SOME STEADY STATE AND TRANSIENT SOLUTIONS
FOR SAMPLED QUEUES

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1. INTRODUCTION

In recent years, queuing problems have received much attention and a considerable amount of literature has been written on the subject. For instance, queuing theory has been used to study efficient systems for economically servicing customers in a grocery store, or at ticket windows, or car wash establishments. Many areas of application exist. There are the queues of cars at traffic signals, of people in front of escalators, or airborne planes waiting to land at an airport, of ships waiting to pass through a canal, of individuals at a restaurant waiting to be served, of water remaining in a dam, of toys in a store waiting to be sold, or of books in a library waiting to be read. With imagination, many problems can be interpreted in terms associated with queuing situations; therefore, the subject has great appeal.

A queue, or waiting line, is a collection or file of objects waiting for service. A sampled queue is a queue which is not continuously monitored, but is sampled, or observed, at specific moments.

For purposes of discussion let some measured feature of a queue, $Q(t)$, be a function of time as shown below:

![Figure 1.1. The dynamic queue operation](image-url)
Ragazzini and Franklin [1958] discuss the sampling process where the value of $Q(t)$ is read or sampled at, say, equal intervals of time duration $\tau$. The function $Q(t)$ is then described by a sequence of numbers $Q(0), Q(\tau), Q(2\tau), \ldots, Q(n\tau), \ldots,$ which is a rather restricted description of the continuous function. That is, information is lost in the process of sampling the time axis. However, if the function is well behaved, the intermediate values of $Q(t)$ can be approximated with reasonable accuracy by interpolation formulas. When $Q(t)$ is not well behaved, large and unpredictable variations in $Q(t)$ may occur between sampling nodes.

Under this latter situation the number sequence will give a very poor representation of the function $Q(t)$. It is important then that the manner of sampling be related to the characteristics of the function being sampled; otherwise, information will be lost in the sampling process. If the sampling is done properly, little if any information need be lost. Ragazzini and Franklin [1958] point out that in the latter circumstance, the use of more samples would merely burden the system by carrying unessential information that could have been obtained by the simplest of interpolative processes. Moran [1954] points out that in the case of sampled systems, the number of equations required for solution is proportional to $\tau^{-1}$ and that the amount of labor required is proportional to $\tau^{-3}$.

Thus, although continuous systems are capable of carrying and transmitting complete information, the information is sometimes excessive for the purposes it will be used. If this is the case, then the
analysis of the discrete situation may be adequate, provided a proper sampling frequency can be found. In addition, when analysing the continuous system it frequently occurs that a set of differential (or integral) equations will result and that numerical (digital) integration must be used. Under these circumstances the system might as well be considered as a sampled system from the beginning.

Whereas the term queue is concerned with the collection of objects, the term queuing process refers to a more extensive and dynamic system, containing the concept of a queue. Most queuing processes are concerned with the study of the waiting line as it fluctuates with time.

Basic to all queuing processes, or systems, are three aspects of the operation which influence the actual queue within the system. The system is then studied in terms of its effect upon the queue. The three operations of the system involve the following:

a. Input Process--This refers to the individuals arriving in the system and is specified by the probability distribution of the number of individuals arriving during a given time interval.

b. Queue Discipline--This refers to the rule or moral code determining the manner in which customers form up in a queue and the manner in which they behave while waiting. A common queue discipline is the so-called first-come-first-served rule, while others involve priority and pre-emptive rules.

c. Service Mechanism--This not only refers to the manner in which a unit is serviced, but also to the time required to service an individual. That is, many problems consider the case of multiple channel
service situations which assume that there are several servicing agents in parallel, series, or both. This thesis will study the standard queue problem pertaining to a single channel or servicing agent.

Both the input process and the service mechanism may be identified, in part at least, by the distribution respectively associated with the arrival and service of the units. The type of distribution will depend upon whether the study is in terms of discrete time units or continuous time. In the case of continuous time some of the commonly used distributions are:

a) The Deterministic

\[
F(t) = \begin{cases} 
0 & ; t < \theta \\
1 & ; t \geq \theta 
\end{cases} \quad (1.1)
\]

b) The Exponential

\[
F(t) = \begin{cases} 
1 - e^{-t/\theta} & ; t \geq 0 \\
0 & ; t < 0 
\end{cases} \quad (1.2)
\]

\[
f(t)dt = \theta^{-1} e^{-t/\theta} dt; \ t \geq 0; \quad (1.3)
\]

c) The Gamma or Erlangian

\[
f(t)dt = \frac{(r/\theta)^r t^{r-1} e^{-rt/\theta}}{\Gamma(r)} dt; \ t \geq 0, \ r \geq 1, \quad (1.4)
\]

where

- \( t \) is the time between successive events;
- \( \theta \) is the mean time between such events;
- \( F(t) \) is the cumulative distribution of \( t \) and
- \( f(t) \) is the density function of \( t \) \([f(t)dt = dF(t)]\).
In the case of arrival events $\theta = \lambda$ and for service events $\theta = \mu$. For
discrete time, some of the commonly used distributions are:

a) **The Poisson**

\[
f(m) = \frac{e^{-\lambda} \lambda^m}{m!}, \quad m = 0, 1, 2, \ldots; \tag{1.5}
\]

b) **The Geometric**

\[
f(m) = pq^m, \quad m = 0, 1, 2, \ldots; \tag{1.6}
\]

c) **The negative Binomial for fixed $n > 0$**

\[
f(m) = \binom{n + m - 1}{n - 1} p^n q^m, \quad m = 0, 1, 2, \ldots; \tag{1.7}
\]

d) **The Binomial for fixed $n > 0$**

\[
f(m) = \binom{n}{m} p^m q^{n-m}; \quad m = 0, 1, 2, \ldots n, \tag{1.8}
\]

where

- $n$ denotes the number of events which have occurred during a given
discrete time unit,
- $p$ denotes the probability of an event occurring during a given
time period,
- $q$ equals $1 - p$, and
- $f(m)$ is the probability density function.

It is important to note, [Magistad, 1961] that the geometric dis-
tribution is the discrete counterpart of the exponential distribution;
the negative binomial distribution is the discrete counterpart of the
gamma distribution; and that the discrete counterpart of the determin-
istic distribution is the geometric distribution with $p = 1$ or the
binomial distribution with $n = 1$ and $p = 1$. 
Associated with a given queue system are different types of solutions. Three important characteristics of most solutions are:

a) **The queue property under study.** This pertains to the queue size, or the queuing or waiting time of an arriving unit. This thesis used the definition of Smith [1953] in which queuing time is defined as time spent in the queue before service, and waiting time is understood to be queuing time plus service time.

b) **The sampling mechanism.** This is primarily concerned with whether or not the solutions are in terms of discrete or continuous time. In most cases where sampling is utilized, the sampling will occur at constant intervals.

c) **The equilibrium characteristics of the queue.** This refers to whether or not the solutions of the queue system are concerned with the transient or steady state conditions of the system.

This thesis will concern itself only with queue lengths for the following reasons:

a. It appears to involve the cause rather than the effect. That is, waiting times are the result of having a queue present upon arrival and not vice versa.

b. In the case of queue length, one can actually observe and record the length of a queue continuously with time. Waiting time, on the other hand, is not always an observable event, since it can only be observed and recorded at those instances when an individual actually enters the queue. Nevertheless, a non-observable but continuous concept does exist, by supposing that if one were to enter the queue at some specific moment, there would be a waiting time.
c. The waiting times for a given queue length can be estimated using information about the service times per individual; however, one cannot estimate lengths from a knowledge of waiting times, because waiting time is an average figure while queue length represents an individual observation.

The queue length is measured in terms of the number of objects in a queue and is denoted by \( x; x = 0,1,2,\ldots \). Furthermore, the queue lengths will be studied in terms of transient time effects as well as for steady-state situations when \( t \) approaches infinity. Steady-state solutions, when applicable, are often preferred to transient solutions because the equations are usually less complex and are independent of the time variable and the initial conditions. However, in some circumstances, such as where the servicing mechanism is repeatedly turned on and off, the process may oscillate and never reach a steady-state situation, or it may simply diverge indefinitely. Under these circumstances, the transient solutions offer a practical description of the process.

This thesis is concerned with the random variable \textit{queue length} measured in discrete time, and described by the density function \( p_x(k), (x = 0,1,2,\ldots) \), which denotes the probability of \( x \) units in the queue at the end of the \( k \)-th period. The problem is to derive an expression for evaluating \( p_x(k) \) when the input and output parameters of the system are given. Such an expression represents a description of the dynamic queue system, from which many particular properties can be explored. Problems of parameter estimation or hypothesis testing will not be considered here.
2. REVIEW OF LITERATURE

The study of queues has been of interest to mathematicians and engineers for the past fifty years, some extensive bibliographies being given by Kendall [1951], Riley [1956], Doig [1957], and Saaty [1957]. This chapter contains a brief presentation of some of the pertinent literature concerning the theory of queues. It is presented in four sections:

(i) Early development,

(ii) Method of formulation for solution,

(iii) Transient solutions,

(iv) Queue systems and nature of studies.

2.1. Early Development

The first major study, and perhaps the largest single contribution to the field of congestion problems, was undertaken by Erlang (1878-1929) in relation to telephone engineering. Brockmeyer, et al. [1948], give an excellent account of Erlang and his works. Foremost among the achievements was the development of a general technique of using generating functions and differential equations for solving the problems associated with queues. Erlang also initiated the technique of considering that each unit possessed a series of individual phases while in service. The duration of each phase was defined by the exponential distribution; from this, he derived a general class of service distributions known as the Erlang distribution. His many other innovations are still commonly used. The most outstanding is a parameter of special importance called the traffic intensity, $\rho$, which is the expected
service demand per unit of time. The traffic intensity, $\rho$, is measured by a dimensionless unit called the erlang. Brockmeyer, et al. [1948], present a rigorous definition.

Among the most common distribution functions treated in the literature, the exponential and the deterministic functions given in Section 1 are frequently used to describe the time to service a unit. In a like manner the number of individuals, $a$, to arrive during the period $\tau$, or the time, $t_a$, between arrivals is described respectively by the probability function $f(a) = \alpha_a$,

$$\alpha_a = \frac{e^{-\lambda} \lambda^a}{a!}; \quad a = 0, 1, 2, \ldots, \quad (2.1.1)$$

and the distribution function

$$F(t_a) = 1 - e^{-t_a \lambda}; \quad t_a \geq 0. \quad (2.1.2)$$

The exponential distribution for the time between events figures prominently in most of the literature on queue theory for two reasons: 1) it is the distribution function that results from "random arrivals" (or departures); and 2) it has a special property in which the probability of the time to the next event is independent of when the last event occurred. This independence of past events permits one to express the time series aspects of queuing theory in terms of a Markov process. A Markov process is a process $x(t)$ that has the property that the conditional distribution of $x(t_2)$ given $x(t_1)$, $t_1 < t_2$, is independent of $x(t)$ for all $t < t_1$. This simple dependence, only on the
previous condition of the queue, simplifies the required theory considerably and explains why the Markov property is so desirable.

A third type of service distribution introduced by Erlang and later by Kendall [1948] permits a wider class of distributions than the exponential such that the Markov property is still maintained. This is done by considering the concept of phases for each unit. In terms of queues it implies that when a new unit arrives for service, it passes through a series of phases, r in number, and only after it has attained the r-th phase can it leave the queue. Thus, in effect, since the duration of each phase is assumed to be exponentially distributed, the total service time has a gamma type or Erlangian distribution. With the proper formulation of this concept, the Markov properties of the queue are retained. Steady-state solutions for random arrivals and the three service distributions above are due to Erlang.

Another type of approach was first introduced by Palm [1943], and later developed by Feller [1949a, 1949b, 1950] in his theory of recurrent events. Although the stochastic process describing the fluctuations in queue-size is not, in general, Markovian, it is possible to work instead with an enumerable Markov chain if attention is directed to the epochs, or regeneration points, at which individual customers depart. Thus, it is desirable to consider the stochastic process only at points, or epochs, where past history can be neglected and a new service period begins. Kendall [1951] pointed out that if the assumption of an exponential service time distribution is dropped, the above simplification is not possible. That is, with perfectly general input and output distributions, the only regeneration points are:
1. Epochs at which an arrival and a departure occur simultaneously;

2. Epochs at which a new customer arrives and finds the server free.

However, Kendall [1953] further showed that by the proper choice of the set of points over which probabilities are measured, an imbedded Markov chain can be defined. That is, by properly choosing the epoch or regeneration point, we may always define a stochastic system in terms of a Markov chain, at least in a trivial way.

Takacs [1955] explained the above concept in the following manner. Let the state, \( S_x \), denote that the number of people waiting and being served is \( x \). Then if and only if the service times are identically and independently exponentially distributed, the changes of states actually form a Markov process. It becomes a Markov type, however, by extending the notion of the state to be \( S_{x, \tau_s} \), where \( \tau_s \) is defined as a continuous variable which measures the time elapsed, since service began on the individual currently being served. Takacs then went on to explain that the state of the system can be characterized by \( S_w \), which considers only a single variable, \( w \), used to define the waiting time for a person arriving at that moment. The process is still Markovian even for an arbitrary service distribution. The advantage here is important because each variable defining \( S \) implied an added dimension of an infinite number of equations to be solved simultaneously. This means

---

1The usual notation for designating a state is \( E \), but since this may be confused with the symbol for an expectation, this thesis will use \( S \) to denote a state.
that with \( S_v \) just one infinite set of equations is required, while with
\( S_n, \tau_s \), an infinite number of such infinite sets, is required.

Cox [1955a] in his paper on non-Markovian stochastic processes
summarized the three methods available to handle such processes that
arise in queuing theory when the distributions of arrival time and
service time are not exponential:

(i) The use of regeneration points, i.e., of an imbedded Markov
chain. The behaviour is considered at a discrete set of
time instances, chosen so that the resulting process is
Markovian. The regeneration points may determine those
properties of the whole process which are of physical
interest.

(ii) Erlang's method, in which the process is divided into
fictitious stages, the time spent in each stage having an
exponential distribution. The whole process is Markovian
provided that the specification of the state of the system
includes an account of which stage has been reached.

(iii) The inclusion of sufficient supplementary variables, ex-
pended life-times, in the specification of the state of the
system to make the whole process Markovian in continuous
time.

In this paper Cox was particularly concerned with method (iii) and its
relation to (ii).

Aside from the problem of the basic process, there may occur the
problem of observation and identification of measurements with the proper
states. This subject was recently discussed by Gilbert [1959].
It is apparent, therefore, that all queue situations are not Markovian, and that special techniques must be employed so that the process can be treated as such.

2.2. Method of Formulation for Solution

The reduction of the problem to a Markovian model is simplified because Markov chains may be studied by several rather general and well-established methods. One of the most general methods available, assuming constant transitions in time, is the use of transition matrices discussed by Erlang [Brockmeyer, et al., 1948], and developed in more detail by Feller [1950]. Ledermann and Reuter [1954] studied the spectral theory of simple birth and death processes, of which queues are a special case, by a powerful analytical method based on regarding an infinite matrix equation as the limit of a sequence of finite matrix equations of increasing dimensions. This method, however, is rather complicated to apply even in the simple special cases.

In the case of finite chains one may develop distributions in discrete time by the recursive equation

\[ P[k + 1] = T' P[k] \]  \hspace{1cm} (2.2.1)

where

- \( P[k] \) is a column vector of probabilities, \( p_{x'} \) of units in the queue at time \( k \), and

- \( T' \) is the transpose of the transition matrix \( T \).

Obviously, if \( T \) is of any size, the arithmetic becomes alarmingly large. If one is interested in the limiting case as \( k \) approaches infinity, such a technique is not very useful.
Another method used by Erlang [Brockmeyer, et al., 1948] was that of setting up simultaneous difference equations and also, for limiting cases, a set of simultaneous differential equations for continuous time. This is the most common approach employed because it lends itself to explicit analytic solutions. Particularly useful, is the simple method of obtaining equilibrium solutions by recognizing that

\[
\lim_{t \to \infty} \frac{\partial}{\partial t} p_x(t) = 0 ; \quad x = 0, 1, 2, \ldots ,
\]

(2.2.2)

when equilibrium situations exist. Thus, the solution is easily obtained by reducing the simultaneous differential equation, dependent on \( t \), to a simple set of algebraic homogeneous equations, independent of \( t \). This approach is almost always easier than obtaining a transient solution and letting \( t \) approach infinity.

An important modification of this technique was given by Crommelin [1932] and explained in some detail by Jensen [Brockmeyer, et al., 1948]. Considering the case of telephone circuits in which the point of view was focused on the servers (circuits) rather than customers, Jensen considered the simultaneous equations which are reduced to a simple time independent equation by means of a generating function.

\[
f(s) = \sum_{x=0}^{\infty} s^x p(x,n),
\]

(2.2.3)

where \( p(x,n) \) is the probability of \( x \) out of \( n \) servers being occupied under steady-state conditions (when \( x > n \), there are \( x-n \) customers waiting). Assuming a Poisson distribution for the number of arrivals during the time \( kt \), Jensen reviewed how Crommelin then obtained the expression
\[ f(s) = \frac{Q_n(s) - s^n P(n,n)}{1 - s^n e^{t\lambda(1 - s)}} \]  

(2.2.4)

where

\[ Q_n(s) = \sum_{x=0}^{n} s^x p(x,n) ; \]

and

\[ P(n,n) = \sum_{x=0}^{n} p(x,n) ; \]

and

\[ \lambda t = \text{Poisson parameter at time } t. \]

Recognizing that \( f(s) \) is analytic in the region \( |s| < 1 \), it is evident that if the denominator has any zeros, the numerator must have the same zeros. Thus, the explicit zeros of the denominator are used to study the zeros of the numerator.

With attention focused on transient solutions for a queuing situation, Bailey [1954] used the same technique for a solution of the single server queue with random arrival and random service.

An entirely different approach to queuing theory was the integral equation method by Lindley [1952], which was concerned with queuing times. Lindley showed that if certain general independence conditions are satisfied, and if the mean service time is less than the mean arrival interval, then the distribution function turns out to be a solution of an integral equation of the Wiener-Hopf type. The equation obtained for the distribution of queuing times of the \((r+1)\)-st customer was

\[ F_{r+1}(x) = \int_{x < u_r} F_r(x - u_r) \, dG(u_r) , \]  

(2.2.5)
where

\( F_r(x) \) is the distribution function of the random variable \( x_r \), continuous to the right,

\( x_r \) is the queuing time of the \( r \)-th customer,

\( G(u_r) \) is the distribution function of any \( u_r \),

\[ u_r = s_r - \tau_r, \]

\( s_r \) = service time of \( r \)-th customer,

\( \tau_r \) = time interval between arrival of \( r \)-th and \((r+1)\)-st customer.

He pointed out that his development is not restricted to the assumption of random customer arrivals, that the queuing time distribution depends only on the distribution of the difference between service time and inter-arrival time and not on their individual distributions.

Smith [1953] used the same type of integral approach to show that the service time distribution exerts a strong influence over the analytical character of the distribution of waiting times. For exponentially distributed service times, it was shown that the waiting times will be exponentially distributed for a wide class of arrival-time distributions. While certain general results are developed, the main interest was in determining under what circumstances the queuing time distribution are of a simple nature. A class of explicitly solvable queues was also considered: distributions of the service times and arrival times both of chi-squared type with \( 2n \) and \( 2m \) degrees of freedom, scale factors \( \lambda \) and \( \mu \), respectively. Then if \( n = \alpha m \) where \( \alpha = 1, 2, \) or \( 3 \), or if \( m = \beta n \), where \( \beta = 2 \) or \( 3 \), he obtained explicit
solutions for the required roots from which one can solve for the
distribution of queuing times under equilibrium conditions.

Takacs [1955] introduced a powerful new approach that leads to an
integro-differential equation. He considered the case of Poisson
arrival and service times which are identically distributed, mutually
independent random variables. Thus, the distribution of service times
is rather general, and even though it was stipulated that the arrival
distribution is Poisson, he also treated the more general case where
the parameter $\lambda(t)$ is not constant with time. Thus, his basic assump-
tions were much broader than those of previous authors. All results,
however, were in terms of the waiting time of an arriving unit except
for one section where he derived from previous results the queue length
when $\lambda(t) = \lambda$ is homogeneous in time and the system has reached equi-
librium conditions.

Still another basic technique for studying queues is by means of
the theories of random walk, recurrent events, and renewal processes.
Renewal theory was introduced by Palm [1943] who considered the regen-
eration and renewal processes in application to telephone traffic theo-
ry. Feller [1948, 1949a, 1950] further developed the theory of
recurrent events and Smith [1958b] gave an excellent review of renewal
theory and its ramifications. Smith considered a renewal process as
the continuous analogue of recurrent events, the former being concerned
with continuous time, and the latter with discrete time. Thus a renewal
process is a random walk with positive increments. Such an approach
to queuing problems is certainly an important one, for the regeneration
points referred to by Kendall [1951] are simply points of renewal. As
Smith pointed out, when regeneration points are established, renewal
theory is directly applicable to studies such as (a) total busy time
since \( t = 0 \), (b) the total man hours lost by customers and (c) the
number of customers who encounter a queue of size greater than \( N \). As
will be seen later, the knowledge of recurrent events may also be used
to generate probabilities of queue length.

Of course, one must recognize that queuing problems are but a sub-
class of the set known commonly as the birth-death type of stochastic
problem. Thus, any general work done in this area will often apply to
queuing theory. For instance, consider the generalized birth-death
model,

\[
\frac{\partial}{\partial t} P(s, t) = (\lambda_1 s - \mu_1)(s - 1) \frac{\partial P(s, t)}{\partial s} = s^{-1}(\lambda_0 s - \mu_0)(s - 1)P(s, t) \\
+ \mu_0 (1 - s^{-1})P_0(t),
\]

(2.2.6)

where

\[
\begin{align*}
\lambda_1 dt &= \text{probability of birth in interval } dt; \\
\lambda_0 dt &= \text{probability of immigration in interval } dt; \\
\mu_1 dt &= \text{probability of death in interval } dt; \\
\mu_0 dt &= \text{probability of emigration in interval } dt; \\
p_x(t) &= \text{probability of } x \text{ individuals present at time } t; \\
P(s, t) &= \sum_{x=0}^{\infty} s^x p_x(t).
\end{align*}
\]

This model gives rise to the partial differential equation for the prob-
ability generating function of the random variable \( x(t) \). The number of
individuals present at time t may be used to set up the usual queuing
equation by letting $\mu_1 = \lambda_1 = 0$. The most general references for
birth-death processes are Feller [1948], Doob [1953], and Bartlett
[1956].

Meisling [1958] used a slightly different approach to the usual
single-server queuing system where time is treated as a discrete vari-
able. First, he obtained steady-state solutions when the number of
customers arriving in a fixed time interval is binomial and the service
times are assumed to be identically distributed and statistically in-
dependent but otherwise unrestricted. Then he obtained limiting
results as the time interval tends to zero; these results were checked
with the steady-state solution using random arrivals and a general
service distribution.

There are also other studies, such as the one by Gaver [1959], in
which customers arrive in groups, or bunches, and are serviced one at a
time. Gaver assumed that the bunches arrive with exponential arrival
times and that service times are also exponential.

A final method which will not be elaborated is the well-known
Monte Carlo or simulation type of approach.

2.3. The Transient Solution

This thesis is particularly concerned with transient solutions.
As mentioned previously, the continuous time solution of the problem of
the single channel queue with random arrivals and negative exponential
service time was first presented by Ledermann and Reuter [1954] as a
particular case of their spectral theory of simple birth and death
processes. Bailey [1954] showed that the results previously obtained by Ledermann and Reuter, using a complicated spectral analysis, could be derived in a comparatively simple way by means of a standard generating function technique similar to that used by Crommelin [1932]. Bailey's results for the probability of $x(t)$ units in the queue at time $t$ are

$$
\frac{\partial}{\partial t} p_x(t) = \left( \frac{\mu}{\lambda} \right)^{1/2} \left[ u-x(t) \right] e^{-\left( \lambda+\mu \right)t} \left\{ -\left( \lambda+\mu \right) I_{u-x(t)} + (\lambda\mu)^{1/2} I_{u-x(t)-1} 
+ (\lambda\mu)^{1/2} I_{u-x(t)+1} + \lambda I_{u+x(t)+2} - 2(\lambda\mu)^{1/2} I_{u+x(t)+1} 
+ \mu I_{u+x(t)} \right\},
$$

(2.3.1)

where

- $\lambda$ is the parameter of a random arrival distribution,
- $\mu$ is the parameter of a random service distribution,
- $u$ is the number of units in the queue at $t = 0$,
- $I_n$ is the suppressed Bessel function of order $n$ with argument $2(\lambda\mu)^{1/2} t$; defined by $I_n(x) = i^{-n} J_n(ix)$ with $i = \sqrt{-1}$.

Thus, $p_x(t)$ can be obtained by integrating the above equation using the boundary conditions

$$
p_x(0) = \begin{cases} 
1, & u = x(t) \\
0, & u \neq x(t) 
\end{cases}.
$$

(2.3.2)

Independent of Ledermann and Reuter [1954] and Bailey [1954], another solution was derived by Morse [1955]. Morse used still another approach by trying to express the time series in terms of its
auto-correlation function and its related frequency spectrum. Since
the service function is non-linear, the usual method of computing the
auto-correlation function by averaging over the distribution of arriv-
als is not applicable. He approaches the problem then by computing the
transient behaviour of the probabilities $p_x(t)$ for various initial
conditions. Once the transient solutions are obtained, the auto-
correlation function can be computed from these transient solutions.
He worked out in detail, for one channel, exponential service and
Poisson arrivals, the transient solutions. The basic solutions for the
$m$ channel exponential service case are also given. His solution for
the one channel case is

$$p_x(t) = \frac{\mu}{\pi}(\lambda/\mu)^{x(t)-u} \int_0^{2\pi} [\sin u\theta - (\lambda/\mu)^{1/2} \sin(u + 1) \theta]$$

$$\cdot [\sin x(t)\theta - (\lambda/\mu)^{1/2} \sin (x(t) + 1)\theta] w^{-1} e^{-\lambda t} d\theta ,$$

(2.3.3)

where

$$w = \mu + \lambda - 2(\lambda \mu) \cos \theta ,$$

and the other parameters are defined as before.

Later Clarke [1956] considered the same random queuing process but
deviated from the usual assumptions of constant Poisson parameters $\mu$
and $\lambda$ in time. He considered the very general and interesting case
where $\lambda$ and $\mu$ are functions of time, $\lambda(t)$ and $\mu(t)$. The problem of
finding an exact solution for the probability distribution of the queue
length as a function of time was reduced to the solution of an integral
equation of the Volterra type. When the ratio of the incoming and
outgoing traffic is constant, the equation can be solved explicitly and
the required distribution obtained. The author then proceeded to study
the process for large values of \( t \) and particularly for the unstable
case where \( \rho(t) > 1 \). For the special case of time-invariant parameters
\( \lambda \) and \( \mu \), Clarke's transient solution is

\[
P_x(t) = e^{-(\lambda+\mu)t} \left[ \left( \frac{\mu}{\lambda} \right)^{\frac{u-x(t)}{2}} P_{u-x(t)} + \left( \frac{\mu}{\lambda} \right)^{\frac{u-x(t)-1}{2}} P_{u-x(t)+1} \right]
+ (1 - \frac{\lambda}{\mu}) \left( \frac{\lambda}{\mu} \right)^{x(t)} \sum_{k=u+x(t)+2}^{\infty} \left( \frac{\mu}{\lambda} \right)^k \text{I}_k,
\]

(2.3.4)

where the previous notation is maintained.

Champernowne [1956], using a different technique of random walks
with an absorbing barrier, also derived the above transient results
when the input probability density is Poisson and the service distribu-
tion is exponential.

Simultaneously, Luchak [1956] obtained the continuous time solution
of the problem of the single channel queue, using time-dependent random
arrivals and a general class of service times. The general class of
service times is obtained by the technique of allowing a unit to possess
\( k \) phases before it is serviced. For this he used infinite matrices.
Later Luchak [1958] simplified the previous formulae at the expense of
restricting the random arrivals to be time-independent. In this later
article he derived the Laplace transform of the generating function and
obtained
\[ P^*(s, S) = \frac{s^{u+1} + \mu(s-1)p_0^*(S)}{(\lambda + \mu + S)s - \mu - \lambda s \sum_{j=1}^{\infty} d_j s^j} \]  

(2.3.5)

where the previous symbols are again used and

- \( d_j \) is the probability that the element consists of \( j \) phases,
- \( s \) is the argument of the generating function,
- \( P^*(s, S) \) indicates the Laplace transform of the function, \( P(s, t) \),
- and

\( p_0^*(S) \) is the Laplace transform of the function, \( p_0(t) \).

Using the same method as Crommelin [1932] and Bailey [1954], Luchak [1958] then proceeded to solve for the unknown \( p_0^*(S) \) by means of the roots of the denominator of \( P^*(s, S) \). Upon inversion, he obtained

\[ p_0(t) = \frac{1}{\gamma} \sum_{\sigma=u+1}^{\infty} \sigma \left \{ \frac{\gamma^{\sigma}}{\sigma!} + \sum_{i=1}^{\infty} \frac{(\rho \gamma)^i}{i!} \left \{ \sum_{j=0}^{\infty} h_{i,j} \frac{\gamma^j + 1 + \sigma}{(j + 1 + \sigma)!} \right \} \right \} e^{-(1 + \rho)\gamma} \]  

(2.3.6)

where

- \( \gamma = \mu t \),
- \( \rho = \lambda/\mu \),
- \( \sigma \) is an arbitrary integral constant,
- \( h_{i,j} \) is a probability obtained by equating coefficients in the relationship,

\[ \left ( \sum_{j=1}^{\infty} d_j s^{j-1} \right )^i = \sum_{j=0}^{\infty} h_{i,j} s^j. \]

The above equation is a function of \( j \), the number of phases that an
element contains, and \( i \), the total number of phases in the queue either waiting or being processed. Two special cases for \( d_j \) were considered and solutions for a Pearson Type III as well as another special service time distribution are given.

Moran [1954], Gani [1957, 1958], and Yeo [1960, 1961] have considered the storage problems of infinite dams which is but another formulation of the queuing situation. Gani [1957] gave an excellent review of the general problems of storage and their analogy to queuing situations. In the most recent paper, Yeo [1961] obtained the discrete time-dependent solution for a queue (infinite dam) with discrete additive inputs, releasing a unit at the end of each time interval unless the queue is empty. This is done by obtaining the probability of first emptiness after \( k \) unit time periods, given initially \( u \) units in the queue. This technique is based upon a result by Kendall [1937], although Kendall did not formulate it as an occupancy problem. Using the same techniques to obtain the probability of first emptiness, the discrete time-dependent solution was then derived by means of transform methods which rely on the probability of emptiness. The results are:

\[
P_x(k) = x_{x+k-u}(k) + \alpha_0 \sum_{j=0}^{k-u-2} p_0(u+j) \left\{ \alpha_{k-u-1+j}(k-u-1-j) \right. \\
\left. \quad - \alpha_{k-u+x-j}(k-u-1-j) \right\}; \tag{2.3.7}
\]

\[
P_0(k) = \alpha_0^{-1} \sum_{j=u+1}^{k+1} \frac{1}{j+1} \alpha_{k+1-j}(k+1); \quad (k \geq u)
\]

\[
= 0; \quad (k < u),
\]

\[
\{ \alpha > \max(0, u-k-1); \quad k = 1, 2, \ldots \}.
\]
where

- \( k \) is the discrete unit time period,
- \( u \) is the number of units in the queue initially \( (k = 0) \),
- \( \alpha_i \) = probability of \( i \) inputs during a single period,
- \( \alpha_i(k) \) = probability of \( i \) inputs during \( k \) periods.

Specific time-dependent solutions were then obtained for the negative binomial and Poisson inputs. The results for the negative binomial,

\[
\alpha_i = (1 - \xi)^n \left( \frac{n-1+i}{n+1} \right)^i ; \quad (0 \leq \xi \leq 1, \ i = 0,1,2,\ldots)
\]

are

\[
p_x(k) = (1-\xi)^{nk} \left( \frac{nk-l+x+k-u}{nk-1} \right) + \sum_{j=0}^{k-u-2} \frac{u(j+1)n(k-u-j)^x}{j!} \xi^{k-u+x-j-1}
\]

\[
\left\{ \left[ \frac{(n+1)(k-u-l-j)+x+1}{n(k-u-l-j)-1} \right] - \xi \left( \frac{(n+1)(k-u-l-j)+x}{n(k-u-l-j)-1} \right) \right\} ; \quad (2.3.8)
\]

\[
p_0(k) = (1-\xi)^{nk} \sum_{j=u+1}^{k+1} \left( \frac{(n+1)(k+n-j)}{n(n+1)-1} \right)^{k+1-j} ; \quad (k \geq u) \quad (2.3.9)
\]

\[
= 0 ; \quad (k < u)
\]

\[
\left\{ x > \max(0,u-k-1) ; \quad k = 1,2,\ldots \right\}.
\]

We note an apparent printing error in the first terms of (2.3.8); \( \xi^{x+k-u} \) was omitted.

When the inputs are Poisson such that \( \alpha_i = e^{-\lambda} \frac{\lambda^i}{i!} ; (\lambda > 0, \ i=0,1,2,\ldots) \), the results are
\[ p_x(k) = e^{-\lambda k} \frac{(\lambda k)^x}{x!} \left(\frac{(x+k-u)}{(x+u)}\right)^{-1} \]

\[ + \sum_{j=0}^{k-u-2} p_0(u+j) e^{-\lambda (k-u-j)} \frac{\lambda^{k-u-1+x-j}}{(k-u+1+x-j)!} \left(\frac{(k-u-1-j)}{(k-u+1+x-j)!}\right)^{-1} \]

\[ \cdot \left\{ \frac{(k-u-1-j)^{k-u-1+x-j}}{(k-u+1+x-j)!} - e^{-\lambda(k-u-j)} \frac{\lambda^{k-u+1+x-j}}{(k-u+1+x-j)!} \right\} \quad (2.3.10) \]

\[ p_0(k) = e^{-\lambda k} \sum_{j=\max(0,u-k-1)}^{\min(k+1,u)} \frac{\lambda^{k+1-j}}{(k+1)(k+l-j)!} \quad ; \quad (k \geq u) \]

\[ = 0 \quad ; \quad (k < u), \]

\[ \{ x > \max(0,u-k-1); \quad k = 1,2,\ldots \} . \]

The steady-state solution based on Poisson inputs is shown to be

\[ p_x = p_0 \phi_x \quad (2.3.11) \]

\[ p_0 = \left\{ 1 + \sum_{x=1}^{\infty} \phi_x \right\}^{-1} ; \quad x = 1,2,3,\ldots \quad (2.3.12) \]

where

\[ \phi_x = \sum_{j=1}^{\infty} e^{-\lambda(j+1)} \frac{(\lambda j)^{x+j}}{(x+j)!} - \frac{(\lambda j)^{x+j+1}}{(x+j+1)!} \]

It becomes apparent now that the majority of queue studies deal with systems in equilibrium, and those systems are complicated. Because of these complicated relations, Cox [1955b] gave a few simple formulae, in the appendix, for predicting approximately the behaviour
of systems not in equilibrium. For instance, if there are initially $u$ units in a queue where the traffic intensity, $\rho$, exceeds unity, then the expected queue size will grow. If $u$ is not too small, and $\rho$ is appreciably greater than unity, then the queue size $x(t)$ at time $t$ will have the approximate mean and variance

$$E[x(t)] = u + \frac{t}{\mu} (\rho^{-1}) \quad ;$$

$$V[x(t)] = V_a(t) + \frac{t}{\mu} C_s \quad ,$$

where

$\mu$ and $C_s$ are, respectively, the mean and coefficient of variation for the service time,

$\lambda$ is the mean interval of time between successively arriving customers,

$E[x(t)]$ is the expected number of units in the queue at time $t$,

$V[x(t)]$ is the variance of the number of units in the queue at time $t$,

$V_a(t)$ is the variance of the number of customers arriving in time $t$. [For random arrivals $V_a(t) = t/\lambda$.]

Similar approximate relations were also given for queuing times, and approximate relations for expressing the mean and variance of the time to dissipate a queue when $u$ is large but $\rho < 1$.

2.4. Queue Systems and Nature of Studies

In this section a listing of the author, queue system, and nature of the study is given for the cases just mentioned. This is done to provide a better perspective of the various queuing systems and
solutions which have been discussed. The type of queuing system is classified by Kendall's [1953] method, which identifies the type of input and service mechanism assuming that the queue discipline is first-come-first-serve. The classification system specifies in order:

(i) The distribution associated with the input;
(ii) The distribution associated with the service mechanism;
(iii) The number of servers associated with the service mechanism.

To identify the different distributions, Kendall uses the following code letters (see also Section 1):

<table>
<thead>
<tr>
<th>Code</th>
<th>Letter</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td></td>
<td>Deterministic</td>
</tr>
<tr>
<td>M</td>
<td></td>
<td>Exponential</td>
</tr>
<tr>
<td>E_k</td>
<td></td>
<td>Erlangian</td>
</tr>
<tr>
<td>G</td>
<td></td>
<td>General unspecified type distribution</td>
</tr>
<tr>
<td>GI</td>
<td></td>
<td>A general distribution subject to the restriction that successive times between events are statistically independent.</td>
</tr>
</tbody>
</table>

In the case of discrete time situations this same notation will be used by recognizing the nature of the discrete and continuous counterparts. This is done because in most cases the queue system is of a continuous nature even though it is studied in discrete time. Occasionally a queue system will be defined in a discrete manner. One such case is shown in the listing where the distribution of arrivals in a given period is assumed to be binomial. For this case the code letter $B_n$ is used to indicate the binomial distribution for fixed $n > 1$. 
With these conventions a particular type of queuing system can be identified by giving a label such as D/G/3 (regular arrivals/no special assumption about the service distribution/three servers).

In a similar manner this thesis will identify the nature of the study with respect to

(i) The queue property;

(ii) The sampling mechanism;

(iii) The equilibrium characteristics.

The two properties most often studied with respect to a queue are the length of the queue and the waiting time or queuing time of a new arrival. The letter x will denote that solutions are in terms of the queue length while the letter w will indicate that the solutions are in terms of either the waiting time or the queuing time.

The sampling mechanism will be assumed to be either of a discrete or a continuous nature. The discrete case will be identified by the letter k, and the continuous case will be identified by the letter t.

The equilibrium characteristics are concerned with whether or not the study considers only steady state solutions or whether it includes the transient conditions. The code letter E is used to represent the equilibrium or steady-state situations while the letter T is used to represent the transient situations. Because steady state solutions can be obtained from transient type solutions, a hierarchy is established such that the code letter T does not preclude the presence of steady state solutions.
With these conventions a particular type of study can be identified by giving the study a label such as $x/k/E$ (length of queue/discrete time intervals/equilibrium conditions).

The following list contains only those situations where the queue discipline is first-come-first-serve; therefore, it does not include priority or pre-emptive type systems. The list is ordered first by year and then alphabetically by author within a year and includes the comparable classification for this thesis.

<table>
<thead>
<tr>
<th>Author</th>
<th>Reference Date</th>
<th>Queuing System</th>
<th>Nature of Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crommelin</td>
<td>1932</td>
<td>$M/D/S$</td>
<td>$w/t/E$</td>
</tr>
<tr>
<td>Erlang (1908-29)</td>
<td>1948</td>
<td>$M/M/S$</td>
<td>$x,w/t/E$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$M/D/S$</td>
<td>$w/t/E$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$M/E_k/1$</td>
<td>$w/t/E$</td>
</tr>
<tr>
<td>Lindley</td>
<td>1952</td>
<td>$G/I/G/1$</td>
<td>$w/t/T$</td>
</tr>
<tr>
<td>Kendall</td>
<td>1953</td>
<td>$G/I/M/S$</td>
<td>$x,w/t/E$</td>
</tr>
<tr>
<td>Pollaczek</td>
<td>1953</td>
<td>$G/I/G/S$</td>
<td>$w/t/T$</td>
</tr>
<tr>
<td>Smith</td>
<td>1953</td>
<td>$G/I/M/1$</td>
<td>$w/t/T$</td>
</tr>
<tr>
<td>Bailey</td>
<td>1954</td>
<td>$M/M/1$</td>
<td>$x/t/T$</td>
</tr>
<tr>
<td>Moran</td>
<td>1954</td>
<td>$E_k/D/1$</td>
<td>$x/k/E$</td>
</tr>
<tr>
<td>Morse</td>
<td>1955</td>
<td>$M/M/S$</td>
<td>$x/t/T$</td>
</tr>
<tr>
<td>Takacs</td>
<td>1955</td>
<td>$M/GI/1$</td>
<td>$w/t/T$</td>
</tr>
<tr>
<td>Champernowne</td>
<td>1956</td>
<td>$M/M/1$</td>
<td>$x/k/T$</td>
</tr>
<tr>
<td>Clarke</td>
<td>1956</td>
<td>$M/M/1$</td>
<td>$w/t/T$</td>
</tr>
<tr>
<td>Luchak</td>
<td>1956</td>
<td>$M/E_k/1$</td>
<td>$x,w/t/E$</td>
</tr>
<tr>
<td>Author</td>
<td>Reference Date</td>
<td>Queuing System</td>
<td>Nature of Study</td>
</tr>
<tr>
<td>----------</td>
<td>----------------</td>
<td>----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Luchak</td>
<td>1958</td>
<td>M/E_k/1</td>
<td>x/t/T</td>
</tr>
<tr>
<td>Meisling</td>
<td>1958</td>
<td>B_n/GI/1</td>
<td>x,w/k/E</td>
</tr>
<tr>
<td>Yeo</td>
<td>1960</td>
<td>M/D/1</td>
<td>x/k/T</td>
</tr>
<tr>
<td>Yeo</td>
<td>1961</td>
<td>GI/D/1</td>
<td>x/k/T</td>
</tr>
<tr>
<td>Magistad</td>
<td>This Thesis</td>
<td>GI/M/1</td>
<td>x/k/T</td>
</tr>
</tbody>
</table>

In addition, there are several expository papers worthy of mention. Cox (1955b) discussed the integro-differential equation and methods of simplification by using complex transition probabilities. He showed that the integro-differential equations are simplified considerably when distributions associated with the process have rational Laplace transforms. This agrees with the results of Pollaczek (1953) in which he revealed that manageable solutions may be expected for the GI/G/S system whenever the Laplace transform of the service distribution is rational. Cani (1957) gave an excellent review of problems and the importance of solutions for the finite and infinite dam. He reviewed also steady state solutions of the M/D/1 type for both the discrete and continuous time.

For those interested in the related field of renewal processes, Smith (1958b) provided an excellent expository statement of related theory. Feller (1949b) provided the basis for the theory of recurrent events in which he revealed the simplicity that can be achieved by finding a renewal process buried or imbedded in an apparently more complicated stochastic process.
It is important to remind the reader that the studies of Takacs and Clarke are more general than indicated in the listing. This arises because the parameter of the exponential distribution was not constant as normally assumed, but was allowed to vary with time. That is, Takacs considered the case $M/GI/1$ where $\lambda(t)$ is the parameter of the arrival distribution. Clarke considered the case $M/M/1$ where $\lambda(t)$ and $\mu(t)$ are respectively the parameters of the arrival and service distributions.

It is interesting to note from the above list that the use of random arrival and service distributions is predominant. In those cases involving the more general types of queuing systems, the solutions appear to be in terms of the waiting times. This thesis considers a queue of the type $GI/M/1$ which has not been previously treated except in terms of the waiting times.
3. STATEMENT OF THE PROBLEM AND NOTATION

This thesis considers the queue size at the end of each discrete time period. The number of arrivals during each time period is assumed to be independent of the queue size and time period, and the number of departures during each period is either zero or one with respective probabilities \( \beta_0 \) and \( \beta_1 \) \( \beta_0 + \beta_1 = 1 \), independent of the time period and queue size, unless the queue is empty. The time-dependent solution for the probability of queue size is desired as well as the steady-state solution.

The infinite dam problem of Moran [1954], Gani [1958] and Yeo [1961] is, in essence, a queuing problem. Yeo considers in discrete time a dam (queue) which is fed by independent additive inputs and which releases a unit at the end of each time interval unless the queue is empty. This paper will extend the above situation to the case where at the end of each period a unit leaves the queue with probability \( \beta_1 \), not necessarily equal to unity. It is assumed that during each period, \( k = 1, 2, 3, \ldots \), the number of units arriving is a random variable, \( a(k) = 0, 1, 2, \ldots \), independent of the queue size \( x(k - 1) \) at the end of the previous period. The number of departures from the queue at the end of each period is also a random variable, \( b(k) = 0, 1 \), independent of the number of arrivals \( a(k) \) and previous queue size \( x(k - 1) \) unless \( x(k - 1) + a(k) = 0 \), in which case \( b(k) = 0 \). Therefore, the queue size is a Markov chain specified by the equation

\[
x(k) = x(k - 1) + a(k) - b(k) .
\] (3.1)
The change in queue size during the $k$-th period is $a(k) - b(k)$, which has as possible values, $-1, 0, 1, ...$. In order to have only non-negative random variables, we will define a new random variable

$$c(k) = 1 + a(k) - b(k),$$

as the net change in queue size during the $k$-th period. This random variable is one greater than the actual change in queue size; hence, the queue size at the end of the $k$-th period is

$$x(k) = x(k-1) + c(k) - 1.$$  

(3.3)

The random variables thus far are $x(k)$, $a(k)$, $b(k)$ and $c(k)$, $k = 1, 2, 3, ...$, with the following respective probabilities, except for the $\beta$'s and $\gamma$'s when $x(k-1) + a(k) = 0$:

$$p_x(k) = \text{prob}\{x(k) = x\}, \quad x = 0, 1, 2, ...;$$

(3.4)

$$\alpha_a(k) = \alpha_a = \text{prob}\{a(k) = a\}, \quad a = 0, 1, 2, ...;$$

(3.5)

$$\beta_b(k) = \beta_b = \text{prob}\{b(k) = b\}, \quad b = 0, 1$$

(3.6)

$$\gamma_c(k) = \gamma_c = \text{prob}\{c(k) = c\}, \quad c = 0, 1, 2, ... .$$

(3.7)

In (3.6), $\beta_1(k) = \beta_1$ and $\beta_0(k) = \beta_0 = 1 - \beta_1$; however, if $x(k-1) + a(k)$ $= 0$, $\beta_1(k) = 0$ and $\beta_0(k) = 1$. The situation for (3.7) is even more complicated if $x(k-1) + a(k) = 0$, as indicated below.


<table>
<thead>
<tr>
<th>$c(k)$</th>
<th>$x(k-1) + a(k) &gt; 0$</th>
<th>$x(k-1) + a(k) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\gamma_0$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\gamma_1$</td>
<td>$\gamma_1 + \gamma_0$</td>
</tr>
<tr>
<td>$c \geq 2$</td>
<td>$\gamma_c$</td>
<td>$\gamma_c$</td>
</tr>
</tbody>
</table>
Hence, from the standpoint of studying the queue process in terms of \( c(k) \), the situation is complex. We may, however, study certain random walk situations in which \( c(k) \) does not have these restrictions. Consequently certain derivations will be made within the framework of random walk situations rather than the actual queue process.

In terms of the random walk situation, the respective probability generating functions (p.g.f.) for all cases are denoted as:

\[
P(s,k) = \sum_{x=0}^{\infty} p_x(k)s^x, \quad |s| \leq 1; \tag{3.8}
\]

\[
A(s) = \sum_{a=0}^{\infty} \alpha_a s^a, \quad |s| \leq 1; \tag{3.9}
\]

\[
B(s) = \sum_{b=0}^{\infty} \beta_b s^{-b}, \quad 0 < \epsilon \leq |s| \leq 1; \tag{3.10}
\]

\[
C(s) = \sum_{c=0}^{\infty} \gamma_c s^c = sA(s)B(s), \quad |s| \leq 1. \tag{3.11}
\]

A new random variable \( r(k) \) is called the net change after \( k \) periods, such that \( r(k) = \sum_{i=1}^{k} c(i) \). The probability of a net change, \( r(k) \) after \( k \) periods is denoted by

\[
\gamma_r(k) = \begin{cases}
\gamma_r(k) = \text{prob}[c(1) + c(2) + \ldots + c(k) = r(k) = r]; & k=1,2,3, \ldots; \\
\gamma_r(0) = 0, \quad \text{for all } r,
\end{cases} \tag{3.12}
\]

which may be obtained from the \( k \)-fold convolution of the distribution of \( c(k) \). In this case, the p.g.f. for \( \gamma_r(k) \) is

\[
[C(s)]^k = \sum_{r=0}^{\infty} \gamma_r(k)s^r; \quad |s| \leq 1; \quad k = 1,2,3, \ldots. \tag{3.13}
\]
Furthermore, let $\gamma_r^0$ be the coefficient of $s^r$ in the expansion of

$$\frac{s}{k} \frac{\partial}{\partial s} [c(s)]^k - [c(s)]^k + \gamma_0 [c(s)]^{k-1}$$

$$= s[c(s)]^{k-1} c'(s) - [c(s)]^k + \gamma_0 [c(s)]^{k-1}$$

$$= [c(s)]^{k-1} [sc'(s) - c(s) + \gamma_0]$$

$$= [c(s)]^{k-1} \sum_{i=2}^{\infty} (i-1) \gamma_i s^i.$$

By expanding and collecting terms, the above equation becomes

$$\sum_{r=2}^{\infty} \sum_{i=2}^{r} (i-1) \gamma_i \gamma_{r-i} (k-1) s^r$$

$$= \sum_{r=2}^{\infty} \gamma_r^0 s^r.$$

For computational purposes, the direct expansion and collection of terms from (3.14) yields the equivalent relation

$$\gamma_r^0 = [(\frac{r-k}{k}) \gamma_r (k) + \gamma_0 \gamma_r (k-1)]; \quad r \geq 2, \quad k \geq 1,$$

$$\gamma_1^0 = \gamma_0^0 = 0.$$
4. METHODS OF SOLUTION

Two methods of solution are given here: a direct method of solution, and a solution by generating functions. Both methods are discussed together with a suggestion and an aid for selecting one method of solution in preference to the other.

4.1. Solution by Direct Method

The direct solution for $p_x(k)$ will be derived with the aid of path diagrams, and will be expressed in terms of the probability of first emptiness, $g(u,k)$. The probability of first emptiness, $g(u,k)$ states that at the end of the $k$-th period the queue, which contained initially $u$ units, is empty for the first time. Appendices 9.1 and 9.2 provide the proof that

$$g(u,k) = \begin{cases} 
\gamma_1(1) = \gamma_1, & u = 0, \ k = 1; \\
\frac{\gamma_k}{(k-1)} = \frac{\gamma_0 \gamma_k (k-1)}{k-1}, & u = 0, \ k > 1; \\
\frac{u}{k} \gamma_{k-u}(k), & 1 \leq u \leq k; \\
0 & \text{otherwise},
\end{cases} \quad (4.1.1)$$

when at most one unit is allowed to leave the queue each period.

Figure 4.1 demonstrates the concept of a path diagram showing the queue size as a function of time. It illustrates (4.1.1) in terms of the probability of a monotonically nondecreasing path from the point $(0,u)$ to the point $(k,k)$ such that the path is always above the line $Y(t) = t$ and touches the solid line for the first time at $k$. The path
Figure 4.1. Graph showing path locus of observed cumulative net change at the end of each period, compared with the line, $Y(t) = t$, of maximum possible departures.
represents the locus of cumulated net changes, \( Y(t) = u + r(t) \). The solid line \( Y(t) = t \) indicates the maximum number of units that could have left the queue by the end of some period \( k \). Thus the vertical distance between the path and the solid line at the end of any period \( k \) is \( x(k) \), the number of units in the queue at time \( k \), i.e., \( Y(k) = k + x(k) \). If the path touches the solid line at one of the grid points, then the queue is empty. Thus the probability of a path from \( (0, u) \) to \( (k, k) \) such that it does not touch the solid line \( Y(t) = t \) prior to \( t = k \) is \( g(u, k) \) and represents the probability of first emptiness at \( t = k \). In other words, \( g(u, k) \) is the sum of the probabilities of all configurations, such as the one in Figure 4.1, for which \( Y(0) = u, t < Y(t) \leq Y(t+1), t = 1, 2, \ldots, k-1; Y(k) = k \). These probabilities obviously can be based on (3.11), (3.13), (3.15), and (3.17) because \( x(t-1) + a(t) > 0, t < k \). If we set up an alternative computing procedure which does not change the probability of first emptiness at \( t = k \), this procedure is also acceptable. The desired probabilities are reformulated as follows:

\[
 g(u, k) = \frac{\Pr \{ Y(t) > t \mid Y(0) = u, Y(t+1) \geq Y(t), Y(k) = k \}}{\Pr \{ Y(k) = k \mid Y(0) = u, Y(t+1) \geq Y(t) \}}, \quad t = 1, 2, \ldots, k-1.
\] 

(4.1.2)

\( Y(t) \) may be less than \( t \) for some values of \( t \) between \( 1 \) and \( k \), i.e., not meet the requirement that a queue cannot have negative occupancy; however, this will not affect the probability of configurations favorable to first emptiness at \( t = k \), because all favorable configurations meet the queue requirement. The validity of these statements is developed with greater rigor in Appendix 9.1. The requirement \( Y(t+1) \geq Y(t) \) is the same as \( c(t) \geq 0 \). Based on (4.1.2), we can use
(3.11), (3.13), (3.15), and (3.17) to evaluate \( g(u, k) \). This is accomplished in Appendices 9.1 and 9.2.

Now let us return to the problem of determining \( p_x(k) \), the probability that the queue contains \( x \) units at time \( k \). In this context, \( x \geq 0 \). For \( k = 1, 2, \ldots, u, \) \( x \) is simply related to the sum of the net changes, since the queue could not possibly be empty prior to \( k = u \).

Thus \( p_x(k) = \gamma_{x+k-u}(k) \) when \( k = 1, 2, \ldots, u \). The solution for \( p_x(k) \) when \( k = u + 1, u + 2, \ldots \), is more complex because of the possibilities of an empty queue, which in turn nullify the otherwise normal operating rules for possible departures. That is, when a queue is empty, units cannot leave.

A key requisite for the solution of \( p_x(k) \), \( k = u + 1, u + 2, \ldots \), is a solution for \( p_0(k) \), \( k = u + 1, u + 2, \ldots \), denoting the probability of an empty queue at the end of the \( k \)-th period. The probability \( p_0(k) \) differs from the probability \( g(u, k) \) in that the latter implies that the queue is empty for the first time at the end of the \( k \)-th period, while the former allows that the queue may have been empty several times prior to the \( k \)-th period. As before, the queue cannot possibly be empty until \( u \) periods have elapsed and subsequently may be empty at \( k = u + 1, u + 2, \ldots \). Since the queue may be empty prior to the \( k \)-th period, emptiness is empty at \( k \) results with a cumulative net change of \( r(k) = k-u \) by the end of the \( k \)-th period.

To obtain the probability \( p_x(k) \), \( k = u + 1, u + 2, \ldots \), Yeo [1961] includes an extra time period, stating that during this period there is a single departure but no arrivals for a zero net change with probability \( 70 \). The last requirement of a zero net change states that the path from
k to \( k+1 \) is nonincreasing. This means that because the over-all path must be nondecreasing it has to pass through the point \((k, k)\) to reach \((k+1, k)\). That is, the queue has to be empty at time \( k \) in order to reach the artificial point \((k+1, k)\). This property is also shown in Figure 4.1 where the dashed line \( Y(t) = t - 1 \) represents the locus of an extra artificial time period. The probability of a path from \((0, u)\) to \((k+1, k)\) which does not touch the dashed line except at \((k+1, k)\) can be computed based on the probability of first emptiness, because even though the queue were empty prior to \( t = k \), the true path could not touch the line \( Y(t) = t - 1 \).

In Appendix 9.1, the probability of first emptiness was developed subject to the restrictions \( Y(0) = u \), \( Y(k) = k \), and \( r(k) = k-u, \) i.e., \( u = k - r(k) \). The implication in this section is that if \( k \) is replaced by \( k+1 \), with a net change of zero during the \((k+1)\)st period, then \( k+1 - r(k) \) is equivalent to \( u \); hence, the probability of first emptiness at \( t = k+1 \), given a total net change of \( r(k) = r, \) is \( g(k+1 - r, k+1) \) for \( r = 0, 1, \ldots, k-u \). Thus,

\[
g(k+1-r, k+1) = p_0(k; r)\gamma_0; \quad r = 0, 1, 2, \ldots, k-u , \quad (4.1.3)
\]

where

\[
p_x(k; r) \text{ denotes the probability of } x \text{ units in the queue at the end of the } k-\text{th period, given that the cumulative net change at the } k-\text{th period is } r(k) = r.
\]

From (4.1.3), it follows that

\[
p_0(k; r) = \frac{g(k+1-r, k+1)}{\gamma_0}; \quad r = 0, 1, 2, \ldots, k-u . \quad (4.1.4)
\]
The path diagram illustrated by Figure 4.1 recognizes only those acceptable paths \([Y(0) = u; Y(k) = k; Y(t+1) \geq Y(t), t = 1, 2, \ldots, k-1]\) which do not touch the dashed line. Yet paths that do touch the dashed line, i.e., \(x(j-1) + a(j) = 0\), do not necessarily represent unacceptable situations; it means simply that the queue is still empty and that \(r(k)\) will never equal \(k-u\). It is still possible to satisfy the condition of an empty queue at \(t = k\), since \(k-j\) periods remain in which the proper combination of arrivals and departures may occur. One evaluates this situation as if starting over again with the pseudo diagram assuming \(u = 1\) and \(k+1-j\) time periods. This has been effectively done by evaluating \(P_0(k; r), r = 0, 1, 2, \ldots, k-u\). Therefore, the probability of an empty queue at time \(k\) is obtained by summing the probabilities associated with all acceptable situations such that

\[
P_0(k) = \sum_{r=0}^{k-u} P_0(k; r).
\]

Consider next the determination of \(P_x(k; r), x \geq 0\). The probability of a path from \((0, u)\) to \((k+2, k+1)\) which does not touch the dashed line except at \((k+2, k+1)\) can be written in terms of the probability of first emptiness and the number of units in the queue at time \(k\). The basic relationship is

\[
g(k+1-r, k+2) = p_1(k; r)g(2, 2) + p_0(k; r)g(1, 2);
\]

\[
r = 0, 1, 2, \ldots, k-u.
\]

The right hand side sums the probabilities of being at the point \((k, j)\), \(j = k, k+1, \ldots\) at time \(k\) and then moving to the point \((k+2, k+1)\) by way
of \((k+1, k+1)\). The general relationship is

\[
g(k+1-r, k+j) = \sum_{\ell=0}^{j-1} g(\ell+1, j)p_\ell(k; r); \tag{4.1.6}
\]

where \(x\) is some arbitrary integer less than or equal to \(r\). The left hand side represents a path from \((0, u)\) to \((k+j, k+j-1)\) by way of \((k+j-1, k+j-1)\) such that the path does not touch the dotted line prior to time \((k+j)\).

By interchanging limits, the sequence \(\{g(k+1-r, k+j)\}\) for \(r = 0, 1, 2, \ldots, k-u\) can be written as \(\{g(u+1+r, k+j)\}\) for \(r = k-u, k-u-1, \ldots, 0\). In some cases this latter presentation is preferred to the former because it contains \(u\) explicitly as a reminder that the probability of emptying the queue depends on the size of the queue at \(t = 0\).

This set of equations (4.1.6) can then be represented in matrix notation as:

\[
P(k; r) = g(k; r), \tag{4.1.7}
\]

where
\[
P(k; r) = \begin{bmatrix}
p_x(k; r) \\
p_{x-1}(k; r) \\
\vdots \\
p_1(k; r) \\
p_0(k; r) \\
(x+1, 1)
\end{bmatrix}, \quad g(k; r) = \begin{bmatrix}
g(u + 1 + r, k + x + 1) \\
g(u + 1 + r, k + x) \\
\vdots \\
g(u + 1 + r, k + 2) \\
g(u + 1 + r, k + 1) \\
(x + 1, 1)
\end{bmatrix} \\
(4.1.8)
\]

\[
G = \begin{bmatrix}
\xi_{x+1,x+1} & \xi_{x,x+1} & \cdots & \xi_{2,x+1} & \xi_{1,x+1} \\
0 & \xi_{x,x} & \cdots & \xi_{2,x} & \xi_{1,x} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \xi_{2,2} & \xi_{1,2} & \cdots & \xi_{1,1}
\end{bmatrix}
\]

and \( g_{ij} = g(i,j) = \frac{i}{j} \gamma_{j-1}(j) \).

Thus

\[
P(k; r) = G^{-1} g(k; r), \quad r = 0, 1, \ldots, k-u, \quad (4.1.9)
\]

from which one can solve for \( p_x(k; r) \). It should be noted that the matrix \( G \) is not dependent upon either \( k \) or \( r \); hence, it need not be inverted for many different situations.

The probability \( p_x(k) \) is then obtained by summing \( p_x(k; r) \) over all \( r \)'s to obtain

\[
P(k) = G^{-1} g(k), \quad k = u + 1, \ldots, \quad (4.1.10)
\]

\((x + 1, x + 1)(x + 1, 1)\)
where
\[
P(k) = [p_x(k), p_{x-1}(k), \ldots, p_0(k)]';
\]
\[
g(k) = \left[ g(k; 0), g(k; 1), g(k; 2), \ldots, g(k; k-u) \right]_{(x+1, k-u+1)}^{(k-u+1, 1)}
\]
\[1 = [1, 1, 1, \ldots, 1]' .
\]

When \( k \leq u \), there has not been sufficient time for the queue to be empty previously; hence,
\[
P(k) = [\gamma_{x-u+k}(k), \gamma_{x-u+k-1}(k), \ldots, \gamma_{1-u+k}(k), \gamma_{k-u}(k)]'.
\]

The above results can be written in the recursive form
\[
p_x(k) = \gamma_{x-u+k}(k), \quad k = 1, 2, \ldots, u;
\]
\[
p_0(k) = \sum_{r=0}^{k-u} g(u+1+r, k+1)/\gamma_0 , \quad k \geq u + 1 ;
\]
\[
p_x(k) = \sum_{r=0}^{k-u} \left\{ g(u+1+r, k+x+1) - \sum_{\ell=0}^{x-1} p_\ell(k+\ell)g(\ell+1, x+1) \right\} / g(x+1, x+1),
\]
\[k \geq u + 1, \quad x \geq 1 ,
\]

which does not require inverting the matrix \( G \). In (4.1.12), if \( x-u+k < 0 \), \( \gamma_{x-u+k}(k) = 0 \); also since \( p_u(0) = 1 \) and \( p_i(0) = 0 \) for \( i \neq u \), \( \gamma_0(0) = 1 \) and \( \gamma_i(0) = 0 \) for \( i \neq 0 \). As a comparison with later results, it will be noted that the total number of terms involving the basic quantities \( g(u, k) \) that must be summed is:
\[
1, \quad k = 1, 2, \ldots, u;
\]
\[
(k-u+1)2^x, \quad k = u + 1, u + 2, \ldots,
\]

where a basic quantity \( g(u, k) \) is counted as often as it appears.
The total number of terms \((k-u+1)2^x\) was derived from the equivalent expression

\[ p_x(k) = \sum_{r=0}^{k-u} p_x(k; r). \]  \hspace{1cm} \text{(4.1.14)}

Let \(N_x\) denote the number of terms to be summed in computing \(p_x(k)\), and \(N_x(r)\) denote the number of terms to be summed in computing \(p_x(k; r)\). Then

\[ N_x(r) = 1 + N_0(r) + N_1(r) + \ldots + N_{x-1}(r). \]

Hence

\[ N_x(r) = 2N_{x-1}(r) = 2^x N_0(r) = 2^x. \]

Thus

\[ N_x = \sum_{r=0}^{k-u} N_x(r) = (k-u+1)2^x. \]

\section*{4.2. Solution by Generating Function}

Another method of solution which will provide results in a different form is by the use of generating functions. The queue process may be described by the following difference equations:

\[ p_0(k+1) = p_0(k) \left[ \gamma_0 + \gamma_1 \right] + p_1(k) \gamma_0, \]  \hspace{1cm} \text{(4.2.1)}

\[ p_x(k+1) = \sum_{i=0}^{x+1} p_i(k) \gamma_{x+1-i}; \ x = 1, 2, \ldots. \]

Multiplying the \(x\)-th equation on both sides by \(s^{x+1}\) and summing over all equations, the following p.g.f. is obtained for the difference equations:
\[ sF(s, k+1) = P(s, k)C(s) + (s-1)p_0(k) \gamma_0. \] (4.2.2)

Consider then the \( z \) transform of this difference equation.

Let

\[ P^*(s, z) = \sum_{k=0}^{\infty} P(s, k)z^{-k}; \] (4.2.3)

\[ p^*_1(z) = \sum_{k=0}^{\infty} p_1(k)z^{-k} \]

denote the \( z \)-transforms of \( P(s, k) \) and \( p_1(k) \), respectively, where \( p_u(0) = 1, p_1(0) = 0 \) for \( i \neq u \) and \( P(s, 0) = s^u \). Using the translation theorem of Gardner and Barnes [1942] (see also Jury [1958, page 301]), the \( z \)-transform of \( P(s, k+1) \) can be written in terms of \( P^*(s, z) \) and \( P(s, 0) \) by the relation

\[ \sum_{k=0}^{\infty} P(s, k+1)z^{-k} = zP^*(s, z) - zP(s, 0). \] (4.2.4)

The \( z \)-transform of the difference equation then becomes

\[ szP^*(s, z) - szP(s, 0) = P^*(s, z)C(s) + (s-1)p_0(z)\gamma_0; \] (4.2.5)

hence,

\[ P^*(s, z) = \frac{zP(s, 0) + s^{-1}(s-1)p_0(z)\gamma_0}{z - s^{-1}C(s)}. \] (4.2.6)

Expanding in decreasing powers of \( z \) or by inverse transform tables, e.g., Ragazzini and Franklin [1958], the p.g.f. for the queue size at time \( k \) (\( k \geq 1 \)) is
\[ P(s, k) = s^{-1} C(s)^k + \sum_{j=0}^{k-1} s^{-1} C(s)^{k-1-j} P_0(j) \]  

(4.2.7)

Expanding both sides in increasing powers of \( s, |s| \leq 1 \), and equating coefficients gives

\[ p_x(k) = \gamma_{x-u+k}(k) + \sum_{j=0}^{k-1} p_0(j) \gamma_0 [\gamma_{x+k-j-1}(k-1-j) - \gamma_{x+k-j}(k-1-j)] . \]  

(4.2.8)

In Section 4.1, it was shown that \( p_0(k) = 0 \) for \( k = 1, 2, \ldots, u-1 \); therefore, the above equation can be written as:

\[ p_x(k) = \gamma_{x-u+k}(k), \quad k = 1, 2, 3, \ldots, u, \quad x-u+k \geq 0; \]  

(4.2.9)

\[ = \gamma_{x-u+k}(k) + \sum_{j=u}^{k-1} p_0(j) \gamma_0 [\gamma_{x+k-j-1}(k-1-j) - \gamma_{x+k-j}(k-1-j)] , \]

\[ k = u+1, u+2, \ldots; x-u+k \geq 0 . \]

It will be noted that not all of the terms in the summation of (4.2.9) are non-zero. For example, the last part of equation (4.2.9) can be written in the form

\[ p_x(k) = \gamma_{x-u+k}(k) + p_0(k-1) \gamma_0 [\gamma_x(0) - \gamma_{x+1}(0)] \]  

(4.2.10)

\[ + \gamma_0 \sum_{j=u}^{k-2} p_0(j)[\gamma_{x+k-j-1}(k-1-j) - \gamma_{x+k-j}(k-1-j)] . \]

In those cases where \( k = u+1 \) it is understood that the last term, requiring summation in negative order, will be equated to zero. For \( x > 0 \), the second part vanishes because \( \gamma_x(0) = 0 \); however, since \( \gamma_0(0) = 1 \), \( p_0(k) \) will also include \( \gamma_0 p_0(k-1) \).
Furthermore, consider the restriction that $x-u+k \geq 0$ or that $x+k \geq u$. Let $x+k = d$, $d = u$, $u+1$, ..., so that the bracketed expression of (4.2.10) becomes

$$[\gamma_{d-1-j}(k-1-j) - \gamma_{d-j}(k-1-j)]$$  \hspace{1cm} (4.2.11)

for values of $j = u$, $u+1$, ..., $k-2$. Consider the case where $j = d + \Delta$, $\Delta \geq 0$, then (4.2.11) becomes

$$[\gamma_{-1-\Delta}(k-1-d-\Delta) - \gamma_{-\Delta}(k-1-d-\Delta)].$$  \hspace{1cm} (4.2.12)

By definition $\gamma_c(k) = 0$ for negative values of $c$ so that part or all of the bracketed term will vanish under certain conditions. Yeo [1961] states that the validity of the inversion shown by equation (4.2.9) rests on proving that the right hand side of (4.2.9) vanishes for $x < \max(0, u-k)$ or $-x > \min(0, k-u)$ and proves that this is true.

Although some terms may be zero, it is convenient to maintain all such terms for the resultant consistency and simplicity of expression. Changing the limits of summation, the probabilities of queue size become

$$p_x(k) = \begin{cases} 
\gamma_{x-1+k}(k), & x \geq u-k; \\
0, & x < u-k; 
\end{cases}$$  \hspace{1cm} (4.2.13)

$$p_0(k) = \gamma_{k-u}(k) + \gamma_0 p_0(k-1) + \gamma_0 \sum_{j=0}^{k-2} p_0(j)[\gamma_{k-j-1}(k-j-1) - \gamma_{k-j}(k-j-1)]$$

$$= \sum_{r=0}^{k-u} \frac{g(k+1-r, k+1)}{\gamma_0};$$  \hspace{1cm} (4.2.14)
\[ p_x(k) = \gamma_{x+k-u}(k) + \gamma_0 \sum_{j=0}^{k-u-2} p_0(u+j)(\gamma_{x+k-u-j}(k-u-j-1) - \gamma_{x+k-u-j}(k-u-j-1)), \]
\[ x > 0 . \quad (4.2.15) \]

The second expression for \( p_0(k) \) of (4.2.14) can also be written as
\[ p_0(k) = \sum_{r=0}^{k-u} g(u+r+1, k+1)/\gamma_0 , \quad (4.2.16) \]

which can be verified by noting that the terms to be added are the same but in reverse order.

The results shown above agree with those of Yeo when \( \gamma_c(k) \) is appropriately evaluated for the special case when \( \beta_0 = 0 \) and \( \beta_1 = 1 \).

In expanded form, involving either \( \gamma_c(k) \) or \( g(u, k) \) the total number of terms to be summed is no more than
\[ l, \quad k = 1, 2, \ldots, u; \quad (4.2.17) \]
\[ l + (k-u)(k-u+1); \quad k = u, u+1, u+2, \ldots . \]

The total number of terms in (4.2.17) was derived by writing (4.2.15) as
\[ p_x(k) = \gamma_{x+k-u}(k) + \sum_{j=0}^{k-u-1} \left\{ (\gamma_{x+k-u-j}(k-u-j-1) - \gamma_{x+k-u-j}(k-u-j-1)) \right. \]
\[ \left. \cdot \left( \sum_{r=0}^{j} g(u+r+1, u+j+1) \right) \right\}, \quad (4.2.18) \]

which includes the second term shown in (4.2.10).
The number of terms to be summed is then no more than

\[ 1 + \sum_{j=0}^{k-u-1} \frac{1}{2} \sum_{r=0}^{j} 1 \]

\[ = 1 + 2 \sum_{j=0}^{k-u-1} (j+1) \]

\[ = 1 + (k-u)(k-u+1). \]  

(4.2.19)

4.3. Choice of Solutions

In a given situation the question may arise as to which method of solution is preferred. If the preferred method is the one requiring the least number of terms to be added, then the following results can be stated. If \( k \leq u \) use the relationship \( p_x(k) = \gamma_{x-u+k}(k) \). If \( k > u \), such that

\[ n \cdot 2^x < n(n-1) + 1, \]

where

\[ n = k - u + 1, \]

then use the direct method of Section 4.1; otherwise, use the recursive method of Section 4.2. The reader is reminded that the number of terms to be added may be overestimated because of the expected occurrence of zero valued terms.

Furthermore, it should be noted that the quantity \( n \cdot 2^x \) is a linear function of \( n \) involving the queue size \( x \) while the quantity \( 1 + n(n-1) \) is a quadratic function of \( n \) not involving \( x \). Thus, since \( n = k - u + 1 \), when \( k \) is small and \( x \) is large, the solution by generating functions is preferred; whereas for large \( k \) and small \( x \), the direct method is preferred.
If the whole family of distributions is desired, it will be easier to construct and invert the matrix G, then proceed as in Section 4.1 using matrices.
5. STEADY-STATE SOLUTION

It has been proven by Lindley [1952] and Klefer and Wolfowitz [1955] that the stationary probabilities \( p_x = \lim_{k \to \infty} p_x(k) \) exist when

\[
\rho = \frac{\text{mean of arrival distribution}}{\text{mean of departure distribution}} = \frac{\lambda}{\mu} < 1 .
\]  

(5.1)

Consider then the situation where \( \rho < 1 \) and write the solution (4.2.15) in the following form:

\[
p_x(k) = \gamma_{x-u+k} + \gamma_0 \sum_{j=0}^{k-1} p_0(k-1-j)[\gamma_{x+j} - \gamma_{x+j+1}], \ x > 0 .
\]  

(5.2)

As \( k \to \infty \) the term \( \gamma_{x-u+k} \) approaches zero so that there remains

\[
p_x = p_0 \gamma_0 \sum_{j=0}^{\infty} [\gamma_{x+j} - \gamma_{x+j+1}] .
\]  

(5.3)

The term \( \gamma_{x-u+k} \) may be shown to approach zero as \( k \) approaches infinity by Chebychev's inequality. Consider the density \( \gamma_x \) and evaluate the first and second moments \( E(r) \) and \( E(r^2) \) as shown below:

\[
E(r) = \frac{\partial}{\partial s} [c(s)]^k \bigg|_{s=1} = k[c(s)]^k c'(s) = k \frac{c'(1)}{1+\lambda-\mu} .
\]

\[
E(r^2) = \frac{\partial}{\partial s} \left\{ s \frac{\partial}{\partial s} [c(s)]^k \right\} \bigg|_{s=1} = k s [c(s)]^k c'(s) \bigg|_{s=1} + k s^2 [c'(s)]^2 + k c''(s) \bigg|_{s=1}
\]

\[
= k c'(1) + k(k-1)[c'(1)]^2 + k c''(1) .
\]
The variance, \( V(r) \), is then

\[
V(r) = E(r^2) - [E(r)]^2
\]

\[
= k \left\{ C'(1) + C''(1) - [C'(1)]^2 \right\},
\]

where \( C'(1) = 1 + A'(1) + B'(1) = 1 + \lambda - \mu \)

\( C''(1) = A''(1) + B''(1) + 2[A'(1) + B'(1) + A'(1)B'(1)] \).

Using Chebychev's inequality, one may then show that

\[
\text{Prob} \left\{ |r - E(r)| \geq x - u + k(\mu - \lambda) \right\} \leq \frac{k \left\{ C''(1) + C'(1) - [C'(1)]^2 \right\}}{(x - u + k(\mu - \lambda))^2}.
\]

As \( k \) approaches infinity the right hand side approaches zero, provided that the second term of the numerator is finite and \( \mu \neq \lambda \). Thus,

\[
\lim_{k \to \infty} \gamma_{x-u+k}(k) = 0.
\]

It should be noted that under the steady state condition, \( 0 < \lambda < \mu \leq 1 \).

The solution \( p_0 \), required for (5.3), is obtained from the p.g.f. of the difference equation (4.2.2) which has been modified by subtracting \( sp(s,k) \) from both sides to give:

\[
s[P(s, k+1) - P(s, k)] = P(s, k)[C(s) - s] + (s-1)p_0(k)\gamma_0. \tag{5.4}
\]

If the time-independent situation has been achieved, then \( P(s, k+1) = P(s, k) \) and the left-hand side can be equated to zero to obtain

\[
0 = P(s, \infty)[C(s) - s] + (s-1)p_0\gamma_0. \tag{5.5}
\]

Solving for \( P(s, \infty) \),
\[
P(s, \infty) = \frac{(1-s)\gamma_0 p_0}{C(s) - s} = \frac{(1-s)\gamma_0 p_0}{s[A(s)B(s) - 1]} \quad (5.6)
\]

Let \( s = 1 \), for which it is known that \( P(1, \infty) = 1 \). The use of L'Hospital's rule gives

\[
P(s, \infty) \bigg|_{s=1} = \frac{-\gamma_0 p_0}{[A(s)B(s) - 1] + s[A'(s)B(s) + B'(s)A(s)]} \bigg|_{s=1} \quad (5.7)
\]

\[
1 = \frac{-\gamma_0 p_0}{\lambda - \mu} \quad (5.8)
\]

If \( \lambda \geq \mu \), it is apparent that a steady-state solution does not exist. If \( \lambda < \mu \), the steady-state condition exists, all of which agrees with (5.1). Hence,

\[
p_0 = \frac{\mu - \lambda}{\gamma_0} \quad (5.9)
\]

provided \( \gamma_0 \neq 0 \).

Thus an explicit form of the steady-state solution is

\[
P_x = (\mu - \lambda) \sum_{j=0}^{\infty} [\gamma_{x+j}(j) - \gamma_{x+j+1}(j)] \quad (5.10)
\]

From (4.2.1), the steady-state solution is

\[
P_{x-1} = \sum_{i=0}^{x} p_i \gamma_{x-1} \quad (5.11)
\]
or

\[
P_x = \frac{1}{\gamma_0} [p_{x-1} - \sum_{j=0}^{x-1} p_j \gamma_{x-j}] \quad x = 1, 2, 3, \ldots
\]

which will perhaps be easier to evaluate in many numerical situations.
6. SOME SPECIFIC DISTRIBUTIONS

The equations derived thus far have assumed a general probability density function, \( \alpha_a \), describing the probability of "a" arrivals during a given period. A binomial probability density function, \( \beta_b \), was also assumed to describe the probability of "b" departures during a given period. The following sections consider special functions for \( \alpha_a \) and \( \beta_b \).

6.1. Poisson Arrival Distribution

If the random variables \( a \), the number of arrivals during each unit time interval, are independently and identically distributed according to a Poisson law with parameter \( \lambda \), then

\[
A(s) = e^{\lambda(s-1)} ;
\]

hence, the p.g.f. for the net change, \( C(s) \), is

\[
C(s) = sA(s)B(s) = se^{\lambda(s-1)}(\beta_0 + \beta_1 s^{-1}) .
\]

Equating coefficients for like powers of \( s \) in the equation \([C(s)]^k\) gives

\[
\gamma_r(k) = e^{-\lambda k} \sum_{i=0}^{r} \binom{k}{i} \beta_0^{r-i} \beta_1^{k-r+i} \left( \frac{\lambda k}{i!} \right)^{i} ; \quad r = 0,1,2,\ldots,k-1
\]

\[
= e^{-\lambda k} \sum_{i=0}^{k} \binom{k}{i} \beta_0^{k-i} \beta_1^{i} \left( \frac{\lambda k}{(r+i-k)!} \right) ; \quad r = k, k+1, \ldots .
\]

From these results together with equation (9.1.4) the probability of first emptiness can be written as:
\[ g(u,k) = \frac{u^k}{k!} e^{-\lambda k} \sum_{i=0}^{k-u} \binom{k}{k-u-i} \beta_0^{k-u-i} \beta_1^{u+i} \frac{(\lambda k)^i}{i!} , \quad u > 0 . \quad (6.1.4) \]

The above equations are then used with (4.2.16) to obtain the probability of emptiness

\[ p_0(k) = e^{\lambda k} \frac{\sum_{r=0}^{k-u} \sum_{i=0}^{u+r+1} \frac{u+r+1}{k+1} \beta_0^{k-u-r-i} \beta_1^{u+r+1+i} \frac{(\lambda (k+1))^{i}}{i!}}{\beta_1} ; \]

\[ k = u, u+1, u+2, \ldots \quad (6.1.5) \]

\[ = 0 ; \quad \text{otherwise}. \]

Given the solutions for \( \gamma_x(k) \), \( g(u,k) \), and \( p_0(k) \), one can determine the probability of a given queue size, \( x \), from any of the equations (4.1.10), (4.1.12), or (4.2.13, 4.2.14, 4.2.15).

The steady-state solutions are

\[ p_0 = \frac{e^{\lambda (\beta_1 - \lambda)}}{\beta_1} \]

\[ p_x = (\beta_1 - \lambda) \sum_{j=0}^{\infty} \left[ \gamma_{x+j}(j) - \gamma_{x+j+1}(j) \right] ; \quad x = 1, 2, \ldots \]

where

\[ \gamma_x(k) \text{ is defined by equation (6.1.3)}. \]

6.2. Poisson Arrival and Certain Departure

In this thesis the departures in successive time periods have been assumed to follow independent binomial distributions with probability \( \beta_0 \) that a unit does not depart and probability \( \beta_1 = 1 - \beta_0 \) that a unit does depart in a given time period. The special case where a unit is certain to depart, i.e., \( \beta_0 = 0, \beta_1 = 1 \), when the queue is not empty,
is referred to here as the case of certain departure. This special case enables the prior equations to be simplified considerably, giving

\[ \gamma_r(k) = \frac{e^{-\lambda k} (\lambda k)^r}{r!} ; \quad r = 0,1,2, \ldots \quad (6.2.1) \]

\[ g(u,k) = \frac{u}{k} e^{-\lambda k} \frac{(\lambda k)^{k-u}}{(k-u)!} ; \quad u = 1,2, \ldots, k \quad (6.2.2) \]

\[ p_0(k) = e^{-\lambda k} \sum_{r=0}^{k-u} \frac{u+r+1}{k+r+1} \frac{[\lambda(k+1)]^{k-u-r}}{(k-u-r)!} ; \quad k = u, u+1, \ldots \quad (6.2.3) \]

\[ = 0 \text{ ; otherwise .} \]

Using equations (4.2.13) and (4.2.15) for \( p_k(x) \) gives

\[ p_k(x) = e^{-\lambda k} \frac{(\lambda k)^{x+k-u}}{(x+k-u)!} + \sum_{j=0}^{k-u-2} p_0(u+j) e^{-\lambda (k-u-j)} \frac{\lambda (k-u-j)^{k-u-1+x-j}}{(k-u-1+x-j)!} \quad (6.2.4) \]

\[ = 1,2,3, \ldots \]

\[ x = u-k, u-k+1, \ldots; \quad k < u \]

\[ = 1,2,3, \ldots \quad k \geq u \]

which agrees with the solution (2.3.10) given by Yeo [1961].

The steady-state solution is

\[ p_0 = (1 - \lambda) e^\lambda \]

\[ p_k(x) = (1 - \lambda) \sum_{j=0}^{\infty} e^{-\lambda j} \frac{\lambda^j}{(x+j)!} \left\{ 1 - \left[ \frac{\lambda j}{(x+j+1)} \right] \right\} \quad (6.2.5) \]

which also agrees with the results (2.3.11) of Yeo.
6.3. Negative Binomial Arrival Distribution

If the probability, \( a_a \), of "a" arrivals during each unit time interval follows a negative binomial distribution with parameters \( \xi \) and \( n \), then

\[
\alpha_a = \binom{n-1+a}{n-1} (1-\xi)^n \xi^a; \quad a = 0, 1, 2, \ldots, 0 \leq \xi \leq 1, \quad (6.3.1)
\]

\[
A(s) = (1-\xi)^n (1-\xi s)^{-n}, \quad (6.3.2)
\]

and the p.g.f. for the net change, \( C(s) \), is

\[
C(s) = sA(s)B(s) = s(1-\xi)^n (1-\xi s)^{-n} (\beta_0 + \beta_1 s^{-1}). \quad (6.3.3)
\]

Equating coefficients for like powers of \( s \) in the equation \([C(s)]^k \) gives

\[
\gamma_r(k) = (1-\xi)^{nk} \sum_{i=0}^{r} \binom{k}{i} (nk+r-i-1)\beta_{i}^{k-i} \beta_{0}^{i} r^{-i}; \quad r = 0, 1, 2, \ldots, k, \quad (6.3.4)
\]

\[
= (1-\xi)^{nk} \sum_{i=0}^{k} \binom{k}{i} (nk+r-i-1)\beta_{i}^{k-i} \beta_{0}^{i} r^{-i}; \quad r = k+1, k+2, \ldots
\]

From these results together with equation (9.1.4), the probability of first emptiness can be equated as:

\[
g(u, k) = \frac{u}{k} (1-\xi)^{nk} \sum_{i=0}^{k-u} \binom{k-u}{i} (nk+k-u-i-1)\beta_{i}^{k-i} \beta_{0}^{i} s^{k-u-i}; \quad u = 1, 2, 3, \ldots, k. \quad (6.3.5)
\]

The above equations are then used with (4.2.16) to obtain the probability of emptiness.
\[ p_0(k) = \frac{(1-\xi)^{nk}}{\beta_1^{(k+1)}} \sum_{r=0}^{k-u} \sum_{i=0}^{k+1+r} \binom{k+1}{r+1}(nk+n+k-u-r-1)_{nk+n-1} \]  \hspace{1cm} (6.3.6)

\[ \beta_1^{k+1-i} \beta_0^{n+k-u-r-i} \]  \hspace{1cm} (6.3.6)

\[ = 0 \]  \hspace{1cm} = 0 \; \text{otherwise.} \]

Given the solutions for \( \gamma_x(k), g(u,k), \) and \( p_0(k) \), one can determine the probability of a given queue size, \( x \), from any of the equations (4.1.10), (4.1.12), or (4.2.13, 4.2.14, 4.2.15).

The steady-state solutions are

\[ p_0 = \frac{[\beta_1-n\xi(1-\xi)]^{-1}}{(1-\xi)^n \beta_1} ; \]  \hspace{1cm} (6.3.7)

\[ p_x = [\beta_1-n\xi(1-\xi)]^{-1} \sum_{j=0}^{\infty} [\gamma_{x+j}(j) - \gamma_{x+j+1}(j)] , \]

where \( \gamma_x(k) \) is defined in equation 6.3.4.

6.4. Negative Binomial Arrival and Certain Departure

In the special case where a unit is certain to depart, i.e., \( \beta_0 = 0, \beta_1 = 1 \), when the queue is not empty, enables the equations in Section 6.3 to be simplified giving:

\[ \gamma_x(k) = \binom{nk+r-1}{nk-1}\xi^r(1-\xi)^nk ; \]  \hspace{1cm} r = 0,1,2,\ldots , \hspace{1cm} (6.4.1)

\[ g(u,k) = \frac{u}{k} \binom{nk+k-u-1}{nk-1}\xi^{k-u}(1-\xi)^nk ; \]  \hspace{1cm} u = 1,2,3,\ldots , \hspace{1cm} (6.4.2)
\[ p_0(k) = \begin{cases} \frac{(1-\xi)^{nk}}{k+1} \sum_{r=0}^{k} \left( \frac{n^{k+u-r}}{n^{k+u-1}} \right)^{k-u-r} ; & k \geq u, \\ 0; & \text{otherwise,} \end{cases} \]

which can also be written as

\[ p_0(k) = \begin{cases} \frac{(1-\xi)^{nk}}{k+1} \sum_{i=u+1}^{k+1} \left( \frac{n^{k+u-i}}{n^{k+u-1}} \right)^{k+1-i} ; & k \geq u, \\ 0; & \text{otherwise.} \end{cases} \] (6.4.3)

Using equations (4.2.13) and (4.2.15), the results for \( p_x(k) \) are:

\[ p_x(k) = \left( \frac{n^{k+x-u-1}}{nk-1} \right) (1-\xi)^{nk} x-u+k \]

\[ + \sum_{i=0}^{k-u-2} p_0(u+i)(1-\xi)^{n(k-u-i)} x^{k-u-i-1} \]

\[ \cdot \left[ \left( \frac{(n+1)(k-u-i-1)+x-1}{n(k-u-i-1)} \right)^{n(k-u-i-1)-1} \right] \]

\[ k = 1, 2, 3, \ldots \]

\[ x = u-k, u-k+1, \ldots ; k < u \]

\[ = 1, 2, 3, \ldots ; k \geq u. \]

Equation (6.4.3) agrees with the results given by Yeo [1961], equation (2.3.9), for certain departures. Equation (6.4.4) should also agree with (2.3.8) but differs by what is believed to be a typographical error in Yeo's paper.
The steady-state solution is

\[ p_0 = \frac{1-n\xi(1-\xi)^{-1}}{(1-\xi)^n}, \]  
\[ (6.4.5) \]

\[ p_x = \left[1-n\xi(1-\xi)^{-1}\right] \sum_{j=0}^{\infty} (1-\xi)^{nj+1} x^j \left[ (\frac{n_{j+1}}{n_{j-1}} - \xi) \frac{n_{j+1}}{n_{j-1}} \right] ; \]

\[ x = 1, 2, 3, \ldots \]
7. SUMMARY

This thesis considers the problem of deriving the probability, \( p_x(k) \), that at the end of the \( k \)-th time interval there are \( x \) units in a queue, assuming a general distribution for arrivals and a Bernoulli distribution for departures. The problem differs from that of Yeo [1961] in that it permits a Bernoulli distribution for departures instead of certain departure as assumed by Yeo.

The solution is approached by determining the probability of first emptiness \( g(u,k) \) after \( k \) time periods when initially there were \( u \) units present. It is shown that in equation (4.1.1)

\[
g(u,k) = \begin{cases} 
\gamma_1(1) = \gamma_1; & u = 0, k = 1 \\
\gamma_k^0 \frac{\gamma_k^0 \gamma(k-1)}{k-1}; & u = 0, k > 1 \\
u \gamma_{k-u}(k); & 1 \leq u \leq k \\
0; & \text{otherwise},
\end{cases}
\]

(7.1)

where \( \gamma_1(k) \) is the probability of a net change of \( 1 \) units in \( k \) time periods (net change is defined in Section 3), \( \gamma_1(1) = \gamma_1 \) and

\[
\gamma_k^0 = \frac{1}{k} \sum_{i=2}^{r} (i-1) \gamma_1 \gamma_{i-1}(k-1) = \gamma_0 \gamma_k(k-1).
\]

The results for \( u=0 \) are special cases not included by Yeo [1961].

The probability \( p_x(k) \) is then derived by two separate techniques. The first is obtained from the relation

\[
p(k) = G^{-1} g(k); \quad k = u+1, u+2, \ldots
\]

(7.2)
defined by (4.1.10). This method is simply an extension of the process 
used for deriving \( g(u, k) \) and requires the inversion of a matrix.

The above results can be written in the recursive form

\[
\begin{align*}
px(k) &= \gamma_{x-u+k}(k) ; & k = 1, 2, \ldots, u, \\
p_0(k) &= \sum_{r=0}^{k-u} g(u+1+r, k+1)/\gamma_0; & k \geq u+1, \\
p_x(k) &= \sum_{r=0}^{k-u} \left\{ g(u+1+r, k+x+1) - \sum_{\ell=0}^{x-1} p_x(\ell; r)g(\ell+1, x+1) \right\}/g(x+1, x+1) ; & k \geq u+1, x \geq 1,
\end{align*}
\] (7.3)

which is shown by (4.1.12); \( \gamma_0 = \alpha_0 \beta_1 \) is defined in Section 3.

The second method utilizes generating functions to give an explicit 
expression

\[
\begin{align*}
px(k) &= \gamma_{x-u+k}(k) ; & k = 1, 2, 3, \ldots, u; x-u+k \geq 0, \\
&= \gamma_{x-u+k}(k) + \sum_{j=u}^{k-2} p_0(j)\gamma_0[\gamma_{x+k-j-1}(k-1-j) - \gamma_{x+k-j}(k-1-j)]; & k = u+1, u+2, \ldots; x-u+k \geq 0 ,
\end{align*}
\] (7.4)

which is given by equations (4.2.9) and (4.2.10).

A measure of the computational effort required by each of the two 
methods has been assessed. It is concluded that when one is interested 
in small queue sizes and/or small values of \( k \), the first method involves 
fewer terms to be summed, and is therefore more advantageous. When \( k 
\) and/or \( x \) is large, the second method derived from generating functions 
involves fewer summations and is more advantageous for evaluating \( px(k) \).
Two specific input distributions were then considered, the Poisson and negative binomial, and the general solution, \( p_x(k) \), is given in explicit form for each case. The solutions so derived were then restricted further by assuming that a departure is certain in each period when there is at least one unit in the queue. The restricted results were then simplified and checked with those of Yeo [1961].

Steady-state solutions are derived and the conditions required for such solutions are discussed. Specifically, if the expected number of arrivals, \( \lambda \), is less than the expected number of departures, \( \mu \), within a given period, a steady-state solution will exist and

\[
\lim_{k \to \infty} \gamma_{x-u+k}(k) = 0.
\] (7.5)

A steady-state solution may then be obtained from (7.4) by allowing \( k \) to approach infinity yielding the relation

\[
p_x = p_0 \gamma_0 \sum_{j=0}^{\infty} \left[ \gamma_{x+j}(j) - \gamma_{x+j+1}(j) \right].
\] (7.6)

The solution for \( p_0 \),

\[
p_0 = \frac{\mu - \lambda}{\gamma_0},
\] (7.7)

given by (5.9) is obtained from the steady-state generating function (5.6).

The two special cases mentioned above are then evaluated to give the steady-state solutions. The only steady-state solution given by Yeo was for Poisson arrival and certain departure and was checked with
the solution derived in this thesis. The solution for $p_0$ given in (7.7) is easier to use than that given by Yeo for this special case.

A steady-state solution for $p_x$ may also be obtained from the recursive relationship

$$p_x = \frac{1}{\gamma} [p_{x-1} - \sum_{j=0}^{x-1} p_j \gamma^{x-j}]; \quad x = 1, 2, 3, \ldots$$

(7.8)
given in (5.11).

As a suggestion for future study, one may consider the probability of first emptiness. The relationship

$$s(u, p) = \frac{u}{p} \gamma^{p-u}(p),$$

(7.9)
as proven, holds only for those cases where at most one unit is allowed to leave the queue each period. Kemperman [1950] gives a general relationship for those cases where the number of arrivals is a random variable from a general and independent distribution and the departures each period is a random variable which can exceed unity. Since most of the subsequent derivations required to obtain $p_x(k)$ can be extended without difficulty, it may be advantageous to derive and simplify the more general results.

It should be noted that a general class of explicitly solvable queues can be described using the same techniques of Smith [1953]. This class is defined by the relation $n = \alpha m$ (or $m = \beta n$) where $\alpha = 1, 2, \text{ or } 3$ (or $\beta = 2, \text{ or } 3$); $n$ is the maximum number of binomial arrivals within a period and $m$ the maximum number of binomial departures during the same period. The transient solution is given by Magistad [1960].
In view of the steady-state solutions (2.3.11) and (2.3.12) given by Yeo, and by definition, it is required that

$$P_0 = 1 - \sum_{x=1}^{\infty} P_x.$$  \hspace{1cm} (7.10)

The solutions (5.9) and (5.10) derived from the generating functions are not written in a form where it is obvious that (7.10) is true. A direct proof of the relationship

$$\frac{\mu - \lambda}{\gamma_0} = 1 - (\mu - \lambda) \sum_{x=1}^{\infty} \sum_{j=0}^{\infty} [\gamma_{x+j}(j) - \gamma_{x+j+1}(j)]$$  \hspace{1cm} (7.11)

would serve as a helpful check.

Another approach to the derivation of the formula for $g(u,k)$, $0 \leq u \leq k$, is the following.

A queue which started with $u$ units at time zero and is empty at time $k$ was first empty at either time $k$, $k-1$, ..., $u+1$, or $u$. Hence we can write

$$\gamma_{k-u}(k) = u \sum_{j=0}^{k-u} g(u, k-j) \gamma_j(j),$$  \hspace{1cm} (7.12)

where $\gamma_0(0) = 1$. Using (9.1.4), $g(u, k-j) = \frac{u}{k-j} \gamma_{k-u}(k-j)$, then (7.12) becomes

$$u \sum_{j=1}^{k-u} \left[ \frac{\gamma_j(j)}{k-j} \right] (k-j) = (k-u) \gamma_{k-u}(k).$$  \hspace{1cm} (7.13)

Hence an alternative proof of (9.1.4) is to verify (7.13).
8. LIST OF REFERENCES


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2 This thesis may be purchased from the Mathematical Center, 2e Boerhaavestraat 49, Amsterdam (O), Netherlands.


9. APPENDICES

9.1. Probability of First Emptiness, $g(u,k)$ when $u > 0$

Utilizing the concept of the path diagram shown in Figure 4.1, one may transform the physical aspects of a queue situation into an equivalent formulation regarding random walk with an absorbing barrier. Such a transformation is advantageous in many cases for it often permits a comparison between a random walk with an absorbing barrier and an unrestricted random walk. This is the method used by Kemperman [1950]. For situations in continuous time, one may refer to proofs by Smith and Kendall [1957]. Kemperman considered the case where, in essence, the number of arrivals and departures during each time period is allowed to be any positive integer equal to or greater than zero. The solution is surprisingly simple when at most only one unit is allowed to leave each period, which is the situation considered in this thesis. The following development deviates from that of Kemperman [1950] in that it is much simpler, it is an induction proof, and the notation has been modified to conform with this thesis.

Designate the various steps of a path by $c = 0, 1, 2, \ldots$, and let $N_c$ denote the number of steps of magnitude $c$, where $c$ has the same meaning as before: one plus the difference between the number of arrivals and the number of departures. Consider only those paths which are initiated at $(0, u)$ and which terminate at $(k, k)$, $0 < u \leq k$.

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3 Unpublished lecture notes of Dr. W. L. Smith. Portions of these notes may be obtained by contacting Dr. W. L. Smith, Department of Statistics, University of North Carolina, Chapel Hill, North Carolina.
In the case where \( u = 1 \) and \( k = 3 \) the unrestricted random walk allows six paths:

\[
(011), (101), (110) \text{ and } (200), (020), (002).
\]

The restricted random walk (i.e., the ensemble of those paths of the queuing process which start at \((0, 1)\), end at \((3, 3)\) and do not touch the barrier before) \( t = 3 \) allows two paths:

\[(110) \text{ and } (200)\].

Each of the paths \((011), (101), (110)\) can be characterized by \(\{N_0 = 1, N_1 = 2, N_2 = 0\}\). Each of the paths \((200), (020), (002)\) can be characterized by \(\{N_0 = 2, N_1 = 0, N_2 = 1\}\). In general two paths from \((0, u)\) to \((k, k)\) are said to be of the same type if they have the same sequence of numbers \(\{N_0, N_1, N_2, \ldots, N_{k-u}\}\).

Let \(m_1[u, k; \{N_0, N_1, \ldots, N_{k-u}\}]\) denote the number of paths of type \(\{N_0, N_1, \ldots, N_{k-u}\}\) which start at \((0, u)\), and end at \((k, k)\) and which do not touch the boundary until the end of the \(k\)-th step; the probability of any individual non-touching path of a given type is

\[
P[u, k; \{N_0, N_1, \ldots, N_{k-u}\}] = \prod_{c=0}^{k-u} \gamma_c \quad ; \tag{9.1.1}
\]

\(m_2[u, k; \{N_0, N_1, \ldots, N_{k-u}\}]\) denote the number of unrestricted paths of type \(\{N_0, N_1, \ldots, N_{k-u}\}\) which start at \((0, u)\) and end at \((k, k)\); the probability of any individual unrestricted path of a given type is again

\[
P[u, k; \{N_0, N_1, \ldots, N_{k-u}\}] = \prod_{c=0}^{k-u} \gamma_c \quad .
\]
Now $g(u, k)$ can be obtained by summing over all non-touching paths of various types

$$g(u, k) = \sum_{T} m_{1}(u, k; T) \cdot P(u, k; T), \quad (9.1.2)$$

where $T$ stands for $\{N_0, N_1, \ldots, N_{k-u}\}$. At the same time $\gamma_{k-u}(k)$ is obtained by summing over all unrestricted paths of various types

$$\gamma_{k-u}(k) = \sum_{T} m_{2}(u, k; T) \cdot P(u, k; T). \quad (9.1.3)$$

In the above example with $u = 1$, $k = 3$

$$m_{1}[1,3; 1,2,0] = 1$$
$$m_{2}[1,3; 1,2,0] = 3$$
$$m_{1}[1,3; 2,0,1] = 1$$
$$m_{2}[1,3; 2,0,1] = 3.$$

Hence in this case,

$$\frac{m_{1}(1,3; T)}{m_{2}(1,3; T)} = \frac{1}{3} = \frac{u}{k},$$

regardless of the type $T$. We shall prove this to be true in general for $0 < u \leq k$. As a consequence,

$$g(u, k) = \sum_{T} m_{1}(u, k; T) \cdot P(u, k; T) = \frac{u}{k} \sum_{T} m_{2}(u, k; T) \cdot P(u, k; T)$$

$$= \frac{u}{k} \gamma_{k-u}(k), \quad 0 < u \leq k. \quad (9.1.4)$$
Lemma. For $0 < u \leq k$ we have

$$\frac{m_1[u,k; \{N_0, N_1, \ldots, N_{k-u}\}]}{m_2[u,k; \{N_0, N_1, \ldots, N_{k-u}\}]} = \frac{u}{k}. \quad (9.1.5)$$

Proof

In the first place the number of unrestricted paths of type

$$\{N_0, N_1, \ldots, N_{k-u}\}$$

is given by

$$m_2[u,k; \{N_0, N_1, \ldots, N_{k-u}\}] = \frac{k!}{N_0! N_1! \ldots N_{k-u}!}, \quad (9.1.6)$$

where

$$\sum_c N_c = k, \quad (9.1.7)$$

and

$$\sum_c c N_c = k - u. \quad (9.1.8)$$

In the second place consider $m_1(u,k; T)$ in case $k = 1$. When $k = 1$, the only possible value for $u$, $0 < u \leq k$, is $u = 1$, and the only possible step (whether unrestricted or non-touching) $c$, $c = 0, 1, \ldots, k-u$, is $c = 0$, which defines a path of the type $N_0$, where $N_0 = 1$. It is apparent then that the relationship

$$m_1[1,1; \{N_0=1\}] = 1 = m_2[1,1; \{N_0=1\}]$$

satisfies the lemma for the case.

In the third place consider $m_1(u,k; T)$, assuming (9.1.5) holds for $k-1$. Subdivide the restricted paths according to the first step. Then the problem of the remaining steps is to evaluate
\[ m_1[u+c-1, k-1; \{ N_0^{-\delta_0}, c, N_1^{\delta_1}, c, \ldots, N_{k-u}^{\delta_{k-u}}, c \} ] \] (9.1.9)

where

\[
\delta_{i,c} = \begin{cases} 
1 & \text{if } i = c; \\
0 & \text{otherwise}
\end{cases}
\]

The total number of restricted paths from \((0, u)\) to \((k, k)\) of type \(\{N_0, N_1, \ldots, N_{k-u}\}\) may then be written as

\[
m_1[u,k; \{ N_0, N_1, \ldots, N_{k-u} \} ] = \sum_{c=0}^{k-u} m_1[u+c-1, k-1; \{ N_0^{\delta_0}, c, \ldots, N_{k-u}^{\delta_{k-u}}, c \} ]
\]

\[
= \sum_{c=0}^{k-u} \frac{u+c-1}{k-1} \cdot \frac{(k-1)!}{(N_0^{\delta_0}, c)! \cdots (N_{k-u}^{\delta_{k-u}}, c)!}
\]

as a consequence of (9.1.6) and the assumption that (9.1.5) holds for \((k-1)\). Since only the factor \((N_c^{a+c})\) is affected by any particular value of \(c\) and because this simply decreases its value by unity, we may write the last member of (9.1.10) as

\[
\sum_{c=0}^{k-u} \frac{u+c-1}{k-1} \cdot \frac{(k-1)! N_c}{N_0^{\delta_0}, c! \cdots N_{k-u}^{\delta_{k-u}}, c!}
\]

\[
= \sum_{c=0}^{k-u} \frac{(k-1)! N_c}{(u+c-1)N_c}
\]

(9.1.11)

Using (9.1.7) and (9.1.8), the expression (9.1.11) becomes

\[
\frac{(k-1)! (u+k-u-k)}{k-u} = \frac{u(k-1)!}{(k-1) \prod_{c=0}^{k-u} N_c!}
\]

(9.1.12)
By multiplying the numerator and denominator of (9.1.12) by $k$, we obtain

$$m_1[u,k; \{ N_0, N_1, \ldots, N_{k-u} \}] = \frac{u^k}{k} \prod_{c=0}^{k-u} N_c!$$

$$= \frac{u^k}{k} m_2[u,k; \{ N_0, N_1, \ldots, N_{k-u} \}], \quad 0 < u \leq k,$$

which satisfies (9.1.4) and completes the proof.

If $u > k$, the queue cannot be empty after $k$ periods, when at most only one unit is allowed to leave each period. Thus $g(u,k) = 0$ when $u > k$.

### 9.2. Probability of Subsequent First Emptiness when $u = 0$

From the standpoint of solving the queue problem for cases of $u \geq 0$, one need only consider values of $g(u,k)$ where $u > 0$, since the development utilizes the technique of an artificial barrier, $Y(t) = t - 1$.

Although not required, we proceed to solve for $g(u,k)$ when $u = 0$. If $u = 0$ and $k > 1$, during the first period $2 \leq c(1) \leq k$, which establishes a situation where $1 \leq u \leq (k-1)$ for the next $(k-1)$ periods. Thus

$$g(0,k) = \sum_{c=2}^{k} \gamma_c g(c-1, k-1). \quad (9.2.1)$$

From (9.1.4), equation (9.2.1) can be written as

$$g(0,k) = \frac{1}{k-1} \sum_{c=2}^{k} (c-1) \gamma_c \gamma_{k-c} (k-1); \quad (9.2.2)$$

so that by (3.15) and (3.17),

$$g(0,k) = \frac{\gamma_k^0}{(k-1)} = \frac{\gamma_k (k-1)}{k-1}, \quad k > 1. \quad (9.2.3)$$
If \( u = 0 \) and \( k = 1 \) then either \( a(1) = b(1) = 1 \) or \( a(1) = b(1) = 0 \), i.e., \( c(1) = 1 \), for which \( \text{Prob} \{ c(1) = 1 \} = \gamma_1(1) = \gamma_1 \); hence,

\[
g(0,1) = \gamma_1(1) = \gamma_1. \tag{9.2.4}
\]

In summary,

\[
g(0,k) = \begin{cases} 
\gamma_1(1) = \gamma_1, & k = 1; \\
\gamma_0^k/(k-1), & k > 1,
\end{cases}
\]

where \( \gamma_x(k) \) is the coefficient of \( s^x \) in the expansion of \( [C(s)]^k \) and \( \gamma_0^k \) the coefficient of \( s^r \) in the expansion of

\[
\left\{ \frac{s}{k} \frac{\partial}{\partial s} [C(s)]^k - [C(s)]^k + \gamma_0 [C(s)]^{k-1} \right\}.
\]