gamma_{11})^2}{2 \cdot \frac{p+q-1}{pq} \cdot \frac{\sigma^2}{r}} \right\}
\]

\[
h_2(z_{11})h_3(U)h_4(s^2)dz_{11}dUd(s^2)dt.
\]

4.7. Expected Value and MSE of the Estimator of a Cell Mean, and AMSE

From the above density function, we obtain, putting

\[
J_k = \iint_{R} z_{11}^k h_2(z_{11})h_3(U)h_4(s^2)dz_{11}dUd(s^2) \tag{4.7.1}
\]

\((k \text{ is a positive integer}), \)

\[
E_\theta(T_{11}) = \mu_{11} - J_1 \tag{4.7.2}
\]

and

\[
E_\theta(T_{11} - \mu_{11})^2 = \sigma^2/r + 2\gamma_{11}J_1 - J_2. \tag{4.7.3}
\]

These results are analogous to those in Chapter 2, except for the variate \( U \) in (4.7.1), which was zero in the earlier case.
Now using \( \psi_1, \psi_2 \) and \( \psi \), as defined in equations (4.6.6), (4.6.7) and (4.3.7), we make the following reductions:

\[
J_k = \exp\left(-\psi_1/2\right)(2\pi)^{-1/2} \iint \exp(-u/2) \sum_{m=-\infty}^{\infty} \left(\frac{pq\gamma_{11}}{v_1\sigma^2}\right)^m x \\
\left\{\frac{v_1 u}{pq\sigma^2} + \left(-1\right)^{m+k}\left(\frac{v_1 u}{pq\sigma^2}\right)^2\right\} h_3(u)h_4(s^2) \frac{du}{2\sqrt{u}} dU d(s^2) . \quad (4.7.4)
\]

For \( k=1 \), terms under the summation sign corresponding to even values of \( m \) vanish, and so

\[
J_1 = \exp\left(-\psi_1/2\right)(2\pi)^{-1/2} \iint \exp(-u/2) \sum_{m=-\infty}^{\infty} \frac{\left(\frac{pq\gamma_{11}}{v_1\sigma^2}\right)^m (\frac{v_1 u}{pq\sigma^2})^{m+1}}{(2m+1)!} x \\
h_3(u)h_4(s^2) \frac{du}{\sqrt{u}} dU d(s^2) \\
= \gamma_{11} \exp(-\psi_1/2) \iint \sum_{m=-\infty}^{\infty} \frac{(\psi_{1/2})^m}{m!} G(\frac{1}{2}, m + \frac{3}{2}, u) h_3(u)h_4(s^2) du dU d(s^2)
\]

Also (cf. (4.6.7))

\[
h_3(u) = \exp(-\psi_2/2) \sum_{n=0}^{\infty} \frac{(\psi_{2/2})^n}{n!} \alpha_n \left(\frac{1}{2}, n + \frac{1}{2}, u\right).
\]

Hence

\[
J_1 = \gamma_{11} \exp(-\psi/2) \iint \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(\psi_{1/2})^m (\psi_{2/2})^n}{m! n!} \alpha_n \left(\frac{1}{2}, m + \frac{3}{2}, n + \frac{1}{2}, u\right) x h_4(s^2) du dU d(s^2) .
\]

If we put

\[
U' = u + U,
\]

then this reduces to
\[ J_1 = \gamma_{11} \exp(-\psi/2) \int_0^\infty \int_0^\infty \frac{v_1 s^2 F_p^2/\sigma^2}{\Gamma(\psi/2)} \frac{1}{p!} \Gamma\left(\frac{1}{2}, p + \frac{v_1}{2} + 1, u\right) \rho_4(s^2) du'd(s^2) \]

\[ = \gamma_{11} \exp(-\psi/2) \int_0^\infty \int_0^\infty \frac{v_1 s^2 F_p^2/\sigma^2}{\Gamma(\psi/2)} \frac{1}{p!} \Gamma\left(\frac{1}{2}, p + \frac{v_1}{2} + 1, U'\right) \Gamma\left(\frac{1}{2}, \frac{v}{2}, \frac{v s^2}{\sigma^2}\right), \]

\[ x \, du' \, d(v s^2/\sigma^2) \]

\[ = \gamma_{11} \exp(-\psi/2) \sum_{p=0}^{\infty} \frac{\psi/2}{p!} I_x(\alpha) (p + \frac{v_1}{2} + 1, \frac{v}{2}), \quad (4.7.5) \]

in a manner similar to that in Section 2.4. The validity of the change of the order of integration and summation follows from Theorem 27B in Halmos (1950).

When \( k \in \mathbb{C} \), terms in (4.7.4) that correspond to odd values of \( m \) vanish and so

\[ J_2 = \exp(\psi_{1/2})(2\pi)^{-\frac{3}{2}} \iiint_R \exp(-u/2) \sum_{m=0}^{\infty} \frac{v_1 \sigma^2}{m!} \left(\frac{v_1 \sigma^2}{u}\right)^m \frac{1}{pqr} \frac{1}{\Gamma(2m+1)} \frac{1}{2m} \]

\[ \times h_3(u) \rho_4(s^2) du \frac{du}{\sqrt{u}} \, dU \, d(s^2) \]

\[ = \frac{v_1 \sigma^2}{pqr} \exp(-\psi_{1/2}) \iiint_R \sum_{m=0}^{\infty} \frac{\psi_{1/2}^m}{m!} (2m+1) \Gamma\left(\frac{1}{2}, m + \frac{3}{2}, u\right) \]

\[ \times h_3(u) \rho_4(s^2) du \frac{du}{\sqrt{u}} \, dU \, d(s^2) \]
\[
\frac{\nu_1 \sigma^2}{pqr} \exp(-\psi/2) \int \int \int \left[ \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} G(\frac{1}{2}, m + \frac{3}{2}, u) \right. \\
+ \sum_{m=0}^{\infty} \frac{(\psi/2)^m}{m!} G(\frac{1}{2}, m + \frac{5}{2}, u) \left. \int h_2(u) h_4(s^2) du \right] du \left( s^2 \right),
\]

and on applying the same type of algebra as we used for \(J_1\), we find that this equals

\[
\exp(-\psi/2) \int \int \sum_{\mathclap{p=0}}^{\infty} \left[ \frac{\nu_1 \sigma^2}{pqr} \sum_{\mathclap{p=0}}^{\infty} \frac{(\psi/2)^p}{p!} G(\frac{1}{2}, \frac{1}{2} + 1, u') \right. \\
+ \left. \frac{2}{\gamma_{11}} \sum_{\mathclap{p=0}}^{\infty} \frac{(\psi/2)^p}{p!} G(\frac{1}{2}, p + \frac{1}{2} + 2, u') \int h_4(s^2) du' \left( s^2 \right) \right] \left( s^2 \right)du \left( s^2 \right)
\]

\[
= \frac{\nu_1 \sigma^2}{pqr} \exp(-\psi/2) \sum_{\mathclap{p=0}}^{\infty} \frac{(\psi/2)^p}{p!} I_{x/2} (p + \frac{1}{2} + 1, \frac{v}{2}) \\
+ \frac{2}{\gamma_{11}} \exp(-\psi/2) \sum_{\mathclap{p=0}}^{\infty} \frac{(\psi/2)^p}{p!} I_{x/2} (p + \frac{1}{2} + 2, \frac{v}{2}). \\
\tag{4.7.6}
\]

Using the symbol \(\mathcal{Q}\), defined in (2.4.5), we may write

\[
J_1 = \gamma_{11} \mathcal{Q}_{v/2} (\frac{\psi}{2}, \frac{v}{2}, x_1) \tag{4.7.7}
\]

and

\[
J_2 = \frac{\nu_1 \sigma^2}{pqr} \mathcal{Q}_{\frac{v}{2} + 1} (\frac{\psi}{2}, \frac{v}{2}, x_1) + \gamma_{11} \mathcal{Q}_{v/2 + 2} (\frac{\psi}{2}, \frac{v}{2}, x_1). \\
\tag{4.7.8}
\]

Substituting these in (4.7.2) and (4.7.3), we have

\[
E_c(T_{11}) = \mu_{11} - \gamma_{11} \mathcal{Q}_{v/2 + 1}
\]
and
\[ E_\theta (T_{1i} - \mu_{1i})^2 = \frac{\sigma^2}{r} \left[ 1 - \frac{v_1}{pq} Q_{v_1 + 1}^{0.3} + \frac{r\gamma_{11}^2}{\sigma^2} (2Q_{v_3 + 1}^{0.3} - Q_{v_3 + 2}^{0.3}) \right] \] (4.7.10)

The expectation and \text{MSE} of the estimator of any other cell mean have similar expressions. Generally, for any \( \mu_{ij} (i=1,2, \ldots, p; j=1,2, \ldots, q) \), whose estimator is \( T_{ij} \), we have
\[ E_\theta (T_{ij} - \mu_{ij})^2 = \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} Q_{v_3 + 1}^{0.3} + \frac{r\gamma_{ij}^2}{\sigma^2} (2Q_{v_3 + 1}^{0.3} - Q_{v_3 + 2}^{0.3}) \right], \]

and hence
\[ \text{AMSE}_\theta = \sum_{i=1}^{p} \sum_{j=1}^{q} \omega_{ij} E_\theta (T_{ij} - \mu_{ij})^2 \]
\[ = \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} Q_{v_3 + 1}^{0.3} + \frac{\psi}{pq} (Q_{v_3 + 1}^{0.3} - Q_{v_3 + 2}^{0.3}) \right], \] (4.7.11)

where \( \psi^* \) is as defined in 4.3.5).

4.8 General Observations

Note that expressions (4.7.9) and (4.7.11) are similar to the corresponding expressions in Chapter 2, viz., (2.4.9) and 2.4.10).

The points of difference are that now we have \( Q_{v_1}^{0.3} \) and \( Q_{v_3}^{0.3} \) instead of \( Q_{\gamma/2}^{0.3} \) and \( Q_{\gamma/2}^{0.3} \); \( (p-1)(q-1)/pq \) instead of \( 1/4 \), and \( \psi^* \) as well as \( \psi \). In fact, the expressions in Chapter 2 may be looked upon as special forms of (4.7.9) and (4.7.11), for in the 2x2 case
\[ v_1 = 1, \]
\[ (p-1)(q-1)/pq = 1/4, \]
and

$$\psi^* = \psi.$$ 

Hence the observations made in Section 2.6 have a straightforward generalization to the present situation.

Thus for $0 < \alpha < 1$, 

$$0 < |E_{\theta}(T_{ij}) - \mu_{ij}| < |\gamma_{ij}| Q_{V^*_\theta + 1},$$

and the bias tends to zero as $\psi$ tends to $\infty$.

Also

$$\frac{AMSE_{Q}}{\sigma^2 / r} - \left\{ 1 - \frac{(p-1)(q-1)}{pq} + \frac{\psi^*}{pq} \right\}$$

$$= \frac{(p-1)(q-1)}{pq} \left[ 1 - Q_{V^*_\theta + 1} \right] + \frac{\psi^*}{pq} \left[ 2Q_{V^*_\theta + 1} - Q_{V^*_\theta + 2} \right]$$

$$= \frac{1}{pq} \left[ (p-1)(q-1) - \psi^* \right] \left[ 1 - Q_{V^*_\theta + 1} \right] + \frac{\psi^*}{pq} \left[ Q_{V^*_\theta + 1} - Q_{V^*_\theta + 2} \right]$$

$$> 0, \text{ if } 0 \leq \psi^* \leq (p-1)(q-1)$$

and

$$\frac{AMSE_{Q}}{\sigma^2 / r} - 1 = \frac{1}{pq} \left[ \psi^* - (p-1)(q-1) \right] Q_{V^*_\theta + 1}$$

$$+ \frac{\psi^*}{pq} \left[ Q_{V^*_\theta + 1} - Q_{V^*_\theta + 2} \right]$$

$$> 0, \text{ if } \psi^* \geq (p-1)(q-1)$$

This means (cf. equations (4.3.3) and 4.3.4)) that in the domain

$$0 \leq \psi^* \leq (p-1)(q-1),$$

the set of estimators $T_{ij}$ is necessarily worse.
than the set of estimators $\hat{\mu}_{ij}$. On the other hand, in the domain
$\Psi^* \geq (p-1)(q-1)$, the set of estimators $T_{ij}$ is necessarily worse than
the set of estimators $\hat{\mu}_{ij}$.

For $\Psi^* = 0$,

$$\frac{\text{AMSE}_0}{\sigma^2 / r} = 1 - \frac{(p-1)(q-1)}{pq} Q_{\Psi^*} + 1. \quad (4.8.4)$$

Because of the properties of the $Q$ function (see Section (2.5)), this
means that the ratio at $\Psi^* = 0$ is smaller, the larger the value of $x_\alpha$,
for a particular $\psi$. However, with varying $\psi$ (but $\Psi^* = 0$), for a given
$x_\alpha$, the ratio increases as $\psi$ increases.

4.9 An Alternative Approach

For the simultaneous estimation of all cell means, we have
considered above the minimization of the AMSE as our goal. An
alternative approach would be to choose a procedure that keeps the
largest of the $pq$ MSE's

$$E_0 (T_{ij} - \mu_{ij})^2 \quad (i=1,2,\ldots,p; j=1,2,\ldots,q)$$

at a low level.

Now, for the estimators $\hat{\mu}_{ij}$ all the MSE's are equal:

$$\text{MSE}_0 (\hat{\mu}_{ij}) = \sigma^2 / r \quad (4.9.1)$$

for $i=1,2,\ldots,q$.

On the other hand, for the estimators $\tilde{\mu}_{ij}$ the largest MSE is

$$\max_{i,j} \gamma_{ij}^2 + \frac{\sigma^2}{r} \left[ 1 - \frac{(p-1)(q-1)}{pq} \right]. \quad (4.9.2)$$
The expressions (4.9.1) and (4.9.2) suggest the following TE procedure:

1. First, make a test for the hypothesis

\[ H_0 : \max_{i,j} \gamma_{ij}^2 \leq \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq} \quad (4.9.3) \]

against the alternative

\[ H_1 : \max_{i,j} \gamma_{ij}^2 > \frac{\sigma^2}{r} \cdot \frac{(p-1)(q-1)}{pq} \quad (4.9.4) \]

2. In case \( H_0 \) is rejected in step (1), take \( \hat{\mu}_{ij} \) as the estimates of \( \mu_{ij} \); otherwise take \( \tilde{\mu}_{ij} \) as the estimates.

However, here too we have the problem of developing a suitable test for \( H_0 \). Even the heuristic test given by the rejection region

\[ \max_{i,j} z_{ij}^2 > C \quad (4.9.5) \]

raises too difficult a distribution problem, especially so because the distribution of the statistic

\[ \max_{i,j} z_{ij}^2 \]

is not dependent solely on the parameter

\[ \max_{i,j} \gamma_{ij}^2 \].
5. FURTHER STUDY OF pxq CASE WITH SOME NUMERICAL RESULTS

5.1 Percentage Points of x

In the pxq case, with the preliminary test given in Section 4.5, the study of the bias for a single cell mean will go along the same lines as for the 2x2 case.

Of greater interest is the variation of the AMSE with varying p and q, varying ψ and ψ* and also with varying r and α.

For the numerical part of the work, we have considered two pairs of values of p and q, viz., p=q=3 and p=q=5, two values of r, viz., r=3 and r=5, and three non-trivial levels of significance, viz., α=.25, .50 and .75. The critical values of the test statistic x corresponding to α=.25, .50 and .75, for the chosen values of p, q and r, are shown in Table 5.1.1. Here, again, the two trivial values of α, viz., α=0 and α=1, have not been considered, since the corresponding percentage points are

\[ x_0 = 1 \tag{5.1.1} \]

and

\[ x_1 = 0 \], \tag{5.1.2} \]

irrespective of p, q and r(r ≥ 2).

5.2 Bias of \( T_{ij} \) as a Proportion of \( \gamma_{ij} \)

From (4.7.9) the estimator \( T_{ij} \) of the cell mean \( \mu_{ij} \) has expectation

\[ E(T_{ij}) = \mu_{ij} - \gamma_{ij} Q \frac{(\psi, \psi', x_\alpha)}{2 + 1} \]
for the 2x2 case by 4 and 16, respectively, to obtain comparable
\(\psi\)-values for the 3x3 and 5x5 cases.

### Table 5.2.1. Values of \(\gamma^{-1}_{1j}E(\mu_{1j}-T_{1j})\)  
\( (p=q=5) \)

<table>
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<tr>
<th>(\psi)</th>
<th>(\alpha = .25)</th>
<th>(\alpha = .50)</th>
<th>(\alpha = .75)</th>
<th>(\alpha = .25)</th>
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### Table 5.2.2. Values of \(\gamma^{-1}_{1j}E(\mu_{1j}-T_{1j})\)  
\( (p=q=5) \)

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Table 5.1.1. Upper $\alpha$-points of the statistic $x$

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<td>.238</td>
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<td>.107</td>
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<td>.386</td>
<td>.328</td>
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<tr>
<td>$r=5$</td>
<td>.281</td>
<td>.238</td>
<td>.198</td>
</tr>
</tbody>
</table>

Hence $Q_v$ is the proportion of $\gamma_{ij}$ that constitutes the bias of $T_{ij}$. $\frac{1}{v} + 1$

We present the values of this relative bias

$$\gamma_{ij}^{-1} E(\mu_{ij} - T_{ij}) = Q_v + 1$$

(which is the same for all $i,j$) for our chosen values of $p$, $q$, $r$ and $\alpha$ in Tables 5.2.1 and 5.2.2.

Note that for $\alpha=0$, the bias of $T_{ij}$ is $-\gamma_{ij}$, and for $\alpha=1$, the bias is zero, irrespective of $p$, $q$ and $r$.

As in the 2x2 case, here also we have chosen eleven values of $\psi$. Note that a $\psi$ for the 2x2 case is not comparable to the same $\psi$-value for, say, the 3x3 case. However, the values

$$\frac{\psi}{(p-1)(q-1)}$$

would seem to be comparable (cf. formula for expectation of interaction mean square; also the null-hypothesis is $\psi \leq 1$ (see (2.2.1)) in the 2x2 case, as opposed to the null hypothesis $\psi \leq (p-1)(q-1)$ (see (4.3.5) and (4.3.7)) in the pxq case). We have, therefore, multiplied the $\psi$-values
On examining these tables, we find that the observations we made in connection with the bias in Section 3.3 hold here as well. Thus the bias decreases as \( \psi \) increases. The change in the bias for changing \( r \) is only slight. On the other hand, a change in the level of significance effects a substantial change in the bias. The magnitude of the relative bias seems to be dependent on the numbers of factor levels, \( p \) and \( q \). Thus, while for \( p=q=3 \) the relative bias corresponding to \( \alpha = .75 \) may be said to be only moderate, for \( p=q=5 \) even with such a high level of significance the relative bias remains quite high, at least towards the beginning of the range of \( \psi \)-values. It would thus seem that, if keeping the relative bias under control is one of our objectives, then \( \alpha \) should be increased progressively with \( p \) and \( q \).

5.3 Relative AMSE: \( \text{AMSE}/(\sigma^2/r) \)

As we stated earlier, our main goal is to minimize the average mean square error (AMSE) when we are trying to estimate all \( pq \) cell means simultaneously. Here we shall study the variation of the AMSE or rather of the ratio (see equation (4.7.11))

\[
\frac{\text{AMSE}}{\sigma^2/r} = 1 - \frac{(p-1)(q-1)}{pq} Q_{v_2} \left( \frac{\psi - \beta}{\sigma}, \frac{\nu}{2}, x_\alpha \right) + \frac{\psi^*}{pq} \left[ 2Q_{v_2 + 1} \left( \frac{\psi}{2}, \frac{\nu}{2}, x_\beta \right) - Q_{v_2 + 2} \left( \frac{\psi}{2}, \frac{\nu}{2}, x_\alpha \right) \right].
\]

(5.3.1)

The values of this ratio corresponding to our chosen values of \( p \), \( q \), \( r \) and \( \alpha \) are shown in Tables 5.3.1 - 5.3.4. Note that the AMSE depends on

\[
\psi^* = pq r \Sigma \Sigma \omega_{ij} \gamma_{ij}^2 / \sigma^2
\]
as well as on

\[ \psi = r \sum \sum \frac{\gamma_{ij}^2}{\sigma^2}. \]

Hence, we need, for each set of \( p, q, r \) and \( \alpha \) values, a two-way table, with \( \psi \) varying between the columns, say, and \( \psi^* \) varying between the rows. For \( \psi^* \) we have taken the same eleven values as for \( \psi \). Not all the \( 11 \times 11 \) combinations are possible, however, as shown in Section 4.4. The cells corresponding to the inadmissible combinations have been left blank in the tables. \(^1\)

Besides \( \alpha = .25, .50 \) and \( .75 \), the two trivial values, \( \alpha = 0 \) and \( \alpha = 1 \), may also be taken into account. However, for \( \alpha = 0 \),

\[ Q_{\psi_{1/2}^*} + 1 = Q_{\psi_{1/2}^*} = 1 \]

and hence

\[
\frac{\text{AMSE}}{\sigma^2/r} = \begin{cases} 
5/9 + \psi^*/9 & \text{if } p=q=3 \\
9/25 + \psi^*/25 & \text{if } p=q=5
\end{cases}, \quad (5.3.2)
\]

irrespective of \( \psi \) and \( r \), while for \( \alpha = 1 \),

\[ Q_{\psi_{1/2}^*} + 1 = Q_{\psi_{1/2}^*} = 0 \]

and hence

\[
\frac{\text{AMSE}}{\sigma^2/r} = 1, \quad (5.3.3)
\]

for all \( p, q, r, \psi \) and \( \psi^* \).

\(^{1/}\) In Tables 5.3.1 - 5.3.4 the columns for \( \psi = 40.0 \) are omitted for lack of space and because only in two columns some values are different from 1.000. These columns correspond to \( p=q=3, \alpha = .25 \), and even there the largest difference is .009 for \( r=5 \) and .004 for \( r=5 \).
Table 5.3.1. Values of the ratio AMSE ($\sigma^2 \psi$)

\[ (p=q=3, r=3) \]

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Table 5.3.2. Values of the ratio \( \text{AMSE}/(\sigma^2/r) \)

\[ (p=q=3, \ r=5) \]

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5.4. Comments on the Values of AMSE

Let us now examine the values tabulated in Section 5.3 to see how the AMSE is affected by changing $r$, $\alpha$, $\psi$ and $\psi^*$. 

First, consider the variation of the AMSE with $\psi$ and $\psi^*$, $\alpha (0 < \alpha < 1)$ and $r$ being kept fixed. If $\psi$ is kept fixed and $\psi^*$ allowed to vary, the AMSE increases monotonically with $\psi^*$. This, of course, is apparent from equation (4.7.11) as well. On the other hand, if $\psi^*$ is kept fixed and $\psi$ allowed to vary, then the behavior of AMSE is a bit irregular. It is seen that for small $\psi^*$, the AMSE increases monotonically with $\psi$ and tends to unity; for large $\psi^*$, the AMSE decreases monotonically as $\psi$ increases; while the tables seem to indicate that for some intermediate $\psi^*$-values, as $\psi$ increases the AMSE increases for a time, reaches a maximum and then tends to decrease. Corresponding to Theorem 3.5.1, here also we have the following theorem, which can be proved using Lemma 3.5.1:

**Theorem 5.4.1.** \( r \text{AMSE}(T|\alpha)/\sigma^2 \to 1 \) as $\psi$ or $\psi^* \to \infty$, provided $\alpha > 0$.

**Proof:** That $r \text{AMSE}/\sigma^2 \to 1$ for $\psi \to \infty$ is obtained directly from Lemma 3.5.1. On the other hand, if $\psi^* \to \infty$, then $\psi \to \infty$ (cf. equation (4.4.1)) and from (4.4.1)

$$\psi^8 q_j \left( \frac{\psi}{2}, q, x_\alpha \right) \leq (pq\psi)^8 q_j \left( \frac{\psi}{2}, q, x_\alpha \right) \to 0,$$

for $a > 0$, so that, because of (4.7.11), $r \text{AMSE}/\sigma^2 \to 1$.

Secondly, as was true for the 2x2 case, the change in $r \text{AMSE}/\sigma^2$ corresponding to a change in $r$ is rather negligible.

Lastly, for fixed $r$, $\psi$, $\psi^*$, the AMSE is highly sensitive to a change in $\alpha$. The general effect of a higher level of significance is to raise
the value of the AMSE toward the lower part of the range of ψ*-values, i.e., for \( ψ* \leq (p-1)(q-1) \), and to reduce the value of the AMSE toward the other part of the range, i.e., for \( ψ* > (p-1)(q-1) \).

If one uses equal weights, then \( ψ* = ψ \), and the behavior of the ratio \( \frac{AMSE}{σ^2} \) can be seen from the principal diagonals (marked for convenience) of the above tables. It will be found that in this particular case the ratio behaves very much like the quantity \( \frac{RMSE}{σ^2} \) for a 2x2 experiment (see Section 3.5).

5.5. Optimal Choice of \( α \)

Here, too, some suggestions may be made as to the choice of a suitable level of significance for the preliminary test. One may again consider the four approaches we talked about in Section 3.6:

(a) Bayes Approach: Note that since the ratio

\[
\frac{AMSE}{σ^2/r}
\]

is dependent on both \( ψ \) and \( ψ* \), to use this approach we must start with a joint prior distribution of \( ψ \) and \( ψ* \). We do not attempt to say anything more than the following, because the choice of the prior distribution is bound to be arbitrary: It is apparent from Tables 5.3.1 - 5.3.4 that if one's prior knowledge indicates that \( ψ* \) is very likely to be less than \( (p-1)(q-1) \), then a relatively small value of \( α \) should be used for the preliminary test; while in case the prior information indicates that \( ψ* \) is not so likely to be small, then a high value, perhaps one near to unity, should be selected.

(b) Minimax Approach: In the absence of any prior knowledge about \( ψ \) and \( ψ* \), one might like to choose \( α \) in such a way that
\[
\begin{align*}
\max_{\psi, \psi^*} & \quad \frac{r \text{ AMSE}}{\sigma^2} \\
\end{align*}
\]
is a minimum. The result, (4.8.3) implies that this \text{minimax level of significance is } \alpha=1, so that according to this approach one should choose the estimator \(\hat{\mu}_{ij}\), irrespective of the sample observations. A remark similar to the one made at the end of Subsection (b) of Section 3.6 pertains to this result.

(c) \textbf{Minimizing the }L_1\text{-norm of the Regret Function:} This approach will be discussed for the case \(\psi = \psi^*\). The regret function analogous to (3.6.6) is now

\[
\frac{r \text{ AMSE}_\psi(T|\alpha)}{\sigma^2} - \min_{\alpha} \frac{r \text{ AMSE}_\psi(T|\alpha)}{\sigma^2} = s(\psi, \alpha)
\]

(5.5.1)

and it equals

\[
\frac{r \text{ AMSE}_\psi(T|\alpha)}{\sigma^2} - \left[ 1 - \frac{(p-1)(q-1)}{pq} + \frac{\psi}{pq} \right] \text{ for } \psi \leq (p-1)(q-1)
\]

(5.5.2)

As in the 2x2 case, this approach leads to the choice of \(\alpha=1\) again.

(d) \textbf{Minimax Regret Approach:} We shall take the special case of equal weights, when \(\psi = \psi^*\). The result (4.8.2) shows that for any \(\psi \leq (p-1)(q-1)\)

\[
\inf_{\alpha} \frac{r \text{ AMSE}_\psi(T|\alpha)}{\sigma^2} = 1 - \frac{(p-1)(q-1)}{pq} + \frac{\psi}{pq},
\]

which value is attained in case \(\alpha=0\). The result (4.8.3) shows that for any \(\psi > (p-1)(q-1)\)
\[
\inf_{\alpha} \frac{r \text{ AMSE}_{\psi}(T|\alpha)}{\sigma^2} = 1,
\]
which value is attained in case \(\alpha=1\). Table 5.3.5 lists the numerical values of these expressions for \(p=q=5\) and the same eleven \(\psi\)-values as were used in Tables 5.3.1 - 5.3.4.

<table>
<thead>
<tr>
<th>(\psi)</th>
<th>(p=q=3)</th>
<th>(\psi)</th>
<th>(p=q=5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.556</td>
<td>0</td>
<td>.360</td>
</tr>
<tr>
<td>0.8</td>
<td>.644</td>
<td>3.2</td>
<td>.488</td>
</tr>
<tr>
<td>1.6</td>
<td>.733</td>
<td>6.4</td>
<td>.616</td>
</tr>
<tr>
<td>2.4</td>
<td>.823</td>
<td>9.6</td>
<td>.744</td>
</tr>
<tr>
<td>4.0</td>
<td>1.000</td>
<td>16.0</td>
<td>1.000</td>
</tr>
<tr>
<td>5.6</td>
<td>1.000</td>
<td>22.4</td>
<td>1.000</td>
</tr>
<tr>
<td>8.0</td>
<td>1.000</td>
<td>32.0</td>
<td>1.000</td>
</tr>
<tr>
<td>12.0</td>
<td>1.000</td>
<td>48.0</td>
<td>1.000</td>
</tr>
<tr>
<td>16.0</td>
<td>1.000</td>
<td>64.0</td>
<td>1.000</td>
</tr>
<tr>
<td>24.0</td>
<td>1.000</td>
<td>96.0</td>
<td>1.000</td>
</tr>
<tr>
<td>40.0</td>
<td>1.000</td>
<td>160.0</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Considering the differences between the entries in the principal diagonals of Tables 5.3.1 - 5.3.4 and the corresponding entries in Table 5.3.5, we find that the level of significance minimizing

\[
\max_{\psi} \psi s(\psi, \alpha)
\]

is some value between \(0.25\) and \(0.50\) (closer to \(0.50\)). On a comparison of this result and the result in Section 3.4, it would seem that the minimax regret \(\alpha\) is at least approximately independent of \(p\) and \(q\).
A fortunate aspect, in the case of equal weights, of this choice of \( \alpha \) between .25 and .50 (nearer to .50) is that not only is the maximum regret minimized, but also the resulting maximum is brought to a low level, so that a sufficient degree of control over the AMSE throughout the range of \( \psi \)-values is obtained. (The situation is similar to that in the 2x2 case, see Section 3.6). By looking at the tables, one can say that such a choice would be useful even when unequal weights are used, provided larger weights go with smaller interactions (implying \( \psi^* < \psi \)). But in case \( \psi^* \) is much larger than \( \psi \), such an \( \alpha \) will make the AMSE too high. In a practical situation one is not likely to know about the relative magnitudes of \( \psi \) and \( \psi^* \). The proper course then would be to control the AMSE throughout the \( \psi-\psi^* \) table, by choosing a very high value of \( \alpha \). Evidently, in this case, the higher \( pq \) is, the higher should be the value of \( \alpha \).

Let us now convert the value of \( \alpha \), related to our test for

\[
H_0: \psi \leq (p-1)(q-1)
\]

against

\[
H_1: \psi > \nu_1
\]

into levels of significance to be used in corresponding ANOVA tests for

\[
H_0: \psi = 0
\]

against

\[
H_1: \psi > 0
\]

The level of significance of the ANOVA test, corresponding to an \( \alpha \) used in our test for material significance, is
\[ 1 - Q_{v_1/2}(0, \nu/2, x_\alpha) \]

\[ = 1 - I_{x_\alpha}(v_1/2, \nu/2) \]  

(5.5.3)

These levels of significance are shown in Table 5.5.1 for the set of values of \( p, q \) and \( r \) considered in this chapter.

Table 5.5.1. Levels of significance for ANOVA test corresponding to \( \alpha = .25, .50 \) and .75

<table>
<thead>
<tr>
<th>Size of the experiment</th>
<th>( \alpha = .25 )</th>
<th>( \alpha = .50 )</th>
<th>( \alpha = .75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p=q=3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r=3 )</td>
<td>.05</td>
<td>.17</td>
<td>.40</td>
</tr>
<tr>
<td>( r=5 )</td>
<td>.04</td>
<td>.15</td>
<td>.38</td>
</tr>
<tr>
<td>( p=q=5 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r=3 )</td>
<td>.007</td>
<td>.04</td>
<td>.13</td>
</tr>
<tr>
<td>( r=5 )</td>
<td>.004</td>
<td>.02</td>
<td>.10</td>
</tr>
</tbody>
</table>

We may conclude that for the equal-weights case our preliminary test may be conducted as an ANOVA test at about the .15 level for a 3x3 experiment and the .02 level for a 5x5 experiment.

However, when unequal weights are used, an ANOVA test at even as high a level as .25, as suggested by Anderson (1960), may not keep the AMSE under proper control. This situation arises presumably because the weights are not being taken into account in the preliminary test procedure. The development of a better test procedure is thus called for to deal with the case of unequal weights.
6. SOME DECISION-THEORETIC RESULTS

6.1. Introduction

In the course of this work, we have at various times thought about
the question how TE procedures, in general, would fit into a decision-
theoretic framework and whether some concepts of that theory (other than
those discussed in previous chapters) might be helpful in constructing
TE procedures. Since we feel that our particular problem might benefit
from such an approach, we have collected some of our thoughts in this
chapter.

6.2. Fixed-sample Decision Problem

Although the theory of statistical decisions can be treated in a
more general set-up, we shall here be concerned with problems connected
with samples of a fixed size, so that we have to deal only with terminal
decisions (vide Wald (1950)).

We have a random variable (possibly a vector) \( x \), taking values in
a space \( \mathcal{X} \) which is the underlying sample space with Borel field \( \mathcal{A} \). On
\( (\mathcal{X}, \mathcal{A}) \) is defined \( \mathcal{F} \), a family of probability measures \( P \). In most
situations,

\[
\mathcal{F} = \{ P_\theta \mid \theta \in \Theta \},
\]

where \( \Theta \) is a parameter space. Together with \( \mathcal{X} \), we have also a space \( \mathcal{D} \)
of possible decisions regarding the probability measures in \( \mathcal{F} \). A
typical member of \( \mathcal{D} \) will be denoted by \( d \). There is also a loss function
\( L \) such that

\[
L(\theta, d)
\]
is the loss incurred through taking the decision \( d \) when \( \theta \) is the true parameter.

A decision rule (function) \( \delta \) is a measurable function from \( \mathcal{X} \) to \( \mathcal{D} \). Thus \( \delta \) is a rule for making a decision on the basis of an observation on \( x \). The choice between any two rival decision functions is made on the basis of the loss function \( L \), or rather the risk function \( r \), where

\[
r(\theta, \delta) = \int L(\theta, \delta(x)) \, dP_\theta(x) .
\]

This choice may be made from various standpoints as illustrated by the following definitions:

**Definition 6.2.1.** A decision function \( \delta_0 \) is said to be admissible if there exists no other decision rule \( \delta \) such that

\[
r(\theta, \delta) \leq r(\theta, \delta_0) , \quad \text{all } \theta \in \Theta ,
\]

and the strict inequality holds for at least one \( \theta \in \Theta \).

**Definition 6.2.2.** Let \( \mathcal{F} \) be a probability measure on \( (\Theta, \mathcal{G}) \), where \( \mathcal{G} \) is a \( \sigma \)-field of subsets of \( \Theta \). Let the average risk relative to \( \mathcal{F} \) of a rule \( \delta \) be denoted by \( r^*(\mathcal{F}, \delta) \), i.e., let

\[
r^*(\mathcal{F}, \delta) = \int_{\Theta} r(\theta, \delta) \, dF(\theta) ,
\]

Then a decision rule \( \delta_{\mathcal{F}} \) is said to be a Bayes rule relative to \( \mathcal{F} \) if

\[
r^*(\mathcal{F}, \delta_{\mathcal{F}}) \leq r^*(\mathcal{F}, \delta)
\]

for all \( \delta \).

**Definition 6.2.3.** A decision function \( \delta_0 \) is said to be minimax if
for all \( \delta \).

**Definition 6.2.4.** A decision function \( \delta_0 \) is said to be minimax regret (or most stringent) if

\[
\sup_{\theta} r(\theta, \delta_0) - \inf_{\delta} r(\theta, \delta) \leq \sup_{\theta} r(\theta, \delta) - \inf_{\delta} r(\theta, \delta)
\]

for all \( \delta \).

For easy reference, we now state some theorems from Wald (1950).

**Theorem 6.2.1.** Suppose, for every \( \theta \in \Theta \), \( P_\theta \) is absolutely continuous w.r.t. a \( \sigma \)-finite measure \( \lambda \) on \((\mathcal{X}, \mathcal{A})\) and denote the corresponding density function by \( f_\theta(x) \). Assume further that:

1. the functions \( L(\theta, \delta(x)) \) and \( f_\theta(x) \) are measurable \((\mathcal{C})\), where \( \mathcal{C} \) is the smallest \( \sigma \)-field generated by the sets \( A \times B \), where \( A \in \mathcal{A} \), \( B \in \mathcal{B} \);

2. the loss function is non-negative: \( L(\theta, \delta) \geq 0 \).

If there exists a rule \( \delta_0 \) satisfying these conditions and for which

\[
\int_{\Theta} L(\theta, \delta_0(x)) f_\theta(x) \, d\sigma(\theta) \leq \int_{\Theta} L(\theta, \delta) f_\theta(x) \, d\sigma(\theta),
\]

for all \( \delta \in \mathcal{D} \),

or, equivalently,

\[
\int_{\mathcal{A}} L(\theta, \delta_0(x)) \, d\mu_x(\theta) \leq \int_{\mathcal{A}} L(\theta, \delta) \, d\mu_x(\theta),
\]

for all \( \delta \in \mathcal{D} \),

\( \mu_x \) being the posterior probability measure on \((\mathcal{A}, \mathcal{B})\) relative to \( \sigma \), then

\( \delta_0 \) is a Bayes rule relative to \( \sigma \).
Definition 6.2.5. Two decision rules, $\delta_1$ and $\delta_2$, are said to be $\lambda$-equivalent if $\delta_1(x) = \delta_2(x)$ a.e. [\lambda], where $\lambda$ has the same meaning as in Theorem 6.2.1. They are said to be risk-equivalent if

$$r(\theta, \delta_1) = r(\theta, \delta_2), \quad \text{all } \theta \in \Theta.$$

[Note that $\lambda$-equivalence implies risk-equivalence.]

Definition 6.2.6. Consider, e.g., a Bayes rule $\delta_\varphi$ relative to $\varphi$.

$\delta_\varphi$ is said to be unique except for $\lambda$-equivalence if any other Bayes rule relative to $\varphi$ is $\lambda$-equivalent to $\delta_\varphi$.

Theorem 6.2.2. A Bayes rule relative to a prior distribution $\varphi$ is admissible if it is unique except for $\lambda$-equivalence.

Theorem 6.2.3. Let $\delta_0$ be a decision rule such that there exists a prior distribution $\varphi$ relative to which $\delta_0$ is a Bayes rule, then if

$$r^*(\varphi, \delta_0) \geq \sup_{\theta} r(\theta, \delta_0),$$

then $\delta_0$ will be a minimax decision rule.

Corollary 6.2.3a. If $\delta_\varphi$ be a Bayes rule relative to some prior distribution $\varphi$, such that the risk $r(\theta, \delta_\varphi)$ is constant over $\varphi$, then $\delta_\varphi$ is a minimax decision rule.

Theorem 6.2.4. Suppose that there is a sequence of prior distributions $\varphi_k (k = 1, 2, \ldots)$ and that $\delta_k$ is a Bayes decision rule relative to $\varphi_k$. If there exists a decision rule $\delta_0$ for which

$$\lim_{k \to \infty} r^*(\varphi_k, \delta_k) = \sup_{\theta} r(\theta, \delta_0),$$

then $\delta_0$ is a minimax decision rule.
6.3. Decision-theoretic Formulation of a TE Procedure

The problem of estimation of a parameter after a test of significance may be considered in the framework of decision theory.

Suppose the element of the parameter vector \( \theta \) that we are interested in is \( \theta_1 \). From other considerations two rival estimators, \( T_1 \) and \( T_2 \), are available to us. Having taken a random observation \( x \in \gamma \) we want to decide which of \( T_1(x) \) and \( T_2(x) \) is to be taken as the estimate of \( \theta_1 \).

We are thus interested in the class \( \Delta \) of decision functions (with values in \( \Theta \)), a typical member of which is \( \delta \) defined by

\[
\delta(x) = [1 - \phi(x)] T_1(x) + \phi(x) T_2(x), \tag{6.3.1}
\]

where \( \phi(x) \) is assumed to be a measurable function taking values 0 or 1.

The fact that we are using small MSE as the criterion of a good estimator implies that we have the following squared-error loss:

\[
L(\theta, \delta) = c(\theta) [\delta(x) - \theta_1]^2
\]

\[
= c(\theta) [(1 - \phi(x)) [T_1(x) - \theta_1]^2 + \phi(x) [T_2(x) - \theta_1]^2], \tag{6.3.2}
\]

where \( c(\theta) > 0 \) is independent of \( \theta_1 \) but may depend on other elements of \( \theta \). The risk function is \( r \), given by

\[
r(\theta, \delta) = c(\theta) \int [(1 - \phi(x)) [T_1(x) - \theta_1]^2 + \phi(x) [T_2(x) - \theta_1]^2] dF_\theta(x), \tag{6.3.3}
\]

which is the MSE of \( \delta \) at \( \theta \), except for the multiplier \( c(\theta) \).

The question might very well be asked why we are taking this restricted class \( \Delta \) of decision rules (estimators) instead of taking the
class of all estimators. The purpose is to set up a framework in which the question raised in Section 2.2 can be discussed. Finding the best possible function $\varphi(x)$ for (6.3.1) (if a 'best' function exists in some sense or other) amounts to finding the optimum partition of the sample space for a TE procedure. Since our objective is to control the MSE of the estimator, the notions of power and level of significance of the preliminary test are clearly irrelevant.

6.4. Bayes Rules for Simple TE Problems

Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample of size $n$ from $N(\mu, \sigma^2)$, where $\mu$ is unknown and is to be estimated. For estimating $\mu$ on the basis of this sample, we may say that we have two available estimators, $T_1 = 0$ and $T_2 = \bar{x}$. Now

$$\text{MSE}_{\theta}(T_1) = \mu^2$$

and

$$\text{MSE}_{\theta}(T_2) = \sigma^2/n,$$

so that if

$$\mu^2 \leq \sigma^2/n,$$

one would prefer $T_1$, and if

$$\mu^2 < \sigma^2/n,$$

one would prefer $T_2$. A TE procedure would be:

1. Test the hypothesis

$$H_0: \, n\mu^2/\sigma^2 \leq 1$$

2. If significance is obtained in step (1), take $T_2(x)$ as the estimate; otherwise take $T_1(x)$. 
In the above notation, we now have

\[ g(x) = [1 - \varphi(x)] \cdot \bar{x} + \varphi(x) \cdot \bar{x} \]  \hspace{1cm} (6.4.1) 

and

\[ L(\theta, \delta) = \frac{1}{\sigma^2} \left[ (1 - \varphi(x))^2 + \varphi(x)(\bar{x} - \mu)^2 \right] . \]  \hspace{1cm} (6.4.2) 

As the prior distribution let us take the conjugate prior distribution corresponding to \( N(\mu, \sigma^2) \). Some arguments in support of conjugate priors have been given by Raiffa and Schlaifer (1961), Chapter 3. One mathematically quite cogent argument is that in this way we can work within a closed system (in the sense that the posterior distributions are of the same type as the prior distributions).

**Case 1: \( \sigma^2 \) Known**

Here

\[ \Theta = \{ \mu \mid - \infty < \mu < \infty \} , \]

and we need consider the prior distribution of \( \mu \). The conjugate prior distribution of \( \mu \) may be put in the form:

\[ d_\Theta(\mu) = \left( 2\pi \sigma^2 / n' \right)^{1/2} \exp[-n' (\mu - m')^2 / 2\sigma^2] d\mu , \]  \hspace{1cm} (6.4.3) 

\[ - \infty < \mu < \infty ; \]

\[ - \infty < m' < \infty , \sigma^2 > 0 , n' > 0 \]

(vide Raiffa and Schlaifer (1961), p. 54).

The posterior distribution of \( \mu \) given \( x \) is

\[ d_{\Theta X}(\mu) = \left( 2\pi \sigma^2 / n'' \right)^{-1/2} \exp[-n'' (\mu - m'')^2 / 2\sigma^2] d\mu , \]  \hspace{1cm} (6.4.4) 

where

\[ n'' = n' + n \]

and

\[ m'' = (n'm' + n \bar{x})/n'' . \]
Theorem 6.2.1 implies that a Bayes solution is defined by:

$$\delta_{\overline{x}}(x) = \begin{cases} 0 & \text{if } \int_{-\infty}^{\infty} \mu^2 \, d\Phi_x(\mu) < \int_{-\infty}^{\infty} (x-\mu)^2 \, d\Phi_x(\mu) \\ \bar{x} & \text{if } \int_{-\infty}^{\infty} \mu^2 \, d\Phi_x(\mu) > \int_{-\infty}^{\infty} (x-\mu)^2 \, d\Phi_x(\mu) \end{cases}$$

i.e.,

$$\delta_{\overline{x}}(x) = \begin{cases} 0 & \text{if } 2\bar{x}m'' < \bar{x}^2, \text{ i.e., } (n' - n)\bar{x}^2 - 2n'm'\bar{x} > 0 \\ \bar{x} & \text{if } 2\bar{x}m'' > \bar{x}^2, \text{ i.e., } (n' - n)\bar{x}^2 - 2n'm'\bar{x} < 0 \end{cases}$$

i.e.,

$$\delta_{\overline{x}}(x) = \begin{cases} 0 \text{ if } \bar{x} > 0 \text{ and } (n' - n)\bar{x} - 2n'm' > 0 \\ \text{or } \bar{x} < 0 \text{ and } (n' - n)\bar{x} - 2n'm' < 0 \\ \bar{x} \text{ if } \bar{x} > 0 \text{ and } (n' - n)\bar{x} - 2n'm' < 0 \\ \text{or } \bar{x} < 0 \text{ and } (n' - n)\bar{x} - 2n'm' > 0 \end{cases}$$

(6.4.5)

We have nine different situations according to the values of $n$, $n'$, $m'$, and for each of these situations $\delta_{\overline{x}}(x) = \bar{x}$ in those subsets of the sample space which are listed in the last column:

1. $n' > n$ and $m' > 0 : \bar{x} < \frac{2n'm'}{n' - n}$
2. $n' > n$ and $m' < 0 : \frac{2n'm'}{n' - n} < \bar{x} < 0$
3. $n' > n$ and $m' = 0 : \emptyset$, the empty set
4. $n' < n$ and $m' > 0 : \frac{2n'm'}{n-n} < \bar{x} < 0$ or $\bar{x} > 0$
5. $n' < n$ and $m' < 0 : \bar{x} < 0$ or $\bar{x} > -\frac{2n'm'}{n-n}$
6. $n' < n$ and $m' = 0 : \emptyset$
(7) \( n' = n \) and \( m' > 0 \) : \( \bar{x} > 0 \)

(8) \( n' = n \) and \( m' < 0 \) : \( \bar{x} < 0 \)

(9) \( n' = n \) and \( m' = 0 \) = any measurable subset of \( \mathcal{L} \) (including \( \emptyset \) and \( \mathcal{U} \))

Some features of these results deserve special attention. Thus, if the prior information regarding \( \mu \) is comparatively precise \((n' > n)\), we would choose \( \bar{x} \) over a small part of the real line. On the other hand, if this is comparatively diffuse \((n' < n)\), we would choose \( \bar{x} \) over the greater part of the real line. When we take any particular case in each group, the result would again seem to support common sense. E.g., take case (1) above. When \( \mu \) has a positive mean \( m' \) and a variance smaller than that of \( \bar{x} \), a negative \( \bar{x} \) will generally underestimate \( \mu \), while if \( \bar{x} \) is much larger than \( m' \) it will be too much of an overestimate, compared to 0.

Note that, for the same problem, a TE procedure, such as those discussed in the previous chapters, would lead to the choice of 0 when

\[
\frac{n\bar{x}^2}{\sigma^2} < \chi^2_{\alpha}
\]

and to the choice of \( \bar{x} \) otherwise (where \( \chi^2_{\alpha} \) is the upper \( \alpha \)-point of a non-central \( \chi^2 \) with 1 d.f. and non-centrality parameter \( n \mu^2/\sigma^2 = 1 \)).

**Case 2:** \( \sigma^2 \) Unknown

Here

\[
\Theta = \{ \mu, \sigma^2 \mid -\infty < \mu < \infty, \sigma^2 > 0 \}.
\]
Any conjugate prior distribution has the following normal-gamma form
(see Naiffa and Schlaifer (1961), p. 55):

\[
d \pi(\mu, \sigma^2) = (2\pi \sigma^2)^{-1/2} \exp\left[ -\frac{n'(\mu-m')(\mu-m')}{2\sigma^2} \right] \exp\left[ -\frac{1}{2} \frac{\nu'}{\sigma^2} \right] \frac{1}{\sigma^2} d\mu d\sigma^2, \tag{6.4.6}
\]

\[-\infty < \mu < \infty, \sigma^2 > 0 ;
\]

\[-\infty < m' < \infty, \nu', n', \nu' > 0 ,
\]

and the corresponding posterior distribution is

\[
d \pi_x(\mu, \sigma^2) = (2\pi \sigma^2)^{-1/2} \exp\left[ -\frac{n''(\mu-m'')}{2\sigma^2} \right] \exp\left[ -\frac{1}{2} \frac{\nu''}{\sigma^2} \right] \frac{1}{\sigma^2} x d\mu d\sigma^2, \tag{6.4.7}
\]

where

\[
n'' = n' + n, \quad m'' = (n'm' + n\bar{x})/n'',
\]

\[
\nu'' = \nu' + \nu + 1, \quad \nu'' = \left[ \left( \nu'\nu' + n'\bar{m}'^2 \right) + \left( \nu\nu + n\bar{x}^2 \right) - n''m''^2 \right]/\nu'',
\]

and

\[
\nu = n - 1, \quad \nu = \sum_{i=1}^{n} (x_i - \bar{x})^2/\nu .
\]

Theorem 6.2.1 implies that a Bayes solution is defined by:

\[
\delta_x(x) = \begin{cases} 
0 & \text{if } \int_{-\infty}^{\infty} \mu^2 \, d\pi_x(\mu) < \int_{-\infty}^{\infty} (\bar{x} - \mu)^2 \, d\pi_x(\mu) \\
\bar{x} & \text{if } \int_{-\infty}^{\infty} \mu^2 \, d\pi_x(\mu) > \int_{-\infty}^{\infty} (\bar{x} - \mu)^2 \, d\pi_x(\mu)
\end{cases}
\]

Here \( \pi_x(\mu) \) is the posterior marginal distribution of \( \mu \), and it has density

\[
\infty \quad [\nu'' + (\mu-m'')^2 n''/\nu'']^{-1/2}. \]
Since this distribution has mean $m''$, the above Bayes solution comes out to be

$$
\delta_\varepsilon(x) = \begin{cases} 
0 & \text{if } 2 \tilde{x}m'' < x^2 \\
\frac{\tilde{x}}{x} & \text{if } 2 \tilde{x}m'' > x^2 
\end{cases},
$$

(6.4.8)

and thus is the same as the Bayes solution (6.4.5) for the case of known $\sigma^2$.

Note that in both cases the Bayes solutions are unique except for $\lambda$-equivalence ($\lambda = $ Lebesgue measure) and hence, from Theorem 6.2.2, are admissible.

6.5. Bayes Rule for Simultaneous Estimation of Cell Means in a Factorial Experiment

Let us now take up for consideration the problem of simultaneous estimation of the cell means of a $p \times q$ factorial experiment. Using the set-up and notation of Chapter 4, we have as estimators of $\frac{1}{\mu}$ the two rival sets:

$$
\hat{\mu}, \tilde{\mu},
$$

and

$$
\hat{\mu}, \tilde{\mu}.
$$

We are looking for a decision rule which will, on the basis of a set of random observations $x_{ijk}(i = 1, \ldots, p; j = 1, \ldots, q; k = 1, \ldots, r)$, estimate $\mu$ according to the estimator $\tilde{\mu}$ on one part of the sample space and according to the estimator $\hat{\mu}$ on the remaining part.

$$
\frac{1}{\mu} = (\mu_{11}, \mu_{12}, \ldots, \mu_{1q}, \mu_{21}, \ldots, \mu_{pq}), \text{ and } \mu \text{ is the corresponding column vector. Similarly for } \tilde{\mu}, \hat{\mu}, \text{ etc.}$$
We are thus considering the class of decision rules \( \Delta \) of which a typical member is a vector \( \underline{\theta} \) defined by

\[
\Delta(x) = [1 - \omega(x)]\underline{\mu}^\perp + \omega(x)\underline{\mu}^\perp ,
\]

(6.5.1)

where \( \omega(x) = 0 \) or 1.

Suppose, as in Chapter 4, we are employing low AMSE as a criterion of good estimators. We then have a loss function of the type:

\[
L(\theta, \Delta) = c(\rho) \sum_i \sum_j \omega_{ij} \left[ [1 - \omega(x)](\tilde{\mu}_{ij} - \mu_{ij})^2 + \omega(x)(\hat{\mu}_{ij} - \mu_{ij})^2 \right],
\]

(6.5.2)

where \( c(\rho) > 0 \) is independent of the \( \mu_{ij} \)'s but may depend on the other element of \( \theta \), viz., \( \sigma^2 \). We shall here take \( c(\rho) = r/\sigma^2 \). The risk function is

\[
r(\theta, \Delta) = c(\rho) \sum_i \sum_j \omega_{ij} \left[ [1 - \omega(x)](\tilde{\mu}_{ij} - \mu_{ij})^2 + \omega(x)(\hat{\mu}_{ij} - \mu_{ij})^2 \right] d\rho(x)
\]

(6.5.3)

and is the AMSE of \( \Delta \) except for the multiplier \( c(\rho) \).

The conjugate prior density of \( \mu \), in case \( \sigma^2 \) is known, is (cf. Raiffa and Schlaifer (1961), p. 56)

\[
(2\pi\sigma^2/r^2)^{-pq/2} \exp[- r^T \sum_i \sum_j (\mu_{ij} - m'_{ij})^2 / 2\sigma^2] ,
\]

(6.5.4)

\[-\infty < m_{ij} < \infty , \]
\[-\infty < m'_{ij} < \infty ; v^i, r^i, v^i > 0 . \]

As in Section 6.4, in either case the Bayes solution would be the same. It leads to the choice of \( \tilde{\mu} = \bar{\mu} \) or \( \hat{\mu} = \bar{\mu} + \bar{z} = \bar{x} \), according as

\[
\sum_i \sum_j \omega_{ij} \tilde{\mu}_{ij}^2 - 2 \sum_i \sum_j \omega_{ij} \tilde{\mu}_{ij} z_{ij} \frac{r^i m'_{ij} + r^i \tilde{x}_{ij}^i}{r + r^i} + 2 \sum_i \sum_j \omega_{ij} z_{ij} \tilde{w}_{ij} > 0 ,
\]

\[
\sum_i \sum_j \omega_{ij} \hat{\mu}_{ij}^2 - 2 \sum_i \sum_j \omega_{ij} \hat{\mu}_{ij} z_{ij} \frac{r^i m'_{ij} + r^i \hat{x}_{ij}^i}{r + r^i} + 2 \sum_i \sum_j \omega_{ij} z_{ij} \hat{w}_{ij} > 0 ,
\]
i.e., according as

\[(r' - r) \sum_{i,j} \omega_{ij} z_{ij}^2 + 2r' \sum_{i,j} \omega_{ij} z_{ij}(w_{ij} - m_{ij}') > 0 . \]  \hspace{1cm} (6.5.6)

This follows from Theorem 6.2.1.

If equal weights are used for all cell means, i.e., if \( \omega_{ij} = 1/pq \),
then the above takes on a simpler form because

\[\sum_{j} \sum_{ij} z_{ij} w_{ij} = 0 .\]

One would then choose \( \hat{\mu} \) or \( \hat{\mu} \) according as

\[(r' - r) \sum_{i} \sum_{ij} z_{ij}^2 - 2r' \sum_{i} \sum_{ij} z_{ij} m_{ij}' > 0 . \]  \hspace{1cm} (6.5.7)

Here again these Bayes rules are unique except for \( \chi \)-equivalence
(\( \chi = \text{Lebesgue measure} \)) and hence are admissible.

6.6. Minimax Decision Rules

Let us now study the nature of minimax decision rules for the
problems discussed in Sections 6.4 and 6.5. It seems that in general
the minimax estimators are the conventional (unbiased and minimum-
variance) estimators.

Thus for the problem of Section 6.4 the minimax estimator in the
class \( \Delta \) is \( \bar{x} \). For consider a conjugate prior distribution with \( n' < n \)
and \( m = 0 \), i.e., case (6) of Section 6.4. As we have seen, for such a
prior distribution \( g(x) = \bar{x} \) is a Bayes solution. And the fact that
\( r(g, \bar{x}) = \text{constant} \) (i.e., unity) implies, by virtue of Corollary 6.2.3a,
that \( \bar{x} \) is a minimax estimator in the class \( \Delta \). Being a Bayes rule unique
except for \( \chi \)-equivalence, it is also admissible.
For the problem of simultaneous estimation of all pq cell means in a pq factorial experiment, discussed in Section 6.5, which is also the main topic of our investigation, we can easily show that the usual estimators \( \hat{\mu} \) constitute a set of minimax estimators in the class \( \hat{A} \), in case the weights \( \omega_{ij} \) are all equal. For in that case we find from (6.5.7) that, for the conjugate prior distribution with \( m'_{ij} = 0 \), \( r' < r \), \( \hat{A}(x) = \hat{\mu} \) is a Bayes decision rule. Also, for this rule \( r(\varepsilon, \hat{A}) = \text{constant (i.e., unity)} \), which implies, by virtue of Corollary 6.2.3a, that \( \hat{\mu} \) is minimax. (We have not used our Bayes estimators corresponding to \( r' > r \), \( m'_{ij} = 0 \), since they have a risk that increases linearly with \( \psi^* \) and hence violate the condition of Theorem 6.2.3.)

For the general case of possibly unequal weights, too, it is conjectured that \( \bar{x} = \hat{\mu} \) is minimax in the class \( \hat{A} \). From (6.5.6), it is seen that for the sequence of prior distributions \( (\varepsilon_k) \) for which \( m'_{ij} = 0 \) (all \( i, j \)) and \( r'_{ik} \to 0 \), the Bayes rules (6.5.6) also tend to

\[ \hat{A} = \hat{\mu} \]

It seems plausible that the nice properties of normal distributions will ensure that the sequence of average risks \( r^*(\varepsilon_k, \delta_k) \) for these Bayes solutions will also tend to the average risk of \( \bar{x} \), viz., unity. This being the same as \( r(\varepsilon, \bar{x}) \), all \( \varepsilon \in \Theta \), Theorem 6.2.4 would imply that \( \bar{x} \) is minimax.

---

The difference between these results and the results in Subsection (b) of Section 3.6 and Subsection (b) of Section 5.5 is that the functions \( \Theta \) in the earlier results were restricted to UMP invariant tests, whereas in the class \( \hat{A} \) there is no such restriction on \( \Theta \).
7. SUMMARY AND CONCLUSIONS

7.1. Summary

It has been our purpose in this investigation to study the consequences (in terms of biases and mean square errors (MSE's) of estimators) of estimating the cell means of a factorial experiment according to either one of two different formulas, depending on the results of a preliminary test for interactions, whereby the preliminary test and the final estimation procedure are carried out on the same set of data. Such procedures combining preliminary testing and subsequent estimation will be called test-estimation (TE) procedures; the resulting estimators, TE estimators.

In Chapter 1 the general ideas were introduced, and it was pointed out that we would confine our attention to two-factor experiments with an equal number of replications per cell. It was assumed that for our experiments Eisenhart's Model I is appropriate and that usual normality assumptions are valid. Related literature was also cited in this chapter.

The special case of factors with two levels each was considered in Chapters 2 and 3. In Chapter 2 the distribution of the TE estimator of a cell mean corresponding to the UMP invariant test for interactions was obtained, and expressions were given, in terms of a function Q, for its bias and MSE. In Chapter 3 numerical values of 'the relative bias' and 'the relative MSE' were obtained. Different approaches to the choice of a suitable significance level for the preliminary test were investigated and various theorems proved, showing that two of these
approaches lead to a degenerate TE estimator, viz., the least squares estimator. An alternative formula for the MSE, which, at least for smaller numbers of replications, will cut down substantially on the numerical work, was also derived.

Chapter 4 is concerned with the general case of pxq factorial experiments (p > 2 and/or q > 2). Two modes (see Sections 4.3 and 4.9) of combining the MSE's of the individual estimators of the pq cell means, and thereby obtaining a single criterion for the goodness of the estimators, were discussed. For a special case of one of these modes, viz., taking the average of the MSE's (AMSE) with equal weights, the distribution, the bias and the MSE of the estimator of any particular cell mean were obtained. The formulas were found to be closely analogous to those in Chapter 2. In Chapter 5 we reported numerical results for two special cases, viz., p = q = 3 and p = q = 5. The magnitude of 'the relative bias' was examined numerically as a function of a non-centrality parameter ψ. The AMSE was studied both for equal weights and for unequal weights. It appeared that in the case of unequal weights a second parameter, ψ*, had to be introduced along with ψ. The question of what would be a suitable significance level for the preliminary test was investigated for this more general case, and generalizations of the theorems of Chapter 3 were presented.

Chapter 6 explored the possibility of studying TE procedures in the framework of decision theory. Here our findings have been tentative but are expected to help future work on the choice of a good preliminary test procedure. The test procedures that we proposed in Chapters 2 and
are based on the notions of significance level and power, hence are somewhat unsatisfactory, because these notions seem irrelevant when our ultimate purpose is to control the MSE's or AMSE of the estimators.

7.2. Conclusions

Now supposing the preliminary test suggested in Chapter 2 or Chapter 4 is used in a TE procedure, the MSE's of the estimators can be controlled by a proper choice of \( r \), the number of replications, and \( \alpha \), the level of significance of the test. The value of \( r \) will in general be determined by the resources available to the experimenter, and there is not much of a choice in this respect. As regards \( \alpha \), the choice will depend on the context of the estimation problem.

If we have some prior information regarding the likely magnitude of the non-centrality parameters \( \psi \) and \( \psi^* \), we may utilize this information to choose a suitable \( \alpha \). Thus, if it is known that \( \psi^* \) is very likely to be less than \( \psi_1 = (p-1)(q-1) \), then a small value of \( \alpha \) should be chosen; while in case indications are that \( \psi^* \) is not so likely to be less than \( \psi_1 \), then a rather high value, perhaps near unity, ought to be selected.

In the absence of any such prior knowledge, a minimax approach might be adopted. This approach leads to the choice of the usual least-squares estimators, \( \hat{\mu}_{ij} \), and so does the approach of minimizing the \( L_1 \)-norm of the regret function in the equal-weights case. However, we set out to find alternatives to these estimators, and so these approaches do not seem helpful to us.
Another approach which would seem quite appropriate in the absence of any prior knowledge about the variability of $\psi$ and $\psi^*$ would be to choose $\alpha$ in such a manner as to minimize the maximum regret. For the 2x2 case, and also for the pxq case when equal weights are used, this approach leads to the choice of an $\alpha$ between .25 and .50 (nearer .50). (These $\alpha$-values refer to a preliminary test for material significance; for conversion into significance levels for customary ANOVA tests, see Tables 3.6.1 and 5.5.1.)

However, in the pxq case, when unequal weights are used, an $\alpha$ of this type does not effect a satisfactory degree of control over the AMSE for values of $\psi^*$ that are too large in comparison to $\psi$ (see Tables 5.3.1-5.3.4). It appears that in such a situation $\alpha$ should be progressively increased as $p$ and/or $q$ increase. Thus while an $\alpha = .75$ might be suitable for the 3x3 case, an $\alpha = .90$ might be required for the 5x5 case. On the other hand, the use of such a high value of $\alpha$ will offset any gain that might accrue from the use of a TE procedure for values of $\psi^*$ that are small compared to $\psi$. Presumably, the development of a better preliminary test procedure is called for to deal with the case of unequal weights.

7.3. Suggestions for Further Research

In our investigation, we have been concerned with two factors only. For dealing with more than two factors, there is a simple, straightforward generalization of our procedure. Thus (whatever the number of factors) we may like to use one type of estimates if the interactions of the highest order are 'materially significant' and
another type of estimates otherwise. Consider, e.g., an experiment with three factors A, B and C, having a, b, and c levels respectively, with r replications for each of the abc treatment combinations. Here the model analogous to (2.2) is

$$
\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk},
$$

(7.3.1)

where

- $\mu$ = general effect;
- $\alpha_i, \beta_j, \gamma_k$ = main effects;
- $(\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}$ = interaction effects of 1st order;
- $(\alpha\beta\gamma)_{ijk}$ = interaction effects of 2nd order.

Let us write the corresponding sample mean as

$$
\bar{y}_{ijk} = m + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk},
$$

(7.3.2)

where the components are the least squares estimates of the corresponding parts of (7.3.1). Taking low average mean square error as the criterion of a good set of estimators, we may choose as estimates of $\mu_{ijk}$

$$
m + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} = \bar{y}_{ijk}
$$

if

$$
r \sum_i \sum_j \sum_k (\alpha\beta\gamma)_{ijk}^2 > (a-1)(b-1)(c-1),
$$

and

$$
m + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} = \bar{y}_{ijk} - (abc)_{ijk}$$
otherwise. The corresponding TE procedure being similar to those discussed in Chapters 2 and 4, our theoretical findings in those chapters may be expected to be valid here, too.

However, in case there are more than two factors, one might want to have a choice between more than two types of estimates. Thus for the case of three factors, we might like to choose among

\[ m + a_i + b_j + c_k, \]
\[ m + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk}, \] and
\[ m + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk}, \]

depending on the outcome of a preliminary test concerning their AMSE's. A TE procedure for this problem will presumably need more complicated mathematics and, indeed, more complicated statistical thinking, for the preliminary test now takes the form of a multiple decision problem.

The case of non-balanced factorial experiments, where the numbers of replications for different combinations are unequal and possibly disproportionate, has not been considered by us. Although our findings, e.g., regarding the optimal choice of significance level, are expected to hold broadly in this context also, a detailed mathematical treatment ought to be made.

Also, a more extensive decision-theoretic treatment of our problem, than was made in Chapter 6, would seem to be in order.

Even in the more elementary treatment of Chapters 2-5, a different family of tests, which should aim at keeping under control the MSE or AMSE, needs to be devised.
In the pxq case, we should try to find a better way of handling the case of possibly unequal weights for the AMSE. If in the definition of the AMSE unequal weights are attached to the cells, a redefinition of main effects and interactions might also be appropriate.

A generalization of Huntsberger's (1955) method of estimation to the case of factorial experiments may be studied. This is expected to lead to a greater overall control of the MSE (or AMSE) here as well.
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