OPTIMAL DESIGNS TO ESTIMATE VARIANCE COMPONENTS AND TO REDUCE PRODUCT VARIABILITY FOR NESTED CLASSIFICATIONS.

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Institute of Statistics Mimeo #313
ABSTRACT

PRAIRIE, RICHARD ROLAND. Optimal Designs to Estimate Variance Components and to Reduce Product Variability for Nested Classifications. (Under the direction of RICHARD LOREE ANDERSON.)

Two separate problems were considered in this thesis, both involving the design of experiments for estimating parameters for nested classifications.

The first problem was concerned with the importance of experimental design for the estimation of functions of variance components for a two stage nested classification when there is a specified objective. The objective considered was that of efficiently reducing product, or total, variability through the expenditure of a fixed amount of funds.

A model was proposed that related the reduction in total variance to the amount of funds expended. With the model considered it was found that the optimal allocation of funds is a function of \( \rho = \sigma^2_A / \sigma^2_B \), where \( \sigma^2_A \) and \( \sigma^2_B \) are the variance components associated with the two assignable sources of variation.

It was assumed that an estimate of \( \rho \) is obtained from an experiment based on the model

\[
x_{ij} = m + A_i + B_{ij}, \quad j = 1, 2, \ldots, n_i;
\]

\[
\sum_{i=1}^{a} n_i = N,
\]

where \( m \) is constant and \( A_i \) and \( B_{ij} \) are normally and independently distributed (NID) with means zero and respective variances \( \sigma^2_A \) and \( \sigma^2_B \).

A method of design construction was given so that different designs are
obtained by varying the number of classes $a$. Two estimators of $\rho$ based on the usual analysis of variance were considered.

A criterion was developed for judging the influence of design and estimator. Some numerical results were presented which indicated that for most situations an intermediate value of $a$, say $N/4 < a < N/2$, will give results that are quite close to the optimal.

The second aspect of this thesis dealt with the design of experiments for estimating parameters from a three stage nested classification, for which $C_{ijk}$ (NID with mean zero and variance $\sigma^2_C$) is added to (1) with $J = 1, 2, \ldots, b_i$; $k = 1, 2, \ldots, n_{ij}$; $\sum_i \sum_j n_{ij} = N$. The estimators of the variance components used were those based on the usual analysis of variance.

Variances of the estimators of $\sigma^2_A$, $\sigma^2_B$ and $\sigma^2_C$ were derived and studied for various designs. Designs were presented for the estimation of $m$, $\sigma^2_C$ and $\sigma^2_C + \sigma^2_B + \sigma^2_A$ that minimized the variances of these estimators. Designs were also given for the estimation of $\sigma^2_B$, $\sigma^2_A$ and $\rho = \sigma^2_A / (\sigma^2_C + \sigma^2_B)$ that appeared to yield estimates whose variances were near minimal.

Attention was also devoted to the simultaneous estimation of $\sigma^2_A$, $\sigma^2_B$ and $\sigma^2_C$. A method of design was suggested and the variances of the estimates from this design were compared numerically with other types of design.
OPTIMAL DESIGNS TO ESTIMATE VARIANCE COMPONENTS AND
TO REDUCE PRODUCT VARIABILITY FOR NESTED CLASSIFICATIONS

by

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To W. S. Mallios I express my thanks for acting as a sounding board for many of the concepts that are, and are not, contained in this thesis.

I thank my wife for putting up with the whole thing.
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1.0 INTRODUCTION

In many situations in which a result from an experimental unit is subject to several sources of variation the problem is to identify the sources and to estimate the variation associated with each. Generally estimation and identification are accomplished through the use of experimental procedures. While considerable effort has been devoted to the efficient design of experiments for the estimation of treatment contrasts and regression parameters, little has been done on designing experiments for the estimation of variance components. Crump (1954) proposed designs for the two stage nested classification that provide optimal estimates of the between class variance component, the within class variance component, or the ratio of the two components. Gaylor (1960) presented optimal designs for the estimation of components of variance and certain functions of the components for the two-way crossed classification. Anderson and Bancroft (1952) and Anderson (1960) introduced a staggered design for the estimation of parameters from a three or more stage nested classification. No known work has been done on the problem of designing experiments for the estimation of variance components with the objective of using the estimates for the efficient planning of future experiments or courses of action.

Part of this thesis is concerned with the design of experiments for the efficient estimation of functions of the components of variance with the objective of reducing product, or process, variability. Only the case in which a result is subject to two sources of variation will be considered. It will be assumed that a single observation can
be represented by

\[ x_{ij} = m + A_i + B_{ij} \quad j = 1, 2, \ldots, n_i \]

\[ \sum n_i = N \quad (1.1) \]

where \( m \) is a constant, \( A_i \) and \( B_{ij} \) are both normally independently distributed (NID) with means zero and variances \( \sigma^2_A \) and \( \sigma^2_B \), respectively.

Under this model the product variance, i.e., the variance of an observation, may be represented by

\[ \sigma^2_T = \sigma^2_A + \sigma^2_B. \]

An analysis of variance for data obeying (1.1) is given in Table 1.1.

Table 1.1. An analysis of variance for a two stage nested classification

| Source of variation | Degrees of freedom | Mean square | Expectation of \( \Sigma \)
<table>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Classes</td>
<td>( a - 1 )</td>
<td>( M_A )</td>
<td>( \sigma^2_B + K\sigma^2_A )</td>
</tr>
<tr>
<td>Within classes</td>
<td>( N - a )</td>
<td>( M_B )</td>
<td>( \sigma^2_B )</td>
</tr>
<tr>
<td>Total</td>
<td>( N - 1 )</td>
<td></td>
<td></td>
</tr>
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</table>

\[ \sigma^2_T = (N - \frac{\Sigma n_i^2}{N})/(a-1) \]

The other part of this thesis is devoted to the study of experimental designs for the purpose of estimating parameters from a three stage nested classification. It will be assumed that each observation can be represented by
\[ x_{ijk} = m + A_i + B_{ij} + C_{ijk} \quad i = 1,2,\ldots,a \\
\quad j = 1,2,\ldots,b_i \\
\quad k = 1,2,\ldots,n_{ij} \\
\quad \Sigma n_{ij} = n_i, \Sigma \Sigma n_{ij} = N \] (1.2)

where \( m \) is a constant, \( A_i, B_{ij} \) and \( C_{ijk} \) are all NID with means zero and variances, \( \sigma^2_A, \sigma^2_B \) and \( \sigma^2_C \), respectively. In this work the emphasis will be placed on design rather than method of estimation. The estimators used throughout are the usual analysis of variance estimators obtained by equating mean squares to their expectations. Optimal designs for the estimation of single functions of components of variance will be given. The analogy between these designs and those proposed by Crump for the two stage classification will also be shown.

Some effort will also be devoted to the study of the influence of design on the simultaneous estimation of the three components of variance from a three stage nested classification.
2.0 REVIEW OF LITERATURE

As was mentioned in Chapter 1, no known work has been done on determining the influence of the experimental design when estimates of variance components are to be used for planning future experiments or courses of action. One of the most common uses of variance components is in the planning of an experiment in which it is desired to minimize the cost of obtaining a sample estimate when the precision is fixed, or conversely, to maximize the precision of an estimate obtained for a given cost. An excellent discussion of this use of variance components is given by Marcuse (1949). To point out the lack of concern for the design of the experiment from which the estimates of the variance components are to be obtained, we quote Marcuse:

"This dilemma may be evaded by first carrying out a preliminary experiment in nested sampling using a set of arbitrarily chosen class frequencies." (Page 192)

It certainly seems that one could do better than to arbitrarily choose the design as Marcuse suggests.

Some work has been done on designing experiments for the specific purpose of estimating variance components. The first such work was published by Hammersley (1949). He considered the two stage nested classification in which an observation may be represented by (1.1). An analysis of variance for such observations is given in Table 1.1. Hammersley used as an estimator of $\sigma_A^2$, $\hat{\sigma}_A^2 = (M_A - M_B)/K$. He derived the variance of $\hat{\sigma}_A^2$, $\text{var}(\hat{\sigma}_A^2)$, and proposed a design to minimize $\text{var}(\hat{\sigma}_A^2)$. For $N$ and $a$ fixed he showed that $\text{var}(\hat{\sigma}_A^2)$ is minimized by choosing a design that has an equal number of units in each class; i.e., requiring
that \( N/a \) be an integer. For \( N/a \) an integer he proved that the number of classes that minimizes \( \text{var}(\hat{\sigma}_A^2) \) is

\[
a_1 = N \left[ \frac{N\sigma_A^2 + 2\sigma_B^2}{N(\sigma_A^2 + \sigma_B^2) + \sigma_B^2} \right]
\]

and therefore the optimum number of units per class is

\[
b_1 = N/a_1 = \frac{N\sigma_A^2 + (N+1)\sigma_B^2}{N\sigma_A^2 + 2\sigma_B^2}.
\]

For either \( a_1 \) or \( b_1 \) not an integer Hammersley did not prove an optimum procedure for determining a design, he only suggested trying the closest integers.

Crump (1954) considered the problem of efficiently designing experiments for the two stage nested classification but did not restrict himself to integer values of \( N/a \). He proved for fixed \( N \) and \( a \), where \( N/a = p + s/a(0 \leq s < a) \), that the optimal design for the estimation of \( \sigma_A^2 \) consists of \( s \) classes with \( p+1 \) units and \( a-s \) classes with \( p \) units. Crump proposed a scheme, which depended upon the ratio \( \rho = \sigma_A^2/\sigma_B^2 \), for determining a design which is optimum. He was unable to prove that his scheme always leads to an optimal design, but for numerous examples his procedure always minimized the variance of \( \hat{\sigma}_A^2 \). Crump's results are elaborated on in Chapter 5 of this thesis.

Baines (1944) considered the biased estimator of \( \rho \) given by

\[
\hat{\rho} = (M_A - M_B) / K M_B.
\]

He determined the design that minimizes \( \text{var}(\hat{\rho}) \); however, he considered only designs with equal allocation to the classes.
Using an unbiased estimator of $\rho$, Crump extended the work of Baines to situations in which $N/a$ is not an integer. Crump proved for a fixed, and with $N/a = p + s/a(0 \leq s < a)$, that the optimal design consists of $a$ classes with $p+1$ observations and $a-s$ classes with $p$ observations. He also suggested a procedure, again depending upon the parameter $\rho$, for determining a design which would minimize the variance of his estimator.

Through the use of numerical methods Crump indicated that the guess or previous estimate of $\rho$ required for his procedure is not very critical and that it may differ from the true value by a considerable amount without greatly affecting the efficiency of the estimator of $\sigma^2_A$ or $\rho$.

Gaylour (1960) extended Crump's work to the two-way crossed classification in which a result may be represented by

$$x_{ijk} = m + R_i + C_j + (RC)_{ij} + E_{ijk}$$

where $m$ is a constant, $R_i$, $C_j$, $(RC)_{ij}$ and $E_{ijk}$ are NID with means zero and variances, $\sigma^2_R$, $\sigma^2_C$, $\sigma^2_{RC}$ and $\sigma^2_E$, respectively. Gaylour found designs which yield estimates of $\sigma^2_E$, $\sigma^2_R$, $\sigma^2_C$, $\sigma^2_{RC}$, $\sigma^2_E + \sigma^2_C$, and $\sigma^2 + \sigma^2_{RC} + \sigma^2_{E}$, with minimum possible variance.

If $\sigma^2_1$ is the linear function of these variance components which is to be estimated, the minimum possible variance is $2\sigma^2_1/(N-1)$. This minimum can be attained only if a design can be constructed so that all
of the information is used to estimate \( m \) and \( \sigma^2_{1} \). The designs he presented for the estimation of the above mentioned functions of the variance components are as follows:

1. \( \sigma^2_E \), select all \( N \) observations from one cell, \( r = c = 1 \) and \( n_{11} = N \);

2. \( \sigma^2_E + \sigma^2_{RC} + \sigma^2_R \), use only one column with \( N \) rows and one observation per cell, \( r = N, c = 1 \) and \( n_{1i} = 1 \) (i = 1, 2, ..., \( N \));

3. \( \sigma^2_E + \sigma^2_{RC} + \sigma^2_C \), same as in (2) with rows and columns interchanged;

4. \( \sigma^2_E + \sigma^2_{RC} + \sigma^2_C + \sigma^2_R \), select each observation from a different row and column, so that \( r = c = N \), \( n_{ij} = 1 \) for i = j and \( n_{ij} = 0 \) for i \( \neq \) j.

For other functions of the variance components, in particular for \( \sigma^2_i = \sigma^2_R \) or \( \sigma^2_C \), it is necessary to estimate other variance components as well as \( \sigma^2_i \), hence, the minimum attainable variance is greater than \( 2\sigma^2_i/(N-1) \). Unfortunately, it was not possible to determine this minimal attainable variance. It was found that the optimum design depends on the estimator used and upon the values of the parameters themselves.

As an estimator of \( \sigma^2_R \) he used an analysis of variance estimator based on the arithmetic of the method of fitting constants which assumes only \( E_{ijk} \) to be random. Because the variance of his estimator was extremely complex for designs with unequal \( n_{ij} \), he considered only designs with \( n_{ij} \) equal to 0 or \( N \). He proved that for \( n_{ij} = 0 \) or \( N \) the variance of the estimator is minimized by taking \( n_{ij} \) equal to 0 or 1. An analysis of variance for such a situation is given in Table 2.1.
Table 2.1. An analysis of variance for a two-way classification

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Columns</td>
<td>c-1</td>
<td>C</td>
<td>$\sigma^2_E + \sigma^2_{RC} + c_1\sigma^2_R + r\sigma^2_C$</td>
</tr>
<tr>
<td>Rows (adjusted for columns)</td>
<td>r-1</td>
<td>R</td>
<td>$\sigma^2_E + \sigma^2_{RC} + c_0\sigma^2_R$</td>
</tr>
<tr>
<td>Interaction (adjusted for rows and columns)</td>
<td>N-r-c+1</td>
<td>I</td>
<td>$\sigma^2_E + \sigma^2_{RC}$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>N-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The estimator of $\sigma^2_R$ Gaylor considered was

$$\hat{\sigma}^2_R = (R^* - I^*)/c_0$$

where $c_0 = (N-c)/(r-1)$. For a given number of rows, r, Gaylor shows that if $N = rc$, where c is an integer, the best design is the balanced $r \times c$ design with one unit per cell. If $N = r(c-1) + s$ where $1 < s < r$, the best design is either the balanced $r \times (c-1)$ design which uses only $N-s$ observations, or the unbalanced design with r rows by c-1 columns plus one column with s of the r rows. On the basis of this result it was proposed that future research might use as a criterion of efficiency the maximum information per observation.

To determine the number of rows that minimizes the exact variance of $\hat{\sigma}^2_R$, Gaylor used an approximate variance of the estimator $\hat{\sigma}^2_R$ to obtain a first approximation $\tilde{r}$, to r. The value of $\tilde{r}$ he obtained was

$$\tilde{r} = (N - \tilde{c} + \tilde{c}_o)/\tilde{c}_o,$$

in which $\tilde{c}$ is the smallest integer greater than or equal to $\tilde{c}_o$ and
\[ \hat{c}_o = \frac{(N - 1/2) \rho_R + (N - 1/2) + 1}{(N - 1/2) \rho_R + 1} \]

where \( \rho_R = \sigma^2_R / (\sigma^2_E + \sigma^2_{RC}) \). He suggested trying integers below and above \( \hat{r} \) until the value of \( r \) is found that minimizes the exact variance of \( \hat{\sigma}^2_R \). Since this exact variance changes very little for small changes in \( r \) near \( \hat{r} \), it is generally recommended that one use as \( r \) the closest integer to \( \hat{r} \). The estimator Gaylor suggested for \( \rho_R \) is

\[
\hat{\rho}_R = \left[ \frac{N-r-c-1}{N-r-c+1} \frac{R^*}{I^*} - 1 \right] / \hat{c}_o.
\]

Again he considered only designs for which \( n_{ij} = 0 \) or \( 1 \). His procedure for determining the optimal design for the estimation of \( \rho_R \) is similar to that given for the estimation of \( \sigma^2_R \) except that the first trial value for \( r \) is determined from

\[
\hat{c}_o = \frac{2\rho_R(N - 1/2) + (N - 1/2) + 1}{\rho_R(N - 1/2) + \rho_R + 2}.
\]

The above results apply to the estimation of \( \sigma^2_C \) and \( \rho_C \) by interchanging \( r \) and \( c \).

Gaylor concluded with a brief study of designs for the simultaneous estimation of \( \sigma^2_R \) and \( \sigma^2_C \). He considered two special designs, called the L-design and Balanced Disjoint Rectangles design, and compared them for the case \( \sigma^2_R = \sigma^2_C \).

In accordance with the results of Crump, Gaylor found that the guess or previous estimate of \( \rho_R \) required for determining a design may vary considerably from the true value, and yet the estimates will be of high efficiency.
For the two stage nested classification, the two-way crossed classification and the three stage nested classification, Searls (1956), (1958) and (1961), respectively, presented a method of obtaining variances of estimated variance components by matrix procedures.

Anderson and Bancroft (1952) and Anderson (1960) considered a staggered design for the simultaneous estimation of variance components from a three or more stage nested classification. The purpose of this design was to distribute more evenly the available degrees of freedom to the sources of variation and thus to obtain more precise estimates of some of the variance components than is possible with a balanced design. Through the use of a numerical example, it was shown that the precision of certain of the estimates may be considerably improved by use of a staggered design.
3.0 REDUCTION OF PRODUCT VARIABILITY THROUGH
THE USE OF VARIANCE COMPONENT ANALYSIS

3.1 Introduction

In this chapter we consider those situations in which a result
from an experimental unit is subject to two independent sources of
random variation, and the problem is to reduce the total variance.
When a result is the sum of two independent parts, the variance of the
result, the total variance, may be expressed as

\[ \sigma_T^2 = \sigma_A^2 + \sigma_B^2, \]

where \( \sigma_A^2 \) and \( \sigma_B^2 \) are the components of variance associated with sources
A and B, respectively. Often a composite of many sources may be en-
compassed in a single source. For example, under a given sampling
scheme, the variation observed for a certain type of electron tube may
be attributable to variation among lots of tubes plus variation among
tubes within lots. If A represents lots and B represents tubes in lots,
the variation associated with B may be due to both measurement error and
inherent variation among tubes.

If one is given the task of reducing \( \sigma_T^2 \) he could quite conceivably
proceed by affecting individually the sources that give rise to \( \sigma_A^2 \) and
\( \sigma_B^2 \). In the usual circumstance, only a limited amount of funds would be
available for reducing \( \sigma_T^2 \) and one would attempt to devise a program
that would yield the greatest reduction in \( \sigma_T^2 \) for a given expenditure
of funds. The crucial matter here is the proper apportionment of funds
toward reduction of the two sources of variation.
Two factors would determine the specific apportionment of funds. They are: (1) the relative rate at which \( \sigma_A^2 \) can be reduced compared to \( \sigma_B^2 \) and, (2) the magnitude of \( \rho = \sigma_A^2 / \sigma_B^2 \). In many experimental situations the relative costs of reducing \( \sigma_A^2 \) and \( \sigma_B^2 \) may be adequately known but \( \rho \) may not be. Often the estimate of \( \rho \) is obtained by performing an experiment with a fixed sample size with units arranged in a specified design.

The primary purpose of this chapter will be to investigate, under a given cost model, the influence of the design of the experiment on the apportionment of funds and therefore on the effectiveness of the reduction of the total variance. The effect of using various estimators of \( \rho \) will be briefly investigated. It will be assumed that the cost of sampling for the experiment is directly proportional to the sample size \( N \).

In order to arrive at an explicit solution for the optimum apportionment of funds, it is necessary to adopt a model which relates funds expended to reduction in variance. Two such cost models will be discussed in Section 3.3.

### 3.2 Notation

- \( a \) = total number of classes
- \( 100A \) = rate of percentage reduction in \( \sigma_A^2 \) to that of \( \sigma_B^2 \) for the same expenditure of funds on each
- \( A_1 \) refers to those classes with \( p+1 \) units per class
- \( A_2 \) refers to those classes with \( p \) units per class
- \( \beta = \) a fixed number between \( \min (\sigma_R^2 | \hat{A}_1) \) and \( \max (\sigma_R^2 | \hat{A}_1) \)
\[ B_1 = \frac{k_1}{C_2} \]
\[ B_2 = \frac{k_2}{C_2} \]
\[ C_1 = \frac{Dk_1 + \ln k_2 - \ln k_1}{k_1 + k_2} \]
\[ C_2 = k_1 + k_2 \]
\[ D = \text{total amount of funds expended to reduce the total variance} \]
\[ = d_A + d_B \]
\[ d_A = \text{amount of funds expended to reduce } \sigma_A^2 \]
\[ d_B = \text{amount of funds expended to reduce } \sigma_B^2 \]
\[ d_1 = \text{value of } d_A \text{ that minimizes } \sigma_R^2 \text{ for fixed } D \text{ [see equation (3.7)]} \]
\[ \hat{d}_1 = \text{estimate of } d_1 \text{ [see equation (3.8)]} \]
\[ E = \text{expectation or average value operator} \]
\[ F(h,N-a) = \text{Snedecor's } F \text{ with } h \text{ and } N-a \text{ degrees of freedom} \]
\[ g_i = \text{scaling factor for the approximate distribution used with the estimator } \hat{\rho}_i, \ i = 1,2 \text{ [see equations (3.13) and (3.19)]} \]
\[ h_i = \text{degrees of freedom for the approximate distribution used with the estimator } \hat{\rho}_i, \ i = 1,2 \text{ [see equations (3.14) and (3.20)]} \]
\[ I_x(m,n) = \int_0^x \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \, dx \]
\[ k_1 = -\ln(.99) \]
\[ k_2 = -\ln(1-A) \]
\[ K_1 = \frac{N(N-2p-1) + ap(p+1)}{N(a-1)} \]
\[ K_2 = (N-2p-1)/(a-2) \]
\[ \ell_i = (a-1)(K_1 \rho_i + 1)/g_i h_1, \ i = 1,2 \]
\( L_i \) = lower limit of integration on \( \mathbb{P} \) when the estimator \( \hat{\rho}_i \) is used

\[
\text{[see equations (3.30) and (3.33)]}
\]

\( N = \text{total number of units in experiment} = \sum_{i=1}^{a} n_i \)

\( n_i = \text{number of units in } i^{th} \text{ class} \) \( (\sum n_i = N) \)

\( M_A, M_{A_1}, M_{A_2}, M_{A_3}, M_B = \text{mean squares in an analysis of variance} \)

\( \text{(see Table 3.1)} \)

\( p = \text{number of units in each of } a\text{-s classes} \) \( (N = ap + s, 0 \leq s < a) \)

\( P = \text{Prob} \left[ \left( \sigma_R^2 | \hat{\rho}_1 \right) \leq \beta \right] \) \( \text{[see equation (3.24)]} \)

\[
\rho = \frac{\sigma_A^2}{\sigma_B^2}
\]

\[
\rho_1 = e^{\frac{C_1 C_2}{k_1}} e^{-\frac{k_1 D}{k_2}}
\]

\[
\rho_2 = e^{\frac{C_2 (D-C_1)}{k_1}} e^{-\frac{k_2 D}{k_2}}
\]

\( \hat{\rho}_1 = \text{estimator of } \rho, i = 1,2 \) \( \text{[see equations (3.10) and (3.11)]} \)

\( \rho_L^* = \text{smallest root of the equation } V_2(\hat{\rho}) = \beta \)

\( \rho_U^* = \text{largest root of the equation } V_2(\hat{\rho}) = \beta \)

\( s = \text{number of classes with } p+1 \text{ units per class} \) \( \text{(see Section 3.4)} \)

\( S_A, S_{A_1}, S_{A_2}, S_{A_3}, S_B = \text{sums of squares in an analysis of variance} \)

\( \text{(see Table 3.1)} \)

\( \sigma_A^2 = \text{component of variance associated with source } A \)

\( \sigma_B^2 = \text{component of variance associated with source } B \)

\[
\sigma_T^2 = \sigma_A^2 + \sigma_B^2
\]

\[
\sigma_T^2 = \text{reduced total variance} = \sigma_B^2 (1-A) + \sigma_A^2 (1-A)
\]

\[
= \sigma_B^2 e^{-k_1 (D-d_A)} + \sigma_A^2 e^{-k_2 d_A}
\]
\[ \sigma_R^2 = \text{reduced variance ratio} = \frac{\sigma_T^2}{\sigma_B^2} = e^{-k_1(D-d_A)} + \rho e^{-k_2d_A} \]

\[ \sigma_{R,1}^2 \text{= reduced variance ratio when } \hat{d}_1 \text{ units have been expended on source A and } D-\hat{d}_1 \text{ on source B [see equation (3.9)]} \]

\[ \sigma_1^2, \sigma_{11}^2, \sigma_{12}^2, \sigma_{13}^2 \text{ = expected values of the mean squares for sources A, A_1, A_2 and A_1 vs. A_2, respectively} \]

\[ U_1 \text{ = upper limit of integration for P when the estimator } \hat{\rho}_1, i = 1,2 \text{ is used [see equations (3.41) and (3.44)]} \]

\[ V_0 = \sigma_R^2 | \hat{\rho}_1 \]

\[ V_1 = e^{-k_1D} (V_0 \text{ when } \hat{\rho} \leq \rho_1) \]

\[ V_2 = e^{-k_1(D-C_1)} \frac{k_1}{\rho} \frac{c_1}{\rho} + \rho e^{-k_2C_1} \hat{\rho} \frac{k_2}{c_2} (V_0 \text{ when } \rho_1 \leq \hat{\rho} \leq \rho_2) \]

\[ V_3 = 1 + \rho e^{-k_2D} (V_0 \text{ when } \hat{\rho} \geq \rho_2) \]

\[ V_4 = \max (V_1, V_3) \]

\[ W_1 = e^{-k_1(D-C_1)} \]

\[ W_2 = \rho e^{-k_2C_1} \]

\[ x_{ij} \text{ = observed value for } j^{th} \text{ unit in } i^{th} \text{ class} = m + A_{ij} + B_{ij} \]

3.3 A Cost Model

There are several different models that could be used to express reduction in variance as a function of funds expended. One of the simplest is the additive model. Under this model the variance obtained after expending d units is equal to the original variance minus an amount proportional to d. For the situation at hand, the reduced total
variance, symbolized by \( \sigma_T^2 \), attained after expending \( d_A \) units on source A and \( d_B \) units on source B is given by

\[
\sigma_T^2 = \sigma_B^2 - K_B d_B + \sigma_A^2 - K_A d_A
\]

where \( K_A \) and \( K_B \) are constants of proportionality.

Such a model is quite unrealistic for depicting an actual real world situation. According to this model the amount that a variance is reduced does not depend on the magnitude of the variance. Generally, more effort would be required to reduce a small variance a given amount than to reduce a large variance by the same amount. Also the model indicates that if enough funds are made available a variance can be reduced to zero and even made negative, a circumstance which is quite absurd.

The basic model that is proposed here is of the power function type. Under this model it is assumed that the expenditure of one unit of funds on source A reduces \( \sigma_A^2 \) by 100 \( A' \) per cent and on source B reduces \( \sigma_B^2 \) by 100 \( B' \) per cent. Hence, the reduced total variance attained after expending \( d_A \) units on source A and \( d_B \) units on source B is

\[
\sigma_T^2 = \sigma_B^2 (1-B')^d_B + \sigma_A^2 (1-A')^d_A . \tag{3.1}
\]

In order to eliminate one parameter, the model proposed above will be slightly modified. If it is assumed that one unit of funds expended reduces \( \sigma_B^2 \) by one per cent and reduces \( \sigma_A^2 \) by 100 \( A \) per cent, the reduced total variance may be written as

\[
\sigma_T^2 = \sigma_B^2 (.99)^{d_B} + \sigma_A^2 (1-A)^{d_A} . \tag{3.2}
\]
Because it is more convenient to work with an exponential than with a power function, $\sigma_T^2$ will be expressed in exponential form. By setting $k_1 = -\ln(0.99)$ and $k_2 = -\ln(1-A)$ we have

$$
\sigma_T^2 = \sigma_B^2 e^{-k_1 d_B} + \sigma_A^2 e^{-k_2 d_A}.
$$

(3.3)

The model as given by (3.3) shall be the model used throughout the remainder of this chapter.

When the total amount of funds is fixed at, say D units, $\sigma_T^2$ may be written as

$$
\sigma_T^2 = \begin{cases}
\sigma_B^2 e^{-k_1 d_B} + \sigma_A^2 e^{-k_2 d_A}, & d_A = 0 \quad (d_B = D) \\
\sigma_B^2 e^{-k_1 (D-d_A)} + \sigma_A^2 e^{-k_2 d_A}, & 0 \leq d_A \leq D \\
\sigma_B^2 + \sigma_A^2 e^{-k_2 D}, & d_A = D(d_B = 0)
\end{cases}
$$

(3.4a, 3.4b, 3.4c)

In determining the optimal allocation of funds to minimize $\sigma_T^2$, we first find that value of $d_A$ which minimizes (3.4b). This value is

$$
\frac{D k_1 + \ln k_2 - \ln k_1 + \ln \rho}{k_1 + k_2}
$$

(3.5)

$$
= c_1 + \frac{\ln \rho}{c_2}
$$

(3.6)

If $d_1 < 0$, we set $d_A = 0$; and if $d_1 > 0$, we set $d_A = D$. Hence the complete solution is

$$
\begin{cases}
0, & 0 \leq \rho \leq \rho_1 \\
c_1 + \frac{\ln \rho}{c_2}, & \rho_1 \leq \rho \leq \rho_2 \\
D, & \rho \geq \rho_2
\end{cases}
$$

(3.7)
where
\[ \rho_1 = e^{-C_1 C_2} = \frac{k_1}{k_2} e^{-k_1 D} \]
and
\[ \rho_2 = e^{C_2 (D-C_1)} = \frac{k_1}{k_2} e^{k_2 D} . \]

For this work it is more convenient to use the ratio \( \sigma_{R}^{*2} = \sigma_{T}^{*2}/\sigma_{B}^{*2} \) rather than \( \sigma_{T}^{*2} \). Since \( d_1 \), as given by (3.7), also minimizes \( \sigma_{R}^{*2} \) and since the statistical properties of both are the same, working with \( \sigma_{R}^{*2} \) is equivalent to working with \( \sigma_{T}^{*2} \), and \( \sigma_{R}^{*2} \) will be used in all future development. The quantity \( \sigma_{R}^{*2} \) will be referred to as the reduced total variance ratio.

As was mentioned previously, \( A \) is often known, but an estimate of \( \rho \) is required before the allocation of funds can be completely specified. If an estimator of \( \rho \), say \( \hat{\rho} \), is substituted into (3.7) for \( \rho \), then as an estimator of \( d_1 \), say \( \hat{d}_1 \), we have

\[
\hat{d}_1 = \begin{cases} 
0 , & L \leq \hat{\rho} \leq \rho_1 \\
C_1 + \frac{\ln \hat{\rho}}{C_2} , & \rho_1 \leq \hat{\rho} \leq \rho_2 \\
D , & \hat{\rho} \geq \rho_2 
\end{cases} \tag{3.8}
\]

where \( L \), which may be negative, depends on the particular estimator of \( \rho \) used.

Therefore the true reduced total variance ratio that exists after expending \( D - \hat{d}_1 \) on source B and \( \hat{d}_1 \) on source A is

\[
\sigma_{R}^{*2} | \hat{d}_1 = \begin{cases} 
-k_1 D \frac{1}{e} + \rho , & L \leq \hat{\rho} \leq \rho_1 \\
-k_1 (D-C_1) \frac{k_1}{C_2} \hat{\rho} , & \rho_1 \leq \hat{\rho} \leq \rho_2 \\
1 + \rho e^{-k_2 D} , & \hat{\rho} \geq \rho_2 
\end{cases} \tag{3.9}
\]
It is important to realize that $\sigma^2_{R|d_1}$, where $\hat{d}_1$ is obtained by experimental methods, may be considered as either a random variable or as a constant, according as $\hat{d}_1$ is considered as a random variable or a constant. The reduced total variance ratio is a random variable in the sense that if we were to perform a large number of experiments, they would yield differing values of $\hat{\rho}$ and hence possibly differing values of $\sigma^2_{R|d_1}$. Conversely, once the experiment has been performed, the allocation determined and the funds actually expended on the two sources contributing to $\sigma^2_B$ and $\sigma^2_A$, $\sigma^2_{R|d_1}$ is a constant associated with the product or process. We shall be concerned with $\sigma^2_{R|d_1}$ as a random variable.

In the next section an experiment for the estimation of $\rho$ is described and two different estimators of $\rho$ are presented.

3.4 Description of the Experiment for Estimating $\rho$

The situation shall be considered where there are $N$ units available for an experiment and the units may be arranged in a two stage nested classification with $a$ classes. The model that represents the data collected according to such a scheme is taken to be of the form given by (1.1). As previously stated it will be assumed that the cost of sampling the units is proportional to $N$ regardless of the type of design. Further, it will be assumed that for any $a$ the $N$ units are allocated as equally as possible to the $a$ classes. Specifically for $N/a = p + s/a$ ($0 \leq s < a$) there are $p+1$ units assigned to each of $s$ classes (referred to as $A_{\perp}$) and $p$ units to each of $a-s$ classes (referred
to as $A_2'$. The motivation for this allocation scheme is given by Crump (1954). In Table 3.1 an analysis of variance is given for such data.

**Table 3.1.** An analysis of variance for a two stage nested classification with $p+1$ units in each of $s$ classes and $p$ units in each of $a-s$ classes.

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classes</td>
<td>$a-1$</td>
<td>$S_A$</td>
<td>$M_A$</td>
<td>$\sigma^2_1 = \sigma^2_B[1 + K_1p]$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$s-1$</td>
<td>$S_{A_1}$</td>
<td>$M_{A_1}$</td>
<td>$\sigma^2_{11} = \sigma^2_B[1 + (p+1)p]$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$a-s-1$</td>
<td>$S_{A_2}$</td>
<td>$M_{A_2}$</td>
<td>$\sigma^2_{12} = \sigma^2_B[1 + pp]$</td>
</tr>
<tr>
<td>$A_1$ vs. $A_2$</td>
<td>1</td>
<td>$S_{A_3}$</td>
<td>$M_{A_3}$</td>
<td>$\sigma^2_{13} = \sigma^2_B[1 + \frac{ap(p+1)}{N}]$</td>
</tr>
<tr>
<td>Within</td>
<td>$N-a$</td>
<td>$S_B$</td>
<td>$M_B$</td>
<td>$\sigma^2_B$</td>
</tr>
<tr>
<td>Total</td>
<td>$N-1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$\frac{a}{K_1} = \frac{[N(N-2p-1) + ap(p+1)]}{N(a-1)}$$

It should be apparent that the partition of the class sum of squares as given in Table 3.1 is relevant only for situations in which $0 < s < a$. When $s = 0$, $a = N/p$, $ap(p+1)/N = p+1$ and the expectation of the mean square for classes becomes $E(M_A) = \sigma^2_B (1+pp)$.

From the manner in which the units are assigned to the classes, i.e., $p+1$ units are assigned to each of $s$ classes and $p$ units to each of $a-s$ classes, it is realized that designating $a$ completely specifies
the design. Therefore, varying the design would be accomplished by varying $a$.

As was previously stated, the concern of this work is where $\rho$, and hence $d_1$, must be estimated. The estimator of $\rho$ that we shall be primarily concerned with is

$$\hat{\rho}_1 = \frac{(M_A - M_B)}{K_1 M_B}.$$  \hspace{1cm} (3.10)

This estimator will be considered because it is the one that is used most extensively in practice. One other estimator will also be considered,

$$\hat{\rho}_2 = \frac{(S_{A_1} + S_{A_2})/(a-2) - M_B}{K_2 M_B}$$  \hspace{1cm} (3.11)

where $K_2 = (N-2p-1)/(a-2)$.

The estimator $\hat{\rho}_2$ is the same as $\hat{\rho}_1$ except that the one degree of freedom due to $A_1$ vs. $A_2$ is not used. This estimator was tried to learn if the exclusion of the one degree of freedom greatly detracted from the worth of an estimator. It was felt that for other than very small $N$ and $a$ the exclusion of the one degree of freedom would have little influence on the worth of an estimator.

The estimator $\hat{\rho}_2$ will be considered only for situations where the experiment is not balanced, i.e., $N/a$ is noninteger. As there is no partition of the class sum of squares, the estimator $\hat{\rho}_2$ does not exist for balanced designs.

In the next section some results will be presented for $\hat{\rho}_1$ and $\hat{\rho}_2$. 
3.5 Some Mathematical Development for the Estimators of \( \rho \)

The development for both estimators is very similar, therefore, the details of the development are given only for \( \hat{\rho}_1 \).

From (3.10) it is noted that

\[
\hat{\rho}_1 = \frac{1}{K_1} \left[ \frac{M_A - M_B}{M_B} \right] = \frac{1}{K_1} \left[ \frac{M_A}{M_B} - 1 \right],
\]

and therefore a study of the properties of \( \hat{\rho}_1 \) becomes a study of \( M_A/M_B \).

Now, \( M_B \) is distributed as \( \sigma_B^2 \chi^2_{N-a}/(N-a) \), where \( \chi^2_v \) is distributed as chi square with \( v \) degrees of freedom. If the design is balanced \( M_A \) is distributed as \( \sigma_B^2 \left[ 1 + \frac{N}{a} \rho \right] \chi^2_{(a-1)}/(a-1) \). Hence, if \( N/a \) is an integer \( F = M_A/M_B(1 + \frac{N}{a} \rho) \) follows Student's F distribution and the distribution of \( \hat{\rho}_1 \) can be linearly transformed to that of \( F \) so that all the known properties of \( F \) can be utilized. However, this work will not be limited to considering only balanced designs. When \( N/a \) is not an integer, \( M_A \) is not distributed as \( \sigma_B^2 \left[ 1 + \frac{N}{a} \rho \right] \chi^2_{(a-1)}/(a-1) \) and it is not possible to make a simple transformation from the distribution of \( \hat{\rho}_1 \) to that of \( F \). To circumvent this difficulty use was made of an approximate distribution of a linear function of chi square variables [studied by Satterthwaite (1946)]. The approximate distribution is set up such that its mean and variance are equal to the mean and variance of the linear function of chi square variables.

Consider the weighted sum of independent chi square variables

\[
Q = \Sigma \lambda_i \chi^2_{(v_i)}.
\]

Let \( Q \) be represented by \( Z = g \chi^2_{(h)} \) where \( g \) and \( h \) are chosen such that
E(Q) = E(Z) and \( \text{var}(Q) = \text{var}(Z) \). It is easily seen that \( E(Q) = \Sigma \lambda_1 v_1 \), \( E(Z) = gh \), \( \text{var}(Q) = 2 \Sigma \lambda_1^2 v_1 \), and \( \text{var}(Z) = 2 g^2 h \). Setting \( E(Q) = E(Z) \) and \( \text{var}(Q) = \text{var}(Z) \) we obtain

\[
g = \Sigma \lambda_1^2 v_1 / \Sigma \lambda_1 v_1
\]

\[
h = (\Sigma \lambda_1 v_1)^2 / \Sigma \lambda_1^2 v_1.
\]

Hence, \( Q = \Sigma \lambda_1^2 v_1 \) is approximately distributed as \( g \chi^2_h \) where \( g \) and \( h \) are given above.

Assuming that the allocation of units to classes is as described in Section 3.4 and using information contained in Table 3.1 we note that \( \hat{\rho}_1 \) may be written as

\[
\hat{\rho}_1 = \frac{1}{K_1} \left[ \frac{\sum A_1}{M_B} - 1 \right]
\]

\[
= \frac{1}{K_1} \left[ \frac{(S_{A_1} + S_{A_2} + S_{A_3})/(a-1)}{M_B} - 1 \right]
\]

\[
= \frac{1}{K_1} \left[ \frac{Q_1/(a-1)}{M_B} - 1 \right]. \quad (3.12)
\]

It is observed that \( S_{A_1} = \sigma_{11}^2 \chi^2_{(s-1)} \), \( S_{A_2} = \sigma_{12}^2 \chi^2_{(a-s-1)} \), \( S_{A_3} = \sigma_{13}^2 \chi^2_{(1)} \) and \( M_B = \sigma_B^2 \chi^2_{(N-a)}/(N-a) \), where \( \sigma_{11}^2 = \sigma_B^2 \left[ 1 + (p+1) \rho \right] \), \( \sigma_{12}^2 = \sigma_B^2 \left[ 1 + p \rho \right] \) and \( \sigma_{13}^2 = \sigma_B^2 \left[ 1 + \frac{ap(p+1)}{N} \rho \right] \). Therefore, \( Q_1 \) in (3.12) can be written as a linear function of chi square variables

\[
Q_1 = \sigma_{11}^2 \chi^2_{(s-1)} + \sigma_{12}^2 \chi^2_{(a-s-1)} + \sigma_{13}^2 \chi^2_{(1)}.
\]
Using the result concerning the approximate distribution of a linear function of chi square variables it is seen that \( Q_1 \) is approximately distributed as \( \sigma_B^2 g_1 x_{(h_1)}^2 \) where

\[
ge_1 = \frac{(s-1) \left[ \sigma_{11}^2 \right]^2 + (a-s-1) \left[ \sigma_{12}^2 \right]^2 + \left[ \sigma_{13}^2 \right]^2}{(s-1) \sigma_{11}^2 + (a-s-1) \sigma_{12}^2 + \sigma_{13}^2} \sigma_B^2
\]

\[
= \frac{(s-1) \left[ 1 + (p+1) \rho \right]^2 + (a-s-1) \left[ 1 + \rho \right]^2 + \left[ 1 + \frac{ap(p+1)}{N} \right]^2 \rho^2}{(s-1) \left[ 1 + (p+1) \rho \right] + (a-s-1) \left[ 1 + \rho \right] + \left[ 1 + \frac{ap(p+1)}{N} \right]} = \frac{T_1}{R_1} \tag{3.13}
\]

\( h_1 = \frac{R_1^2}{T_1} \).

The substitution of \( \sigma_B^2 g_1 x_{(h_1)}^2 \) for \( Q_1 \) and \( \sigma_B^2 x_{(N-a)/(N-a)}^2 \) for \( M_B \) in (3.12) gives

\[
\hat{\rho}_1 = \frac{1}{K_1} \left[ \frac{g_1 x_{(h_1)}^2/(a-1)}{x_{(N-a)/(N-a)}} - 1 \right]
\]

\[
= \frac{1}{K_1} \left[ \frac{g_1 h_1 x_{(h_1)}^2/(a-1) h_1}{x_{(N-a)/(N-a)}} - 1 \right] \tag{3.15}
\]

where \( \approx \) means approximately equal. Recall that

\[
F = \frac{x_{(v_1)/v_1}^2}{x_{(v_2)/v_2}^2}
\]

follows Snedecor's F distribution with \( v_1 \) and \( v_2 \) degrees of freedom.
Let

\[ F_1 = \frac{X_{(h_1)}^2/h_1}{X_{(N-a)}^2/(N-a)} \]  

(3.16)

which is distributed as F with \( h_1 \) and \( N-a \) degrees of freedom. Hence,

\[ \hat{\rho}_1 = \frac{1}{K_1} \left[ g_1 h_1 F_1 \frac{1}{a-1} - 1 \right]. \]  

(3.17)

It may be noted that when \( N/a = p = K_1 \) is an integer, \( h_1 = a-1 \),

\[ g_1 = \left( 1 + \frac{N}{a} \rho \right), \]

and

\[ \hat{\rho}_1 = \frac{a}{N} \left( 1 + \frac{N}{a} \rho \right) F_1 - 1. \]

Using similar techniques, the following results were obtained for \( \hat{\rho}_2 \):

\[ \hat{\rho}_2 = \frac{(S_{A_1} + S_{A_2})/(a-2) - M_B}{K_2 M_B} \]

\[ = \frac{1}{K_2} \left[ g_2 h_2 F_2 \frac{1}{a-2} - 1 \right] \]  

(3.18)

where

\[ g_2 = \frac{(s-1) [1+(p+1)p]^2 + (a-s-1) [1+pp]^2}{(s-1) [1+(p+1)p] + (a-s-1) [1+pp]} = \frac{T_2}{R_2} \]  

(3.19)

\[ h_2 = \frac{F_2^2}{R_2} \]  

(3.20)

and \( F_2 \) is distributed as Snedecor's F with \( h_2 \) and \( N-a \) degrees of freedom.
In the next section the preceding results will be used in the
formulation of a criterion. This criterion will be used to determine
the influence of the design of the experiment and the effect of the two
different estimators on the effectiveness of reducing product variabil-
ity.

3.6 Criteria for Assessing the Influence of the
Design of Experiment on the Reduction of
Total Variance

The reduced total variance ratio, $\sigma^2_R | \hat{d}_1$, has a distribution whose
lower limit is the minimum attainable variance ratio, namely,

$$\min (\sigma^2_R | \hat{d}_1) = e^{-k_1(D-d_1)} + \rho e^{-k_2d_1}. \quad (3.21)$$

Therefore, it is reasonable that one would seek a design or estimator
which would generate a distribution for $\sigma^2_R | \hat{d}_1$ that has the largest
possible density near $\min (\sigma^2_R | \hat{d}_1)$. The first criterion that is sug-
gested concerns the expectation of $\sigma^2_R | \hat{d}_1$. Ostensibly, the best design
or estimator would be one that yields the smallest expectation of
$\sigma^2_R | \hat{d}_1$. Therefore an expression will be developed for the expectation
of $\sigma^2_R | \hat{d}_1$. The development will be given only for the estimator $\rho_1$.

From consideration of $\sigma^2_R | \hat{d}_1$ as given by (3.9) and from (3.10)
it is apparent that the expectation of $\sigma^2_R | \hat{d}_1$ is
\[ E(\sigma_R^2 | \hat{\theta}_1) = (e^{-k_1D} + \rho) \int_{\frac{1}{K_1}}^{\rho_1} f(\hat{\theta}_1) d\hat{\theta}_1 \]

\[ + \int_{\rho_1}^{\rho_2} \left[ e^{-k_1(D-C_1)\hat{\theta}_1/C_2} + \rho e^{-k_2C_1\hat{\theta}_1/C_2} \right] f(\hat{\theta}_1) d\hat{\theta}_1 \]

\[ + (1 + \rho e^{-k_2D}) \int_{\rho_2}^{\infty} f(\hat{\theta}_1) d\hat{\theta}_1 . \]  

(3.22)

Making the linear transformation

\[ \hat{\rho}_1 = \frac{1}{K_1} \left[ \frac{g_{h_1F_1}}{a-l} - 1 \right] \]

the expectation may be expressed as

\[ E(\sigma_R^2 | \hat{\theta}_1) = (e^{-k_1D} + \rho) \int_{0}^{\rho_1} g(F_1) dF_1 \]

\[ + \int_{\rho_1}^{\rho_2} \left\{ e^{-k_1(D-C_1)\left[ \frac{1}{K_1} \left( \frac{g_{h_1F_1}}{a-l} - 1 \right) \right]} \right\} g(F_1) dF_1 \]

\[ + \rho e^{-k_2C_1\left[ \frac{1}{K_1} \left( \frac{g_{h_1F_1}}{a-l} - 1 \right) \right]} g(F_1) dF_1 \]

\[ + (1 + \rho e^{-k_2D}) \int_{\rho_2}^{\infty} g(F_1) dF_1 , \]  

(3.23)

where
\[ f_1 = \frac{a-1}{g_1 h_1} (K_1 \rho_1 + 1) \]

\[ f_2 = \frac{a-1}{g_1 h_1} (K_1 \rho_2 + 1) \]

and \( g(F_1) \) is the density function for the F-statistic with \( h_1 \) and \( N-a \) degrees of freedom.

Ideally, one would carry out the integration required in (3.23) and then specify numerical values for \( \rho, A, D \) and \( N \). The effect of the design would be determined by substituting various values of \( g \) and observing the numerical results of \( E(\sigma_R^2 | \hat{d}_1) \). A major difficulty with using this procedure was that we were unable to evaluate explicitly the second integral on the RHS of (3.23). Numerical integration could have been used; however, we preferred to use a simpler criterion which may be even more useful than that involving average variance ratio.

This criterion concerns the quantity

\[ P = \text{Prob} \left[ (\sigma_R^2 | \hat{d}_1) \leq \beta \right] , \quad (3.24) \]

where \( \beta \) is a fixed number between \( \min (\sigma_R^2 | \hat{d}_1) \) and \( \max (\sigma_R^2 | \hat{d}_1) \). By careful selection of \( \beta \), such a criterion should do well to pinpoint the effect of design and estimator. Obviously, extreme values of \( \beta \) either small or large, would yield \( P \) quite insensitive to variation in design or estimator regardless of their effects on the distribution of \( \sigma_R^2 | \hat{d}_1 \).

An expression for \( P \) is now developed.

First recourse is made to investigating \( \sigma_R^2 | \hat{d}_1 \) as a function of \( \rho_1 \). Consider the function
\[ V_o(\hat{\rho}_1) = \sigma_R^2 \hat{d}_1. \]

Referring to (3.9) and (3.10) it is seen that \( V_o \) takes on three forms,

\[
V_o = \begin{cases} 
V_1 = e^{-k_1 D} \hat{\rho}_1 + \rho, & -\frac{1}{k_1} \leq \hat{\rho}_1 \leq \rho_1 \\
V_2 = e^{-k_1(D-C_1) + \frac{k_1}{c_2}} \hat{\rho}_1, & \rho_1 \leq \hat{\rho}_1 \leq \rho_2 \\
V_3 = 1 + \rho e^{-k_2 D}, & \hat{\rho}_1 \geq \rho_2 
\end{cases}
\]

(3.25)

where \( \rho_1 = e^{-C_1 C_2} = \frac{k_1}{k_2} e^{1} \) and \( \rho_2 = e^{C_2(D-C_1)} = \frac{k_1}{k_2} e^{2} \).

Let \( \rho_o = (1-e^{-1})/(1-e^{-2}) \). An inspection of \( V_1 \) and \( V_3 \) shows that

\[ V_1 \preceq V_3 \text{ if } \rho \preceq \rho_o. \]

(3.26)

In addition we see that

\[ V_2(\rho_1) = V_1; \quad V_2(\rho_2) = V_3. \]

(3.27)

Also it is apparent that \( V_2 \) is continuous in \([\rho_1, \rho_2]\). The continuity of \( V_2 \) plus the equalities given by (3.27) establish that \( V_o \) is continuous over its entire range.

Next consider the behavior of \( V_2 \) in \([\rho_1, \rho_2]\). First consider \( V_2 \) at its end points. To do this the first derivative of \( V_2 \) with respect
to $\hat{\rho}_1$ is needed. The derivative is

$$V_2'(\hat{\rho}_1) = \frac{k_1}{c_2} e^{-k_1(D-C_1)} \hat{\rho}_1 \frac{(k_1-C_2)/c_2}{\hat{\rho}_1} - \frac{k_2}{c_2} e^{-k_2C_1} \hat{\rho}_1 \frac{(k_2+C_2)/c_2}{\hat{\rho}_1}$$

(3.28)

From (3.28) it is easily verified that

$$V_2'(\rho_1) > 0 \text{ if } \rho < \rho_1; \quad V_2'(\rho_2) < 0 \text{ if } \rho > \rho_2.$$  

It is well known that a necessary but not sufficient condition for a function $f$ to have a relative extremum at a point $x_0$ is that $f'(x_0)=0$. Using this result for (3.27) we note that $V_2'(\hat{\rho}_1)$ is zero when

$$\hat{\rho}_1 = \frac{k_2}{k_1} e^{k_1D-C_1(k_1+C_2)/c_2}.$$  

Since $\frac{Dk_1+\ln k_2-\ln k_1}{k_1+k_2}$ and $C_2 = k_1 + k_2$ this reduces to

$$\hat{\rho}_1 = \rho.$$  

(3.29)

Hence, $V_2(\hat{\rho}_1)$ has at most one extremum, namely, $V_2(\rho)$.

The second derivative at $\hat{\rho}_1 = \rho$ is

$$V_2''(\rho) = -\frac{k_1k_2}{c_2} e^{-k_1D+k_1C_1} \rho -\frac{(2C_2-k_1)/c_2}{C_2}$$

$$+ \frac{k_2(k_2+C_2)}{c_2} e^{-k_2C_1} \rho -\frac{(k_2+C_2)/c_2}{C_2}$$

$$= \frac{k_1k_2}{c_2} \rho \frac{(k_1-2C_2)/c_2}{C_2} e^{k_1(C_1-D)} \left[ \frac{k_2+C_2}{k_2} - 1 \right].$$  

(3.30)
The second derivative will always be positive because $k_1, k_2 > 0$ and

$$\frac{k_2 + c_2}{k_2} - 1 = \frac{c_2}{k_2} = 1 + \frac{k_1}{k_2} > 0;$$

hence, $f'(x_0) = 0$ is a sufficient as well as necessary condition and the extremum will be a minimum.

Collecting these results we see that $V_0$ can be characterized by three different profiles, depending on the value of $\rho$, as shown in Figures 3.1, 3.2, and 3.3. These three ways in which $V_0$ behaves are described below:

1. When $\rho \leq \rho_1$, $V_0$ increases from a minimum of $V_0 = V_1$ to a maximum of $V_0 = V_3$.
2. When $\rho > \rho_2$, $V_0$ decreases from a maximum of $V_0 = V_1$ to a minimum of $V_0 = V_3$.
3. When $\rho_1 < \rho < \rho_2$, $V_0$ decreases from a value of $V_0 = V_1$ to a minimum of $V_0 = V_2(\rho)$ and then increases to a value of $V_0 = V_3$.

In Figure 3.3 it is noted that $V_1 > V_3$ if $\rho > \rho_0$, $V_1 < V_3$ if $\rho < \rho_0$; regardless of whichever $V_1$ or $V_3$ is the smaller, $V_2$ reaches a minimum value which is less than the smaller of $V_1$ and $V_3$. The minimum of $V_2$ equals $V_1$ or $V_3$ only when $\rho = \rho_1$ or $\rho = \rho_2$.

Therefore, if $V_0$ were graphed and then a horizontal line were drawn at $V = \beta$, $\min \left[ V_0(\hat{\rho}_1) \right] < \beta < \max \left[ V_0(\hat{\rho}_1) \right]$, the line would intersect the graph of $V_0$ at either one or two places. In Figures 3.1 and 3.2 there would be exactly one point of intersection. In Figure 3.3 the line would intersect the graph of $V_0$ at exactly two places if both
Figure 3.1. Profile of $V_o$ when $\rho \leq \rho_1$

Figure 3.2. Profile of $V_o$ when $\rho \geq \rho_2$

Figure 3.3. Profile of $V_o$ when $\rho_1 < \rho < \rho_2$
\( V_1 > \beta \) and \( V_3 > \beta \), and would intersect at exactly one place if \( V_1 > \beta \) and \( V_3 \leq \beta \) or \( V_1 \leq \beta \) and \( V_3 > \beta \). It should be observed that in Figure 3.3, if \( V_1 = \beta \) and \( V_3 > \beta \), the lower point of intersection is taken to be \( \hat{\rho}_1 = -1/K_1 \), and if \( V_1 > \beta \) and \( V_3 = \beta \) the upper point of intersection is taken to be at positive infinity.

Also, in each of the Figures 3.1-3.3 the line \( V = \beta \) is shown and the values of \( \hat{\rho}_1 \), \( \rho_L^* \) and \( \rho_U^* \), at which \( V_o(\hat{\rho}_1) = \beta \) are indicated.

Using the information regarding the behavior of \( V_o(\hat{\rho}_1) = \sigma^2 | \hat{d}_1 \) it is found that \( P \) may be expressed as

\[
P = \begin{cases} 
    P_1 = \text{Prob}(\hat{\rho}_1 \leq \rho_U^*) & \text{if } \rho \leq \rho_1 \text{ or if } \rho_1 < \rho < \rho_2 \text{ and } V_1 \leq \beta, \ V_3 > \beta \\
    P_2 = \text{Prob}(\hat{\rho}_1 > \rho_L^*) & \text{if } \rho \geq \rho_2 \text{ or if } \rho_1 < \rho < \rho_2 \text{ and } V_1 > \beta, \ V_3 \leq \beta \\
    P_3 = \text{Prob}(\rho_L^* \leq \hat{\rho}_1 \leq \rho_U^*) & \text{if } \rho_1 < \rho < \rho_2 \text{ and } V_1 > \beta, \ V_3 > \beta
\end{cases}
\]  

(3.31)

where \( \rho_L^* \) and \( \rho_U^* \) are the smallest and largest roots, respectively, of the equation

\[
V_2(\hat{\rho}_1) = \beta.
\]

(3.32)

\( P \) is expressed in integral form as

\[
P = \begin{cases} 
    P_1 = \int_{\frac{1}{K_1}}^{\rho_U^*} f(\hat{\rho}_1) \, d\hat{\rho}_1 \\
    P_2 = \int_{\rho_L^*}^{\infty} f(\hat{\rho}_1) \, d\hat{\rho}_1 \\
    P_3 = \int_{\rho_L^*}^{\rho_U^*} f(\hat{\rho}_1) \, d\hat{\rho}_1
\end{cases}
\]

(3.33) - (3.35)
To evaluate the integrals (3.33)-(3.35) use is made of the relations given by (3.17). Then, using the density function of $P$ the general form for $P$ can be approximated by

$$
P = \int_{R^*} \frac{a-1}{(N-a)g_1} \frac{(h_1-2)/2}{(N-a)g_1} \frac{(a-1)K_1}{(N-a)g_1} (\hat{K}_1 \hat{\rho}_1 + 1) \left[ 1 + \frac{a-1}{(N-a)g_1} (K_1 \hat{\rho}_1 + 1) \right] (h_1 + N-a)/2 \, d\hat{\rho}_1$$

(3.36)

where $R^*$ is the appropriate domain of integration. Note that $P$ as given by (3.36) is exact when $N/a$ is an integer. By making the transformation of variable

$$\frac{a-1}{(N-a)g_1} (K_1 \hat{\rho}_1 + 1) = \frac{x}{1-x}$$

(3.37)

(3.36) is transformed into the incomplete beta form

$$
P = \int_R \frac{x^{h_1/2 - 1}}{B(\frac{h_1}{2}, \frac{N-a}{2})} \left(1-x\right)^{(N-a)/2 - 1} \, dx$$

(3.37)

where $R$ is the transformed domain of integration.

Each of the integrals (3.33)-(3.35) can thus be approximated by incomplete beta functions and $P$ can be expressed as

$$P = \begin{cases} 
    P_1 = I_{\frac{h_1}{2}, \frac{N-a}{2}} \\
    P_2 = 1 - I_{\frac{h_1}{2}, \frac{N-a}{2}} \\
    P_3 = I_{\frac{h_1}{2}, \frac{N-a}{2}} - I_{\frac{h_1}{2}, \frac{N-a}{2}} 
\end{cases}$$

(3.38)

(3.39)

(3.40)
where

\[
U_1 = \frac{\frac{a-1}{g_1(N-a)}(K_1 \rho_U^* + 1)}{1 + \frac{a-1}{g_1(N-a)}(K_1 \rho_U^* + 1)}
\]

\[ (3.41) \]

\[
L_1 = \frac{\frac{a-1}{g_1(N-a)}(K_1 \rho_L^* + 1)}{1 + \frac{a-1}{g_1(N-a)}(K_1 \rho_L^* + 1)}
\]

\[ (3.42) \]

\[
I_x(m,n) = \int_0^x \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \, dx.
\]

\[ (3.43) \]

As was previously stated, the general development of this section as given for \( \hat{\rho}_1 \) applies also to \( \hat{\rho}_2 \). The only differences in the results that occur for the different estimators are in the parameter \( h \) and the limits of integration for \((3.38)-(3.40)\). The parameter \( h_1 \) is replaced by \( h_2 \) as given by \((3.20)\). The limits of integration \( U_1 \) and \( L_1 \) are replaced by \( U_2 \) and \( L_2 \), whichever is appropriate, where

\[
U_2 = \frac{\frac{a-2}{g_2(N-a)}(K_2 \rho_U^* + 1)}{1 + \frac{a-2}{g_2(N-a)}(K_2 \rho_U^* + 1)}
\]

\[ (3.44) \]

\[
L_2 = \frac{\frac{a-2}{g_2(N-a)}(K_2 \rho_L^* + 1)}{1 + \frac{a-2}{g_2(N-a)}(K_2 \rho_L^* + 1)}
\]

\[ (3.45) \]

After we have an estimator and a value for \( \beta \) and have selected a set of values for the system parameters \( D, A \) and \( \rho \), the influence of the
design can then be ascertained by varying the number of classes $a$ for fixed $N$, over a specified range and computing for each value of $a$ the quantity $P$ as given by use of the forms (3.38)-(3.40). The merit of the two estimators would be determined by recomputing a set of $P$'s for each estimator.

Hence, an experimenter could determine for himself which design he should use for his particular situation, i.e., for a specific $D$, $A$, $\rho$ and sample of $N$. To make the results of this chapter more useful it was decided to present some numerical results.

3.7 Some Numerical Results for

$$P = \text{Prob} \left( \frac{\hat{\sigma}_{R}}{\hat{\sigma}_{A}} \leq \beta \right)$$

3.7.1 General Remarks. Using the results derived in Section 3.6, a series of tables of values of $P$ were prepared. It was hoped that consideration of some numerical results would aid in understanding the effect of design and estimator on the specific objective considered, viz., reducing product variability.

3.7.2 Values of $\rho$, $100A$, $D$, $N$, $a$ and $\beta$ Used in Computing $P$. To compute $P$ it is necessary first to specify the system parameters $\rho$, $100A$, and $D$; the design parameters $N$ and $a$, and the fixed number $\beta$. Unfortunately, one is not able to examine all the combinations of system and design parameters that he would like. A set of parameters were chosen that should represent many situations encountered in actual practice. The values of $\rho$ selected were: $1/10$, $1/4$, $1/2$, $1$, $2$, $4$ and $10$. The values of $100A$ selected were: $1/10$, $1/4$, $1/2$, $1$, $2$, $4$, and $10$. Note, for example, that $\rho = 10$ means that $\sigma_{A}^2$ is ten times larger than
\( \sigma_B^2 \), and \( 100A = 10 \) means that \( \sigma_A^2 \) is reduced at ten times the rate at which \( \sigma_B^2 \) is reduced. For \( D \), the two values \( D = 25 \) and \( D = 100 \) were selected. The majority of the results are presented for \( D = 100 \).

Two sample sizes were considered: a relatively small sample, \( N = 24 \), and a relative large sample, \( N = 72 \). With \( N = 24 \) the values of \( \alpha \) used were: \( 2, 4, 6, 8, 12, 16 \) and \( 20 \); with \( N = 72 \) they were: \( 2, 4, 8, 12, 18, 24, 36, 48, \) and \( 60 \).

Two values of \( \beta \) were used:

\[
\beta(.90) = \left[ 9 \min (\sigma_R^{12} | \hat{a}_1) + \max (\sigma_R^{12} | \hat{a}_1) \right] / 10;
\]

\[
\beta(.75) = \left[ 3 \min (\sigma_R^{12} | \hat{a}_1) + \max (\sigma_R^{12} | \hat{a}_1) \right] / 4.
\]

The value \( \beta(.90) \) represents a reduction in total variance of 90 per cent of the maximum attainable amount, and \( \beta(.75) \) represents a reduction of 75 per cent of the maximum attainable amount.

For the 49 combinations of \( \rho \) and \( 100A \), with \( D = 100 \), and using \( \beta = \beta(.90) \), the values of \( P \) are presented in Tables 3.2 - 3.8. Also included in these tables for interpretative purposes are \( \min (\sigma_R^{12} | \hat{a}_1) \) and \( \max (\sigma_R^{12} | \hat{a}_1) \). Because the results obtained for \( D = 25 \), in terms of the influence of design and estimator, are very similar to those obtained for \( D = 100 \) only a small portion of the results are presented for \( D = 25 \), viz., those for \( 100A = 1 \). The values of \( P \) for \( D = 25 \) and \( 100A = 1 \) are presented in Table 3.9. Of course the differences between \( \min (\sigma_R^{12} | \hat{a}_1) \) and \( \max (\sigma_R^{12} | \hat{a}_1) \) are considerably greater with \( D = 100 \) than with \( D = 25 \).

For three combinations of \( \rho \) and \( 100A \), with \( D = 100 \), values of \( P \) are presented using \( \beta(.75) \) in addition to \( \beta(.90) \). The three combinations
Table 3.2. Values of $P = \text{Prob}[\left(\sigma_{R}^{2} | \hat{\beta}_{1} \right) \leq \beta]$ for $100A = 1/10$ and $D = 100$

<table>
<thead>
<tr>
<th>$n$ estimator$^a/$</th>
<th>1/10</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>$N = 24$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (1)</td>
<td>1.00</td>
<td>1.00</td>
<td>.99</td>
<td>.95</td>
<td>.85</td>
<td>.75</td>
<td>.39</td>
<td></td>
</tr>
<tr>
<td>4 (1)</td>
<td>1.00</td>
<td>1.00</td>
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| $\min (\sigma_{R}^{2} | \hat{\beta}_{1} )$ | .466 | .616 | .866 | 1.366 | 2.366 | 4.366 | 10.043 |
| $\max (\sigma_{R}^{2} | \hat{\beta}_{1} )$ | 1.090 | 1.226 | 1.452 | 1.905 | 2.809 | 4.619 | 10.366 |
| $\beta(.90)$        | .528 | .677 | 1.925 | 1.420 | 2.410 | 4.390 | 10.076 |

$^a$/The symbols (1) and (2) refer to the estimators $\hat{\beta}_{1}$ and $\hat{\beta}_{2}$, respectively.
Table 3.3. Values of $P = \text{Prob}\left[\sigma_R^2|\hat{\alpha}_1| \leq \beta\right]$ for $100A = 1/4$ and $D = 100$

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$\text{min} (\sigma_R^2|\hat{\alpha}_1) = 0.466 \quad 0.616 \quad 0.866 \quad 1.366 \quad 2.349 \quad 4.092 \quad 8.786$

$max (\sigma_R^2|\hat{\alpha}_1) = 1.078 \quad 1.195 \quad 1.389 \quad 1.779 \quad 2.557 \quad 4.366 \quad 10.366$

$\beta(0.90) = 0.527 \quad 0.674 \quad 0.918 \quad 1.407 \quad 2.370 \quad 4.119 \quad 8.944$

\(a\) The symbols (1) and (2) are defined in Table 3.2.
Table 3.4. Values of $P = \text{Prob} \left( \frac{\sigma_R^2}{\hat{\sigma}_1^2} \leq \beta \right)$ for $100A = 1/2$ and $D = 100$

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$N = 24$

$\frac{\sigma_R^2}{\hat{\sigma}_1^2}$

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</table>

$N = 72$

$\frac{\sigma_R^2}{\hat{\sigma}_1^2}$

| $\min (\sigma_R^2/\hat{\sigma}_1^2)$ | .466 | .616 | .866 | 1.352 | 2.147 | 3.423 | 7.058 |
| $\max (\sigma_R^2/\hat{\sigma}_1^2)$ | 1.061 | 1.151 | 1.303 | 1.606 | 2.366 | 4.366 | 10.366 |
| $\beta(.90)$                          | .525 | .670 | .910 | 1.377 | 2.202 | 2.169 | 3.517 | 7.389 |

$^{a/}$The symbols (1) and (2) are defined in Table 3.2.

$^{b/}$The values under this column that are in parentheses are for $\beta(.75)$. 
Table 3.5. Values of \( P = \text{Prob}\left[\sigma^2_R|\hat{a}_1| \leq \beta\right] \) for \( 100A = 1 \) and \( D = 100 \)

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<td>.17</td>
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</table>

\( N = 24 \)

| 2 (1)           | .96    | .86    | .80    | \( .23 \) | .15    | .45    | .52    | .66    |
| 4 (1)           | .99    | .90    | .82    | \( .41 \) | .27    | .63    | .74    | .90    |
| 8 (1)           | .99    | .93    | .85    | \( .57 \) | .38    | .75    | .88    | .98    |
| 12 (1)          | .99    | .93    | .86    | \( .64 \) | .44    | .81    | .93    | 1.00   |
| 18 (1)          | .99    | .93    | .87    | \( .69 \) | .48    | .86    | .96    | 1.00   |
| 24 (1)          | .98    | .91    | .86    | \( .70 \) | .49    | .88    | .98    | 1.00   |
| 36 (1)          | .94    | .87    | .82    | \( .68 \) | .47    | .89    | .98    | 1.00   |
| 48 (1)          | .88    | .76    | .76    | \( .59 \) | .41    | .87    | .98    | 1.00   |
| 48 (2)          | .88    | .80    | .76    | \( .58 \) | .40    | .87    | .98    | 1.00   |
| 60 (1)          | .76    | .69    | .66    | \( .45 \) | .30    | .82    | .95    | 1.00   |
| 60 (2)          | .76    | .69    | .66    | \( .44 \) | .30    | .82    | .95    | 1.00   |

\[\min (\sigma^2_R|\hat{a}_1)\]  \(= .466\)  \(= .616\)  \(= .856\)  \(= 1.210\)  \(= 1.711\)  \(= 2.464\)  \(= 4.660\)

\[\max (\sigma^2_R|\hat{a}_1)\]  \(= 1.037\)  \(= 1.092\)  \(= 1.183\)  \(= 1.366\)  \(= 2.366\)  \(= 4.366\)  \(= 10.366\)

\(\beta(.90)\)  \(= .523\)  \(= .664\)  \(= .888\)  \(= (1.249)\)  \(= 1.226\)  \(= 1.777\)  \(= 2.654\)  \(= 5.231\)

\(a\)/The symbols (1) and (2) are defined in Table 3.2.

\(b\)/The values under this column that are in parentheses are for \(\beta(.75)\).
Table 3.6. Values of $P = \text{Prob} \left[ (\sigma_{R}^{12} | \hat{d}_{\perp}) \leq \beta \right]$ for $100A = 2$ and $D = 100$

<table>
<thead>
<tr>
<th>$a_{\text{estimator}}^{a/}$</th>
<th>$p$</th>
<th>$N = 24$</th>
<th>$N = 72$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1/10</td>
<td>1/4</td>
<td>1/2</td>
</tr>
<tr>
<td>2 (1)</td>
<td>.86</td>
<td>.82</td>
<td>(.36)$^{b/}$</td>
</tr>
<tr>
<td>4 (1)</td>
<td>.85</td>
<td>.81</td>
<td>(.53)</td>
</tr>
<tr>
<td>6 (1)</td>
<td>.82</td>
<td>.80</td>
<td>(.58)</td>
</tr>
<tr>
<td>8 (1)</td>
<td>.79</td>
<td>.78</td>
<td>(.59)</td>
</tr>
<tr>
<td>12 (1)</td>
<td>.72</td>
<td>.72</td>
<td>(.56)</td>
</tr>
<tr>
<td>16 (1)</td>
<td>.65</td>
<td>.66</td>
<td>(.46)</td>
</tr>
<tr>
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<td>.65</td>
<td>.66</td>
<td>(.45)</td>
</tr>
<tr>
<td>20 (1)</td>
<td>.55</td>
<td>.56</td>
<td>(.33)</td>
</tr>
<tr>
<td>20 (2)</td>
<td>.55</td>
<td>.56</td>
<td>(.33)</td>
</tr>
</tbody>
</table>

| $\min (\sigma_{R}^{12} | \hat{d}_{\perp})$ | .466 | .609 | .767 | .965 | 1.215 | 1.530 | 2.326 |
| $\max (\sigma_{R}^{12} | \hat{d}_{\perp})$ | 1.013 | 1.033 | 1.067 | 1.366 | 2.366 | 4.366 | 10.366 |
| $\beta(.90)$                 | .521 | .651 | (.842) | .797 | 1.005 | 1.330 | 1.814 | 3.130 |

\[a/^\text{The symbols (1) and (2) are defined in Table 3.2.}\]

\[b/^\text{The values under this column that are in parentheses are for } \beta(.75).\]
Table 3.7. Values of $P = \text{Prob} \left[ (\hat{\sigma}_R^2 | \hat{d}_1) \leq \beta \right]$ for $100A = 4$ and $D = 100$

<table>
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<th>$1/4$</th>
<th>$1/2$</th>
<th>$1$</th>
<th>$2$</th>
<th>$4$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>.75</td>
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<td>.64</td>
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<td>.89</td>
<td>.95</td>
</tr>
<tr>
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<td>.97</td>
</tr>
<tr>
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<td>.56</td>
<td>.74</td>
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<td>.73</td>
<td>.91</td>
<td>.98</td>
<td>1.00</td>
</tr>
<tr>
<td>16 (1)</td>
<td>.67</td>
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<td>.44</td>
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<td>.89</td>
<td>.98</td>
<td>1.00</td>
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<td>16 (2)</td>
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<td>.86</td>
<td>.95</td>
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<td>.93</td>
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<td>1.00</td>
</tr>
<tr>
<td>24 (1)</td>
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<td>.82</td>
<td>.94</td>
<td>.99</td>
<td>1.00</td>
<td>1.00</td>
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<td>1.00</td>
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<td>.69</td>
<td>.89</td>
<td>.99</td>
<td>1.00</td>
<td>1.00</td>
</tr>
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<td>.58</td>
<td>.69</td>
<td>.89</td>
<td>.99</td>
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<td>1.00</td>
</tr>
<tr>
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<td>.43</td>
<td>.53</td>
<td>.78</td>
<td>.96</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>60 (2)</td>
<td>.68</td>
<td>.43</td>
<td>.53</td>
<td>.78</td>
<td>.96</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

*min $\left( \hat{\sigma}_R^2 | \hat{d}_1 \right)$: .466 .558 .640 .734 .842 .965 1.157

$max \left( \hat{\sigma}_R^2 | \hat{d}_1 \right)$: 1.002 1.004 1.008 1.366 2.366 4.366 10.366

$\beta(.90)$: .519 .603 .677 .797 .994 1.305 2.077

$^a$/The symbols (1) and (2) are defined in Table 3.2.
Table 3.8. Values of $P = \text{Prob}\left[\left(\sigma^2_R|\hat{\phi}_1\right) \leq \beta\right]$ for $100A = 10$ and $D = 100$

| $\rho$ estimator$^a/|$ | 1/10 | 1/4  | 1/2  | 1    | 2    | 4    | 10  |
|-------------------------|------|------|------|------|------|------|-----|
| 2 (1)                   | .98  | .49  | .52  | .60  | .69  | .76  | .84 |
| 4 (1)                   | .98  | .64  | .71  | .81  | .90  | .95  | .98 |
| 6 (1)                   | .98  | .67  | .76  | .86  | .94  | .98  | 1.00|
| 8 (1)                   | .97  | .68  | .77  | .88  | .96  | .99  | 1.00|
| 12 (1)                  | .94  | .65  | .76  | .89  | .97  | 1.00 | 1.00|
| 16 (1)                  | .87  | .60  | .71  | .86  | .96  | .99  | 1.00|
| 16 (2)                  | .87  | .60  | .70  | .85  | .96  | .99  | 1.00|
| 20 (1)                  | .73  | .48  | .59  | .79  | .93  | .99  | 1.00|
| 20 (2)                  | .73  | .48  | .58  | .78  | .92  | .98  | 1.00|

$N = 24$

| $\rho$ estimator$^a/|$ | 1/10 | 1/4  | 1/2  | 1    | 2    | 4    | 10  |
|-------------------------|------|------|------|------|------|------|-----|
| 2 (1)                   | .99  | .56  | .57  | .63  | .70  | .77  | .84 |
| 4 (1)                   | 1.00 | .76  | .79  | .86  | .93  | .96  | .99 |
| 8 (1)                   | 1.00 | .84  | .89  | .95  | .99  | 1.00 | 1.00|
| 12 (1)                  | 1.00 | .86  | .92  | .98  | 1.00 | 1.00 | 1.00|
| 18 (1)                  | 1.00 | .85  | .93  | .99  | 1.00 | 1.00 | 1.00|
| 24 (1)                  | 1.00 | .84  | .93  | .99  | 1.00 | 1.00 | 1.00|
| 36 (1)                  | 1.00 | .80  | .91  | .98  | 1.00 | 1.00 | 1.00|
| 48 (1)                  | .98  | .75  | .86  | .97  | 1.00 | 1.00 | 1.00|
| 48 (2)                  | .98  | .75  | .86  | .97  | 1.00 | 1.00 | 1.00|
| 60 (1)                  | .90  | .67  | .78  | .92  | .99  | 1.00 | 1.00|
| 60 (2)                  | .90  | .67  | .78  | .92  | .99  | 1.00 | 1.00|

$N = 72$

| $\min(\sigma^2_R|\hat{\phi}_1)$ | .439 | .476 | .506 | .537 | .570 | .606 | .656 |
| $\max(\sigma^2_R|\hat{\phi}_1)$ | 1.000 | 1.000 | 1.000 | 1.366 | 2.366 | 4.366 | 10.366 |
| $\beta(.90)$              | .495 | .528 | .555 | .620 | .750 | .982 | 1.627 |

$^a/$ The symbols (1) and (2) are defined in Table 3.2.
Table 3.9. Values of \( P = \text{Prob} \left[ \left( \sigma_R^2 \hat{\delta} \right) \right] \leq \beta \) for 100A = 1 and D = 25

<table>
<thead>
<tr>
<th>( \beta ) estimator</th>
<th>1/10</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
</tr>
</thead>
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<td>.78</td>
<td>(.05)</td>
<td>.90</td>
<td>.61</td>
<td>.80</td>
</tr>
<tr>
<td>4 (1)</td>
<td>.97</td>
<td>.90</td>
<td>.76</td>
<td>(.09)</td>
<td>.94</td>
<td>.71</td>
<td>.99</td>
</tr>
<tr>
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<td>.89</td>
<td>.75</td>
<td>(.11)</td>
<td>.98</td>
<td>.71</td>
<td>.91</td>
</tr>
<tr>
<td>8 (1)</td>
<td>.95</td>
<td>.88</td>
<td>.73</td>
<td>(.11)</td>
<td>.91</td>
<td>.71</td>
<td>.99</td>
</tr>
<tr>
<td>12 (1)</td>
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<td>.83</td>
<td>.69</td>
<td>(.11)</td>
<td>.94</td>
<td>.74</td>
<td>.94</td>
</tr>
<tr>
<td>16 (1)</td>
<td>.83</td>
<td>.75</td>
<td>.63</td>
<td>(.09)</td>
<td>.93</td>
<td>.73</td>
<td>1.00</td>
</tr>
<tr>
<td>16 (2)</td>
<td>.84</td>
<td>.75</td>
<td>.63</td>
<td>(.09)</td>
<td>.93</td>
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<tr>
<td>20 (1)</td>
<td>.69</td>
<td>.63</td>
<td>.54</td>
<td>(.06)</td>
<td>.91</td>
<td>.73</td>
<td>.99</td>
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<tr>
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<td>.63</td>
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<td>(.06)</td>
<td>.90</td>
<td>.72</td>
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</table>

\( N = 24 \)

<table>
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<tr>
<th>( \beta ) estimator</th>
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<th>1/4</th>
<th>1/2</th>
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<td>.80</td>
<td>(.06)</td>
<td>.94</td>
<td>.58</td>
<td>.73</td>
</tr>
<tr>
<td>4 (1)</td>
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<td>.96</td>
<td>.80</td>
<td>(.11)</td>
<td>.91</td>
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<td>.82</td>
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<tr>
<td>8 (1)</td>
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<td>.83</td>
<td>(.16)</td>
<td>.74</td>
<td>.94</td>
<td>1.00</td>
</tr>
<tr>
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<td>.85</td>
<td>(.18)</td>
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<td>.84</td>
<td>(.21)</td>
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<td>.81</td>
<td>(.20)</td>
<td>.87</td>
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<td>1.00</td>
</tr>
<tr>
<td>48 (1)</td>
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<td>.90</td>
<td>.75</td>
<td>(.17)</td>
<td>.85</td>
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<td>1.00</td>
</tr>
<tr>
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<td>.90</td>
<td>.75</td>
<td>(.16)</td>
<td>.85</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
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<td>.79</td>
<td>.65</td>
<td>(.12)</td>
<td>.81</td>
<td>.98</td>
<td>1.00</td>
</tr>
<tr>
<td>60 (2)</td>
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<td>.65</td>
<td>(.12)</td>
<td>.80</td>
<td>.98</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\( N = 72 \)

\( \min (\sigma_R^2 \hat{\delta}) \)
| 1.028 | 1.278 | 1.764 | 2.556 | 4.111 | 8.778 |

\( \max (\sigma_R^2 \hat{\delta}) \)
| 1.078 | 1.389 | 1.778 | 2.778 | 4.778 | 10.778 |

\( \beta(.90) \)
| .898  | 1.044 | 1.289 | (1.767) | 1.765 | 2.578 | 4.178 | 8.978 |

\( ^a/ \) The symbols (1) and (2) are defined in Table 3.2.

\( ^b/ \) The values under this column that are in parentheses are for \( \beta(.75) \).
are: \( \rho = 2, 100A = 1/2; \rho = 1, 100A = 1; \) and \( \rho = 1/2, 100A = 2. \) These results are included in Tables 3.4, 3.5 and 3.6. For \( \rho = 1 \) and \( 100A = 1 \) with \( D = 25, \beta(.75) \) is also used in addition to \( \beta(.90). \) These results are included in Table 3.9.

3.7.3 Outline of Computational Procedure. The computations required to obtain the results were carried out on an IBM 650. All the programming was done in the FORTRAN language. A brief outline of the program is as follows:

1. For each of the 56 combinations of \( \rho, 100A, \) and \( D, \) numerical values were computed for \( k_1, k_2, C_1, C_2, \rho_1, \rho_2, V_1, V_2(\rho), V_3, V_4, W_1, W_2, B_1 \) and \( B_2. \)

2. A test on \( \rho \) was made and \( \beta(.90) \) and \( \beta(.75) \) were computed in one of three ways depending on the outcome of the test.

If

\[
\rho \leq \rho_1, \beta(.90) = (9V_1 + V_3)/10 \\
\beta(.75) = (3V_1 + V_3)/4
\]

\[
\rho \geq \rho_2, \beta(.90) = (V_1 + 9V_3)/10 \\
\beta(.75) = (V_1 + 3V_3)/4
\]

\[
\rho_1 < \rho < \rho_2, \beta(.90) = \left[9V_2(\rho) + V_4\right]/10 \\
\beta(.75) = \left[3V_2(\rho) + V_4\right]/4
\]

Note that if \( \rho = \rho_1, V_2(\rho) = V_1 \) and \( V_4 = V_3; \) if \( \rho = \rho_2, V_2(\rho) = V_3 \) and \( V_4 = V_1. \)

3. The Newton iterative method was used to compute the root or roots of the equation.
\[ W_1 \hat{\rho}_1^{B_1} + W_2 \hat{\rho}_2^{B_2} - \beta = 0 \]

The number of iterations carried for each root was such that

\[ |\rho_{i+1}^* - \rho_i^*| < 10^{-k} \]

where \( \rho_i^* \) is the root from the \( i \)th iteration. For \( \rho \leq \rho_1 \) and \( \rho \geq \rho_2 \) one root was computed using as the first approximations the values \( \rho_2 \) and \( \rho_1 \), respectively. For \( \rho_1 < \rho < \rho_2 \) and \( V_1, V_2 > \beta \) two roots were computed. As a first approximation to the smaller root, \( \rho_L^* \), the value \( \rho_1 \) was used, for the larger root, \( \rho_U^* \), the value \( \rho_2 \) was used.

(4) For \( N = 24: \) \( a = 2, 4, 6, 8 \) and 12, and for \( N = 72: \) \( a = 2, 4, 8, 12, 18, 24 \) and 36, the limits of integration \( L_1 \) and \( U_1 \) were computed for each combination of \( \rho \), 100A and D. Note that all of these designs are balanced; hence, \( K_1 = N/a \) and

\[ g_1 = 1 + \frac{N}{a} \rho; L_1 = (a-1)(\frac{N}{a} \rho_L^* + 1)/(N-a)(1 + \frac{N}{a} \rho) \text{ and} \]

\[ U_1 = (a-1)(\frac{N}{a} \rho_U^* + 1)/(N-a)(1 + \frac{N}{a} \rho). \]

(5) For \( N = 24: \) \( a = 16 \) and 20, and for \( N = 72: \) \( a = 48 \) and 60, the limits of integration, \( L_1 \) and \( U_1 \), and the parameters, \( g_1 \) and \( h_4 \), were computed.

(6) The values of \( P = \text{Prob} \left[ (\sigma_R^2 | \hat{d}_1^1) \leq \beta \right] \) were obtained by evaluating the integrals in (3.38)-(3.40). A subroutine developed at VPI for the IBM 650 was used to evaluate these integrals.

3.7.4 Discussion of Results. Study of the results indicates that the difference between \( \max (\sigma_R^2 | \hat{d}_1^1) \) and \( \min (\sigma_R^2 | \hat{d}_1^1) \) is very sensitive to the parameters \( \rho \) and 100A. With some combinations of \( \rho \) and 100A there
is little difference between $\max (\sigma_R^2 | \hat{a}_1)$ and $\min (\sigma_R^2 | \hat{a}_1)$. For these situations the type of design or estimator is not very important because the worst possible allocation of funds results in a total variance which is only slightly larger than that resulting from the best possible allocation of funds. Particular combinations of 100A and $\rho$ where the allocation makes little difference are: $100A = 1/10$, $\rho = 10$; $100A = 1/4$, $\rho = 4$; $100A = 1/2$, $\rho = 2$; and $100A = 1$, $\rho = 1$.

It is also obvious that the magnitude of $P$ depends on the specific combination of $\rho$ and 100A.

For situations where 100A and $\rho$ are both very large, or both very small, the values of $P$ are near one. The reason for this may be understood by considering a specific case. Consider the combination $\rho = 1/10$ and $100A = 1/10$. With $100A = 1/10$ we would want to allocate all our funds to source B unless $\rho$ were quite large. This condition is reflected in the value for $\rho_1$ which is 3.67, i.e., we would want to allocate all our funds to source B unless $\rho$ were greater than 3.67. Hence, $\Pr[(\sigma_R^2 | \hat{a}_1) \leq \beta(.90)] = \Pr[(\hat{\rho} \leq 4.39) | (\rho = 1/10)]$ is very large no matter what design or estimator is used.

For some combinations of $\rho$ and 100A, viz., $\rho = 2$, $100A = 1/2$; $\rho = 1$, $100A = 1$ and $\rho = 1/2$, $100A = 2$, the values of $P$ using $\beta(.90)$ are relatively low. To study the results in a somewhat higher range of $P$, values of $P$ were computed for the above mentioned combinations of $\rho$ and 100A using $\beta(.75)$.

From general consideration of the manner in which $P$ varies as a function of $a$, it appears that if one uses a design that is moderately
near the optimal he will do quite well in achieving his objective of reducing total variance.

The results may be summarized according to sample size as follows:

(1) For $N = 24$ use $a = 8$ if $100A > 1/2$; $a = 12$ if $100A \leq 1/4$; and $a = 8$ or $12$ if $1/4 \leq 100A \leq 1/2$. Never use $a = 2$ if $100A > 1$ or $a \leq 4$ if $100A \leq 1/2$. Never use $a \geq 20$ if $100A \geq 1$. It is probably best not to use $a \geq 16$.

(2) For $N = 72$ use $a = 24$ or $36$ if $100A \leq 1/2$, $a = 18$ for $100A = 1/2$ and $a = 12$ or $18$ for $100A = 4, 10$. Never use $a \leq 4$. Also never use $a = 8$ for $100A \leq 1/4$ or never use $a \geq 48$ for $100A \geq 1$.

The results may also be summarized according to cost, i.e., according to value of $100A$, as follows:

(1) $100A = 1/10$: for $N = 24$ use $a = 24$; do not use $a = 2, 4$ for $\rho = 10$. For $N = 72$ use $a = 24-36$; do not use $a = 2, 4, 8$ for $\rho = 10$.

(2) $100A = 1/4$: for $N = 24$ use $a = 8-12$; do not use $a = 2, 4$ for $\rho \geq 4$. For $N = 72$ use $a = 24-36$; do not use $a = 2, 4$ for $\rho \geq 4$.

(3) $100A = 1/2$: For $N = 24$ use $a = 8-12$; do not use $a = 2, 4$ for $\rho \geq 2$. For $N = 72$ use $a = 24-36$; do not use $a = 2, 4$ for $\rho \geq 2$.

(4) $100A = 1$: for $N = 24$ use $a = 8$; do not use $a = 2$ for $\rho > 1$ and do not use $a \geq 16$ for $\rho \leq 1$. For $N = 72$ use $a = 18-24$; do not use $a = 2, 4$ for $\rho \geq 1$ and do not use $a \geq 48$ for $\rho \leq 1$. 
(5) $100A = 2$: for $N = 24$ use $a = 8$; do not use $a = 2$ for $\rho \geq 1/2$ and do not use $a \geq 16$ for $\rho \leq 1$. For $N = 72$ use $a = 18-24$; do not use $a = 2, 4$ for $\rho \geq 1/2$ and do not use $a \geq 48$ for $\rho \leq 1$.

(6) $100A = 4$: for $N = 24$ use $a = 8$; do not use $a = 2$ for $\rho \geq 1/4$ and do not use $a = 20$ for $\rho \leq 2$ or $a \geq 16$ for $\rho \leq 1$. For $N = 72$ use $a = 18-24$; do not use $a = 2, 4$ for $\rho \geq 1/4$ and do not use $a \geq 48$ for $\rho \leq 1$.

(7) $100A = 10$: for $N = 24$ use $a = 8$; do not use $a = 2$ for $\rho \geq 1/4$ and do not use $a = 20$ for $\rho \leq 1/2$. For $N = 72$ use $a = 12-24$; do not use $a = 2$ for $\rho \geq 1/4$ and do not use $a = 60$ for $\rho \leq 1/2$.

The results seem to indicate that for most situations an intermediate value of $a$, say $N/4 \leq a \leq N/2$, will give results that are quite close to the optimal. If one value of $a$ were to be recommended it would be $a = N/3$.

It is of interest to compare the results concerning the optimal $a$ in the situation considered here with the results given by Crump (1954) for the estimation of $\hat{\rho}$. As the value of $a$ which minimizes $\text{var}(\hat{\rho})$ he gives the value

$$a_1 = 1 + \frac{(N-5)(N_0 + 1)}{2N_0 + N - 3}.$$

Using $N = 72$, and substituting for $\rho$ the values: $1/10$, $1$, and $10$, we obtain for $a_1$ the values: $7.6$, $24$ and $33$, respectively. As expected, comparison of the results for optimal $a$ using Crump's formula indicates close agreement with the results given in Tables 3.5 and 3.9 with $100A = 1$. 


Comparison of results from the estimator $\hat{\rho}_2$ with those from $\hat{\rho}_1$ indicate very little difference between the two. Hence, the discarding of the information from the one degree of freedom due to $A_1$ vs. $A_2$ appears to have had little effect for the situations considered. It should be noted that the one degree of freedom was discarded only for designs in which $a \geq \frac{2}{3} N$ and therefore only a relatively small percentage of the available degrees of freedom for classes was lost. For situations in which $a$ is small it is likely that the discarding of one degree of freedom would have a greater effect.
4.0 VARIANCES OF ESTIMATED VARIANCE COMPONENTS  
FOR A THREE STAGE NESTED CLASSIFICATION

4.1 Introduction

The variances of the estimated variance components for the three stage nested classification given by model (1.2) will be derived in this chapter. An analysis of variance for data described by (1.2) is given in Table 4.1.

Table 4.1. An analysis of variance for a three stage nested classification

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>A classes</td>
<td>a - 1</td>
<td>( Q_1/(a-1) )</td>
<td>( \sigma_C^2 + k_1^2 \sigma_B^2 + k_2^2 \sigma_A^2 )</td>
</tr>
<tr>
<td>B in A classes</td>
<td>( \Sigma b_i - a )</td>
<td>( Q_2/(\Sigma b_i - a) )</td>
<td>( \sigma_C^2 + k_1 \sigma_B^2 )</td>
</tr>
<tr>
<td>Within B classes</td>
<td>( N - \Sigma b_i )</td>
<td>( Q_3/(N-\Sigma b_i) )</td>
<td>( \sigma_C^2 )</td>
</tr>
<tr>
<td>Total</td>
<td>N - 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
\mathbf{\text{(a) }} & Q_1 = \Sigma \Sigma \Sigma x_{i,j,k}^2 - \Sigma \Sigma n_{i,j} \bar{x}_{i,j}^2, \quad \bar{x}_{i,j} = \Sigma x_{i,j,k}/n_{i,j} \\
\mathbf{\text{(b) }} & Q_2 = \Sigma \Sigma n_{i,j} \bar{x}_{i,j}^2 - \Sigma n_{i,j} \bar{x}_i^2, \quad \bar{x}_i = \Sigma \Sigma x_{i,j,k}/n_i \\
\mathbf{\text{(c) }} & Q_3 = \Sigma n_i \bar{x}_i^2 - n \bar{x}^2, \quad \bar{x} = \Sigma \Sigma x_{i,j,k}/N \\
\mathbf{\text{(d) }} & K_1 = \left( N - \Sigma \Sigma \frac{n_{i,j}}{n_i} \right)/(\Sigma b_i - a), \quad K_1' = \left( \Sigma \Sigma \frac{n_{i,j}}{n_i} - \Sigma \Sigma \frac{n_{i,j}^2}{N} \right)/(a - 1) \\
\mathbf{\text{(e) }} & K_2 = \left( N - \Sigma \frac{n_i^2}{N} \right)/(a - 1)
\end{align*} \]
In this work the usual analysis of variance estimators of the variance components will be considered. The estimators, obtained by equating mean squares to their expectations, are given by

\[
\hat{\sigma}_C^2 = \frac{Q_2}{N - \sum b_i} \\
\hat{\sigma}_B^2 = \frac{Q_2/(\sum b_i - a) - Q_3/(N - \sum b_i)}{K_1} \\
\hat{\sigma}_A^2 = \frac{K_1 Q_1/(a-1) - K_1 Q_2/(\sum b_i - a) + (K_1' - K_1) Q_3/(N - \sum b_i)}{K_1 K_2}
\] (4.1)

The variances of these estimators are

\[
\text{var}(\hat{\sigma}_C^2) = \text{var}(Q_2)/(N - \sum b_i)^2 \\
\text{var}(\hat{\sigma}_B^2) = \left[\text{var}(Q_2)/(\sum b_i - a)^2 + \text{var}(Q_3)/(N - \sum b_i)^2 - 2 \text{cov}(Q_2 Q_3)/(\sum b_i - a)(N - \sum b_i)\right]/K_1^2
\] (4.2)

\[
\text{var}(\hat{\sigma}_A^2) = \left[K_1^2 \text{var}(Q_1)/(a-1)^2 + K_1^2 \text{var}(Q_2)/(\sum b_i - a)^2 + (K_1' - K_1)^2 \text{var}(Q_3)/(N - \sum b_i)^2 - 2K_1 K_1' \text{cov}(Q_1 Q_2)/(a-1)(\sum b_i - a) + 2K_1(K_1' - K_1) \text{cov}(Q_1 Q_3)/(a-1)(N - \sum b_i)ight]
\]

\[- 2K_1 K_1' \text{cov}(Q_2 Q_3)/(\sum b_i - a)(N - \sum b_i)/(K_1 K_2)^2.
\]

Therefore, to obtain the variances as given by (4.2) it will be necessary to procure the variances of, and covariances among, \(Q_1, Q_2\).
and $Q_j$. The technique used to obtain the variances and covariances of these quadratic forms depends on a well known result from statistical theory [see Whittle (1953)].

For the situation where $x$ has a multinormal distribution with covariance matrix $V$ it can be shown that the expectation and variance of the quadratic form $y = x'N_x$ are

$$E(y) = \text{tr}(VM)$$
$$\text{var}(y) = 2\text{tr}(VM)^2.$$  \hspace{1cm} (4.3)

Also, the covariance between two quadratic forms $y_1 = x'N_x$ and $y_2 = x'N_x$ is

$$\text{cov}(y_1, y_2) = 2\text{tr}(VMVN).$$  \hspace{1cm} (4.4)

Let $J_{ij}$ denote a column vector with $n_i$ elements. This vector has $n_{ij}$ elements that are "ones" corresponding to the $x$'s in the $i^{th}$ A-class and $j^{th}$ B-class, and the remaining $n_i - n_{ij}$ elements are zeros. If we let $O_{ij}$ denote a null column vector with $n_{ij}$ elements and $U_{ij}$ a column vector with $n_{ij}$ elements all "ones", the transpose $J'_{ij}$ is

$$J'_{ij} = (0'_{i1}, 0'_{i2}, \ldots, 0'_{i,j-1}, U'_{ij}, 0'_{i,j+1}, \ldots, 0'_{in_j}).$$

A vector with $n_i$ elements all "ones" will be indicated as $U_i$, and one with $N$ elements all "ones" will be simply $U_N$. Certain useful properties of these vectors are listed below:

$$J'_{ij}J_{jm} = \begin{cases} 
n_{ij} & \text{if } m = j \\
0 & \text{if } m \neq j \end{cases}$$  \hspace{1cm} (4.5)

$$J'_{ij}U_i = n_{ij}; U_i'U_i = n_i; U_N'U_N = N.$$
Some general properties of the trace of matrices are indicated in (4.6)

\[ \text{tr}(AB) = \text{tr}(BA) \]

\[ \text{tr} [(A^2 + (CD)] = \text{tr}(AB) + \text{tr}(CD) \]

(4.6)

provided the numbers of rows and columns of the matrices permit multiplication and addition. The relations given by (4.5) and (4.6) will be used extensively throughout the development of this chapter.

The covariance matrix \( \Sigma \), required in (4.3) and (4.4) of the vector of observations from a three stage nested classification is given by

\[
\Sigma = \text{diag}(\Sigma_1) =
\begin{bmatrix}
\Sigma_1 \\
\Sigma_2 \\
\vdots \\
\Sigma_a
\end{bmatrix}
\]

(4.7)

where

\[ \Sigma_1 = \sigma_C^2 I (n_1) + \sigma_B^2 \sum_{j} j_{ij} + \sigma_A^2 u_1 u_1' \]

As usual \( I(n_1) \) is the identity matrix.

4.2 Derivation of Variances of \( Q_1 \), \( Q_2 \) and \( Q_3 \)

To obtain the variances of \( Q_1 \), \( Q_2 \) and \( Q_3 \), matrix products of the form \( (\Sigma M)^2 \) will be needed for each of the three sums of squares, \( Q_1 \), \( Q_2 \) and \( Q_3 \). First consider \( Q_1 \), the sum of squares for the A-classes.
The sum of squares $Q_1$ may be written as

$$Q_1 = x'Ax$$

where

$$A = \text{diag}(\frac{1}{n_1}U_1U_1') - \frac{1}{N} U_NN'N.$$  \hspace{1cm} (4.8)

Letting $r_i = (\frac{1}{n_i} - \frac{1}{N})$ the matrix product $VA$ is represented by

$$VA = \begin{bmatrix}
  r_{1i}V_iU_iU_i' - \frac{1}{N} V_iU_iU_i' & \ldots & - \frac{1}{N} V_iU_iU_a' \\
  - \frac{1}{N} V_2U_2U_2' & r_{22}V_2U_2U_2' & \ldots \\
  \vdots & \vdots & \ddots \\
  - \frac{1}{N} V_aU_aU_a' & \ldots & r_{aa}V_aU_aU_a'
\end{bmatrix}. \hspace{1cm} (4.9)$$

At this point it is easy to obtain the expectation of $Q_1$. Referring to (4.3) and (4.9) we have

$$E(Q_1) = \text{tr}(VA) = \sum_i \text{tr}(r_{ii}V_iU_iU_i').$$

(4.10)

Substitution of $V_1$, as given in (4.7), into (4.10) gives

$$E(Q_1) = \sum_i r_i \text{tr}\left(\sigma_C^2 \bar{I}(n_i) + \sigma_B^2 \sum_j \hat{J}_{ij}' \hat{J}_{ij} + \sigma_A^2 U_iU_i'U_iU_i'\right). \hspace{1cm} (4.11)$$

Then by making use of (4.7) and (4.8) we find that $E(Q_1)$ is readily reduced to
\[ E(Q_1) = \sum_i r_i \left[ \sigma_i^2 + \sum_j n_{ij} \sigma_j^2 + n_i \sigma_B^2 \right] \]
\[ = (a-1) \sigma_c^2 + (\Sigma \Sigma \frac{n_{ij}}{n_i} - \Sigma \Sigma \frac{n_{ij}}{N}) \sigma_B^2 + (N - \Sigma \frac{n_i}{N}) \sigma_A^2. \]  

(4.12)

To obtain the variance of \( Q_1 \) we require

\[ \text{tr}(\mathbf{VA})^2 = \sum_i \text{tr}(r_i \mathbf{V}_i \mathbf{U}_i^\prime \mathbf{U}_i^\prime) + \frac{1}{N} \sum_i \sum_{i \neq j} \text{tr}(V_i U_i U_i^\prime)(V_j U_j U_j^\prime). \]  

(4.13)

We shall proceed with the evaluation of \( \text{tr}(\mathbf{VA})^2 \) by considering the two terms on the right hand side (RHS) of (4.13) separately. Using (4.5) and (4.6) and substituting \( V_i \) from (4.7) we have that

\[ \sum_i \text{tr}(r_i V_i U_i U_i^\prime)^2 \]
\[ = \sum_i r_i^2 \text{tr}\left[ \sigma_c^2 U_i U_i^\prime + \sigma_B^2 \sum_j n_{ij} J_{ij} U_i^\prime + \sigma_A^2 n_i U_i U_i^\prime \right]. \]  

(4.14)

For fixed "i" the coefficients of the squares and cross products of the components of (4.14) are derived below,

\[ \sigma_c^4: \quad r_i^2 \text{tr}(U_i U_i^\prime U_i U_i^\prime) = r_i^2 n_i; \]
\[ \sigma_B^4: \quad r_i^2 \text{tr}\left[ (\Sigma \Sigma n_{ij} J_{ij} U_i^\prime)(\Sigma n_{im} J_{im} U_i^\prime) \right] \]
\[ = r_i^2 \text{tr}\left[ \Sigma n_{ij}^2 J_{ij} J_{ij} U_i^\prime U_i^\prime + \Sigma \Sigma n_{ij} n_{im} J_{ij} J_{im} U_i^\prime U_i^\prime \right] \]
\[ = r_i^2 [\sum_j n_{ij}^2 + \sum_{j \neq m} n_{ij} n_{im}] = r_i^2 [\sum_j n_{ij}^2] \] ;
\[ \sigma_C^2 \sigma_B^2: \quad 2r_1^2 \operatorname{tr} \left[ u_{1 \ell}^u u_{1 \ell}^t \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t \right] \]

\[ = 2r_1^2 \operatorname{tr} \left[ u_{1 \ell} \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t \right] = 2r_1^2 n_{1 \ell} \sum_{\ell \ell} n_{1 \ell} \quad ; \]

\[ \sigma_A^2: \quad r_1^2 n_{1 \ell} \operatorname{tr} \left( u_{1 \ell} u_{1 \ell}^t u_{1 \ell}^t \right) = r_1^2 n_{1 \ell} \quad ; \]

\[ \sigma_C^2 \sigma_A^2: \quad 2r_1^2 \operatorname{tr} \left( n_{1 \ell} u_{1 \ell}^t u_{1 \ell}^t u_{1 \ell}^t \right) = 2r_1^2 n_{1 \ell}^3 \]

\[ \sigma_B^2 \sigma_A^2: \quad 2r_1^2 \operatorname{tr} \left[ \sum_{\ell \ell} \left( \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t \right) \left( n_{1 \ell} u_{1 \ell}^t \right) \right] \]

\[ = 2r_1^2 \operatorname{tr} \left[ \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} n_{1 \ell} u_{1 \ell}^t \right] = 2r_1^2 n_{1 \ell} \sum_{\ell \ell} n_{1 \ell} \quad . \]

Next, for fixed "i" and "\ell" consider the second term on the RHS of (4.13). Substituting for \( V_{\ell} \) and \( Y_{\ell} \) as given in (4.7) gives

\[ \operatorname{tr} \left[ (V_{\ell} u_{1 \ell}^t)(Y_{\ell} u_{1 \ell}^t) \right] \]

\[ = \operatorname{tr} \left\{ \left[ \sigma_C^2 u_{1 \ell}^t + \sigma_B^2 \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t + \sigma_A^2 n_{1 \ell} u_{1 \ell}^t \right] \right. \]

\[ \left. \cdot \left[ \sigma_C^2 u_{1 \ell}^t + \sigma_B^2 \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t + \sigma_A^2 n_{1 \ell} u_{1 \ell}^t \right] \right\} \quad . \]

The coefficients of the squares and cross products of the components of (4.15) are obtained below,

\[ \sigma_C^4: \quad \operatorname{tr} \left( u_{1 \ell} u_{1 \ell}^t u_{1 \ell}^t u_{1 \ell}^t \right) = n_{1 \ell} n_{1 \ell} \quad ; \]

\[ \sigma_B^4: \quad \operatorname{tr} \left[ \sum_{\ell \ell} \left( \sum_{\ell \ell} n_{1 \ell} J_{\ell \ell} u_{1 \ell}^t \right) \left( n_{1 \ell} u_{1 \ell}^t \right) \right] = \sum_{\ell \ell} n_{1 \ell} n_{1 \ell} \quad . \]
\[ \sigma_C^2 \sigma_B^2: \text{tr} \left[ U_i^\top A_i J_{\mu \nu} U_i^\top + U_i^\top \Sigma n_i \eta_{i j k} \right] \]
\[ = \text{tr} \left[ U_i^\top \Sigma n_i \eta_{i j k} + U_i^\top \Sigma n_i \eta_{i j k} \right] \]
\[ = n_i \Sigma n_i \eta_{i j k} + n_i \Sigma n_i \eta_{i j k} \; \]
\[ \sigma_A^2: \text{tr} \left( n_i n_i \eta_{i j k} U_i^\top U_i U_i^\top \right) = n_i^2 n_i \eta_{i j k} \; \]
\[ \sigma_C^2 \sigma_A^2: \text{tr} \left[ U_i^\top A_i J_{\mu \nu} U_i^\top + U_i^\top \Sigma n_i \eta_{i j k} \right] \]
\[ = n_i^2 + n_i \Sigma n_i \eta_{i j k} \; \]
\[ \sigma_B^2 \sigma_A^2: \text{tr} \left[ \Sigma n_i \eta_{i j k} U_i^\top U_i U_i^\top \right] \]
\[ = \text{tr} \left[ \Sigma n_i \eta_{i j k} U_i^\top U_i U_i^\top \right] \]
\[ = n_i^2 \Sigma n_i \eta_{i j k} + n_i \Sigma n_i \eta_{i j k} \; \]

Finally, to obtain the coefficients of the squares and cross products of the components in \( \text{var}(Q) \) the results from the first and second terms of the RHS of (4.13) are combined and then summed over "\( i \)" and "\( \eta \)" as indicated. The coefficients are given below,

\[ \sigma_C^4: \Sigma \left[ \frac{N-n_i}{N n_i} \right]^2 n_i^2 + \frac{1}{N^2} \Sigma \Sigma n_i n_i \eta_{i j k} \]
\[ = \frac{1}{N^2} \left[ \Sigma \left( N^2 - 2Nn_i + n_i^2 \right) + \Sigma \Sigma n_i n_i \eta_{i j k} \right] \]
\[ = \frac{1}{N^2} \left[ aN^2 - 2N^2 + N^2 \right] = a - 1; \]
\[ o^4_B: \sum_{i} \left[ \frac{N-n-1_i}{N_n_i} \right]^{\frac{1}{2}} \left[ \Sigma \frac{n^2_{i,j}}{n} \right]^{\frac{1}{2}} \frac{1}{N^2} \sum_{i} \sum_{j} \sum_{m} n^2_{i,j} n^2_{i,m} \]

\[ = \frac{1}{N^2} \left\{ \sum_{i} \left[ N^2 \frac{1}{n^2_i} \left( \Sigma n^2_{i,j} \right)^2 - 2N \frac{1}{n^2_i} \left( \Sigma n^2_{i,j} \Sigma n^2_{i,j} \right) \right] \right\} \]

\[ + \sum_{i} \sum_{j} \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \frac{1}{N^2} \]

\[ = \frac{1}{N^2} \left[ N^2 \sum_{i} \frac{1}{n^2_i} \left( \Sigma n^2_{i,j} \right)^2 - 2N \sum_{i} \frac{1}{n^2_i} \left( \Sigma n^2_{i,j} \Sigma n^2_{i,j} \right) \right] ; \]

\[ o^2_C o^2_B: \sum_{i} \left[ \frac{N-n-1_i}{N_n_i} \right]^{\frac{1}{2}} \frac{1}{n^2_i} \left[ \Sigma \frac{n^2_{i,j}}{n} \right]^{\frac{1}{2}} \frac{1}{N^2} \sum_{i} \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \frac{1}{N^2} \left[ n^2_i \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \right] \]

\[ = \frac{1}{N^2} \left\{ 2N^2 \frac{1}{n^2_i} \Sigma n^2_{i,j} - 4N \Sigma n^2_{i,j} n^2_{i,j} + 2 \Sigma n^2_{i,j} n^2_{i,j} \right\} \]

\[ + \sum_{i} \sum_{j} \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \frac{1}{N^2} \left[ n^2_i \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \right] \]

\[ = \frac{1}{N^2} \left[ 2N^2 \Sigma \frac{1}{n^2_i} n^2_{i,j} - 4N \Sigma n^2_{i,j} n^2_{i,j} + 2N \Sigma n^2_{i,j} n^2_{i,j} \right] \]

\[ = \frac{2}{N} \left[ N \Sigma \frac{1}{n^2_i} n^2_{i,j} - \Sigma n^2_{i,j} \right] ; \]

\[ o^4_A: \sum_{i} \left[ \frac{N-n-1_i}{N_n_i} \right]^{\frac{1}{2}} \frac{1}{n^2_i} \sum_{i} \sum_{j} \sum_{\not{j} n^2_{i,j} n^2_{i,j} \not{j}} \frac{1}{N^2} \sum_{i} \sum_{j} \sum_{m} \sum_{\not{m} n^2_{i,j} n^2_{i,m} \not{m}} \frac{1}{N^2} \left[ n^2_i \sum_{j} \sum_{\not{j} n^2_{i,j} n^2_{i,j} \not{j}} \right] \]

\[ = \frac{1}{N^2} \left[ n^2_i \sum_{j} \sum_{\not{j} n^2_{i,j} n^2_{i,j} \not{j}} \right] \]

\[ = \frac{1}{N^2} \left[ n^2_i \sum_{j} \sum_{\not{j} n^2_{i,j} n^2_{i,j} \not{j}} \right] \]
\( \sigma_C^2 \sigma_A^2: 2 \sum_{i} \left[ \frac{n_{i1}^2}{N} \right] + \sum_{i \neq j} \left[ n_{ij}^2 \right] \)

\[ = \frac{1}{N^2} \left[ 2N^2 \sum_{i} n_i^2 - 4N \sum_{i} n_i^2 + 2 \sum_{i} n_i^3 + \sum_{i \neq j} \left( n_{ij}^2 + n_{ij}^2 \right) \right] \]

\[ = \frac{1}{N^2} \left[ 2N^3 - 4N \sum_{i} n_i^2 + 2N \sum_{i} n_i^2 \right] \]

\[ = \frac{2}{N} \left[ n_i^2 - \sum_{i} n_i^2 \right] ; \]

\( \sigma_B^2 \sigma_A^2: 2 \sum_{i} \left[ \frac{n_{i1}^2}{N} \right] \left( n_{ij}^2 + \sum_{i \neq j} \left( n_{ij}^2 + n_{ij}^2 \right) \right) \)

\[ = \frac{1}{N^2} \left[ 2N^2 \sum_{i} \sum_{j} n_{ij}^2 - 2N \sum_{i} n_i^2 \sum_{j} n_{ij}^2 + \sum_{i} n_i^2 \right] \]

\[ + \sum_{i \neq j} \left( n_{ij}^2 + \sum_{i} n_i^2 \right) \]

\[ = \frac{2}{N^2} \left[ N^2 \sum_{i} n_i^2 - 2N \sum_{i} n_i^2 \sum_{j} n_{ij}^2 + \sum_{i} n_i^2 \sum_{j} n_{ij}^2 \right] \].

After collecting the preceding results the variance of \( Q_1 \) is given by

\[ \text{Var}(Q_1) = 2 \text{tr}(VA)^2 \]

\[ = 2 \left( (a-1) \sigma_C^4 + \frac{1}{N^2} \left[ N^2 \sum_{i} \frac{1}{n_i^2} \left( \sum_{i \neq j} n_{ij}^2 \right) - 2N \sum_{i} \frac{1}{n_i^2} \right] \right) \]

\[ + \sum_{i \neq j} \left( n_{ij}^2 \right) \sigma_B^4 + \frac{2}{N} \left[ N \sum_{i} \frac{1}{n_i^2} \sum_{j} n_{ij}^2 - \sum_{i} \sum_{j} n_{ij}^2 \right] \sigma_C^2 \sigma_B^2 \]

\[ + \frac{1}{N^2} \left[ N^2 \sum_{i} n_i^2 - 2N \sum_{i} n_i^3 \right] \sigma_A^2 + \frac{2}{N} \left[ N^2 \sum_{i} n_i^2 \right] \sigma_C^2 \sigma_A^2 \]

\[ + \frac{2}{N^2} \left[ N^2 \sum_{i} n_i^2 - 2N \sum_{i} n_i^2 \sum_{j} n_{ij}^2 + \sum_{i} n_i^2 \sum_{j} n_{ij}^2 \right] \sigma_B^2 \sigma_A^2 \]

\[ + \frac{2}{N^2} \left[ N^2 \sum_{i} n_i^2 - 2N \sum_{i} n_i^2 \sum_{j} n_{ij}^2 + \sum_{i} n_i^2 \sum_{j} n_{ij}^2 \right] \sigma_B^2 \sigma_A^2 \]

\[ (4.16) \]
Next consider the sum of squares for the B-classes. This sum of squares is

\[ Q_2 = x' B x. \]

The matrix \( B \) is

\[ B = \text{diag}(B_1), \]

where

\[ B_1 = \sum_j \frac{1}{n_{1j}} J_{1j} J_{1j}' - \frac{1}{n_i} U_1 U_1'. \tag{4.17} \]

From (4.7) and (4.17) one sees that

\[ V_B = \text{diag}(V_i B_1) \tag{4.18} \]

and

\[
V_i B_1 = \sigma^2_C \left[ \sum_j \frac{1}{n_{1j}} J_{1j} J_{1j}' - \frac{1}{n_i} U_1 U_1' \right] \\
+ \sigma^2_B \left[ \sum_j \frac{1}{n_{1j}} J_{1j} J_{1j}' - \frac{1}{n_i} \sum_j n_{1j} J_{1j} U_1' \right] \\
+ \sigma^2_A \left[ \sum_j U_1 J_{1j}' - U_1 U_1' \right] \\
= \sigma^2_C \left[ \sum_j \frac{1}{n_{1j}} J_{1j} J_{1j}' - \frac{1}{n_i} U_1 U_1' \right] \\
+ \sigma^2_B \left[ \sum_j \frac{1}{n_{1j}} J_{1j} J_{1j}' - \frac{1}{n_i} \sum_j n_{1j} J_{1j} U_1' \right].
\]

Again by making use of (4.5) and (4.6) it is seen that

\[ \text{tr}(V_i B_1) = \sigma^2_C (b_i - 1) + \sigma^2_B (n_i - \frac{1}{n_i} \sum_j n_{1j}^2). \]
Hence

\[
E(Q_2) = \sum_i \text{tr}(Y_i B_i)
\]

\[
= (\Sigma b_i - a_i) \sigma_C^2 + (N - \Sigma \Sigma n_{ij}^2) \sigma_B^2.
\]  

(4.19)

It is noted that

\[
\text{tr}(YB)^2 = \text{tr diag}(Y_i B_i)^2 = \Sigma \text{tr}(Y_i B_i)^2.
\]  

(4.20)

To obtain the coefficients of the squares and cross products of the components in (4.20) we first fix "i", and then take the trace of

\[(Y_i B_i)^2.\]

The coefficients are derived below,

\[
\sigma_C^2: \text{tr} \left[ \frac{1}{2} J_{ij}J'_{ij}J_{ij}J'_{ij} - \frac{2}{n_{ij}} \Sigma J_{ij}J'_{ij}U_i U_i' \right] = (b_i - 2 + 1) = b_i - 1;
\]

\[
\sigma_B^2: \text{tr} \left\{ \Sigma J_{ij}J'_{ij}J_{ij}J'_{ij} - \frac{2}{n_{ij}} \Sigma n_{ij}J_{ij}J'_{ij}J_{ij}J'_{ij} \right. \\
+ \left. \frac{1}{n_{ij}^2} \left( \Sigma n_{ij}^2 J_{ij}U_i U_i' + \Sigma \Sigma n_{im} J_{ij}J_{ij}U_i U_i' \right) \right\}
\]

\[
= \Sigma n_{ij}^2 - \frac{2}{n_{ij}} \Sigma n_{ij}^3 + \frac{1}{n_{ij}^2} \left( \Sigma n_{ij}^4 + \Sigma \Sigma n_{ij}^2 n_{im}^2 \right)
\]

\[
= \Sigma \Sigma (n_{ij}^2 - \frac{2}{n_{ij}} n_{ij}^3 + \frac{1}{n_{ij}^2} (\Sigma n_{ij}^2)^2).
\]
\[ \sigma_C^2 \sigma_B^2 = 2 \text{tr} \left( \sum_{m} \frac{1}{n_{im}} J_{im} J_{im} - \frac{1}{n_{j}} U_{ij} U_{ij} \right) \left[ \sum_{j} \frac{1}{n_{ij}} J_{ij} J_{ij} - \frac{1}{n_{ij}} U_{ij} U_{ij} \right] \]

\[ = 2 \text{tr} \left[ \sum_{j} \frac{1}{n_{ij}} J_{ij} J_{ij} - \frac{1}{n_{ij}} U_{ij} U_{ij} \right] - \frac{1}{n_{ij}} \sum U_{ij} J_{ij} J_{ij} + \frac{1}{n_{ij}} \sum U_{ij} J_{ij} J_{ij} 

\[ = 2 \left[ \frac{1}{n_{ij}} \sum \frac{n_{ij}^2}{n_{ij}} - \frac{1}{n_{ij}} \sum \frac{n_{ij}^2}{n_{ij}} \right] . \]

The variance of \( \bar{Q} \) is found by summing the coefficients just derived over "i" and multiplying by two, this gives

\[ \text{Var}(\bar{Q}) = 2 \left( \sum_{i} \left( \frac{1}{n_{ij}} - a \right) \sigma_C^2 + 2(N - \sum_{i} \frac{n_{ij}^2}{n_{ij}}) \sigma_C^2 \sigma_B^2 \right) \]

\[ + \left[ \sum_{i} \sum_{j} \frac{n_{ij}^3}{n_{ij}} - 2 \sum_{i} \sum_{j} \frac{n_{ij}^3}{n_{ij}} + \sum_{i} \frac{n_{ij}^2}{n_{ij}} \left( \sum_{j} \frac{n_{ij}^2}{n_{ij}} \right)^2 \right] \sigma_B^4 \right\} . \] (4.21)

The sum of squares for the C-classes is

\[ Q = x' \Sigma x \]

where \( \Sigma = \text{diag}(C_1) \) and

\[ C_1 = \mathbb{I}(n_{ij}) - \frac{1}{n_{ij}} J_{ij} J_{ij} \cdot \] (4.22)

Then \( \Sigma = \text{diag}(V_1) \) and from (4.7) and (4.22)

\[ V_1 C_1 = \sigma_C^2 \left[ \mathbb{I}(n_{ij}) - \frac{1}{n_{ij}} J_{ij} J_{ij} \right] \]

\[ + \sigma_B^2 \left[ \sum_{j} J_{ij} J_{ij} - \sum_{j} J_{ij} J_{ij} \right] + \sigma_A^2 \left[ U_{ij} U_{ij} - \sum_{j} U_{ij} U_{ij} \right] \] (4.23)

\[ = \sigma_C^2 \left[ \mathbb{I}(n_{ij}) - \frac{1}{n_{ij}} J_{ij} J_{ij} \right] . \]
The variance of $Q_3$ is

$$\text{var}(Q_3) = 2 \text{tr} (V C)^2 = 2 \sum_i \text{tr} (V_i C_i)^2.$$ 

From (4.23) we obtain

$$(V_i C_i)^2 = \sigma_C^2 \left[ I(n_i) - \frac{1}{n_{ij}} J_{ij} J_{ij}^T \right]$$

and

$$\text{tr} (V_i C_i)^2 = \sigma_C^4 (n_i - b_i).$$

Hence

$$\text{var}(Q_3) = 2(N - \sum_i b_i) \sigma_C^4. \quad (4.24)$$

4.3 Derivation of Covariances among $Q_1$, $Q_2$ and $Q_3$

To obtain the covariances among $Q_1$, $Q_2$ and $Q_3$ it is necessary to obtain matrix products of the form $V W V^T$ for each of the three pairwise combinations among the quadratic forms. First consider the covariance between $Q_1$ and $Q_2$.

To derive $\text{cov}(Q_1 Q_2)$, as given by (4.4), we require

$$2 \text{tr}(V W V B) = 2 \sum_i \text{tr}(r_i V_i U_i U_i^T V_i B_i). \quad (4.25)$$

Substituting $V_i$ from (4.7) and $B_i$ from (4.17) into (4.25) gives

$$\text{tr}(V W V B) = \sum_i r_i \text{tr} \left\{ \left[ \sigma_C^2 U_i U_i^T + \sigma_B^2 \sum_j n_{ij} J_{ij} J_{ij}^T + \sigma_A^2 n_{ii} U_i U_i^T \right] \right. \left. \cdot \left[ \sigma_C^2 \left( \sum_j \frac{1}{n_{ij}} J_{ij} J_{ij}^T - \frac{1}{n_i} U_i U_i^T \right) + \sigma_B^2 \left( \sum_j J_{ij} J_{ij}^T - \frac{1}{n_i} \sum_j n_{ij} J_{ij} U_i^T \right) \right] \right\}. \quad (4.26)$$
For fixed "i" the coefficients of the squares and crossproducts of the components of (4.26) are derived below,

\[
\sigma_C^4: \quad r_i \text{tr} \left[ \Sigma U_i J_i^t - U_i U_i^t \right] = 0 ;
\]

\[
\sigma_B^4: \quad r_i \text{tr} \left[ (\Sigma n_i m_j U_i^t) (\Sigma n_i m_j U_i^t) - (\Sigma n_i m_j U_i^t) \left( \frac{1}{n_i} \Sigma n_i m_j U_i^t \right) \right]
\]

\[
= r_i \text{tr} \left[ \Sigma n_i m_j U_i^t - \frac{1}{n_i} \Sigma n_i m_j U_i^t \right]
\]

\[
= r_i \left[ \Sigma n_i^2 - \frac{1}{n_i} (\Sigma n_i^2)^2 \right] ;
\]

\[
\sigma_C^2 \sigma_B^2: \quad r_i \text{tr} \left[ n_i U_i J_i^t - n_i U_i U_i^t \right] + \Sigma n_i U_i J_i^t - \Sigma n_i U_i J_i^t = 0
\]

\[
\sigma_C^2 \sigma_A^2: \quad r_i \text{tr} [n_i U_i J_i^t - n_i U_i U_i^t] = 0
\]

\[
\sigma_B^2 \sigma_A^2: \quad r_i \text{tr} [n_i U_i J_i^t - n_i U_i U_i^t] = 0
\]

\[
Cov(Q_1 Q_2) \text{ is obtained by summing (4.26) over "i" and multiplying by two, the result is}
\]

\[
\text{cov}(Q_1 Q_2) = 2 \sum_i \left[ \frac{N-n_i}{N n_i} \right] \left[ \Sigma n_i^2 - \frac{1}{n_i} (\Sigma n_i^2)^2 \right] \sigma_B^4
\]

\[
(4.27)
\]

Proceeding in a similar manner it is found that \( \text{cov}(Q_1 Q_3) = 0 \) and \( \text{cov}(Q_2 Q_3) = 0 \).
4.4 Variances of $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$ and $\hat{\sigma}_C^2$

The variances of $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$ and $\hat{\sigma}_C^2$ are obtained by substituting the expressions for $\operatorname{var}(Q_1)$, $\operatorname{var}(Q_2)$, $\operatorname{var}(Q_3)$ and $\operatorname{cov}(Q_1 Q_2)$, as given by (4.16), (4.21), (4.24) and (4.27), respectively, into (4.2).

4.5 Covariances among $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$ and $\hat{\sigma}_C^2$

Once the variances and covariances of $Q_1$, $Q_2$ and $Q_3$ have been derived it is a simple matter to derive the covariances among the estimated components of variance. Consider the covariance between $\hat{\sigma}_C^2$ and $\hat{\sigma}_B^2$. Covariance may be expressed as

$$\operatorname{cov}(\hat{\sigma}_C^2, \hat{\sigma}_B^2) = \frac{1}{2} \left[ \operatorname{var}(\hat{\sigma}_C^2 + \hat{\sigma}_B^2) - \operatorname{var}(\hat{\sigma}_C^2) - \operatorname{var}(\hat{\sigma}_B^2) \right]. \quad (4.28)$$

From (4.1) we have that

$$\operatorname{var}(\hat{\sigma}_C^2 + \hat{\sigma}_B^2) = \operatorname{var} \left[ \frac{Q_3}{N-\Sigma b_i} + \frac{1}{K_1} \left( \frac{Q_2}{\Sigma b_i - a} - \frac{Q_3}{\Sigma b_i} \right) \right]$$

$$= \frac{1}{K_1} \left[ \frac{(K_1 - 1)^2}{(\Sigma b_i)^2} \operatorname{var}(Q_3) + \frac{\operatorname{var}(Q_2)}{(\Sigma b_i - a)^2} \right]. \quad (4.29)$$

Substitution of $\operatorname{var}(\hat{\sigma}_C^2 + \hat{\sigma}_B^2)$ as given by (4.29), of $\operatorname{var}(\hat{\sigma}_C^2)$ and $\operatorname{var}(\hat{\sigma}_B^2)$ as given by (4.2) into (4.28) yields

$$\operatorname{cov}(\hat{\sigma}_C^2, \hat{\sigma}_B^2) = -\frac{\operatorname{var}(Q_3)}{(N-\Sigma b_i)^2 K_1}. \quad (4.30)$$

Proceeding in a similar manner the other two covariances are obtained.
They are:

$$\text{cov}(\hat{o}_C, \hat{o}_A) = -\frac{(K_1-K_1')}{(N-U b_i)K_1K_2} \text{var}(Q_2) \quad (4.31)$$

$$\text{cov}(\hat{o}_B, \hat{o}_A) = -\frac{1}{K_1K_2} \left[ \frac{K_1' \text{var}(Q_2)}{(\Sigma b_i-a)^2} + \frac{(K_1'-K_1) \text{var}(Q_2)}{(N-U b_i)^2} \right] + \frac{\text{cov}(Q_1Q_2)}{K_1K_2(\Sigma b_i-a)(a-1)} \quad (4.32)$$
5.0 OPTIMAL DESIGNS FOR ESTIMATING PARAMETERS FOR A THREE STAGE NESTED CLASSIFICATION

5.1 Introduction

Crump (1954) considered a two stage nested classification where the data can be represented by the model (1.1). A form of an analysis of variance for data obeying (1.1) is given in Table 1.1. He indicated the optimum designs for estimating \( m, \sigma_B^2, \sigma_A^2 \) and \( \rho = \sigma_A^2 / \sigma_B^2 \) for a fixed total sample of \( N \), when the usual analysis of variance estimators are used. For estimating \( m \) the optimum design has \( a = N \) with \( n_1 = 1 \), and for estimating \( \sigma_B^2 \) it has \( a = 1 \) with \( n_1 = N \).

The estimator of \( \sigma_A^2 \) is given as

\[
\hat{\sigma}_A^2 = \frac{M_A - M_B}{K_1}
\]  

(5.1)

with variance

\[
\text{var}(\hat{\sigma}_A^2) = 2 \left[ \frac{N \sum n_i^2 - 2N \sum n_i^3 + (\sum n_i^2)^2}{(N^2 - \sum n_i^2)^2} \frac{1}{\sigma_A^2} \right] + \frac{2N}{N^2 - \sum n_i^2} \sigma_A^2 \sigma_B^2 + \frac{N^2(N-1)(a-1)}{(N-a)(N^2 - \sum n_i^2)^2} \sigma_B^4 .
\]  

(5.2)

For fixed \( a \) the variance of \( \hat{\sigma}_A^2 \) is minimized when \( n_1 = \frac{N}{a} \) \( (i = 1, 2, \ldots, a) \).

When the \( n_i \) are equal,

\[
\text{var}(\hat{\sigma}_A^2) = \frac{2}{N^2} \left[ \frac{(a \sigma_B^2 + N \sigma_A^2)^2}{a-1} + \frac{(a \sigma_B^2)^2}{N-a} \right] .
\]  

(5.3)
Var($\hat{\sigma}_A^2$) is minimized by taking the number of classes to be

$$a_1 = \frac{N(N\rho + 2)}{N\rho + N + 1}.$$  \hspace{1cm} (5.4)

For the situation where $N/a$ is not an integer but

$$N/a = p + s/a, \quad 0 < s < a,$$

(5.5)

Crump proved that var($\hat{\sigma}_A^2$) is minimized by putting $p+1$ units in each of $s$ classes and $p$ units in each of $a-s$ classes. He hypothesized the following procedure to determine a design

1) find $a_1$ from (5.4)

ii) if $a_1$ is nonintegral, choose as $a$ the closest integer to $a_1$

and allocate as given above: $[p+1$ units in each of $s$ classes and $p$ units in each of $a-s$ classes].

Because this procedure may not always give the minimum variance, he suggested checking both the closest integer above and below $a_1$.

As an estimator of $\rho$, Crump used the unbiased estimator

$$\hat{\rho} = \frac{N}{N^2 - \Sigma n_i^2} \left[ \frac{(N-a-2)(a-1)}{N-a} \frac{M_A}{M_B} - (a-1) \right].$$  \hspace{1cm} (5.6)

The variance of this estimator is

$$\text{var}(\hat{\rho}) = \frac{2}{N-a-4} \left\{ \begin{array}{c}
(N-a-2) \frac{N^2 \Sigma n_i^2 - 2N \Sigma \Sigma n_i^3 + (\Sigma n_i^2)^2}{(N^2 - \Sigma n_i^2)^2} \\
+ \frac{2N(N-3)}{N^2 - \Sigma n_i^2} \rho + \frac{N^2(N-3)(a-1)}{(N^2 - \Sigma n_i^2)^2} \end{array} \right\}.$$ \hspace{1cm} (5.7)
Crump showed that \( \text{var}(\hat{\rho}) \) is also minimized for fixed \( a \) by having the \( n_i \) equal. When the \( n_i \) are equal,

\[
\text{var}(\hat{\rho}) = \frac{2(N-3)}{(N-a-4)(a-1)} \left[ \frac{a}{N} + \rho \right]^2, \tag{5.8}
\]

and \( \text{var}(\hat{\rho}) \) is minimized by taking the number of classes to be

\[
a_2 = 1 + \frac{(N-5)(Np+6)}{2N \rho + N-3}. \tag{5.9}
\]

If \( a_2 \) is nonintegral he suggested having \( a \) classes, where \( a \) is the closest integer to \( a_2 \), with \( p+1 \) units in each of \( a \) classes and \( p \) units in each of \( a \) classes.

In the following sections we shall consider analogous problems of design for a three stage nested sampling plan, where an observation can be represented by (1.2). The estimators used will be the same type as considered by Crump, i.e., the usual analysis of variance estimators.

5.2 Estimation of Parameters for a Three Stage Nested Classification

5.2.1 Estimation of \( m \). An unbiased estimator of \( m \) is given by

\[
\hat{m} = \frac{\Sigma \Sigma \Sigma x_{i,j,k}}{N} \tag{5.10}
\]

with variance

\[
\text{var}(\hat{m}) = \frac{\Sigma n_i^2 \sigma_A^2}{N} + \frac{\Sigma \Sigma n_{i,j}^2 \sigma_B^2}{N^2} + \frac{\sigma_C^2}{N}. \tag{5.11}
\]

It is obvious that \( \text{var}(\hat{m}) \) is a minimum when \( n_i = n_{i,j} = 1 \) for all \( i \) and \( j \). Hence the optimum design has one unit in each of \( N \) \( A \)-classes.
5.2.2 Estimation of $\sigma^2_C$. An unbiased estimator of $\sigma^2_C$ is given by (4.1) and the variance of the estimator is given in Section 4.4. It is obvious that $\text{var}(\hat{\sigma}^2_C)$ is a minimum when $\sum_i b_i = b$ is a minimum, i.e., when $a = b_i = 1$ and $n_{ij} = N$. Hence the optimum design has all $N$ units in one B-class in one A-class.

5.2.3 Estimation of $\sigma^2_B$. An unbiased estimator of $\sigma^2_B$ is given by (4.1) and the variance of the estimator in Section 4.4. For fixed $a$ and $b$ it can be shown that $\text{var}(\hat{\sigma}^2_B)$ attains a minimum when $n_i = \frac{N}{a}$ and $n_{ij} = \frac{N}{b}$ for all $i$ and $j$. When there is equal allocation to the A-classes and to the B-classes, $\hat{\sigma}^2_B = b(M_B - M_C)/N$ and $\text{var}(\hat{\sigma}^2_B)$ is given by

$$\text{var}(\hat{\sigma}^2_B) = \frac{2}{N^2} \left[ \frac{(b \sigma^2_C + N \sigma^2_B)^2}{b-a} + \frac{(b \sigma^2_B)^2}{N-b} \right].$$ (5.12)

Obviously the value of $a$ that minimizes the above expression is $a = 1$. When $a = 1$, $\text{var}(\hat{\sigma}^2_B)$ has exactly the same form as that given by (5.3) for $\text{var}(\hat{\sigma}^2_A)$ in the two stage nested plan. Hence the suggested procedure for obtaining the design that minimizes $\text{var}(\hat{\sigma}^2_B)$ is to choose $a = 1$ and proceed as described in Section 5.1 with $\sigma^2_B$ substituted for $\sigma^2_A$ and $\sigma^2_C$ for $\sigma^2_B$.

5.2.4 Estimation of $\sigma^2_A$. An unbiased estimator of $\sigma^2_A$ is given by (4.1) and the variance of the estimator in Section 4.4. We have not been able to prove for fixed $a$ and $b$ that $\text{var}(\hat{\sigma}^2_A)$ is minimized when a completely balanced design is used, i.e., when $n_i = N/a$ is an integer, and $n_{ij} = N/b$ is an integer for all $i$ and $j$. However, it does appear that a balanced design is sufficiently close to optimal for the estimation
of \( \sigma_A^2 \) to warrant its consideration. Also, analytical treatment of the expression for \( \text{var}(\sigma_A^2) \) as given in Section 4.4 appears to be very difficult. Because a balanced design appears to be near optimal and because the expression for \( \text{var}(\sigma_A^2) \) from a balanced design is in a form that can be manipulated, we shall first consider designs for which \( n_i = N/a, \) \( n_j = N/b \) and both are integers for all \( i \) and \( j \). With a balanced design \( \sigma_A^2 = \frac{a}{N} (M_A - M_B) \) and \( \text{var}(\sigma_A^2) \) may be expressed as

\[
\text{var}(\sigma_A^2) = \frac{2a^2}{N^2} \left[ \frac{(\sigma_C^2 + \frac{N}{b} \sigma_B^2 + \frac{N}{a} \sigma_A^2)^2}{a - 1} + \frac{(\sigma_C^2 + \frac{N}{b} \sigma_B^2)^2}{b - a} \right].
\]  

(5.13)

It is obvious that \( \text{var}(\sigma_A^2) \) is a minimum when \( b \) is a maximum, that is when \( b = N \). When \( b = N \), the expression for \( \text{var}(\sigma_A^2) \) reduces to

\[
\text{var}(\sigma_A^2) = \frac{2}{N^2} \left\{ \frac{a \left( \sigma_C^2 + \sigma_B^2 + N \sigma_A^2 \right)^2}{a - 1} + \frac{a \left( \sigma_C^2 + \sigma_B^2 \right)^2}{N - a} \right\}.
\]  

(5.14)

The expression for \( \text{var}(\sigma_A^2) \) given above is in exactly the same form as that given by (5.3) for \( \text{var}(\sigma_A^2) \) in a two stage nested plan. Hence, the procedure suggested for obtaining a design is to set \( b = N \) and then proceed as described in Section 5.1 with \( \sigma_C^2 + \sigma_B^2 \) substituted for \( \sigma_B^2 \).

5.2.5 Estimation of \( \sigma_C^2 + \sigma_B^2 + \sigma_A^2 \). An unbiased estimator, \( \sigma^2 \), of \( \sigma^2 = \sigma_C^2 + \sigma_B^2 + \sigma_A^2 \) is

\[
\sigma^2 = \frac{N}{\Sigma_{i=1}^{N} (x_{i\text{ll}} - \bar{x})^2/N - 1},
\]  

(5.15)

where

\[
\bar{x} = \frac{\Sigma_{i=1}^{N} x_{i\text{ll}}}{N}.
\]
The variance of this estimator is

\[
\text{var}(\hat{\rho}^2) = 2(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)^2/(N-1) = 2\sigma^4/(N-1). \tag{5.16}
\]

Since this variance attains the lower bound as given by Gaylor (1960) the optimal design has \(a = b = N\).

5.2.6 Estimation of \(\rho = \sigma_A^2/(\sigma_C^2 + \sigma_B^2)\). For designs which are not balanced the estimator of \(\rho\) would be exceedingly complicated. Because the components \(\sigma_C^2\) and \(\sigma_B^2\) appear in \(\rho\) only as a simple sum, \(\sigma_C^2 + \sigma_B^2\), there is no need to provide separate estimates of them. Hence, it appears that the appropriate design will be similar to the type suggested for the estimation of \(\sigma_A^2\) insofar as only one unit should be assigned to each B-class. When \(b = N\), an unbiased estimator of \(\rho\) is

\[
\hat{\rho} = \frac{N}{N^2 - \Sigma n_i^2} \left[ (N-a-s) \frac{Q_1}{Q_2} - (a-l) \right]; \tag{5.18}
\]

the variance of this estimator is

\[
\text{var}(\hat{\rho}) = \frac{2}{N-a-l} \left\{ \left[ (N-a-2) \frac{N^2 \Sigma n_i^2 - 2N \Sigma n_i^3 + (\Sigma n_i^2)^2}{(N^2 - \Sigma n_i^2)^2} + 1 \right] \rho^2 + \frac{2N(N-3)}{N^2 - \Sigma n_i^2} \rho + \frac{N^2(N-3)(a-l)}{(N^2 - \Sigma n_i^2)^2} \right\} \tag{5.19}
\]

We note that \(\text{var}(\hat{\rho})\) as given by (5.19) is in the same form as that given by (5.7). Hence the procedure for obtaining a design is to let \(b = N\) and then proceed as described in Section 5.1 where the estimation of \(\rho\) is discussed for a two stage nested classification.
6.0 SOME ASPECTS OF THE SIMULTANEOUS ESTIMATION OF VARIANCE COMPONENTS FOR A THREE STAGE NESTED CLASSIFICATION

6.1 Introduction

In this chapter we shall be concerned with the designs of experiments for the simultaneous estimation of variance components where a result or observation may be represented by (1.2). An analysis of variance for data obeying (1.2) is given in Table 4.1. As before, only estimators of the variance components based on the usual analysis of variance will be considered.

The designs presented in Chapter 5 are for the estimation of single functions of the variance components. Unfortunately, a design which is presented for the estimation of a single component may be very poor for, or may actually preclude the estimation of, other components. For example, the design that is suggested for the estimation of \( \sigma_A^2 \) (see Section 5.2.4) does not allow the separate estimation of \( \sigma_B^2 \) and \( \sigma_C^2 \); only the sum \( \sigma_C^2 + \sigma_B^2 \) is estimable.

Presently it is the usual practice to design balanced experiments for the joint estimation of the three variance components. There are several reasons for designing balanced experiments, among them are:

(1) The calculation of the sums of squares, and the coefficients of the components in the expected mean squares are somewhat easier for data collected from a balanced than from an unbalanced plan.

(2) Evaluation of the precision of the estimates from an unbalanced plan is vastly more complicated than from a balanced
plan. With a balanced design and under the assumptions of (1.2), the sums of squares associated with the sources of variation are uncorrelated and each is distributed as chi square. The estimators $\hat{\sigma}_A^2$ and $\hat{\sigma}_B^2$, given by (4.1) are linear functions of uncorrelated chi square variables and thus the variances of these estimators are relatively easy to obtain.

For the general unbalanced design the sum of squares associated with sources A and B are correlated and neither is distributed as chi square. Hence, the variances of $\hat{\sigma}_A^2$ and $\hat{\sigma}_B^2$ are very tedious to derive. This may be appreciated by consulting Chapter 4.

(3) With an assignment of degrees of freedom for which a balanced design exists, the allocation of units to classes is uniquely specified, viz., equal allocation to the classes. For an assignment for which a balanced design does not exist there is a multitude of possible allocations.

Unfortunately, requiring that a plan be balanced greatly restricts the manner in which degrees of freedom may be assigned to the sources of variation. Particularly bad is the circumstance that one may assign, at most, one-half of the available degrees of freedom to sources A and B, together, while to source C must be assigned at least one-half of the available degrees of freedom. Thus, if $\sigma_A^2$ and $\sigma_B^2$ are large and important compared to $\sigma_C^2$, one is forced to estimate $\sigma_A^2$ and $\sigma_B^2$ with too little precision. Therefore, it appears that in situations of the type just described it may be desirable to sacrifice balance in order to
gain precision in the estimates of $\sigma_A^2$ and $\sigma_B^2$. However, as soon as an assignment of degrees of freedom is proposed for which a balanced design does not exist, one is faced with the dilemma of choosing a method for allocating units to classes. Contrary to a two stage nested plan where units are allocated as equally as possible to the classes [see Crump (1954)] there is no obvious method for allocating units with a three stage nested classification.

Designs have been suggested by Anderson and Bancroft (1952), Anderson (1960) and by Calvin and Miller (1961) for the situation where the degrees of freedom are assigned as $n-1$, $n$ and $n$ to the sources A, B and C, respectively.

The purpose of this chapter will be to study methods of designing experiments for the simultaneous estimation of variance components for a three stage nested classification. It is not intended that a final solution to this problem of design will be presented here; hopefully, this work will be a starting point for future research.

6.2 Designs for Estimating Variance Components for a Three Stage Nested Classification

Anderson's so-called staggered design was proposed for a five stage nested classification where the assignment of degrees of freedom to sources of variation was $n-1$ to the main classes and $n$ to each of the subclasses. In this chapter the staggered design will be considered for a three stage nested classification.

The design would be composed of two basic configurations, as illustrated below:
Type (1) has two C-classes in each B-class and two B-classes in each A-class. Type (2) has one C-class in each B-class and two B-classes in each A-class. The design would consist of \( n/2 \) configurations of type (1), and \( n/2 \) of type (2). For this design, as with the usual unbalanced design, the sums of squares due to A-classes and due to B-classes are not distributed as chi square. However, unlike the usual unbalanced design, the sums of squares due to A-classes and due to B-classes are uncorrelated. For this particular situation, the sum of squares due to A-classes may be partitioned into three independent parts; \( A_1 \), \( A_2 \) and \( A_1 \) vs. \( A_2 \) with respective degrees of freedom: \( (n-2)/2 \), \( (n-2)/2 \) and 1. The sum of squares due to B-classes may be partitioned into two independent parts, B in \( A_1 \), and B in \( A_2 \), each with \( n/2 \) degrees of freedom. The usual analysis of variance for the Anderson design with the above described partition is given in Table 6.1.

From Table 6.1 two analysis of variance type estimators of \( \sigma_A^2 \) are suggested. The first is \( \hat{\sigma}_A^2 \) as given by (4.1). The second is the weighted sum

\[
\hat{\sigma}_A^2 = \frac{W_1 \hat{\sigma}_{A_1}^2 + W_2 \hat{\sigma}_{A_2}^2}{W_1 + W_2}, \tag{6.1}
\]

where

\[
\hat{\sigma}_{A_1}^2 = \left[ 2Q_{11}/(n-2) - 2Q_{21}/n \right]/4,
\]

\[
\hat{\sigma}_{A_2}^2 = \left[ 2Q_{12}/(n-2) - 2Q_{22}/n \right]/2.
\]
Table 6.1. The usual analysis of variance for the Anderson design

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>A classes</td>
<td>n-1</td>
<td>$Q_1/(n-1)$</td>
<td>$\sigma_C^2 + \frac{2n-10}{6(n-1)} \sigma_B^2 + \frac{9n-10}{3(n-1)} \sigma_A^2$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>(n-2)/2</td>
<td>$2Q_{11}/(n-2)$</td>
<td>$\sigma_C^2 + 2\sigma_B^2 + 4\sigma_A^2$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(n-2)/2</td>
<td>$2Q_{12}/(n-2)$</td>
<td>$\sigma_C^2 + \sigma_B^2 + 2\sigma_A^2$</td>
</tr>
<tr>
<td>$A_1$ vs. $A_2$</td>
<td>1</td>
<td>$Q_{13}$</td>
<td>$\sigma_C^2 + \frac{4}{3} \sigma_B^2 + \frac{8}{3} \sigma_A^2$</td>
</tr>
<tr>
<td>B in A classes</td>
<td>n</td>
<td>$Q_2/n$</td>
<td>$\sigma_C^2 + \frac{3}{2} \sigma_B^2$</td>
</tr>
<tr>
<td>B in $A_1$ ($B_1$)</td>
<td>n/2</td>
<td>$2Q_{21}/n$</td>
<td>$\sigma_C^2 + 2\sigma_B^2$</td>
</tr>
<tr>
<td>B in $A_2$ ($B_2$)</td>
<td>n/2</td>
<td>$2Q_{22}/n$</td>
<td>$\sigma_C^2 + \sigma_B^2$</td>
</tr>
<tr>
<td>Within B classes</td>
<td>n</td>
<td>$Q_3/n$</td>
<td>$\sigma_C^2$</td>
</tr>
<tr>
<td>Total</td>
<td>3n-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and $W_1$ and $W_2$ are predetermined weights. Because $\sigma_A^2$ is a linear function of uncorrelated chi square variables, it would be relatively easy to derive its variance. The difficulties with this estimator are that it does not use the information provided by the one degree of freedom due to $A_1$ vs. $A_2$, and that there is no obvious way to choose the weights $W_1$ and $W_2$. Anderson (1961) has suggested use of iterated linear estimates (linear in the mean squares) for such situations where the weights for the $j+1$st iteration are based on the estimates obtained from the $j$th iteration.
As an alternative to be compared with the staggered design a second design will be considered. For the assignment of degrees of freedom n-1, n and n this design will consist of n configurations of the type

That is, there are two C-classes in one B-class and one C-class in the other B-class and there are two B-classes in each A-class. This is the design given by Calvin and Miller (1961) for the three stage nested classification.

The usual analysis of variance for such a design is given in Table 6.2. The estimator of $\sigma_A^2$ is

$$\frac{\sigma_A^2}{n} = \left[ \frac{Q_1}{n-1} - \frac{5}{4} \frac{Q_2}{n} + \frac{Q_3}{4n} \right] / 3. \quad (6.2)$$

Unlike the circumstance for the Anderson design, the sums of squares due to A-classes and due to B-classes are each distributed as chi square. However, contrary to the result given by Calvin and Miller, the sums of squares due to the A-classes and due to the B-classes are correlated. Referring to (4.27) the covariance between the two mean squares $Q_1/(n-1)$ and $Q_2/n$ is calculated to be

$$\text{cov} \left[ \frac{Q_1}{n-1}, \frac{Q_2}{n} \right] = \frac{4}{9n} \sigma_B^4. \quad (6.3)$$

Therefore, the variance, as given by (4.2), of the estimator $\sigma_A^2$ from such a design would be relatively easy to obtain. Using the fact that $Q_1, Q_2$ and $Q_3$ are distributed as chi square, along with the result for
Table 6.2. The usual analysis of variance for the Calvin-Miller design

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>A classes</td>
<td>n-1</td>
<td>( Q_1/(n-1) )</td>
<td>( \sigma_C^2 + 5/3 \sigma_B^2 + 3\sigma_A^2 )</td>
</tr>
<tr>
<td>B in A classes</td>
<td>n</td>
<td>( Q_2/n )</td>
<td>( \sigma_C^2 + 4/3 \sigma_B^2 )</td>
</tr>
<tr>
<td>Within B classes</td>
<td>n</td>
<td>( Q_3/n )</td>
<td>( \sigma_C^2 )</td>
</tr>
<tr>
<td>Total</td>
<td>3n-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the covariance of \( Q_1/(n-1) \) and \( Q_2/n \) as given by (6.3), the variance of \( \sigma_A^2 \) is given as

\[
\frac{\text{var}(\sigma_A^2)}{2\sigma_C^2} = \frac{1}{9} \left[ \left( 1 + \frac{5}{3} \rho_2 + 3\rho_1 \right)^2 \frac{n-1}{n} + \frac{25}{16} \left( 1 + \frac{4}{3} \rho_2 \right)^2 \frac{n}{n} + \frac{1}{16n} - \frac{5}{9} \rho_2^2 \right],
\]

(6.4)

where \( \rho_1 = \sigma_A^2/\sigma_C^2 \) and \( \rho_2 = \sigma_B^2/\sigma_C^2 \).

For assignments of degrees of freedom other than n-1, n and n, where balance cannot be achieved, there is no known procedure for constructing the best design. We shall consider a specific procedure for constructing a design, called the \( D_2 \)-design:

1. Write

\[
N/a = q_1 + r_1/a \quad 0 \leq r_1 < a
\]

and assign \( q_1 + 1 \) units to each of \( r_1 \) A-classes (this group of A-classes will be designated by \( C_1 \)) and \( q_1 \) units to each of the remaining \( a - r_1 \) A-classes (designated by \( C_2 \)).
(2) Write
\[ \frac{b}{a} = q_2 + \frac{r_2}{a}, \quad 0 \leq r_2 < a. \]

To each A-class assign \( q_2 \) B-classes and then one extra B-class to each of \( r_2 \) A-classes. Make sure that \( b > a \).

(3) Within each A-class, assign the units to the B-classes as equally as possible.

It may be noted that when \( b \geq N/2 \) and a \( D_2 \)-design is constructed, the design will usually satisfy the relations
\[
|n_i - n_j| = 0 \text{ or } 1 \quad (6.5)
\]
\[ |n_{ij} - n_{jm}| = 0 \text{ or } 1. \]

For the assignment of degrees of freedom \( n-1 \), \( n \) and \( n \) this procedure will yield a design which consists of \( n \) configurations of the type

```
  |
  |
  |
  |
```

and is the same as the Calvin-Miller design mentioned previously.

In order to gain insight as to how the \( D_2 \)-design compares with some other methods of design, including the Anderson design where the comparison is feasible; it was decided to make use of some numerical examples. For these examples, a sample size of \( N = 48 \) was selected. Eight assignments of degrees of freedom were chosen. They were:
<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>7</td>
<td>15</td>
<td>23</td>
<td>7</td>
<td>15</td>
<td>23</td>
<td>31</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>16</td>
<td>8</td>
<td>32</td>
<td>24</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>16</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

It is noted for the assignment (11, 12, 24) that a completely balanced design may be constructed. This particular assignment was included as a reference point. The eight assignments listed above are intended to encompass the range of assignments for which \( b \geq N/2 \). No assignments where \( b < N/2 \) were considered. There were two reasons for this:

1. When \( b < N/2 \), there are many assignments for which a balanced design could be constructed.

2. We are mainly concerned with the situation where \( \sigma_A^2 \) and \( \sigma_B^2 \) are large and important compared to \( \sigma_C^2 \); therefore, we presumably are not interested in designs where more than half the available degrees of freedom are assigned to source C.

Two designs were considered for each allocation of degrees of freedom. The first was constructed by dividing the A-classes into groups and then assigning the B-classes and C-classes so that balance would be maintained within each group. To simplify presentation, this design will be referred to as a \( D_1 \)-design. For those assignments where it was possible, a design of the type proposed by Anderson was constructed. Care was taken so that a \( D_1 \)-design was not given that was the same as a \( D_2 \)-design. The designs considered for each of the assignments (2)-(8) are given below: (A, B in A, and C in B in A with the number of A-classes for each design indicated below the design.)
(2) $7, 24, 16$

\[ D_1: \]
\[ 4 \]
\[ D_2: \]
\[ 8 \]

(3) $15, 16, 16$

\[ D_1: \]
\[ 8 \]
\[ D_2: \]
\[ 16 \]

(4) $23, 8, 16$

\[ D_1: \]
\[ 8 \]
\[ D_2: \]
\[ 16 \]

(5) $7, 32, 8$

\[ D_1: \]
\[ 4 \]
\[ D_2: \]
\[ 8 \]
(6) 15, 24, 8

\[ D_1: \]

\[ D_2: \]

\[ 4 \]

\[ 8 \]

(7) 23, 16, 8

\[ D_1: \]

\[ D_2: \]

\[ 4 \]

\[ 12 \]

\[ 8 \]

\[ 8 \]

\[ 16 \]

(8) 31, 8, 8

\[ D_1: \]

\[ D_2: \]

\[ 4 \]

\[ 4 \]

\[ 24 \]

\[ 8 \]

\[ 8 \]

\[ 16 \]

The usual analysis of variance estimators given by (4.1), with one exception, were the only estimators considered in this study. Criteria
used to compare the designs were the quantities,

\[ E_A = \frac{\text{var}(\sigma_A^2) \text{ for the assignment in question}}{\text{var}(\sigma_A^2) \text{ for } (11,12,24)} \]

\[ E_B = \frac{\text{var}(\sigma_B^2) \text{ for the assignment in question}}{\text{var}(\sigma_B^2) \text{ for } (11,12,24)} \]

\[ E_C = \frac{\text{var}(\sigma_C^2) \text{ for the assignment in question}}{\text{var}(\sigma_C^2) \text{ for } (11,12,24)} \]

The results given in Section 4.4 were used for the calculation of \( \text{var}(\sigma_A^2), \text{var}(\sigma_B^2) \) and \( \text{var}(\sigma_C^2) \). As the above criteria are functions of the parameters, \( \rho_1 = \sigma_A^2/\sigma_C^2 \) and \( \rho_2 = \sigma_B^2/\sigma_C^2 \), it was necessary to specify numerical values for these parameters. The 49 combinations of \( \rho_1 \) and \( \rho_2 \) each equal to 1/10, 1/4, 1/2, 1, 2, 4 and 10 were selected.

The numerical results are presented in Tables 6.3-6.12. Study of the tables indicates, as expected, that the manner in which the degrees of freedom are assigned to the sources of variation has a large effect on the variances of the estimated variance components. For example, with \( \rho_1 = 10 \) and \( \rho_2 = 2 \) the non-balanced assignment \((31, 8, 8)\) yields for the two designs \( D_1 \) and \( D_2 \), respectively, \( E_A = .59 \) and \( .53 \), \( E_B = 1.88 \) and \( 2.35 \) and \( E_C = 3.00 \). In this situation, the assignment \((31, 8, 8)\) naturally is much better than \((11, 12, 24)\) for the estimation of \( \sigma_A^2 \), much poorer for the estimation of \( \sigma_B^2 \) and very much poorer for the estimation of \( \sigma_C^2 \).

On the basis of the results some general observations can be made:

(1) The result in Tables 6.4 and 6.5 can be used to compare the Anderson principle of design and the \( D_2 \)-design. From comparison of the results in Table 6.4 with those in Table 6.5 it is noted that the \( D_2 \)-design does better relative to the Anderson design at the assignment \((7,24,16)\) than at the assignment \((15,16,16)\).
Table 6.3. Values of $V_A$, $V_B$ and $V_C$ for the balanced design with degrees of freedom: 11, 12, 24\(^a\)/

Values of $V_A$ for $\rho_2$ equal

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>1/10</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>.0220</td>
<td>.0322</td>
<td>.0536</td>
<td>.113</td>
<td>.296</td>
<td>.924</td>
<td>4.90</td>
</tr>
<tr>
<td>1/4</td>
<td>.0350</td>
<td>.0472</td>
<td>.0720</td>
<td>.138</td>
<td>.335</td>
<td>.990</td>
<td>5.05</td>
</tr>
<tr>
<td>1/2</td>
<td>.0657</td>
<td>.0813</td>
<td>.112</td>
<td>.189</td>
<td>.409</td>
<td>1.11</td>
<td>5.30</td>
</tr>
<tr>
<td>1</td>
<td>.161</td>
<td>.184</td>
<td>.225</td>
<td>.325</td>
<td>.590</td>
<td>1.38</td>
<td>5.84</td>
</tr>
<tr>
<td>2</td>
<td>.488</td>
<td>.525</td>
<td>.589</td>
<td>.734</td>
<td>1.09</td>
<td>2.06</td>
<td>7.08</td>
</tr>
<tr>
<td>4</td>
<td>1.69</td>
<td>1.75</td>
<td>1.86</td>
<td>2.10</td>
<td>2.64</td>
<td>3.97</td>
<td>10.1</td>
</tr>
<tr>
<td>10</td>
<td>9.65</td>
<td>9.80</td>
<td>10.0</td>
<td>10.6</td>
<td>11.6</td>
<td>14.1</td>
<td>23.4</td>
</tr>
</tbody>
</table>

$V_B$ = .0404, .0573, .0937, .198, .531, 1.70, 9.20

$V_C = .0417$

\(^a\)/$V_A = \text{var}(\hat{\sigma}_A^2)/2\sigma_C^4$

$V_B = \text{var}(\hat{\sigma}_B^2)/2\sigma_C^4$

$V_C = \text{var}(\hat{\sigma}_C^2)/2\sigma_C^4$
Table 6.4. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: 7, 24, 16

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$1/10$</th>
<th>$1/4$</th>
<th>$1/2$</th>
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\[ E_B = \begin{array}{c} 1.30 \\ 1.39 \end{array} \]

\[ E_C = \begin{array}{c} .91 \\ .96 \end{array} \]

\[ E_A = \begin{array}{c} .75 \\ .77 \end{array} \]

\[ E_A = \begin{array}{c} .65 \\ .65 \end{array} \]

\[ E_A = \begin{array}{c} .59 \\ .59 \end{array} \]

\[ E_A = \begin{array}{c} .57 \\ .57 \end{array} \]

\[ \frac{E_A}{E_A} = \frac{\text{var}(\sigma_A^2) \text{ for } (7,24,16)}{\text{var}(\sigma_A^2) \text{ for } (11,12,24)} \]

\[ E_B = \frac{\text{var}(\sigma_B^2) \text{ for } (7,24,16)}{\text{var}(\sigma_B^2) \text{ for } (11,12,24)} \]

\[ E_C = \frac{\text{var}(\sigma_C^2) \text{ for } (7,24,16)}{\text{var}(\sigma_C^2) \text{ for } (11,12,24)} \]
**Table 6.5. Values of \( E_A \), \( E_B \) and \( E_C \) for the \( D_1 \) and \( D_2 \) designs with degrees of freedom: 15, 16, 16**

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**Values**

\[ E_B = 1.99 \]

\[ E_C = 1.50 \]

\[ a/E_A = \frac{\text{var}(\sigma_A^2 \text{ for } (15,16,16))}{\text{var}(\sigma_A^2 \text{ for } (11,12,24))} \]

\[ E_B = \frac{\text{var}(\sigma_B^2 \text{ for } (15,16,16))}{\text{var}(\sigma_B^2 \text{ for } (11,12,24))} \]

\[ E_C = \frac{\text{var}(\sigma_C^2 \text{ for } (15,16,16))}{\text{var}(\sigma_C^2 \text{ for } (11,12,24))} \]
Table 6.6. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: 23, 8, 16.

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\[ E_C = 1.50 \]

\[ E_A = \frac{\text{var}(\hat{\sigma}_A^2) \text{ for } (23, 8, 16)}{\text{var}(\hat{\sigma}_A^2) \text{ for } (11, 12, 24)} \]

\[ E_B = \frac{\text{var}(\hat{\sigma}_B^2) \text{ for } (23, 8, 16)}{\text{var}(\hat{\sigma}_B^2) \text{ for } (11, 12, 24)} \]

\[ E_C = \frac{\text{var}(\hat{\sigma}_C^2) \text{ for } (23, 8, 16)}{\text{var}(\hat{\sigma}_C^2) \text{ for } (11, 12, 24)} \]
Table 6.7. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: 7, 32, 8

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</table>

$E_B$     |      |     |     |   |   |   |    |
| $D_1$    | 3.19 | 2.43| 1.71| 1.08| .70| .52| .44|
| $D_2$    | 2.97 | 2.27| 1.60| 1.02| .67| .51| .43|

$E_C = 3.00$

$E_A^\text{a/} = \left[\text{var}(\sigma^2_A) \text{ for } (7,32,8)\right]/\left[\text{var}(\sigma^2_A) \text{ for } (11,12,24)\right]$  

$E_B = \left[\text{var}(\sigma^2_B) \text{ for } (7,32,8)\right]/\left[\text{var}(\sigma^2_B) \text{ for } (11,12,24)\right]$  

$E_C = \left[\text{var}(\sigma^2_C) \text{ for } (7,32,8)\right]/\left[\text{var}(\sigma^2_C) \text{ for } (11,12,24)\right]$
Table 6.8. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: $15, 24, 8^a/6$

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<tr>
<td>$E_B$</td>
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<td>2.72</td>
<td>1.95</td>
<td>1.27</td>
<td>.66</td>
<td>.65</td>
<td>.56</td>
</tr>
</tbody>
</table>

$E_C = 3.00$

$^a/6E_A = \left[ \text{var}(\hat{\sigma}_A^2) \text{ for } (15, 24, 8) \right] / \left[ \text{var}(\hat{\sigma}_A^2) \text{ for } (11, 12, 24) \right]$

$E_B = \left[ \text{var}(\hat{\sigma}_B^2) \text{ for } (12, 24, 8) \right] / \left[ \text{var}(\hat{\sigma}_B^2) \text{ for } (11, 12, 24) \right]$

$E_C = \left[ \text{var}(\hat{\sigma}_C^2) \text{ for } (15, 24, 8) \right] / \left[ \text{var}(\hat{\sigma}_C^2) \text{ for } (11, 12, 24) \right]$
Table 6.9. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: 23, 16, 8\(^a\)

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>1/10</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.09</td>
<td>1.02</td>
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<td>1.94</td>
<td>1.65</td>
<td>1.44</td>
<td>1.30</td>
<td>1.20</td>
</tr>
<tr>
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<td>1.28</td>
<td>1.20</td>
<td>1.12</td>
<td>1.04</td>
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<td>.95</td>
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<td>1.35</td>
<td>1.25</td>
<td>1.19</td>
</tr>
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<td>1.05</td>
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<td>1.01</td>
<td>.98</td>
<td>.95</td>
<td>.94</td>
</tr>
<tr>
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<td>1.32</td>
<td>1.30</td>
<td>1.29</td>
<td>1.26</td>
<td>1.22</td>
<td>1.19</td>
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<td>.61</td>
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<td>.75</td>
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<td>.62</td>
<td>.62</td>
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<td>$D_2$</td>
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<td>.51</td>
<td>.52</td>
<td>.52</td>
<td>.55</td>
<td>.59</td>
<td>.70</td>
</tr>
</tbody>
</table>

$E_B$  

| $E_B$   | 3.24 | 2.60 | 2.01 | 1.46 | 1.13 | .96 | .88 |  
| $D_2$   | 4.98 | 3.89 | 2.84 | 1.89 | 1.29 | .99 | .84 |  

$E_C = 3.00$

\[\frac{E_A}{\bar{E}_A} = \frac{\var(\hat{\sigma}_A^2) \text{ for } (23,16,8)}{\var(\hat{\sigma}_A^2) \text{ for } (11,12,24)}\]

\[E_B = \frac{\var(\hat{\sigma}_B^2) \text{ for } (23,16,8)}{\var(\hat{\sigma}_B^2) \text{ for } (11,12,24)}\]

\[E_C = \frac{\var(\hat{\sigma}_C^2) \text{ for } (23,16,8)}{\var(\hat{\sigma}_C^2) \text{ for } (11,12,24)}\]
Table 6.10. Values of $E_A$, $E_B$ and $E_C$ for the $D_1$ and $D_2$ designs with degrees of freedom: 31, 8, 88/

<table>
<thead>
<tr>
<th>$\rho_1$</th>
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<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
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<td>3.88</td>
<td>3.34</td>
<td>3.00</td>
<td>2.76</td>
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<td>2.22</td>
<td>2.23</td>
<td>2.24</td>
<td>2.30</td>
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<tr>
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<td>3.83</td>
<td>3.58</td>
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<td>3.04</td>
<td>2.84</td>
<td>2.71</td>
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<td>2.07</td>
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<tr>
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<td>2.59</td>
<td>2.59</td>
<td>2.59</td>
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<tr>
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<td>.79</td>
<td>.90</td>
<td>1.10</td>
<td>1.40</td>
<td>1.79</td>
</tr>
<tr>
<td>$D_2$</td>
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<td>.81</td>
<td>.89</td>
<td>1.05</td>
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<td>1.67</td>
<td>2.09</td>
</tr>
<tr>
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<td>.60</td>
<td>.62</td>
<td>.67</td>
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<td>1.00</td>
<td>1.44</td>
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<tr>
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<td>1.11</td>
<td>1.63</td>
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<tr>
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<td>.54</td>
<td>.55</td>
<td>.59</td>
<td>.67</td>
<td>.94</td>
</tr>
<tr>
<td>$D_2$</td>
<td>.43</td>
<td>.44</td>
<td>.45</td>
<td>.47</td>
<td>.53</td>
<td>.64</td>
<td>.99</td>
</tr>
</tbody>
</table>

| $E_B$ | 3.19 | 2.81 | 2.44 | 2.11 | 1.86 | 1.76 | 1.71 |
| $E_B$ | 6.83 | 5.58 | 4.33 | 3.16 | 2.35 | 1.91 | 1.66 |

$E_C = 3.00$

$E_A = \left[ \text{var}(\hat{\sigma}^2_A) \text{ for (31, 8, 8)} \right] / \left[ \text{var}(\hat{\sigma}^2_A) \text{ for (11, 12, 24)} \right]$

$E_B = \left[ \text{var}(\hat{\sigma}^2_B) \text{ for (31, 8, 8)} \right] / \left[ \text{var}(\hat{\sigma}^2_B) \text{ for (11, 12, 24)} \right]$

$E_C = \left[ \text{var}(\hat{\sigma}^2_C) \text{ for (31, 8, 8)} \right] / \left[ \text{var}(\hat{\sigma}^2_C) \text{ for (11, 12, 24)} \right]$
For both $\rho_1$, $\rho_2 > 1$ there is not a large difference between the two designs. The $D_2$-design tends to be somewhat better when $\rho_2 < \rho_1$ and slightly poorer when $\rho_2 > \rho_1$. For example, at $\rho_1 = 10$ and $\rho_2 = 2$ the $D_2$-design is about 9 per cent better for the assignment $(7, 24, 16)$ and 8 per cent better for the assignment $(15, 16, 16)$ than the Anderson design. At $\rho_1 = 2$ and $\rho_2 = 10$ the Anderson design is about 3 per cent better for the assignment $(7, 24, 16)$ and 2 per cent better for the assignment $(15, 16, 16)$ than the $D_2$-design.

For both $\rho_1, \rho_2 \leq 1$ the Anderson design with the assignment $(15, 16, 16)$ is generally better than the $D_2$-design while with the assignment $(7, 24, 16)$ the two designs yield very similar results.

For $\rho_1 < 1$ and $\rho_2 > 1$ the Anderson design is generally a little better than the $D_2$-design. The greatest difference is 10 per cent with the assignment $(15, 16, 16)$ at $\rho_1 = 1/10$ and $\rho_2 = 2$.

For $\rho_1 > 1$ and $\rho_2 \leq 1$ the $D_2$-design is generally a little better than the Anderson design. The greatest difference is 10 per cent with the assignment $(7, 24, 16)$ at $\rho_1 = .10$ and $\rho_2 = 1/10$.

The $D_2$-design appears better for the estimation of $\sigma_B^2$ if $\rho_2 > 2$ and the Anderson design better if $\rho_2 < 2$.

(2) A comparison between the $D_1$-design and $D_2$-design for the assignments $(23, 8, 16)$, $(23, 16, 8)$ and $(31, 8, 8)$ should show the
same general characteristics. This is because the $D_1$-designs are very much alike for the three assignments, as are the $D_2$-designs. Consideration of Tables 6.6, 6.9 and 6.10 indicates a distinct pattern in the comparison between the $D_1$-design and $D_2$-design. The $D_2$-design tends to be better only for large $\rho_1 \geq 4$ with $(23,8,16)$ and $(31,8,8)$ and $\rho_1 \geq 2$ with $(23,16,8)$ and when $\rho_2 < \rho_1$. The difference between the two designs is the greatest with the assignment $(23,8,16)$. It may be noted that the two designs have no configurations in common with the assignment $(23,8,16)$, but do have some in common with the assignments $(23,16,8)$ and $(31,8,8)$.

For the estimation of $\sigma_B^2$ the $D_2$-design is always poorer with the single exception of $\rho_2 = 10$ for the assignment $(23,16,8)$.

(3) The $D_2$-design for the assignment $(7,32,8)$ is similar in its basic features to the $D_2$-design given for the assignments $(7,24,16)$ and $(15,16,16)$. However, the $D_1$-design is quite different from that given for the assignments $(7,24,16)$ and $(15,16,16)$. For the assignment $(7,32,8)$ the $D_2$-design is always better than the $D_1$-design for the estimation of both $\sigma_A^2$ and $\sigma_B^2$.

(4) Both the $D_1$-design and $D_2$-design for the assignment $(15,24,8)$ are quite similar in their basic configurations to the designs given for the assignment $(15,16,16)$. As expected,
a comparison between designs with the assignment \((15, 24, 8)\)
yields results for the estimation of both \(\sigma_A^2\) and \(\sigma_B^2\) that are
very similar to those obtained for the assignment \((15, 16, 16)\).
The major difference is that the \(D_2\)-design is somewhat better
relative to the \(D_1\)-design for \((15, 24, 8)\) than for \((15, 16, 16)\).

Consideration also may be given to selecting assignments
which are best in general. We will not consider situations
where \(\rho_1\) and \(\rho_2\) are both small, i.e., where both \(\rho_1, \rho_2 < 1\).

For both \(\rho_1, \rho_2 > 1\), it appears that the \(D_2\)-design with
the assignment \((15, 24, 8)\) is quite good. There are specific
combinations of \(\rho_1\) and \(\rho_2\) for which another assignment is
better, but generally the assignment \((15, 24, 8)\) does the best
job. In comparison with the balanced assignment \((11, 12, 24)\)
it is noted that \(E_A\) ranges from \(.66\) to \(.70\).

For \(\rho_1 \leq 1\) and \(\rho_2 > 1\) the \(D_2\)-design with the assignment
\((7, 32, 8)\) appears to be very good. In comparison with the
balanced assignment \((11, 12, 24)\) it is noted that \(E_A\) ranges
from \(.21\) to \(.65\).

For \(\rho_1 > 1\) and \(\rho_2 \leq 1\) the \(D_2\)-design with the assignment
\((23, 16, 8)\) appears to be very good. In comparison with the
balanced assignment \((11, 12, 24)\) it is noted that \(E_A\) ranges
from \(.51\) to \(.75\).

The assignments and designs mentioned above for the
estimation of \(\sigma_A^2\) are also quite good for the estimation of \(\sigma_B^2\).
when $\rho_2 \geq 1$. When $\rho_2$ is small, $\rho_2 < 1$, the assignment suggested to estimate $\sigma^2_A$ is rather poor for estimating $\sigma^2_B$. However, this may not be of particular concern as a good estimate of $\sigma^2_B$ may not be necessary when $\sigma^2_B$ is small compared to $\sigma^2_A$ and $\sigma^2_C$.

It is rather interesting to note that each of the three suggested designs is a $D_2$-design in combination with a given assignment of degrees of freedom.

One important point that evolves from this brief study of designs for the simultaneous estimation of variance components is the need for defining some meaningful criteria. It is apparent with regard to assignment of degrees of freedom that changes which increase the precision of one component invariably reduce the precision of one or both the other components. This often also is true with regard to the two different types of designs considered. Hence, an important problem is one of stating the objectives for which the estimates are to be used and then determining the experimental design that does the best job of accomplishing the stated objectives.

An actual experiment using a staggered design for the simultaneous estimation of variance components is given by Calvin and Miller (1961). The experiment was performed on field grown, flue-cured tobacco with five fertility treatments replicated four times. The response measured was the concentration of nitrogen in plants. Two groups of five plants each were randomly selected from each plot. From the first group one sample was drawn while from the second group duplicate samples were
drawn. One chemical determination was made on the sample from the first group and one was made on one sample from the second group. The other sample from the second group was analyzed in duplicate. Hence, there were twenty configurations of the type

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```

The analysis of variance obtained from the actual data is given in Table 6.11.

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Mean square</th>
<th>Expectation of mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>4</td>
<td>2.125084</td>
<td></td>
</tr>
<tr>
<td>Replications</td>
<td>3</td>
<td>0.163398</td>
<td></td>
</tr>
<tr>
<td>Treatments x replications</td>
<td>12</td>
<td>0.367238</td>
<td>$\sigma_C^2 + 3/2 \sigma_B^2 + 5/2 \sigma_A^2 + 4\sigma_A^2$</td>
</tr>
<tr>
<td>Plant groups in plots (A)</td>
<td>20</td>
<td>0.096897</td>
<td>$\sigma_C^2 + 7/6 \sigma_B^2 + 3/2 \sigma_A^2$</td>
</tr>
<tr>
<td>Field samples in plant groups (B)</td>
<td>20</td>
<td>0.050459</td>
<td>$\sigma_C^2 + 4/3 \sigma_B^2$</td>
</tr>
<tr>
<td>Determinations in field samples (C)</td>
<td>20</td>
<td>0.001767</td>
<td>$\sigma_C^2$</td>
</tr>
</tbody>
</table>
The estimates of the components of variance obtained were

\[ \hat{\sigma}_C^2 = .001767 \]
\[ \hat{\sigma}_B^2 = .03652 \]
\[ \hat{\sigma}_A^2 = .03502 \]
\[ \hat{\sigma}_{A_0}^2 = .05579 \]

To fit these results into the framework of this chapter we shall consider the results only from stages A, B and C. With a three stage nested classification the D$_2$-design which allocates the degree of freedom nearly equally to the sources is n configurations of the type

```
  /\   \
 / \  / \ 
|   |   |
```

as suggested by Calvin and Miller. To match nearly the experimental situation described by Calvin and Miller, let n = 20, and assign the degrees of freedom as (19,20,20). If we take the estimates of the components to be their true values then \( \rho_1 = 19.8 \) and \( \rho_2 = 20.7 \), thus, for purposes of computation, we shall consider \( \rho_1 = \rho_2 = 20 \). With the above information we may compute \( V_A = \text{var}(\hat{\sigma}_A^2)/2\sigma_C^4 \) as given by (6.4). The calculated value of \( V_A \) is \( V_A = 57.6 \). If the covariance between the sums of squares due to A and due to B is neglected, as Calvin and Miller do, the calculated value of \( V_A \) is \( V_A = 58.9 \). For this case the exclusion of the covariance term does not have much effect on \( V_A \). Although the covariance term had little effect in this particular case one should
be very careful for other situations, particularly for higher stage
classifications where more than one covariance may not be zero.

For purposes of comparison, $V_A$ was computed for $\rho_1 = \rho_2 = 20$ with
the assignment of degrees of freedom $(14, 15, 30)$, an assignment for
which a balanced design is available; $V_A = 76.6$.

In addition to the work concerning design, brief consideration was
given to a second method of estimation which involves using a weighted
linear function of mean squares. Anderson (1961) presents a weighted
least squares procedure and gives a numerical example. He finds the
variance of the weighted estimator to be very close to that of the
usual analysis of variance estimator. For the assignment of degrees
of freedom $(15, 16, 16)$ used with the Anderson design, we shall consider
the weighted estimator $\hat{\sigma}_A^2$ as given by (6.1).

The independent estimators $\hat{\sigma}_{A1}^2$ and $\hat{\sigma}_{A2}^2$ as given by (6.1) have
variances

$$\text{var}(\hat{\sigma}_{A1}^2)/2\sigma_C^4 = \frac{1}{16} \left[ \frac{(1 + 2\rho_2 + 4\rho_1)^2}{(n-1)/2} + \frac{(1 + 2\rho_2)^2}{n/2} \right]$$

$$\text{var}(\hat{\sigma}_{A2}^2)/2\sigma_C^4 = \frac{1}{4} \left[ \frac{(1 + \rho_2 + 2\rho_1)^2}{(n-2)/2} + \frac{(1 + \rho_2)^2}{n/2} \right].$$

For large $\rho_1$ and/or $\rho_2$, $\text{var}(\hat{\sigma}_{A1}^2)/2\sigma_C^4$ is approximately equal to
$\text{var}(\hat{\sigma}_{A2}^2)/2\sigma_C^4$. Therefore, as $\hat{\sigma}_{A1}^2$ and $\hat{\sigma}_{A2}^2$ are each based on the same
number of degrees of freedom the estimator suggested for large $\rho_1$ and/or
$\rho_2$ is

$$\frac{\hat{\sigma}_A^2}{\sigma_A} = (\hat{\sigma}_{A1}^2 + \hat{\sigma}_{A2}^2)/2.$$
Values of \( \text{var}(\hat{\sigma}_A^2) / \text{var}(\hat{\sigma}_A^2) \) for these combinations of \( \rho_1 \) and \( \rho_2 \) were given previously but are repeated in Table 6.12. Consideration of the results indicates that the weighted estimator is better than the usual analysis of variance estimator when either \( \rho_1 \) or \( \rho_2 \) is greater than four. However, when an advantage exists for the weighted estimator it is slight, never exceeding four per cent. As expected, the weighted estimator is quite poor for small \( \rho_1 \) and \( \rho_2 \). One should keep in mind that \( \hat{\sigma}_A^2 \) is based on one less degree of freedom than is \( \sigma_A^2 \).

Although this study of a weighted estimator was confined to one simple example, the results tend to support the use of the usual analysis of variance estimator for situations where the degrees of freedom are evenly distributed to the sources of variation.

Table 6.12. Values of \( \text{var}(\hat{\sigma}_A^2) / \text{var}(\hat{\sigma}_A^2) \) for the D_1-design with degrees of freedom: 15, 16, 16

<table>
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<th>( \rho_1 )</th>
<th>1/10</th>
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<th>1/2</th>
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<th>2</th>
<th>4</th>
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<td>.76</td>
<td>.80</td>
<td>.84</td>
<td>.90</td>
<td>.96</td>
<td>1.00</td>
<td>1.04</td>
</tr>
<tr>
<td>1/4</td>
<td>.80</td>
<td>.83</td>
<td>.86</td>
<td>.91</td>
<td>.96</td>
<td>1.00</td>
<td>1.04</td>
</tr>
<tr>
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<td>.86</td>
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<td>.89</td>
<td>.92</td>
<td>.97</td>
<td>1.00</td>
<td>1.04</td>
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<td>.95</td>
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<td>1.04</td>
</tr>
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<td>.97</td>
<td>.98</td>
<td>.99</td>
<td>1.00</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.01</td>
<td>1.03</td>
</tr>
<tr>
<td>10</td>
<td>1.02</td>
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7.0 SUMMARY AND CONCLUSIONS

7.1 The Problems

Two separate problems were considered in this dissertation:

(1) The importance of experimental design for the estimation of functions of variance components for a two stage nested classification when the objective is to economically reduce product variability.

(2) Comparison of a number of experimental designs for estimating parameters for a three stage nested classification.

7.2 The Influence of Experimental Design in Reducing Product Variability for a Two Stage Nested Classification

When a result from an experimental unit is the sum of two independent random parts, the variance of the result, the total variance, may be expressed as

$$\sigma_T^2 = \sigma_A^2 + \sigma_B^2$$

where $\sigma_A^2$ and $\sigma_B^2$ are the variance components associated with sources A and B, respectively.

The objective considered was that of reducing $\sigma_T^2$. It was assumed that a fixed amount of funds, say D, are available to expend on sources A and B in order to reduce $\sigma_T^2$. The model was proposed that related reduction in product variance to the amount of funds expended. Under the model it was assumed that the expenditure of one unit of funds on source B reduces $\sigma_B^2$ by one per cent and on source A reduces $\sigma_A^2$ by 100A per cent. With the above model, it was determined that the optimal allocation of funds is a function of $\rho = \frac{\sigma_A^2}{\sigma_B^2}$.
It was assumed that an experiment is to be performed to estimate \( \rho \) in a two-stage classification with \( N \) units assigned to \( a \) classes, based on a model of the form

\[
x_{ij} = m + A_i + B_{ij} \quad j = 1, 2, \ldots, n_i
\]

\[
\sum n_i = N
\]

where \( m \) is constant, \( A_i \) and \( B_{ij} \) are normally and independently distributed (NID) with means zero and respective variances \( \sigma_A^2 \) and \( \sigma_B^2 \). The type of experiment considered was such that, for \( N/a = p + s/a \) \( (0 \leq s < a) \) there are \( p+1 \) units assigned to each of \( s \) classes and \( p \) units to each of \( a-s \) classes. Under such an experimental setup with \( N \) fixed, different designs are obtained by varying the number of classes \( a \).

Two estimators of \( \rho \) were considered, the most important of which was

\[
\hat{\rho}_1 = (M_A - M_B)/K \cdot M_B
\]

where \( M_A \) is the source A mean square, \( M_B \) is the source B mean square, and \( K \) is the coefficient of \( \sigma_A^2 \) in the expectation of \( M_A \) in the usual analysis of variance.

The other estimator was introduced to determine the effect of not using the one degree of freedom that represents the contrast between classes with \( p+1 \) units per class and classes with \( p \) units per class.

As a criterion for judging the influence of design and estimator on the reduction of \( \sigma_T^2 \), the quantity considered was

\[
P = \text{Prob}[\sigma_T^2 | \hat{\alpha}_1] \leq \beta,
\]
where $\sigma^2_R | \hat{d}_1$ is the reduced total variance, divided by $\sigma^2_B$, that is attained after $\hat{d}_1$ units of funds have been expended on source A and $D - \hat{d}_1$ on source B, and $\beta$ is a fixed number in the interval $\min(\sigma^2_R | \hat{d}_1), \max(\sigma^2_R | \hat{d}_1)$. It was found that $P$ could be expressed approximately in the form of an incomplete beta function whose parameters and limits of integration are functions of $A$, $\rho$, $D$, $N$ and $a$. It was then noted that the influence of design and estimator could be ascertained by specifying $A$, $\rho$, $D$, and $N$, and evaluating the appropriate incomplete beta function for various values of $a$.

To aid in appreciating the influence of design and estimator some numerical results were presented. The results indicated that there is a significant decline in $P$ only for designs deviating quite severely from the optimal. For most situations an intermediate value of $a$, say $N/4 \leq a \leq N/2$, will give results that are quite close to the optimal.

Comparison of the results from the two different estimators indicated the values of $P$ from the two estimators to be very close.

### 7.3 Designs for Estimating Parameters for a Three Stage Nested Classification

The mathematical model considered here was

$$x_{ijk} = m + A_i + B_{ij} + C_{ijk}$$

$$i = 1, 2, \ldots, a$$

$$j = 1, 2, \ldots, b_i$$

$$k = 1, 2, \ldots, n_{ij}$$

$$\Sigma n_{ij} = n_i, \Sigma \Sigma n_{ij} = N$$

where $m$ is constant, $A_i$, $B_{ij}$, and $C_{ijk}$ are all NID with means zero and
respective variances, $\hat{\sigma}^2_A$, $\hat{\sigma}^2_B$, and $\hat{\sigma}^2_C$. Emphasis was placed on design rather than on the method of estimation. The estimators used were those that are obtained from the usual analysis of variance by equating mean squares to their expectations.

Variances of the estimated components $\hat{\sigma}^2_A$, $\hat{\sigma}^2_B$, and $\hat{\sigma}^2_C$ were studied for various designs. It was found that the expression for $\text{var}(\hat{\sigma}^2_B)$ was complex and that for $\text{var}(\hat{\sigma}^2_A)$ was very complex.

Designs were presented for the estimation of $m$, $\hat{\sigma}^2_C$, and $\sigma^2 = \hat{\sigma}^2_C + \hat{\sigma}^2_B + \hat{\sigma}^2_A$ that minimized the variances of the estimates. Designs were also given for the estimation of $\hat{\sigma}^2_B$, $\hat{\sigma}^2_A$, and $\rho = \hat{\sigma}^2_A/(\hat{\sigma}^2_C + \hat{\sigma}^2_B)$ that appeared to yield estimates whose variances were near minimal. The designs turned out to be analogous to those given by Crump (1954) for the estimation of components for a two stage nested classification.

Some attention was also devoted to the simultaneous estimation of $\hat{\sigma}^2_C$, $\hat{\sigma}^2_B$, and $\hat{\sigma}^2_A$. A method of design referred to as a $D_2$-design was suggested such that

$$|n_i - n_j| = 0 \text{ or } 1$$

$$|n_{ij} - n_{jm}| = 0 \text{ or } 1$$

whenever the number of B-classes, $b$, is greater than or equal to $N/2$, i.e., $b \geq N/2$. For the assignment of degrees of freedom n-1, n, n this design turned out to be the same as that presented by Calvin and Miller (1961). To learn how the $D_2$-design compares with other designs some numerical examples were considered. For each of seven assignments of degrees of freedom with $N = 48$ one additional design was considered.
along with the $D_2$-design. Where it was possible this additional design was the type of design suggested by Anderson and Bancroft (1952) and Anderson (1960). For the balanced assignment $(11,12,24)$, the quantities $\text{var}(\hat{\sigma}_A^2)/\sigma_C^4$, $\text{var}(\hat{\sigma}_B^2)/\sigma_C^4$ and $\text{var}(\hat{\sigma}_C^2)/\sigma_C^4$ were computed over a range of values of $\rho_1 = \sigma_A^2/\sigma_C^2$ and $\rho_2 = \sigma_B^2/\sigma_C^2$. For the seven unbalanced assignments the quantities

$$E_A = \frac{\text{var}(\hat{\sigma}_A^2)\text{ for the assignment in question}}{\text{var}(\hat{\sigma}_A^2)\text{ for } (11,12,24)}$$

$$E_B = \frac{\text{var}(\hat{\sigma}_B^2)\text{ for the assignment in question}}{\text{var}(\hat{\sigma}_B^2)\text{ for } (11,12,24)}$$

$$E_C = \frac{\text{var}(\hat{\sigma}_C^2)\text{ for the assignment in question}}{\text{var}(\hat{\sigma}_C^2)\text{ for } (11,12,24)}$$

were computed.

As expected, it was noted that assignment of degrees of freedom has a large effect on the variances of the estimated variance components. For those assignments of degrees of freedom where an Anderson design was possible, it was observed that the relative merit of the $D_2$-design depended upon the parameters $\rho_1$ and $\rho_2$. For some combinations of $\rho_1$ and $\rho_2$, the Anderson design was better than the $D_2$-design and for other combinations the $D_2$-design was better.

The results were summarized by suggesting assignments and designs for the three situations:

$$\rho_1 > 1, \rho_2 > 1$$

$$\rho_1 \leq 1, \rho_2 > 1$$

and

$$\rho_1 > 1, \rho_2 \leq 1.$$
7.4 Suggested Future Research

(1) Consideration of methods of estimation other than those based on the usual analysis of variance. In particular, weighted least squares and maximum-likelihood estimators might be compared with the usual analysis of variance estimators.

(2) Consideration of additional objectives for which the estimated functions of the variance components are to be used.

(3) Extending to the multi-stage nested classification the concept of reducing product variability through the use of variance components.

(4) Consideration of sequential estimation procedures, using results from the first stage to construct designs for later stages.

(5) Further study of methods of design for the simultaneous estimation of variance components for multi-stage classifications, with special attention to the development of realistic criteria for optimization.
LIST OF REFERENCES


